Graded Decagon of Opposition with Fuzzy Quantifier-Based Concept-Forming Operators

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Abstract. We introduce twelve operators called *fuzzy quantifier-based operators*. They are proposed as a new tool to help to deepen the analysis of data in fuzzy formal concept analysis. Moreover, we employ them to construct a graded extension of Aristotle's square, namely the *graded decagon of opposition*.

Keywords: Fuzzy formal concept analysis · Evaluative linguistic expressions · Square of opposition · Łukasiewicz MV-algebra

1 Introduction

Formal Concept Analysis (FCA) is a mathematical theory applied to the analysis of data (see [6]). The input of FCA is a triple called *formal context* that consists of a set of objects, a set of attributes, and a binary relation between objects and attributes. FCA techniques extract a collection of *formal concepts* from every formal context.

Formal concepts are special clusters that correspond to concepts such as "numbers divisible by 5", or "white roses in the garden". *Fuzzy Formal Concept Analysis* (FFCA) generalizes formal concept analysis to include also vague information. The input of FFCA is an L-context (X, Y, I) where L is a support of an algebra of truth values, X is a set of objects, Y a set of attributes, and I is a fuzzy relation $I : X \times Y \longrightarrow L$.

A *fuzzy concept* is a pair (A, B) where A, B are fuzzy sets $A : X \longrightarrow L$, $B : Y \longrightarrow L$. A is called *extent* and it is a fuzzy set of all objects $x \in X$ that have all attributes of B, and B is called *intent* and it is a fuzzy set of all attributes $y \in Y$ being satisfied by all objects of A. Namely, $A(x)$ is the degree to which "x has all attributes of B", and $B(y)$ is the degree to which "the attribute y is satisfied by all objects of A".

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In this article, we define twelve special operators as a tool to deepen the analysis of data. To explain their function, let us consider the following situation.

Let (X, Y, I) be an L-context, where X is a set of students, Y are their skills, $I(x, y)$ is the degree to which "a student x has the skill y". Thus, given a fuzzy concept (A, B), we know that A is the fuzzy set representing all students with all skills in B.

Let us now ask, how many students share "*almost all* skills in B" ("*most* skills in B", or "*many* skills in B"). Additionally, we may need to classify students with respect to how many skills of B they *do not have* and exactly, to consider the following fuzzy sets of X: students sharing "*few* skills in B", students who do *not* have "*most* skills in B", or students "do *not* have *many* skills in B". Similarly, we can also consider a fuzzy set of Y formed of all skills shared by "almost all" ("most", "many", or "few") students of A, and the fuzzy set of Y made of all skills that are *not shared* by "most" (or "many") students in A. Each of the previous sets is generated by a fuzzy quantifier-based concept-forming operator, that allows us to introduce an extended notion of fuzzy concept.

Fuzzy quantifier-based operators are defined taking into account expressions of natural language *extremely big*, *very big*, and *not small* that are formalized within the *theory of evaluative linguistic expressions* [8]. Finally, starting from the Łukasiewicz MV-algebra, we employ the fuzzy quantifier-based operators to represent a graded decagon of opposition, which is a graded extension of *Aristotle's square* (see Fig. 1).

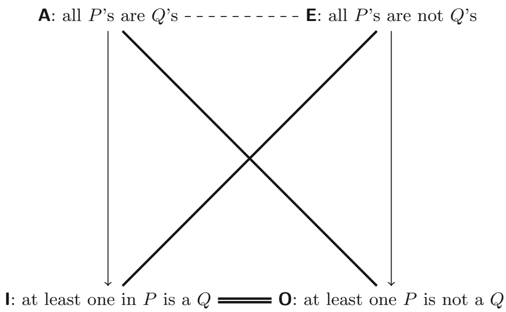

Fig. 1. Aristotle's square. The lines ━━, ══, ⟶, - - - denote that the corresponding propositions are contradictories, sub-contraries, sub-alterns, and contraries, respectively.

The article is organized as follows. Section 2 reviews some basic notions and results regarding MV-algebras, fuzzy formal concept analysis, and the graded square of opposition. Section 3 introduces the fuzzy quantifier-based operators and the corresponding new notions of fuzzy concepts. In Sect. 4, we construct a graded decagon of opposition using the former. Finally, in the last section we discuss further possible development of our results.

2 Preliminaries

This section describes some fundamental notions and results regarding MV-algebras, fuzzy formal concept analysis, and the graded square of opposition.

2.1 MV-Algebras

Definition 1. *A lattice $\langle L, \vee, \wedge\rangle$ is* complete *if and only if all subsets of L have both supremum and infimum.*

Definition 2. *A* residuated lattice *is an algebra $\langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1}\rangle$ where*

(i) $\langle L, \wedge, \vee, \mathbf{0}, \mathbf{1}\rangle$ is a bounded lattice,
(ii) $\langle L, \otimes, \mathbf{1}\rangle$ is a commutative monoid, and
(iii) $a \otimes b \leq c$ iff $a \leq b \rightarrow c$, for all $a, b, c \in L$ (adjunction property).

Definition 3 ([3,11]). *An* MV-algebra *is a residuated lattice*

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1}\rangle$$

where $a \vee b = (a \rightarrow b) \rightarrow b$, for each $a, b \in L$. We will also work with the following additional operations on L:

(i) $\neg a = a \rightarrow \mathbf{0}$ (negation),
(ii) $a \oplus b = \neg(\neg a \otimes \neg b)$ (strong disjunction),
(iii) $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$ (biresiduation).

Example 1. A special MV-algebra is the *standard Łukasiewicz MV-algebra*

$$\mathcal{L}_{Ł} = \langle [0,1], \vee, \wedge, \otimes, \rightarrow, 0, 1\rangle$$

where $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$, $a \otimes b = \max(0, a + b - 1)$ and $a \rightarrow b = \min(1, 1 - a + b)$, $\neg a = 1 - a$ and $a \oplus b = \min\{1, a + b\}$, for all $a, b \in L$.

In the following lemma, we list some properties of complete MV-algebras[1] that will be used below.

Lemma 1. *Let $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1}\rangle$ be a complete MV-algebra. Then the following holds for all $a, b, c, d, e \in L$:*

(a) If $a \leq b$ and $c \leq d$, then $a \wedge c \leq b \wedge d$.
(b) Let I be any index set. Then for each $k \in I$, $\bigwedge_{i \in I} a_i \leq a_k$ and $a_k \leq \bigvee_{i \in I} a_i$.
(c) If $a_i \leq b_i$ for each $i \in I$, then $\bigvee_{i \in I} a_i \leq \bigvee_{i \in I} b_i$.
(d) $a \oplus \neg a = \mathbf{1}$ and $a \otimes \neg a = \mathbf{0}$.
(e) If $a \otimes b \leq e$, then $(a \wedge c) \otimes (b \wedge d) \leq e$.
(f) If $a \leq b$ and $c \leq d$, then $a \otimes c \leq b \otimes d$ and $a \oplus c \leq b \oplus d$.

[1] More generally, the properties (a), (b) and (c) hold in any complete lattice.

2.2 Fuzzy Formal Concept Analysis

In this subsection, we recall the definition of two pairs of fuzzy concept-forming operators ($^{\uparrow}$, $^{\downarrow}$), and ($^{\cap}$, $^{\cup}$) existing in literature. Given a complete residuated lattice L, by a fuzzy set of the universe X we mean a function $A : X \longrightarrow L$. If A is a fuzzy set on X, then we write $A \underset{\sim}{\subset} X$. For each $A, B \underset{\sim}{\subset} X$, we put $\mathcal{S}_X(A, B) = \bigwedge_{x\in X}(A(x) \to B(x))$, which represents the degree of inclusion of A in B[2].

Definition 4 ([1,12]). *Let* (X, Y, I) *be an* L*-context and* $A \underset{\sim}{\subset} X$, $B \underset{\sim}{\subset} Y$. *We put*

$$A^{\uparrow}(y) = \bigwedge_{x\in X} (A(x) \to I(x, y)) \quad \textit{and} \quad B^{\downarrow}(x) = \bigwedge_{y\in Y} (B(y) \to I(x, y)),$$

for all $x \in X$ *and* $y \in Y$.

The $A^{\uparrow}(y)$ and $B^{\downarrow}(x)$ correspond to the truth degrees of the statements "an attribute y is shared by *all* objects of A" and "an object x has *all* attributes of B", respectively.

Definition 5 ([13]). *Let* (X, Y, I) *be an* L*-context. If* $A \underset{\sim}{\subset} X$ *and* $B \underset{\sim}{\subset} Y$, *then*

$$A^{\cap}(y) = \bigvee_{x\in X} (A(x) \otimes I(x, y)) \quad \textit{and} \quad B^{\cup}(x) = \bigwedge_{y\in Y} (I(x, y) \to B(y)),$$

for all $x \in X$ *and* $y \in Y$.

The operators $^{\cap}$ and $^{\cup}$ are borrowed from the rough set theory. Namely, $A^{\cap}(y)$ and $B^{\cup}(x)$ correspond to the truth degrees of the statements "an attribute y is shared by at least one object of A" and "an object x has no attributes outside B", respectively.

Each pair $(A, B) \in L^X \times L^Y$ such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$ is called *standard* L*-concept*. Analogously, each pair $(A, B) \in L^X \times L^Y$ such that $A^{\cap} = B$ and $B^{\cup} = A$ is called *property-oriented* L*-concept*.

Theorem 1. *The pair of mappings* $^{\uparrow} : L^X \to L^Y$ *and* $^{\downarrow} : L^Y \to L^X$ *forms an* antitone Galois connection *between* X *and* Y, *i.e.* $\mathcal{S}_X(A, B^{\downarrow}) = \mathcal{S}_Y(B, A^{\uparrow})$, *for each* $A \underset{\sim}{\subset} X$ *and* $B \underset{\sim}{\subset} Y$.

Theorem 2. *The pair of mappings* $^{\cap} : L^X \to L^Y$ *and* $^{\cup} : L^Y \to L^X$ *forms an* isotone Galois connection *between* X *and* Y, *i.e.* $\mathcal{S}_X(A, B^{\cup}) = \mathcal{S}_Y(A^{\cap}, B)$, *for each* $A \underset{\sim}{\subset} X$ *and* $B \underset{\sim}{\subset} Y$.

Definition 6. *Given a set* X *and a complete residuated lattice* L, *by a* fuzzy preposet *we mean a pair* $(X, \mathcal{R})$ *where* $\mathcal{R}$ *is a fuzzy relation on* X *that is reflexive, i.e.* $\mathcal{R}(x, x) = 1$ *for each* $x \in X$, *and* $\otimes$*-transitive, i.e.* $\mathcal{R}(x, y) \otimes \mathcal{R}(y, z) \leq \mathcal{R}(x, z)$, *for each* $x, y, z \in X$.

[2] Note that this formula is interpretation of the logical formula $(\forall x)(A(x) \Rightarrow B(x))$ defining classical inclusion between (fuzzy) sets in a model of fuzzy predicate logic.

2.3 Graded Square of Opposition and Fuzzy Concept-Forming Operators

In this subsection, we define graded square of opposition referring to [5], and we enunciate a theorem that shows how this square can be obtained using the fuzzy concept-forming operators introduced in Subsect. 2.2.

Definition 7. *Let P_A and P_B be properties represented by $A, B \subsetapprox X$, then we say that*

1. *P_A and P_B are* contraries *if and only if $A(x) \otimes B(x) = 0$ for each $x \in X$,*
2. *P_A and P_B are* sub-contraries *if and only if $A(x) \oplus B(x) = 1$ for each $x \in X$,*
3. *P_A and P_B are* sub-alterns *if and only if $A(x) \rightarrow B(x) = 1$ for each $x \in X$,*
4. *P_A and P_B are* contradictories *if and only if $A(x) = \neg B(x)$ for each $x \in X$.*

Definition 8. *In a* graded square of opposition *the vertices* **A**, **E**, **I**, *and* **O** *are fuzzy sets representing the propositions $P_{\mathbf{A}}$, $P_{\mathbf{E}}$, $P_{\mathbf{I}}$, and $P_{\mathbf{O}}$ such that the following conditions hold:*

1. *$P_{\mathbf{A}}$ and $P_{\mathbf{E}}$ are* contraries*;*
2. *$P_{\mathbf{I}}$ and $P_{\mathbf{O}}$ are* sub-contraries*;*
3. *$P_{\mathbf{A}}$ and $P_{\mathbf{I}}$ are* sub-alterns*, as well as $P_{\mathbf{E}}$ and $P_{\mathbf{O}}$;*
4. *$P_{\mathbf{A}}$ and $P_{\mathbf{O}}$ are* contradictories*, as well as $P_{\mathbf{E}}$ and $P_{\mathbf{I}}$.*

From now, given the L-contexts (X, Y, I), we suppose that L is the Łukasiewicz MV-algebra, because we will need the *double negation law*, i.e. $\neg\neg a = a$ for each $a \in L$. Moreover, we put $(\neg I)(x, y) = \neg I(x, y)$. In the standard Łukasiewicz algebra, $\neg I(x, y) = 1 - I(x, y)$, for all $x \in X$ and $y \in Y$.

This lemma follows from the results found in [5].

Lemma 2. *Let $A \subsetapprox X$ be a normal fuzzy set[3], then*

1. *$A_I^{\uparrow}(y) \otimes A_{\neg I}^{\uparrow}(y) = 0$,*
2. *$A_I^{\cap}(y) \oplus A_{\neg I}^{\cap}(y) = 1$,*
3. *$A_I^{\uparrow}(y) \leq A^{\cap_I}(y)$, and $A_{\neg I}^{\uparrow}(y) \leq A_{\neg I}^{\cap}(y)$,*
4. *$\neg A_I^{\uparrow}(y) = A_{\neg I}^{\cap}(y)$, and $\neg A_I^{\cap}(y) = A_{\neg I}^{\uparrow}(y)$,*

for each $y \in Y$.

Theorem 3. *Let $A \subsetapprox X$. If A is normal, then $A_I^{\uparrow}$, $A_{\neg I}^{\uparrow}$, $A_I^{\cap}$ and $A_{\neg I}^{\cap}$ are the vertices of a graded square of opposition, and they represent proprieties that are in relation of contrary, sub-contrary, sub-altern, and contradictory as shown in Fig. 2.*

Observe that we obtain the graded square of opposition defined in [4] when fixing $y \in Y$.

Example 2. Let (X, Y, I) be an L-context, where $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3, y_4\}$, and $I(x_1, y_1) = 0.25$, $I(x_2, y_1) = 0.6$, $I(x_3, y_1) = 1$, $I(x_4, y_1) = 0.25$. The graded square of opposition associated to $A = \{x_1, 0.5/x_2, 0.6/x_3, 0.5/x_4\}$ and y_1 is depicted in Fig. 3.

[3] There exists $x \in X$ such that $A(x) = 1$.

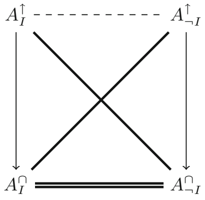

Fig. 2. Graded square of opposition

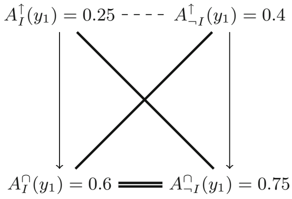

Fig. 3. Example of graded square of opposition

3 Fuzzy Quantifier-Based Operators

In this section, we introduce the *fuzzy quantifier based-operators* extending the notion of fuzzy concept. Our theory is based on the theory of *intermediate quantifiers* presented in [7,9] and elsewhere. The theory is based on the concept of *evaluative linguistic expression*. These are expressions of natural language such as "small, very big, rather medium", etc. In this paper we confine only to "not small", "very big" and "extremely big" and use a simplified model in which we consider only extensions in the (linguistic) context $\langle 0, 0.5, 1 \rangle$[4] that are fuzzy sets $\mathsf{BiEx}, \mathsf{BiVe}, \neg\mathsf{Sm}\nu$ depicted in Fig. 4. For justification of this model, see [8,10].

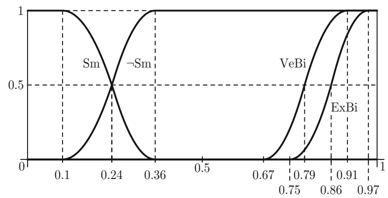

Fig. 4. Shapes of the fuzzy sets BiEx, $\mathsf{BiVe}, \neg\mathsf{Sm}\nu$.

Remark 1. It is clear that $\mathsf{BiEx}(x) \leq \mathsf{BiVe}(x) \leq \neg\mathsf{Sm}\nu(x)$ holds for all $x \in [0, 1]$.

The *cardinality* of $A \underset{\sim}{\subset} X$ is defined by $|A| = \sum_{x \in X} A(x)$. Furthermore, given $A, B \underset{\sim}{\subset} X$, we consider the following measure that expresses how large the size of A is w.r.t. the size of B (see [9])

$$\mu_B(A) = \begin{cases} 1 & \text{if } B = \emptyset \text{ or } A = B, \\ \frac{|A|}{|B|} & \text{if } B \neq \emptyset \text{ and } A \subseteq B, \\ 0 & \text{otherwise.} \end{cases}$$

For our further reasoning, we need a special operation called *cut of a fuzzy set*. It is motivated by the need to form a new fuzzy set from a given one by extracting several elements together with their membership degrees and putting the other membership degrees equal to 0.

[4] By a linguistic context for evaluative expressions, we understand a triple of numbers $\langle v_L, v_S, v_R \rangle$ that determines an interval $[v_L, v_S] \cup [v_S, v_R]$ in which all values range. For the more detailed explanation, see [10].

Definition 9 ([7]). *Let $A, B \underset{\sim}{\subset} X$. The* cut of A with respect to B *is the fuzzy set*

$$(A|B)(x) = \begin{cases} A(x) & \text{if } A(x) = B(x), \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Now, we give the definition of positive and negative fuzzy quantifier based-operators that are based on the relation I, and on the functions $\neg\mathsf{Smv}, \mathsf{BiVe}$ and BiEx. Our aim is to capture positive, or negative information in (X, Y, I).

Definition 10 (Fuzzy quantifier-based operators). *Let us consider an L-context (X, Y, I), $A \underset{\sim}{\subset} X$, $B \underset{\sim}{\subset} Y$, $x \in X$, and $y \in Y$. Let $Ev \in \{\neg\mathsf{Smv}, \mathsf{BiVe}, \mathsf{BiEx}\}$. Then we put:*

(i) Positive fuzzy quantifier based-operators

$$A^{\uparrow}_{I,Ev}(y) = \bigvee_{Z \underset{\sim}{\subset} X} \big(\bigwedge_{x \in X} ((A|Z)(x) \rightarrow I(x, y)) \wedge Ev(\mu_A(A|Z))\big), \quad (2)$$

and

$$B^{\downarrow}_{I,Ev}(x) = \bigvee_{Z \underset{\sim}{\subset} Y} \big(\bigwedge_{y \in Y} ((B|Z)(y) \rightarrow I(x, y)) \wedge Ev(\mu_B(B|Z))\big), \quad (3)$$

(ii) Negative fuzzy quantifier based-operators

$$A^{\uparrow}_{\neg I,Ev}(y) = \bigvee_{Z \underset{\sim}{\subset} X} \big(\bigwedge_{x \in X} ((A|Z)(x) \rightarrow \neg I(x, y)) \wedge Ev(\mu_A(A|Z))\big), \quad (4)$$

and

$$B^{\downarrow}_{\neg I,Ev}(x) = \bigvee_{Z \underset{\sim}{\subset} Y} \big(\bigwedge_{y \in Y} ((B|Z)(y) \rightarrow \neg I(x, y)) \wedge Ev(\mu_B(B|Z))\big), \quad (5)$$

Informal explanation of the formulas in Definition 10 is the following:

(i) $A^{\uparrow}_{I,Ev}(y)$ is the truth degree to which *there exists a cut of A such that "all its objects have the attribute y" and "its size is Ev (not small, very big or extremely big) w.r.t. the size of A"*. Analogous statement holds for $B^{\downarrow}_{I,Ev}(y)$.

(ii) $A^{\uparrow}_{\neg I,Ev}(x)$ is the truth degree to which *there exists a cut of A such that "all its objects do not have the attribute y" and "its size is Ev (not small, very big or extremely big) w.r.t. the size of A"*. Analogous statement holds for and $B^{\downarrow}_{\neg I,Ev}(y)$.

Remark 2. (a) If $Z \underset{\sim}{\subset} X$ and $y \in Y$, then $\bigwedge_{x \in X}((A|Z)(x) \rightarrow I(x, y)) = (A|Z)^{\uparrow}_{I}(y)$ and $\bigwedge_{x \in X}((A|Z)(x) \rightarrow \neg I(x, y)) = (A|Z)^{\uparrow}_{\neg I}(y)$.

(b) If $Z \underset{\sim}{\subset} Y$ and $x \in X$, then $\bigwedge_{y \in Y}((B|Z)(y) \rightarrow I(x, y)) = (B|Z)^{\downarrow}_{I}(y)$ and $\bigwedge_{y \in Y}((B|Z)(y) \rightarrow \neg I(x, y)) = (B|Z)^{\downarrow}_{\neg I}(x)$.

Since BiEx, BiVe and $\neg\mathsf{Smv}$ lay behind the definition of the intermediate quantifiers *almost*, *most* and *many* (cf. [9]), formulas $A^{\uparrow}_{I,Ev}(y)$, $A^{\uparrow}_{\neg I,Ev}(y)$, $B^{\downarrow}_{I,Ev}(x)$ and $B^{\downarrow}_{\neg I,Ev}(x)$ can be understood as interpretation of the linguistic expressions summarized in Table 1.

Table 1. Verbal description of the fuzzy quantifier-based operators

Truth degree	Statement
$A^{\uparrow}_{I,\mathsf{BiEx}}(y)$	y is shared by *almost all* objects of A
$B^{\downarrow}_{I,\mathsf{BiEx}}(x)$	x has *almost all* attributes of B
$A^{\uparrow}_{I,\mathsf{BiVe}}(y)$	y is shared by *most* objects of A
$B^{\downarrow}_{I,\mathsf{BiVe}}(x)$	x has *most* attributes of B
$A^{\uparrow}_{I,\neg\mathsf{Smv}}(y)$	y is shared by *many* objects of A
$B^{\downarrow}_{I,\neg\mathsf{Smv}}(x)$	x has *many* attributes of B
$A^{\uparrow}_{\neg I,\mathsf{BiEx}}(y)$	y is shared by *few* objects of A
$B^{\downarrow}_{\neg I,\mathsf{BiEx}}(x)$	x has *few* attributes of B
$A^{\uparrow}_{\neg I,\mathsf{BiVe}}(y)$	y is not shared by *most* objects of A
$B^{\downarrow}_{\neg I,\mathsf{BiVe}}(x)$	*most* attributes of B are not satisfied by x
$A^{\uparrow}_{\neg I,\neg\mathsf{Smv}}(y)$	y is not shared by *many* objects of A
$B^{\downarrow}_{\neg I,\neg\mathsf{Smv}}(x)$	*many* attributes of B are not satisfied by x

In the sequel, new notions of fuzzy concepts are introduced considering additional information generated by the fuzzy-quantifier-based operators.

Definition 11. *Let* $Ev \in \{\neg\mathsf{Smv}, \mathsf{BiVe}, \mathsf{BiEx}\}$ *and* $H \in \{I, \neg I\}$*. For each* $A, \tilde{A} \underset{\sim}{\subset} X$*, and* $B, \tilde{B} \underset{\sim}{\subset} Y$*, we set*

(i) $A^{\Uparrow}_{H,Ev} = (A^{\uparrow}_{H}, A^{\uparrow}_{H,Ev})$ *and* $(B, \tilde{B})^{\Downarrow}_{H,Ev} = B^{\downarrow}_{H}$*,*
(ii) $(A, \tilde{A})^{\triangle}_{H,Ev} = A^{\uparrow}_{H}$ *and* $B^{\triangledown}_{H,Ev} = (B^{\downarrow}_{H}, B^{\downarrow}_{H,Ev})$*.*

Definition 12 (Extended fuzzy concepts). *Let* $Ev \in \{\neg\mathsf{Smv}, \mathsf{BiVe}, \mathsf{BiEx}\}$, $A, \tilde{A} \underset{\sim}{\subset} X$*, and* $B, \tilde{B} \underset{\sim}{\subset} Y$*. Then, we say that*

(i) $(A, (B, \tilde{B}))$ *is a* positive concept with Ev-attributes *if and only if* $A = (B, \tilde{B})^{\Downarrow}_{I,Ev}$ *and* $(B, \tilde{B}) = A^{\Uparrow}_{I,Ev}$*.*
(ii) $(A, (B, \tilde{B}))$ *is a* negative concept with Ev-attributes *if and only if* $A = (B, \tilde{B})^{\Downarrow}_{\neg I,Ev}$ *and* $(B, \tilde{B}) = A^{\Uparrow}_{\neg I,Ev}$*.*
(iii) $((A, \tilde{A}), B)$ *is a* positive concept with Ev-objects *if and only if* $(A, \tilde{A}) = B^{\triangledown}_{I,Ev}$ *and* $B = (A, \tilde{A})^{\triangle}_{I,Ev}$*.*
(iv) $((A, \tilde{A}), B)$ *is a* negative concept with Ev-objects *if and only if if* $(A, \tilde{A}) = B^{\triangledown}_{\neg I,Ev}$ *and* $B = (A, \tilde{A})^{\triangle}_{\neg I,Ev}$*.*

The following theorems state that the pairs of operators given by Definition 11 are both Galois connections between fuzzy preposets (see Definition 6). Given a set X, for each $A, B, C, D \subsetsim X$, we set

$$\mathcal{R}_X((A,B),(C,D)) = \mathcal{S}_X(C,A). \tag{6}$$

Theorem 4. *Let $Ev \in \{\neg\mathsf{Smv}, \mathsf{BiVe}, \mathsf{BiEx}\}$ and $H \in \{I, \neg I\}$. Then,*

(a) the pair of mappings $^{\Uparrow}_{H,Ev} : L^X \to L^Y \times L^Y$ and $^{\Downarrow}_{H,Ev} : L^Y \times L^Y \to L^X$ is a Galois connection between the fuzzy preposets $(L^X, \mathcal{S}_X)$ and $(L^Y \times L^Y, \mathcal{R}_Y)$, i.e. $\mathcal{S}_X(A, (B,\tilde{B})^{\Downarrow}_{H,Ev}) = \mathcal{R}_Y(A^{\Uparrow}_{H,Ev}, (B,\tilde{B}))$ for each $A \subsetsim X$ and $B, \tilde{B} \subsetsim Y$,
(b) the pair of mappings $^{\vartriangle}_{H,Ev} : L^Y \times L^Y \to L^X$ and $^{\triangledown}_{H,Ev} : L^X \to L^Y \times L^Y$ is a Galois connection between the fuzzy preposets $(L^Y \times L^Y, \mathcal{R}_Y)$ and $(L^X, \mathcal{S}_X)$, i.e. $\mathcal{R}_Y((A,\tilde{A}), B^{\triangledown}_{H,Ev}) = \mathcal{S}_X((A,\tilde{A})^{\vartriangle}_{H,Ev}, B)$ for each $A, \tilde{A} \subsetsim X$ and $B \subsetsim Y$.

Proof. We prove only item (a), because item (b) can be proved analogously.

Let $A \subsetsim X$, and $B, \tilde{B} \subsetsim Y$. By Definition 11(i), $\mathcal{S}_X(A, (B,\tilde{B})^{\Downarrow}_{H,Ev}) = \mathcal{S}_X(A, B^{\downarrow}_H)$. Moreover, by Theorem 1, we know that $\mathcal{S}_X(A, B^{\downarrow}_H) = \bigwedge_{x\in X}(A(x) \to B^{\downarrow}_H(x))$ is equal to $\mathcal{S}_Y(B, A^{\uparrow}_H) = \bigwedge_{y\in Y}(B(y) \to A^{\uparrow}_H(y))$. Eventually, by (6), $\mathcal{S}_Y(B, A^{\uparrow}_H) = \mathcal{R}_Y(A^{\Uparrow}_{H,Ev}, (B,\tilde{B}))$. Then, we conclude that $\mathcal{S}_X(A, (B,\tilde{B})^{\Downarrow}_{H,Ev}) = \mathcal{R}_Y(A^{\Uparrow}_{H,Ev}, (B,\tilde{B}))$. □

4 Graded Decagon of Opposition with Fuzzy Quantifier-Based Operators

In this section, we introduce the definition of graded decagon of opposition, which is a generalization of the graded square of opposition given in Definition 8. Moreover, we construct a graded decagon of opposition using some fuzzy quantifier-based operators.

Definition 13 (Graded decagon of opposition). *A graded decagon of opposition consists of vertices $A_1, \dots, A_5 \subsetsim X$, and $N_1, \dots, N_5 \subsetsim X$ representing the propositions $P_{A_1}, \dots, P_{A_5}$, $P_{N_1}, \dots, P_{N_5}$ such that:*

1. *P_{A_i} and P_{N_j} are* contraries*, for each $i, j \in \{1 \dots, 4\}$,*
2. *P_{A_5} and P_{N_5} are* sub-contraries*,*
3. *P_{A_i} and $P_{A_{i+1}}$ are* sub-alterns*, as well as P_{N_i} and $P_{N_{i+1}}$, for each $i \in \{1, \dots, 4\}$,*
4. *P_{A_1} and P_{N_5} are* contradictories*, as well as P_{A_5} and P_{N_1}.*

The graded decagon of opposition is depicted in Fig. 5.

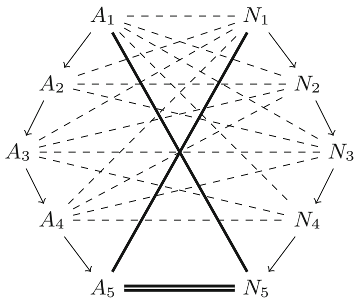

Fig. 5. Graded decagon of opposition

In the sequel, we prove a few lemmas in order to construct a graded decagon of opposition with the fuzzy quantifier-based operators.

Lemma 3. *For each* $A \underset{\sim}{\subset} X$ *and* $y \in Y$*, the following properties hold:*

(a) $A^{\uparrow}_{I}(y) \leq A^{\uparrow}_{I,\mathsf{BiEx}}(y) \leq A^{\uparrow}_{I,\mathsf{BiVe}}(y) \leq A^{\uparrow}_{I,\neg\mathsf{Smv}}(y)$,
(b) $A^{\uparrow}_{\neg I}(y) \leq A^{\uparrow}_{\neg I,\mathsf{BiEx}}(y) \leq A^{\uparrow}_{\neg I,\mathsf{BiVe}}(y) \leq A^{\uparrow}_{\neg I,\neg\mathsf{Smv}}(y)$.

Proof. We give the proof of item (a) only. The proof of item (b) is analogous.

Let $Ev \in \{\neg\mathsf{Smv}, \mathsf{BiVe}, \mathsf{BiEx}\}$. Trivially, $A^{\uparrow}_{I}(y) = (A|A)^{\uparrow}_{I}(y) \wedge Ev(\mu_A(A|A))$. By Lemma 1(b),

$$(A|A)^{\uparrow}_{I}(y) \wedge Ev(\mu_A(A|A)) \leq \bigvee_{Z \underset{\sim}{\subset} X} ((A|Z)^{\uparrow}_{I}(y) \wedge Ev(\mu_A(A|Z))),$$

namely $A^{\uparrow}_{I}(y) \leq A^{\uparrow}_{I,Ev}(y)$. By Remark 1, for each $Z \underset{\sim}{\subset} X$,

$$\mathsf{BiEx}(\mu_A(A|Z)) \leq \mathsf{BiVe}(\mu_A(A|Z)) \leq \neg\mathsf{Smv}(\mu_A(A|Z)).$$

Consequently, by Lemma 1(a),

$$(A|Z)^{\uparrow}_{I}(y) \wedge \mathsf{BiEx}(\mu_A(A|Z)) \leq (A|Z)^{\uparrow}_{I}(y) \wedge \mathsf{BiVe}(\mu_A(A|Z)) \leq (A|Z)^{\uparrow}_{I}(y) \wedge \neg\mathsf{Smv}(\mu_A(A|Z)).$$

Finally, by Lemma 1(c), $A^{\uparrow}_{I,\mathsf{BiEx}}(y) \leq A^{\uparrow}_{I,\mathsf{BiVe}}(y) \leq A^{\uparrow}_{I,\neg\mathsf{Smv}}(y)$. □

In some relations, it is necessary to add the assumption that the fuzzy set in concern is non-empty. In classical logic, we add the formula $(\exists x)A(x)$ that assures us that "there exists at least one element x" and speak about *existential import* (or presupposition). In fuzzy logic, the quantifier $\exists$ is interpreted by supremum. This leads us to the following definition.

Definition 14. *Let* $A \underset{\sim}{\subset} X$*,* $y \in Y$*,* $Ev \in \{\neg\mathsf{Smv}, \mathsf{BiVe}, \mathsf{BiEx}\}$*, and* $H \in \{I, \neg I\}$*. Then the following formulas have existential import:*

(i) $(A^{\uparrow}_{H}(y))^{*} = \bigwedge_{x\in X}(A(x) \to H(x,y)) \otimes \bigvee_{x\in X} A(x)$,
(ii) $(A^{\uparrow}_{H,Ev}(y))^{*} = \bigvee_{Z\underset{\sim}{\subset}X}[(\bigwedge_{x\in X}((A|Z)(x) \to H(x,y)) \wedge Ev(\mu_A(A|Z))) \otimes \bigvee_{x\in X}(A|Z)(x)]$.

The existential import is used in the following lemmas.

Lemma 4. *Let* $A \underset{\sim}{\subset} X$*,* $y \in Y$*,* $Ev \in \{\neg\mathsf{Smv}, \mathsf{BiVe}, \mathsf{BiEx}\}$*, and* $H \in \{I, \neg I\}$*. Then,*

$$(A^{\uparrow}_{H,Ev}(y))^{*} \leq \left(\bigvee_{x\in X} A(x)\right) \to A^{\cap}_{H}(y).$$

Proof. By Lemma 1(b), the following inequality holds: for each $Z \underset{\sim}{\subset} X$ and $x \in X$

$$(A|Z)^{\uparrow}_{H}(y) \leq (A|Z)(x) \to H(x,y).$$

By the adjunction property, $(A|Z)^{\uparrow}_{H}(y) \otimes (A|Z)(x) \leq H(x,y)$. By Lemma 1(f),

$$(A|Z)^{\uparrow}_{H}(y) \otimes (A|Z)(x) \otimes A(x) \leq A(x) \otimes H(x,y).$$

Hence,

$$(A|Z)^{\uparrow}_{H}(y) \otimes \bigvee_{x\in X}(A|Z)(x) \otimes \bigvee_{x\in X} A(x) \leq \bigvee_{x\in X} A(x) \otimes H(x,y).$$

By Lemma 1(e),

$$((A|Z)^{\uparrow}_{H}(y) \wedge Ev(\mu_A(A|Z))) \otimes \bigvee_{x\in X}(A|Z)(x) \otimes \bigvee_{x\in X} A(x) \leq \bigvee_{x\in X} A(x) \otimes H(x,y).$$

Using the adjunction property, we conclude that $(A^{\uparrow}_{H,Ev}(y))^{*} \leq (\bigvee_{x\in X} A(x)) \to A^{\cap}_{H}(y)$. □

Lemma 5. *Let* $A \underset{\sim}{\subset} X$*,* $y \in Y$*, and* $Ev_1, Ev_2 \in \{\neg\mathsf{Smv}, \mathsf{BiVe}, \mathsf{BiEx}\}$*. Then,*

$$(A^{\uparrow}_{I,Ev_1}(y))^{*} \ \otimes \ (A^{\uparrow}_{\neg I,Ev_2}(y))^{*} = 0.$$

Proof. Let $y \in Y$, $x \in X$ and $Z_1, Z_2 \underset{\sim}{\subset} X$. By Definition 4, and by Lemma 1(b),

$$(A|Z_1)^{\uparrow}_{I}(y) \leq (A|Z_1)(x) \to I(x,y), \text{ and } (A|Z_2)^{\uparrow}_{\neg I}(y) \leq (A|Z_2)(x) \to \neg I(x,y).$$

Then, by the adjunction property,

$$(A|Z_1)^{\uparrow}_{I}(y) \otimes (A|Z_1)(x) \leq I(x,y), \text{ and } (A|Z_2)^{\uparrow}_{\neg I}(y) \otimes (A|Z_2)(x) \leq \neg I(x,y).$$

By Lemma 1(e), $((A|Z_1)^{\uparrow}_{I}(y) \wedge Ev_1(\mu_A(A|Z_1))) \otimes (A|Z_1)(x) \leq I(x,y)$, and $((A|Z_2)^{\uparrow}_{\neg I}(y) \wedge Ev_2(\mu_A(A|Z_2))) \otimes (A|Z_2)(x) \leq \neg I(x,y)$.

By Lemma 1(d), (f),

$$((A|Z_1)^{\uparrow}_{I}(y) \wedge Ev_1(\mu_A(A|Z_1))) \otimes (A|Z_1)(x) \otimes ((A|Z)^{\uparrow}_{\neg I}(y) \wedge Ev_2(\mu_A(A|Z))) \otimes (A|Z_1)(x) = 0,$$

Finally,

$$\bigvee_{Z_1 \subsetsim X} \left((A|Z_1)^{\uparrow}_{I}(y) \wedge Ev_1(\mu_A(A|Z_1)) \otimes \bigvee_{x \in X} (A|Z_1)(x) \right) \otimes \bigvee_{Z_2 \subsetsim X} \left(((A|Z_2)^{\uparrow}_{\neg I}(y) \wedge Ev_2(\mu_A(A|Z_2))) \otimes \bigvee_{x \in X} (A|Z_2)(x) \right) = 0,$$

and hence, $(A^{\uparrow}_{I,Ev_1}(y))^* \otimes (A^{\uparrow}_{\neg I,Ev_2}(y))^* = 0.$ □

Lemma 6. *Let $A \subsetsim X$, $y \in Y$, and $Ev \in \{\neg\mathsf{Smv}, \mathsf{BiVe}, \mathsf{BiEx}\}$. Then,*

$$(A^{\uparrow}_{I,Ev}(y))^* \otimes (A^{\uparrow}_{\neg I}(y))^* = 0 \quad \textit{and} \quad (A^{\uparrow}_{I}(y))^* \otimes (A^{\uparrow}_{\neg I,Ev}(y))^* = 0.$$

Proof. The proof is similar to that of Lemma 5. □

The following theorem shows that we can obtain a decagon of oppositions starting from our operators.

Theorem 5. *Let (X, Y, I) be an L-context, where L is the standard Łukasiewicz MV-algebra, and let $A \subsetsim X$. If A is normal, then*

$$A^{\uparrow}_{I}, A^{\uparrow}_{I,\mathsf{BiEx}}, A^{\uparrow}_{I,\mathsf{BiVe}}, A^{\uparrow}_{I,\neg\mathsf{Smv}}, A^{\cap}_{I}, A^{\uparrow}_{\neg I}, A^{\uparrow}_{\neg I,\mathsf{BiEx}}, A^{\uparrow}_{\neg I,\mathsf{BiVi}}, A^{\uparrow}_{\neg I,\neg\mathsf{Smv}}, A^{\cap}_{\neg I}$$

are the vertices of a graded decagon of opposition, and they represent proprieties that are in relation of contrary, sub-contrary, sub-altern, and contradictory as shown in Fig. 6.

Proof. The proof follows by Theorem 3, Lemma 3, Lemma 4, Lemma 5, and Lemma 6. □

Example 3. Let (X, Y, I) be an L-context, where $X = \{x_1, \ldots, x_{24}\}$, $Y = \{y_1, \ldots, y_{10}\}$, and the L-relation I between the objects of X and the attribute y_1 of Y is defined by Table 2. Let us fix the context $\langle 0, 0.5, 1\rangle$. Then the functions $\neg\mathsf{Smv} : [0,1] \to [0,1]$, $\mathsf{BiVe} : [0,1] \to [0,1]$, and $\mathsf{BiEx} : [0,1] \to [0,1]$ are defined in [10] (cf. also Fig. 4). Furthermore, put

$$A = \{{}^{1}/x_1, \ldots, {}^{1}/x_7, {}^{0.6}/x_8, {}^{0.93}/x_9, {}^{0.5}/x_{10}, {}^{1}/x_{11}, {}^{0.7}/x_{12}, {}^{0.98}/x_{13}, {}^{1}/x_{14}, \ldots, {}^{1}/x_{16}, {}^{0.8}/x_{17}, {}^{1}/x_{18}, \ldots, {}^{1}/x_{20}, {}^{0.5}/x_{21}, {}^{1}/x_{22}, {}^{1}/x_{23}, {}^{0.66}/x_{24}, {}^{1}/x_{25}, {}^{1}/x_{26}\}.$$

Then we obtain the graded decagon of opposition depicted in Fig. 7.

Table 2. The fuzzy relation I between the objects of X and attribute y_1.

I	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
y_1	0.5	0.15	0.31	0.5	0.66	0.5	0.5	0	0.73	0	0.5	0.8

I	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}
y_1	0.98	0.25	0.5	0.5	0.27	0.5	0.5	0.6	0	0.37	0.5	0.02	0.5	0.6

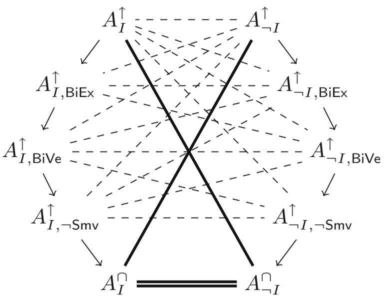

Fig. 6. Graded decagon of opposition

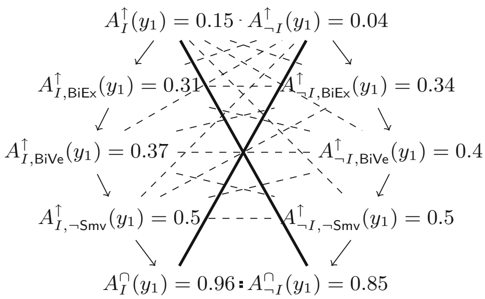

Fig. 7. Example of graded decagon of opposition

5 Future Directions

In this article, a graded decagon of opposition is introduced as a graded generalization of Aristotle's square, and it is constructed using some fuzzy quantifier-based operators. As future work, we intend to analyze more deeply the role that the fuzzy quantifier-based operators could have in fuzzy formal concept analysis. Moreover, fixed an evaluative linguistic expression Ev_1, we will find another evaluative linguistic expression Ev_2 such that the pair of operators ${}^{\uparrow}_{I,Ev_1}$ and ${}^{\downarrow}_{I,Ev_2}$ forms a Galois connection. Finally, we would like to propose our operators as fuzzy generalizations of the scaling quantifiers used in Relational concept analysis [2].

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