





On Solving the Quadratic Sum-of-Ratios Problems

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Abstract. This paper addresses the numerical solution of fractional programs with quadratic functions in the ratios. Instead of considering a sum-of-ratios problem directly, we developed an efficient global search algorithm, which is based on two approaches to the problem. The first one adopts a reduction of the fractional minimization problem to the solution of an equation with an optimal value of a parametric d.c. minimization problem. The second approach reduces the original problem to the optimization problem with nonconvex (d.c.) constraints. Hence, the fractional programs can be solved by applying the Global Search Theory of d.c. optimization.

The global search algorithm developed for sum-of-ratios problems was tested on the examples with quadratic functions in the numerators and denominators of the ratios. The numerical experiments demonstrated that the algorithm performs well when solving rather complicated quadratic sum-of-ratios problems with up to 100 variables or 1000 terms in the sum.

Keywords: Fractional optimization · Nonconvex problem · Difference of two convex functions · Quadratic functions · Global search algorithm · Computational testing

1 Introduction

The global minimization of the sum of fractional functions has attracted the interest of researchers and practitioners for a number of years, because these problems have a large number of important real-life applications. From the theoretical viewpoint, the solution of these problems implies facing significant challenges, because, in general, fractional programs are nonconvex problems, i.e., they generally have several (often a huge number of) local optimal solutions that are not globally optimal [18]. It was proven that the sum-of-ratios program is NP-complete [6]. Various specialized methods and algorithms have been proposed for solving these problems globally (see, for example, surveys in [3, 18]), but the development of new efficient methods for the following sum-of-ratios problem [3, 7, 18]

$$(\mathcal{FP}) : \quad f(x) := \sum_{i=1}^m \frac{\psi_i(x)}{\varphi_i(x)} \downarrow \min_x, \quad x \in S,$$

where $S \subset \mathbb{R}^n$ is a closed convex set and $\psi_i, \varphi_i : \mathbb{R}^n \rightarrow \mathbb{R}$, are continuous function such that

$$(\mathcal{H}_0) : \quad \psi_i(x) > 0, \varphi_i(x) > 0 \quad \forall x \in S, \quad i = 1, \dots, m;$$

still remains an important field of research in the mathematical optimization. If functions $\psi_i(\cdot), \varphi_i(\cdot), i = 1, \dots, m$ are quadratic functions, then we classify Problem (\mathcal{FP}) as a quadratic fractional program.

On the other hand, the following general problem of d.c. optimization

$$(\mathcal{P}) : \quad \begin{cases} f_0(x) := g_0(x) - h_0(x) \downarrow \min_x, & x \in S, \\ f_j(x) := g_j(x) - h_j(x) \leq 0, & j = 1, \dots, m, \end{cases}$$

where the functions $g_j(\cdot), h_j(\cdot), j = 1, \dots, m$, are convex on \mathbb{R}^n , S is a closed and convex set, and $S \subset \mathbb{R}^n$, remained over recent years one of the attractive objects in nonconvex optimization [13–15, 19, 27, 28].

It is worth noting that any continuous optimization problem can be approximated by a d.c. problem with any prescribed accuracy [13, 14, 28]. In addition, the space of d.c. functions, being a linear space, is then closed under most of operations usually considered in optimization (see, e.g., [19, 28]). Moreover, any twice differentiable function belong to the space of d.c. functions [13–15, 19, 27, 28]. The convexity of the two convex components g and h of the d.c. function f is widely used to develop appropriate theoretical and algorithmic tools.

We develop a new efficient global search method for the fractional optimization problems, which is based on the two following ideas [9, 11, 12]. First, generalizing the Dinkelbach's approach [4], we propose to reduce the sum-of-ratios problem with d.c. functions to solving an equation with the optimal value function of an auxiliary parametric problem with the vector parameter that satisfies the nonnegativity assumption. Secondly, we also use the reduction of the fractional program to a problem of type (\mathcal{P}) , where $f_0(x)$ is a linear function, i.e. an optimization problem over nonconvex feasible set given by d.c. inequality constraints.

Furthermore, based on the Global Search Theory for d.c. optimization [19, 22] and on the solution of these two particular cases of the general d.c. optimization problem (\mathcal{P}) , we develop a two-method technology for solving a sum-of-ratios problem and verify it on test problems with nonconvex quadratic functions in the numerators and denominators of the ratios.

Most of the approaches and techniques for solving fractional programs are designed for problems with affine functions in the numerators and denominators of the ratios [1, 5, 16, 17]. Problems with nonconvex quadratic (d.c.) functions are more complex problems since it concerns a finding a global solution.

The outlines of the paper are as follows. In Sect. 2 and 3, we recall two approaches to solving Problem (\mathcal{FP}) using auxiliary d.c. minimization problems and problems with d.c. inequality constraints, respectively. In Sect. 4, we show how to represent explicitly the nonconvex functions, describing the goal function and the constraints of the auxiliary d.c. problems as differences of two convex

functions (the d.c. representation). Further, in Sect. 5, we propose the algorithm which combines the two approaches to solving the fractional program via d.c. optimization. The final section offers computational testing of the developed algorithm on fractional program instances with up to 100 variables or 1000 terms in the sum generated by the approach from [2].

2 Reduction to the D.C. Minimization Problem

Consider the following auxiliary problem

$$(\mathcal{P}_\alpha) : \quad \Phi_\alpha(x) := \sum_{i=1}^m [\psi_i(x) - \alpha_i \varphi_i(x)] \downarrow \min_x, \quad x \in S,$$

where $\alpha = (\alpha_1, \dots, \alpha_m)^\top \in \mathbb{R}_+^m$ is the vector parameter and the set $S \subset \mathbb{R}^n$ is, as above closed and convex.

First, let us recall some results from [9] about the relations between Problems (\mathcal{FP}) and (\mathcal{P}_α) . Further introduce the function $\mathcal{V}(\alpha)$ of the optimal value to Problem (\mathcal{P}_α) :

$$\mathcal{V}(\alpha) := \inf_x \{ \Phi_\alpha(x) \mid x \in S \} = \inf_x \left\{ \sum_{i=1}^m [\psi_i(x) - \alpha_i \varphi_i(x)] : x \in S \right\}. \quad (1)$$

In addition, suppose that the following assumptions are fulfilled:

$$(\mathcal{H}_1) : \quad \begin{cases} (a) \ \mathcal{V}(\alpha) > -\infty \ \forall \alpha \in \mathcal{K}, \text{ where } \mathcal{K} \text{ is a convex set from } \mathbb{R}^m; \\ (b) \ \forall \alpha \in \mathcal{K} \subset \mathbb{R}^m \text{ there exists a solution } z = z(\alpha) \text{ to Problem } (\mathcal{P}_\alpha). \end{cases}$$

In what follows, we say that a given parameter vector $\alpha = (\alpha_1, \dots, \alpha_m)^\top \in \mathbb{R}^m$ satisfies “the nonnegativity condition” in Problem (\mathcal{FP}) , if the following inequalities hold

$$(\mathcal{H}(\alpha)) : \quad \psi_i(x) - \alpha_i \varphi_i(x) \geq 0 \ \forall x \in S, \quad i = 1, \dots, m.$$

Theorem 1. [9] *Suppose that in Problem (\mathcal{FP}) the assumptions (\mathcal{H}_0) , (\mathcal{H}_1) are satisfied. In addition, let there exist a vector $\alpha_0 = (\alpha_{01}, \dots, \alpha_{0m})^\top \in \mathcal{K} \subset \mathbb{R}^m$ at which “the nonnegativity condition” $(\mathcal{H}(\alpha_0))$ holds.*

Finally, suppose that in Problem (\mathcal{P}_{α_0}) the following equality takes place:

$$\mathcal{V}(\alpha_0) \triangleq \min_x \left\{ \sum_{i=1}^m [\psi_i(x) - \alpha_{0i} \varphi_i(x)] : x \in S \right\} = 0. \quad (2)$$

Then, any solution $z = z(\alpha_0)$ to Problem (\mathcal{P}_{α_0}) is a solution to Problem (\mathcal{FP}) , so that $z \in \text{Sol}(\mathcal{P}_{\alpha_0}) \subset \text{Sol}(\mathcal{FP})$.

Hence, in order to check up the equality (2), we should be able to solve globally Problem (\mathcal{P}_α) at a current $\alpha \in \mathbb{R}_+^m$. Since $\psi_i(\cdot)$, $\varphi_i(\cdot)$, $i = 1, \dots, m$, are convex or d.c. functions it can be readily seen that Problem (\mathcal{P}_α) turns

out to be a parametric d.c. minimization problem. As a consequence, in order to solve the auxiliary Problem (\mathcal{P}_α) , we can apply the global search strategy for d.c. minimization problems [19, 22]. Its theoretical foundation can be provided by the following Global Optimality Condition written here in the terms of Problem (\mathcal{P}_α) under the assumption that $\psi_i(\cdot)$, $\varphi_i(\cdot)$, $i = 1, \dots, m$, are convex functions and therefore $g(x) = \sum_{i=1}^m \psi_i(x)$, $h_\alpha(x) = \sum_{i=1}^m \alpha_i \varphi_i(x)$,

$$\Phi_\alpha(x) = \sum_{i=1}^m (\psi_i(x) - \alpha_i \varphi_i(x)) = g(x) - h_\alpha(x).$$

If $\psi_i(\cdot)$, $\varphi_i(\cdot)$, $i = 1, \dots, m$, are d.c. functions, it is easy to see that the goal function of (\mathcal{P}_α) is d.c., as well, but having another d.c. decomposition.

Theorem 2. [19, 22] *Suppose, $z(\alpha)$ is a global solution to (\mathcal{P}_α) , $\zeta := \Phi_\alpha(z(\alpha))$. Then,*

$$(\mathcal{E}) : \quad \begin{cases} (a) & \forall (y, \beta) \in \mathbb{R}^n \times \mathbb{R} : h_\alpha(y) = \beta - \zeta, \\ (b) & g(x) - \beta \geq \langle \nabla h_\alpha(y), x - y \rangle \quad \forall x \in S. \end{cases} \quad (3)$$

The meaning of Theorem 2 lies in the fact that by selecting the ‘‘perturbation parameters’’ (y, β) (satisfying (3a)) and solving the linearized problem

$$(\mathcal{P}_\alpha \mathcal{L}) : \quad g(x) - \langle \nabla h_\alpha(y), x \rangle \downarrow \min_x, \quad x \in S,$$

we try to violate the principal inequality (3b) (where $y \in \mathbb{R}^n$ is not obligatory feasible).

Furthermore, according to the Theorem 1, we are able to avoid the direct solution of Problem (\mathcal{FP}) and address the parametrized problem (\mathcal{P}_α) with $\alpha \in \mathbb{R}_+^m$. Hence, we propose to combine a solution of Problem (\mathcal{P}_α) with a search of the parameter $\alpha \in \mathbb{R}_+^m$ in order to find $\alpha_0 \in \mathbb{R}_+^m$ such that $\mathcal{V}(\alpha_0) = 0$. This idea can be implemented by the following algorithm. Let $[\alpha_-^0, \alpha_+^0]$ be an initial segment for varying α , and $x_0 \in S$ stands for the starting point.

Algorithm 1 for solving the fractional problem via d.c. minimization

- Stage 0. (Initialization) $k := 0$. Set $x^k := x_0$, $\alpha_-^k := 0$, $\alpha_+^k := \alpha_+^0$, $\alpha^k := \frac{\alpha_+^0}{2} \in [\alpha_-^0, \alpha_+^0]$.
- Stage I. (Local search) Starting at x^k find a critical point to Problem $(\mathcal{P}_k) := (\mathcal{P}_{\alpha^k})$ using the special local search method for d.c. minimization [19].
- Stage II. (Global search) Find a global solution $z(\alpha^k)$ to Problem (\mathcal{P}_k) using the global search scheme for the parametric d.c. minimization [19, 22] Problem (\mathcal{P}_k) .
- Stage III. (Stopping criterion) If $\mathcal{V}(\alpha^k) = 0$ and $\min_i \{ \psi_i(z(\alpha^k)) - \alpha_i^k \varphi_i(z(\alpha^k)) \} \geq 0$, then STOP: $z(\alpha^k) \in \text{Sol}(\mathcal{FP})$.
- Stage IV. (Parameter variation) Find new parameters α^{k+1} , α_-^{k+1} and α_+^{k+1} ; $k := k + 1$ and go to Stage I.

Remark 1. The algorithm for solving the fractional program (\mathcal{FP}) consists of three basic stages: (a) a local and (b) a global searches in Problem (\mathcal{P}_α) with a fixed vector parameter α (Stages I, II) and (c) the method for finding the vector α (Stage IV).

Let $[\alpha_-, \alpha_+]$ be a segment for varying α . In addition, assume that we are able to compute the value $\mathcal{V} := \mathcal{V}(\alpha)$ and let a solution $z(\alpha)$ to Problem (\mathcal{P}_α) be given.

In order to calculate a new parameter α^{new} and new boundaries α_-^{new} and α_+^{new} at the current iteration k for a segment for varying α on Stage IV of Algorithm 1 one can use the following procedure, where $\alpha := \alpha^k$ from Stage III of Algorithm 1.

Stage IV. Parameter variation algorithm

- Step 1. If $\mathcal{V} > 0$, then set $\alpha_-^{new} := \alpha$, $\alpha^{new} := \frac{1}{2}(\alpha_+ + \alpha)$, $\alpha_+^{new} := \alpha_+$.
- Step 2. If $\mathcal{V} < 0$, then set $\alpha_+^{new} := \alpha$, $\alpha^{new} := \frac{1}{2}(\alpha_- + \alpha)$, $\alpha_-^{new} := \alpha_-$.
- Step 3. If $\mathcal{V} = 0$ and $\min_i \Phi_i(z(\alpha), \alpha) < 0$, then set

$$\begin{aligned} \alpha_i^{new} &:= \frac{\psi_i(z(\alpha))}{\varphi_i(z(\alpha))} \quad \forall i : \psi_i(z(\alpha)) - \alpha_i \varphi_i(z(\alpha)) < 0; \\ \alpha_i^{new} &:= \alpha_i \quad \forall i : \psi_i(z(\alpha)) - \alpha_i \varphi_i(z(\alpha)) \geq 0. \end{aligned}$$

In addition, set $\alpha_-^{new} := 0$, $\alpha_+^{new} := t\alpha^{new}$, where $t = \frac{\max\{\alpha_{+1}^0; \dots; \alpha_{+m}^0\}}{\max\{\alpha_1; \dots; \alpha_m\}}$.

- Step 4. $\alpha^{k+1} := \alpha^{new}$, $\alpha_-^{k+1} := \alpha_-^{new}$ and $\alpha_+^{k+1} := \alpha_+^{new}$; $k := k + 1$ and go to Stage I of Algorithm 1.

Remark 2. To choose an initial segment $[\alpha_-^0, \alpha_+^0]$ for varying α , we should take into account the following considerations. According to “the nonnegativity condition” ($\mathcal{H}(\alpha)$) and the assumption (\mathcal{H}_0), we have

$$\alpha_i \leq \frac{\psi_i(x)}{\varphi_i(x)} \leq \sum_{i=1}^m \frac{\psi_i(x)}{\varphi_i(x)} \quad \forall x \in S, \quad \forall i = 1, \dots, m,$$

therefore one can choose $\alpha_{+i}^0 := \frac{\psi_i(x^0)}{\varphi_i(x^0)}$, $\alpha_{-i}^0 = 0$, $i = 1, \dots, m$.

The performed computational experiment [8,9,11] showed that solving fractional problem via d.c. minimization takes a large number of iterations, generated by an ineffective work of Stage IV. Therefore, it is very important to choose a suitable parameter α in order to reduce the total number of iterations and, therefore, the corresponding run-time of Algorithm 1.

Using the reduction of the sum-of-ratios problem to the optimization problem with nonconvex constraints, we will look not only for a starting value of the parameter α for Problem (\mathcal{P}_α), but also for a better vector for Problem (\mathcal{FP}).

3 Reduction to the Problem with D.C. Constraints

In this section we consider the following optimization problem

$$(\mathcal{DCC}) : \quad \begin{cases} \sum_{i=1}^m \alpha_i \downarrow \min_{(x, \alpha)}, & x \in S, \\ \psi_i(x) - \alpha_i \varphi_i(x) \leq 0, & i \in I = \{1, \dots, m\}. \end{cases}$$

The relations between Problems (\mathcal{FP}) and (\mathcal{DCC}) are as follows.

Proposition 1. [10] *For any global solution $(x_*, \alpha_*) \in \mathbb{R}^n \times \mathbb{R}^m$ to Problem (\mathcal{DCC}) , the point x_* will be a global solution to Problem (\mathcal{FP}) and $\alpha_{*i} = \frac{\psi_i(x_*)}{\varphi_i(x_*)}$, $i \in I$.*

Remark 3. It is clear that, in contrast to Theorem 1 from Sect. 2, the vector $\alpha_* = (\alpha_{*1}, \dots, \alpha_{*m})^\top \in \mathbb{R}^m$ must be found simultaneously with the solution vector x_* .

It is easy to see that Problem (\mathcal{DCC}) is a nonconvex optimization problem with the linear goal function and the nonconvex feasible set (see, e.g., [14, 25]). So, we can solve Problem (\mathcal{DCC}) using the exact penalization approach for d.c. optimization developed in [20, 23, 24]

However, the computational experiments showed [11, 12] that solving fractional program via problem with d.c. constraints (\mathcal{DCC}) took more run-time than using the parametric d.c. minimization, i.e. Problem (\mathcal{P}_α) .

Notwithstanding, in low-dimensional test problems of fractional programming the known global solutions were found just by the local search method (LSM) [21] for Problem (\mathcal{DCC}) (see [10]). Therefore, we apply here only the LSM, based on the classical idea of linearization with respect to the basic nonconvexity of the problem [19, 21, 26].

The LSM for Problem (\mathcal{DCC}) is based on the consecutive solutions of the following partially linearized (at the point (x^s, α^s)) problem [10, 21, 26]:

$$(\mathcal{DCC}\mathcal{L}_s) : \quad \begin{cases} \sum_{i=1}^m \alpha_i \downarrow \min_{(x, \alpha)}, & x \in S, \\ g_i(x, \alpha_i) - \langle \nabla h_i(x^s, \alpha_i^s), (x, \alpha_i) - (x^s, \alpha_i^s) \rangle - h_i(x^s, \alpha_i^s) \leq 0, & i \in I, \end{cases}$$

where the functions $g_i(\cdot)$ and $h_i(\cdot)$, $i \in I$, are the convex functions obtained by the d.c. representation of the constraint functions $\psi_i(x) - \alpha_i \varphi_i(x)$, $i \in I$ (see, for example, Sect. 4 for quadratic $\psi_i(\cdot)$ and $\varphi_i(\cdot)$) and where $\nabla h_i(x, \alpha_i) = (\nabla_{x_1} h_i(x, \alpha_i), \dots, \nabla_{x_n} h_i(x, \alpha_i), \nabla_{\alpha_i} h_i(x, \alpha_i))^\top \in \mathbb{R}^{n+1}$.

Due to the consecutive solutions of linearized convex problems $(\mathcal{DCC}\mathcal{L}_s)$ starting at the point (x^0, α^0) we generate the sequence $\{(x^s, \alpha^s)\}: (x^{s+1}, \alpha^{s+1}) \in \text{Sol}(\mathcal{DCC}\mathcal{L}_s)$. As it was proven in [21], the cluster point $(x_*, \alpha_*) \in \{x \in S \mid g_i(x, \alpha_i) - \langle \nabla h_i(x_*, \alpha_{*i}), (x, \alpha_i) - (x_*, \alpha_{*i}) \rangle - h_i(x_*, \alpha_{*i}) \leq 0, i \in I\}$ of the sequence $\{(x^s, \alpha^s)\}$ generated by the LSM, is a solution to the linearized Problem $(\mathcal{DCC}\mathcal{L}_*)$ (which is Problem $(\mathcal{DCC}\mathcal{L}_s)$ with (x_*, α_*) instead of (x_s, α_s)), and a critical point of Problem (\mathcal{DCC}) : $(x_*, \alpha_*) \in \text{Sol}(\mathcal{DCC}\mathcal{L}_*)$.

In order to implement the LSM [21, 26] and the global search scheme for d.c. minimization [19, 22] (Stage II of Algorithm 1), we need an explicit d.c. representation of functions $\psi_i(x) - \alpha_i\varphi_i(x)$, $i \in I$. As well-know the d.c. decomposition of function is not unique. The next section presents several possible d.c. representations for the goal function of Problem (\mathcal{P}_α) and constraint's functions of Problem (\mathcal{DCC}) (in the case of quadratic $\psi_i(\cdot)$ and $\varphi_i(\cdot)$).

4 D.C. Representations of the Goal Function and Constraint's Functions

Consider the following quadratic functions $\psi_i(x) > 0$, $\varphi_i(x) > 0 \ \forall x \in S$:

$$\psi_i(x) = \langle x, A^i x \rangle + \langle a^i, x \rangle + \xi_i, \quad \varphi_i(x) = \langle x, B^i x \rangle + \langle b^i, x \rangle + \gamma_i,$$

where the matrices A^i and B^i are $(n \times n)$ positive definite, $a^i, b^i \in \mathbb{R}^n$, $\xi_i, \gamma_i \in \mathbb{R}$, $i \in I$. Therefore, the functions $\psi_i(\cdot)$ and $\varphi_i(\cdot)$ are convex functions, $i \in I$.

In this case, the d.c. representation of the goal function $\Phi_\alpha(x)$ of Problem (\mathcal{P}_α) (where $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}_+^m$ is the vector parameter) can be rather simple: $\Phi_\alpha(x) = g(x) - h_\alpha(x)$, where $g(x) = \sum_{i=1}^m \psi_i(x)$, $h_\alpha(x) := \sum_{i=1}^m \alpha_i \varphi_i(x)$,

or, in another way, $g_\alpha^1(x) = \sum_{i=1}^m [\langle x, A^i x \rangle + \langle (a^i - \alpha_i b^i), x \rangle] + \sum_{i=1}^m (\xi_i - \alpha_i \gamma_i)$,

$$h_\alpha^1(x) = \sum_{i=1}^m \alpha_i \langle x, B^i x \rangle.$$

Remark 4. If the symmetric matrices A^i or/and B^i are indefinite, then they can be represented as the difference of two symmetric positive definite matrices $A^i = A_1^i - A_2^i$, $A_1^i, A_2^i > 0$, $B^i = B_1^i - B_2^i$, $B_1^i, B_2^i > 0$, using, for example, the method from [19]. After this it is possible to construct functions $g(\cdot)$ and $h(\cdot)$ by adding for all $i \in I$ a convex part with the matrix A_1^i or/and B_1^i into $g(\cdot)$ and a nonconvex part with the matrix A_2^i or/and B_2^i into $h(\cdot)$, i.e.

$$g_\alpha^2(x) = \sum_{i=1}^m [\langle x, A_1^i x \rangle + \alpha_i \langle x, B_2^i x \rangle + \langle (a^i - \alpha_i b^i), x \rangle] + \sum_{i=1}^m (\xi_i - \alpha_i \gamma_i),$$

$$h_\alpha^2(x) = \sum_{i=1}^m [\langle x, A_2^i x \rangle + \alpha_i \langle x, B_1^i x \rangle].$$

A more complicated d.c. representation of functions appears in Problem (\mathcal{DCC}), because $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}_+^m$ is a variable and the problem has the following nonconvex term

$$\alpha_i \varphi_i(x) = \alpha_i \langle x, B^i x \rangle + \alpha_i \langle b^i, x \rangle + \alpha_i \gamma_i, \tag{4}$$

which generates the bilinearity, and as a consequence, the nonconvexity in every constraint ($i \in I$).

The term $\alpha_i \langle b^i, x \rangle$ in (4) can be represented in the d.c. form as follows

$$\langle \alpha_i b^i, x \rangle = \frac{1}{4} \| \alpha_i b^i + x \|^2 - \frac{1}{4} \| \alpha_i b^i - x \|^2, \quad i \in I. \quad (5)$$

Further, the product $\alpha_i \langle x, B^i x \rangle$ can be expressed by formula (5):

$$\alpha_i \langle x, B^i x \rangle = \frac{1}{4} (\alpha_i + \langle x, B^i x \rangle)^2 - \frac{1}{4} (\alpha_i - \langle x, B^i x \rangle)^2, \quad i \in I,$$

if B^i , $i \in I$, are positive definite matrices and the following conditions hold

$$\alpha_i + \langle x, B^i x \rangle \geq 0, \quad \alpha_i - \langle x, B^i x \rangle \geq 0 \quad \forall x \in S, \quad i \in I. \quad (6)$$

Then

$$g_i(x, \alpha_i) = \psi_i(x) + \frac{1}{4} (\alpha_i - \langle x, B^i x \rangle)^2 + \frac{1}{4} \| \alpha_i b^i - x \|^2 - \alpha_i \gamma_i,$$

$$h_i(x, \alpha_i) = \frac{1}{4} (\alpha_i + \langle x, B^i x \rangle)^2 + \frac{1}{4} \| \alpha_i c^i + x \|^2$$

are convex functions and their difference present the constraints in Problem (DCC) in the following d.c. form:

$$\psi_i(x) - \alpha_i \varphi_i(x) = g_i(x, \alpha_i) - h_i(x, \alpha_i) \leq 0, \quad i \in I. \quad (7)$$

5 Global Search Scheme for Solving the Sum-of-Ratios Problems

The previous computational experiments [9, 11, 12] have demonstrated that Algorithm 1 developed for solving fractional programs via d.c. minimization (see Sect. 2) is quite efficient when applied to problems with affine functions in the ratios. The algorithm has also shown its effectiveness for problems with quadratic numerators and linear denominators [9, 11, 12]. However, the algorithm wastes a lot of run-time on finding the vector parameter α at which the optimal value of Problem (\mathcal{P}_α) is equal to zero. The shortcoming of the approach of the reduction of the sum-of-ratios problems (\mathcal{FP}) to problems with d.c. constraints (DCC) is a lot of run-times spent on solving Problem (DCC).

The results of computational experiments suggest a combination of the two approaches for solving the fractional programs. So, we propose to use the local search for Problem (DCC) to find a starting value α^0 of the parameter α , which takes less time to reduce the optimal value function of Problem (\mathcal{P}_α) to zero. This idea could be implemented by the method which consists of 3 basic parts: the local search in Problem (DCC) with d.c. inequality constraints, the global search in d.c. minimization Problem (\mathcal{P}_α) with a fixed vector parameter α (found by the LSM for Problem (DCC)) and the method for finding the vector α at which the optimal value of Problem (\mathcal{P}_α) is equal to zero.

We denote $g^k(\cdot) = g_{\alpha^k}(\cdot)$, $h^k(\cdot) = h_{\alpha^k}(\cdot)$, $\Phi_k(\cdot) = \Phi_{\alpha^k}(\cdot)$.

Let an initial point $x_0 \in S$, a vector $\alpha_0 = (\alpha_{01}, \dots, \alpha_{0m}) \in \mathbb{R}_+^m$ and an initial segment $[\alpha_-^0, \alpha_+^0]$ for varying α be given. In addition, let there

be given the number sequences $\{\tau_s\}$, $\{\delta_s\}$, $\{\varepsilon_s\}$, such that $\tau_s, \delta_s, \varepsilon_s > 0$, $s = 0, 1, 2, \dots$, $\tau_s \downarrow 0$, $\delta_s \downarrow 0$, $\varepsilon_s \downarrow 0$ ($s \rightarrow \infty$).

Global search scheme for fractional program (F-GSS)

- Stage 0. (**Initialization**) $k := 0$, $v^k := (x_0, \alpha_0)$, $\alpha_-^k := 0$, $\alpha_+^k := \alpha_+^0$.
- Stage I. (**Local search**) Starting from the point v^k , find by the LSM from [21] a critical point $(x(\alpha^k), \alpha^k)$ in the d.c. constrained Problem (\mathcal{DCC}).
- Stage II. (**Global search**) Starting from the point $x(\alpha^k)$ find a solution $z(\alpha^k)$ to Problem (\mathcal{P}_k) with the help of the global search scheme for d.c. minimization [19,22], which consists of the following steps:

- Step 0. $s := 0$, $z^s := x(\alpha^k)$, $\zeta_s := \Phi_k(z^s)$.
- Step 1. Choose a number $\beta \in [\beta_-, \beta_+]$, where the numbers $\beta_- = \inf(g^k, S)$, $\beta_+ = \sup(g^k, S)$ can be approximated by rather rough estimates. Set $\beta_0 := g^k(z^0)$.
- Step 2. Construct an approximation $\mathcal{A}(\beta) = \{y^1, \dots, y^N \mid h^k(y^j) = \beta - \zeta_s, j = 1, \dots, N = N(\beta)\}$ of the level surface $\{y \in \mathbb{R}^n \mid h^k(y) = \beta - \zeta_s\}$ of the function $h^k(\cdot)$ and, according to Theorem 2, form a collection of indices J_s defined as follows $J_s = J_s(\beta) = \{j \in \{1, \dots, N_s\} \mid g^k(y^j) \leq \beta\}$.
- Step 3. If $J_s = \emptyset$, then set $\beta := \beta + \Delta\beta \in [\beta_-, \beta_+]$, and loop to Step 2.
- Step 4. For every $j \in J_s$ find a global $2\delta_s$ -solution $\bar{u}^j \in S$ to the following linearized convex problem

$$g^k(x) - \langle \nabla h^k(y^j), x \rangle \downarrow \min_x, \quad x \in S.$$

and after that, starting at $\bar{u}^j \in S$, apply the LSM from [19] to produce a $2\tau_s$ -critical vector $u^j \in S$, so that $g^k(u^j) - \langle \nabla h^k(u^j), u^j \rangle - 2\tau_s \leq \inf_x \{g^k(x) - \langle \nabla h^k(u^j), x \rangle \mid x \in S\}$.

- Step 5. For every $j \in J_s$ find a global $2\delta_s$ -solution $v^j: h^k(v^j) = \beta - \zeta_s$, to the following level problem (see Global Optimality Condition (\mathcal{E}))

$$\langle \nabla h^k(v), u^j - v \rangle \uparrow \max_v, \quad h^k(v) = \beta - \zeta_s.$$

Note that for a quadratic function $h^k(\cdot)$, this problem can be solved manually.

- Step 6. Compute the number $\eta_s(\beta) := \eta_s^0(\beta) - \beta$, where $\eta_s^0(\beta) := g^k(u^p) - \langle \nabla h^k(v^p), u^p - v^p \rangle := \min_{j \in J_s} \{g^k(u^j) - \langle \nabla h^k(v^j), u^j - v^j \rangle\}$.

- Step 7. If $\eta_s(\beta) < 0$, then set $s := s + 1$, $z^s := u^p$ and loop to Step 2.

- Step 8. (Else) set $\beta := \beta + \Delta\beta \in [\beta_-, \beta_+]$ and go to Step 2.
 If $\eta_s(\beta) \geq 0 \forall \beta \in [\beta_-, \beta_+]$ (i.e. the one-dimensional search on β is terminated) and $\delta_s \leq \delta_*$, $\tau_s \leq \tau_*$, $\varepsilon_s \leq \varepsilon_*$, where $\delta_* > 0$, $\tau_* > 0$, $\varepsilon_* > 0$ are the fixed accuracies of corresponding computations, then the global search (Stage II) has been terminated: $z(\alpha^k) := z^s$.

- Stage III. (**Stopping criterion**) If $\mathcal{V}(\alpha^k) = 0$ and $\min_i \{\psi_i(z(\alpha^k)) - \alpha_i^k \varphi_i(z(\alpha^k))\} \geq 0$, then STOP: $z(\alpha^k) \in \text{Sol}(\mathcal{FP})$.

Stage IV. (**Parameter variation**) Implement the parameter variation algorithm to find new parameters α^{k+1} , α_-^{k+1} and α_+^{k+1} . Set $\vartheta^{k+1} = (z(\alpha^k), \alpha^{k+1})$, $k = k + 1$ and go to Stage I.

6 Computational Simulations

The algorithm developed on the basis of Global search scheme F-GSS from Sect. 5 combining two approaches for solving the fractional programs (\mathcal{FP}) via d.c. optimization problems was coded in C++ language and applied to solve sum-of-ratios problems with quadratic functions in the ratios.

The set of test examples was generated by the technique from [2]. The method of generation was based on the Calamai's and Vicente's idea [29] to construct a nonconvex quadratic problem with known local and global solutions and on the reduction Theorem 1 to obtain fractional problem with quadratic functions in the numerators and denominators of the ratios.

All computational experiments were performed on the Intel Core i7-4790K CPU 4.0 GHz. All auxiliary convex (linearized) problems arising not only during the implementation of the LSM for a d.c. constrained problem (Stage I of the F-GSS) but also during the global search procedures for a d.c. minimization problem (Stage II of the F-GSS) were solved by the software package IBM ILOG CPLEX 12.6.2.

Table 1 shows the results of computational testing of the F-GSS and employs the following denotations: n is the number of variables (problem's dimension); m is the number of terms in the sum; $f(x_0)$ is the value of the goal function of Problem (\mathcal{FP}) at the starting point x_0 ; f_{glob} stands for the value of the function at the solution provided by the F-GSS; St is the number of critical points passed by the algorithm; $it-\alpha$ is the number of variation of the parameter α in the F-GSS; PL is the number of solved auxiliary (linearized) problems; T stands for the CPU time; f_M is the value of the goal function of Problem (\mathcal{FP}) provided by the `fmincon` solver of MATLAB.

The test problems (with known global solution) constructed with up to 100 variables and 1000 terms in the sum were successfully solved (see Table 1). We got the global solution in all tests. At the same time, the `fmincon` solver of MATLAB fails to find the global solution in all test problems (the starting point was the same for both algorithms) and the gap was from 1% ($n = 100$, $m = 70$) to 4.3% ($n = 10$, $m = 10$), and 1.8% on average.

Thus, we conclude that new computational results on solving of the sum-of-ratios problems with quadratic functions in the ratios are rather promising. The computational experiment showed that the algorithm developed for solving fractional programs is quite efficient.

Table 1. Results of computational testing of the F-GSS

n	m	$f(x_0)$	f_{glob}	St	it	$-\alpha$	PL	T(hh:mm:ss)	f_M
10	10	12.412	10	3	62		742	00:00:00.96	10.444
10	20	24.184	20	4	87		984	00:00:01.27	20.747
10	30	36.600	30	3	135		1241	00:00:01.68	31.185
10	40	45.905	40	6	160		1967	00:00:02.68	41.055
10	50	59.338	50	5	163		1961	00:00:02.38	51.579
10	100	117.097	100	5	125		2171	00:00:03.00	102.925
20	10	10.985	10	7	73		2850	00:00:07.11	10.163
20	50	53.140	50	8	175		6853	00:00:17.13	50.923
20	100	109.900	100	7	216		6070	00:00:14.86	101.779
20	200	217.532	200	8	231		12648	00:00:33.45	203.212
20	300	331.148	300	8	437		18183	00:00:52.05	305.493
30	300	317.672	300	6	256		41390	00:02:58.92	303.219
30	400	429.763	400	8	501		36858	00:02:22.43	405.392
30	500	540.361	500	3	405		27851	00:02:48.37	507.485
30	600	648.150	600	7	477		134532	00:12:01.93	609.279
30	700	760.107	700	5	307		444514	00:59:51.79	711.176
30	800	843.491	800	4	498		674305	01:39:07.62	808.296
30	900	912.228	900	6	495		3691396	09:21:47.13	912.228
30	1000	1089.367	1000	3	500		2165762	06:25:22.73	1016.486
50	100	106.017	100	5	181		71973	00:11:37.92	101.043
50	200	213.638	200	7	276		56980	00:12:06.29	203.447
50	300	320.066	300	8	324		133878	00:25:20.29	303.882
50	400	429.090	400	6	298		85923	00:17:37.12	405.265
50	500	532.842	500	9	308		189331	00:42:18.27	506.385
50	1000	1059.655	1000	53	104		369255	01:38:07.78	1013.276
70	100	105.464	100	8	214		65055	00:22:52.85	102.133
70	200	210.726	200	8	226		188620	01:08:06.39	203.113
70	300	316.092	300	6	214		233466	01:32:50.30	303.042
70	400	425.458	400	6	214		115923	00:51:24.14	404.868
100	70	73.042	70	110	150		281305	03:26:19.46	70.693
100	90	95.764	90	6	181		139464	01:42:55.57	92.530
100	100	105.456	100	8	253		248367	04:47:31.88	101.156

7 Conclusions

In this paper, we showed how fractional programs with nonconvex quadratic functions can be solved by applying the Global Search Theory for d.c.

optimization problems. Instead of considering a sum-of-ratios problem directly, we developed an efficient global search algorithm, which is based on two approaches. The first one adopts a reduction of the fractional minimization problem to the solution of an equation with an optimal value of the d.c. minimization problem with a vector parameter. The second method is based on the reduction of the sum-of-ratios problem to the optimization problem with nonconvex constraints.

The global search algorithm developed for fractional program was tested on examples with nonconvex quadratic functions in the numerators and denominators of the ratios. The numerical experiments demonstrated that the algorithm performs well when solving rather complicated sum-of-ratios problems with up to 100 variables or 1000 terms in the sum.

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