Chapter 20
Numerical Evaluation of Highly Oscillatory Integrals of Arbitrary Function Using Gauss-Legendre Quadrature Rule

K. T. Shivaram and H. T. Prakasha

20.1 Introduction

The numerical integration of a highly oscillating function is one of the most difficult parts for solving applied problems in signal processing, image analysis, electrodynamics, quantum mechanics, fluid dynamics, Fourier transforms, plasma transport, Bose-Einstein condensates, etc. Analytical or numerical calculation of these integrals are difficult when the parameter Ω is increased, In most of the cases, lower-order quadrature methods are failures such as trapezoidal rule, Simpson's rule, etc. The numerical quadrature method for oscillatory integrals was first implemented by Louis Napoleon George Filon [1]; Filon-type methods show the efficiently computing aspect of the Fourier integral computation of moments where something other than x is itself a difficult task. Levin and Sidi [2] evaluate the first few oscillations of integrand using a standard process, David Levin [3]. the modified method that does not require the calculation of the moment. Iserles [4] developed a similar method by the use of higher-order derivatives of the integrand. Evans and Chung [5] proposed a numerical integration method for computing the oscillatory integrals; recently Ihsan Hascelik [6] evaluate the numerical integrals with integrands of the form on 0, 1. by n-point Gauss rule of three-term recurrence relation method. The integration rule proposed in this paper requires the zeros of P_{2n} (x) and computed associated weights. The integration points are increased in order to improve the accuracy of the numerical solution. The reminder of this paper is presented as follows. In Sect. 20.1, mathematical preliminaries are required for the understanding concept of the derivation and also calculated Gauss-Legendre

K. T. Shivaram (✉) · H. T. Prakasha
Department of Mathematics, Dayananda Sagar College of Engineering, Bangalore, India

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quadrature sampling points and its weights of order N = 20, 50, 100. Section 20.2 provides the mathematical formulas and illustrations with numerical examples (Fig. 20.1).

20.2 Gauss-Legendre Quadrature Formula over Oscillating Function

If $\omega = 1, r = 2$, numerical integration of an arbitrary function f is described as

$$\begin{aligned} I1 &= \int_0^1 f(x)\cos\left(\frac{1}{x^2}\right)dx = \int_0^1 f\left(\sqrt{t}\right)\cos\left(\frac{1}{t}\right)\frac{dt}{2\sqrt{t}} \\ &= \sum_{i=0}^{m} w_i \frac{1}{2} f\left(\sqrt{t_i}\right)\cos\left(\frac{1}{t_i}\right)\frac{1}{\sqrt{t_i}} \\ &= \sum_{i=0}^{m} w_i \frac{1}{2} f\left(\sqrt{x_i}\right)\cos\left(\frac{1}{x_i}\right)\frac{1}{\sqrt{x_i}} \end{aligned} \tag{20.1}$$

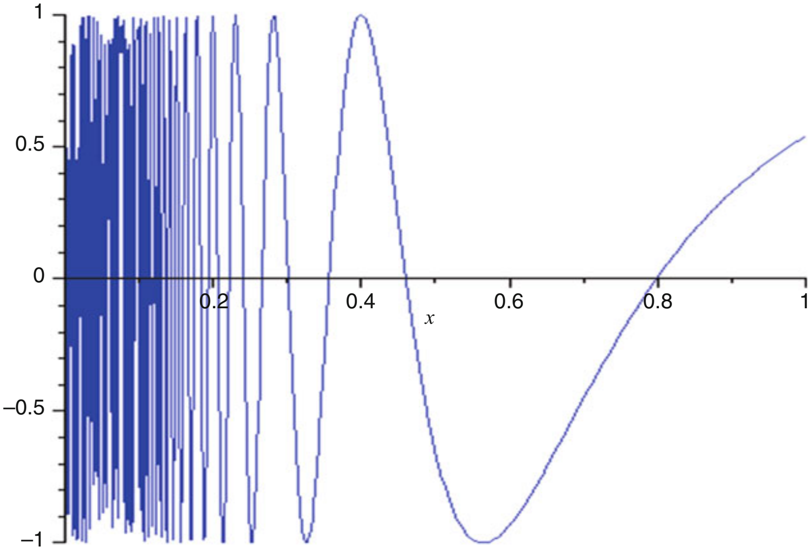
If $\omega = 2, r = 200$, numerical integration of an arbitrary function f is described as

$$\begin{aligned} I_2 &= \int_0^1 f(x)\cos\left(\frac{2}{x^{200}}\right)dx = \int_0^1 f\left(t^{\frac{1}{200}}\right)\cos\left(\frac{2}{t}\right)\frac{t^{\frac{-199}{200}}dt}{200} \\ &= \sum_{i=0}^{m} w_k f\left(t_i^{\frac{1}{200}}\right)\cos\left(\frac{2}{t_i}\right)\frac{t^{\frac{-199}{200}}}{200} = \sum_{i=0}^{m} w_i f\left(x_i^{\frac{1}{200}}\right)\cos\left(\frac{2}{x_i}\right)\frac{x_i^{\frac{-199}{200}}}{200} \end{aligned} \tag{20.2}$$

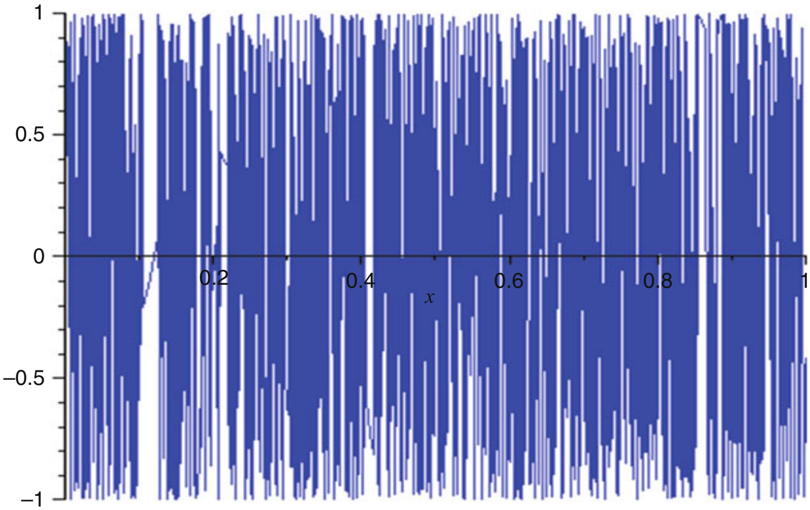
If $\omega = 2, r = 1$, numerical integration of an arbitrary function f is described as

$$\begin{aligned} I_3 &= \int_0^1 f(x)\sin\left(\frac{2}{x}\right)dx = \int_0^1 f(t)\sin\left(\frac{2}{t}\right)dt \\ &= \sum_{i=0}^{m} w_k \frac{1}{2} f(t_i)\sin\left(\frac{2}{t_i}\right) = \sum_{i=0}^{m} w_i \frac{1}{2} f(x_i)\sin\left(\frac{2}{x_i}\right) \end{aligned} \tag{20.3}$$

If $\omega = 1, r = 200$, the numerical integration of an arbitrary function f is described as

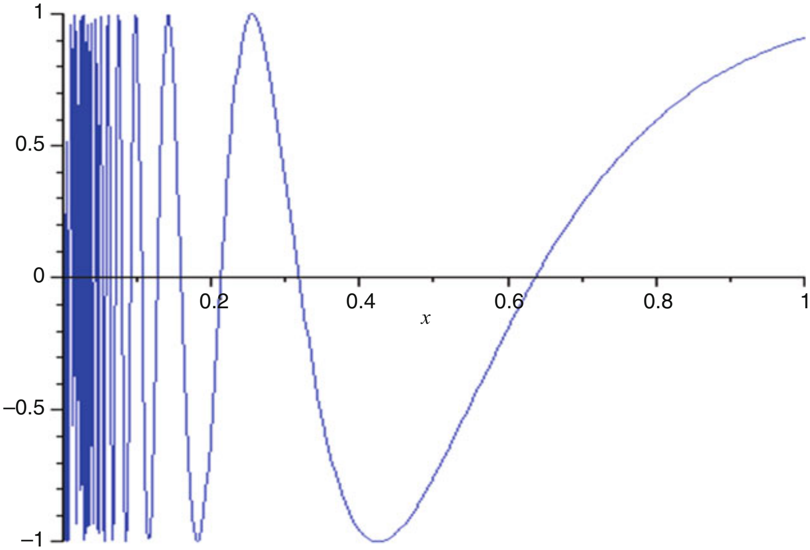


(a)

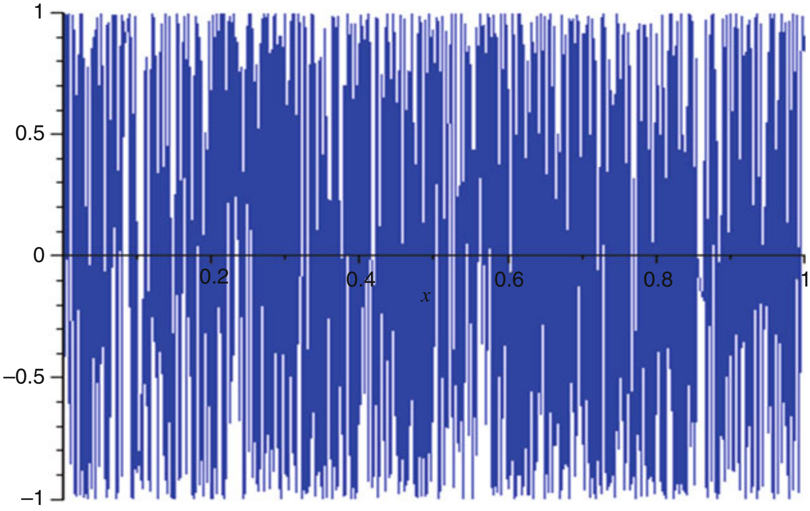


(b)

Fig. 20.1 Oscillation of weighted functions. (**a**) $\mathrm{w}\,(\mathrm{x}) = \cos\left(\frac{1}{\mathrm{x}^2}\right)$. (**b**) $w(x) = \cos\left(\frac{2}{x^{200}}\right)$. (**c**) $w(x) = \sin\left(\frac{2}{\mathrm{x}}\right)$. (**d**) $w(x) = \sin\left(\frac{1}{x^{200}}\right)$



(c)



(d)

Fig. 20.1 (continued)

$$I_4 = \int_0^1 f(x)\ \sin\left(\frac{1}{x^{200}}\right)dx = \int_0^1 f\left(t^{\frac{1}{200}}\right)\sin\left(\frac{1}{t}\right)\frac{t^{\frac{-199}{200}}dt}{200}$$
$$= \sum_{i=0}^{m} w_k\, f\left(t_i^{\frac{1}{200}}\right)\sin\left(\frac{1}{t_i}\right)\frac{t^{\frac{-199}{200}}}{200} = \sum_{i=0}^{m} w_i\, f\left(x_i^{\frac{1}{200}}\right)\sin\left(\frac{1}{x_i}\right)\frac{x_i^{\frac{-199}{200}}}{200} \quad (20.4)$$

where ξ_i and η_j are sampling points and w_i and w_j are corresponding weights. We can rewrite Eq. (20.1) as where ξ_i and η_j are sampling points and w_i and w_j are corresponding weights. We can rewrite Eq. (20.1) as

$$I_1 = \sum_{i=0}^{m} w_k\ f\ (\ x_k\ ) \quad (20.5)$$

where $W_k = \frac{1}{2\sqrt{x_i}}\ \cos\left(\frac{1}{x_i}\right) * w_i$ and $x_k = \sqrt{x_i}$. We have demonstrated the algorithm to calculate sampling points and weights of Eq. (20.5) as follows:

Step 1. $k \to 1$
Step 2. $i = 1, m.$
Step 3. $W_k = \frac{1}{2\sqrt{x_i}}\cos\left(\frac{1}{x_i}\right) * w_i$

$x_k = \sqrt{x_i}$

Step 4. compute step 3.
Step 5. compute step 2

Computed sampling points and corresponding weights for different values of N are based on the above algorithm.

20.3 Numerical Results

Compare the numerical results obtained with that of the exact value of various order N = 20, 50, 100 by Gauss-Legendre quadrature rule; these are tabulated in Table 20.1, and results are accurate in order to increase the order L.

20.4 Conclusion

In this paper, numerical integration of the form $\int_0^1 f(x)\sin\left(\frac{\omega}{x^r}\right)dx$ and $\int_0^1 f(x)\cos\left(\frac{\omega}{x^r}\right)\ dx$ are evaluated numerically with different values of ω and r

Table 20.1 Compare the numerical results by using Gauss-Legendre quadrature rule

Exact values	Order	Computed value
$\int_0^1 \frac{-8x}{x^4+4} \cos\left(\frac{1}{x^2}\right) dx$ $= 0.0946528064$ Hascelik 6	L = 20 L = 50 L = 100	0.0894310923 0.0946122970 0.0946528381
$\int_0^1 - x^2 \cos\left(\frac{2}{x^{200}}\right) dx$ $= 0.002110004128$	L = 20 L = 50 L = 100	0.00160917532 0.00229053172 0.00211090975
$\int_0^1 \frac{x}{(3x+2)} \sin\left(\frac{2}{x}\right) dx$ $= 0.0182548954$	L = 20 L = 50 L = 100	0.01624179037 0.01816533218 0.01825483641
$\int_0^1 x^2 \sin\left(\frac{1}{x^{200}}\right) dx$ $= 0.003117835926$	L = 20 L = 50 L = 100	0.001821673088 0.003127131233 0.003117361087
$\int_0^1 \frac{20}{(x^2+4)} \sin\left(\frac{2}{x}\right) dx$ $= 0.193783272144$ Hascelik 6	L = 20 L = 50 L = 100	0.134418903421 0.192247814349 0.193780915316
$\int_0^1 e^{5x} \sin\left(\frac{1}{x^{200}}\right) dx$ $= 0.461841915645$ Hascelik 6	L = 20 L = 50 L = 100	0.561087771890 0.461889135655 0.461841783027

. We have applied Gauss-Legendre quadrature rules of order 2 L to evaluate the typical numerical integration of highly oscillating function.

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