Most Favorable Russell Measures of Efficiency: Properties and Measurement

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Abstract. Conventional radial efficiency measurement models in data envelopment analysis are unable to produce appropriate efficiency scores for production units lying outside the cone generated by the convex hull of the extreme efficient production units. In addition, in the case of production technologies with variable returns to scale, the efficiency scores measured from the input and output sides are usually different. To solve these problems, the Russell measure of efficiency, which takes both the inputs and outputs into account, has been proposed. However, the conventional Russell efficiency is measured under the least favorable conditions, rather than the general custom of measuring under the most favorable ones. This paper develops a model to measure Russell efficiency under the most favorable conditions in two forms, the average and the product. They can be transformed into a second-order cone program and a mixed integer linear program, respectively, so that the solution can be obtained efficiently. A case of Taiwanese commercial banks demonstrates that they are more reliable and representative than the radial measures. Since the most favorable measures are higher than the least favorable measures, and the targets for making improvements are the easiest to reach, they are more acceptable to the production units to be evaluated.

Keywords: Data envelopment analysis · Russell measure · Radial measure · Slacks-based measure

1 Introduction

Efficiency measurement is an important management task because it reveals the extent to which the performance of a production unit, or more generally, a decision making unit (DMU), has been unsatisfactory in the past and provides a direction for making improvements in the future. Many ideas for measuring efficiency have been proposed [14]. Since the seminal work of Charnes et al. [11], data envelopment analysis (DEA) has been considered an effective technique for measuring the relative efficiency of a set of DMUs that applies multiple inputs to produce multiple outputs.

Charnes et al.'s model [11], usually referred to as the CCR model, is applied to production technologies with constant returns to scale (CRS). Banker et al. [7] developed a modified model that allows for technologies with variable returns to scale (VRS). This model is commonly referred to as the BCC model in the literature. The efficiencies

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measured from the CCR and BCC models are a form of radial measure. One weakness of this type of efficiency measure is that the efficiency scores of the DMUs lying outside the cone generated by the convex hull of the extreme efficient DMUs cannot be appropriately assigned. The radial efficiency can be measured from either the input or output side. In the case of the BCC model, there is another weakness. While the efficiencies measured from the input and output sides are the same for the CCR model, they are usually different for the BCC model. Which model should be used between the input and output sides depends on the purpose of the evaluation. When there is no specific purpose, there is no rule to follow in deciding which model to use.

One way to solve these problems is to use the Russell measure of efficiency [12, 13] to take all the inputs and outputs into account. The corresponding model is nonlinear. To obtain a linear model, Pastor et al. [20] proposed an enhanced Russell efficiency measure. Tone [23] termed this measure the slacks-based measure (SBM).

One feature of the DEA methodology is it allows the DMUs being evaluated to select the most favorable conditions by which to measure efficiency. This feature makes this methodology widely accepted for performance evaluation. While the Russell measures can solve the problems of inappropriate efficiency scores being assigned to certain DMUs and different results being obtained from the input and output models, they are calculated under the least favorable conditions for inefficient DMUs. In other words, the target on the production frontier selected for measuring efficiency is the farthest, rather than the general custom of being the closest, point to the DMU being evaluated. The results are thus unfair to inefficient DMUs.

Various approaches for measuring efficiency based on the closest targets have been proposed in the literature, starting with the works of Briec [8, 9]. The major differences of the approaches are the ways in which the production frontier and distance are defined. For example, Aparicio et al. [6] developed a mixed integer linear programming model to find the closest target in the conventional production possibility set. Aparicio and Pastor [4, 5] searched for the closest target in the extended facet production possibility set defined by Olesen and Petersen [19]. Fukuyama et al. [15] investigated the least-distance p-norm measures on an extended free disposable set based on the work of Ando et al. [1]. Petersen [21] developed a model to find the direction with the shortest distance to the production frontier. Aparicio [2] conducted a survey of the literature on this topic.

In this paper, we develop a model to measure the most favorable Russell efficiency based on the frontier used in the conventional way of measuring the least favorable Russell efficiency. Gonzaléz and Álvarez [16] initiated this study with an input-oriented Russell measure. Aparicio et al. [6] developed a model in the primal (envelopment form) and dual (multiplier form) combined spaces to measure the non-oriented Russell measure. However, Aparicio et al. [3] showed that, while this model works correctly for non-oriented measures, it cannot be successfully applied to input- or output-oriented measures, and they proposed a bilevel linear programming model. The model developed in the current study has two forms, the average and the product. The former can be transformed into a second-order cone program and the latter into a mixed integer linear program such that both can be solved efficiently. Since more favorable efficiency measures imply higher efficiency scores and closer targets for inefficient DMUs to reach with less effort, the results are more persuasive and acceptable to the DMUs being evaluated.

2 Conventional Efficiency Measures

Suppose a set of n DMUs that applies m inputs X_i, $i = 1, \ldots, m$ to produce s outputs Y_r, $r = 1, \ldots, s$. Let X_{ij} and Y_{rj} denote the ith input and rth output, respectively, of DMU j, $j = 1, \ldots, n$. The production possibility set constructed from these DMUs under variable returns to scale is $T = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \sum_{j=1}^{n} \lambda_j X_{ij} \leq x_i,\ i = 1, \ldots, m,\ \sum_{j=1}^{n} \lambda_j Y_{rj} \geq y_r,\ r = 1, \ldots, s,\ \sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geq 0, j = 1, \ldots, n\}$. The strongly efficient frontier of this set is $\partial^{S}(T) = \{(\boldsymbol{x}, \boldsymbol{y}) \in T \mid \hat{\boldsymbol{x}} \leq \boldsymbol{x}, \hat{\boldsymbol{y}} \geq \boldsymbol{y}, \text{ and } (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \neq (\boldsymbol{x}, \boldsymbol{y}) \Rightarrow (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \notin T\}$, which is the set of strongly efficient points of T. Theoretically, a DMU should select a point on the strongly efficient frontier to measure efficiency. The BCC model [7] for measuring the efficiency of DMU k in the envelopment form can be formulated from the input or output side, as follows:

Input-orientation

$$\theta_k^{\mathrm{I}} = \min.\ \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \tag{1a}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j X_{ij} + s_i^- = \theta X_{ik}, \quad i = 1, \ldots, m$$

$$\sum_{j=1}^{n} \lambda_j Y_{rj} - s_r^+ = Y_{rk}, \quad r = 1, \ldots, s$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j,\ s_i^-,\ s_r^+ \geq 0, \quad j = 1, \ldots, n,\ i = 1, \ldots,\ m,\ r = 1, \ldots, s$$

$$\theta \text{ unrestricted in sign.}$$

Output-orientation

$$\frac{1}{\theta_k^{\mathrm{O}}} = \max.\ \varphi + \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \tag{1b}$$

$$\text{s.t.} \ \sum_{j=1}^{n} \lambda_j X_{ij} + s_i^- = X_{ik},\ i = 1, \ldots, m$$

$$\sum_{j=1}^{n} \lambda_j Y_{rj} - s_r^+ = \varphi Y_{rk}\ ,\ r = 1, \ldots, s$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j,\ s_i^-,\ s_r^+ \geq 0, \quad j = 1,\ldots,n,\ i = 1,\ldots,\ m,\ r = 1,\ \ldots,\ s,$$

$$\varphi \text{ unrestricted in sign,}$$

where ε is a small non-Archimedean number used to avoid ignoring the unfavorable factors when measuring efficiency. The input efficiency θ_k^{I} and output efficiency θ_k^{O} need not be the same. When the convexity constraint $\sum_{j=1}^{n} \lambda_j = 1$ is deleted, the BCC model becomes the CCR model [11]. In this case, the input and output models produce the same efficiency score, which is denoted as θ_k^{CCR} in this paper.

One way to solve the problems caused by the non-Archimedean number and input-output difference in efficiency measurement is to apply a non-radial measure, such as the Russell measure of efficiency. The Russell measure under variable returns to scale is calculated via the following model [12]:

$$R_k^{\min} = \min.\ \frac{1}{m+s}\left(\sum_{i=1}^{m} \theta_i + \sum_{r=1}^{s} \frac{1}{\varphi_r}\right) \tag{2}$$

$$\text{s.t.} \sum_{j=1}^{n} \lambda_j X_{ij} \leq \theta_i X_{ik},\ \theta_i \leq 1, \quad i = 1,\ldots,m$$

$$\sum_{j=1}^{n} \lambda_j Y_{rj} \geq \varphi_r Y_{rk},\ \varphi_r \geq 1,\ r = 1,\ldots,s$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \geq 0, \qquad j = 1,\ldots,n.$$

The efficiency is defined as the average of individual factor efficiencies. The constraints $\theta_i \leq 1$ and $\varphi_r \geq 1$ are imposed to restrict the target points for evaluating efficiency to those that dominate the DMU being evaluated. If an assumption of constant returns to scale is desired, then one simply deletes the convexity constraint $\sum_{j=1}^{n} \lambda_j = 1$.

The Russell measure defines efficiency as the average of the efficiencies of all input and output factors. Pastor et al. [20] and Tone [23] defined efficiency as the product of the arithmetic average of the efficiencies of the *m* inputs and the harmonic average of the efficiencies of the *s* outputs in the form of:

$$Q_k^{\min} = \min. \ \frac{\frac{1}{m}\sum_{i=1}^{m}\theta_i}{\frac{1}{s}\sum_{r=1}^{s}\varphi_r},$$

subject to the same constraints as those in Model (2). Substituting θ_i with $(X_{ik} - s_i^-)/X_{ik}$ and φ_r with $(Y_{rk} + s_r^+)/Y_{rk}$, one obtains the following equivalent model:

$$Q_k^{\min} = \min. \ \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^-/X_{ik}}{1 + \frac{1}{s}\sum_{r=1}^{s} s_r^+/Y_{rk}} \tag{3}$$

$$\text{s.t.} \ \sum_{j=1}^{n} \lambda_j X_{ij} + s_i^- = X_{ik}, \ i = 1, \ldots, m$$

$$\sum_{j=1}^{n} \lambda_j Y_{rj} - s_r^+ = Y_{rk}, \ r = 1, \ldots, s$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j, \ s_i^-, \ s_r^+ \geq 0, \quad j = 1, \ldots, n, \ i = 1, \ldots, m, \ r = 1, \ldots, s.$$

This model is called the slacks-based measure (SBM) model in Tone [23]. The advantage of this model over Model (2) is that Model (2) is a nonlinear program, while this model is a fractional linear program, which can be linearized by applying a variable substitution technique proposed in Charnes and Cooper [10].

Different from the radial measure that requires either all inputs to be reduced in the same proportion θ as in Model (1a) or all outputs to be expanded in the same proportion φ as in Model (1b), the Russell measure takes the inputs and outputs into account at the same time, and the proportions θ_i and φ_r can be different for different factors. More importantly, the projection point used to measure efficiency is on the strongly efficient frontier. Pastor et al. [20] and Tone [23] proved that the Russell efficiency measure of the product form is less than or equal to both the input and output radial efficiency measures. In symbols, it is $Q_k^{\min} \leq \theta_k^{\mathrm{I}}$ and $Q_k^{\min} \leq \theta_k^{\mathrm{O}}$.

The objective of Model (2) or Model (3) is to find the greatest rates for reducing the inputs and expanding the outputs of the DMU being evaluated within the production possibility set at the same time. The purpose of the model is actually to identify the production frontier, rather than measuring efficiencies. The objective value, known as the efficiency of the DMU, is a by-product of this frontier identification process. However, since the objective function has a minimization direction, the efficiency measured from this model is the lowest among all possible measures, which contradicts the basic idea of DEA suggesting that efficiency is measured under the most favorable conditions.

3 Most Favorable Measures

The envelopment form of the BCC input model (1a) is intended to find the minimum value for θ to reduce the inputs of the DMU being evaluated such that the resulting point is still in the production possibility set. The purpose is to identify a frontier facet from the production possibility set based on which efficiency of this DMU is measured. If this DMU lies in the cone generated by the convex hull of the extreme efficient DMUs in the input space so that the slack variables are zero, then the target point $(\sum_{j=1}^{n} \lambda_j \boldsymbol{X}_j, \sum_{j=1}^{n} \lambda_j \boldsymbol{Y}_j) = (\theta_k^{\mathrm{I}} \boldsymbol{X}_k, \boldsymbol{Y}_k)$, where $\boldsymbol{X}_j = (X_{1j}, \ldots, X_{mj})$ and $\boldsymbol{Y}_j = (Y_{1j}, \ldots, Y_{sj})$, reflects that its efficiency is θ_k^{I}, which is a by-product of this process. Conceptually, one should find the maximum value for θ to be the most favorable efficiency measure after all the frontier facets are identified. Due to the geometric property of the radial measures, the minimum and maximum values for θ are the same. Consider six DMUs, labelled as $A \sim F$ in Fig. 1, which apply different combinations of inputs X_1 and X_2 to produce one output Y. In measuring the efficiency of DMU D, the idea of the BCC input model is to identify a frontier facet by reducing X_{1D} and X_{2D} in the same proportion of θ along the ray $\overrightarrow{OD}$ until it reaches the boundary of the production possibility set at $\hat{D}$. The minimum value for θ, or the largest extent of contraction, is θ_k^{I}, which is the ratio of $O\hat{D}$ to OD. After all the frontier facets are identified, the strongly efficient frontier is then determined, and the efficiency is measured as the largest value for θ such that θD on the ray $O\hat{D}$ intercepts the strongly efficient frontier in the region of D' to D''. Since the intersection of the ray $\overrightarrow{OD}$ with the strongly efficient frontier in the region of D' to D'' is the unique point $\hat{D}$, the minimum and maximum values of θ are the same.

In measuring the Russell efficiency, all inputs and outputs are allowed to contract and expand in different proportions, respectively. The minimum and maximum values for the efficiency in this case may not be the same. More specifically, the target point found in the process of identifying the frontier facet via minimizing the distance parameters may not be the same as that found in the process of maximizing the parameters. For example, the efficiency of DMU D in Fig. 1 calculated from Model (2) is actually the lowest that can be obtained by using the points on the strongly efficient frontier in the region of D' to D'' as the target. The idea of the DEA technique, however, is to measure the efficiency under the most favorable conditions. Following this idea, one should search for a target in the region of D' to D'' that can produce the highest efficiency. The procedure for accomplishing this task can be separated into two phases, where Phase I is to identify the strongly efficient frontier and Phase II is to find a point on the strongly efficient frontier that will produce the highest efficiency.

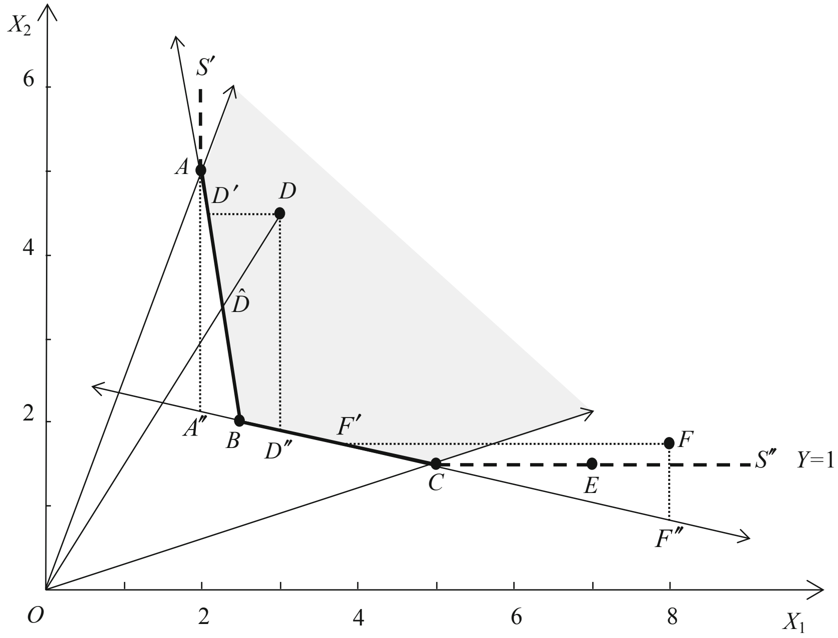

Fig. 1. Geometric interpretation of the efficiency measurement of various models.

To construct the strongly efficient frontier, all extreme efficient DMUs of the production possibility set that span the full dimensional efficient facets are identified first by applying any DEA model, e.g., Model (1a). Each strongly efficient frontier facet is the convex hull of a set of $m + s$ neighboring extreme efficient DMUs, provided the hyperplane extended from all sides of this facet envelops all DMUs. Let E_0 denote the set of the indices of $m + s$ extreme efficient DMUs such that the convex hull of these $m + s$ DMUs is a frontier facet F_0. The frontier facet F_0 can be expressed as $F_0 = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \sum_{j \in E_0} \lambda_j X_{ij} = x_i,\ i = 1, \ldots, m,\ \sum_{j \in E_0} \lambda_j Y_{rj} = y_r,\ r = 1, \ldots, s,\ \sum_{j \in E_0} \lambda_j = 1,\ \lambda_j \geq 0,\ j \in E_0\}$. The frontier hyperplane extended from this frontier facet is $H_0 = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \sum_{j \in E_0} \lambda_j X_{ij} = x_i,\ i = 1, \ldots, m,\ \sum_{j \in E_0} \lambda_j Y_{rj} = y_r,\ r = 1, \ldots, s,\ \sum_{j \in E_0} \lambda_j = 1,\ \lambda_j$ unrestricted in sign, $j \in E_0\}$. The mathematical expression of the frontier hyperplane H_0 differs from the frontier facet F_0 only in that the values of λ_j are allowed to be negative. Since F_0 is a frontier facet, the corresponding frontier hyperplane H_0 must envelop all n DMUs. In this case, every DMU d must have a projection point (target) on the hyperplane H_0 that dominates itself. The projection point for DMU d can be expressed as $(\sum_{j \in E_0} \lambda_j^{(d)} X_{ij}, \sum_{j \in E_0} \lambda_j^{(d)} Y_{rj})$, where $\sum_{j \in E_0} \lambda_j^{(d)} = 1$ and $\lambda_j^{(d)}$ are unrestricted in sign. Since every DMU d is dominated by its projection point $(\sum_{j \in E_0} \lambda_j^{(d)} X_{ij}, \sum_{j \in E_0} \lambda_j^{(d)} Y_{rj})$, we have $\sum_{j \in E_0} \lambda_j^{(d)} X_{ij} + s_i^{(d)-} = X_{id},\ i = 1, \ldots, m,\ d = 1, \ldots, n$ and $\sum_{j \in E_0} \lambda_j^{(d)} Y_{rj} - s_r^{(d)+} = Y_{rd},\ r = 1, \ldots, s,\ d = 1, \ldots, n$, where $s_i^{(d)-}, s_r^{(d)+} \geq 0$.

Using the DMUs in Fig. 1 to explain this, line segment $\overline{BC}$ is a frontier facet that can be expressed as $F = \{Z \,|\, Z = \lambda_B B + \lambda_C C, \lambda_B + \lambda_C = 1, \lambda_B, \lambda_C \geq 0\}$. The entire line $\overleftrightarrow{BC}$ is expressed as $H = \{Z \mid Z = \lambda_B B + \lambda_C C,\ \lambda_B + \lambda_C = 1,\ \lambda_B,\ \lambda_C$ unrestricted in sign$\}$. Since $\overline{BC}$ is a frontier facet, the corresponding line $\overleftrightarrow{BC}$ must envelop all DMUs by having nonnegative slacks. Consider four DMUs, A, D, C, and F, which can be projected to A", D", C" = C, and F" on Line $\overleftrightarrow{BC}$, respectively, by fixing X_1 and Y at their current values. For DMU D, we have $D" = \lambda_B^{(D)}B + \lambda_C^{(D)}C = (3,\ 1.9;\ 1)^T$, with $\lambda_B^{(D)} = 0.8$ and $\lambda_C^{(D)} = 0.2$. Positive values for $\lambda_B^{(D)}$ and $\lambda_C^{(D)}$ indicate that D" is located on line segment $\overline{BC}$. The corresponding slack variables have nonnegative values of $s_1^{(D)-} = 0$, $s_2^{(D)-} = 4.5 - 1.9 = 2.6$, and $s_1^{(D)+} = 0$. For DMU C, we have $C" = C$, with $\lambda_B^{(C)} = 0$ and $\lambda_C^{(C)} = 1$, and all the slacks are zero. For DMU A, we have $A" = \lambda_B^{(A)}B + \lambda_C^{(A)}C = (2,\ 2.1;\ 1)^T$, with $\lambda_B^{(A)} = 1.2$ and $\lambda_C^{(A)} = -0.2$, where $\lambda_C^{(A)}$ is negative. Positive $\lambda_B^{(A)}$ and negative $\lambda_C^{(A)}$ indicate that A" is located to the left of DMU B on line $\overleftrightarrow{BC}$. The corresponding slacks are $s_1^{(A)-} = 0$, $s_2^{(A)-} = 5 - 2.1 = 2.9$, and $s_1^{(A)+} = 0$, which are nonnegative. Finally, for DMU F, we have $F" = \lambda_B^{(F)}B + \lambda_C^{(F)}C = (8,\ 0.9;\ 1)^T$, with $\lambda_B^{(F)} = -1.2$ and $\lambda_C^{(F)} = 2.2$, where $\lambda_B^{(F)}$ is negative. Negative $\lambda_B^{(F)}$ and positive $\lambda_C^{(F)}$ indicate that F" is located to the right of DMU C on line $\overleftrightarrow{BC}$. The corresponding slacks are $s_1^{(F)-} = 0$, $s_2^{(F)-} = 1.75 - 0.9 = 0.85$, and $s_1^{(F)+} = 0$, which, again, are nonnegative.

Since it is not known beforehand which frontier facet of the strongly efficient frontier will be selected by DMU k to find the target to measure efficiency, all frontier facets must be considered. Let E denote the set of the indices of the extreme efficient DMUs. We use the binary variable B_j to indicate whether or not an extreme efficient DMU j is used to span the frontier facet. The conditions for DMU k to consider all possible frontier facets to measure efficiency can be expressed as:

$$\sum_{j \in E} \lambda_j^{(d)} X_{ij} + s_i^{(d)-} = X_{id},\ i = 1, \ldots, m, \quad d = 1, \ldots, n \tag{4.1}$$

$$\sum_{j \in E} \lambda_j^{(d)} Y_{rj} - s_r^{(d)+} = Y_{rd}, \quad r = 1, \ldots, s, \quad d = 1, \ldots, n \tag{4.2}$$

$$\sum_{j \in E} \lambda_j^{(d)} = 1, \qquad d = 1, \ldots, n \tag{4.3}$$

$$s_i^{(d)-},\ s_r^{(d)+} \geq 0, \qquad i = 1, \ldots, m, \quad r = 1, \ldots, s, \quad d = 1, \ldots, n \tag{4.4}$$

$$\lambda_j^{(k)} \geq 0, \quad j \in E \tag{4.5}$$

$$\lambda_j^{(d)} \text{ unrestricted in sign}, \ \ j \in E, \qquad d = 1, \ldots, n \quad d \neq k \tag{4.6}$$

$$-MB_j \le \lambda_j^{(d)} \le MB_j, \quad j \in E, \quad d = 1, \ldots, n \tag{4.7}$$

$$\sum\nolimits_{j \in E} B_j \le m + s \tag{4.8}$$

$$B_j \in \{0,\ 1\}, \quad j \in E, \tag{4.9}$$

where M is a large number for allowing all possible $\lambda_j^{(d)}$ values to appear. The frontier facet spanned by the efficient DMUs corresponding to $B_j = 1$ is the facet for DMU k to measure efficiency. Constraints (4.1)–(4.4), for $d = k$, and (4.5) require the assessed DMU k to select a point on this facet to calculate efficiency. Constraints (4.1)–(4.4) and (4.6) ensure that all DMUs are enveloped by the hyperplane extended from the frontier facet.

The frontier hyperplane has a dimension of $m + s$. The sum of B_j is thus equal to $m + s$. However, to account for the degenerate case where the number of extreme efficient DMUs is less than $m + s$, we require $\sum_{j \in E} B_j \le m + s$ in Constraint (4.8).

For cases of constant returns to scale, the convexity constraint $\sum_{j \in E} \lambda_j^{(d)} = 1$ is not needed. Moreover, since the frontier facets must pass through the origin, this implies that the origin must always be used with the $m + s - 1$ of other efficient DMUs to constitute the frontier facet. Thus, $m + s$ in constraint (4.8) is changed to $m + s - 1$.

To measure the Russell efficiency based on the closest target to the assessed DMU, one first applies Model (2), or any DEA model, to identify the extreme efficient DMUs, with their indices comprising the set E. One then uses the following mathematical program to calculate the efficiency of DMU k:

$$R_k^{\max} = \max.\ \frac{1}{m+s}\left[\sum_{i=1}^{m}\left(\frac{X_{ik} - s_i^{(k)-}}{X_{ik}}\right) + \sum_{r=1}^{s}\left(\frac{Y_{rk}}{Y_{rk} + s_r^{(k)+}}\right)\right] \tag{5}$$

$$\text{s.t.}\quad \text{Constraint Set } (4).$$

The objective function is nonlinear, and the constraints are linear, in that some binary variables are involved. This model can be solved efficiently by transforming it into a second-order cone program, as proposed in Sueyoshi and Sekitani [22] or a semidefinite program, as discussed in Halická and Trnovská [17].

Similarly, the Russell measure of the product form, i.e., the slacks-based measure, with the target closest to the assessed DMU can be calculated via the following model:

$$Q_k^{\max} = \max.\ \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^{(k)-}/X_{ik}}{1 + \frac{1}{s}\sum_{r=1}^{s} s_r^{(k)+}/Y_{rk}} \tag{6}$$

$$\text{s.t.}\quad \text{Constraint Set } (4).$$

This model is a fractional mixed integer program. By applying the variable substitution technique proposed in Charnes and Cooper [10], with $1/(1+\frac{1}{s}\sum_{r=1}^{s} s_r^{(k)+}/Y_{rk}) = w$, $w\lambda_j^{(d)} = \mu_j^{(d)}$, $ws_i^{(d)-} = t_i^{(d)-}$, and $ws_r^{(d)+} = t_r^{(d)+}$, a linear mixed integer program for the VRS case is obtained as follows:

$$Q_k^{\max} = \max.\ w - \frac{1}{m}\sum_{i=1}^{m}\frac{t_i^{(k)-}}{X_{ik}} \tag{7}$$

$$\text{s.t.}\ \ w + \frac{1}{s}\sum_{r=1}^{s}\frac{t_r^{(k)+}}{Y_{rk}} = 1$$

$$\sum\nolimits_{j\in E}\mu_j^{(d)}X_{ij} + t_i^{(d)-} = wX_{id} \qquad i=1,\ldots,m,\quad d=1,\ldots,n$$

$$\sum\nolimits_{j\in E}\mu_j^{(d)}Y_{rj} - t_r^{(d)+} = wY_{rd},\quad r=1,\ldots,s,\quad d=1,\ldots,n$$

$$\sum\nolimits_{j\in E}\mu_j^{(d)} = w,\qquad d=1,\ldots,n$$

$$t_i^{(d)-}, t_r^{(d)+} \geq 0,\qquad i=1,\ldots,m,\ r=1,\ldots,s,\qquad d=1,\ldots,n$$

$$\mu_j^{(k)} \geq 0,\ j\in E$$

$$\mu_j^{(d)}\text{unrestricted in sign},\qquad j\in E, d=1,\ldots,n,\qquad d\neq k$$

$$-MB_j \leq \mu_j^{(d)} \leq MB_j,\quad j\in E,\quad d=1,\ldots,n$$

$$\sum\nolimits_{j\in E} B_j \leq m+s$$

$$w \geq 0$$

$$B_j \in \{0,\ 1\},\qquad\qquad j\in E,$$

This model is much easier than Model (6) to solve.

4 Some Properties

The most favorable Russell measures of efficiency have several properties. First, it is noted that the conventional Russell measure of the product form, $R_k^{\min}$, is calculated based on the target that is the farthest to the assessed DMU. It can be calculated by changing the direction of optimization in Model (5) from maximization to minimization although the model in this case is more complicated than Model (2). This is also true for the product form of Model (6). We thus have the following theorem:

Theorem 1. The most favorable Russell measures of efficiency, both the average and product forms, are greater than or equal to the least favorable measures.

Second, every DMU uses a point on the strongly efficient frontier as the target to measure efficiency. The constraints (4.1)–(4.4) for $d = k$ and (4.5) in models (5) and (6), where all X_{ik} and Y_{rk} are positive, ensure that their most favorable Russell measures, both the average and product forms, are always positive. This leads to the following theorem:

Theorem 2. The most favorable Russell measures are always positive.

Third, models (2) and (3) use the average and product, respectively, of the input and output efficiencies as the DMU efficiency. The output efficiency in Model (3) is a harmonic average of the efficiencies of individual outputs, instead of the usual arithmetic average. To make the two measures comparable, the Russell efficiency of the average form can be defined as the average of the arithmetic average of the input efficiencies and the harmonic average of the output efficiencies, that is,

$$\hat{R}_k^{\max} = \frac{1}{2}[(\frac{1}{m}\sum_{i=1}^{m}\theta_i) + (\frac{1}{\frac{1}{s}\sum_{r=1}^{s}\varphi_r})].$$

Based on the arithmetic-geometric mean inequality stipulating that the arithmetic mean is greater than or equal to the geometric mean, we have the following relationship:

$$\begin{aligned}\hat{R}_k^{\max} &= \frac{1}{2}[(\frac{1}{m}\sum_{i=1}^{m}\hat{\theta}_i) + (\frac{1}{\frac{1}{s}\sum_{r=1}^{s}\hat{\varphi}_r})] \geq \frac{1}{2}[(\frac{1}{m}\sum_{i=1}^{m}\theta_i^*) + (\frac{1}{\frac{1}{s}\sum_{r=1}^{s}\varphi_r^*})]\\ &\geq \left(\frac{\frac{1}{m}\sum_{i=1}^{m}\theta_i^*}{\frac{1}{s}\sum_{r=1}^{s}\varphi_r^*}\right)^{1/2} \geq \frac{\frac{1}{m}\sum_{i=1}^{m}\theta_i^*}{\frac{1}{s}\sum_{r=1}^{s}\varphi_r^*} = Q_k^{\max},\end{aligned}$$

where $(\hat{\theta}_i, \hat{\varphi}_r)$ and $(\theta_i^*, \varphi_r^*)$ are the optimal solutions corresponding to $\hat{R}_k^{\max}$ and $Q_k^{\max}$, respectively. The last inequality is obtained due to the fact that the value in the parentheses is less than or equal to one, and its square has a smaller value. This proves the following theorem:

Theorem 3. When the output efficiency in the Russell measure of the average and product forms is defined as the same, the Russell measure of the average form, $\hat{R}_k^{\max}$, is greater than or equal to that of the product form, $Q_k^{\max}$.

Finally, in radial measures, the efficiency scores are difficult to interpret when some slack variables have positive values. Geometrically, if a DMU lies in the cone generated by the extreme efficient DMUs, then all the slack variables will be zero when using the radial model to measure efficiency. In this case, the radial input efficiency of a DMU k can be measured via Model (5) with the constraints corresponding to DMU k replaced with $\sum_{j\in S}\lambda_j^{(k)}X_{ij} = \theta X_{ik}$, $i = 1, \ldots, m$ and $\sum_{j\in S}\lambda_j^{(k)}Y_{rj} = Y_{rk}$, $r = 1, \ldots, s$, and the objective function replaced with min θ. If we change the direction of optimization from minimization to maximization, we still obtain the same objective value because the ray emanating from the origin to DMU k intersects the frontier facet at only one point. We thus have $\theta^* = \min\ \theta = \max\ \theta$.

To compare the most favorable Russell measure with the radial input measure, we can formulate the constraints corresponding to DMU k in Model (5) as $\sum_{j \in S} \lambda_j^{(k)} X_{ij} = \theta_i X_{ik}$, $\theta_i \leq 1$, $i = 1,\ldots, m$ and $\sum_{j \in S} \lambda_j^{(k)} Y_{rj} = \varphi_r Y_{rk}$, $\varphi_r \geq 1$, $r = 1, \ldots, s$, with the objective function of

$$R_k^{\max} = \max \frac{1}{m+s} \left(\sum_{i=1}^{m} \theta_i + \sum_{r=1}^{s} \frac{1}{\varphi_r}\right).$$

Since BCC input efficiency θ_k^{I} is a special case of the average form of the Russell measure for $\theta_i = \theta$ for all i, and $\varphi_r = 1$ for all r, we have

$$R_k^{\max} = \max \frac{1}{m+s}\left(\sum_{i=1}^{m} \theta_i + \sum_{r=1}^{s} \frac{1}{\varphi_r}\right) \geq \max \frac{1}{m+s}(m\theta + s) \geq \max\ \theta = \theta_k^{\mathrm{I}}.$$

Similarly, since BCC output efficiency θ_k^{O} is a special case of the average form of the Russell measure for $\theta_i = 1$ for all i, and $\varphi_r = \varphi$ for all r, we have

$$R_k^{\max} = \max \frac{1}{m+s}\left(\sum_{i=1}^{m} \theta_i + \sum_{r=1}^{s} \frac{1}{\varphi_r}\right) \geq \max \frac{1}{m+s}\left(m + \frac{s}{\varphi}\right) \geq \max \frac{1}{\varphi} = \theta_k^{\mathrm{O}}.$$

We thus have the following theorem:

Theorem 4. For DMUs lying in the cone generated by the convex hull of the extreme efficient DMUs, the most favorable Russell measure $R_k^{\max}$ is greater than or equal to both the radial input measure θ_k^{I} and output measure θ_k^{O}.

This theorem also holds for production technologies of constant returns to scale. Combined with the property where the conventional least favorable Russell measure $Q_k^{\min}$ is less than or equal to both BCC input efficiency θ_k^{I} and output efficiency θ_k^{O}, we have the following result for DMU k:

$$Q_k^{\min} \leq \left\{ \begin{array}{c} \theta_k^{\mathrm{I}} \\ \theta_k^{\mathrm{O}} \end{array} \right\} \leq R_k^{\max}.$$

Note that the second inequality holds only for DMUs lying in the cone generated by the extreme efficient DMUs, while the first holds for all situations.

Table 1. Efficiencies measured from different models for the twelve inefficient banks.

Bank	Radial efficiency	Russell efficiency			
	CCR	Average form		Product form (SBM)	
	θ_k^{CCR} (rank)	R_k^{min} (rank)	R_k^{max} (rank)	Q_k^{min} (rank)	Q_k^{max} (rank)
1	0.9960 (1)	0.9571 (1)	0.9960 (1)	0.8964 (2)	0.9920 (1)
2	0.9498 (5)	0.9182 (3)	0.9905 (2)	0.8388 (3)	0.9810 (2)
5	0.9933 (2)	0.8532 (6)	0.8634 (8)	0.6985 (5)	0.7143 (7)
7	0.8894 (8)	0.7662 (9)	0.8705 (7)	0.3389 (11)	0.6462 (9)
8	0.7328 (12)	0.6837 (11)	0.7577 (12)	0.2642 (12)	0.5721 (12)
9	0.9877 (4)	0.9477 (2)	0.9644 (3)	0.8971 (1)	0.9290 (3)
11	0.9379 (6)	0.8592 (5)	0.9558 (4)	0.7271 (4)	0.9122 (4)
12	0.9910 (3)	0.8900 (4)	0.9501 (5)	0.6252 (7)	0.8763 (5)
15	0.8607 (9)	0.7445 (10)	0.8233 (9)	0.4195 (9)	0.6654 (8)
17	0.9333 (7)	0.7764 (8)	0.8076 (10)	0.4536 (8)	0.6269 (10)
21	0.8548 (10)	0.8176 (7)	0.9131 (6)	0.6280 (6)	0.8279 (6)
23	0.7593 (11)	0.6835 (12)	0.7819 (11)	0.3979 (10)	0.5804 (11)
Ave.	0.9072	0.8248	0.8895	0.5988	0.7770

5 Taiwanese Commercial Banks

In a study predicting the performance of banks, Kao and Liu [18] measured the efficiencies of twenty-four Taiwanese commercial banks using total deposits, interest expenses, and non-interest expenses as the inputs and total loans, interest income, and non-interest income as the outputs.

By applying the conventional CCR model to the data in Kao and Liu [18], the efficiencies of the twenty-four banks under constant returns to scale are calculated. There are twelve banks that are efficient. Column two of Table 1 shows the results for the twelve inefficient banks. The numbers in parentheses are the ranks of the banks among those that are inefficient. In calculating the CCR efficiency, it is noted that only Bank No. 2 of these twelve inefficient banks lies in the cone generated by the extremely efficient banks. In other words, the other eleven banks have at least one slack variable with positive values. Their efficiency scores are dependent on the values assigned to the non-Archimedean number ε. The rankings obtained from the CCR efficiency scores are thus not reliable.

By applying Model (5) under constant returns to scale with the objectives of minimization and maximization, the least and most favorable Russell measures of efficiency of the average form for the twelve inefficient banks are calculated, respectively. The results are shown in columns three and four of Table 1. As expected, the most favorable measures are greater than the least favorable measures for all twelve banks. The average of the most favorable measures of 0.8895, as shown in the last row, is 7.84% higher than that of the least favorable measures of 0.8248. The rankings based on the two measures are slightly different, with a mean absolute difference of 1.17 ranks.

According to Theorem 4, the most favorable Russell measures of those DMUs lying in the cone generated by the extreme efficient DMUs are greater than or equal to their radial measures. This implies that $R_k^{\max}$ in column four of Table 1 must be greater than or equal to the corresponding θ_k^{CCR} in column two. However, since only Bank No. 2 is in the defined cone, this relationship only holds for six of the twelve inefficient banks. The rankings based on $R_k^{\max}$ are quite different from those based on θ_k^{CCR}. The largest difference between the two rankings occurs for Bank No. 5, with a difference of six ranks. The mean absolute difference between the two rankings is 1.83 ranks. Due to the effect of the positive slack values on the efficiency scores, the rankings based on $R_k^{\max}$ are more reliable than those based on θ_k^{CCR}.

The product form of the least and most favorable Russell measures for the twelve inefficient banks under CRS can be calculated via Model (7), with the objectives of minimization and maximization, respectively. The results are shown in the last two columns of Table 1. The latter is obviously greater than the former for every bank. The average scores shown in the last row of Table 1 indicate that the latter is 29.76% higher than the former. The rankings based on these two measures are also different, with a mean absolute difference of 1.16 ranks.

Based on the theorem proved in Pastor et al. [20] and Tone [23], the least favorable Russell measures of the DMUs must be less than or equal to their radial measures. By comparing the numbers in columns two and five, this property is confirmed, and their averages show that the latter is 34% lower than the former.

Another pair of measures worth comparing is the least favorable Russell measures of the product form, $Q_k^{\min}$, and the most favorable Russell measures of the average form, $R_k^{\max}$. The former is the conventional SBM, which has the lowest efficiency measures among all types of Russell measures, while the latter, in contrast, has the highest efficiency measure. This is actually a consequence of Theorem 4. The numbers in columns four and five show that $R_k^{\max}$ is indeed greater than $Q_k^{\min}$ for every bank, and the average of the former, 0.8895, is 48.5% higher than that of the latter, 0.5988. The rankings based on the two measures differ not by much, with a mean absolute difference of 1.33 ranks. Since the efficiency measure of the former is higher, and the corresponding target is closer to the assessed bank, making it easier to reach, it is more acceptable to the banks being evaluated.

All the discussions in this example are based on the assumption of constant returns to scale. Similar discussions can be made under the assumption of variable returns to scale.

6 Conclusion

The Russell measure of efficiency was proposed to solve the problems of radial measures of efficiency that cannot provide appropriate efficiency scores for inefficient DMUs lying outside the cone generated by the convex hull of the extreme efficient DMUs and different scores produced from the input and output models under variable returns to scale. While the conventional Russell measures can be used to solve these problems, they are the least favorable measures, which contradict the idea of DEA that efficiency should be measured under the most favorable conditions. Moreover, the targets associated with the measures are more difficult for inefficient DMUs to reach to become efficient. To amend this drawback, a model is developed in this paper to calculate Russell measures based on the target that is closest to the assessed DMU.

Two forms of the Russell measure are considered, the average and the product. It is proven that, first, the average form produces higher efficiency scores than the product form when the output efficiency is defined in the same way. Second, the most favorable Russell measures of the average form are greater than or equal to the radial measures for DMUs lying in the cone generated by the extreme efficient DMUs. A case of Taiwanese commercial banks confirms these findings. In real world applications, the most favorable efficiency measures produce a target that requires least effort for an inefficient DMU to reach to become efficient. The corresponding rankings provide better information for the top management to make appropriate decisions. For these reasons, the most favorable Russell measures are more reliable and representative, and are more acceptable to the DMUs to be evaluated as their efficiency scores.

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