The Continuous Hotelling Pure Location Game with Elastic Demand Revisited

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Abstract. The Hotelling pure location game has been revisited. It is assumed that there are two identical players, strategy sets are one-dimensional, and demand as a function of distance is constant or strictly decreasing. Besides qualitative properties of conditional payoff functions, attention is given to the structure of the equilibrium set, best-response correspondences and the existence of potentials.

Keywords: Hotelling game · Potential game · Pure Nash equilibrium existence · Principle of Minimum Differentiation

1 Introduction

In mathematics, a strategy to make progress is by studying concrete examples and thereby trying to find out what drives the results. This in particular holds in game theory for the topic of Nash equilibria of games in strategic form. An important example in this context is Cournot oligopoly games. In the present article we consider another one: Hotelling games.

By 'Hotelling games', one understands a variety of games that appeared in the literature after the seminal article of Hotelling [1].[1] The focus of interest of the present article is pure location Hotelling games. In fact we consider the pure location part of the model in [1] dealing with two sellers of a homogeneous product locating a single plant on a finite one-dimensional geographic market.

The aim of our article is to further develop the theory for the Hotelling pure location game with elastic demand. The bulk of articles presupposes inelastic demand, meaning that the demand function f is constant; the elastic case was first treated by [2]. Our article deals with the more difficult elastic case. The game we consider is a generalisation of that in [7] in the sense that more general demand functions are allowed.

The article is organized as follows. In Sect. 2, we fix the setting. Section 3 makes some useful observations about Nash equilibria of games with location and player symmetry. Section 4 reviews the inelastic case. Before proceeding in Sect. 6 to the equilibrium structure of the elastic case, Sect. 5 establishes properties of game-theoretic fundamental objects for this case. Section 7 investigates in which sense the game is a potential game. Finally, Sect. 8 provides some concluding remarks.

[1] For an overview and discussion of the literature we refer to [5] and [6].

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2 Setting

Below we fix the setting for the Hotelling game that we are going to consider. Well, strategy sets are one-dimensional, there are two identical players and demand may be inelastic. We will allow for a non-continuous demand function. In order to distinguish the game in the present article from discrete variants (see [14] and references therein), we simply refer to it as the cHg ('continuous Hotelling game').

Throughout the whole article, S denotes a real interval $[0, L]$ with $L > 0$ and $f : S \to \mathbb{R}$ is a positive function which is constant or is strictly decreasing; without loss of generality we assume $f = 1$ in the case f is constant. The case of constant f is referred to as the inelastic case and the other one as elastic case.

In this article by a continuous Hotelling game (cHg), we understand a two-person game in strategic form with player set $N = \{1, 2\}$, common strategy set S and defining the function $\mathcal{L} : S \to \mathbb{R}$ by

$$\mathcal{L}(x) := \int_0^x f(z)\,dz$$

payoff functions $u_1, u_2 \colon S \times S \to \mathbb{R}$ given by,

$$u_i(x_1, x_2) = \begin{cases} \mathcal{L}(x_i) + \mathcal{L}(\frac{|x_1 - x_2|}{2}) & \text{if } x_i < x_j, \\ \mathcal{L}(L - x_i) + \mathcal{L}(\frac{|x_1 - x_2|}{2}) & \text{if } x_i > x_j, \\ \frac{1}{2}\big(\mathcal{L}(x_i) + \mathcal{L}(L - x_i)\big) & \text{if } x_i = x_j. \end{cases}$$

In the case where f even is continuously differentiable, the cHg becomes the game in [7]. We refer to f as a demand function.[2]

The functions $f_- : [0, L\,[\to \mathbb{R}$ and $f_+ : [0, L\,[\to \mathbb{R}$ are well-defined by

$$f_-(z) := \lim_{w \uparrow z} f(w), \quad f_+(z) := \lim_{w \downarrow z} f(w).$$

So f_- and f_+ are decreasing, $f_- \geq f \geq f_+$. One also knows that f_- is right continuous and that f_+ is left continuous.

$\mathcal{L}$ has the following simple properties:

A. $\mathcal{L} \geq 0$ and $\mathcal{L}(0) = 0$.
B. $\mathcal{L}$ is strictly increasing.
C. $\mathcal{L}$ is continuous.
D. $\mathcal{L}$ is linear if f is constant and $\mathcal{L}$ is strictly concave if f is strictly decreasing.
E. f is semidifferentiable: $D_-\mathcal{L}(x) = f_-(x)$ and $D_+\mathcal{L}(x) = f_+(x)$. And if f is continuous at x, then $\mathcal{L}$ is differentiable at x and $D\mathcal{L}(x) = f(x)$.

[2] Its standard interpretation in location theory concerns two competing vendors on a beach. The vendors simultaneously and independently select a position. Customers go to the closest vendor and split themselves evenly if the vendors choose an identical position. Each vendor wants to maximize his number of customers. One can reframe the interpretation as two candidates placing themselves along an ideological spectrum, with citizens voting for whichever one is closest (see e.g. [4]).

F. For $x' > x$ the inequalities $(x' - x)f(x') \leq \mathcal{L}(x') - \mathcal{L}(x) \leq (x' - x)f(x)$ hold and these inequalities are strict if f is strictly decreasing.

There is an interesting principle for the cHg, the so called Principle of Minimum Differentiation. For our (interpretation of the) cHg, this principle, coined by Boulding [3], comes down to that firms liking[3] to locate together. We formalize this principle for the cHg as follows: the Principle of Minimum Differentiation holds if the game has $(\frac{L}{2}, \frac{L}{2})$ as unique (pure) Nash equilibrium.

3 Games with Player and Location Symmetry

The content of this section is borrowed from [14].

In this section we consider a game in strategic form with two players 1, and 2, with common strategy set $S = [0, L]$ with $L > 0$, and with payoff functions $g_1, g_2 : S \times S \to \mathbb{R}$. Assume, player symmetry, i.e.

$$g_2(x_1, x_2) = g_1(x_2, x_1) \;\; (x_1, x_2 \in S).$$

Also assume,

$$g_i(x_1, x_2) = g_i(L - x_1, L - x_2) \;\; (x_1, x_2 \in S),$$

i.e. location symmetry. The cHg is an example of such a game.

We denote the conditional payoff function of player i where his opponent plays $x_j \in S$ by $g_i^{(x_j)}$; so $g_1^{(x_2)} : S \to \mathbb{R}$ is defined by $g_1^{(x_2)}(x_1) = g_1(x_1, x_2)$ and $g_2^{(x_1)} : S \to \mathbb{R}$ of player 2 is defined by $g_2^{(x_1)}(x_2) = g_2(x_1, x_2)$. With B_i we denote the best-response correspondence of player i; so $B_i : S \multimap S$.

The location symmetry implies the formulas[4]

$$g_i^{(L-z)}(x) = g_i^{(z)}(L - x) \text{ and } B_i(x) = \{L\} - B_i(L - x).$$

And player symmetry implies

$$g_1^{(z)} = g_2^{(z)} \text{ and } B_1 = B_2 \;=:\; B.$$

Denoting the Nash equilibrium set by E, player symmetry also implies for every $(e_1, e_2) \in E$ that $\{(e_1, e_2), (e_2, e_1)\} \subseteq E$ and location symmetry implies that $\{(e_1, e_2), (L - e_1, L - e_2)\} \subseteq E$. Thus for every $(e_1, e_2) \in E$

$$\{(e_1, e_2), (e_2, e_1), (L - e_1, L - e_2), (L - e_2, L - e_1)\} \subseteq E. \tag{1}$$

Having this, we like to see $(e_1, e_2), (e_2, e_1), (L - e_1, L - e_2), (L - e_2, L - e_1)$ as the same equilibrium. We formalize this by defining on E the relation $\sim$ by

$$(e_1, e_2) \sim (e_1', e_2') \text{ means:}$$

[3] However, see concluding remark 3 in Sect. 8.
[4] Here $\{L\} - B_i(L - x)$ is the Minkowski sum of the sets $\{L\}$ and $-B_i(L - x)$.

$$(e_1, e_2) \in \{(e'_1, e'_2), (e'_2, e'_1), (L - e'_1, L - e'_2), (L - e'_2, L - e'_1)\}.$$

It is straightforward to check that this relation is an equivalence relation. Denote by $[E]$ the set of its equivalence classes, to be called equilibrium classes, and by $[(e_1, e_2)]$ the equilibrium class of $(e_1, e_2) \in E$. We have

$$[(e_1, e_2)] = \{(e_1, e_2), (e_2, e_1), (L - e_1, L - e_2), (L - e_2, L - e_1)\}.$$

We define the multiplicity of an equilibrium as the number of elements of its equilibrium class. Of course, if the game has a unique equilibrium (e_1, e_2), then there is just one equilibrium class consisting of this equilibrium and (e_1, e_2) has multiplicity 1. Note that with the action distance of an action profile (x_1, x_2) defined by $|x_2 - x_1|$, each element of a given equilibrium class has the same action distance. Also note that (1) implies:

$$\#E = 1 \;\Rightarrow\; E = \{(\frac{L}{2}, \frac{L}{2})\}.$$

Theorem 1. *If* $(e_1, e_2) \in E$, *then* (e_1, e_2) *has multiplicity* $1, 2$ *or* 4 *and*

$$\#[(e_1, e_2)] = 1 \;\Leftrightarrow\; e_1 = e_2 \wedge e_1 + e_2 = L \;\Leftrightarrow\; e_1 = e_2 = \frac{L}{2};$$

$$\#[(e_1, e_2)] = 2 \;\Leftrightarrow\; [e_1 = e_2 \wedge e_1 + e_2 \neq L] \vee [e_1 \neq e_2 \wedge e_1 + e_2 = L];$$

$$\#[(e_1, e_2)] = 4 \;\Leftrightarrow\; [e_1 \neq e_2 \wedge e_1 + e_2 \neq L]. \qquad \diamond$$

Proof. It is easy to prove the three displayed statements. They in turn imply, as desired, that $\#[(e_1, e_2)] \neq 3$. Q.E.D.

We shall freely use all the results in this section together with the results A–F for the function $\mathcal{L}$ in Sect. 2. Because of player symmetry, we often only present results for player 1.

4 Inelastic Case

Let us start our investigation of the continuous Hotelling game by considering the well-known inelastic case, i.e. the case where f is constant. Without loss of generality, we assume $f = 1$.

First it may good to have a look to the simple results in Lemma 1 and Proposition 1 below. In addition to these results, the following simple results hold for the inelastic case (but not with exception of those in parts 2a and 2b for the elastic case):

Theorem 2. *1. (a)* $u_1^{(x_2)}$ *is on* $[0, x_2[$ *strictly increasing and on* $]x_2, L]$ *strictly decreasing.*

(b) If $x_2 \leq \frac{L}{2}$, *then* $u_1^{(x_2)}$ *is on* $[0, x_2]$ *strictly increasing.*

(c) If $x_2 \geq \frac{L}{2}$, *then* $u_1^{(x_2)}$ *is on* $[x_2, L]$ *strictly decreasing.*

2. (a) *If* $x_2 < \frac{L}{2}$, *then* $u_1^{(x_2)}$ *is on* $[0, x_2[$ *concave and on* $[x_2, L]$ *concave.*
   (b) *If* $x_2 > \frac{L}{2}$, *then* $u_1^{(x_2)}$ *is on* $[0, x_2]$ *concave and on* $]x_2, L]$ *concave.*
   (c) $u_1^{(\frac{L}{2})}$ *is concave.*
   (d) $u_1^{(x_2)}$ *is strictly quasi-concave.*
3. $B(x) = \begin{cases} \emptyset \text{ if } x \neq \frac{L}{2}, \\ \{\frac{L}{2}\} \text{ if } x = \frac{L}{2}. \end{cases}$
4. $E = \{(\frac{L}{2}, \frac{L}{2})\}$. *Thus there is one equilibrium class, this class contains one element and the Principle of Minimum Differentiation holds.* ◇

5 Properties of Fundamental Objects

5.1 Smoothness

Lemma 1. *1. If* $0 < x_2 < \frac{L}{2}$, *then* $\lim_{x_1 \uparrow x_2} u_1^{(x_2)}(x_1) = \mathcal{L}(x_2) < u_1^{(x_2)}(x_2) < \mathcal{L}(L - x_2) = \lim_{x_1 \downarrow x_2} u_1^{(x_2)}(x_1)$. *And* $u_1^{(0)}(0) < \mathcal{L}(L) = \lim_{x_1 \downarrow 0} u_1^{(0)}(x_1)$.
2. If $L > x_2 > \frac{L}{2}$, *then* $\lim_{x_1 \uparrow x_2} u_1^{(x_2)}(x_1) = \mathcal{L}(x_2) > u_1^{(x_2)}(x_2) > \mathcal{L}(L - x_2) = \lim_{x_1 \downarrow x_2} u_1^{(x_2)}(x_1)$. *And* $u_1^{(L)}(L) < \mathcal{L}(L) = \lim_{x_1 \uparrow L} u_1^{(0)}(x_1)$.
3. $\lim_{x_1 \to \frac{L}{2}} u_1^{(\frac{L}{2})}(x_1) = u_1^{(\frac{L}{2})}(\frac{L}{2}) = \mathcal{L}(\frac{L}{2})$. ◇

Proof. 1. $\lim_{x_1 \uparrow x_2} u_1^{(x_2)}(x_1) = \lim_{x_1 \uparrow x_2}(\mathcal{L}(x_1) + \mathcal{L}(\frac{|x_1 - x_2|}{2})) = \mathcal{L}(x_2) + \mathcal{L}(0) = \mathcal{L}(x_2) < \frac{\mathcal{L}(x_2) + \mathcal{L}(L - x_2)}{2} = u_1^{(x_2)}(x_2)$ and $\lim_{x_1 \downarrow x_2} u_1^{(x_2)}(x_1) = \lim_{x_1 \uparrow x_2}(\mathcal{L}(L - x_1) + \mathcal{L}(\frac{|x_1 - x_2|}{2})) = \mathcal{L}(L - x_2) + \mathcal{L}(0) = \mathcal{L}(L - x_2) > \frac{\mathcal{L}(x_2) + \mathcal{L}(L - x_2)}{2} = u_1^{(x_2)}(x_2)$.

2. Analogous to part 1.

3. $\lim_{x_1 \downarrow \frac{L}{2}} u_1^{(\frac{L}{2})}(x_1) = \lim_{x_1 \downarrow \frac{L}{2}}(\mathcal{L}(L - x_1) + \mathcal{L}(\frac{|x_1 - \frac{L}{2}|}{2}) = \mathcal{L}(\frac{L}{2}) = u_1^{(\frac{L}{2})}(\frac{L}{2})$. With this $\lim_{x_1 \uparrow \frac{L}{2}} u_1^{(\frac{L}{2})}(x_1) = \lim_{x_1 \uparrow \frac{L}{2}} u_1^{(\frac{L}{2})}(L - x_1) = \lim_{x_1 \downarrow \frac{L}{2}} u_1^{(\frac{L}{2})}(x_1) = u_1^{(\frac{L}{2})}(\frac{L}{2})$. So the desired result follows. Q.E.D.

Proposition 1. *1.* $u_1^{(x_2)}$ *is continuous at every* $x_1 \neq x_2$ *and discontinuous at every* $x_1 = x_2 \neq \frac{L}{2}$.
2. If $x_2 \not\in \{0, \frac{L}{2}, L\}$, *then* $u_1^{(x_2)}$ *is at* $x_1 = x_2$ *neither upper-semicontinuous nor lower-semicontinuous.* $u_1^{(0)}$ *is at* 0 *lower-semicontinuous, but not upper-semicontinuous.* $u_1^{(L)}$ *is at* L *upper-semicontinuous, but not lower-semicontinuous.*
3. For $x_2 < \frac{L}{2}$, $u_1^{(x_2)}$ *is left-upper-semicontinuous at* x_2 *and right-lower-semicontinuous at* x_2. *For* $x_2 > \frac{L}{2}$, $u_1^{(x_2)}$ *is right-upper-semicontinuous at* x_2 *and left-lower-semicontinuous at* x_2.
4. $u_1^{(\frac{L}{2})}$ *is continuous.*
5. $u_1^{(x_2)}$ *is semidifferentiable at each* $x_1 \neq x_2$. *If* f *is continuous, then* $u_1^{(x_2)}$ *even is differentiable at each* $x_1 \neq x_2$. ◇

Proof. 2. This follows from Lemma 1(1, 2).
1. First statement: clear. Second statement: from part 2.
3. By Lemma 1(1, 2).
4. By Lemma 1(3) together with part 1.
5. Clear. Q.E.D.

Here is an improvement of Proposition 1(5).

Proposition 2. *1.* $D_\pm u_1^{(x_2)}(x_1) = f_\pm(x_1) - \frac{1}{2} f_\mp(\frac{x_2 - x_1}{2})$ $(x_1 < x_2)$ *and* $D_\pm u_1^{(x_2)}(x_1) = -f_\mp(L - x_1) + \frac{1}{2} f_\pm(\frac{x_1 - x_2}{2})$ $(x_1 > x_2)$.[5]
2. $u_1^{(\frac{L}{2})}$ *is semidifferentiable at* $\frac{L}{2}$ *and* $D_\pm u_1^{(\frac{L}{2})}(\frac{L}{2}) = \mp f_-(\frac{L}{2}) \pm \frac{1}{2} f_+(0)$. ◇

Proof. 1. First statement: for $x_1 \in [0, x_2[$, we have $u_1^{(x_2)}(x_1) = \mathcal{L}(x_1) + \mathcal{L}(\frac{x_2 - x_1}{2})$. This implies $D_\pm u_1^{(x_2)}(x_1) = \mathcal{L}_\pm(x_1) + D_\pm(\mathcal{L}(\frac{x_2 - x_1}{2})) = f_\pm(x_1) - \frac{1}{2} f_\mp(\frac{x_2 - x_1}{2})$.
Second statement: in the same way.
2. Suppose f is continuous at 0 and $\frac{L}{2}$. We have

$$D_+ u_1^{(\frac{L}{2})}(\frac{L}{2}) = \lim_{h \downarrow 0} \frac{u_1^{(\frac{L}{2})}(\frac{L}{2} + h) - u_1^{(\frac{L}{2})}(\frac{L}{2})}{h} = \lim_{h \downarrow 0} \frac{\mathcal{L}(\frac{L}{2} - h) + \mathcal{L}(\frac{h}{2}) - \mathcal{L}(\frac{L}{2})}{h}$$

$$= \lim_{h \downarrow 0} \frac{\mathcal{L}(\frac{L}{2} - h) - \mathcal{L}(\frac{L}{2})}{h} + \lim_{h \downarrow 0} \frac{\mathcal{L}(\frac{h}{2})}{h} = \lim_{h \uparrow 0} - \frac{\mathcal{L}(\frac{L}{2} + h) - \mathcal{L}(\frac{L}{2})}{h} + \lim_{h \downarrow 0} \frac{\mathcal{L}(h) - \mathcal{L}(0)}{2h}$$

$$= -D_- \mathcal{L}(\frac{L}{2}) + \frac{1}{2} D_+ \mathcal{L}(0) = -f_-(\frac{L}{2}) + \frac{1}{2} f_+(0).$$

From this $D_- u_1^{(\frac{L}{2})}(\frac{L}{2}) = \lim_{h \uparrow 0} \frac{u_1^{(\frac{L}{2})}(\frac{L}{2} + h) - u_1^{(\frac{L}{2})}(\frac{L}{2})}{h} = \lim_{h \downarrow 0} \frac{u_1^{(\frac{L}{2})}(\frac{L}{2} - h) - u_1^{(\frac{L}{2})}(\frac{L}{2})}{-h} = \lim_{h \downarrow 0} - \frac{u_1^{(\frac{L}{2})}(\frac{L}{2} + h) - u_1^{(\frac{L}{2})}(\frac{L}{2})}{h} = -D_+ u_1^{(\frac{L}{2})}(\frac{L}{2})$. Q.E.D.

5.2 Monotonicity

Lemma 2. *1. For all* $x_2, x_1, x_1' \in S$ *with* $x_2 < x_1 < x_1'$

$$u_1^{(x_2)}(x_1) - u_1^{(x_2)}(x_1') \begin{cases} \geq (x_1' - x_1)(f(L - x_1) - \frac{1}{2} f(\frac{x_1 - x_2}{2})), \\ \leq (x_1' - x_1)(f(L - x_1') - \frac{1}{2} f(\frac{x_1' - x_2}{2})). \end{cases}$$

2. For all $x_2, x_1, x_1' \in S$ *with* $x_1 < x_1' < x_2$

$$u_1^{(x_2)}(x_1) - u_1^{(x_2)}(x_1') \begin{cases} \geq (x_1' - x_1)(\frac{1}{2} f(\frac{x_2 - x_1}{2}) - f(x_1)), \\ \leq (x_1' - x_1)(\frac{1}{2} f(\frac{x_2 - x_1'}{2}) - f(x_1')). \end{cases}$$ ◇

[5] So, if f is continuous, then by Proposition 1(5), these formulas become $Du_1^{(x_2)}(x_1) = f(x_1) - \frac{1}{2} f(\frac{x_2 - x_1}{2})$ $(x_1 < x_2)$ and $Du_1^{(x_2)}(x_1) = -f(L - x_1) + \frac{1}{2} f(\frac{x_1 - x_2}{2})$ $(x_1 > x_2)$.

Proof. 1. We have $u_1^{(x_2)}(x_1) - u_1^{(x_2)}(x_1')$

$$= \Big(\mathcal{L}(L - x_1) - \mathcal{L}(L - x_1')\Big) - \Big(\mathcal{L}(\frac{x_1' - x_2}{2}) - \mathcal{L}(\frac{x_1 - x_2}{2})\Big).$$

From this, as desired, $u_1^{(x_2)}(x_1) - u_1^{(x_2)}(x_1') \geq (x_1' - x_1) f(L - x_1) - \frac{x_1' - x_1}{2} f(\frac{x_1 - x_2}{2})$ and $u_1^{(x_2)}(x_1) - u_1^{(x_2)}(x_1') \leq (x_1' - x_1) f(L - x_1') - \frac{x_1' - x_1}{2} f(\frac{x_1' - x_2}{2})$.

2. Analogous to part 1. Q.E.D.
   Notation:

$$V := \{x_1 \in S \mid f(L - x_1) \geq \frac{1}{2} f(0)\}. \tag{2}$$

Note that V is a real interval containing L.

Lemma 3. $u_1^{(x_2)}$ *is strictly decreasing on* $V \cap \,]x_2, L]$. ◇

Proof. Suppose $x_1, x_1' \in V \cap \,]x_2, L]$ with $x_1 < x_1'$. We prove that $u_1^{(x_2)}(x_1) > u_1^{(x_2)}(x_1')$. Well, as $f(L - x_1) \geq \frac{1}{2} f(0)$, we obtain with Lemma 2(1), as desired,

$$u_1^{(x_2)}(x_1) - u_1^{(x_2)}(x_1') \geq (x_1' - x_1)(f(L - x_1) - \frac{1}{2} f(\frac{x_1 - x_2}{2})) \geq$$

$$(x_1' - x_1)(f(L - x_1) - \frac{1}{2} f(0)) \geq 0.$$

Finally, note that here the last inequality is strict if f is constant and otherwise the second inequality is strict. Q.E.D.

Lemma 4. *Suppose f is upper-semicontinuous at $L - x_2$ and $f(L - x_2) < \frac{1}{2} f(0)$. Then there exists a punctured right neighbourhood of x_2 on which $u_1^{(x_2)}$ is strictly increasing.* ◇

Proof. Note that $x_2 < L$. Let $d = \frac{1}{2} f(0) - f(L - x_2)$. As f is upper-semicontinuous at $L - x_2$, we can fix x_1' with $L > x_1' > x_2$ such that

$$f(L - x_1') < f(L - x_2) + \frac{d}{2} \; (x_2 < x_1' < x_1'').$$

We have $\frac{1}{2} f(\frac{x_1'' - x_2}{2}) > 0 > \frac{d}{2} - \frac{1}{2} f(0)$. With Lemma 2(1), we obtain for $x_1, x_1' \in \,]x_2, x_1'\,[$ with $x_1 < x_1'$, as desired that $u_1^{(x_2)}(x_1) - u_1^{(x_2)}(x_1') \leq (x_1' - x_1)(f(L - x_1') - \frac{1}{2} f(\frac{x_1' - x_2}{2})) < (x_1' - x_1)(f(L - x_2) + \frac{d}{2} + (\frac{d}{2} - \frac{1}{2} f(0))) = 0$. Q.E.D.

Proposition 3. *Suppose* $f(\frac{L}{2}) \geq \frac{1}{2} f(0)$.

1. If $x_2 \leq \frac{L}{2}$, then $u_1^{(x_2)}$ is strictly increasing on $[0, x_2]$.
2. If $x_2 \geq \frac{L}{2}$, then $u_1^{(x_2)}$ is strictly decreasing on $[x_2, L]$.
3. $u_1^{(\frac{L}{2})}$ has $\frac{L}{2}$ as unique maximiser. ◇

Proof. 1, 2. By location symmetry, it is sufficient to prove part 2. Fix $x_2 \geq \frac{L}{2}$. We prove that $u_1^{(x_2)}$ is strictly decreasing on $]x_2, L]$; together with Lemma 1(2) and Proposition 1(4) the desired result follows. Well, with V as in (2), $[\frac{L}{2}, L] \subseteq V$. This implies $]x_2, L] \subseteq V \cap]x_2, L]$. Finally, apply Lemma 3.

3. By parts 1 and 2. Q.E.D.

The statements in Proposition 3 are no longer valid for the situation $f(\frac{L}{2}) < \frac{1}{2}f(0)$. For example, Lemma 4 shows (by taking $x_2 = \frac{L}{2}$), that then Proposition 3(2) is no longer valid.

Proposition 4. *Suppose f is continuous.*

1. *If $x_2 < \frac{L}{2}$, then $u_1^{(x_2)}$ is strictly decreasing on $[\frac{5}{6}L, L]$.*
2. *If $x_2 > \frac{L}{2}$, then $u_1^{(x_2)}$ is strictly increasing on $[0, \frac{1}{6}L]$.* ◇

Proof. By location symmetry, it is sufficient to prove part 1. So suppose $x_2 < \frac{L}{2}$. Note that $\frac{x_1 - x_2}{2} > \frac{2x_1 - L}{4}$ and for $x_1 \geq \frac{5}{6}L$ that $L - x_1 \leq \frac{2L - x_1}{4}$. Finally, note that, together with Proposition 2(1), for $x_1 \geq \frac{5}{6}L$

$$Du_1^{(x_2)}(x_1) = -f(L - x_1) + \frac{1}{2}f(\frac{x_1 - x_2}{2}) \leq -f(\frac{2L - x_1}{4}) + \frac{1}{2}f(\frac{2L - x_1}{4}) < 0. \text{ Q.E.D.}$$

5.3 Concavity

Proposition 5. *Suppose f is strictly decreasing.*

1. *If $x_2 < \frac{L}{2}$, then $u_1^{(x_2)}$ is on $[0, x_2[$ strictly concave and on $[x_2, L]$ strictly concave.*
2. *If $x_2 > \frac{L}{2}$, then $u_1^{(x_2)}$ is on $[0, x_2]$ strictly concave and on $]x_2, L]$ strictly concave.*
3. *$u_1^{(\frac{L}{2})}$ is on $[0, \frac{L}{2}]$ strictly concave and on $[\frac{L}{2}, L]$ strictly concave.*
4. *If $f(\frac{L}{2}) \geq \frac{1}{2}f(0)$, then $u_1^{(x_2)}$ is strictly quasi-concave.* ◇

Proof. 1. First statement: for the function $u_1^{(x_2)} : [0, x_2[\to \mathbb{R}$ we have $u_1^{(x_2)}(x_1) = \mathcal{L}(x_1) + \mathcal{L}(\frac{x_2 - x_1}{2})$. So this function is a sum of strictly concave functions and therefore strictly concave.

Second statement: for the function $u_1^{(x_2)} :]x_2, L] \to \mathbb{R}$ we have $u_1^{(x_2)}(x_1) = \mathcal{L}(L - x_1) + \mathcal{L}(\frac{x_1 - x_2}{2})$. So this function is a sum of strictly concave functions and therefore strictly concave. As the function $u_1^{(x_2)} : [x_2, L] \to \mathbb{R}$ is, by Proposition 1(1, 3) right lower-semicontinuous, it follows, as desired, that also this function is strictly concave.

2. Analogous to part 1.

3. For the function $u_1^{(\frac{L}{2})} : [0, \frac{L}{2}[\to \mathbb{R}$ we have $u_1^{(\frac{L}{2})}(x_1) = \mathcal{L}(x_1) + \mathcal{L}(\frac{\frac{L}{2} - x_1}{2})$. So this function is a sum of strictly concave functions and therefore strictly concave. As, by Proposition 1(2), $u_1^{(\frac{L}{2})}$ is continuous, it follows that $u_1^{(\frac{L}{2})}$ is on

$[0, \frac{L}{2}]$ strictly concave. By location symmetry, it follows that $u_1^{(\frac{L}{2})}$ is on $[\frac{L}{2}, L]$ strictly concave.

4. Suppose $x_2 \geq \frac{L}{2}$. By Proposition 3(2), $u_1^{(x_2)}$ is strictly decreasing on $[x_2, L]$. By parts 2 and 3, $u_1^{(x_2)}$ is on $[0, x_2]$ strictly concave. If $x_2 = \frac{L}{2}$, then $u_1^{(x_2)}$ is continuous by Proposition 1(4) and it follows that $u_1^{(x_2)}$ is strictly concave. If $x_2 > \frac{L}{2}$, then by Lemma 1(2), $\lim_{x_1 \uparrow x_2} u_1^{(x_2)}(x_1) > u_1^{(x_2)}(x_2) > \lim_{x_1 \downarrow x_2} u_1^{(x_2)}(x_1)$ and it follows that $u_1^{(x_2)}$ is strictly quasi-concave.

So the statement holds for $x_2 \geq \frac{L}{2}$. Noting that $u_1^{(x_2)}(x_1) = u_1^{(L-x_2)}(L - x_1)$ the statement now also holds for $x_2 \leq \frac{L}{2}$. Q.E.D.

5.4 Best-Response Correspondences

Notation: by $\mathrm{fix}(B)$ we denote the set of fixed points of the best-response correspondence B, i.e. the set $\{x \in S \mid x \in B(x)\}$.

Proposition 6. *$0 \notin B(x_2)$ $(x_2 \in S)$ and $L \notin B(x_2)$ $(x_2 \in S)$.* ◇

Proof. By location symmetry, it is sufficient to prove the first statement. By Lemma 1(1), this statement holds for $x_2 = 0$. Now suppose $0 < x_2 < \frac{L}{2}$. We have $u_1^{(x_2)}(0) = \mathcal{L}(0) + \mathcal{L}(\frac{x_2}{2}) < \mathcal{L}(\frac{3}{4}x_2) < \mathcal{L}(\frac{3}{4}x_2) + \mathcal{L}(\frac{x_2}{4}) = u_1^{(x_2)}(\frac{3}{4}x_2)$. Thus $0 \notin B(x_2)$. Q.E.D.

Lemma 5. $B(x_2) \subseteq \begin{cases} \mathrm{argmax}_{x_1 \in\,]x_2, L]}\, u_1^{(x_2)}(x_1)\ (0 \leq x_2 < \frac{L}{2}), \\ \mathrm{argmax}_{x_1 \in [0, x_2\, [}\, u_1^{(x_2)}(x_1)\ (\frac{L}{2} < x_2 \leq L). \end{cases}$ ◇

Proof. We prove the statement for $0 \leq x_2 < \frac{L}{2}$); then the other statement follows by location symmetry. Well, part 1 implies that the statement is true for $x_2 = 0$. Now suppose $0 < x_2 < \frac{L}{2}$. For every $h > 0$ with $0 \leq x_2 - h < x_2 < x_2 + h \leq L$ we have $u_1^{(x_2)}(x_2 - h) = \mathcal{L}(x_2 - h) + \mathcal{L}(\frac{h}{2}) < \mathcal{L}(L - x_2 - h) + \mathcal{L}(\frac{h}{2}) = u_1^{(x_2)}(x_2 + h)$. This implies $B(x_2) \subseteq [x_2, L]$. By Lemma 1(1), $u_1^{(x_2)}(x_2) < \lim_{x_1 \downarrow x_2} u_1^{(x_2)}(x_1)$. Therefore $B(x_2) \subseteq\,]x_2, L]$. The desired result now follows. Q.E.D.

Proposition 7. *1. $x_2 \neq \frac{L}{2} \Rightarrow x_2 \notin B(x_2)$. Thus $\mathrm{fix}(B) \subseteq \{\frac{L}{2}\}$.*
2. If f is continuous at 0 and $\frac{L}{2}$ and $f(\frac{L}{2}) < \frac{1}{2}f(0)$, then $\frac{L}{2} \notin \mathrm{fix}(B)$. ◇

Proof. 1. By Lemma 5.

2. For $x_1 \in\,]0, \frac{L}{2}\,[$ we obtain

$$u_1^{(\frac{L}{2})}(x_1) = \mathcal{L}(x_1) + \mathcal{L}(\frac{L}{4} - \frac{x_1}{2}) = \mathcal{L}(\frac{L}{2}) + \mathcal{L}(x_1) - \mathcal{L}(\frac{L}{2}) + \mathcal{L}(\frac{L}{4} - \frac{x_1}{2})$$

$$= u_1^{(\frac{L}{2})}(\frac{L}{2}) - (\mathcal{L}(\frac{L}{2}) - \mathcal{L}(x_1)) + \mathcal{L}(\frac{L}{4} - \frac{x_1}{2})$$

$$\geq u_1^{(\frac{L}{2})}(\frac{L}{2}) - (\frac{L}{2} - x_1)f(x_1) + (\frac{L}{4} - \frac{x_1}{2})f(\frac{L}{4} - \frac{x_1}{2})$$

$$= u_1^{(\frac{L}{2})}(\frac{L}{2}) + \frac{1}{2}(\frac{L}{2} - x_1)(-2f(x_1) + f(\frac{L-2x_1}{4})).$$

As $-2f(\frac{L}{2}) + f(\frac{L-2\frac{L}{2}}{4}) = -2f(\frac{L}{2}) + f(0) > 0$ and f is continuous at 0 and $\frac{L}{2}$, there exists $\delta > 0$ such that $-2f(x_1) + f(\frac{L-2x_1}{4}) > 0$ for every $x_1 \in]\frac{L}{2} - \delta, \frac{L}{2}[$. So for these x_1 we obtain $u_1^{(\frac{L}{2})}(x_1) > u_1^{(\frac{L}{2})}(\frac{L}{2})$. It follows that $\frac{L}{2} \not\in B(\frac{L}{2})$. Q.E.D.

Terminology: given a correspondence $F : A \multimap B$, we call F proper if $F(a) \neq \emptyset$ $(a \in A)$ and call F at most single-valued if $\#F(a) \leq 1$ $(a \in A)$.

Proposition 8. *1. Suppose* $f(\frac{L}{2}) > \frac{1}{2}f(0)$.
- *(a)* B *is at most single-valued.*
- *(b)* $B(\frac{L}{2}) = \{\frac{L}{2}\}$.
- *(c) Suppose* f *is lower-semicontinuous at* $\frac{L}{2}$. *Then there exists a punctured open interval around* $\frac{L}{2}$ *on which* B *is empty-valued, thus* B *is not proper.*

2. Suppose $f(\frac{L}{2}) = \frac{1}{2}f(0)$.
- *(a)* B *is single-valued.*
- *(b)* $B(\frac{L}{2}) = \{\frac{L}{2}\}$.

3. Suppose $f(\frac{L}{2}) < \frac{1}{2}f(0)$.
- *(a) Suppose* f *is upper-semicontinuous. Then* B *is proper and* B *is on* $S\setminus\{\frac{L}{2}\}$ *single-valued.*
- *(b) Suppose* f *is continuous. Then* $B(\frac{L}{2}) = \{x_1, L - x_1\}$ *with* x_1 *the unique solution* $y \in]0, \frac{L}{2}[$ *of the equation* $f(y) = \frac{1}{2}f(\frac{\frac{L}{2}-y}{2})$.

4. Suppose $f(\frac{L}{2}) \leq \frac{1}{2}f(0)$, f *continuous and* $x_2 \in]\frac{L}{2}, L]$. *Then* $B(x_2) = \{x_1\}$ *with* x_1 *the unique solution* $y \in]0, x_2[$ *of the equation* $f(y) = \frac{1}{2}f(\frac{x_2-y}{2})$. ⋄

Proof. 1b, 2b. By Proposition 3(3).

1a. As $u_1^{(x_2)}$ is, by Proposition 5, strictly quasi-concave.

1c. By location symmetry, it is sufficient to prove that there exists $\delta > 0$ such that $B(x_2) = \emptyset$ for all $x_2 \in]\frac{L}{2} - \delta, \frac{L}{2}[$. Well, as f is lower-semicontinuous at $\frac{L}{2}$, we can take $\delta > 0$ such that $f(\frac{L}{2} + \delta) > \frac{1}{2}f(0)$. Next fix $x_2 \in]\frac{L}{2} - \delta, \frac{L}{2}[$. As $\frac{L}{2} - \delta < x_2$, we have $L - x_2 < \frac{L}{2} + \delta$ and therefore for $x_1 \in]x_2, L]$ it follows that $f(L - x_1) \geq f(L - x_2) \geq f(\frac{L}{2} + \delta) > \frac{1}{2}f(0)$. So, with V as in (2), $]x_2, L] \subseteq V \cap]x_2, L]$. By Lemma 3, $u_1^{(x_2)}$ is strictly decreasing on $]x_2, L]$. As $x_2 < \frac{L}{2}$, Proposition 7(1) guarantees $B(x_2) \subseteq]x_2, L]$. It follows that $B(x_2) = \emptyset$.

2a. As $u_1^{(x_2)}$ is, by Proposition 5, strictly quasi-concave, we have $\#B(x_2) \leq 1$ $(x_2 \in S)$. So we still need to prove that $B(x_2) \neq \emptyset$ $(x_2 \in S)$. By part 2b and location symmetry, it is sufficient to show that $B(x_2) \neq \emptyset$ for $x_2 < \frac{L}{2}$.

Fix $x_2 < \frac{L}{2}$. It is sufficient to show that $u_1^{(x_2)}$ has a maximiser on $[0, x_2]$ and on $]x_2, L]$. Well, by Proposition 1(1, 3, 4), $u_1^{(x_2)}$ is on $[0, x_2]$ upper-semicontinuous and therefore, by the Lemma of Weierstrass-Lebesgue, has a maximiser on this segment. As $f(L - x_2) < f(\frac{L}{2}) = \frac{1}{2}f(0)$, Lemma 4 guarantees that there exists $\delta > 0$ such that $u_1^{(x_2)}$ is strictly increasing on $]x_2, x_2 + \delta]$. Also $u_1^{(x_2)}$ is continuous on $[x_2 + \delta, L]$. It follows that $u_1^{(x_2)}$ has a maximiser on $]x_2, L]$.

3a. First statement: by location symmetry, it is sufficient to show that $B(x_2) \neq \emptyset$ for $x_2 \leq \frac{L}{2}$. Well, for $x_2 = \frac{L}{2}$, this follows from the Weierstrass Theorem, as $u_1^{(\frac{L}{2})}$ is continuous by Proposition 1(4). Now fix $x_2 < \frac{L}{2}$. The rest of the proof is the same as that in part 3a after 'Fix $x_2 < \frac{L}{2}$'.

Second statement: by the above is sufficient to prove that $\#B(x_2) \leq 1$ for all $x_2 \neq \frac{L}{2}$. By location symmetry it is sufficient to prove this inequality for $x_2 < \frac{L}{2}$. So suppose $x_2 < \frac{L}{2}$. By Proposition 7(1), $B(x_2) \subseteq \,]x_2, L]$. As, by Proposition 5(1), $u_1^{(x_2)} : \,]x_2, L] \to \mathbb{R}$ is strictly concave, this function has at most one maximiser. This implies $\#B(x_2) \leq 1$.

3b. By Propositions 5(3) and 1(4), the function $u_1^{(\frac{L}{2})} : [0, \frac{L}{2}] \to \mathbb{R}$ is strictly concave and continuous. It follows that this function has a unique maximiser, say x_1. Noting that $u_1^{(\frac{L}{2})}(x) = u_1^{(\frac{L}{2})}(L - x)$ $(x \in S)$, it follows that $B(\frac{L}{2}) = \{x_1, L - x_1\}$. By Propositions 6 and 7(2) we have $0 < x_1 < \frac{L}{2}$. As f is continuous, the function $u_1^{(\frac{L}{2})} : [0, \frac{L}{2}] \to \mathbb{R}$ is by Proposition 1(5) differentiable at its interior maximiser x_1, Fermat's theorem gives $Du_1^{(\frac{L}{2})}(x_1) = 0$. Proposition 2(1) implies $f(x_1) = \frac{1}{2} f(\frac{\frac{L}{2} - x_1}{2})$. As the function $y \mapsto f(y) - \frac{1}{2} f(\frac{\frac{L}{2} - y}{2})$ is strictly increasing on $]0, \frac{L}{2}\,[$, the proof is complete.

4. By parts 2b and 3c, $\#B(x_2) = 1$. Let $B(x_2) = \{x_1\}$. Now, $0 < x_1 < x_2$ by Lemma 5. As $u_1^{(x_2)}$ is differentiable at its interior maximiser x_1, Fermat's theorem gives $Du_1^{(x_2)}(x_1) = 0$. Proposition 2(1) implies $f(x_1) = \frac{1}{2} f(\frac{x_2 - x_1}{2})$. As the function $y \mapsto f(y) - \frac{1}{2} f(\frac{x_2 - y}{2})$ is strictly increasing on $]0, x_2\,[$, the proof is complete. Q.E.D.

For the inelastic case $B(x) = \emptyset$ holds for all $x \neq \frac{L}{2}$. Proposition 8(1c) shows that this property continues to hold in case of a continuous demand function f with $f(\frac{L}{2}) > \frac{1}{2} f(0)$ for x in a punctured neighbourhood of $\frac{L}{2}$.

Proposition 9. *Suppose f is upper-semicontinuous and $f(\frac{L}{2}) \leq \frac{1}{2} f(0)$. Then*

$$B(x_2) = \begin{cases} \operatorname{argmax}_{x_1 \in]x_2, L]} u_1^{(x_2)}(x_1) & (0 \leq x_2 < \frac{L}{2}), \\ \operatorname{argmax}_{x_1 \in [0, x_2[} u_1^{(x_2)}(x_1) & (\frac{L}{2} < x_2 \leq L). \end{cases} \qquad \diamond$$

Proof. By Lemma 5, we still have to prove '$\supseteq$'. We prove the statement for $0 \leq x_2 < \frac{L}{2}$; then the other statement follows by location symmetry. So fix $x_2 \in [0, \frac{L}{2}\,[$ and suppose $\tilde{x}_1 \in \operatorname{argmax}_{x_1 \in]x_2, L]} u_1^{(x_2)}(x_1)$. By Proposition 8(2a, 3a), $u_1^{(x_2)}$ has a maximiser, say $\overline{x}_1$. By Lemma 5, $\overline{x}_1 \in \operatorname{argmax}_{x_1 \in]x_2, L]} u_1^{(x_2)}(x_1)$. It follows that $u_1^{(x_2)}(\tilde{x}_1) = u_1^{(x_2)}(\overline{x}_1)$. This implies $\tilde{x}_1 \in B(x_2)$. Q.E.D.

Concerning the statement in the next theorem, note that we have Proposition 8(2a, 3a) on single-valuedness of the correspondence B.

Theorem 3. *Suppose $f(\frac{L}{2}) \leq \frac{1}{2} f(0)$ and f is continuously differentiable with $Df < 0$. Then the functions $B : [0, \frac{L}{2}\,[\to \mathbb{R}$ and $B : \,]\frac{L}{2}, L] \to \mathbb{R}$ are continuously differentiable and strictly increasing.* $\diamond$

Proof. By location symmetry, it is sufficient to prove the second statement. Well, by Proposition 8(4), $B(x_2)$ is the unique solution y of the equation $f(y) = \frac{1}{2}f(\frac{x_2-y}{2})$ $(0 < y < x_2)$. So we have $f(B(x_2)) = \frac{1}{2}f(\frac{x_2-B(x_2)}{2})$. The implicit function theorem applies and implies that $B : [0, \frac{L}{2}[\rightarrow \mathbb{R}$ is continuously differentiable with $DB(x_2) = \frac{Df(\frac{x_2-B(x_2)}{2})}{4Df(B(x_2))+Df(\frac{x_2-B(x_2)}{2})} < 0$. Q.E.D.

6 Equilibria

In this section we provide results for the Nash equilibrium set E.

Proposition 10. *If f is continuous at* 0 *and* $\frac{L}{2}$ *and* $f(\frac{L}{2}) < \frac{1}{2}f(0)$, *then* $(\frac{L}{2}, \frac{L}{2}) \notin E$. ◇

Proof. By Proposition 7(2). Q.E.D.

Theorem 4. *If f is continuous and $(e_1, e_2) \in E$, then $e_1 + e_2 = L$.* ◇

Proof. Suppose f is continuous. If f is constant, then, by Theorem 2(3), $E = \{(\frac{L}{2}, \frac{L}{2})\}$ and so the statement is true. Now assume that f is strictly decreasing. First we prove by contradiction that $e_1 + e_2 \geq L$ for each equilibrium (e_1, e_2). So suppose (e_1, e_2) is an equilibrium with $e_1 + e_2 < L$. By player symmetry, we may assume that $e_2 \leq e_1$. Proposition 7(1) and $e_1 + e_2 < L$ imply $e_2 \neq e_1$. So $e_2 < e_1$ holds. By Proposition 6, $0 < e_2 < e_1 < L$. As $u_1^{(e_2)}$ is differentiable at e_1 and $u_2^{(e_1)}$ is differentiable at e_2, it follows $Du_1^{(e_2)}(e_1) = Du_2^{(e_1)}(e_2) = 0$. So, by Proposition 2(1), noting that $Du_2^{(e_1)}(e_2) = Du_1^{(e_1)}(e_2)$

$$0 = -f(L - e_1) + \frac{1}{2}f(\frac{e_1 - e_2}{2}),\ 0 = f(e_2) - \frac{1}{2}f(\frac{e_1 - e_2}{2}).$$

We obtain $f(L - e_1) = f(e_2)$. As $L - e_1 \neq e_2$, this contradicts the strict decreasingness of f. Thus $e_1 + e_2 \geq L$ for each equilibrium (e_1, e_2). By location symmetry, $(L - e_1, L - e_2)$ is also an equilibrium. Therefore $(L - e_1) + (L - e_2) \geq L$. Hence $e_1 + e_2 = L$ follows. Q.E.D.

The next example shows that Theorem 4 no longer holds if we allow f therein to be discontinuous.

Example 1. This example is taken from [11]. Consider the case $L = 1$ with the following discontinuous demand function

$$f(z) = \begin{cases} 2 - z \text{ if } 0 \leq z \leq \frac{5}{24}, \\ \frac{1}{2} - \frac{1}{4}z \text{ if } \frac{5}{24} < z \leq 1. \end{cases}$$

Note that f is upper-semicontinuous and that $f(\frac{L}{2}) < \frac{1}{2}f(0)$.

We now prove that $(\frac{2}{3}, \frac{1}{4})$ is an equilibrium.

By Proposition 9, $B(\frac{1}{4}) = \mathrm{argmax}_{x_1 \in]\frac{1}{4}, 1]} u_1^{(\frac{1}{4})}(x_1)$ and by Proposition 5(1), the function $u_1^{(\frac{1}{4})}$ is on $[\frac{1}{4}, 1]$ strictly concave. By Proposition 2(1), we have $D_- u_1^{(\frac{1}{4})}(\frac{2}{3}) = -f_+(\frac{1}{3}) + \frac{1}{2} f_-(\frac{5}{24}) = -\frac{5}{12} + \frac{43}{48} = \frac{23}{48} > 0$. And $D_+ u_1^{(\frac{1}{4})}(\frac{2}{3}) = -f_-(\frac{1}{3}) + \frac{1}{2} f_+(\frac{5}{24}) = -\frac{5}{12} + \frac{43}{192} = -\frac{37}{192} < 0$. This implies $\frac{2}{3} \in B(\frac{1}{4})$.

By Proposition 9, $B(\frac{2}{3}) = \mathrm{argmax}_{x_1 \in [0, \frac{2}{3}[} u_1^{(\frac{1}{4})}(x_1)$ and by Proposition 5(2), the function $u_1^{(\frac{2}{3})}$ is on $[0, \frac{2}{3}]$ strictly concave. By Proposition 2(1), we have $D_- u_1^{(\frac{2}{3})}(\frac{1}{4}) = f_-(\frac{1}{4}) - \frac{1}{2} f_+(\frac{5}{24}) = \frac{42}{96} - \frac{43}{192} = \frac{41}{192} > 0$. And $D_+ u_1^{(\frac{2}{3})}(\frac{1}{4}) = f_+(\frac{1}{4}) - \frac{1}{2} f_-(\frac{5}{24}) = \frac{42}{96} - \frac{43}{48} = -\frac{44}{96} < 0$. This implies $\frac{1}{4} \in B(\frac{2}{3})$. ⋄

Define the function $H : [0, \frac{L}{2}] \to \mathbb{R}$ by

$$H(x_1) := f(x_1) - \frac{1}{2} f(\frac{L}{2} - x_1).$$

Note that $H(0) > 0$, H is decreasing, and strictly decreasing if f is not constant. Thus H has at most one zero. If f is continuous and $f(\frac{L}{2}) \leq \frac{1}{2} f(0)$, then H has a unique zero; we denote this zero by

$$x^\star.$$

As $H(\frac{L}{4}) = \frac{1}{2} f(\frac{L}{4}) > 0$, we obtain

$$x^\star \in \begin{cases}]\frac{L}{4}, \frac{L}{2}[\text{ if } f(\frac{L}{2}) < \frac{1}{2} f(0), \\ = \frac{L}{2} \text{ if } f(\frac{L}{2}) = \frac{1}{2} f(0). \end{cases} \tag{3}$$

Theorem 5. *Suppose f is continuous. Then the game has a Nash equilibrium. Even:*

1. *if $f(\frac{L}{2}) \geq \frac{1}{2} f(0)$, then $E = \{(\frac{L}{2}, \frac{L}{2})\}$.*
2. *if $f(\frac{L}{2}) < \frac{1}{2} f(0)$, then $E = \{(x^\star, L - x^\star), (L - x^\star, x^\star)\}$.* ⋄

Proof. 1. Suppose $f(\frac{L}{2}) \geq \frac{1}{2} f(0)$. By Theorem 2(3), we may suppose that f is strictly decreasing. By Proposition 8(1b, 2b), we have $\{(\frac{L}{2}, \frac{L}{2})\} \subseteq E$. Now suppose $(e_1, e_2) \in E$. We have to prove that $(e_1, e_2) = (\frac{L}{2}, \frac{L}{2})$. This we do by contradiction. So suppose $(e_1, e_2) \neq (\frac{L}{2}, \frac{L}{2})$. By player symmetry, we may suppose $e_1 \leq e_2$. By Theorem 4, $e_2 - L = e_1$. By Proposition 6, $e_1 \neq 0$ and $e_2 \neq L$. It follows that $0 < e_1 < \frac{L}{2} < e_2 < L$. As $(e_1, L - e_1) \in E$, we have $Du_1^{(L-e_1)}(e_1) = 0$. By Proposition 2(1), $f(e_1) - \frac{1}{2} f(\frac{L}{2} - e_1) = 0$. Thus $f(\frac{L}{2}) < f(e_1) = \frac{1}{2} f(\frac{L}{2} - e_1) \leq \frac{1}{2} f(0)$, a contradiction.

2. Suppose $f(\frac{L}{2}) < \frac{1}{2} f(0)$. '$\subseteq$': suppose $(e_1, e_2) \in E$. By location symmetry, we may suppose $e_1 \leq e_2$. By Theorem 4, $e_1 + e_2 = L$. Propositions 6 and 10 now imply $0 < e_1 < \frac{L}{2} < e_2 < L$. By Proposition 8(3a), $e_1 = B(e_2)$. By Proposition 8(4), $f(e_1) = \frac{1}{2} f(\frac{e_2 - e_1}{2}) = \frac{1}{2} f(\frac{L}{2} - e_1)$. Thus e_1 is a zero of H, and therefore $e_1 = x^\star$. We see $(e_1, e_2) = (x^\star, L - x^\star) \in \{(x^\star, L - x^\star), (L - x^\star, x^\star)\}$.

'$\supseteq$': by location symmetry, it is sufficient to prove that $(x^\star, L - x^\star) \in E$. By definition of $x^\star$ we have $0 = f(x^\star) - \frac{1}{2} f(\frac{L}{2} - x^\star) = f(x^\star) - \frac{1}{2} f(\frac{(L-x^\star)-x^\star}{2})$. Therefore Proposition 8(4) guarantees that $x^\star = B(L-x^\star)$. This implies $L-x^\star = B(x^\star)$. It follows that $(x^\star, L - x^\star) \in E$. Q.E.D.

Thus for a continuous demand function, the Principle of Minimum Differentiation holds if and only if $f(\frac{L}{2}) \geq \frac{1}{2} f(0)$.

Corollary 1. *Suppose f is continuous. Then the game has one equilibrium class and this class contains one or two elements.* ◇

In the next section we shall prove by a completely different approach that each cHg has a Nash equilibrium (even if f is not continuous).

Proposition 11. *Suppose f is continuous. Then for all $(e_1, e_2) \in E$ it holds that $e_1, e_2 \in \,]\frac{L}{4}, \frac{3}{4}L\,[$.* ◇

Proof. By Theorem 5 and (3). Q.E.D.

7 Potentials

In this section we review the results in [12] on potentials for the cHg. We shall encounter the notions of generalized ordinal potential, best-response potential, a weak quasi-potential and quasi-potential; again we denote with E the Nash equilibrium set.[6] We note that each generalized ordinal potential game and each best-response potential game is a weak quasi potential game.

Define the function $P^\bullet \colon S \times S \to \mathbb{R}$ by

$$P^\bullet(x_1, x_2) := \mathcal{L}(\min\{x_1, x_2\}) + \mathcal{L}(L - \max\{x_1, x_2\}) + \mathcal{L}(\frac{|x_2 - x_1|}{2}). \tag{4}$$

Note that $P^\bullet$ is continuous irrespective of the continuity of f. In the case of $f = 1$, i.e. elastic demand, we have

$$P^\bullet(x_1, x_2) = \begin{cases} L - \frac{x_2 - x_1}{2} \text{ if } x_1 < x_2, \\ L \text{ if } x_1 = x_2, \\ L - \frac{x_1 - x_2}{2} \text{ if } x_1 > x_2. \end{cases}$$

Theorem 6. *1. Suppose f is continuous.*
(a) If $f(\frac{L}{2}) \leq \frac{1}{2} f(0)$, then $P^\bullet$ is a continuous best-response potential.

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[6] For the cHg, a function $P : S \times S \to \mathbb{R}$ is (1) a generalized ordinal potential if for every $a_1, b_1, z \in [0, L]$ it holds that $u_1(a_1, z) < u_1(b_1; z) \;\Rightarrow\; P(a_1, z) < P(b_1, z)$ and for every $a_2, b_2, z \in [0, L]$ it holds that $u_2(z, a_1) < u_2(z, b_1) \;\Rightarrow\; P(z, a_1) < P(z, b_1)$; (2) a best-response potential if $B_1(x_2) = \mathrm{argmax}_{x_1 \in S} P(x_1, x_2)$ $(x_2 \in S)$ and $B_2(x_1) = \mathrm{argmax}_{x_2 \in S} P(x_1, x_2)$ $(x_1 \in S)$; (3) a quasi potential if $\mathrm{argmax}\, P = E$. (4) a weak quasi potential if $\mathrm{argmax}\, P \subseteq E$. In this case, one calls the game a 'generalized ordinal potential game' (etc.).

   (b) If $f(\frac{L}{2}) > \frac{1}{2} f(0)$, then there does not exist a continuous best-response potential.
   (c) If f is strictly decreasing, then $P^\bullet$ is a continuous quasi potential.
2. *If f is strictly decreasing, then $P^\bullet$ is a continuous weak quasi potential.*
3. *If f is constant, then $P(x_1, x_2) = -(|\frac{L}{2} - x_1| + |\frac{L}{2} - x_2|)$ is a continuous quasi potential.*
4. *The game may not have a generalized ordinal potential, even if f is continuous.* ◇

Proof. 1a. See Proposition 3.1 in [12].

1b. Suppose $f(\frac{L}{2}) > \frac{1}{2} f(0)$. By contradiction, suppose P is a continuous best-response potential. By the Weierstrass theorem this would imply that B is proper. But, by Proposition 8(1c), B is not proper.

1c. See Proposition 3.2 in [12].

2. See concluding remark 3 in [12].

3. By Theorem 2(3), $E = \{(\frac{L}{2}, \frac{L}{2})\}$. Thus, as desired, $\mathrm{argmax}(P) = E$.

4. See Proposition 3.3 in [12]. Q.E.D.

Theorem 6(2, 3) implies:

Corollary 2. *Each cHg has a Nash equilibrium.* ◇

In addition to Theorem 6(1b), we have:

Proposition 12. *In the case f is constant, $P : [0, L] \times [0, L] \to \mathbb{R}$ defined by*

$$P(x_1, x_2) := \begin{cases} -|x_1 - x_2| \text{ if } x_1 = \frac{L}{2} \vee x_2 = \frac{L}{2}, \\ \frac{1}{|x_1 - \frac{L}{2}|} + \frac{1}{|x_2 - \frac{L}{2}|} \text{ if } x_1 \neq \frac{L}{2} \wedge x_2 \neq \frac{L}{2} \end{cases}$$

is a (discontinuous) best-response potential. ◇

Proof. As P is symmetric, P being a quasi-potential comes down to

$$\text{for all } x_2 \in [0, L] : \quad B(x_2) = \mathrm{argmax}_{x_1 \in [0,L]} P(x_1, x_2).$$

We have $\mathrm{argmax}_{x_1 \in [0,L]} P(x_1, x_2) = \begin{cases} \emptyset \text{ if } x_2 \neq \frac{L}{2}, \\ \{\frac{L}{2}\} \text{ if } x_2 = \frac{L}{2}. \end{cases}$ So $P(x_1, \frac{L}{2}) = -|x_1 - \frac{L}{2}|$; thus, as desired, $\mathrm{argmax}_{x_1 \in [0,L]} P(x_1, \frac{L}{2}) = \{\frac{L}{2}\}$. Now fix $x_2 \neq \frac{L}{2}$. We have $P(x_1, x_2) = \begin{cases} \frac{1}{|x_1 - \frac{L}{2}|} + \frac{1}{|x_2 - \frac{L}{2}|} \text{ if } x_1 \neq \frac{L}{2}, \\ -|\frac{L}{2} - x_2| \text{ if } x_1 = \frac{L}{2}. \end{cases}$ From this formula, one sees, as desired, that $\mathrm{argmax}_{x_1 \in [0,L]} P(x_1, x_2) = \emptyset$. Q.E.D.

8 Concluding Remarks

1. We presented results for the Hotelling pure location game with two identical players, one-dimensional strategy sets and a demand function f which is constant (inelastic case) or strictly decreasing (elastic case). The elastic case has been poorly studied in the literature. For the elastic case we tried to derive the results without further smoothness assumptions on f.

2. Although the inelastic case for our one-dimensional case of the cHg is simple to analyse, this is no longer true for the two-dimensional case (see [9]).
3. We have shown that for a continuous f, the Principle of Minimum Differentiation holds if and only if $f(\frac{L}{2}) \geq \frac{1}{2}f(0)$. This is in accordance with the observations in the literature (e.g. [5]) that this principle is not so robust.
4. In Corollary 1 we have shown that in the case of a continuous f, the game has at most one equilibrium class and this class contains one or two elements. In Example 1 we have shown for a specific cHg with a discontinuous demand function, that it has $(\frac{2}{3}, \frac{1}{4})$ as equilibrium. Therefore also $(\frac{1}{4}, \frac{2}{3})$, $(\frac{1}{3}, \frac{3}{4})$ and $(\frac{3}{4}, \frac{1}{3})$ are equilibria and this game has an equilibrium class with four elements. An interesting question is whether there exists a cHg with more than one equilibrium class.
5. We have shown 'by hand' that the cHg has a Nash equilibrium in the case of a continuous demand function. One might want to have a deeper reason for this existence. Concerning this Theorem 6 shows that each cHg admits a continuous weak quasi potential (and therefore has an equilibrium).
6. As the cHg is a game with discontinuous payoff functions, it may be interesting to find out in which sense general equilibrium existence results for games in strategic form with discontinuous payoff functions apply. We here only mention that the result in [8] does not apply as it assumes quasi-concave conditional payoff functions.
7. The type of strict quasi-concavity in Proposition 5(5) has been studied in more detail in [10], where it was called 'semi-strict demi-concavity'.
8. A direction for further research concerns the comparison of the results in the present article with those for the, also poorly studied, discrete variant of the cHg [13,14].

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