The Lang–Vojta Conjectures on Projective Pseudo-Hyperbolic Varieties

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1 Introduction

These notes grew out of a mini-course given from May 13th to May 17th at UQÀM in Montréal during a workshop on Diophantine Approximation and Value Distribution Theory.

1.1 What Is in These Notes?

We start with an overview of Lang–Vojta's conjectures on pseudo-hyperbolic *projective* varieties. These conjectures relate various different notions of hyperbolicity. We start with Brody hyperbolicity and discuss conjecturally related notions of hyperbolicity in arithmetic geometry and algebraic geometry in subsequent sections. We slowly work our way towards the most general version of Lang–Vojta's conjectures and provide a summary of all the conjectures in Section 12.

After having explained the main conjectures with the case of curves and closed subvarieties of abelian varieties as our guiding principle, we collect recent advances on Lang–Vojta's conjectures and present these in a unified manner. These results are concerned with endomorphisms of hyperbolic varieties, moduli spaces of maps into a hyperbolic variety, and also the

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behaviour of hyperbolicity in families of varieties. The results presented in these sections are proven in [15, 49, 50, 55, 56].

We also present results on the Shafarevich conjecture for smooth hypersurfaces obtained in joint work with Daniel Litt [52]. These are motivated by Lawrence–Venkatesh's recent breakthrough on the non-density of integral points on the moduli space of hypersurfaces [63], and are in accordance with Lang–Vojta's conjecture for *affine* varieties. Our results in this section are proven using methods from Hodge theory, and are loosely related to Bakker–Tsimerman's chapter in this book [12].

In the final section we sketch a proof of the fact that being groupless is a Zariski-countable open condition, and thus in particular stable under generization. To prove this, we follow [55] and introduce a non-archimedean notion of hyperbolicity. We then state a non-archimedean analogue of the Lang–Vojta conjectures which we prove under suitable assumptions. These results suffice to prove that grouplessness is stable under generization.

1.2   Anything New in These Notes?

The main contribution of these notes is the systematic presentation and comparison between different notions of hyperbolicity, and their "pseudofications". As it is intended to be a broad-audience introduction to the Lang–Vojta conjectures, it contains all definitions and well-known relations between these. Also, Lang–Vojta's original conjectures are often only stated for varieties over $\overline{\mathbb{Q}}$, and we propose natural extensions of their conjectures to varieties over arbitrary algebraically closed fields of characteristic zero. We also define for each notion appearing in the conjecture the relevant "exceptional locus" (which Lang only does for some notions of hyperbolicity in [62]).

The final version of Lang–Vojta's conjecture as stated in Section 12 does not appear anywhere in the literature explicitly. Furthermore, the section on groupless varieties (Section 4) contains simple proofs that do not appear explicitly elsewhere. Also, we have included a thorough discussion of the a priori difference between being arithmetically hyperbolic and Mordellic for a projective variety in Section 7. This difference is not addressed anywhere else in the literature.

1.3   Rational Points over Function Fields

We have not included any discussion of rational points on projective varieties over function fields of smooth connected curves over a field k, and unfortunately ignore the relation to Lang–Vojta's conjecture throughout these notes.

1.4   Other Relevant Literature

Lang stated his conjectures in [62]; see also [23, Conjecture XV.4.3] and [1, §0.3]. In [85, Conj. 4.3] Vojta extended this conjecture to quasi-projective varieties. In [62] Lang "pseudofied" the notion of Brody hyperbolicity. Here he was inspired by Kiernan–Kobayashi's extension of the notion of Kobayashi hyperbolicity introduced in [58].

There are several beautiful surveys of the Green–Griffiths and Lang–Vojta conjectures. We mention [24–26, 31, 38, 86].

The first striking consequence of Lang–Vojta's conjecture was obtained by Caporaso–Harris–Mazur [19]. Their results were further investigated by Abramovich, Ascher–Turchet, Hassett, and Voloch; see [1–4, 9, 42].

Campana's conjectures provide a complement to Lang–Vojta's conjectures, and first appeared in [17, 18]; see also Campana's chapter in this book [16]. In a nutshell, the "opposite" of being pseudo-hyperbolic (in any sense of the word "hyperbolic") is conjecturally captured by Campana's notion of a "special" variety.

Conventions. Throughout these notes, we will let k be an algebraically closed field of characteristic zero. If X is a locally finite type scheme over $\mathbb{C}$, we let X^{an} be the associated complex-analytic space [39, Expose XII]. If K is a field, then a variety over K is a finite type separated K-scheme.

If X is a variety over a field K and L/K is a field extension, then $X_L := X \times_{\mathrm{Spec}\,K} \mathrm{Spec}\,L$ will denote the base-change of $X \to \mathrm{Spec}\,K$ along $\mathrm{Spec}\,L \to \mathrm{Spec}\,K$. More generally, if $R \to R'$ is an extension of rings and X is a scheme over R, we let $X_{R'}$ denote $X \times_{\mathrm{Spec}\,R} \mathrm{Spec}\,R'$.

If K is a number field and S is a finite set of finite places of K, then $\mathcal{O}_{K,S}$ will denote the ring of S-integers of K.

2   Brody Hyperbolicity

We start with the classical notion of Brody hyperbolicity for complex varieties.

Definition 2.1. A complex-analytic space X is *Brody hyperbolic* if every holomorphic map $\mathbb{C} \to X$ is constant. A locally finite type scheme X over $\mathbb{C}$ is *Brody hyperbolic* if X^{an} is Brody hyperbolic.

If X is a complex-analytic space, then a non-constant holomorphic map $\mathbb{C} \to X$ is commonly referred to as an entire curve in X. Thus, to say that X is Brody hyperbolic is to say that X has no entire curves.

We recall that a complex-analytic space X is *Kobayashi hyperbolic* if Kobayashi's pseudo-metric on X is a metric [59]. It is a fundamental result

of Brody that a *compact* complex-analytic space X is Brody hyperbolic if and only if it is Kobayashi hyperbolic; see [59, Theorem 3.6.3].

Remark 2.2 (Descending Brody Hyperbolicity). Let $X \to Y$ be a proper étale (hence finite) morphism of varieties over $\mathbb{C}$. It is not hard to show that X is Brody hyperbolic if and only if Y is Brody hyperbolic. (It is crucial that $X \to Y$ is finite **and** étale.)

Fundamental results in complex analysis lead to the following classification of Brody hyperbolic projective curves.

Theorem 2.3 (Liouville, Riemann, Schwarz, Picard). *Let X be a smooth projective connected curve over $\mathbb{C}$. Then X is Brody hyperbolic if and only if* genus$(X) \geq 2$.

More generally, a smooth quasi-projective connected curve X over $\mathbb{C}$ is Brody hyperbolic if and only if X is not isomorphic to $\mathbb{P}^1_{\mathbb{C}}$, $\mathbb{A}^1_{\mathbb{C}}$, $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$, nor a smooth proper connected genus one curve over $\mathbb{C}$.

Remark 2.4. It is implicit in Theorem 2.3 that elliptic curves are not Brody hyperbolic. More generally, a non-trivial abelian variety A of dimension g over $\mathbb{C}$ is not Brody hyperbolic, as its associated complex-analytic space is uniformized by $\mathbb{C}^g$. Since A even has a dense entire curve, one can consider A to be as far as possible from being Brody hyperbolic. We mention that Campana conjectured that a projective variety has a dense entire curve if and only if it is "special". We refer the reader to Campana's article in this book for a further discussion of Campana's conjecture [16].

By Remark 2.4, an obvious obstruction to a projective variety X over $\mathbb{C}$ being Brody hyperbolic is that it contains an abelian variety. The theorem of Bloch–Ochiai–Kawamata says that this is the only obstruction if X can be embedded into an abelian variety (see [57]).

Theorem 2.5 (Bloch–Ochiai–Kawamata). *Let X be a closed subvariety of an abelian variety A over $\mathbb{C}$. Then X is Brody hyperbolic if and only if IT does not contain the translate of a positive-dimensional abelian subvariety of A.*

Throughout these notes, we mostly focus on closed subvarieties of abelian varieties, as in this case the results concerning Lang–Vojta's conjectures are complete; see Section 13 for details.

The theorem of Bloch–Ochiai–Kawamata has been pushed further by work of Noguchi–Winkelmann–Yamanoi; see [76–78, 87, 88]. Other examples of Brody hyperbolic varieties can be constructed as quotients of bounded domains, as we explain now.

Remark 2.6 (Bounded Domains). Let D be a bounded domain in the affine space $\mathbb{C}^N$, and let X be a reduced connected locally finite type scheme over $\mathbb{C}$. Then, any holomorphic map $X^{\mathrm{an}} \to D$ is constant; see [55,

Remark 2.9] for a detailed proof. In particular, the complex-analytic space D is Brody hyperbolic (take $X = \mathbb{A}^1_{\mathbb{C}}$).

It follows from Remark 2.6 that a (good) quotient of a bounded domain is Brody hyperbolic. This observation applies to locally symmetric varieties, Shimura varieties, and thus moduli spaces of abelian varieties. We conclude this section by recording the fact that the moduli space of abelian varieties (defined appropriately) is a Brody hyperbolic variety.

Example 2.7. Let $g \geq 1$ and let $N \geq 3$ be integers. Then, the (fine) moduli space of g-dimensional principally polarized abelian varieties with level N structure is a smooth quasi-projective variety over $\mathbb{C}$ which is Brody hyperbolic. Indeed, its universal cover is biholomorphic to a bounded domain in $\mathbb{C}^{g(g+1)/2}$, so that we can apply Remark 2.6. (As the coarse moduli space of elliptic curves is given by the j-line $\mathbb{A}^1_{\mathbb{C}}$, we see that it is not Brody hyperbolic. This is the reason for which we consider the moduli space of abelian varieties with level structure.)

3   Mordellic Varieties

What should correspond to being Brody hyperbolic in arithmetic geometry? Lang was the first to propose that a "Mordellic" projective variety over $\overline{\mathbb{Q}}$ should be Brody hyperbolic (over the complex numbers). Roughly speaking, a projective variety over $\overline{\mathbb{Q}}$ is Mordellic if it has only finitely many rational points in any fixed number field. To make this more precise, one has to choose models (see Definition 3.1 below). Conversely, a projective variety over a number field which is Brody hyperbolic (over the complex numbers) should be Mordellic. In this section we will present this conjecture of Lang.

Throughout this section, we let k be an algebraically closed field of characteristic zero. We first clarify what is meant with a model.

Definition 3.1. Let X be a finite type separated scheme over k and let $A \subset k$ be a subring. A *model for X over A* is a pair $(\mathcal{X}, \phi)$ with $\mathcal{X} \to \operatorname{Spec} A$ a finite type separated scheme and $\phi : \mathcal{X}_k \xrightarrow{\sim} X$ an isomorphism of schemes over k. We will often omit ϕ from our notation.

Remark 3.2. What constitutes the data of a model for X over A? To explain this, let X be an affine variety over $\mathbb{C}$, say $X = \operatorname{Spec} R$. Note that the coordinate ring R of X is a finite type $\mathbb{C}$-algebra. Suppose that X is given by the zero locus of polynomials $f_1, \ldots, f_r$ with coefficients in a subring A, so that $R \cong \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. Then $\mathcal{R} := A[x_1, \ldots, x_n]/(f_1, \ldots, f_r) \subset R$ is a finitely generated A-algebra and $\mathcal{R} \otimes_A \mathbb{C} = R$. That is, if $\mathcal{X} = \operatorname{Spec} \mathcal{R}$, then $\mathcal{X}$ is a model for X over A. We will be interested in studying A-valued points on $\mathcal{X}$. We follow common notation and let $\mathcal{X}(A)$ denote the

set $\mathrm{Hom}_A(\mathrm{Spec}\, A, \mathcal{X})$. Note that $\mathcal{X}(A)$ is the set of solutions in A of the polynomial system of equations $f_1 = \ldots = f_r = 0$.

With the notion of model now clarified, we are ready to define what it means for a proper variety to be Mordellic. We leave the more general definition for non-proper varieties to the end of this section.

Definition 3.3. A proper scheme X over k is *Mordellic over* k (or: *has-only-finitely-many-rational-points over* k) if, for every finitely generated subfield $K \subset k$ and every (proper) model $\mathcal{X}$ over K, the set $\mathcal{X}(K) := \mathrm{Hom}_K(\mathrm{Spec}\, K, \mathcal{X})$ is finite.

Remark 3.4 (Independence of Models). We point out that the finiteness property required for a projective variety to be Mordellic can also be tested on a fixed model. That is, a proper scheme X over k is Mordellic over k if and only if there is a finitely generated subfield $K \subset k$ and a proper model $\mathcal{X}$ for X over K such that for all finitely generated subfields $L \subset k$ containing K, the set $\mathcal{X}(L) := \mathrm{Hom}_K(\mathrm{Spec}\, L, \mathcal{X})$ is finite.

We note that Mordellicity (just like Brody hyperbolicity) descends along finite étale morphisms (Remark 2.2).

Remark 3.5 (Descending Mordellicity). Let $X \to Y$ be a finite étale morphism of projective varieties over k. Then it follows from the Chevalley–Weil theorem that X is Mordellic over k if and only if Y is Mordellic over k; see Theorem 7.9 for a proof (of a more general result).

It is clear that $\mathbb{P}^1_k$ is not Mordellic, as $\mathbb{P}^1(\mathbb{Q})$ is dense. A deep theorem of Faltings leads to the following classification of projective Mordellic curves. If $k = \overline{\mathbb{Q}}$, then this theorem is proven in [32]. The statement below is proven in [33] (see also [82]).

Theorem 3.6 (Faltings). *Let X be a smooth projective connected curve over k. Then X is Mordellic over k if and only if* $\mathrm{genus}(X) \geq 2$.

Recall that abelian varieties are very far from being Brody hyperbolic (Remark 2.4). The following remark says that abelian varieties are also very far from being Mordellic.

Remark 3.7. It is not at all obvious that a smooth projective connected curve of genus one over $\overline{\mathbb{Q}}$ is not Mordellic. Indeed, it is not an obvious fact that an elliptic curve over a number field K has positive rank over some finite field extension of K, although this is certainly true and can be proven in many different ways. In fact, by a theorem of Frey–Jarden [36] (see also [49, §3.1] or [41, §3]), if A is an abelian variety over k, then there is a finitely generated subfield $K \subset k$ and an abelian variety $\mathcal{A}$ over K with $\mathcal{A}_k \cong A$ such that $\mathcal{A}(K)$ is dense in A. This theorem is not hard to prove when k is uncountable but requires non-trivial arguments otherwise. Thus, if $\dim A \neq 0$, then one can

consider the abelian variety A to be as far as possible from being Mordellic. This statement is to be compared with the conclusion of Remark 2.4.

By Remark 3.7, an obvious obstruction to a projective variety X over k being Mordellic is that it contains an abelian variety. The following theorem of Faltings says that this is the only obstruction if X can be embedded into an abelian variety; see [34].

Theorem 3.8 (Faltings). *Let X be a closed subvariety of an abelian variety A over k. Then X is Mordellic over k if and only if X does not contain the translate of a positive-dimensional abelian subvariety of A.*

There are strong similarities between the statements in the previous section and the current section. These similarities (and a healthy dose of optimism) lead to the first version of the Lang–Vojta conjecture. To state this conjecture, let us say that a variety X over k is *strongly-Brody hyperbolic over k* if, for every subfield $k_0 \subset k$, every model $\mathcal{X}$ for X over k_0, and every embedding $k_0 \to \mathbb{C}$, the variety $\mathcal{X}_{\mathbb{C}}$ is Brody hyperbolic.

Conjecture 3.9 (Weak Lang–Vojta, I). *Let X be an integral projective variety over k. Then X is Mordellic over k if and only if X is strongly-Brody hyperbolic over k.*

As stated, this conjecture does not predict that, if X is a projective Brody hyperbolic variety over $\mathbb{C}$, then every conjugate of X is Brody hyperbolic. We state this conjecture separately.

Conjecture 3.10 (Conjugates of Brody Hyperbolic Varieties). *If X is an integral variety over k. Then X is strongly-Brody hyperbolic over k if and only if there is a subfield $k_0 \subset k$, a model $\mathcal{X}$ for X over k_0, and an embedding $k_0 \to \mathbb{C}$ such that the variety $\mathcal{X}_{\mathbb{C}}$ is Brody hyperbolic.*

Concretely, Conjecture 3.10 says that, if X is a Brody hyperbolic variety over $\mathbb{C}$ and σ is a field automorphism of $\mathbb{C}$, then the σ-conjugate X^{σ} of X is again Brody hyperbolic.

We briefly discuss the notion of Mordellicity for quasi-projective (not necessarily proper) schemes. We will also comment on this more general notion in Section 7. This notion appears in this generality (to our knowledge) for the first time in Vojta's paper [86], and it is also studied in [56]. It is intimately related to the notion of "arithmetic hyperbolicity" [49, 53]; see Section 7 for a discussion.

In the non-proper case, it is natural to study integral points rather than rational points. Vojta noticed in [86] that, in fact, it is more natural to study "near-integral points". Below we make this more precise.

Definition 3.11. Let $X \to S$ be a morphism of schemes with S integral. We define $X(S)^{(1)}$ to be the set of P in $X(K(S))$ such that, for every point s in S of codimension one, the point P lies in the image of $X(\mathcal{O}_{S,s}) \to X(K(S))$.

Vojta refers to the points in $X(S)^{(1)}$ as "near-integral" S-points. We point out that on an affine variety, there is no difference between the finiteness of integral points and "near-integral" points; see Section 7.

Definition 3.12 (Quasi-Projective Mordellic Varieties). A variety X over k is *Mordellic over k* if, for every $\mathbb{Z}$-finitely generated subring $A \subset k$ and every model $\mathcal{X}$ for X over A, the set $\mathcal{X}(A)^{(1)}$ of near-integral A-points is finite.

The study of near-integral points might seem unnatural at first. To convince the reader that this notion is slightly more natural than the notion of integral point, we include the following remark.

Remark 3.13 (Why "Near-Integral" Points?). Consider a proper scheme $\mathcal{X}$ over $\mathbb{Z}$ with generic fibre $X := \mathcal{X}_{\mathbb{Q}}$. Let K be a finitely generated field of characteristic zero and let $A \subset K$ be a regular $\mathbb{Z}$-finitely generated subring. Then, the set of K-rational points $X(K)$ **equals** the set of near-integral A-points of $\mathcal{X}$. On the other hand, if K has transcendence degree at least one over $\mathbb{Q}$, then it is not necessarily true that every K-point of X is an A-point of $\mathcal{X}$. Thus, studying K-rational points on the proper variety X over $\mathbb{Q}$ is equivalent to studying near-integral points of the proper scheme $\mathcal{X}$ over $\mathbb{Z}$.

With this definition at hand, we are able to state Faltings's finiteness theorem for abelian varieties over number rings as a statement about the Mordellicity of the appropriate moduli space. The analogous statement on its Brody hyperbolicity is Example 2.7.

Theorem 3.14 (Faltings, Shafarevich's Conjecture for Principally Polarized Abelian Varieties). *Let k be an algebraically closed field of characteristic zero. Let $g \geq 1$ and let $N \geq 3$ be integers. Then, the (fine) moduli space $\mathcal{A}_{g,k}^{[N]}$ of g-dimensional principally polarized abelian varieties with level N structure is a smooth quasi-projective Mordellic variety over k.*

Example 2.7 and Theorem 3.14 suggest that there might also be an analogue of Lang–Vojta's conjecture for quasi-projective schemes. It seems reasonable to suspect that an affine variety over k is Mordellic over k if and only if it is strongly-Brody hyperbolic over k; see for example [46] for a discussion of Lang's conjectures in the affine case. However, stating a reasonable conjecture for quasi-projective varieties requires some care, and would take us astray from our current objective. We refer the interested reader to articles of Ascher–Turchet and Campana in this book [8, 16] for a related discussion, and the book by Vojta [85].

Remark 3.15 (From Shafarevich to Mordell). Let us briefly explain how Faltings shows that Theorem 3.14 implies Faltings's finiteness theorem for curves (Theorem 3.6). Let X be a smooth projective connected curve of

genus at least two over k. By a construction of Kodaira [69], there is a finite étale morphism $Y \to X$, an integer $g \geq 1$, and a non-constant morphism $Y \to \mathcal{A}_{g,k}^{[3]}$. Since $\mathcal{A}_{g,k}^{[3]}$ is Mordellic over k and $Y \to \mathcal{A}_{g,k}^{[3]}$ has finite fibres, it follows that Y is Mordellic over k. As Mordellicity descends along finite étale morphisms (Remark 3.5), we conclude that X is Mordellic, as required.

4 Groupless Varieties

To study Lang–Vojta's conjectures, it is natural to study varieties which do not "contain" any algebraic groups. Indeed, as we have explained in Remark 2.4 (resp. Remark 3.7), a Brody hyperbolic variety (resp. a Mordellic variety) does not admit any non-trivial morphisms from an abelian variety. For projective varieties, it turns out that this is equivalent to not admitting a non-constant map from any connected algebraic group (see Lemma 4.4 below).

As before, we let k be an algebraically closed field of characteristic zero. We start with the following definition.

Definition 4.1. A variety X over k is *groupless* if every morphism $\mathbb{G}_{m,k} \to X$ (of varieties over k) is constant, and for every abelian variety A over k, every morphism $A \to X$ is constant.

Remark 4.2. We claim that, for proper varieties, the notion of grouplessness can be tested on morphisms (or even rational maps) from abelian varieties. That is, a proper variety X over k is groupless if and only if, for every abelian variety A over k, every rational map $A \dashrightarrow X$ is constant. To show this, first note that a morphism $\mathbb{G}_{m,k} \to X$ extends to a morphism $\mathbb{P}_k^1 \to X$ and that $\mathbb{P}_k^1$ is surjected upon by an elliptic curve. Therefore, if every morphism from an abelian variety is constant, then X is groupless and has no rational curves. Now, if X is proper over k and has no rational curves, every rational map $A \dashrightarrow X$ with A an abelian variety extends to a morphism (see [50, Lemma 3.5]). Thus, if every morphism $A \to X$ is constant with A an abelian variety, we conclude that every rational map $A \dashrightarrow X$ is constant. This proves the claim. We also conclude that a proper variety is groupless if and only if it is "algebraically hyperbolic" in Lang's sense [62, p. 176].

Remark 4.3 (Lang's Algebraic Exceptional Set). For X a proper variety over k, Lang defines the *algebraic exceptional set* $\mathrm{Exc}_{alg}(X)$ *of* X to be the union of all non-constant rational images of abelian varieties in X. With Lang's terminology at hand, as is explained in Remark 4.2, a proper variety X over k is groupless over k if and only if $\mathrm{Exc}_{alg}(X)$ is empty.

Let us clear up why we refer to this property as groupless.

Lemma 4.4 (Why Call This Groupless?). *A variety X over k is groupless if and only if for all finite type connected group schemes G over k, every morphism $G \to X$ is constant.*

Proof. This follows from Chevalley's structure theorem for algebraic groups over the algebraically closed field k of characteristic zero. A detailed proof is given in [50, Lemma 2.5]. □

The notion of grouplessness is well-studied, and sometimes referred to as "algebraic hyperbolicity" or "algebraic Lang hyperbolicity"; see [43], [62, page 176], [59, Remark 3.2.24], or [60, Definition 3.4]. We will only use the term "algebraically hyperbolic" for the notion introduced by Demailly in [29] (see also [15, 50, 56]). The term "groupless" was first used in [50, Definition 2.1] and [55, Definition 3.1].

Example 4.5. A zero-dimensional variety is groupless. Note that $\mathbb{P}^1_k$, $\mathbb{A}^1_k$, $\mathbb{A}^1_k \setminus \{0\}$ and smooth proper genus one curves over k are not groupless.

Much like Brody hyperbolicity and Mordellicity, grouplessness descends along finite étale morphisms. We include a sketch of the proof of this simple fact.

Lemma 4.6 (Descending Grouplessness). *Let $X \to Y$ be a finite étale morphism of varieties over k. Then X is groupless over k if and only if Y is groupless over k.*

Proof. If Y is groupless, then X is obviously groupless. Therefore, to prove the lemma, we may assume that X is groupless. Let G be $\mathbb{G}_{m,k}$ or an abelian variety over k. Let $G \to Y$ be a morphism. Consider the pull-back $G' := G \times_Y X$ of $G \to Y$ along $X \to Y$. Then, as k is algebraically closed and of characteristic zero, each connected component of G' is (or: can be endowed with the structure of) an algebraic group isomorphic to $\mathbb{G}_{m,k}$ or an abelian variety over k. Therefore, the morphism $G' \to X$ is constant. This implies that $G \to Y$ is constant. □

We include an elementary proof of the fact that the classification of one-dimensional groupless varieties is the same as that of one-dimensional Brody hyperbolic curves.

Lemma 4.7. *A smooth quasi-projective connected curve X over k is groupless over k if and only if X is not isomorphic to $\mathbb{P}^1_k$, $\mathbb{A}^1_k$, $\mathbb{A}^1_k \setminus \{0\}$, nor a smooth proper connected curve of genus one over k.*

Proof. If X is groupless, then X is not isomorphic to $\mathbb{P}^1_k$, $\mathbb{A}^1_k$, $\mathbb{A}^1_k \setminus \{0\}$, nor a smooth proper connected curve of genus one over k; see Example 4.5. Thus to prove the lemma, we may (and do) assume that X is not isomorphic to either of these curves. Let $Y \to X$ be a finite étale cover of X such that the smooth projective model $\overline{Y}$ of Y is of genus at least two. (It is clear that such a cover exists when $X = \mathbb{G}_{m,k} \setminus \{1\}$ or $X = E \setminus \{0\}$ with E an elliptic

curve over k. This is enough to conclude that such a cover always exists.) By Lemma 4.6, the variety X is groupless if and only if Y is groupless. Thus, it suffices to show that $\overline{Y}$ is groupless. To do so, assume that we have a morphism $\mathbb{G}_{m,k} \to \overline{Y}$. By Riemann–Hurwitz, this morphism is constant, as $\overline{Y}$ has genus at least two. Now, let A be an abelian variety over k and let $A \to \overline{Y}$ be a morphism. To show that this morphism is constant, we compose $A \to \overline{Y}$ with the Jacobian map $\overline{Y} \to \mathrm{Jac}(\overline{Y})$ (after choosing some point on $\overline{Y}$). If the morphism $A \to \overline{Y}$ is non-constant, then it is surjective. Since a morphism of abelian varieties is a homomorphism (up to translation of the origin), this induces a group structure on the genus > 1 curve $\overline{Y}$. However, as the automorphism group of (the positive-dimensional variety) $\overline{Y}$ is finite, the curve $\overline{Y}$ cannot be endowed with the structure of an algebraic group. This shows that $A \to \overline{Y}$ is constant, and concludes the proof. $\qquad\square$

Bloch–Ochiai–Kawatama's theorem (Theorem 2.5) and Faltings's analogous theorem for rational points on closed subvarieties of abelian varieties (Theorem 3.8) characterize "hyperbolic" subvarieties of abelian varieties. It turns out that this characterization also holds for groupless varieties, as we explain now.

If X is a closed subvariety of an abelian variety A over k, we define the special locus $\mathrm{Sp}(X)$ of X to be the union of the translates of positive-dimensional abelian subvarieties of A contained in X.

Lemma 4.8. *Let X be a closed integral subvariety of an abelian variety A over k. Then X is groupless over k if and only if $\mathrm{Sp}(X)$ is empty.*

Proof. Clearly, if X is groupless over k, then X does not contain the translate of a positive-dimensional abelian subvariety of A, so that $\mathrm{Sp}(X)$ is empty. Conversely, assume that X does not contain the translate of a non-zero abelian subvariety of A. Let us show that X is groupless. Since the Albanese variety of $\mathbb{P}^1_k$ is trivial, any map $\mathbb{G}_{m,k} \to X$ is constant. Thus, to conclude the proof, we have to show that all morphisms $A' \to X$ are constant, where A' is an abelian variety over k. To do so, note that the image of $A' \to X$ in A is the translate of an abelian subvariety of A, as morphisms of abelian varieties are homomorphisms up to translation. This means that the image of $A' \to X$ is the translate of an abelian subvariety, hence a point (by our assumption). $\qquad\square$

Remark 4.9. Let A be a simple abelian surface. Let $X = A \setminus \{0\}$. Then X is groupless. This remark might seem misplaced, but it shows that "grouplessness" as defined above does not capture the non-hyperbolicity of a quasi-projective variety. The "correct" definition in the quasi-projective case is discussed in Section 6 (and is also discussed in [56, 86]).

Although grouplessness does not capture the non-hyperbolicity of quasi-projective varieties (Remark 4.9), Lang conjectured that grouplessness is equivalent to being Mordellic and to being Brody hyperbolic (up to choosing

a model over $\mathbb{C}$) for *projective* varieties. This brings us to the second form of Lang–Vojta's conjecture.

Conjecture 4.10 (Weak Lang–Vojta, II). *Let X be an integral projective variety over k. Then the following are equivalent.*

(1) *The projective variety X is Mordellic over k.*
(2) *The variety X is strongly-Brody hyperbolic over k.*
(3) *The variety X is groupless over k.*

5 Varieties of General Type

In this section we discuss the role of varieties of general type in Lang–Vojta's conjecture. Recall that a line bundle L on a smooth projective variety S over k is big if there is an ample line bundle A and an effective divisor D such that $L \cong A \otimes \mathcal{O}_S(D)$; see [64, 65]. We follow standard terminology and say that an integral proper variety X over k is of general type if it has a desingularization $X' \to X$ with X' a smooth projective integral variety over k such that the canonical bundle $\omega_{X'/k}$ is a big line bundle. For example, if $\omega_{X'/k}$ is ample, then it is big. Moreover, we will say that a proper variety X over a field k is of general type if, for every irreducible component Y of X, the reduced closed subscheme Y_{red} is of general type.

Varieties of general type are well-studied; see [64, 65]. For the sake of clarity, we briefly collect some statements. Our aim is to emphasize the similarities with the properties presented in the earlier sections.

For example, much like Brody hyperbolicity, Mordellicity, and grouplessness, the property of being of general type descends along finite étale morphisms. That is, if $X \to Y$ is a finite étale morphism of proper schemes over k, then X is of general type if and only if Y is of general type. Moreover, a simple computation of the degree of the canonical bundle of a curve implies that, if X is a smooth projective connected curve over k, then X is of general type if and only if $\mathrm{genus}(X) \geq 2$.

Kawamata and Ueno classified which closed subvarieties of an abelian variety are of general type. To state their result, for A an abelian variety over k and X a closed subvariety of A, recall that the special locus $\mathrm{Sp}(X)$ of X is the union of translates of positive-dimensional abelian subvarieties of A contained in X. Note that Bloch–Ochiai–Kawamata's theorem (Theorem 2.5) can be stated as saying that a closed subvariety X of an abelian variety A over $\mathbb{C}$ is Brody hyperbolic if and only if $\mathrm{Sp}(X)$ is empty. Similarly, Faltings's theorem (Theorem 3.8) can be stated as saying that a closed subvariety of an abelian variety A over k is Mordellic if and only if $\mathrm{Sp}(X)$ is empty. The latter is also equivalent to saying that X is groupless over k by Lemma 4.8. The theorem of Kawamata–Ueno now reads as follows.

Theorem 5.1 (Kawamata–Ueno). *Let A be an abelian variety and let X be a closed integral subvariety of A. Then $\mathrm{Sp}(X)$ is a closed subset of X, and X is of general type if and only if $\mathrm{Sp}(X) \neq X$.*

Note that being of general type and being groupless are not equivalent. This is not a surprise, as the notion of general type is a birational invariant, whereas the blow-up of a smooth groupless surface along a point is no longer groupless. The conjectural relation between varieties of general type and the three notions (Brody hyperbolicity, Mordellicity, and grouplessness) introduced above is as follows.

Conjecture 5.2 (Weak Lang–Vojta, III). *Let X be an integral projective variety over k. Then the following are equivalent.*

(1) *The projective variety X is Mordellic over k.*
(2) *The variety X is strongly-Brody hyperbolic over k.*
(3) *Every integral subvariety of X is of general type.*
(4) *The variety X is groupless over k.*

Note that the notion of general type is a birational invariant, but hyperbolicity is not. What should (conjecturally) correspond to being of general type? The highly optimistic conjectural answer is that being of general type should correspond to being "pseudo"-Brody hyperbolic, "pseudo"-Mordellic, and "pseudo"-groupless. The definitions of these notions are essentially the same as given above, the only difference being that one has to allow for an "exceptional locus". In the following sections we will make this more precise.

6   Pseudo-Grouplessness

Let k be an algebraically closed field of characteristic zero. Roughly speaking, a projective variety X over k is groupless if it admits no non-trivial morphisms from a connected algebraic group. Conjecturally, a projective variety X over k is groupless if and only if every subvariety of X is of general type. To see what should correspond to being of general type, we will require the more general notion of pseudo-grouplessness.

Definition 6.1. Let X be a variety over k and let $\Delta \subset X$ be a closed subset. We say that X is *groupless modulo Δ* (*over k*) if, for every finite type connected group scheme G over k and every dense open subscheme $U \subset G$ with $\mathrm{codim}(G \setminus U) \geq 2$, every non-constant morphism $U \to X$ factors over Δ.

Hyperbolicity modulo a subset was first introduced by Kiernan–Kobayashi [58], and is thoroughly studied in Kobayashi's book [59]. As we will see below, it is quite natural to extend the study of hyperbolic varieties to the study of varieties which are hyperbolic modulo a subset.

For proper schemes, the notion of "groupless modulo the empty set" coincides with the notion of grouplessness introduced before (and studied in [49, 50, 55]). For the reader's convenience, we include a detailed proof of this.

Lemma 6.2. *Let X be a proper scheme over k. Then the following are equivalent.*

(1) *The scheme X is groupless modulo the empty subscheme $\emptyset$ over k.*
(2) *The scheme X is groupless.*
(3) *For every finite type connected group scheme G over k and every dense open subscheme $V \subset G$, every morphism $V \to X$ is constant.*

Proof. It is clear that (1) implies (2). To show that (2) implies (3), let G be a finite type connected group scheme over k, let $V \subset G$ be a dense open subscheme, and let $f : V \to X$ be a morphism of schemes over k. Then, as X is proper over k, there is an open subscheme $U \subset G$ containing V with $\mathrm{codim}(G \setminus U) \geq 2$ such that the morphism $f : V \to X$ extends to a morphism $f' : U \to X$. Since X is groupless and proper, it does not contain any rational curves. Therefore, as the variety underlying G is smooth over k [81, Tag 047N], it follows from [50, Lemma 3.5] (see also [27, Corollary 1.44]) that the morphism $f' : U \to X$ extends (uniquely) to a morphism $f'' : G \to X$. Since X is groupless, the morphism f'' is constant. This implies that f is constant. Finally, it is clear (from the definitions) that (3) implies (1). $\square$

Definition 6.3. A variety X is *pseudo-groupless* (*over* k) if there is a proper closed subset $\Delta \subsetneq X$ such that X is groupless modulo Δ.

The word "pseudo" in this definition refers to the fact that the non-hyperbolicity of the variety is concentrated in a proper closed subset. Note that a variety X is pseudo-groupless if and only if every irreducible component of X is pseudo-groupless.

Example 6.4. Let C be smooth projective connected curve of genus at least two and let X be the blow-up of $C \times C$ in a point. Then X is not groupless. However, its "non-grouplessness" is contained in the exceptional locus Δ of the blow-up $X \to C \times C$. Thus, as X is groupless modulo Δ, it follows that X is pseudo-groupless.

Let us briefly say that an open subset U of an integral variety V is *big* if $\mathrm{codim}(V \setminus U)$ is at least two. Now, the reader might wonder why we test pseudo-grouplessness on maps whose domain is a big open subset of some algebraic group. The example to keep in mind here is the blow-up of a simple abelian surface in its origin. In fact, as we test pseudo-grouplessness on big open subsets of abelian varieties (and not merely maps from abelian varieties), such blow-ups are *not* pseudo-groupless. Also, roughly speaking, one should consider big open subsets of abelian varieties as far as possible from being hyperbolic, in any sense of the word "hyperbolic". For example, much like

how abelian varieties admit a dense entire curve (Remark 2.4), a big open subset of an abelian variety admits a dense entire curve. This is proven using Sard's theorem in [86]. Thus, big open subsets of abelian varieties are also as far as possible from being Brody hyperbolic.

We now show that the statement of Lemma 4.6 also holds in the "pseudo" setting, i.e., we show that pseudo-grouplessness descends along finite étale morphisms. As we have mentioned before, this descent property also holds for general type varieties.

Lemma 6.5. *Let $f : X \to Y$ be a finite étale morphism of varieties over k. Then X is pseudo-groupless over k if and only if Y is pseudo-groupless over k.*

Proof. We adapt the arguments in the proof of [55, Proposition 2.13]. First, if Y is groupless modulo a proper closed subset $\Delta_Y \subset Y$, then clearly X is groupless modulo the proper closed subset $f^{-1}(\Delta_Y)$. Now, assume that X is groupless modulo a proper closed subset $\Delta_X \subsetneq X$. Let G be a finite type connected (smooth quasi-projective) group scheme over k, let $U \subset G$ be a dense open subscheme with $\mathrm{codim}(G \setminus U) \geq 2$ and let $\phi : U \to Y$ be a morphism which does not factor over $f(\Delta_X)$. The pull-back of $G \to Y$ along the finite étale morphism $f : X \to Y$ induces a finite étale morphism $V := U \times_Y X \to U$. Since U is smooth over k, by purity of the branch locus [39, Théorème X.3.1], the finite étale morphism $V \to U$ extends (uniquely) to a finite étale morphism $G' \to G$. Note that every connected component G'' of G' has the structure of a finite type connected group scheme over k (and with this structure the morphism $G'' \to G$ is a homomorphism). Now, since smooth morphisms are codimension-preserving, we see that $\mathrm{codim}(G'' \setminus V) \geq 2$. As the morphism $V \to X$ does not factor over $f^{-1}(f(\Delta_X))$, it does not factor over Δ_X, and is thus constant (as X is groupless modulo Δ_X). This implies that the morphism $U \to Y$ is constant, as required. □

Remark 6.6 (Birational Invariance). Let X and Y be proper schemes over k. Assume that X is birational to Y. Then X is pseudo-groupless over k if and only if Y is pseudo-groupless over k. This is proven in [56]. Thus, as pseudo-grouplessness is a birational invariant among proper varieties, this notion is more natural to study from a birational perspective than grouplessness.

Remark 6.7. Contrary to a hyperbolic proper variety, a proper pseudo-groupless variety could have rational curves. For example, the blow-up of the product of two smooth curves of genus two in a point (as in Example 6.4) contains precisely one rational curve. However, a pseudo-groupless proper variety is not covered by rational curves, i.e., it is non-uniruled, as all rational curves are contained in a proper closed subset (by definition).

Remark 6.8. Let X be a proper scheme over k and let $\Delta \subset X$ be a closed subset. It follows from the valuative criterion of properness that X is groupless

modulo Δ if and only if, for every finite type connected group scheme G over k and every dense open subscheme $U \subset G$, any non-constant morphism $U \to X$ factors over Δ.

Recall that Lemma 4.4 says that the grouplessness of a proper variety entails that there are no non-constant morphisms from *any* connected algebraic group. One of the main results of [56] is the analogue of Lemma 4.4 for pseudo-groupless varieties. The proof of this result (see Theorem 6.9 below) relies on the structure theory of algebraic groups.

Theorem 6.9. *If X is a proper scheme X over k and Δ is a closed subset of X, then X is groupless modulo Δ over k if and only if, for every abelian variety A over k and every open subscheme $U \subset A$ with $\mathrm{codim}(A \setminus U) \geq 2$, every non-constant morphism of varieties $U \to X$ factors over Δ.*

Theorem 6.9 says that the pseudo-grouplessness of a proper variety can be tested on morphisms from big open subsets of abelian varieties (or on rational maps from abelian varieties). A similar, but different, statement holds for affine varieties. Indeed, if X is an affine variety over k, then X is groupless modulo $\Delta \subset X$ if and only if every non-constant morphism $\mathbb{G}_{m,k} \to X$ factors over Δ.

Lang conjectured that a projective variety is pseudo-groupless if and only if it is of general type. Note that, by the birational invariance of these two notions, this conjecture can be reduced to the case of smooth projective varieties by Hironaka's resolution of singularities.

Conjecture 6.10 (Strong Lang–Vojta, I). *Let X be an integral projective variety over k. Then X is pseudo-groupless over k if and only if X is of general type over k.*

Note that this conjecture predicts more than the equivalence of (3) and (4) in Conjecture 5.2. Also, even though it is stated for projective varieties, one could as well formulate the conjecture for proper varieties (or even proper algebraic spaces). The resulting "more general" conjecture actually follows from the above conjecture.

Example 6.11. By Kawamata–Ueno's theorem (Theorem 5.1) and Lemma 4.8, the Strong Lang–Vojta conjecture holds for closed subvarieties of abelian varieties.

Remark 6.12. If X is a proper pseudo-groupless surface, then X is of general type (see [56] for a proof). For higher-dimensional varieties, Conjecture 6.10 predicts a similar statement, but this is not even known for threefolds. However, assuming the Abundance Conjecture and certain conjectures on Calabi–Yau varieties, one can show that every proper pseudo-groupless variety is of general type (i.e., (1) $\implies$ (2) in Conjecture 6.10). Regarding the implication (2) $\implies$ (1), not much is known beyond the one-dimensional case. For example, if X is a proper surface of general type, then

Conjecture 6.10 implies that there should be a proper closed subset $\Delta \subset X$ such that every rational curve $C \subset X$ is contained in Δ. Such statements are known to hold for certain surfaces of general type by the work of Bogomolov and McQuillan; see [30, 71].

7 Pseudo-Mordellicity and Pseudo-Arithmetic Hyperbolicity

In the previous section, we introduced pseudo-grouplessness and stated Lang–Vojta's conjecture that a projective variety is of general type if and only if it is pseudo-groupless. In this section, we explain what the "pseudo" analogue is of the notion of Mordellicity, and explain Lang–Vojta's conjecture that a projective variety is of general type if and only if it is pseudo-Mordellic.

7.1 Pseudo-Arithmetic Hyperbolicity

As we have said before, Lang coined the term "Mordellic". We will now introduce the related (and a priori different) notion of arithmetic hyperbolicity (as defined in [49, 52, 53]); see also [83, §2], and [10, 11]. In Section 3 we ignored that the extension of the notion of Mordellicity over $\overline{\mathbb{Q}}$ to arbitrary algebraically closed fields can actually be done in two a priori different ways. We discuss both notions now and give them *different* names. We refer the reader to Section 3 for our conventions regarding models of varieties, and we continue to let k denote an algebraically closed field of characteristic zero.

Definition 7.1. Let X be a variety over k and let Δ be a closed subset of X. We say that X is *arithmetically hyperbolic modulo Δ over k* if, for every $\mathbb{Z}$-finitely generated subring A and every model $\mathcal{X}$ for X over A, we have that every positive-dimensional irreducible component of the Zariski closure of $\mathcal{X}(A)$ in X is contained in Δ.

Definition 7.2. A variety X over k is *pseudo-arithmetically hyperbolic over k* if there is a proper closed subset $\Delta \subset X$ such that X is arithmetically hyperbolic modulo Δ over k.

Remark 7.3. A variety X over k is arithmetically hyperbolic over k (as defined in [49] and [53, §4]) if and only if X is arithmetically hyperbolic over k modulo the empty subscheme.

Lemma 7.4 (Independence of Model). *Let X be a variety over k and let Δ be a closed subset of k. Then the following are equivalent.*

(1) The finite type scheme X over k is arithmetically hyperbolic modulo Δ.

(2) *There is a $\mathbb{Z}$-finitely generated subring $A \subset k$, there is a model $\mathcal{X}$ for X over A, and there is a model $\mathcal{D} \subset \mathcal{X}$ for $\Delta \subset X$ over A such that, for every $\mathbb{Z}$-finitely generated subring $B \subset k$ containing A, the set*

$$\mathcal{X}(B) \setminus \mathcal{D}(B)$$

is finite.

Proof. This follows from standard spreading out arguments. These type of arguments are used in [53] to prove more general statements in which the objects are algebraic stacks.   ☐

Remark 7.5. We unravel what the notion of arithmetic hyperbolicity modulo Δ entails for affine varieties. To do so, let X be an affine variety over k, and let Δ be a proper closed subset of X. Choose the following data.

- integers $n, \delta, m \geq 1$;
- polynomials $f_1, \ldots, f_n \in k[x_1, \ldots, x_m]$;
- polynomials $d_1, \ldots, d_\delta \in k[x_1, \ldots, x_m]$;
- an isomorphism

$$X \cong \mathrm{Spec}(k[x_1, \ldots, x_m]/(f_1, \ldots, f_n));$$

- an isomorphism

$$\Delta \cong \mathrm{Spec}(k[x_1, \ldots, x_m]/(d_1, \ldots, d_\delta)).$$

Let A_0 be the $\mathbb{Z}$-finitely generated subring of k generated by the (finitely many) coefficients of the polynomials $f_1, \ldots, f_n, d_1, \ldots, d_\delta$. Now, the following statements are equivalent.

(1) The variety X is arithmetically hyperbolic modulo Δ over k.
(2) For every $\mathbb{Z}$-finitely generated subring $A \subset k$ containing A_0, the set

$$\{a \in A^m \mid f_1(a) = \ldots = f_n(a) = 0\} \setminus \{a \in A^m \mid d_1(a) = \ldots = d_\delta(a) = 0\}$$

is finite.

Thus, roughly speaking, one could say that an algebraic variety over k is arithmetically hyperbolic modulo Δ over k if "X minus Δ" has only finitely many A-valued points, for any choice of finitely generated subring $A \subset k$.

7.2   *Pseudo-Mordellicity*

The reader might have noticed a possibly confusing change in terminology. Why do we not refer to the above notion as being "Mordellic modulo Δ"? The precise reason brings us to a subtle point in the study of integral points

valued in higher-dimensional rings (contrary to those valued in $\mathcal{O}_{K,S}$ with S a finite set of finite places of a number field K). To explain this subtle point, let us first define what it means to be pseudo-Mordellic. For this definition, we will require the notion of "near-integral" point (Definition 3.11).

Definition 7.6. Let X be a variety over k and let Δ be a closed subset of X. We say that X is *Mordellic modulo Δ over k* if, for every $\mathbb{Z}$-finitely generated subring A and every model $\mathcal{X}$ for X over A, we have that every positive-dimensional irreducible component of the Zariski closure of $\mathcal{X}(A)^{(1)}$ in X is contained in Δ, where $\mathcal{X}(A)^{(1)}$ is defined in Definition 3.11.

Remark 7.7. Let X be a proper scheme over k and let Δ be a closed subset of X. Then, by the valuative criterion of properness, the proper scheme X is Mordellic modulo Δ if, for every finitely generated subfield $K \subset k$ and every proper model $\mathcal{X}$ over K, the set $\mathcal{X}(K) \setminus \Delta$ is finite.

Definition 7.8. A variety X over k is *pseudo-Mordellic over k* if there is a proper closed subset $\Delta \subset X$ such that X is Mordellic modulo Δ over k.

Note that X is Mordellic over k (as defined in Section 3) if and only if X is Mordellic modulo the empty subset. It is also clear from the definitions that, if X is Mordellic modulo Δ over k, then X is arithmetically hyperbolic modulo Δ over k. In particular, a pseudo-Mordellic variety is pseudo-arithmetically hyperbolic and a Mordellic variety is arithmetically hyperbolic. Indeed, roughly speaking, to say that a variety is arithmetically hyperbolic is to say that any set of integral points on it is finite, and to say that a variety is Mordellic is to say that any set of "near-integral" points on it is finite. The latter sets are a priori bigger. However, there is no difference between these two sets when $k = \overline{\mathbb{Q}}$. That is, a variety X over $\overline{\mathbb{Q}}$ is arithmetically hyperbolic modulo Δ if and only if it is Mordellic modulo Δ over $\overline{\mathbb{Q}}$.

Following the exposition in the previous sections, let us prove the fact that pseudo-arithmetic hyperbolicity (resp. pseudo-Mordellicity) descends along finite étale morphisms of varieties.

Theorem 7.9 (Chevalley–Weil). *Let $f : X \to Y$ be a finite étale surjective morphism of varieties over k. Let $\Delta \subset X$ be a closed subset. If X is Mordellic modulo Δ over k (resp. arithmetically hyperbolic modulo Δ over k), then Y is Mordellic modulo $f(\Delta)$ over k (resp. arithmetically hyperbolic modulo $f(\Delta)$ over k).*

Proof. We assume that X is Mordellic modulo Δ, and show that Y is Mordellic modulo $f(\Delta)$. (The statement concerning arithmetic hyperbolicity is proven similarly.)

Let $A \subset k$ be a regular $\mathbb{Z}$-finitely generated subring, let $\mathcal{X}$ be a model for X over A, let $\mathcal{Y}$ be a model for Y over A, and let $F : \mathcal{X} \to \mathcal{Y}$ be a finite étale surjective morphism such that $F_k = f$. Assume for a contradiction that Y is not Mordellic modulo $f(\Delta)$. Then, replacing A by a larger regular $\mathbb{Z}$-finitely generated subring of k if necessary, for $i = 1, 2, \ldots$, we may choose pairwise

distinct elements a_i of $\mathcal{Y}(A)^{(1)}$ whose closure in Y is an irreducible positive-dimensional subvariety $R \subset Y$ such that $R \not\subset f(\Delta)$. For every $i = 1, 2, \ldots$, choose a dense open subscheme U_i of Spec A whose complement in Spec A has codimension at least two and such that a_i defines a morphism $a_i : U_i \to \mathcal{X}$. Consider $V_i := U_i \times_{\mathcal{Y}, F} \mathcal{X} \to \mathcal{X}$, and note that $V_i \to U_i$ is finite étale. By Zariski–Nagata purity of the branch locus [39, Théorème X.3.1], the morphism $V_i \to U_i$ extends to a finite étale morphism Spec $B_i \to A$. By Hermite's finiteness theorem, as the degree of B_i over A is bounded by $\deg(f)$, replacing a_i by an infinite subset if necessary, we may and do assume that $B := B_1 \cong B_2 \cong B_3 \cong \ldots$. Now, the $b_i : V_i \to \mathcal{X}$ define elements in $\mathcal{X}(B)^{(1)}$. Let S be their closure in X. Note that $R \subset S$. In particular, $S \not\subset \Delta$. This contradicts the fact that X is Mordellic modulo Δ. Thus, we conclude that Y is Mordellic modulo $f(\Delta)$.   □

Corollary 7.10 (Pseudo-Chevalley–Weil). *Let $f : X \to Y$ be a finite étale surjective morphism of finite type separated schemes over k. Then X is pseudo-Mordellic over k if and only if Y is pseudo-Mordellic over k.*

Proof. Since $f : X \to Y$ has finite fibres, the fibres of f are Mordellic over k. Therefore, if Y is pseudo-Mordellic over k, it easily follows that X is pseudo-Mordellic over k. Conversely, if X is pseudo-Mordellic over k, then it follows from Theorem 7.9 that Y is pseudo-Mordellic over k.   □

Corollary 7.11 (Pseudo-Chevalley–Weil, II). *Let $f : X \to Y$ be a finite étale surjective morphism of finite type separated schemes over k. Then X is pseudo-arithmetically hyperbolic over k if and only if Y is pseudo-arithmetically hyperbolic over k.*

Proof. Similar to the proof of Corollary 7.10.   □

Remark 7.12 (Birational Invariance). The birational invariance of the notion of pseudo-Mordellicity is essentially built into the definition. Indeed, the infinitude of the set of near-integral points is preserved under proper birational modifications. More precisely, let X and Y be proper integral varieties over k which are birational. Then X is pseudo-Mordellic over k if and only if Y is pseudo-Mordellic over k.

It is not clear to us whether the notion of pseudo-arithmetic hyperbolicity over k is a birational invariant for proper varieties over k, unless $k = \overline{\mathbb{Q}}$. Similarly, it is not so clear to us whether pseudo-arithmetically hyperbolic proper varieties are pseudo-groupless. On the other hand, this is not so hard to prove for pseudo-Mordellic varieties.

Theorem 7.13. *If X is a pseudo-Mordellic proper variety over k, then X is pseudo-groupless over k.*

Proof. The fact that an arithmetically hyperbolic variety is groupless is proven in [49, §3] using the potential density of rational points on an abelian

variety over a field K of characteristic zero (Remark 3.7). The statement of the theorem is proven in [56] using similar arguments. □

Remark 7.14. Let X be a proper surface over k. If X is pseudo-Mordellic over k, then X is of general type. To prove this, note that X is pseudo-groupless (Theorem 7.13), so that the claim follows from the fact that pseudo-groupless proper surfaces are of general type; see Remark 6.12.

Recall that a closed subvariety X of an abelian variety A is groupless modulo its special locus $\mathrm{Sp}(X)$, where $\mathrm{Sp}(X)$ is the union of translates of non-zero abelian subvarieties of A contained in X. (We are freely using here Kawamata–Ueno's theorem that $\mathrm{Sp}(X)$ is a closed subset of X.) This was proven in Lemma 4.8. In [34] Faltings proved the arithmetic analogue of this statement.

Theorem 7.15 (Faltings). *Let A be an abelian variety over k, and let $X \subset A$ be a closed subvariety. Then X is Mordellic modulo $\mathrm{Sp}(X)$.*

Lang and Vojta conjectured that a projective variety over $\overline{\mathbb{Q}}$ is pseudo-Mordellic if and only if it is of general type. We propose extending this to arbitrary algebraically closed fields of characteristic zero. As we also expect the notions of pseudo-arithmetic hyperbolicity and pseudo-Mordellicity to coincide, we include this in our version of the Lang–Vojta conjecture.

Conjecture 7.16 (Strong Lang–Vojta, II). *Let X be an integral projective variety over k. Then the following statements are equivalent.*

(1) The variety X is pseudo-Mordellic over k.
(2) The variety X is pseudo-arithmetically hyperbolic over k.
(3) The variety X is pseudo-groupless over k.
(4) The projective variety X is of general type over k.

This is a good time to collect examples of arithmetically hyperbolic varieties.

Example 7.17. It follows from Faltings's theorem [34] that a normal projective connected pseudo-groupless surface X over k with $\mathrm{h}^1(X, \mathcal{O}_X) > 2$ is pseudo-Mordellic. Let us prove this claim. To do so, let $\Delta \subset X$ be a proper closed subset such that X is groupless modulo Δ. Moreover, let A be the Albanese variety of X, let $p : X \to A$ be the canonical map (after choosing some basepoint in $X(k)$), and let Y be the image of X in A. Note that $\dim Y \geq 1$. If $\dim Y = 1$, then the condition on the dimension of A implies that Y is not an elliptic curve. In this case, since $\dim X = 2$ and $\dim Y = 1$, the claim follows from Faltings's (earlier) finiteness theorem for hyperbolic curves. However, if $\dim Y = 2$, we have to appeal to Faltings's Big Theorem. Indeed, in this case, the morphism $X \to Y$ is generically finite. Let $X \to X' \to Y$ be the Stein factorization of the morphism $X \to Y$, where $X' \to Y$ is a finite morphism with X' normal. Since X and X' are birational, it suffices to show that X' is pseudo-Mordellic (by the birational

invariance of pseudo-Mordellicity and pseudo-grouplessness). Thus, we may and do assume that $X = X'$, so that $X \to A$ is finite. If the rational points on X are dense, then they are also dense in Y, so that Y is an abelian subvariety of A, contradicting our assumption that $\mathrm{h}^1(X, \mathcal{O}_X) = \dim A > 2$. Thus, the rational points on X are not dense. In particular, every irreducible component of the closure of a set of rational points on X is a curve of genus 1 (as X does not admit any curves of genus zero). Since X is pseudo-groupless, these components are contained in Δ.

Example 7.18. Let X be a smooth projective connected curve over k, let $n \geq 1$ be an integer, and let Δ be a proper closed subset of Sym_X^n. It follows from Faltings's theorem that Sym_X^n is groupless modulo Δ over k if and only if Sym_X^n is arithmetically hyperbolic modulo Δ over k.

Example 7.19 (Moriwaki). Let X be a smooth projective variety over k such that Ω_X^1 is ample and globally generated. Then X is Mordellic by a theorem of Moriwaki [73]; see [7] for the analogous finiteness result in the logarithmic case.

Example 7.20. For every $\mathbb{Z}$-finitely generated normal integral domain A of characteristic zero, the set of A-isomorphism classes of smooth sextic surfaces in $\mathbb{P}_A^3$ is finite; see [54]. This finiteness statement can be reformulated as saying that the moduli stack of smooth sextic surfaces is Mordellic.

Example 7.21. Let X be a smooth proper hyperkaehler variety over k with Picard number at least three. Then X is not arithmetically hyperbolic; see [49].

7.3   Intermezzo: Arithmetic Hyperbolicity and Mordellicity

Let k be an algebraically closed field of characteristic zero. In this section, we show that the (a priori) difference between arithmetic hyperbolicity (modulo some subset) and Mordellicity is quite subtle, as this difference disappears in many well-studied cases.

The following notion of purity for models over $\mathbb{Z}$-finitely generated rings was first considered in [15] precisely to study the a priori difference between arithmetic hyperbolicity and Mordellicity.

Definition 7.22 (Pure Model). Let X be a variety over k and let $A \subset k$ be a subring. A model $\mathcal{X}$ for X over A is *pure over A* (or: *satisfies the extension property over A*) if, for every smooth finite type separated integral scheme T over A, every dense open subscheme $U \subset T$ with $T \setminus U$ of codimension at least two in T, and every A-morphism $f : U \to \mathcal{X}$, there is a (unique) morphism

$\overline{f} : T \to \mathcal{X}$ extending the morphism f. (The uniqueness of the extension $\overline{f}$ follows from our convention that a model for X over A is separated.)

Remark 7.23. Let X be a variety over k, and let $A \subset k$ be a subring. Let $\mathcal{X}$ be a pure model for X over A, and let $B \subset k$ be a subring containing A such that $\operatorname{Spec} B \to \operatorname{Spec} A$ is smooth (hence finite type). Then $\mathcal{X}_B$ is pure over B.

Definition 7.24. A variety X over k has an *arithmetically pure model* if there is a $\mathbb{Z}$-finitely generated subring $A \subset k$ and a pure model $\mathcal{X}$ for X over A.

Remark 7.25. Let X be a proper variety over k which has an arithmetically pure model. Then X has no rational curves. To prove this, assume that $\mathbb{P}^1_k \to X$ is a non-constant (hence finite) morphism, i.e., the proper variety X has a rational curve over k. Then, if we let 0 denote the point $(0 : 0 : 1)$ in $\mathbb{P}^2_k$, the composed morphism $\mathbb{P}^2_k \setminus \{0\} \to \mathbb{P}^1_k \to X$ does not extend to a morphism from $\mathbb{P}^2_k$ to X. Now, choose a $\mathbb{Z}$-finitely generated subring $A \subset k$ and a model $\mathcal{X}$ over A such that the morphism $\mathbb{P}^1_k \to X$ descends to a morphism $\mathbb{P}^1_A \to \mathcal{X}$ of A-schemes. Define $U = \mathbb{P}^2_A \setminus \{0\}$ and $T = \mathbb{P}^2_A$, where we let $\{0\}$ denote the image of the section of $\mathbb{P}^2_A \to \operatorname{Spec} A$ induced by 0 in $\mathbb{P}^2_k$. Since the morphism $U_k \to \mathcal{X}_k$ does not extend to a morphism $T_k \to X_k$, we see that the morphism $U \to \mathcal{X}$ does not extend to a morphism $T \to \mathcal{X}$, so that $\mathcal{X}$ is not pure. This shows that a proper variety over k with a rational curve has no arithmetically pure model.

Remark 7.26. Let X be a proper variety over k. A pure model for X over a $\mathbb{Z}$-finitely generated subring A of k might have rational curves in every special fibre (of positive characteristic). Examples of such varieties can be constructed as complete subvarieties of the moduli space of principally polarized abelian varieties.

Remark 7.27. Let X be a smooth projective variety over k. If $\Omega^1_{X/k}$ is ample, then X has an arithmetically pure model. Indeed, choose a $\mathbb{Z}$-finitely generated subring $A \subset k$ with A smooth over $\mathbb{Z}$ and a smooth projective model $\mathcal{X}$ for X over A such that $\Omega_{\mathcal{X}/A}$ is ample. Then, the geometric fibres of $\mathcal{X} \to \operatorname{Spec} A$ do not contain any rational curves, so that [37, Proposition 6.2] implies that $\mathcal{X}$ is a pure model for X over A.

Remark 7.28. Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero, and let X be a variety over k. Then X has an arithmetically pure model if and only if X_L has an arithmetically pure model.

Theorem 7.29. *Let X be a variety over k which has an arithmetically pure model. Let $\Delta \subset X$ be a closed subset. Then X is Mordellic modulo Δ over k if and only if X is arithmetically hyperbolic modulo Δ over k.*

Proof. We follow the proof of [15, Theorem 8.10]. Suppose that X is arithmetically hyperbolic modulo Δ over k. Let $A \subset k$ be a $\mathbb{Z}$-finitely generated subring and let $\mathcal{X}$ be a pure model for X over A. It suffices to show that, for every $\mathbb{Z}$-finitely generated subring $B \subset k$ containing A, the set $\mathcal{X}(B)^{(1)} \setminus \Delta$ is finite. To do so, we may and do assume that $\operatorname{Spec} B \to \operatorname{Spec} A$ is smooth in which case it follows from the definition of a pure model that $\mathcal{X}(B)^{(1)} = \mathcal{X}(B)$. We conclude that

$$\mathcal{X}(B)^{(1)} \setminus \Delta = \mathcal{X}(B) \setminus \Delta$$

is finite. This shows that X is Mordellic modulo Δ over k. $\qquad\square$

Lemma 7.30 (Affine Varieties). *Let X be an affine variety over k. Then X has an arithmetically pure model.*

Proof. Affine varieties have an arithmetically pure model by Hartog's Lemma. $\qquad\square$

Lemma 7.31. *Let X be a variety over k which admits a finite morphism to some semi-abelian variety over k. Then X has an arithmetically pure model.*

Proof. Let G be a semi-abelian variety and let $X \to G$ be a finite morphism. It follows from Hartog's Lemma that X has an arithmetically pure model if and only if G has an arithmetically pure model. Choose a $\mathbb{Z}$-finitely generated subring and a model $\mathcal{G}$ for G over A such that $\mathcal{G} \to \operatorname{Spec} A$ is a semi-abelian scheme. Then, this model $\mathcal{G}$ has the desired extension property by [72, Lemma A.2], so that G (hence X) has an arithmetically pure model. $\qquad\square$

Remark 7.32. Let X be a projective integral groupless surface over k which admits a non-constant map to some abelian variety. Then X has an arithmetically pure model by [15, Lemma 8.11].

Corollary 7.33. *Let X be an integral variety over k, and let $\Delta \subset X$ be a closed subset. Assume that one of the following statements holds.*

(1) *The variety X is affine over k.*
(2) *There is a finite morphism $X \to G$ with G a semi-abelian variety over k.*
(3) *We have that X is a groupless surface which admits a non-constant morphism $X \to A$ with A an abelian variety over k.*

Then X is arithmetically hyperbolic modulo Δ over k if and only if X is Mordellic modulo Δ over k.

Proof. Assume (1). Then the statement follows from Lemma 7.30 and Theorem 7.29. Similarly, if (2) holds, then the statement follows from Lemma 7.31 and Theorem 7.29. Finally, assuming (3), the statement follows from Remark 7.32 and Theorem 7.29. $\qquad\square$

Remark 7.34. Let $g \geq 1$ and $N \geq 3$ be integers. Now, if X is the fine moduli space of g-dimensional principally polarized abelian schemes over k with level

$\overline{\mathbb{Q}}$ structure, then X has an arithmetically pure model. As is explained in [68], this is a consequence of Grothendieck's theorem on homomorphisms of abelian schemes [40]. The existence of such a model is used by Martin-Deschamps to deduce the Mordellicity of X_k over k from the Mordellicity of X over $\overline{\mathbb{Q}}$ (cf. Theorem 3.14).

8 Pseudo-Brody Hyperbolicity

The notion of pseudo-hyperbolicity appeared first in the work of Kiernan and Kobayashi [58] and afterwards in Lang [62]; see also [59]. We recall some of the definitions.

Definition 8.1. Let X be a variety over $\mathbb{C}$ and let Δ be a closed subset of X. We say that X is *Brody hyperbolic modulo* Δ if every holomorphic non-constant map $\mathbb{C} \to X^{\mathrm{an}}$ factors over Δ.

Definition 8.2. A variety X over $\mathbb{C}$ is *pseudo-Brody hyperbolic* if there is a proper closed subset $\Delta \subsetneq X$ such that X is Brody hyperbolic modulo Δ.

Green–Griffiths and Lang conjectured that a projective integral variety of general type is pseudo-Brody hyperbolic. The conjecture that a projective integral variety is of general type if and only if it is pseudo-Brody hyperbolic is commonly referred to as the Green–Griffiths–Lang conjecture.

Note that the notion of pseudo-Brody hyperbolicity is a birational invariant. More precisely, if X and Y are proper integral varieties over $\mathbb{C}$ which are birational, then X is pseudo-Brody hyperbolic if and only if Y is pseudo-Brody hyperbolic. Furthermore, just like the notions of pseudo-Mordellicity and pseudo-grouplessness, the notion of pseudo-Brody hyperbolicity descends along finite étale morphisms. That is, if $X \to Y$ is finite étale, then X is pseudo-Brody hyperbolic if and only if Y is pseudo-Brody hyperbolic. Also, it is not hard to show that, if a variety X is Brody hyperbolic modulo Δ, then X is groupless modulo Δ.

Note that a variety X is Brody hyperbolic (as defined in Section 2) if and only if X is Brody hyperbolic modulo the empty set. Bloch–Ochiai–Kawamata's theorem classifies Brody hyperbolic closed subvarieties of abelian varieties. In fact, their result is a consequence of the following more general statement (also proven in [57]).

Theorem 8.3 (Bloch–Ochiai–Kawamata). *Let X be a closed subvariety of an abelian variety A. Let* $\mathrm{Sp}(X)$ *be the special locus of X. Then* $\mathrm{Sp}(X)$ *is a closed subset of X and X is Brody hyperbolic modulo* $\mathrm{Sp}(X)$.

We now introduce the pseudo-analogue of Kobayashi hyperbolicity for algebraic varieties. Of course, these definitions make sense for complex-analytic spaces.

Definition 8.4. Let X be a variety over $\mathbb{C}$ and let Δ be a closed subset of X. We say that X is *Kobayashi hyperbolic modulo* Δ if, for every x and y in $X^{\mathrm{an}} \setminus \Delta^{\mathrm{an}}$ with $x \neq y$, the Kobayashi pseudo-distance $d_{X^{\mathrm{an}}}(p,q)$ is positive.

Definition 8.5. A variety X over $\mathbb{C}$ is *pseudo-Kobayashi hyperbolic* if there is a proper closed subset $\Delta \subsetneq X$ such that X is Kobayashi hyperbolic modulo Δ.

It is clear from the definitions and the fact that the Kobayashi pseudo-metric vanishes everywhere on $\mathbb{C}$, that a variety X which is Kobayashi hyperbolic modulo a closed subset $\Delta \subset X$ is Brody hyperbolic modulo Δ. Nonetheless, the notion of pseudo-Kobayashi hyperbolicity remains quite mysterious at the moment. Indeed, we do not know whether a pseudo-Brody hyperbolic projective variety X over $\mathbb{C}$ is pseudo-Kobayashi hyperbolic.

One can show that the notion of pseudo-Kobayashi hyperbolicity is a birational invariant. That is, if X and Y are proper integral varieties over $\mathbb{C}$ which are birational, then X is pseudo-Kobayashi hyperbolic if and only if Y is pseudo-Kobayashi hyperbolic; see [59]. Moreover, just like the notions of pseudo-Mordellicity and pseudo-grouplessness, pseudo-Kobayashi hyperbolicity descends along finite étale morphisms.

Yamanoi proved the pseudo-Kobayashi analogue of Bloch–Ochiai–Kawamata's theorem for closed subvarieties of abelian varieties; see [88, Theorem 1.2].

Theorem 8.6 (Yamanoi). *Let X be a closed subvariety of an abelian variety A. Let $\mathrm{Sp}(X)$ be the special locus of X. Then $\mathrm{Sp}(X)$ is a closed subset of X and X is Kobayashi hyperbolic modulo $\mathrm{Sp}(X)$.*

The Lang–Vojta conjecture and the Green–Griffiths conjecture predict that the above notions of hyperbolicity are equivalent. To state this conjecture, we will need one more definition. (Recall that k denotes an algebraically closed field of characteristic zero.)

Definition 8.7. A variety X over k is *strongly-pseudo-Brody hyperbolic* (resp. *strongly-pseudo-Kobayashi hyperbolic*) if, for every subfield $k_0 \subset k$, every model $\mathcal{X}$ for X over k_0, and every embedding $k_0 \to \mathbb{C}$, the variety $\mathcal{X}_{0,\mathbb{C}}$ is pseudo-Brody hyperbolic (resp. pseudo-Kobayashi hyperbolic).

Conjecture 8.8 (Strong Lang–Vojta, III). *Let X be an integral projective variety over k. Then the following statements are equivalent.*

(1) The variety X is strongly-pseudo-Brody hyperbolic over k.
(2) The variety X is strongly-pseudo-Kobayashi hyperbolic over k.
(3) The projective variety X is pseudo-Mordellic over k.
(4) The projective variety X is pseudo-arithmetically hyperbolic over k.
(5) The projective variety X is pseudo-groupless over k.
(6) The projective variety X is of general type over k.

As stated this conjecture does not predict that every conjugate of a pseudo-Brody hyperbolic variety is again pseudo-Brody hyperbolic. We state this as a separate conjecture, as we did in Conjecture 3.10 for Brody hyperbolic varieties.

Conjecture 8.9 (Conjugates of Pseudo-Brody Hyperbolic Varieties). *If X is an integral variety over k and σ is a field automorphism of k, then the following statements hold.*

(1) *The variety X is pseudo-Brody hyperbolic if and only if X^σ is pseudo-Brody hyperbolic.*
(2) *The variety X is pseudo-Kobayashi hyperbolic if and only if X^σ is pseudo-Kobayashi hyperbolic.*

We conclude this section with a brief discussion of a theorem of Kwack on the algebraicity of holomorphic maps to a hyperbolic variety, and a possible extension of his result to the pseudo-setting.

Remark 8.10 (Borel Hyperbolicity). Let X be a variety over $\mathbb{C}$ and let $\Delta \subset X$ be a closed subset. We extend the notion of Borel hyperbolicity introduced in [51] to the pseudo-setting and say that X is *Borel hyperbolic modulo* Δ if, for every reduced variety S over $\mathbb{C}$, every holomorphic map $f : S^{\mathrm{an}} \to X^{\mathrm{an}}$ with $f(S^{\mathrm{an}}) \not\subset \Delta^{\mathrm{an}}$ is the analytification of a morphism $\varphi : S \to X$. The proof of [51, Lemma 3.2] shows that, if X is Borel hyperbolic modulo Δ, then it is Brody hyperbolic modulo Δ. In [61] Kwack showed that, if X is a proper Kobayashi hyperbolic variety, then X is Borel hyperbolic (modulo the empty set). It seems reasonable to suspect that Kwack's theorem also holds in the pseudo-setting. Thus, we may ask: if X is Kobayashi hyperbolic modulo Δ, does it follow that X is Borel hyperbolic modulo Δ?

The reader interested in investigating further complex-analytic notions of hyperbolicity is also encouraged to have a look at the notion of taut-hyperbolicity modulo a subset introduced by Kiernan–Kobayashi [58]; see also [59, Chapter 5].

9   Algebraic Hyperbolicity

In the following three sections we investigate (a priori) different function field analogues of Mordellicity. Conjecturally, they are all equivalent notions. At this point it is also clear that hyperbolicity *modulo* a subset is more natural to study (especially from a birational perspective) which is why we will give the definitions in this more general context.

The notion we introduce in this section extends Demailly's notion of algebraic hyperbolicity [29, 50] to the pseudo-setting.

Definition 9.1 (Algebraic Hyperbolicity Modulo a Subset). Let X be a projective scheme over k and let Δ be a closed subset of X. We say that X is *algebraically hyperbolic over k modulo Δ* if, for every ample line bundle L on X, there is a real number $\alpha_{X,\Delta,L}$ depending only on X, Δ, and L such that, for every smooth projective connected curve C over k and every morphism $f : C \to X$ with $f(C) \not\subset \Delta$, the inequality

$$\deg_C f^* L \leq \alpha_{X,\Delta,L} \cdot \text{genus}(C)$$

holds.

Definition 9.2. A projective scheme X is *pseudo-algebraically hyperbolic* (*over k*) if there is a proper closed subset Δ such that X is algebraically hyperbolic modulo Δ.

We will say that a projective scheme X is *algebraically hyperbolic* over k if it is algebraically modulo the empty subset. This terminology is consistent with that of [50].

The motivation for introducing and studying algebraically hyperbolic projective schemes are the results of Demailly stated below. They say that algebraic hyperbolicity lies between Brody hyperbolicity and grouplessness. In particular, the Lang–Vojta conjectures as stated in the previous sections imply that groupless projective varieties should be algebraically hyperbolic, and that algebraically hyperbolic projective varieties should be Brody hyperbolic. This observation is due to Demailly and allows one to split the conjecture that groupless projective varieties are Brody hyperbolic into two a priori different parts.

Before stating Demailly's theorems, we note that it is not hard to see that pseudo-algebraic hyperbolicity descends along finite étale maps, and that pseudo-algebraic hyperbolicity for projective schemes is a birational invariant; see [56, §4] for details. These two properties should be compared with their counterparts for pseudo-grouplessness, pseudo-Mordellicity, pseudo-Brody hyperbolicity, and pseudo-Kobayashi hyperbolicity.

Demailly's theorem for projective schemes reads as follows.

Theorem 9.3 (Demailly). *Let X be a projective scheme over $\mathbb{C}$. If X is Brody hyperbolic, then X is algebraically hyperbolic over $\mathbb{C}$.*

A proof of this is given in [29, Theorem 2.1] when X is smooth. The smoothness of X is, however, not used in its proof. We stress that it is *not* known whether a pseudo-Brody hyperbolic projective scheme is pseudo-algebraically hyperbolic. On the other hand, Demailly proved that algebraically hyperbolic projective schemes are groupless, and his proof can be adapted to show the following more general statement.

Theorem 9.4 (Demailly $+ \epsilon$). *Let X be a projective scheme over k and let $\Delta \subset X$ be a closed subset. If X is algebraically hyperbolic modulo Δ, then X is groupless modulo Δ.*

Proof. See [50] when $\Delta = \emptyset$. The more general statement is proven in [56]. The argument involves the multiplication maps on an abelian variety. $\square$

Combining Demailly's theorems with Bloch–Ochiai–Kawamata's theorem, we obtain that a closed subvariety of an abelian variety over k is algebraically hyperbolic over k if and only if it is groupless. The pseudo-version of this theorem is due to Yamanoi (see Section 13 for a precise statement).

10   Boundedness

To say that a projective variety X is algebraically hyperbolic (Definition 9.1) is to say that the degree of any curve C is bounded uniformly and linearly in the genus of that curve. The reader interested in understanding how far we are from proving that groupless projective schemes are algebraically hyperbolic is naturally led to studying variants of algebraic hyperbolicity in which one asks (in Definition 9.1 above) for "weaker" bounds on the degree of a map. This led the authors of [50] to introducing the notion of boundedness. To state their definition, we first recall some basic properties of moduli spaces of morphisms between projective schemes.

Let S be a scheme, and let $X \to S$ and $Y \to S$ be projective flat morphisms of schemes. By Grothendieck's theory of Hilbert schemes and Quot schemes [75], the functor

$$\mathrm{Sch}/S^{op} \to \mathrm{Sets}, \quad T \to S \mapsto \mathrm{Hom}_T(Y_T, X_T)$$

is representable by an S-scheme which we denote by $\underline{\mathrm{Hom}}_S(X, Y)$. Moreover, for $h \in \mathbb{Q}[t]$ a polynomial, the subfunctor parametrizing morphisms whose graph has Hilbert polynomial h is representable by a quasi-projective subscheme $\underline{\mathrm{Hom}}_S^h(Y, X)$ of $\underline{\mathrm{Hom}}_S(Y, X)$. Similarly, the subfunctor of $\underline{\mathrm{Hom}}_S(X, X)$ parametrizing automorphisms of X over S is representable by a locally finite type group scheme $\mathrm{Aut}_{X/S}$ over S. It is imperative to note that this group scheme need not be quasi-compact. In fact, for a K3 surface X over $\mathbb{C}$, the scheme $\mathrm{Aut}_{X/\mathbb{C}}$ is zero-dimensional. Nonetheless, there are K3 surfaces with infinitely many automorphisms. Thus, the automorphism group scheme of a projective scheme over k is not necessarily of finite type (even when it is zero-dimensional).

If $S = \mathrm{Spec}\, k$, $d \geq 1$ is an integer, and $X = Y = \mathbb{P}_k^1$, let $\underline{\mathrm{Hom}}_k^d(\mathbb{P}_k^1, \mathbb{P}_k^1)$ be the subscheme of $\underline{\mathrm{Hom}}_k(\mathbb{P}_k^1, \mathbb{P}_k^1)$ parametrizing morphisms of degree d. In particular, we have that $\underline{\mathrm{Hom}}_k^1(\mathbb{P}_k^1, \mathbb{P}_k^1) = \mathrm{Aut}_{\mathbb{P}_k^1/k} = \mathrm{PGL}_{2,k}$. For every $d \geq 1$,

the quasi-projective scheme $\underline{\mathrm{Hom}}_k^d(\mathbb{P}_k^1, \mathbb{P}_k^1)$ is non-empty (and even positive-dimensional). If we identity the subscheme of $\underline{\mathrm{Hom}}_k(\mathbb{P}_k^1, \mathbb{P}_k^1)$ parametrizing constant morphisms with $\mathbb{P}_k^1$, then

$$\underline{\mathrm{Hom}}_k(\mathbb{P}^1, \mathbb{P}_k^1) = \mathbb{P}_k^1 \sqcup \mathrm{PGL}_{2,k} \sqcup \bigsqcup_{d=2}^{\infty} \underline{\mathrm{Hom}}_k^d(\mathbb{P}_k^1, \mathbb{P}_k^1).$$

It follows that the scheme $\underline{\mathrm{Hom}}_k(\mathbb{P}^1, \mathbb{P}_k^1)$ has infinitely many connected components. It is in particular not of finite type.

It turns out that studying projective varieties X over k for which *every* Hom-scheme $\mathrm{Hom}_k(Y, X)$ is of finite type is closely related to studying algebraically hyperbolic varieties. The aim of this section is to explain the connection in a systematic manner as is done in [15, 50, 56]. We start with the following definitions.

Definition 10.1 (Boundedness Modulo a Subset). Let $n \geq 1$ be an integer, let X be a projective scheme over k, and let Δ be a closed subset of X. We say that X is *n-bounded over k modulo* Δ if, for every normal projective variety Y of dimension at most n, the scheme $\underline{\mathrm{Hom}}_k(Y, X) \setminus \underline{\mathrm{Hom}}_k(Y, \Delta)$ is of finite type over k. We say that X is *bounded over k modulo* Δ if, for every $n \geq 1$, the scheme X is n-bounded modulo Δ.

Definition 10.2. Let $n \geq 1$ be an integer. A projective scheme X over k is *pseudo-n-bounded over k* if there is a proper closed subset Δ such that X is n-bounded modulo Δ.

Definition 10.3. A projective scheme X over k is *pseudo-bounded over k* if it is pseudo-n-bounded over k for every $n \geq 1$.

Remark 10.4. At the beginning of this section we discussed the structure of the scheme $\underline{\mathrm{Hom}}_k(\mathbb{P}_k^1, \mathbb{P}_k^1)$. From that discussion it follows that $\mathbb{P}_k^1$ is not 1-bounded over k. In particular, if X is a 1-bounded projective variety over k, then it has no rational curves. It is also not hard to show that $\mathbb{P}_k^1$ is not pseudo-1-bounded by showing that, for every x in $\mathbb{P}^1(k)$, there is a y in $\mathbb{P}^1(k)$ such that the set of morphisms $f : \mathbb{P}_k^1 \to \mathbb{P}_k^1$ with $f(y) = x$ is infinite. We refer the interested reader to Section 11 for a related discussion.

We say that X is *bounded* if it is bounded modulo the empty subset. We employ similar terminology for n-bounded. This terminology is consistent with that of [15, 50]. Let us start with looking at some implications and relations between these a priori different notions of boundedness.

Boundedness is a condition on moduli spaces of maps from higher-dimensional varieties. Although it might seem a priori stronger than 1-boundedness, Lang–Vojta's conjecture predicts their equivalence. In fact, we have the following result from [50] which shows the equivalence of three a

priori different notions. In this theorem, the implications $(2) \implies (1)$ and $(3) \implies (1)$ are straightforward consequences of the definitions.

Theorem 10.5. *Let X be a projective scheme over k. Then the following are equivalent.*

(1) The projective scheme X is 1-bounded over k.
(2) The projective scheme X is bounded over k.
(3) For every ample line bundle $\mathcal{L}$ and every integer $g \geq 0$, there is an integer $\alpha(X, \mathcal{L}, g)$ such that, for every smooth projective connected curve C of genus g over k and every morphism $f : C \to X$, the inequality

$$\deg_C f^* \mathcal{L} \leq \alpha(X, \mathcal{L}, g)$$

holds.

Proof. The fact that a 1-bounded scheme is n-bounded for every $n \geq 1$ is proven by induction on n in [50, §9]. The idea is that, if $f_i : Y \to X$ is a sequence of morphisms from an n-dimensional smooth projective variety Y with pairwise distinct Hilbert polynomial, then one can find a smooth hyperplane section $H \subset Y$ such that the restrictions $f_i|_H$ of these morphisms f_i to H still have pairwise distinct Hilbert polynomial.

The fact that a bounded scheme satisfies the "uniform" boundedness property in (3) follows from reformulating this statement in terms of the quasi-compactness of the universal Hom-stack of morphisms of curves of genus g to X; see the proof of [50, Theorem 1.14] for details. □

Studying boundedness is "easier" than studying boundedness modulo a subset Δ. Indeed, part of the analogue of this theorem for pseudo-boundedness (unfortunately) requires an assumption on the base field k.

Theorem 10.6. *Let X be a projective scheme over k, and let Δ be a closed subset of X. Assume that k is **uncountable**. Then X is 1-bounded modulo Δ if and only if X is bounded modulo Δ.*

Proof. This is proven in [15], and the argument is similar to the proof of Theorem 10.5. We briefly indicate how the uncountability of k is used.

Assume that X is 1-bounded modulo Δ. We show by induction on n that X is n-bounded modulo Δ over k. If $n = 1$, then this holds by assumption. Thus, let $n > 1$ be an integer and assume that X is $(n-1)$-bounded modulo Δ. Let Y be an n-dimensional projective reduced scheme and let $f_m : Y \to X$ be a sequence of morphisms with pairwise distinct Hilbert polynomial such that, for every $m = 1, 2, \ldots$, we have $f_m(Y) \not\subset \Delta$. Since k is **uncountable**, there is an ample divisor D in Y which is not contained in $f_m^{-1}(\Delta)$ for all $m \in \{1, 2, \ldots\}$. Now, the restrictions $f_m|_D : D \to X$ have pairwise distinct Hilbert polynomial and, for infinitely many m, we have that $f_m(D) \not\subset \Delta$. This contradicts the induction hypothesis. We conclude that X is bounded modulo Δ over k, as required. □

The "pseudo" analogue of the equivalence between (2) and (3) in Theorem 10.5 holds without any additional assumption on k; see [15].

Theorem 10.7. *Let X be a projective scheme over k. Then X is bounded modulo Δ over k if and only if, for every ample line bundle $\mathcal{L}$ and every integer $g \geq 0$, there is an integer $\alpha(X, \mathcal{L}, g)$ such that, for every smooth projective connected curve C of genus g over k and every morphism $f : C \to X$ with $f(C) \not\subset \Delta$, the inequality*

$$\deg_C f^*\mathcal{L} \leq \alpha(X, \mathcal{L}, g)$$

holds.

It is not hard to see that being pseudo-n-bounded descends along finite étale maps. Also, if X and Y are projective schemes over k which are birational, then X is pseudo-1-bounded if and only if Y is pseudo-1-bounded; see [56, §4]. However, in general, it is not clear that pseudo-n-boundedness is a birational invariant (unless $n = 1$ or k is uncountable).

It is shown in [15, 50] that pseudo-algebraically hyperbolic varieties are pseudo-bounded. More precisely, one can prove the following statement.

Theorem 10.8. *If X is algebraically hyperbolic modulo Δ over k, then X is bounded modulo Δ.*

Proof. This is proven in three steps in [15, §9]. First, one chooses an uncountable algebraically closed field L containing k and shows that X_L is algebraically hyperbolic modulo Δ_L. Then, one makes the "obvious" observation that X_L is 1-bounded modulo Δ_L. Finally, as L is uncountable and X_L is 1-bounded modulo Δ_L, it follows from Theorem 10.6 that X_L is bounded modulo Δ_L.   □

Demailly proved that algebraically hyperbolic projective varieties are groupless (Theorem 9.4). His proof can be adapted to show the following more general statement.

Proposition 10.9 (Demailly + ϵ). *If X is 1-bounded modulo Δ over k, then X is groupless modulo Δ.*

11 Geometric Hyperbolicity

In the definition of Mordellicity over $\overline{\mathbb{Q}}$ one considers the "finiteness of arithmetic curves" on some model. On the other hand, the notions of algebraic hyperbolicity and boundedness require one to test "boundedness of curves". In this section we introduce a new notion in which one considers the "finiteness of pointed curves".

Definition 11.1 (Geometric Hyperbolicity Modulo a Subset). Let X be a variety over k and let Δ be a closed subset of X. We say that X is *geometrically hyperbolic over k modulo Δ* if, for every x in $X(k) \setminus \Delta$, every smooth connected curve C over k and every c in $C(k)$, we have that the set $\mathrm{Hom}_k((C, c), (X, x))$ of morphisms $f : C \to X$ with $f(c) = x$ is finite.

Definition 11.2. A variety X over k is *pseudo-geometrically hyperbolic over k* if there is a proper closed subset Δ such that X is geometrically hyperbolic modulo Δ.

We say that a variety X over k is *geometrically hyperbolic over k* if it is geometrically hyperbolic modulo the empty subset. At this point we should note that a projective scheme X over k is geometrically hyperbolic over k if and only if it is "$(1,1)$-bounded". The latter notion is defined in [50, §4], and the equivalence of these two notions is [50, Lemma 4.6] (see also Proposition 11.4 below). The terminology "$(1,1)$-bounded modulo Δ" is used in [15], and also coincides with being geometrically hyperbolic modulo Δ for projective schemes by the results in [15, §9].

Remark 11.3 (Geometric Hyperbolicity Versus Arithmetic Hyperbolicity). Let us say that a scheme T is an *arithmetic curve* if there is a number field K and a finite set of finite places S of K such that $T = \mathrm{Spec}\,\mathcal{O}_{K,S}$. Let X be a variety over $\overline{\mathbb{Q}}$. It is not hard to show that the following two statements are equivalent.

(1) The variety X is arithmetically hyperbolic (or Mordellic) over $\overline{\mathbb{Q}}$.
(2) For every arithmetic curve $\mathcal{C}$, every closed point c in $\mathcal{C}$, every model $\mathcal{X}$ for X over $\mathcal{C}$, and every closed point x of $\mathcal{X}$, the subset

$$\mathrm{Hom}_{\mathcal{C}}((\mathcal{C}, c), (\mathcal{X}, x)) \subset \mathcal{X}(\mathcal{C})$$

of morphisms $f : \mathcal{C} \to \mathcal{X}$ with $f(c) = x$ is finite.

Indeed, if (1) holds, then $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{X})$ is finite by definition, so that clearly the set

$$\mathrm{Hom}_{\mathcal{C}}((\mathcal{C}, c), (\mathcal{X}, x))$$

is finite. Conversely, assume that (2) holds. Now, let $\mathcal{C}$ be an arithmetic curve and let $\mathcal{X}$ be a model for X over $\mathcal{C}$. To show that $\mathcal{X}(\mathcal{C})$ is finite, let c be a closed point of $\mathcal{C}$ and let κ be its residue field. Then κ is finite and c lies in $\mathcal{C}(\kappa)$. In particular, the image of c along any morphism $\mathcal{C} \to \mathcal{X}$ is a κ-point of $\mathcal{X}$. This shows that

$$\mathcal{X}(\mathcal{C}) \subset \bigcup_{x \in \mathcal{X}(\kappa)} \mathrm{Hom}_{\mathcal{C}}((\mathcal{C}, c), (\mathcal{X}, x)).$$

Since $\mathcal{X}(\kappa)$ is finite and every set $\mathrm{Hom}_{\mathcal{C}}((\mathcal{C}, c), (\mathcal{X}, x))$ is finite, we conclude that $\mathcal{X}(\mathcal{C})$ is finite, as required.

The second statement allows one to see the similarity between geometric hyperbolicity and arithmetic hyperbolicity. Indeed, the variety X is geometrically hyperbolic over $\overline{\mathbb{Q}}$ if, for every integral algebraic curve C over $\overline{\mathbb{Q}}$, every closed point c in C, and every closed point x of X, the set

$$\mathrm{Hom}_k((C, c), (X, x)) = \mathrm{Hom}_C((C, c), (X \times C, (x, c)))$$

is finite.

Just like pseudo-grouplessness and pseudo-Mordellicity, it is not hard to see that pseudo-geometric hyperbolicity descends along finite étale morphisms. Also, if X and Y are projective varieties which are birational, then X is pseudo-geometrically hyperbolic if and only if Y is pseudo-geometrically hyperbolic.

The following proposition says that a projective scheme is geometrically hyperbolic if and only if the moduli space of pointed maps is of finite type. In other words, asking for boundedness of all pointed maps is equivalent to asking for the finiteness of all sets of pointed maps.

Proposition 11.4. *Let X be a projective scheme over k and let Δ be a closed subset of X. Then the following are equivalent.*

(1) *For every smooth projective connected curve C over k, every c in $C(k)$ and every x in $X(k) \backslash \Delta$, the scheme $\underline{\mathrm{Hom}}_k((C, c), (X, x))$ is of finite type over k.*
(2) *The variety X is geometrically hyperbolic modulo Δ.*

Proof. This is proven in [15, §9]. The proof is a standard application of the bend-and-break principle. Indeed, the implication (2) $\implies$ (1) being obvious, let us show that (1) $\implies$ (2). Thus, let us assume that X is not geometrically hyperbolic modulo Δ, so that there is a sequence $f_1, f_2, \dots$ of pairwise distinct elements of $\mathrm{Hom}_k((C, c), (X, x))$, where C is a smooth projective connected curve over k, $c \in C(k)$ and $x \in X(k) \backslash \Delta$. Since $\underline{\mathrm{Hom}}_k((C, c), (X, x))$ is of finite type, the degree of all the f_i is bounded by some real number (depending only on X, Δ, c, x and C). In particular, it follows that some connected component of $\underline{\mathrm{Hom}}_k((C, c), (X, x))$ has infinitely many elements. As each connected component of $\underline{\mathrm{Hom}}_k((C, c), (X, x))$ is a finite type scheme over k, it follows from bend-and-break [27, Proposition 3.5] that there is a rational curve in X containing x. This contradicts the fact that every rational curve in X is contained in Δ (by Proposition 11.7). $\square$

This proposition has the following consequence.

Corollary 11.5. *Let X be a projective scheme over k and let Δ be a proper closed subset of X. If X is 1-bounded modulo Δ, then X is geometrically hyperbolic modulo Δ.*

Proof. If X is 1-bounded, then it is clear that, for every smooth projective connected curve C, every c in $C(k)$ and every x in $X(k) \setminus \Delta$, the scheme $\underline{\mathrm{Hom}}_k((C,c),(X,x))$ is of finite type over k. Indeed, the latter scheme is closed in the scheme $\underline{\mathrm{Hom}}_k(C,X)$, and contained in the quasi-projective subscheme $\underline{\mathrm{Hom}}_k(C,X) \setminus \underline{\mathrm{Hom}}_k(C,\Delta)$. Therefore, the result follows from Proposition 11.4. □

Remark 11.6. Urata showed that a Brody hyperbolic projective variety over $\mathbb{C}$ is geometrically hyperbolic over $\mathbb{C}$; see [59, Theorem 5.3.10] (or the original [84]). Note that Corollary 11.5 generalizes Urata's theorem (in the sense that the assumption in Corollary 11.5 is a priori weaker than being Brody hyperbolic, and we also allow for an "exceptional set" Δ). Indeed, as a Brody hyperbolic projective variety is 1-bounded (even algebraically hyperbolic), Urata's theorem follows directly from Corollary 11.5.

Demailly's argument to show that algebraically hyperbolic projective varieties are groupless (Theorem 9.4) can be adapted to show that geometrically hyperbolic projective varieties are groupless; see [56] for a detailed proof.

Proposition 11.7. *Let X be a projective variety over k and let Δ be a closed subset of X. If X is geometrically hyperbolic modulo Δ over k, then X is groupless modulo Δ over k.*

12 The Conjectures Summarized

After a lengthy preparation, we are finally ready to state the complete version of Lang–Vojta's conjecture.
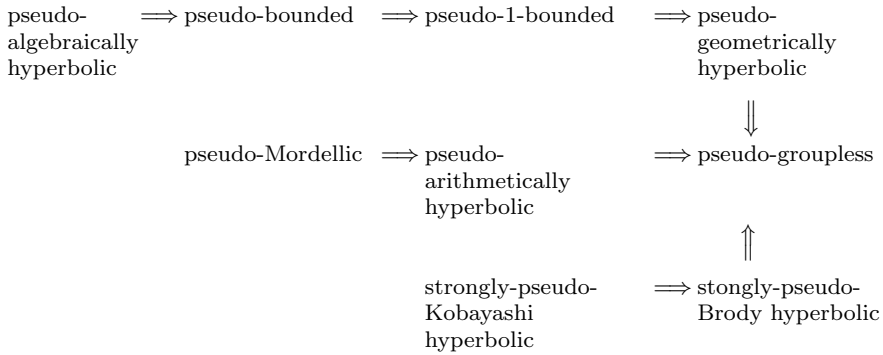
Conjecture 12.1 (Strong Lang–Vojta, IV). *Let X be an integral projective variety over k. Then the following statements are equivalent.*

- *(1) The variety X is strongly-pseudo-Brody hyperbolic over k.*
- *(2) The variety X is strongly-pseudo-Kobayashi hyperbolic.*
- *(3) The projective variety X is pseudo-Mordellic over k.*
- *(4) The projective variety X is pseudo-arithmetically hyperbolic over k.*
- *(5) The projective variety X is pseudo-groupless over k.*
- *(6) The projective variety X is pseudo-algebraically hyperbolic over k.*
- *(7) The projective variety X is pseudo-bounded over k.*
- *(8) The projective variety X is pseudo-1-bounded over k.*
- *(9) The projective variety X is pseudo-geometrically hyperbolic over k.*
- *(10) The projective variety X is of general type over k.*

Conjecture 12.1 is the final version of the Lang–Vojta conjecture for pseudo-hyperbolic varieties, and also encompasses Green–Griffiths's conjecture for projective varieties of general type. We note that one aspect of the Lang–Vojta conjecture and the Green–Griffiths conjecture that is ignored

in this conjecture is whether the conjugate of a Brody hyperbolic variety is Brody hyperbolic (see Conjectures 3.10 and 8.9).

The following implications are known. First, (6) $\implies$ (7), (7) $\implies$ (8), (8) $\implies$ (9), and (9) $\implies$ (5). Also, (3) $\implies$ (4), (3) $\implies$ (5). Finally, (2) $\implies$ (1) and (1) $\implies$ (5). The following diagram summarizes these known implications. The content of the Strong Lang–Vojta conjecture is that all the notions appearing in this diagram are equivalent.

$$
\begin{array}{llll}
\text{pseudo-} & \implies \text{pseudo-bounded} & \implies \text{pseudo-1-bounded} & \implies \text{pseudo-} \\
\text{algebraically} & & & \text{geometrically} \\
\text{hyperbolic} & & & \text{hyperbolic} \\
& & & \Downarrow \\
& \text{pseudo-Mordellic} & \implies \text{pseudo-} & \implies \text{pseudo-groupless} \\
& & \text{arithmetically} & \\
& & \text{hyperbolic} & \\
& & & \Uparrow \\
& & \text{strongly-pseudo-} & \implies \text{stongly-pseudo-} \\
& & \text{Kobayashi} & \text{Brody hyperbolic} \\
& & \text{hyperbolic} &
\end{array}
$$

We stress that the Strong Lang–Vojta conjecture is concerned with classifying projective varieties of general type via their complex-analytic or arithmetic properties. Recall that Campana's special varieties can be considered as being opposite to varieties of general type. As Campana's conjectures are concerned with characterizing special varieties via their complex-analytic or arithmetic properties, his conjectures should be considered as providing another part of the conjectural picture. We refer the reader to [16] for a discussion of Campana's conjectures.

The following conjecture is only concerned with hyperbolic varieties and is, therefore, a priori weaker than the Strong Lang-Vojta conjecture. It is not clear to us whether the Strong Lang–Vojta conjecture can be deduced from the following weaker version, as there are pseudo-hyperbolic projective varieties which are not birational to a hyperbolic projective variety.

Conjecture 12.2 (Weak Lang–Vojta, IV). *Let X be an integral projective variety over k. Then the following statements are equivalent.*

(1) The variety X is strongly-Brody hyperbolic over k.
(2) The variety X is strongly-Kobayashi hyperbolic over k.
(3) The projective variety X is Mordellic over k.
(4) The projective variety X is arithmetically hyperbolic over k.
(5) The projective variety X is groupless over k.
(6) The projective variety X is algebraically hyperbolic over k.

(7) *The projective variety X is bounded over k.*
(8) *The projective variety X is 1-bounded over k.*
(9) *The projective variety X is geometrically hyperbolic over k.*
(10) *Every integral subvariety of X is of general type.*

Remark 12.3 (Strong Implies Weak). Let us illustrate why the strong Lang–Vojta conjecture implies the Weak Lang–Vojta conjecture. To do so, let X be a projective variety. Assume that X is groupless. Then X is pseudo-groupless. Thus, by the Strong Lang–Vojta conjecture, we have that X is Mordellic modulo some proper closed subset $\Delta \subset X$. Now, since X is groupless, it follows that Δ is groupless. Repeating the above argument shows that Δ is Mordellic, so that X is Mordellic.

We know *more* about the Weak Lang–Vojta conjecture than we do about the Strong Lang–Vojta conjecture. Indeed, it is known that (1) $\iff$ (2) by Brody's Lemma. Also, it is not hard to show that (2) $\implies$ (5). Moreover, we know that (3) $\implies$ (4) and (4) $\implies$ (5). Of course, we also have that (6) $\implies$ (7), (7) $\implies$ (8), and (8) $\iff$ (9). In addition, we also have that (1) $\implies$ (6) and that (10) $\implies$ (5). Figure 1 summarizes these known implications.

Figure 2 below illustrates a projective variety which satisfies the Weak Lang–Vojta conjecture. The picture shows that this variety has infinitely many points valued in a number field (in orange), admits an entire curve (in blue), admits algebraic maps of increasing degree from some fixed curve (in red), and admits a non-constant map from an abelian variety (in green). It is therefore a non-Mordellic, non-Brody hyperbolic, non-bounded, and non-groupless projective variety.

12.1   The Conjecture on Exceptional Loci

We now define the exceptional loci for every notion that we have seen so far. As usual, we let k be an algebraically closed field of characteristic zero.

Definition 12.4. Let X be a variety over k.

- We define Δ_X^{gr} to be the intersection of all proper closed subset Δ such that X is groupless modulo Δ. Note that Δ_X^{gr} is a closed subset of X and that X is groupless modulo Δ_X^{gr}. We refer to Δ_X^{gr} as the *groupless-exceptional locus* of X.
- We define Δ_X^{ar-hyp} to be the intersection of all proper closed subsets Δ such that X is arithmetically hyperbolic modulo Δ. Note that X is arithmetically hyperbolic modulo Δ_X^{ar-hyp}. We refer to Δ_X^{ar-hyp} as the *arithmetic-exceptional locus* of X.
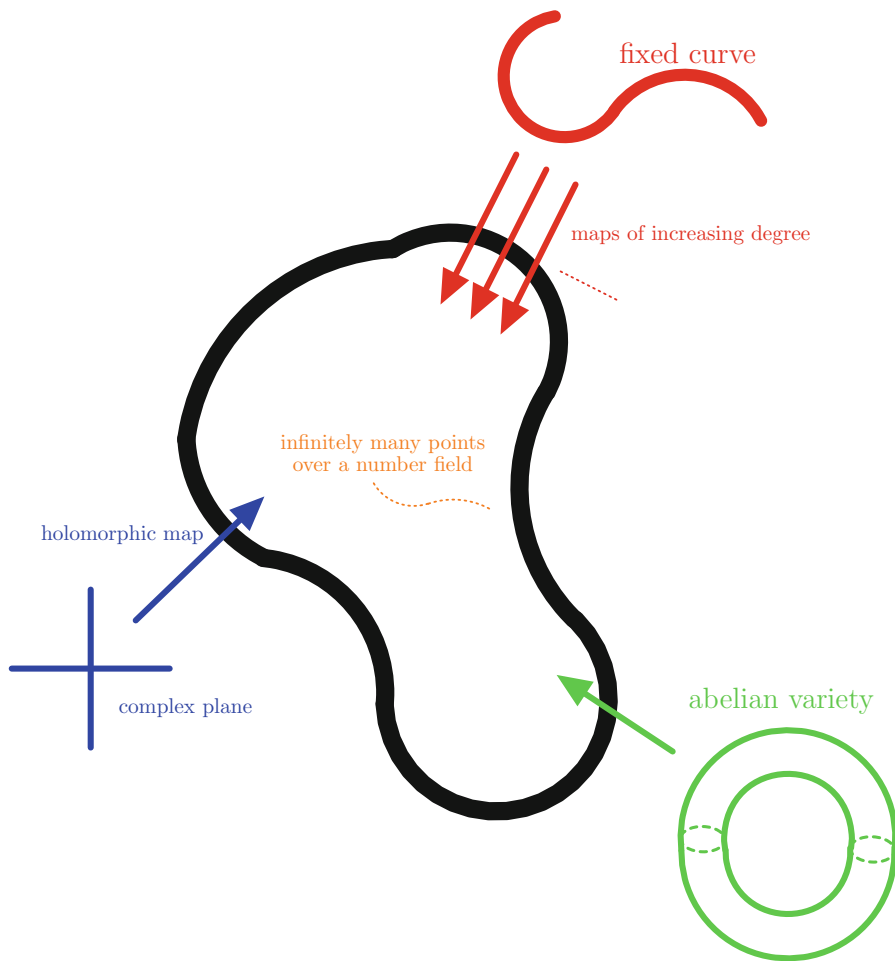
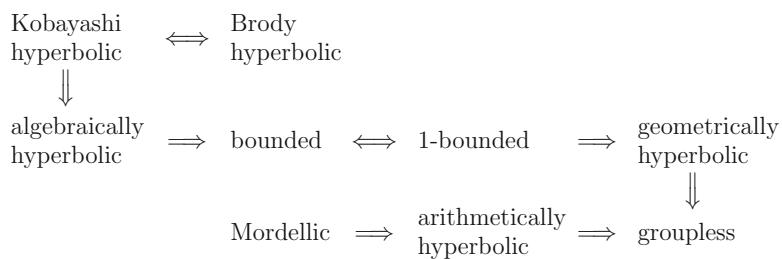**Fig. 1** A projective variety satisfying the Weak Lang-Vojta conjecture

Fig. 2 Known implications between notions of hyperbolicity

- We define Δ_X^{Mor} to be the intersection of all proper closed subsets Δ such that X is Mordellic modulo Δ. Note that X is Mordellic modulo Δ_X^{Mor}. We refer to Δ_X^{Mor} as the *Mordellic-exceptional locus* of X.

Assuming X is a proper variety over k for a moment, it seems worthwhile stressing that Δ_X^{gr} equals the (Zariski) closure of Lang's algebraic exceptional set $\mathrm{Exc}_{alg}(X)$ as defined in [62, p. 160].

Definition 12.5. Let X be a variety over $\mathbb{C}$.

- We let Δ_X^{Br} be the intersection of all closed subsets Δ such that X is Brody hyperbolic modulo Δ. Note that Δ_X^{Br} is a closed subset of X and that X is Brody hyperbolic modulo Δ_X^{Br}. We refer to Δ_X^{Br} as the *Brody-exceptional locus* of X.
- We let Δ_X^{Kob} be the intersection of all closed subsets Δ such that X is Kobayashi hyperbolic modulo Δ. Note that Δ_X^{Kob} is a closed subset of X and that X is Kobayashi hyperbolic modulo Δ_X^{Kob}. We refer to Δ_X^{Kob} as the *Kobayashi-exceptional locus* of X.

We note that Δ_X^{Br} coincides with Lang's analytic exceptional set $\mathrm{Exc}(X)$ (defined in [62, p. 160]). Indeed, $\mathrm{Exc}(X)$ is defined to be the Zariski closure of the union of all images of non-constant entire curves $\mathbb{C} \to X^{\mathrm{an}}$.

Definition 12.6. Let X be a projective scheme over k.

- We define $\Delta_X^{alg-hyp}$ to be the intersection of all proper closed subsets Δ such that X is algebraically hyperbolic modulo Δ. Note that $\Delta_X^{alg-hyp}$ is a proper closed subset of X and that X is algebraically hyperbolic modulo $\Delta_X^{alg-hyp}$. We refer to $\Delta_X^{alg-hyp}$ as the *algebraic exceptional locus* of X.
- For $n \geq 1$, we define $\Delta_X^{n-bounded}$ to be the intersection of all proper closed subsets Δ such that X is n-bounded modulo Δ. Note that $\Delta_X^{n-bounded}$ is a proper closed subset of X and that X is n-bounded modulo $\Delta_X^{n-bounded}$. We refer to $\Delta_X^{n-bounded}$ as the *n-bounded-exceptional locus* of X.
- We define $\Delta_X^{bounded}$ to be the intersection of all proper closed subsets Δ such that X is bounded modulo Δ. Note that $\Delta_X^{bounded}$ is a proper closed subset of X and that X is bounded modulo $\Delta_X^{bounded}$. We refer to $\Delta_X^{bounded}$ as the *bounded-exceptional locus* of X.
- We define $\Delta_X^{geom-hyp}$ to be the intersection of all proper closed subsets Δ such that X is geometrically hyperbolic modulo Δ. Note that $\Delta_X^{geom-hyp}$ is a proper closed subset of X and that X is geometrically hyperbolic modulo $\Delta_X^{geom-hyp}$. We refer to $\Delta_X^{geom-hyp}$ as the *geometric-exceptional locus* of X.

The strongest version of Lang–Vojta's conjecture stated in these notes claims the equality of all exceptional loci. Note that these loci are all, by definition, closed subsets. This is to be contrasted with Lang's definition of his "algebraic exceptional set" (see [62, p. 160]).

Conjecture 12.7 (Strongest Lang–Vojta Conjecture). *Let k be an algebraically closed field of characteristic zero. Let X be an integral projective variety over k. Then the following three statements hold.*

(1) We have that

$$\Delta_X^{gr} = \Delta_X^{Mor} = \Delta_X^{geom-hyp} = \Delta_X^{1-bounded} = \Delta_X^{bounded} = \Delta_X^{alg-hyp}.$$

(2) The projective variety X is of general type if and only if $\Delta_X^{gr} \neq X$.
(3) If $k = \mathbb{C}$, then $\Delta_X^{gr} = \Delta_X^{Br} = \Delta_X^{Kob}$.

Remark 12.8 (Which Inclusions Do We Know?). Let X be a projective scheme over k. We have that

$$\Delta_X^{gr} \subset \Delta_X^{ar-hyp} \subset \Delta_X^{Mor},$$

and

$$\Delta_X^{gr} \subset \Delta_X^{geom-hyp} \subset \Delta_X^{1-bounded} \subset \Delta_X^{bounded} \subset \Delta_X^{alg-hyp}.$$

If k is uncountable, then

$$\Delta_X^{1-bounded} = \Delta_X^{bounded}.$$

If $k = \mathbb{C}$, then

$$\Delta_X^{gr} \subset \Delta_X^{Br} \subset \Delta_X^{Kob}.$$

Remark 12.9 (Reformulating Brody's Lemma). It is not known whether $\Delta_X^{Kob} \subset \Delta_X^{Br}$. Brody's lemma can be stated as saying that, if Δ_X^{Br} is empty, then Δ_X^{Kob} is empty.

Remark 12.10 (Reformulating Demailly's Theorem). It is not known whether $\Delta_X^{alg-hyp} \subset \Delta_X^{Kob}$. Demailly's theorem (Theorem 9.4) can be stated as saying that, if Δ^{Kob} is empty, then $\Delta^{alg-hyp}$ is empty.

13   Closed Subvarieties of Abelian Varieties

We have gradually worked our way towards the following theorem which says that the Strongest Lang–Vojta conjecture holds for closed subvarieties of abelian varieties. Recall that, for X a closed subvariety of an abelian variety A, the subset $\mathrm{Sp}(X)$ is defined to be the union of translates of positive-dimensional abelian subvarieties of A contained in A. It is a fundamental

fact that $\mathrm{Sp}(X)$ is a closed subset of X. It turns out that $\mathrm{Sp}(X)$ is the "exceptional locus" of X in any sense of the word "exceptional locus".

Theorem 13.1 (Bloch–Ochiai–Kawamata,   Faltings,   Yamanoi, Kawamata–Ueno). *Let A be an abelian variety over k, and let $X \subset A$ be a closed subvariety. Then the following statements hold.*

(1) We have that $\mathrm{Sp}(X) \neq X$ of X equals if and only if X is of general type.
(2) We have that

$$\mathrm{Sp}(X) = \Delta_X^{gr} = \Delta_X^{Mor} = \Delta_X^{ar-hyp} = \Delta_X^{geom-hyp} = \Delta_X^{1-bounded}$$
$$= \Delta_X^{bounded} = \Delta_X^{alg-hyp}.$$

(3) If $k = \mathbb{C}$, then $\Delta_X^{gr} = \Delta_X^{Br} = \Delta_X^{Kob}$.

Proof. The fact that $\mathrm{Sp}(X) \neq X$ if and only if X is of general type is due to Kawamata–Ueno (see also Theorem 5.1). Moreover, an elementary argument (see Example 6.11) shows that X is groupless modulo $\mathrm{Sp}(X)$, so that $\Delta_X^{gr} \subset \mathrm{Sp}(X)$. On the other hand, it is clear from the definition that $\mathrm{Sp}(X) \subset \Delta_X^{gr}$. This shows that $\mathrm{Sp}(X) = \Delta_X^{gr}$.

By Faltings's theorem (Theorem 7.15), we have that X is Mordellic modulo $\mathrm{Sp}(X)$. This shows that $\Delta_X^{Mor} = \Delta_X^{ar-hyp} = \Delta_X^{gr} = \mathrm{Sp}(X)$. (One can also show that $\Delta_X^{ar-hyp} = \Delta_X^{Mor}$ without appealing to Faltings's theorem. Indeed, as X is a closed subvariety of an abelian variety, it follows from Corollary 7.33 that X is arithmetically hyperbolic modulo Δ if and only if X is Mordellic modulo Δ.)

It follows from Bloch–Ochiai–Kawamata's theorem that $\Delta_X^{Br} = \mathrm{Sp}(X)$. Yamanoi improved this result and showed that $\Delta_X^{Kob} = \mathrm{Sp}(X)$; see Theorem 8.6 (or the original [88, Theorem 1.2]). In his earlier work [87, Corollary 1.(3)], Yamanoi proved that $\Delta_X^{alg-hyp} = \mathrm{Sp}(X)$. Since

$$\Delta_X^{geom-hyp} \subset \Delta_X^{1-bounded} \subset \Delta_X^{bounded} \subset \Delta_X^{alg-hyp},$$

this concludes the proof.   □

14   Evidence for Lang–Vojta's Conjecture

In the previous sections, we defined every notion appearing in Lang–Vojta's conjecture, and we stated the "Strongest", "Stronger", and "Weakest" versions of Lang–Vojta's conjectures. We also indicated the known implications between these notions, and that the Strongest Lang–Vojta conjecture is known to hold for closed subvarieties of abelian varieties by work of Bloch–Ochiai–Kawamata, Faltings, Kawamata–Ueno, and Yamanoi.

In the following four sections, we will present some evidence for Lang–Vojta's conjectures. The results in the following sections are all in accordance with the Lang–Vojta conjectures.

15 Dominant Rational Self-maps of Pseudo-Hyperbolic Varieties

Let us start with a classical finiteness result of Matsumura [45, §11].

Theorem 15.1 (Matsumura). *If X is a proper integral variety of general type over k, then the set of dominant rational self-maps $X \dashrightarrow X$ is finite.*

Note that Matsumura's theorem is a vast generalization of the statement that a smooth curve of genus at least two has only finitely many automorphisms. Motivated by Lang–Vojta's conjecture, the arithmetic analogue of Matsumura's theorem is proven in [56] (building on the results in [49]) and can be stated as follows.

Theorem 15.2. *If X is a proper pseudo-Mordellic integral variety over k, then the set of rational dominant self-maps $X \dashrightarrow X$ is finite.*

Idea of Proof. We briefly indicate three ingredients of the proof of Theorem 15.2.

(1) First, one can use Amerik's theorem on dynamical systems [5] to show that every dominant rational self-map is a birational self-map of finite order whenever X is a pseudo-Mordellic projective variety.
(2) One can show that, if X is a projective integral variety over k such that $\mathrm{Aut}_k(X)$ is infinite, then $\mathrm{Aut}_k(X)$ has an element of infinite order. (It is crucial here that k is of characteristic zero.) This result is proven in [49].
(3) If X is a projective non-uniruled integral variety over k such that $\mathrm{Bir}_k(X)$ is infinite, then $\mathrm{Bir}_k(X)$ has a point of infinite order. To prove this, one can use Prokhorov–Shramov's notion of quasi-minimal models (see [79]) to reduce to the analogous finiteness result for automorphisms stated in (2). The details are in [56].

Combining (1) and (3), one obtains the desired result for pseudo-Mordellic projective varieties (Theorem 15.2).   □

There is a similar finiteness statement for pseudo-algebraically hyperbolic varieties. This finiteness result is proven in [50] for algebraically hyperbolic varieties, and in [56] for pseudo-algebraically hyperbolic varieties.

Theorem 15.3. *If X is a projective pseudo-algebraically hyperbolic integral variety over k, then the set of dominant rational self-maps $X \dashrightarrow X$ is finite.*

In fact, more generally, we have the following a priori stronger result.

Theorem 15.4. *If X is a projective pseudo-1-bounded integral variety over k, then the set of dominant rational self-maps $X \dashrightarrow X$ is finite.*

Proof. For 1-bounded varieties this is proven in [50]. The more general statement for pseudo-1-bounded varieties is proven in [56] by combining Amerik's theorem [5] and Prokhorov–Shramov's theory of quasi-minimal models [79] with Weil's Regularization Theorem and properties of dynamical degrees of rational dominant self-maps. □

As the reader may have noticed, for pseudo-Mordellic, pseudo-algebraically hyperbolic, and pseudo-1-bounded projective varieties we have satisfying results.

What do we know in the complex-analytic setting? We have the following result of Noguchi [59, Theorem 5.4.4] for Brody hyperbolic varieties.

Theorem 15.5 (Noguchi). *If X is a Brody hyperbolic projective integral variety over $\mathbb{C}$, then $\mathrm{Bir}_{\mathbb{C}}(X)$ is finite.*

First Proof of Theorem 15.5. Since a Brody hyperbolic projective integral variety over $\mathbb{C}$ is bounded by, for instance, Demailly's theorem (Theorem 9.4), this follows from Theorem 15.4. □

Second Proof of Theorem 15.5. Let $Y \to X$ be a resolution of singularities of X. Note that, every birational morphism $X \dashrightarrow X$ induces a dominant rational map $Y \dashrightarrow X$. Since X has no rational curves (as X is Brody hyperbolic) and Y is smooth, by [50, Lemma 3.5], the rational map $Y \dashrightarrow X$ extends uniquely to a surjective morphism $Y \to X$.

Therefore, we have that

$$\mathrm{Bir}_{\mathbb{C}}(X) \subset \mathrm{Sur}_{\mathbb{C}}(Y, X)$$

Noguchi proved that the latter set is finite (see Theorem 16.1 below). He does so by showing that it is the set of $\mathbb{C}$-points on a finite type zero-dimensional scheme over $\mathbb{C}$. We discuss this result of Noguchi in more detail in the next section. □

It is important to note that, in light of Green–Griffiths' and Lang–Vojta's conjectures, one expects an analogous finiteness result for pseudo-Brody hyperbolic varieties (as pseudo-Brody hyperbolic varieties should be of general type). This is, however, not known, and we state it as a separate conjecture.

Conjecture 15.6 (Pseudo-Noguchi, I). *If X is a pseudo-Brody hyperbolic projective integral variety over $\mathbb{C}$, then $\mathrm{Bir}_{\mathbb{C}}(X)$ is finite.*

Remark 15.7 (What Do We Not Know Yet?). First, it is not known whether the automorphism group of a groupless projective variety is finite. Also, it is not known whether a pseudo-Kobayashi hyperbolic projective variety has a finite automorphism group. Moreover, it is not known whether

a geometric hyperbolic projective variety has only finitely many automorphisms. As these problems are unresolved, the finiteness of the set of birational self-maps is also still open.

16 Finiteness of Moduli Spaces of Surjective Morphisms

Our starting point in this section is the following finiteness theorem of Noguchi for dominant rational maps from a fixed variety to a hyperbolic variety (*formerly* a conjecture of Lang); see [59, §6.6] for a discussion of the history of this result.

Theorem 16.1 (Noguchi). *If X is a Brody hyperbolic proper variety over $\mathbb{C}$ and Y is a projective integral variety over $\mathbb{C}$, then the set of dominant rational maps $f : Y \dashrightarrow X$ is finite.*

In light of Lang–Vojta's conjecture, any "hyperbolic" variety should satisfy a similar finiteness property. In particular, one should expect similar (hence more general) results for bounded varieties, and such results are obtained in [50] over arbitrary algebraically closed fields k of characteristic zero.

Theorem 16.2. *If X is a 1-bounded projective variety over k and Y is a projective integral variety over k, then the set of dominant rational maps $f : Y \dashrightarrow X$ is finite.*

In particular, the same finiteness statement holds for bounded varieties and algebraically hyperbolic varieties. Indeed, such varieties are (obviously) 1-bounded.

Corollary 16.3. *If X is a bounded projective variety over k (e.g., algebraically hyperbolic variety over k) and Y is a projective integral variety over k, then the set of dominant rational maps $f : Y \dashrightarrow X$ is finite.*

We now make a "pseudo"-turn. In fact, the finiteness result of Noguchi should actually hold under the weaker assumption that X is only pseudo-Brody hyperbolic. To explain this, recall that Kobayashi–Ochiai proved a finiteness theorem for dominant rational maps from a given variety Y to a fixed variety of general type X which generalizes Matsumura's finiteness theorem for the group $\mathrm{Bir}_k(X)$ (Theorem 15.1).

Theorem 16.4 (Kobayashi–Ochiai). *Let X be a projective variety over k of general type. Then, for every projective integral variety Y, the set of dominant rational maps $f : Y \dashrightarrow X$ is finite.*

In light of Lang–Vojta's conjectures and Kobayashi–Ochiai's theorem, any "pseudo-hyperbolic" variety should satisfy a similar finiteness property. For

example, Lang–Vojta's conjecture predicts a similar finiteness statement for pseudo-Brody hyperbolic projective varieties. We state this as a conjecture. Note that this conjecture is the "pseudo"-version of Noguchi's theorem (Theorem 16.1), and clearly implies Conjecture 15.6.

Conjecture 16.5 (Pseudo-Noguchi, II). *If X is a pseudo-Brody hyperbolic proper variety over $\mathbb{C}$ and Y is a projective integral variety over $\mathbb{C}$, then the set of dominant rational maps $f : Y \dashrightarrow X$ is finite.*

Now, as any "pseudo-hyperbolic" variety is pseudo-groupless, it is natural to first try and see what one can say about pseudo-groupless varieties. For simplicity, we will focus on surjective morphisms (as opposed to dominant rational maps) in the rest of this section.

There is a standard approach to establishing the finiteness of the set of surjective morphisms from one projective scheme to another. To explain this, let us recall some notation from Section 10. Namely, if X and Y are projective schemes over k, we let $\underline{\mathrm{Hom}}_k(Y, X)$ be the scheme parametrizing morphisms $X \to Y$. Note that $\underline{\mathrm{Hom}}_k(Y, X)$ is a countable disjoint union of quasi-projective schemes over k. Moreover, we let $\underline{\mathrm{Sur}}_k(Y, X)$ be the scheme parametrizing surjective morphisms $Y \to X$, and note that $\underline{\mathrm{Sur}}_k(Y, X)$ is a closed subscheme of $\underline{\mathrm{Hom}}_k(Y, X)$.

The standard approach to establishing the finiteness of the set $\mathrm{Sur}_k(Y, X)$ is to interpret it as the set of k-points on the scheme $\underline{\mathrm{Sur}}_k(Y, X)$. This makes it tangible to techniques from deformation theory. Indeed, to show that $\mathrm{Sur}_k(Y, X)$ is finite, it suffices to establish the following two statements:

(1) The tangent space to each point of $\underline{\mathrm{Sur}}_k(Y, X)$ is trivial;
(2) The scheme $\underline{\mathrm{Sur}}_k(Y, X)$ has only finitely many connected components.

It is common to refer to the first statement as a rigidity statement, as it boils down to showing that the objects parametrized by $\underline{\mathrm{Sur}}_k(Y, X)$ are infinitesimally rigid. Also, it is standard to refer to the second statement as being a boundedness property. For example, if Y and X are curves and X is of genus at least two, the finiteness of $\mathrm{Sur}_k(Y, X)$ is proven precisely in this manner; see [70, §II.8]. We refer the reader to [60] for a further discussion of the rigidity/boundedness approach to proving finiteness results for other moduli spaces.

We now focus on the rigidity of surjective morphisms $Y \to X$. The following rigidity theorem for pseudo-groupless varieties will prove to be extremely useful. This result is a consequence of a much more general statement about the deformation space of a surjective morphism due to Hwang–Kebekus–Peternell [44].

Theorem 16.6 (Hwang–Kebekus–Peternell $+\ \epsilon$). *If Y is a projective normal variety over k and X is a pseudo-groupless projective variety over k, then the scheme $\underline{\mathrm{Sur}}_k(Y, X)$ is a countable disjoint union of zero-dimensional smooth projective schemes over k.*

Proof. As is shown in [56], this is a consequence of Hwang–Kebekus–Peternell's result on the infinitesimal deformations of a surjective morphism $Y \to X$. Indeed, since X is non-uniruled (Remark 6.7), for every such surjective morphism $f : Y \to X$, there is a finite morphism $Z \to X$ and a morphism $Y \to Z$ such that f is the composed map $Y \to Z \to X$. Moreover, the identity component $\mathrm{Aut}^0_{Z/k}$ of the automorphism group scheme surjects onto the connected component of f in $\underline{\mathrm{Hom}}_k(Y, X)$. Since X is pseudo-groupless, the same holds for Z. It is then not hard to verify that $\mathrm{Aut}^0_{Z/k}$ is trivial, so that the connected component of f in $\underline{\mathrm{Hom}}_k(Y, X)$ is trivial.   □

Remark 16.7. There are projective varieties X which are **not** pseudo-groupless over k, but for which the conclusion of the theorem above still holds. For example, a K3 surface or the blow-up of a simple abelian surface A in its origin. This means that the rigidity of surjective morphisms follows from properties strictly *weaker* than pseudo-hyperbolicity. We refer to [56] for a more general statement concerning rigidity of surjective morphisms.

When introducing the notions appearing in Lang–Vojta's conjecture, we made sure to emphasize that every one of these is pseudo-groupless. Thus, roughly speaking, any property we prove for pseudo-groupless varieties holds for all pseudo-hyperbolic varieties. This gives us the following rigidity statement.

Corollary 16.8 (Rigidity for Pseudo-Hyperbolic Varieties). *Let X be a projective integral variety over k and let Y be a projective normal variety over k. Assume that one of the following statements holds.*

(1) The variety X is pseudo-groupless over k.
(2) The variety X is pseudo-Mordellic over k.
(3) The projective variety X is pseudo-algebraically hyperbolic over k.
(4) The projective variety X is pseudo-1-bounded over k.
(5) The projective variety X is pseudo-bounded over k.
(6) The variety X is pseudo-geometrically hyperbolic over k.
(7) The field k equals $\mathbb{C}$ and X is pseudo-Brody hyperbolic.

Then the scheme $\underline{\mathrm{Sur}}_k(Y, X)$ is a countable disjoint union of zero-dimensional smooth projective schemes over k.

Proof. Assume that either (1)–(6) or (7) holds. Then X is pseudo-groupless (as explained throughout these notes), so that the result follows from Theorem 16.6.   □

Proving the finiteness of $\mathrm{Sur}_k(Y, X)$ or, equivalently, the boundedness of $\underline{\mathrm{Sur}}_k(Y, X)$, for X pseudo-groupless or pseudo-Mordellic seems to be out of reach currently. However, for pseudo-algebraically hyperbolic varieties the desired finiteness property is proven in [56] and reads as follows.

Theorem 16.9. *If X is a pseudo-algebraically hyperbolic projective variety over k and Y is a projective integral variety over k, then the set of surjective morphisms $f : Y \to X$ is finite.*

A similar result can be obtained for pseudo-bounded varieties. The precise result can be stated as follows.

Theorem 16.10. *If X is a pseudo-bounded projective variety over k and Y is a projective integral variety over k, then the set of surjective morphisms $f : Y \to X$ is finite.*

To prove the analogous finiteness property for pseudo-1-bounded varieties, we require (as in the previous section) an additional uncountability assumption on the base field.

Theorem 16.11. *Assume k is uncountable. If X is a pseudo-1-bounded projective variety over k and Y is a projective integral variety over k, then the set of surjective morphisms $f : Y \to X$ is finite.*

We conclude with the following finiteness result for pseudo-algebraically hyperbolic varieties. It is proven in [56] using (essentially) the results in this section and the fact that pseudo-algebraically hyperbolic varieties are pseudo-geometrically hyperbolic.

Theorem 16.12. *If X is algebraically hyperbolic modulo Δ over k, then for every connected reduced projective variety Y over k, every non-empty closed reduced subset $B \subset Y$, and every reduced closed subset $A \subset X$ not contained in Δ, the set of morphisms $f : Y \to X$ with $f(B) = A$ is finite.*

Note that Theorem 16.12 can be applied with B a point or $B = Y$. This shows that the statement generalizes the finiteness result of this section.

17 Hyperbolicity Along Field Extensions

In this section we study how different notions of pseudo-hyperbolicity appearing in Lang–Vojta's conjectures (except for those that only make sense over $\mathbb{C}$ a priori) behave under field extensions. In other words, we study how the exceptional locus for each notion of hyperbolicity introduced in Section 12 behaves under field extensions.

Let us start with X a variety of general type over a field k, and let $k \subset L$ be a field extension. It is natural to wonder whether X_L is also of general type over L. A simple argument comparing the spaces of global sections of $\omega_{X/k}$ and $\omega_{X_L/L}$ shows that this is indeed the case. This observation is our starting point in this section. Indeed, the mere fact that varieties of general type remain varieties of general type after a field extension can be paired

with the Strong Lang–Vojta conjecture to see that similar statements should hold for pseudo-groupless varieties, pseudo-Mordellic varieties, and so on.

The first three results we state in this section say that this "base-change" property can be proven in some cases. For proofs we refer to [15, 50, 56].

Theorem 17.1. *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero. Let X be a projective scheme over k and let Δ be a closed subset of X. Then the following statements hold.*

(1) If X is of general type over k, then X_L is of general type over L.
(2) If X is groupless modulo Δ, then X_L is groupless modulo Δ_L.
(3) If X is algebraically hyperbolic modulo Δ, then X_L is algebraically hyperbolic modulo Δ_L.
(4) If X is bounded modulo Δ, then X_L is bounded modulo Δ_L.

In this theorem we are missing (among others) the notions of 1-boundedness and geometric hyperbolicity. In this direction we have the following result; see [15, 46].

Theorem 17.2. *Let $k \subset L$ be an extension of **uncountable** algebraically closed fields of characteristic zero. Let X be a projective scheme over k and let Δ be a closed subset of X. Then the following statements hold.*

(1) If X is 1-bounded modulo Δ, then X_L is bounded modulo Δ_L.
(2) If X is geometrically hyperbolic modulo Δ, then X_L is geometrically hyperbolic modulo Δ_L.

If $\Delta = \emptyset$, then we do not need to impose uncountability.

Theorem 17.3. *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero. Let X be a projective scheme over k and let Δ be a closed subset of X. Then the following statements hold.*

(1) If X is 1-bounded, then X_L is bounded.
(2) If X is geometrically hyperbolic, then X_L is geometrically hyperbolic.

The reader will have noticed the absence of the notion of Mordellicity and arithmetic hyperbolicity above. The question of whether an arithmetically hyperbolic variety over $\overline{\mathbb{Q}}$ remains arithmetically hyperbolic over a larger field is not an easy one in general, as should be clear from the following remark.

Remark 17.4 (Persistence of Arithmetic Hyperbolicity). Let $f_1, \ldots, f_r \in \mathbb{Z}[x_1, \ldots, x_n]$ be polynomials, and let $X := Z(f_1, \ldots, f_r) = \mathrm{Spec}(\overline{\mathbb{Q}}[x_1, \ldots, x_n]/(f_1, \ldots, f_r)) \subset \mathbb{A}^n_{\overline{\mathbb{Q}}}$ be the associated affine variety over $\overline{\mathbb{Q}}$. To say that X is arithmetically hyperbolic over $\overline{\mathbb{Q}}$ is to say that, for every *number field* K and every finite set of finite places S of K, the set of $a = (a_1, \ldots, a_n) \in \mathcal{O}^n_{K,S}$ with $f_1(a) = \ldots = f_r(a) = 0$ is finite. On the other hand, to say that $X_\mathbb{C}$ is arithmetically hyperbolic over $\mathbb{C}$ is to say that, for

every $\mathbb{Z}$-*finitely generated* subring $A \subset \mathbb{C}$, the set of $a = (a_1, \ldots, a_n) \in A^n$ with $f_1(a) = \ldots = f_r(a) = 0$ is finite.

Despite the apparent difference between being arithmetically hyperbolic over $\overline{\mathbb{Q}}$ and being arithmetically hyperbolic over $\mathbb{C}$, it seems reasonable to suspect their equivalence. For X a projective variety, the following conjecture is a consequence of the Weak Lang–Vojta conjecture for X. However, as it also seems reasonable in the non-projective case, we state it in this more generality.

Conjecture 17.5 (Persistence Conjecture). *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero. Let X be a variety over k and let Δ be a closed subset of X. If X is arithmetically hyperbolic modulo Δ over k, then X_L is arithmetically hyperbolic modulo Δ_L over L.*

Note that we will focus throughout on arithmetic hyperbolicity (as opposed to Mordellicity) as its persistence along field extensions is easier to study. The reader may recall that the difference between Mordellicity and arithmetic hyperbolicity disappears for many varieties (e.g., affine varieties); see Section 7.3 for a discussion.

This conjecture is investigated in [15, 46, 49, 52]. As a basic example, the reader may note that Faltings proved that a smooth projective connected curve of genus at least two over $\overline{\mathbb{Q}}$ is arithmetically hyperbolic over $\overline{\mathbb{Q}}$ in [32]. He then later explained in [33] that Grauert–Manin's function field version of the Mordell conjecture can be used to prove that a smooth projective connected curve of genus at least two over k is arithmetically hyperbolic over k.

In the rest of this section, we will present some results on the Persistence Conjecture. We start with the following result.

Theorem 17.6. *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero. Let X be an arithmetically hyperbolic variety over k such that X_L is geometrically hyperbolic over L. Then X_L is arithmetically hyperbolic over L.*

Note that Theorem 17.6 implies that the Persistence Conjecture holds for varieties over k which are geometrically hyperbolic over any field extension of L.

Theorem 17.6 is inspired by Martin-Deschamps's proof of the arithmetic Shafarevich conjecture over finitely generated fields (see also Remark 7.34). Indeed, in Szpiro's seminar [82], Martin-Deschamps gave a proof of the arithmetic Shafarevich conjecture by using a specialization argument on the moduli stack of principally polarized abelian schemes; see [68]. This specialization argument resides on Faltings's theorem that the moduli space of principally polarized abelian varieties of fixed dimension over $\mathbb{C}$ is geometrically hyperbolic over $\mathbb{C}$. We note that Theorem 17.6 is essentially implicit in her line of reasoning.

We will present applications of Theorem 17.6 to the Persistence Conjecture based on the results obtained in [49, 52]. However, before we give these applications, we mention the following result which implies that the Persistence Conjecture holds for normal projective surfaces with non-zero irregularity $h^1(X, \mathcal{O}_X)$.

Theorem 17.7. *Let X be a projective surface over k which admits a non-constant morphism to some abelian variety over k. Then X is arithmetically hyperbolic over k if and only if X_L is arithmetically hyperbolic over L.*

The proofs of Theorems 17.6 and 17.7 differ tremendously in spirit. In fact, we cannot prove Theorem 17.7 by appealing to the geometric hyperbolicity of X (as it is currently not known whether an arithmetically hyperbolic projective surface which admits a non-constant map to an abelian variety is geometrically hyperbolic). Instead, Theorem 17.7 is proven by appealing to the "mild boundedness" of abelian varieties; see [15]. More explicitly: in the proof of Theorem 17.7, we use that, for every smooth connected curve C over k, there exists an integer $n > 0$ and points $c_1, \ldots, c_n$ in $C(k)$ such that, for every abelian variety A over k and every $a_1, \ldots, a_n$ in $A(k)$, the set

$$\mathrm{Hom}_k((C, c_1, \ldots, c_n), (A, a_1, \ldots, a_n))$$

is finite. This finiteness property for abelian varieties can be combined with the arithmetic hyperbolicity of the surface X in Theorem 17.7 to show that the surface X is mildly bounded. The property of being mildly bounded is clearly *much* weaker than being geometrically hyperbolic, but it turns out to be enough to show the Persistence Conjecture; see [49, §4.1]. Note that it is a bit surprising that abelian varieties (as they are very far from being hyperbolic) satisfy some "mild" version of geometric hyperbolicity. We refer the reader to [49, §4] for the definition of what this notion entails, and to [15] for the fact that abelian varieties are mildly bounded.

We now focus as promised on the applications of Theorem 17.6. Our first application says that the Persistence Conjecture holds for all algebraically hyperbolic projective varieties.

Theorem 17.8. *Let X be a projective algebraically hyperbolic variety over k. Then X is arithmetically hyperbolic over k if and only if, for every algebraically closed field L containing k, the variety X_L is arithmetically hyperbolic over L.*

Proof. Since X is algebraically hyperbolic over k, it follows from (3) in Theorem 17.1 that X_L is algebraically hyperbolic over L. Since algebraically hyperbolic projective varieties are 1-bounded and thus geometrically hyperbolic (Corollary 11.5), the result follows from Theorem 17.6. $\square$

Our second application involves integral points on the moduli space of smooth hypersurfaces. We present the results obtained in [52] in the following section.

17.1   The Shafarevich Conjecture for Smooth Hypersurfaces

We explain in this section how Theorem 17.6 can be used to show the following finiteness theorem. This explanation will naturally lead us to studying integral points on moduli spaces.

Theorem 17.9. *Let $d \geq 3$ and $n \geq 2$ be integers. Assume that, for every number field K and every finite set of finite places S of K, the set of $\mathcal{O}_{K,S}$-isomorphism classes of smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}_{\mathcal{O}_{K,S}}$ is finite. Then, for every $\mathbb{Z}$-finitely generated normal integral domain A of characteristic zero, the set of A-isomorphism classes of smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}_A$ is finite.*

To prove Theorem 17.9, we (i) reformulate its statement in terms of the arithmetic hyperbolicity of an appropriate moduli space of smooth hypersurfaces, (ii) establish the geometric hyperbolicity of this moduli space, and (iii) apply Theorem 17.6. Indeed, the assumption in Theorem 17.9 can be formulated as saying that the (appropriate) moduli space of hypersurfaces is arithmetically hyperbolic over $\overline{\mathbb{Q}}$ and the conclusion of our theorem is then that this moduli space is also arithmetically hyperbolic over larger fields. To make these statements more precise, let $\mathrm{Hilb}_{d,n}$ be the Hilbert scheme of smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}$. Note that $\mathrm{Hilb}_{d,n}$ is a smooth affine scheme over $\mathbb{Z}$. There is a natural action of the automorphism group scheme PGL_{n+2} of $\mathbb{P}^{n+1}_{\mathbb{Z}}$ on $\mathrm{Hilb}_{d,n}$. Indeed, given a smooth hypersurface H in $\mathbb{P}^{n+1}$ and an automorphism σ of $\mathbb{P}^{n+1}$, the resulting hypersurface $\sigma(H)$ is again smooth.

The quotient of a smooth affine scheme over $\mathbb{Z}$ by a reductive group (such as PGL_{n+2}) is an affine scheme of finite type over $\mathbb{Z}$ by Mumford's GIT. However, for the study of hyperbolicity and integral points, this quotient scheme is not very helpful, as the action of PGL_{n+2} on $\mathrm{Hilb}_{d,n}$ is not free. The natural solution it to consider the stacky quotient, as in [13, 14, 47]. However, one may avoid the use of stacks by adding level structure as in [48]. Indeed, by [48], there exists a smooth affine variety H' over $\mathbb{Q}$ with a free action by $\mathrm{PGL}_{n+2,\mathbb{Q}}$, and a finite étale $\mathrm{PGL}_{n+2,\mathbb{Q}}$-equivariant morphism $H' \to \mathrm{Hilb}_{d,n,\mathbb{Q}}$. Let $U_{d;n} := \mathrm{PGL}_{n+2,\mathbb{Q}} \backslash H'$ be the smooth affine quotient scheme over $\mathbb{Q}$. To prove Theorem 17.9, we establish the following result.

Theorem 17.10. *Let $d \geq 3$ and $n \geq 2$ be integers. Assume that $U_{d;n,\overline{\mathbb{Q}}}$ is arithmetically hyperbolic over $\overline{\mathbb{Q}}$. Then, for every algebraically closed field k*

of characteristic zero, the affine variety $U_{d;n,k}$ is arithmetically hyperbolic over k.

Proof. Let us write $U := U_{d;n,\overline{\mathbb{Q}}}$. The proof relies on a bit of Hodge theory. Indeed, we use Deligne's finiteness theorem for monodromy representations [28], the infinitesimal Torelli property for smooth hypersurfaces [35], and the Theorem of the Fixed Part in Hodge theory [80] to show that U_k is geometrically hyperbolic over k. Then, as U_k is geometrically hyperbolic over k, the result follows from Theorem 17.6. □

We now explain how to deduce Theorem 17.9 from Theorem 17.10.

Proof of Theorem 17.9. Write $U := U_{d;n,\overline{\mathbb{Q}}}$. First, the assumption in Theorem 17.9 can be used to show that U is arithmetically hyperbolic over $\overline{\mathbb{Q}}$. Then, since U is arithmetically hyperbolic over $\overline{\mathbb{Q}}$, it follows from Theorem 17.10 that U_k is arithmetically hyperbolic for every algebraically closed field k of characteristic zero. Finally, to conclude the proof, let us recall that arithmetic hyperbolicity descends along finite étale morphisms of varieties (Remark 3.5). In [53], the analogous descent statement is proven for finite étale morphisms of algebraic stacks, after extending the notion of arithmetic hyperbolicity from schemes to stacks. Thus, by applying this "stacky" Chevalley–Weil theorem to the finite étale morphism $U_{d;n,k} \to [\mathrm{PGL}_{n+2,k}\backslash\mathrm{Hilb}_{d,n,k}]$ of stacks, where $[\mathrm{PGL}_{n+2,k}\backslash\mathrm{Hilb}_{d,n,k}]$ denotes the quotient stack, we obtain that the stack $[\mathrm{PGL}_{n+2,k}\backslash\mathrm{Hilb}_{d,n,k}]$ is arithmetically hyperbolic over k. Finally, the moduli-interpretation of the points of this quotient stack can be used to see that, for every $\mathbb{Z}$-finitely generated normal integral domain A of characteristic zero, the set of A-isomorphism classes of smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}_{\mathcal{O}_{K,S}}$ is finite. This concludes the proof. □

Remark 17.11 (Period Domains). Theorem 17.10 actually follows from a more general statement about varieties with a quasi-finite period map (e.g., Shimura varieties). Namely, in [52] it is shown that a complex algebraic variety with a quasi-finite period map is geometrically hyperbolic. For other results about period domains we refer the reader to the article of Bakker–Tsimerman in this book [12].

18   Lang's Question on Openness of Hyperbolicity

It is obvious that being hyperbolic is not stable under specialization. In fact, being pseudo-groupless is not stable under specialization, as a smooth proper curve of genus two can specialize to a tree of $\mathbb{P}^1$'s. Nonetheless, it seems reasonable to suspect that being hyperbolic (resp. pseudo-hyperbolic) is in fact stable under generization. The aim of this section is to investigate this property for all notions of hyperbolicity discussed in these notes. In fact,

on [62, p. 176] Lang says "I do not clearly understand the extent to which hyperbolicity is open for the Zariski topology". This brings us to the following question of Lang and our starting point of this section.

Question 1 (Lang). Let S be a noetherian scheme over $\mathbb{Q}$ and let $X \to S$ be a projective morphism. Is the set of s in S such that $X_{\overline{k(s)}}$ is groupless a Zariski open subscheme of S?

Here we let $k(s)$ denote the residue field of the point s, and we let $k(s) \to \overline{k(s)}$ be an algebraic closure of $k(s)$. Note that one can ask similar questions for the set of s in S such that $X_{\overline{k(s)}}$ is algebraically hyperbolic or arithmetically hyperbolic, respectively.

Before we discuss what one may expect regarding Lang's question, let us recall what it means for a subset of a scheme to be a Zariski-countable open.

If $(X, \mathcal{T})$ is a noetherian topological space, then there exists another topology $\mathcal{T}^{\mathrm{cnt}}$, or $\mathcal{T}$-countable, on X whose closed sets are the countable union of $\mathcal{T}$-closed sets. If S is a noetherian scheme, a subset $Z \subset S$ is a Zariski-countable closed if it is a countable union of closed subschemes $Z_1, Z_2, \ldots \subset S$.

Remark 18.1 (What to Expect? I). We will explain below that the locus of s in S such that X_s is groupless is a Zariski-countable open of S, i.e., its complement is a countable union of closed subschemes. In fact, we will show similar statements for algebraic hyperbolicity, boundedness, geometric hyperbolicity, and the property of having only subvarieties of general type. Although this provides some indication that the answer to Lang's question might be positive, it is not so clear whether one should expect a positive answer to Lang's question. However, it seems plausible that, assuming the Strongest Lang–Vojta conjecture (Conjecture 12.1), one can use certain Correlation Theorems (see Ascher–Turchet [8]) to show that the answer to Lang's question is positive.

One can also ask about the pseudofied version of Lang's question.

Question 2 (Pseudo-Lang). Let S be a noetherian scheme over $\mathbb{Q}$ and let $X \to S$ be a projective morphism. Is the set of s in S such that $X_{\overline{k(s)}}$ is pseudo-groupless a Zariski open subscheme of S?

Again, one can ask similar questions for the set of s in S such that $X_{\overline{k(s)}}$ is pseudo-algebraically hyperbolic or pseudo-arithmetically hyperbolic, respectively.

Remark 18.2 (What to Expect? II). We will argue below that one should expect (in light of the Strong Lang–Vojta conjecture) that the answer to the Pseudo-Lang question is positive. This is because of a theorem of Siu–Kawamata–Nakayama on invariance of plurigenera.

What do we know about the above questions (Questions 1 and 2)? The strongest results we dispose of are due to Nakayama; see [74, Chapter VI.4].

In fact, the following theorem can be deduced from Nakayama's [74, Theorem VI.4.3]. (Nakayama's theorem is a generalization of theorems of earlier theorems of Siu and Kawamata on invariance of plurigenera.)

Theorem 18.3 (Siu, Kawamata, Nakayama). *Let S be a noetherian scheme over $\mathbb{Q}$ and let $X \to S$ be a projective morphism of schemes. Then, the set of s in S such that X_s is of general type is open in S.*

Thus, by Theorem 18.3, assuming the Strong Lang–Vojta conjecture (Conjecture 12.1), the answer to the Pseudo-Lang question should be positive. Also, assuming the Strong Lang–Vojta conjecture, the set of s in S such that $X_{\overline{k(s)}}$ is pseudo-algebraically hyperbolic should be open. Similar statements should hold for pseudo-Mordellicity and pseudo-boundedness. Although neither of these statements are known, some partial results are obtained in [15, §9].

In fact, as a consequence of Nakayama's theorem and the fact that the stack of proper schemes of general type is a countable union of finitely presented algebraic stacks, one can prove the following result.

Theorem 18.4 (Countable-Openness of Every Subvariety Being of General Type). *Let S be a noetherian scheme over $\mathbb{Q}$ and let $X \to S$ be a projective morphism. Then, the set of s in S such that every integral closed subvariety of X_s is of general type is Zariski-countable open in S.*

The countable-openness of the locus of every subvariety being of general type does not give a satisfying answer to Lang's question. However, it does suggest that every notion appearing in the Lang–Vojta conjecture should be Zariski-countable open. This expectation can be shown to hold for some notions of hyperbolicity. For example, given a projective morphism of schemes $X \to S$ with S a complex algebraic variety, one can show that the locus of s in S such that X_s is algebraically hyperbolic is an open subset of S in the countable-Zariski topology; see [15, 29]. This result is essentially due to Demailly.

Theorem 18.5. *Let S be a noetherian scheme over $\mathbb{Q}$ and let $X \to S$ be a projective morphism. Then, the set of s in S such that X_s is algebraically hyperbolic is Zariski-countable open in S.*

It is worth noting that this is not the exact result proven by Demailly, as it brings us to a subtle difference between the Zariski-countable topology on a variety X over $\mathbb{C}$ and the induced topology on $X(\mathbb{C})$. Indeed, Demailly proved that, if $k = \mathbb{C}$ and $S^{\text{not-ah}}$ is the set of s in S such that X_s is not algebraically hyperbolic, then $S^{\text{not-ah}} \cap S(\mathbb{C})$ is closed in the countable topology on $S(\mathbb{C})$. This, strictly speaking, does not imply that $S^{\text{not-ah}}$ is closed in the countable topology on S. For example, if S is an integral curve over $\mathbb{C}$ and η is the generic point of S, then $\{\eta\}$ is not a Zariski-countable open of S, whereas $\{\eta\} \cap S(\mathbb{C}) = \emptyset$ is a Zariski-countable open of $S(\mathbb{C})$.

In [15] similar results are obtained for boundedness and geometric hyperbolicity. The precise statements read as follows.

Theorem 18.6 (Countable-Openness of Boundedness). *Let S be a noetherian scheme over $\mathbb{Q}$ and let $X \to S$ be a projective morphism. Then, the set of s in S such that X_s is bounded is Zariski-countable open in S.*

Theorem 18.7 (Countable-Openness of Geometric Hyperbolicity). *Let S be a noetherian scheme over $\mathbb{Q}$ and let $X \to S$ be a projective morphism. Then, the set of s in S such that X_s is geometrically hyperbolic is Zariski-countable open in S.*

Remark 18.8 (What Goes into the Proofs of Theorems 18.5, 18.6, and 18.7?). The main idea behind all these proofs is quite simple. Let us consider Theorem 18.5. First, one shows that the set of s in S such that X_s is *not* algebraically hyperbolic is the image of countably many constructible subsets of S. This is essentially a consequence of the fact that the Hom-scheme between two projective schemes is a countable union of quasi-projective schemes. Then, it suffices to note that the set of s in S with X_s algebraically hyperbolic is stable under generization. This relies on compactness properties of the moduli stack of stable curves.

Concerning Lang's question on the locus of groupless varieties, we note that in [55] it is shown that the set of s in S such that X_s is groupless is open in the Zariski-countable topology on S.

Theorem 18.9 (Countable-Openness of Grouplessness). *Let S be a noetherian scheme over $\mathbb{Q}$ and let $X \to S$ be a projective morphism. Then, the set of s in S such that $X_{\overline{k(s)}}$ is groupless is Zariski-countable open in S.*

We finish these notes with a discussion of the proof of Theorem 18.9. It will naturally lead us to introducing a non-archimedean counterpart to Lang–Vojta's conjecture.

18.1   Non-archimedean Hyperbolicity and Theorem 18.9

Let S be a noetherian scheme over $\mathbb{Q}$ and let $X \to S$ be a projective morphism. Define S^{n-gr} to be the set of s in S such that $X_{\overline{k(s)}}$ is not groupless. Our goal is to prove Theorem 18.9, i.e., to show that S^{n-gr} is Zariski-countable closed, following the arguments of [55]. As is explained in Remark 18.8, we prove this in two steps.

First, one shows that S^{n-gr} is a countable union of constructible subsets. This step relies on some standard moduli-theoretic techniques. Basically, to say that X is not groupless over k is equivalent to saying that, there is an

integer g such that the Hom-stack $\underline{\mathrm{Hom}}_{\mathcal{A}_g}(\mathcal{U}_g, X \times \mathcal{A}_g) \to \mathcal{A}_g$ has a non-empty fibre over some k-point of $\mathcal{A}_g$, where $\mathcal{A}_g$ is the stack of principally polarized g-dimensional abelian schemes over k, and $\mathcal{U}_g \to \mathcal{A}_g$ is the universal family. We will not discuss this argument and refer the reader to [55] for details on this part of the proof.

Once the first step is completed, to conclude the proof, it suffices to show that the notion of being groupless is stable under generization. To explain how to do this, we introduce a new notion of hyperbolicity for rigid analytic varieties (and also adic spaces) over a non-archimedean field K of characteristic zero; see [55, §2]. This notion is inspired by the earlier work of Cherry [20] (see also [6, 21, 22, 66, 67]).

If K is a complete algebraically closed non-archimedean valued field of characteristic zero and X is a finite type scheme over K, we let X^{an} be the associated rigid analytic variety over K. We say that a variety over K is K-*analytically Brody hyperbolic* if, for every finite type connected group scheme G over K, every morphism $G^{an} \to X^{an}$ is constant. It follows from this definition that a K-analytically Brody hyperbolic variety is groupless. It seems reasonable to suspect that the converse of this statement holds for projective varieties.

Conjecture 18.10 (Non-archimedean Lang–Vojta). *Let K be an algebraically closed complete non-archimedean valued field of characteristic zero, and let X be an integral projective variety over K. If X is groupless over K, then X is K-analytically Brody hyperbolic.*

In [20] Cherry proves this conjecture for closed subvarieties of abelian varieties. That is, Cherry proved the non-archimedean analogue of the Bloch–Ochiai–Kawamata theorem (Theorem 2.5) for closed subvarieties of abelian varieties.

In [55] it is shown that the above conjecture holds for projective varieties over a non-archimedean field K, assuming that K is of equicharacteristic zero and X is a "constant" variety over K (i.e., can be defined over the residue field of K). This actually follows from the following more general result.

Theorem 18.11. *Let K be an algebraically closed complete non-archimedean valued field of equicharacteristic zero with valuation ring $\mathcal{O}_K$, and let $\mathcal{X} \to \mathrm{Spec}\,\mathcal{O}_K$ be a proper flat morphism of schemes. If the special fibre $\mathcal{X}_0$ of $\mathcal{X} \to \mathrm{Spec}\,\mathcal{O}_K$ is groupless, then the generic fibre $\mathcal{X}_K$ is K-analytically Brody hyperbolic.*

Proof. This is the main result of [55] and is proven in three steps. Write $X := \mathcal{X}_K$.

First, one shows that every morphism $\mathbb{G}_{m,K}^{\mathrm{an}} \to X^{\mathrm{an}}$ is constant by considering the "reduction" map $X^{\mathrm{an}} \to X_0$ and a careful analysis of the residue fields of points in the image of composed map $\mathbb{G}_{m,K}^{\mathrm{an}} \to X^{\mathrm{an}} \to \mathcal{X}_0$; see [55, §5] for details. This implies that X has no rational curves.

Now, one wants to show that every morphism $A^{\mathrm{an}} \to X^{\mathrm{an}}$ with A some abelian variety over K is constant. Instead of appealing to GAGA and trying to use algebraic arguments, we appeal to the uniformization theorem of Bosch–Lütkebohmert for abelian varieties. This allows us to reduce to the case that A has good reduction over $\mathcal{O}_K$. In this reduction step we use that every morphism $\mathbb{G}_{m,K}^{\mathrm{an}} \to X^{\mathrm{an}}$ is constant (which is what we established in the first part of this proof); we refer the reader to [55, Theorem 2.18] for details.

Thus, we have reduced to showing that, for A an abelian variety over K *with good reduction over* $\mathcal{O}_K$, every morphism $A^{\mathrm{an}} \to X^{\mathrm{an}}$ is constant. To do so, as A has good reduction over $\mathcal{O}_K$, we may let $\mathcal{A}$ be a smooth proper model for A over $\mathcal{O}_K$. Note that the non-constant morphism $A^{\mathrm{an}} \to X^{\mathrm{an}}$ over K algebraizes by GAGA, i.e., it is the analytification of a non-constant morphism $A \to X$. By the valuative criterion of properness, there is a dense open $U \subset \mathcal{A}$ whose complement is of codimension at least two and a morphism $U \to \mathcal{X}$ extending the morphism $A \to X$ on the generic fibre. Now, since $\mathcal{X}_0$ is groupless, it has no rational curves. In particular, as $\mathcal{A} \to \operatorname{Spec} \mathcal{O}_K$ is smooth, the morphism $U \to \mathcal{X}$ extends to a morphism $\mathcal{A} \to \mathcal{X}$ by [37, Proposition 6.2]. However, since $\mathcal{X}_0$ is groupless, this morphism is constant on the special fibre. The latter implies (as $\mathcal{A} \to \operatorname{Spec} \mathcal{O}_K$ is proper) that the morphism on the generic fibre is constant; see [55, §3.2] for details. We have shown that, for every abelian variety A over K, every morphism $A \to X$ is constant and that every morphism $\mathbb{G}_m^{\mathrm{an}} \to X^{\mathrm{an}}$ is constant.

Finally, by adapting the proof of Lemma 4.4 one can show that the above implies that, for every finite type connected group scheme G over K, every morphism $G^{\mathrm{an}} \to X^{\mathrm{an}}$ is constant, so that X is K-analytically Brody hyperbolic (see [55, Lemma 2.14] for details). □

To conclude the proof of Theorem 18.9, we point out that a straightforward application of Theorem 18.11 shows that being groupless is stable under generization, as required. □

An important problem in the study of non-archimedean hyperbolicity at this moment is finding a "correct" analogue of the Kobayashi pseudo-metric (if there is any at all). Cherry defined an analogue of the Kobayashi metric but it does not have the right properties, as he showed in [21] (see also [55, §3.5]). A "correct" analogue of the Kobayashi metric in the non-archimedean context would most likely have formidable consequences. Indeed, it seems reasonable to suspect that a K-analytic Brody hyperbolic projective variety is in fact "Kobayashi hyperbolic" over K and that "Kobayashi hyperbolic" projective varieties over K are bounded over K by some version of the Arzelà–Ascoli theorem.

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