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Anatoly G. Kusraev
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Operator Theory and Differential Equations

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Preface

This volume contains the proceedings of the conference *Order Analysis and Related Problems of Mathematical Modeling*, which took place at the Vladikavkaz Scientific Center of the Russian Academy of Sciences (Russia) in July 2019. The conference was jointly organized by the Southern Mathematical Institute of the Vladikavkaz Scientific Centre of the Russian Academy of Sciences and Southern Federal University (Russia) with the support of Ministry of Science and Higher Education of the Russian Federation. Its aim is to make current developments in operator theory and differential equations available to the community as rapidly as possible. Moreover, one of the purposes of this conference was to bring together some young and beginning researchers in order to connect people from different schools and generations, give them the opportunity to exchange ideas, and try to attract more young mathematicians to this fascinating area of research. Among the conference participants were mathematicians from Belarus, China, Germany, Israel, Italy, Iran, Russia, Turkey, UK, USA, and Uzbekistan.

The collection presents a wide range of new and interesting problems in operator theory and its applications reflecting the current state of mathematical research in Southern Russia. We believe that the reader will find this book to be a delightful and valuable state-of-the-art account on some fascinating areas of operator theory ranging from various classes of operators (positive operators, convolution operators, backward shift operators, singular and fractional integral operators, and partial differential operators) to important applications.

The articles presented in this collection can be divided into two approximately equal parts. The first part contains articles on general operator theory related to the following topics: positive operators on vector and Banach lattices (Emel'yanov E. Y., Marabeh M. A. A., Pliev M., and Polat F.); Boolean valued analysis of operators (Kusraev A. G. and Kutateladze S. S.); structural properties of linear operators on spaces of holomorphic and ultradifferentiable functions (Ivanova O. A., Melikhov S. N., and Polyakova D.A.); metric theory of surfaces and Killing vector fields on Riemannian manifolds (Klimentov D. S. and Nikonorov Yu. G.); linear operators in approximation theory (Gadzhimirzaev R. M., Magomed-Kasumov M. G., Shakh-Emirov T. N., and Sultanakhmedov M.). The second part consists of articles devoted to the extinction in a finite time for a singular parabolic equation on a Riemannian manifold (Andreucci D. and Tedeev A.F.); regularity of solutions to the linear singular integral equations (Klimentov S. B.); explicit solutions to Darboux system for the Christoffel symbols (Kulaev R. Ch. and Shabat A. B.); spectral analysis of the boundary value problems of incompressible hydrodynamics (Chernish A., Morgulis A. B., and Il'in K.) and the energy operator of five-electron system (Tashpulatov S. M.); asymptotics of self-oscillations of viscous incompressible fluid (Revina S.V.); properties of fractional integral operator (Shishkina E. L.); inverse problems for evolution equations (Babich P.V. and Levenshtam V.B.); heat conduction and reconstructing the inhomogeneity laws for piecewise gradient functions (Nesterov S. A., Vatulyan A. O., and Yurov V. O.); continuous social stratification models (Kazarnikov A.V.).

We are grateful to the authors of this volume for their contribution and to all the anonymous referees for their professional and time-consuming work.

Vladikavkaz, Russia — Anatoly G. Kusraev
Vladikavkaz, Russia — Zhanna D. Totieva

Contents

Extinction in a Finite Time for Parabolic Equations of Fast Diffusion Type on Manifolds

D. Andreucci and A. F. Tedeev

Abstract We prove extinction in a finite time for a singular parabolic equation on a Riemannian manifold, under suitable assumptions on the Riemannian metric and on the inhomogeneous coefficient appearing in the equation. The result relies on a suitable embedding theorem, of which we present a new proof.

MSC Classification 35B33, 35B40, 35K92, 46E35

1 Introduction

We consider the Cauchy problem for the nonlinear parabolic equation:

$$\rho(x)\frac{\partial u}{\partial t} = \operatorname{div}(u^{m-1}|\nabla u|^{p-2}\nabla u), \qquad (x,t) \in S_T = M \times (0,T), \tag{1}$$

$$u(x,0) = u_0(x), \qquad x \in M. \tag{2}$$

We assume that

$$p + m - 3 < 0, \qquad N > p > 1, \tag{3}$$

that is, we consider the fast diffusion case. Here M is a noncompact complete connected Riemannian manifold of topological dimension N, whose measure is

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denoted here by μ. We denote by $d(x)$ for $x \in M$ the distance from a fixed point $x_0 \in M$, and by $V(R)$ the volume of the geodesic ball $B_R(x_0)$, $R > 0$.

Assume that the following isoperimetrical inequality holds true for all measurable subsets $U \subset M$ with a Lipschitz continuous boundary ∂U

$$|\partial U|_{N-1} \geq g(\mu(U)) , \tag{4}$$

where $g(s)$ is an increasing function for $s > 0$. In addition we assume that the function ω defined as

$$\omega(s) := \frac{s^{\frac{N-1}{N}}}{g(s)} , \qquad s > 0 , \tag{5}$$

is non decreasing.

In what follows we denote with a slight abuse of notation $\rho(x) = \rho(d(x))$, where we assume $\rho(s)$ to be a continuous decreasing function for $s \geq 0$, $\rho(0) = 1$. We use the function $\rho^*(s)$, $s > 0$, defined as the decreasing rearrangement of $\rho(d(x))$.

We also need the following assumption, to prove a kind of Hardy inequality:

$$\int_0^s y^{-p} g(y)^p \,\mathrm{d}y \leq c s^{-p+1} g(s)^p , \qquad s > 0 , \tag{6}$$

for a given constant $c > 0$.

Remark 1.1 In the Euclidean case $g(s) = s^{(N-1)/N}$ it is easily seen that (6) is equivalent to $p < N$.

Taking for example M as one of the manifolds with cylindrical end of [2], whose metric is (out of a compact set) $\mathrm{d}t^2 + t^{2k}\,\mathrm{d}M_0$, where $\mathrm{d}M_0$ is the metric of a compact manifold M_0, $0 < k < 1$, one could see that assumption (6) amounts essentially to the non-parabolicity of M in the sense of [2], i.e., to $k > (p-1)/(N-1)$. In this case, $g(s) = \gamma \min\{s^{\frac{N-1}{N}}, s^{\alpha}\}$, $\alpha = k(N-1)/(k(N-1)+1)$, and such condition is equivalent to the restriction $\alpha > (p-1)/p$.

In this note we prove two results. First we prove the following embedding result. Below we set $p^* = Np/(N-p)$ and $\beta = N(p+m-3)+p$.

Theorem 1.2 *Assume* $1 < p < N$, (4)–(6)*. Then for all* $u \in W^{1,p}(M)$ *we have the weighted Sobolev inequality*

$$\left(\int_M |u|^{p^*} \omega(V(d(x)))^{-p^*} \,\mathrm{d}\mu \right)^{\frac{N-p}{N}} \leq C \int_M |\nabla u|^p \,\mathrm{d}\mu , \tag{7}$$

for a suitable constant $C > 0$ *independent of* u.

The result above was proved in [3], in a different framework. Our proof uses a different technique, relying on a symmetrization approach and on the use of Hardy inequality, which seems to us to be sharp and very straightforward.

Next we apply the weighted inequality in (7) to the proof of our extinction result. Let us note, however, that the embedding may be also employed to prove for example sup bounds and blow up estimates; this will be pursued elsewhere. We refer to [5] for a definition of weak solution to our problem (the definition given there carries over straightforwardly to our setting).

Theorem 1.3 *Let u be a nonnegative weak solution to* (1)–(2)*, where we assume* (3)*,* (4)*,* (5)*,* (6) *and*

$$\int_M \left\{\rho(x)\omega^{\delta}\big(V(d(x))\big)\right\}^{\frac{p^*}{p^*-\delta}} \mathrm{d}\mu < \infty\,. \tag{8}$$

Here $\delta > p$ may be any value in (p, p^*) if $\beta \le 0$, and any value $p < \delta < p/(p+m-2) < p^*$ if $\beta > 0$.

Then u becomes identically zero in M in a finite time.

Similar extinction results are well known in the literature, when $\rho = 1$ and the equation is singular, see [1]. In this note we consider the case of the inhomogeneous fast diffusion equation, for which we quote [6]. Our extinction result reduces to the one there for the special cases of Euclidean metric; that is, when ω is constant, and of $\rho(s) = (1+s)^{-\ell}$, $\beta > 0$; i.e., we have extinction if $\ell > \ell^* = \beta/(p+m-2)$.

2 Proof of the Weighted Sobolev Inequality

We have by Hardy-Littlewood inequality

$$I := \left(\int_M |u|^{p^*}\omega(V(d(x)))^{-p^*}\,\mathrm{d}\mu\right)^{\frac{N-p}{N}} \le \left(\int_0^\infty (u^*(s))^{p^*}\left(\omega(V(d(x)))^{-p^*}\right)^*\,\mathrm{d}s\right)^{\frac{N-p}{N}}. \tag{9}$$

By definition

$$\int_0^\infty (u^*(s))^{p^*}\left(\omega(V(d(x)))^{-p^*}\right)^*\,\mathrm{d}s = \int_0^\infty (u^*(s))^{p^*}(\omega(s))^{-p^*}\,\mathrm{d}s\,.$$

Next we have by the properties of Lorentz spaces (see [4])

$$\left(\int_0^\infty (u^*(s))^{p^*}(\omega(s))^{-p^*}\,\mathrm{d}s\right)^{\frac{N-p}{N}} \le c\int_0^\infty (u^*(s))^p s^{\frac{p}{p^*}-1}(\omega(s))^{-p}\,\mathrm{d}s$$
$$= c\int_0^\infty (u^*(s))^p s^{-p}(g(s))^p\,\mathrm{d}s\,. \tag{10}$$

Next we prove the Hardy inequality

$$\int_0^\infty (u^*(s))^p s^{-p}(g(s))^p\,\mathrm{d}s \le c\int_0^\infty (-u^*_s(s))^p (g(s))^p\,\mathrm{d}s\,. \tag{11}$$

We have

$$\frac{\partial}{\partial s}\left[(u^*(s))^p\left(\int_0^s y^{-p}(g(y))^p\,\mathrm{d}y\right)\right] = p(u^*(s))^{p-1}u^*_s(s)\int_0^s y^{-p}(g(y))^p\,\mathrm{d}y$$
$$+ (u^*(s))^p s^{-p}(g(s))^p\,.$$

Integrating this equality between 0 and ∞ we deduce

$$\int\limits_0^\infty (u^*(s))^p s^{-p}(g(s))^p\,\mathrm{d}s$$
$$= p\int_0^\infty \left((u^*(s))^{p-1}(-u^*_s(s))\int_0^s y^{-p}(g(y))^p\,\mathrm{d}y\right)\mathrm{d}s\,. \tag{12}$$

By the Hölder inequality we obtain

$$p\int_0^\infty \left((u^*(s))^{p-1}(-u^*_s(s))\int_0^s y^{-p}(g(y))^p\,\mathrm{d}y\right)\mathrm{d}s \le \left(\int_0^\infty (u^*(s))^p s^{-p}(g(s))^p\,\mathrm{d}s\right)^{\frac{p-1}{p}}$$
$$\times\left(\int_0^\infty (-u^*_s(s))^p\left(\int_0^s y^{-p}(g(y))^p\,\mathrm{d}y\right)^p [s^{-p}(g(s))^p]^{-(p-1)}\,\mathrm{d}s\right)^{\frac{1}{p}}\,. \tag{13}$$

Applying (6) to the right-hand side of (13) and combining it with (12) we arrive at (11). Finally, from Polia-Szego principle we obtain

$$I \le c\int_0^\infty (-u^*_s(s))^p(g(s))^p\,\mathrm{d}s \le c\int_M |\nabla u|^p\,\mathrm{d}\mu\,, \tag{14}$$

where the last inequality is proved below. Following Talenti's approach we estimate by Hölder inequality

$$\frac{1}{h}\int_{\{t<|u|\le t+h\}} |\nabla u|\,\mathrm{d}\mu \le \left(\frac{1}{h}\int_{\{t<|u|\le t+h\}} |\nabla u|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}} \left(\frac{1}{h}|\{t<|u|\le t+h\}|\right)^{\frac{p-1}{p}} .$$

On letting $h \to 0$, we obtain

$$P(t) := |\{|u|=t\}|_{N-1} = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{t<|u|\}} |\nabla u|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}} \left(-\frac{\mathrm{d}}{\mathrm{d}t}v(t)\right)^{\frac{p-1}{p}} , \tag{15}$$

where $v(t) := |\{t < |u|\}|$. Next, by the isoperimetrical inequality (4) we have

$$P(t) \ge g(v(t)) . \tag{16}$$

When we set $s = v(t)$ we get $t = u^*(s)$, and

$$\frac{\mathrm{d}v}{\mathrm{d}t}(t) = u^*_s(s) .$$

Thus (15), (16) imply

$$g(s)^p(-u^*_s(s))^p \le -\frac{\mathrm{d}}{\mathrm{d}s}\int_{\{u^*(s)<|u|\}} |\nabla u|^p\,\mathrm{d}\mu .$$

Finally, on integrating the last inequality over $(0, \infty)$, we arrive at the desired result.
The proof is complete.

3 Extinction in Finite Time: Proof of Theorem 1.3

On multiplying both the sides of equation (1) by u^θ with $\theta > 0$ such that $p+m+\theta > 2$ and integrating by parts yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M \rho v^\delta\,\mathrm{d}\mu = -\gamma\int_M |\nabla v|^p\,\mathrm{d}\mu , \tag{17}$$

where, owing to (3), we have

$$v = u^{\frac{p+m+\theta-2}{p}} , \qquad \delta = \delta(\theta) = \frac{(1+\theta)p}{p+m+\theta-2} > p .$$

We select θ so that the value of δ is the one given in (8); the requirement $p^* > \delta$ translates into $\theta > \theta_0 := (3-p-m)N/p - 1$; note that $\theta_0 > 2-p-m$ under our assumptions. If $\theta_0 \geq 0$ then no requirement is needed on δ other than $\delta < p^*$; otherwise one must impose $\delta < \delta(0+) = p/(p+m-2)$ which belongs to (p, p^*), when $\theta_0 < 0$.

Applying Hölder inequality and the embedding in (7) to v we get

$$\int_M \rho v^\delta \, \mathrm{d}\mu \leq C \left(\int_M |\nabla v|^p \, \mathrm{d}\mu \right)^{\frac{\delta}{p}} \left(\int_M \big(\rho(x)\omega^\delta\big(V(d(x))\big)\big)^{\frac{p^*}{p^*-\delta}} \, \mathrm{d}\mu \right)^{\frac{p^*-\delta}{p^*}} . \tag{18}$$

On setting

$$E(t) := \int_M \rho(x) v^\delta(x,t) \, \mathrm{d}\mu ,$$

and taking into account (8) we arrive at the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \leq -\gamma E(t)^{\frac{p}{\delta}} .$$

Since $\frac{p}{\delta} < 1$, the last inequality leads to extinction in finite time: $E(t) \to 0$ as $t \to \bar{t}$ for some $\bar{t} < +\infty$.

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Inverse Problems in the Multidimensional Hyperbolic Equation with Rapidly Oscillating Absolute Term

P. V. Babich and V. B. Levenshtam

Abstract The paper is devoted to the development of the theory of inverse problems for evolution equations with terms rapidly oscillating in time. A new approach to setting such problems is developed for the case in which additional constraints are imposed only on several first terms of the asymptotics of the solution rather than on the whole solution. This approach is realized in the case of a multidimensional hyperbolic equation with unknown absolute term.

Keywords Multidimensional hyperbolic equation · Rapidly oscillating absolute term · Asymptotics of solution · Inverse problem

Mathematics Subject Classification Primary 35B40, 35R30; Secondary 35L10, 35L15

1 Introduction

We consider some problems of recovering rapidly oscillating in time absolute term from certain data on a partial asymptotics of the solution. Hence we study some of the coefficient inverse problems. The theory of inverse problems was the subject of many monographs (see, e.g. [1–14]) and papers (see, e.g. [15–17]). But there are almost no problems with rapidly oscillating data in the classical theory of inverse problems.

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This paper as paper [18] was motivated by the paper [15], in which inverse problems for the one-dimensional wave equation with unknown absolute term was posed and solved. In [15] right-hand side represented in the form $f(x)r(t)$, where r is unknown. An additional condition in [17] was the value of $q(t)$ of the solution at a fixed point $x = x_0$. In [18] we have the same form of the right-hand side of multidimensional hyperbolic equation, but the unknown term rapidly oscillate: $r = r(t, \omega t), \omega \gg 1$. This brings up the question, should we impose an additional condition on the whole solution, as in [15]. In paper [18] it was established that the additional condition may be imposed only on several first coefficients of the asymptotics of the solution rather than on the whole solution. In the present paper the following inverse problems are solved: (1) f is unknown; (2) f and fast component of r are unknown.

In conclusion, we mention that, problems with data rapidly oscillating in time model many physical (and other) processes (in particular, related to high-frequency mechanical, electromagnetic, and other actions on a medium) see, for example, [19–24]. The inverse problems with such specificity have been studied in [18, 25, 26] by us.

This paper consist of four section. In Sect. 2 some principal symbols are listed that we use in follows. In Sect. 3 important auxiliary results are given. In Sect. 4 we state the main results. In Sect. 5 the main results are proved.

2 Principal Symbols

Let Ω denote a bounded domain in $\mathbb{R}^n, n \in \mathbb{N}.S$ its boundary. We denote the open cylinder $\Omega \times (0, T) \subset \mathbb{R}^{n+1}$ by Q_T, its closure $\overline{Q}_T$. Consider the following hyperbolic initial boundary-value problem with a large parameter ω:

$$\frac{\partial^2 u}{\partial t^2} = Lu + f(x,t)r(t,\omega t), (x,t) \in Q_T, \tag{2.1}$$

$$u|_{t=0} = 0, \ \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \tag{2.2}$$

$$u|_{x \in S} = 0, \tag{2.3}$$

All functions are real. We consider that the symmetric differential expression

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial u}{\partial x_j}\right] - c(x)u - \tag{2.4}$$

is defined in Ω and satisfies the ellipticity condition, so that

$$a_{ij}(x) = a_{ji}(x), \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \gamma \sum_{i=1}^{n} \xi_i^2, \text{ where } \gamma = const > 0, \tag{2.5}$$

for all $x \in \Omega$ and any real vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$.

We shall assume that the function $r(t, \tau)$ is defined and is continuous on the set $D = \{(t, \tau) : (t, \tau) \in [0, T] \times [0, \infty)\}$ and 2π-periodic in τ. Let us represent it as the sum:

$$r(t, \tau) = r_0(t) + r_1(t, \tau),$$

where $r_0(t)$—is the mean value of $r(t, \tau)$ over τ:

$$r_0(t) = \langle r(t, \cdot)\rangle = \langle r(t, \tau)\rangle_\tau \equiv \frac{1}{2\pi}\int_0^{2\pi} r(t, \tau)d\tau.$$

3 The Auxiliary Results

3.1 The Results of V.A. Il'in [27]

Consider the problem

$$\frac{\partial^2 u}{\partial t^2} = Lu + F(x, t), (x, t) \in Q_T, \tag{3.1}$$

$$u|_{t=0} = \varphi(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x), \tag{3.2}$$

$$u|_{x\in S} = 0, \tag{3.3}$$

Let domain Ω, the coefficients of the expression L (2.4), right-hand side F and initial conditions φ and ψ satisfy the following conditions.

I. Ω is bounded connected domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, contained, together with its boundary S, in an open domain $C \in \mathbb{R}^n$.[1]

[1]Recall that a domain is said to be normal if the Dirichlet problem for the Laplace equation in this domain is solvable for continuous boundary function.

II. Coefficients $a_{ij}(x)$ and $c(x)$ ensure existence of full orthonormal in $L_2(\Omega)$ system classic eigenfunctions of problem

$$\begin{cases} Lu = \lambda u, \\ u|_S = 0. \end{cases}$$

To do this, since [27] it suffices to provide further conditions. Functions $a_{ij}(x), c(x)$ can be continued to domain C so that $a_{ij} \in C^{1+\mu}(C), c \in C^{\mu}(C)$, $\mu \geq 0$. Moreover, $a_{ij} \in C^{[\frac{n}{2}]+2}(\overline{\Omega})$, $c \in C^{[\frac{n}{2}]+1}(\overline{\Omega})$. Let $y_m, \lambda_m, m = 1, 2, \ldots,$ denote eigenfunctions and eigenvalues noted above. We shall assume that $\{\lambda_m\}$ is nondecreasing sequence: $0 < \lambda_1 \leq \lambda_2 \leq \ldots$

III. Initial functions $\varphi \in C^{[\frac{n}{2}]+3}(\overline{\Omega})$, $\psi \in C^{[\frac{n}{2}]+2}(\overline{\Omega})$ and $\varphi|_{x\in S} = L\varphi|_{x\in S} = \ldots = L^{\left[\frac{n+4}{4}\right]}\varphi\Big|_{x\in S} = 0$, $\psi|_{x\in S} = L\psi|_{x\in S} = \ldots = L^{\left[\frac{n+2}{4}\right]}\psi\Big|_{s\in S} = 0$. Let φ_m, ψ_m denote the coefficients of the Fourier expansion of functions $\varphi(x), \psi(x)$ in the basis of y_m.

IV. The right-hand side $F \in C([0, T], C^{[\frac{n}{2}]+2}(\overline{\Omega}))$, $F|_{x\in S} = Lf|_{x\in S} = \ldots = L^{\left[\frac{n+2}{4}\right]}f\Big|_{x\in S} = 0$. Let $F_m(t)$ denote the coefficients of the Fourier expansion of function $F(x, t)$ in the basis of y_m

Theorem 1 (V.A. Il'in) *If conditions I–IV hold, the series*

$$u(x, t) = \sum_{m=1}^{\infty} y_m(x) \left[\varphi_m \cos\sqrt{\lambda_m}t + \frac{\psi_m}{\sqrt{\lambda_m}} \sin\sqrt{\lambda_m}t\right] + \sum_{m=1}^{\infty} y_m(x) \frac{1}{\sqrt{\lambda_m}} \int_0^t F_m(\tau) \sin\sqrt{\lambda_m}(t - \tau) d\tau \tag{3.4}$$

and the series u_t, u_{tt} obtained by single and double differentiation of (3.4) with respect to t are converge uniformly in $\overline{Q_T}$. The series $u_{x_i}, u_{tx_i}, u_{x_ix_j}$ obtained by single and double differentiation of (3.4) with respect to any two variables are converge uniformly in any domain that is strictly contained in Q_T. At the same time, (3.4) is classic solution of (3.1)–(3.3).

This result can be found in [27, Theorems 6, 8].[2]

Lemma 1 *If conditions I, II, III hold, then the bilinear series for eigenfunctions $\sum_{m=1}^{\infty} \frac{y_m^2(x)}{\lambda_m^{[\frac{n}{2}]+1}}$ converge uniformly in $\overline{\Omega}$, the bilinear series $\sum_{m=1}^{\infty} \frac{\left|\frac{\partial v_m(x)}{\partial x_i}\right|^2}{\lambda_m^{[\frac{n}{2}]+2}}$ and*

[2]Here and in what follows, we use results of [27] in "classical" terms (see [27, Remark 3, p. 114 of the Russian original]). In [27], such classical versions are not stated explicitly, but when referring to results of [27], we always mean their classical versions.

$\sum_{m=1}^{\infty} \frac{\left|\frac{\partial^2 y_m(x)}{\partial x_i \partial x_j}\right|^2}{\lambda_m^{[\frac{n}{2}]+3}}$ *converge uniformly in any domain that is strictly contained in* $\Omega' \subset \Omega$.

Lemma 2 *Let the coefficients* $a_{ij}(x)$ *be continuous together with their derivatives up to order k, and* $c(x)$ *is continuous together with its derivatives up to order* $k-1$. *We also assume that a function* $\Phi(x)$, $x \in \overline{\Omega}$ *satisfies the following conditions:*

(1) $\Phi \in C^{k+1}(\overline{\Omega})$,

(2) $\Phi|_{x\in S} = L\Phi|_{x\in S} = \ldots = L^{\left[\frac{k}{2}\right]}\Phi\Big|_{x\in S} = 0$.

Then for Φ *inequality of Bessel type holds true:*

$$\sum_{m=1}^{\infty} \Phi_m^2 \lambda_m^{k+1} \leq \begin{cases} \int\limits_{\Omega} \left[\sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_i}(L^{\frac{k}{2}}\Phi)\frac{\partial}{\partial x_j}(L^{\frac{k}{2}}\Phi) + c(L^{\frac{k}{2}}\Phi)^2\right] dx, \, k \text{ is even}, \\ \int\limits_{\Omega} \left[L^{\frac{k+1}{2}}\Phi\right]^2 dx, \, k \text{ is odd}. \end{cases}$$

3.2 The Problem 1

The Direct Problem 1. The Three-Term Asymptotics

Consider problem (2.1)–(2.3), where domain Ω, elliptic differential expression L are the same as in Theorem 1.

Concerning the function $f(x,t)$ defined at $(x,t) \in \overline{Q_T}$, we assume that there exist continuous functions f, Lf, f_t, f_{tt}, f_{ttt} and Lf_t, such that all of them belong to the space of functions $C_{t,x}^{0,[\frac{n}{2}]+2}(\overline{Q_T})$ and, moreover,

$$f|_{x\in S} = Lf|_{x\in S} = \ldots = L^{\left[\frac{n+6}{4}\right]}f\Big|_{x\in S} = 0.$$

For brevity, we refer to functions r with these properties as functions of class $\mathbf{F_1}$.

We shall assume that the function $r(t,\tau)$ is defined and is continuous on the set $D = \{(t,\tau) : (t,\tau) \in [0,T] \times [0,\infty)\}$ and 2π-periodic in τ. As in Sect. 2 let us represent r as the sum of slow and oscillating components:

$$r(t,\tau) = r_0(t) + r_1(t,\tau);$$

we shall assume that $r_0 \in C([0,T])$, and the functions r_1, r_{1t}, r_{1tt}, and r_{1ttt} belong to the class $C(D)$. We denote function r with such properties as function of class $\mathbf{R_1}$.

In the present paper, by a solution of problem (2.1)–(2.3) we mean its classical solution, i.e., a function $u \in C(\overline{Q}_T)$, which has continuous derivatives $u_t \in C(\overline{Q}_T), u_{tt}$, and $u_{x_i x_j} \in C(Q_T), i, j = \overline{1,n}$, and satisfies relations (2.1)–(2.3). Under our assumptions, the solution of problem (2.1)–(2.3), exists and is unique according to Theorem 1.

Below we define functions and constants needed in what follows:

$$\rho_0(t,\tau) = \int_0^\tau \left(\int_0^p r_1(t,s)ds - \left\langle \int_0^\tau r_1(t,s)ds \right\rangle_\tau \right) dp - \left\langle \int_0^\tau \left(\int_0^p r_1(t,s)ds - \left\langle \int_0^\tau r_1(t,s)ds \right\rangle_\tau \right) dp \right\rangle_\tau \tag{3.5}$$

$$\rho_1(t,\tau) = \left\langle \int_0^\tau \rho_0(t,s)ds \right\rangle_\tau - \int_0^\tau \rho_0(t,s)ds.$$

$$b_{1,m} = -\rho_{0\tau}(0,0) f_m(0), \tag{3.6}$$

$$d_m = -\rho_0(0,0) f_m(0), \tag{3.7}$$

$$b_{2,m} = -(2\rho_1(0,0) + \rho_0(0,0)) f'_m(0) - (2\rho_{1t}(0,0) + \rho_{0t}(0,0)) f_m(0), \tag{3.8}$$

where the $f_m(t)$ are the coefficients of the Fourier expansion of $f(x,t)$ in the basis of y_m.

Let us represent the solution of problem (2.1)–(2.3) in the form:

$$u_\omega(x,t) = U_\omega(x,t) + W_\omega(x,t), \omega \gg 1, \tag{3.9}$$

$$U_\omega(x,t) = u_0(x,t) + \omega^{-1} u_1(x,t) + \omega^{-2}\big[u_2(x,t) + v_2(x,t,\omega t)\big], \omega \gg 1, \tag{3.10}$$

$$u_0(x,t) = \sum_{m=1}^{\infty} \frac{y_m(x)}{\sqrt{\lambda_m}} \int_0^t f_m(s) r_0(s) \sin\sqrt{\lambda_m}(t-s)ds, \tag{3.11}$$

$$u_1(x,t) = \sum_{m=1}^{\infty} \frac{b_{1,m}}{\sqrt{\lambda_m}} y_m(x) \sin\sqrt{\lambda_m}t, \tag{3.12}$$

$$v_2(x,t,\tau) = f(x,t)\rho_0(t,\tau), \tag{3.13}$$

$$u_2(x,t) = \sum_{m=1}^{\infty} y_m(x) \left(d_m \cos\sqrt{\lambda_m}t + \frac{b_{2,m}}{\sqrt{\lambda_m}} \sin\sqrt{\lambda_m}t \right). \tag{3.14}$$

Note that, in view of Theorem 1, the series (3.11)–(3.14) converge uniformly and absolutely.

Theorem 2 *The solution $u_\omega(x,t)$ of problem (2.1)–(2.3) can be expressed in the form (3.9)–(3.14), where*

$$\|W_\omega(x,t)\|_{C(\overline{Q}_T)} = o(\omega^{-2}), \omega \to \infty. \tag{3.15}$$

The Inverse Problem 1

Suppose that the function $f(x,t)$ in the initial boundary-value problem (2.1)–(2.3) is the function of class $\mathbf{F_1}$ and the function $r \in \mathbf{R_1}$ is unknown. Choose a point $x^0 \in \Omega$ at which $f(x^0,t) \neq 0$, $t \in [0,T]$, and functions $\varphi_0(t)$ and $\chi(t,\tau)$ satisfying the conditions:

$$\varphi_0 \in C^1([0,T]),\ \varphi_0(0) = 0,\ \varphi_0'(0) = 0;$$
$$\chi \in C^{3,2}(D),$$

where the function $\chi(t,\tau)$ is 2π-periodic in τ and has zero mean $\big(\langle\chi(t,\cdot)\rangle = 0\big)$. Consider the functions $\varphi_1(t)$ and $\varphi_2(t)$ defined by

$$\varphi_1(t) = \sum_{m=1}^{\infty} \frac{b_{1,m}}{\sqrt{\lambda_m}} y_m(x^0) \sin\sqrt{\lambda_m} t, \tag{3.16}$$

$$\varphi_2(t) = \sum_{m=1}^{\infty} y_m(x^0)\left(d_m \cos\sqrt{\lambda_m} t + \frac{b_{2,m}}{\sqrt{\lambda_m}} \sin\sqrt{\lambda_m} t\right), \tag{3.17}$$

where the $b_{1,m}, b_{2,m}$ and d_m are the same as in (4.12)–(4.13), but $\rho_0(t,\tau)$ is now defined by

$$\rho_0(t,\tau) = \frac{1}{f(x^0,t)}\left(\int_0^\tau \left(\int_0^p \chi_{ss}(t,s)ds - \left\langle \int_0^\tau \chi_{ss}(t,s)ds \right\rangle_\tau \right) dp - \left\langle \int_0^\tau \left(\int_0^p \chi_{ss}(t,s)ds - \left\langle \int_0^\tau \chi_{ss}(t,s)ds \right\rangle_\tau \right) dp \right\rangle_\tau \right).$$

The inverse problem 1 is to find a function $r \in \mathbf{R_1}$ for which the solution $u_\omega(x,t)$ of problem (2.1)–(2.3) satisfies the condition

$$\left\| u_\omega(x^0,t) - \left[\varphi_0(t) + \frac{1}{\omega}\varphi_1(t) + \frac{1}{\omega^2}\big(\varphi_2(t) + \chi(t,\omega t)\big)\right]\right\|_{C([0,T])} = o(\omega^{-2}),\ \omega \to \infty. \tag{3.18}$$

Theorem 3 *For any pair of functions χ, φ_0 and point x^0 satisfying the conditions specified above inverse problem 1 is uniquely solvable.*

Remark Finding the function r_0 reduces to solving a Volterra equation of the second kind

$$f(x^0,t)r_0(t) + \int_0^t K(t,s)r_0(s)\,ds = \varphi_0''(t), \tag{3.19}$$

$$K(t,s) = -\sum_{m=1}^{\infty}\sqrt{\lambda_m}\,f_m(s)\sin\left(\sqrt{\lambda_m}(t-s)\right)y_m(x^0).$$

Function r_1 calculated by

$$r_1(t,\tau) = \frac{1}{f(x^0,t)}\frac{\partial^2}{\partial\tau^2}\chi(t,\tau), \tag{3.20}$$

Remark Theorems 2 and 3 can be found together with their proof in paper [18].

3.3 The Lemma of Krasnosel'skii et al. [28, Sec. 22.1]

Suppose that Ω is bounded connected domain in $\mathbb{R}^n$ and S is its boundary. We denote $k_0 > \frac{n}{4}$ is natural value such that $S \in C^{2k_0}$ and functions $b_{ij}, d \in C^{2k_0-2}(\overline{\Omega})$. Moreover, boundary smoothness meant in the same manner as in [29, Theorem 15.2]. In space $L_2(\Omega)$ consider elliptic differential operator

$$L_0 u = \sum_{i,j=1}^{n} b_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} - d(x)u,\ u \in D(L_0) \equiv \dot{W}_2^2(\Omega), \tag{3.21}$$

where $\dot{W}_2^2(\Omega)$ is closure in $W_2^2(\Omega)$ of set of smooth finite in Ω function. We shall assume that coefficient $d(x)$ is so large that L_0 is invertible operator. Results of [29] imply the estimate

$$\left\| L_0^{k_0} u \right\|_{L_2} \geq c\,\|u\|_{W_2^{2k_0}},\ u \in D(L_0^{k_0}),\ c\text{—positive value.} \tag{3.22}$$

We assume that the domain Ω satisfies Sobolev's imbedding Theorem:

$$\|u\|_{C^l(\overline{\Omega})} \leq c\|u\|_{W_2^s(\Omega)}, \; u \in W_2^s(\Omega), \tag{3.23}$$

where $s - l > \frac{n}{2}$, c is positive value. Classic condition for this Theorem is that Ω is star domain.

The above leads to the following result:

Lemma 3 *For any integer $|r| \in [0, 2k_0 - \frac{n}{2}]$ operator $D^r L_0^{-k_0}$ continuously acts from $L_2(\Omega)$ to $C^{2k_0 - r - \frac{n}{2}}(\overline{\Omega})$, where $D^r u = \frac{\partial^r u}{\partial x_1^{r_1} \dots x_n^{r_n}}$, $r = (r_1, \dots, r_n)$ is multi-index with length $|r| = r_1 + \dots + r_n$.*

Lemma 3 can be found in [28, Sec.22.2] without specialization of some requirements to coefficients and boundary.

4 The Main Results

4.1 The Problem 2

The Direct Problem 2. The Main Term of Asymptotics
Let, as in Sect. 3.2, domain Ω and differential expression L satisfy Theorem 1 conditions.

Let us consider the problem (2.1)–(2.3). From this point onward function $f(x, t)$ is invariant with t h.e. $f(x, t) = f(x)$, $x \in \Omega$. We also assume that $f \in C^{[\frac{n}{2}]+2}(\overline{\Omega})$,

$$f|_{x \in S} = Lf|_{x \in S} = L^2 f\Big|_{x \in S} = \dots = L^{\left[\frac{n+2}{4}\right]} f\Big|_{x \in S} = 0. \tag{4.1}$$

Let us denote the class of such functions by $\mathbf{F_2}$.

We shall also assume that function $r(t, \tau)$ is defined and is continuous on the set $D = \{(t, \tau) : (t, \tau) \in [0, T] \times [0, \infty)\}$ and 2π-periodic in τ. As above let represent it as the sum:

$$r(t, \tau) = r_0(t) + r_1(t, \tau),$$

where r_0 is slow component and r_1 is oscillating component. Let us assume that $r_0 \in C([0, T])$, $r_1 \in C(D)$.

Theorem 4 *The following asymptotic formula holds*

$$\left\|u_\omega - u_0\right\|_{C(\overline{\Pi})} = o(1), \; \omega \to \infty, \tag{4.2}$$

where u_ω is solution of problem (2.1)–(2.3).

The Inverse Problem 2
Consider the problem (2.1)–(2.3) in bounded connected normal domain Ω with boundary $S \in C^{2[\frac{n}{2}]+4}$. Let coefficients of expression L belong to the following Holder classes:

$$a_{ij} \in C^{3[\frac{n}{2}]+6}(\overline{\Omega}),\, c \in C^{3[\frac{n}{2}]+5}(\overline{\Omega}), \text{ where } \alpha \in (0, 1),\, c(x) \geq 0,\, x \in \Omega. \tag{4.3}$$

We shall assume that function $r(t, \tau)$ is known, satisfies Theorem 4 conditions, and, moreover, $r_0 \in C^1([0, T])$. Suppose there exists a point $t_0 \in (0, T]$ such that

$$|r_0(t_0)| > |r_0(0)|. \tag{4.4}$$

Let $\mathbf{R_2}$ denote the class of functions r satisfying the above mentioned conditions. We assume that function f is unknown and belong to the class $\mathbf{F_2}$.

Following lemma holds, where

$$\Lambda_m(t) \equiv \int_0^t r_0(s) \sin \sqrt{\lambda_m}(t-s)ds,\, t \in [0, T].$$

Lemma 4 *For any function $r \in \mathbf{R_2}$ there exist values $c_0 > 0$ and $m_0 \in \mathbb{N}$ such that for every number $m \geq m_0$ we have $\Lambda_m(t_0) > \frac{c_0}{\lambda_m}$.*

For brevity, we shall assume that set $M_0 \equiv \{m : \Lambda_m(t_0) = 0\} = \emptyset$.

Concerning the system (2.1)–(2.3) with unknown function f, we supplement the problem with function ψ such that

$$\psi \in C^{3[\frac{n}{2}]+7}(\overline{\Omega}),\ \psi|_{x\in S} = L\psi|_{x\in S} = L^2\psi\Big|_{x\in S} = \ldots = L^{3[\frac{n}{4}]+3}\psi\Big|_{x\in S} = 0. \tag{4.5}$$

The inverse problem 2 is to find function $f \in \mathbf{F_2}$ for which the solution $u_\omega(x, t)$ of problem (2.1)–(2.3) satisfies the condition:

$$\left\| u_\omega(x, t_0) - \psi(x) \right\|_{C([0,\pi])} = o(1),\, \omega \to \infty. \tag{4.6}$$

Theorem 5 *Let functions r_0, ψ and point t_0 satisfy the conditions specified above. Then inverse problem 2 is uniquely solvable. At the same time, the function $f(x)$ calculated by $f(x) = \sum_{m=1}^{\infty} f_m y_m(x),\ f_m = \frac{\psi_m}{\Lambda_m}$.*

4.2 *The Inverse Problem 3*

In this section we consider again problem (2.1)–(2.3). Assume that domain Ω and coefficients of differential expression L are the same as in previous subsection, e.g.

coefficients satisfying to conditions (4.3), domain Ω is bounded, is connected, is normal, and its boundary $S \in C^{2[\frac{n}{2}]+4}$.

Let function f and r belong to ($\mathbf{F_3}$ and $\mathbf{R_3}$) respectively:

$\mathbf{F_3}$: $f, Lf \in C^{[\frac{n}{2}]+2}(\overline{\Omega}),\ f|_{x\in S} = Lf|_{x\in S} = \ldots = L^{\left[\frac{n+6}{4}\right]} f\Big|_{x\in S} = 0$;

$\mathbf{R_3}$: $r(t,\tau)$ is 2π-periodic in τ. As above let represent it as the sum:

$$r(t,\tau) = r_0(t) + r_1(t,\tau),$$

where

$$r_0 \in C^1([0,T]);\, r_1, r_{1t}, r_{1tt}, r_{1ttt} \in C(D)$$

and a point $t_0 \in (0,T]$ such that $|r_0(t_0)| > |r_0(0)|$ there exist.

We shall assume that function r_0 is known, and functions f and r_1 are unknown. For brevity, as in Sect. 4.1 suppose that set $M_0 \equiv \{m, \Lambda_m(t_0) = 0\} = \emptyset$. Choose a 2π-periodic with zero mean in second variable $\chi(t,\tau)$, $\chi \in C^{3,2}(D)$, $D = [0,T] \times [0,\infty)$, and function $\psi \in C^{3[\frac{n}{2}]+9}(\overline{\Omega})$ satisfying the conditions

$$\psi|_{x\in S} = L\psi|_{x\in S} = L^2\psi\Big|_{x\in S} = \ldots = L^{3[\frac{n}{4}]+4}\psi\Big|_{x\in S} = 0. \tag{4.7}$$

And let $x^0 \in \Omega$ is a point at which $\widetilde{f}(x^0) \neq 0$, where

$$\widetilde{f}(x) = \sum_{m=1}^{\infty} \widetilde{f}_m y_m(x),\ \ \widetilde{f}_m = \frac{\psi_m}{\Lambda_m} \tag{4.8}$$

Consider the functions $\varphi_0(t), \varphi_1(t), \varphi_2(t)$, defined as follows. Function $\varphi_0(t)$ is solution of Cauchy problem

$$\begin{cases} \varphi_0''(t) = \widetilde{f}(x^0) r_0(t) + \int_0^t K(t,s) r_0(s)\, ds, \\ \qquad \varphi_0(0) = \varphi_0'(0) = 0, \end{cases} \tag{4.9}$$

where

$$K(t,s) = -\sum_{m=1}^{\infty} \sqrt{\lambda_m}\, \widetilde{f}_m \sin\sqrt{\lambda_m}(t-s) y_m(x^0).$$

Functions φ_1, φ_2 satisfying the conditions

$$\varphi_1(t) = \sum_{m=1}^{\infty} \frac{\widetilde{b}_{1,m}}{\sqrt{\lambda_m}} y_m(x^0) \sin\sqrt{\lambda_m}\, t, \tag{4.10}$$

$$\varphi_2(t) = \sum_{m=1}^{\infty} y_m(x^0)\left(\widetilde{d}_m \cos\sqrt{\lambda_m}t + \frac{\widetilde{b}_{2,m}}{\sqrt{\lambda_m}}\sin\sqrt{\lambda_m}t\right), \tag{4.11}$$

where

$$\widetilde{b}_{1,m} = -\rho_{0\tau}(0,0)\widetilde{f}_m, \tag{4.12}$$

$$\widetilde{d}_m = -\rho_0(0,0)\widetilde{f}_m, \tag{4.13}$$

$$\widetilde{b}_{2,m} = -(2\rho_{1t}(0,0) + \rho_{0t}(0,0))\widetilde{f}_m. \tag{4.14}$$

The inverse problem 3 is to find a functions f and r_1 such that $f \in \mathbf{F_3}, r_1$ is 2π-periodic in τ and, moreover, $r_1, r_{1t}, r_{1tt}, r_{1ttt} \in C(D)$ for which the solution $u_\omega(x,t)$ of problem (2.1)–(2.3) satisfies the conditions

$$\left\| u_\omega(x^0,t) - \left[\varphi_0(t) + \frac{1}{\omega}\varphi_1(t) + \frac{1}{\omega^2}\big(\varphi_2(t) + \chi(t,\omega t)\big)\right]\right\|_{C([0,T])} = o(\omega^{-2}), \tag{4.15}$$

$$\left\| u_\omega(x,t_0) - \psi(x)\right\|_{C(\overline{\Omega})} = o(1), \omega \to \infty. \tag{4.16}$$

Theorem 6 *Let functions r_0, ψ, χ and points x^0, t_0 satisfy the conditions specified above. Then inverse problem 3 is uniquely solvable. At the same time, the function $f(x) = \widetilde{f}(x)$ calculated by* (4.8)*, and*

$$r_1(t,\tau) = (f(x^0))^{-1}\frac{\partial^2}{\partial\tau^2}\chi(t,\tau). \tag{4.17}$$

5 Proofs of the Main Results

Proof of Theorem 4 Consider the function

$$W_\omega(x,t) = u_\omega(x,t) - u_0(x,t) = \sum_{m=1}^{\infty}\frac{f_m y_m(x)}{\lambda_m}\int_0^t \sin\sqrt{\lambda_m}(t-s)r_1(s,\omega s)ds, \tag{5.1}$$

Note that, in view of Lemmas 1, 2 and Cauchy–Schwarz inequality, the series in right-hand side of (5.1) converges uniformly with respect to $t \in [0, T]$. Represent $W_\omega(x, t)$ in the form

$$W_\omega(x,t) = \sum_{m=1}^{m_0} \frac{f_m y_m(x)}{\lambda_m} \int_0^t \sin\sqrt{\lambda_m}(t-s) r_1(s,\omega s)ds +$$

$$\sum_{m=m_0+1}^{\infty} \frac{f_m y_m(x)}{\lambda_m} \int_0^t \sin\sqrt{\lambda_m}(t-s) r_1(s,\omega s)ds \equiv S_{\omega,1} + S_{\omega,2}, m_0 \in \mathbb{N}.$$

Let ε is arbitrary value. Taking into account uniform convergence of the series (5.1), we take number m_0 sufficiently large such that for all $m, m \geq m_0$, and $\omega > 0$

$$\|S_{\omega,2}\|_{C(\overline{\Omega})} < \frac{\varepsilon}{2}. \tag{5.2}$$

For the estimation of $S_{\omega,1}$ choose $\delta > 0$ so small that

$$\int_0^\delta \sin\sqrt{\lambda_m}(t-s) r_1(s,\omega s)ds < \frac{\varepsilon}{2m_0 s_0}, \tag{5.3}$$

where $s_0 = \max_{1\leq i\leq m_0}\left|\frac{f_i}{\lambda_i}\right| \|y_i\|_{C(\overline{\Omega})}$. Further, considering $t \in (s, T]$, we divide the interval $[\delta, t]$ into k equal parts $[t_j, t_{j+1})$, $j = \overline{0, k-1}$, and apply the relation

$$\int_\delta^t \sin\sqrt{\lambda_m}(t-s) r_1(s,\omega s)ds =$$

$$\sum_{j=0}^{k-1}\left[\int_{t_j}^{t_{j+1}} \sin\sqrt{\lambda_m}(t-s) r_1(s,\omega s)ds - \int_{t_j}^{t_{j+1}} \sin\sqrt{\lambda_m}(t-t_j) r_1(t_j,\omega s)ds\right] +$$

$$\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} \sin\sqrt{\lambda_m}(t-t_j) r_1(t_j,\omega s)ds = S_1 + S_2.$$

Choose $k = k(t)$ so large that

$$|S_1| < \frac{\varepsilon}{4m_0 s_0} \tag{5.4}$$

for all $m : m < m_0$ and $\omega > 0$.

Further, in view of equality $\langle r_1(t, \tau)\rangle_\tau = 0$, we choose ω_0 sufficiently large that

$$|S_2| < \frac{\varepsilon}{4m_0 s_0} \tag{5.5}$$

for given $k, t \in [0, T]$, and any $\omega > \omega_0$.

Since inequalities (5.4), (5.5) there exists a number $\omega_0 > 0$ such that

$$|S_{\omega,1}| < \frac{\varepsilon}{2} \tag{5.6}$$

for any $\omega > \omega_0$. Relations (5.2), (5.6) imply the relation (4.2). This completes the proof of Theorem 4.

Proof of the Lemma 4 Choose t_0 that $|r_0(t_0)| > |r_0(0)|$ and apply the relation

$$\Lambda_m(t_0) = \int_0^{t_0} \frac{\sin\sqrt{\lambda_m}(t_0 - s)}{\sqrt{\lambda_m}} r_0(s)ds = \frac{r_0(t_0) - r_0(0)\cos\sqrt{\lambda_m}t_0}{\lambda_m} + \int_0^{t_0} \frac{\cos\sqrt{\lambda_m}(t_0 - s)}{\lambda_m} r_0'(s)ds.$$

Taking into account the condition (4.4), note that $|r_0(t_0)| \neq |r_0(0)\cos\sqrt{\lambda_m}t_0|$ for all $m \in \mathbb{N}$. Thus there exist positive values c_0 and m_0 such that

$$|\Lambda_m(t_0)| > \frac{c_0}{\lambda_m}$$

for $m > m_0$. The Lemma is proved.

Proof of Theorem 5 Choose t_0 that $|r_0(t_0)| > |r_0(0)|$. We assume that the function $f \in \mathbf{F_2}$ is found. It follows from Theorem 4 and conditions (3.11), (4.6) that

$$\sum_{m=1}^{\infty} f_m y_m(x)\Lambda_m(t_0) = \sum_{m=1}^{\infty} \psi_m y_m(x).$$

For $\Lambda_m \neq 0, m \in \mathbb{N}(M_0 = \emptyset)$ we obtain

$$f(x) = \sum_{m=1}^{\infty} f_m y_m(x),\ f_m = \frac{\psi_m}{\Lambda_m(t_0)}.$$

It remains to show that f belongs to class $\mathbf{F_2}$.

In the first place we shall show that function $f \in C^{[\frac{n}{2}]+2}(\overline{\Omega})$. Let us consider the series

$$L^{[\frac{n}{2}]+2} f(x) = \sum_{m=1}^{\infty} \frac{\psi_m}{\Lambda_m(t_0)} L^{[\frac{n}{2}]+2} y_m(x) =$$

$$\sum_{m=1}^{m_0} \frac{\psi_m}{\Lambda_m(t_0)} L^{[\frac{n}{2}]+2} y_m(x) + \sum_{m=m_0+1}^{\infty} \frac{\psi_m}{\Lambda_m(t_0)} \lambda_m^{[\frac{n}{2}]+2} y_m(x) = Y_1 + Y_2,$$

In view of Lemmas 1, 2, 4 and Cauchy–Schwarz inequality, series Y_2 may be estimate as follows

$$\|Y_2\|_{L_2(\Omega)} \le \frac{1}{c_0} \sqrt{\sum_{m=m_0+1}^{\infty} \frac{y_m^2(x)}{\lambda_m^{[\frac{n}{2}]+1}} \cdot \sum_{m=m_0+1}^{\infty} \psi_m^2 \lambda_m^{3[\frac{n}{2}]+7}},$$

where c_0 and m_0 are the same as in Lemma 4.

Further, let g denote the function $g(x) = L^{[\frac{n}{2}]+2} f(x)$, $g \in L_2(\Omega)$. Thus

$$f = L^{-[\frac{n}{2}]-2} g.$$

As in the Lemma 3 consider $D^{[\frac{n}{2}]+2}$ is the derivative of order $\left[\frac{n}{2}\right] + 2$, and then apply it to the function f, we obtain

$$D^{[\frac{n}{2}]+2} f = D^{[\frac{n}{2}]+2} L^{-[\frac{n}{2}]-2} g.$$

From the Lemma 3 it follows that function $D^{[\frac{n}{2}]+2} f$ is continuous.

Note that, since proved smoothness of the function f and properties of the eigenfunctions $y_m(x)$ it follows that for founded function $f(x)$ conditions (4.1) are hold. This completes the proof of Theorem 5.

Proof of Theorem 6 Let the hypotheses of current theorem holds. Then according to Theorem 5 the inverse problem 2 with given functions r_0, ψ and point t_0 is uniquely solvable, and function $\widetilde{f}$ calculable by (4.8) is the inverse problem 2 solution. Providing similar to Theorem 5 reasoning we obtain that $f \in \mathbf{F_3}$.

Further, consider system (2.1)–(2.3) with $f(x,t) = \widetilde{f}(x)$, and also the inverse problem 1 with given functions χ, φ_i, $i = \overline{0,2}$ and point x^0. In view condition (4.9), the function $r_0(t)$ satisfies Volterra equation of the second kind

$$\varphi_0''(t) = \widetilde{f}(x^0) r_0(t) + \int_0^t K(t,s) r_0(s)\, ds,$$

$$K(t,s) = -\sum_{m=1}^{\infty} \sqrt{\lambda_m}\, \widetilde{f}_m \sin\sqrt{\lambda_m}(t-s) y_m(x^0).$$

From Theorem 3 it follows that the inverse problem 1 with given data is uniquely solvable, moreover, its solution may be represented in form $r(t, \tau) = r_0(t) + r_1(t, \tau)$, where r_1 calculated by (4.17). Because of the conditions on function r_0 the inverse problem 1, e.g. solution r, belongs to the class $\mathbf{R_3}$.

Hence pair of functions $\widetilde{f}, r_1$ is solution of the inverse problem 3. This completes the proof of this Theorem.

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On the Brezis–Lieb Lemma and Its Extensions

E. Y. Emelyanov and M. A. A. Marabeh

Abstract Based on employing the unbounded order convergence instead of the almost everywhere convergence, we identify and study a class of Banach lattices in which the Brezis–Lieb lemma holds true. This gives also a net-version of the Brezis–Lieb lemma in L^p for $p \in [1, \infty)$. We discuss an operator version of the Brezis–Lieb lemma in certain convergence vector lattices.

Keywords $a.e.$-convergence · Brezis–Lieb lemma · Banach lattice · uo-convergence · Brezis–Lieb space · Pre-Brezis–Lieb property

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1 Introduction

Throughout this paper, (Ω, Σ, μ) stands for a measure space in which every set $A \in \Sigma$ of nonzero measure has a subset $A_0 \subseteq A$, $A_0 \in \Sigma$, such that $0 < \mu(A_0) < \infty$. It is known that the Fatou lemma is the following implication

$$f_n \xrightarrow{\text{a.e.}} f \implies \int |f| d\mu \leqslant \liminf \int |f_n| d\mu, \tag{1}$$

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where (f_n) is a sequence in $\mathcal{L}^0(\mu)$. The Brezis–Lieb lemma [2, Thm.2] is a refinement of the Fatou lemma.

Theorem 1 (The Brezis–Lieb Lemma) *Let $j : \mathbb{C} \to \mathbb{C}$ be a continuous function with $j(0) = 0$ such that, for every $\varepsilon > 0$, there exist two continuous functions $\phi_\varepsilon, \psi_\varepsilon : \mathbb{C} \to \mathbb{R}_+$ with*

$$|j(x+y) - j(x)| \leqslant \varepsilon\phi_\varepsilon(x) + \psi_\varepsilon(y) \quad (\forall x, y \in \mathbb{C}). \tag{2}$$

Let f be a $\mathbb{C}$-valued function in $\mathcal{L}^0(\mu)$ and (g_n) be a sequence of $\mathbb{C}$-valued functions in $\mathcal{L}^0(\mu)$ such that $g_n \xrightarrow{\text{a.e.}} 0$; $j(f)$, $\phi_\varepsilon(g_n)$, $\psi_\varepsilon(f) \in \mathcal{L}^1(\mu)$ for all $\varepsilon > 0$, $n \in \mathbb{N}$; and let

$$\sup_{\varepsilon>0,\, n\in\mathbb{N}} \int \phi_\varepsilon(g_n(\omega))d\mu(\omega) \leqslant C < \infty.$$

Then

$$\lim_{n\to\infty} \int |j(f+g_n) - (j(f) + j(g_n))|d\mu = 0. \tag{3}$$

Two measure-free versions of Theorem 1 were proved in vector lattices in [5, 9]. The following fact is a corollary of Theorem 1 (see [2, Thm.1]).

Theorem 2 (The Brezis–Lieb Lemma for $\mathcal{L}^p$ $(0 < p < \infty)$) *Suppose $f_n \xrightarrow{\text{a.e.}} f$ and $\int |f_n|^p d\mu \leqslant C < \infty$ for all n and some $p \in (0, \infty)$. Then*

$$\lim_{n\to\infty} \int (|f_n|^p - |f_n - f|^p)d\mu = \int |f|^p d\mu. \tag{4}$$

Proof We reproduce short and instructive arguments from [2]. Take $j(z) = \phi_\varepsilon(z) := |z|^p$ and $\psi_\varepsilon(z) = C_\varepsilon|z|^p$ for a sufficiently large C_ε. Theorem 1 applied to the sequence (g_n), where $g_n = f_n - f$, gives

$$\lim_{n\to\infty} \int (|f_n|^p - (|f|^p + |f_n - f|^p))d\mu = 0. \tag{5}$$

The uniform boundedness assumption on the sequence (f_n) together with (5) ensure

$$\int |f|^p d\mu \leqslant \limsup_{n\to\infty} \int (|f|^p + |f_n - f|^p)d\mu \leqslant C. \tag{6}$$

Formula (6) allows us to rewrite (5) as (4). □

The Fatou lemma (in the case of a uniformly $\mathcal{L}^p$-bounded sequence (f_n)) follows from Theorem 2, since

$$f_n \xrightarrow{\text{a.e.}} f \;\Rightarrow\; \int |f|^p d\mu = \lim_{n\to\infty} \int (|f_n|^p - |f_n - f|^p) d\mu \leqslant \liminf \int |f_n|^p d\mu$$
$$\Rightarrow \int |f| d\mu \leqslant \liminf \int |f_n| d\mu.$$

The next theorem is an immediate corollary of Theorem 2. Notice that the case $p > 1$ was obtained by Frigyes Riesz [11, p.59].

It is known that almost everywhere equality of measurable functions is an equivalence relation. An equivalence class is denoted by $\mathbf{f}$. The notion L^p means the collection of all equivalence classes $\mathbf{f}$ for which $\int |f|^p < \infty$, $f \in \mathbf{f}$.

Theorem 3 (The Brezis–Lieb Lemma for L^p $(1 \leqslant p < \infty)$**)** *Let* $(\mathbf{f}_n)$ *be a sequence in* $L^p(\mu)$ *such that* $\mathbf{f}_n \xrightarrow{\text{a.e.}} \mathbf{f}$ *in* $L^p(\mu)$ *and* $\|\mathbf{f}_n\|_p \to \|\mathbf{f}\|_p$*, where* $\|\mathbf{f}_n\|_p := \left(\int\limits_\Omega |f_n|^p d\mu\right)^{1/p}$ *with* $f_n \in \mathcal{L}^p(\mu)$ *and* $f_n \in \mathbf{f}_n$*. Then* $\|\mathbf{f}_n - \mathbf{f}\|_p \to 0$.

The fact that Theorem 3 becomes a Banach-lattice-result by replacing $a.e.$-convergence with uo-convergence, motivates investigation of the class of Banach lattices in which Theorem 2 yields for uo-convergence. One more important reason for this investigation lies at the sequential nature of $a.e.$-convergence, which makes obstacles in obtaining net-versions of the Brezis–Lieb lemma. To show this, we include [6, Example 1]. Let μ be the Lebesgue measure on $[0, 1]$, $\mathcal{P}_{fin}[0, 1]$ the family of all finite subsets of $[0, 1]$ ordered by inclusion, and $\mathbb{I}_F$ the indicator function of $F \in \mathcal{P}_{fin}[0, 1]$. Then $\mathbb{I}_F \xrightarrow{\text{a.e.}} \mathbb{I}_{[0,1]}$ and $\int\limits_0^1 |\mathbb{I}_F| d\mu = 0$, however

$$\lim_{F\to\infty} \int\limits_0^1 (|\mathbb{I}_F| - |\mathbb{I}_F - \mathbb{I}_{[0,1]}|) d\mu = \lim_{F\to\infty} \int\limits_0^1 (-|\mathbb{I}_{[0,1]}|) d\mu = -1 \neq 1 = \int\limits_0^1 |\mathbb{I}_{[0,1]}| d\mu.$$

Proposition 2 below may serve as a net extension of Theorem 3.

After introducing Brezis–Lieb spaces, we present and discuss an internal geometric characterization of Brezis–Lieb spaces in Theorem 4 [6, Thm.4]. Possible extensions of Theorem 4 to locally solid vector lattices are also considered. In the last part of the paper, we prove Theorem 5 which is an operator version of Theorem 1 in convergence spaces.

In the paper, we consider normed lattices over the complex field $\mathbb{C}$ which are *complexifications* of uniformly complete real normed lattices. More precisely, the modulus of $z = x + iy \in E = F \oplus iF$ is defined by

$$|z| = \sup_{\theta\in[0,2\pi)} [x\cos\theta + y\sin\theta],$$

and its norm is defined by $\|z\| = \|z\|_E := \|\,|z|\,\|_F$. We also adopt notations $E_+ = F_+$, $z = [z]_r + i[z]_i$, $x = \mathbb{R}\mathrm{e}[z]$, and $y = \mathbb{I}\mathrm{m}[z]$ for $z = x + iy$ in E. A net (v_α) in a vector lattice E is said to be *uo*-convergent to $v \in E$ whenever, for every $u \in E_+$, the net $(|v_\alpha - v| \wedge u)$ converges in order to 0.

2 Brezis–Lieb Spaces

We begin with the following definition [6, Def.1] that is motivated by Theorem 3.

Definition 1 A normed lattice $(E, \|\cdot\|)$ is said to be a *Brezis–Lieb space* (*shortly, a BL-space*) (resp. *σ-Brezis–Lieb space* (*σ-BL-space*)) if, for any net (x_α) (resp, for any sequence (x_n)) in X such that $\|x_\alpha\| \to \|x_0\|$ (resp. $\|x_n\| \to \|x_0\|$) and $x_\alpha \xrightarrow{\mathrm{uo}} x_0$ (resp. $x_n \xrightarrow{\mathrm{uo}} x_0$), there holds $\|x_\alpha - x_0\| \to 0$ (resp. $\|x_n - x_0\| \to 0$).

Clearly, any BL-space is a σ-BL-space, and any finite-dimensional normed lattice is a BL-space. Since the $a.e.$-convergence for sequences in L^p coincides with the uo-convergence [8, Prop.3.1], Theorem 3 says that L^p is a σ-BL-space for $1 \leqslant p < \infty$. The Banach lattice c_0 is not a σ-BL-space. Indeed, let (x_n) be a sequence in c_0 given by $x_n = e_{2n} + \sum\limits_{k=1}^{n} \frac{1}{k} e_k$, and let $x = \sum\limits_{k=1}^{\infty} \frac{1}{k} e_k$ in c_0. Then $\|x\| = \|x_n\| = 1$ for all $n \in \mathbb{N}$, and $x_n \xrightarrow{\mathrm{uo}} x$, however $1 = \|x - x_n\|$ does not converge to 0. A minor change of a BL-space may turn it into a normed lattice which is not even a σ-BL-space [6, EX.4]. Indeed, take any infinite dimensional BL-space E and consider $E_1 = \mathbb{R} \oplus_\infty E$. Take a disjoint sequence (y_n) in E such that $\|y_n\|_E \equiv 1$. Then $y_n \xrightarrow{\mathrm{uo}} 0$ in E [8, Cor.3.6]. For each $n \in \mathbb{N}$, let $x_n = (1, y_n) \in E_1$. Then $\|x_n\|_{E_1} = \sup(1, \|y_n\|_E) = 1$ and $x_n = (1, y_n) \xrightarrow{\mathrm{uo}} (1, 0) =: x$ in E_1, however $\|x_n - x\|_{E_1} = \|(0, y_n)\|_{E_1} = \|y_n\|_E = 1$ and so, (x_n) does not converge to x in $(E_1, \|\cdot\|_{E_1})$. Therefore $E_1 = \mathbb{R} \oplus_\infty E$ is not a σ-Brezis–Lieb space. It could be interesting to construct an example of a σ-BL-space which is not a BL-space. The following result of Vladimir Troitsky gives a condition under which a σ-BL-space is a BL-space (see [6, Prop.2]).

Proposition 1 *A Banach lattice with the countable sup property and a weak unit is a BL-space iff it is a σ-BL-space.*

The next definition [6, Def.2] will be used for characterizing BL-spaces.

Definition 2 A normed lattice $(E, \|\cdot\|)$ is said to have the *pre-Brezis–Lieb property* (*shortly, pre-BL property*), whenever $\limsup\limits_{n\to\infty} \|u_0 + u_n\| > \|u_0\|$ for any disjoint normalized sequence $(u_n)_{n=1}^{\infty}$ in E_+ and for any $u_0 \in E$, $u_0 > 0$.

Every finite dimensional normed lattice has the pre-BL property. The Banach lattice c_0 obviously does not possess the pre-BL property. The mentioned modification of the norm in an infinite-dimensional Banach lattice E as above turns it to a Banach lattice $E_1 = \mathbb{R} \oplus_\infty E$ without pre-BL property. Indeed, take a disjoint normalized sequence $(y_n)_{n=1}^\infty$ in E_+. Let $u_0 = (1, 0)$ and $u_n = (0, y_n)$ for $n \geqslant 1$. Then $(u_n)_{n=0}^\infty$ is a disjoint normalized sequence in $(E_1)_+$ with $\limsup_{n\to\infty} \|u_0 + u_n\| = 1 = \|u_0\|$.

The real version of the following result is included in [6, Thm.4]. Here we provide its complex version.

Theorem 4 *For a σ-Dedekind complete Banach lattice E, the following conditions are equivalent*:

(1) *E is a BL-space*;
(2) *E is a σ-BL-space*;
(3) *E possesses the pre-BL property and has order continuous norm.*

Proof (1) $\Rightarrow$ (2) It is trivial.

(2) $\Rightarrow$ (3) We show first that E has the pre-BL property. Suppose that there exist a disjoint normalized sequence $(u_n)_{n=1}^\infty$ in E_+ and $u_0 \in E_+$ with $\limsup_{n\to\infty} \|u_0 + u_n\| = \|u_0\|$. Since $\|u_0 + u_n\| \geqslant \|u_0\|$, then $\lim_{n\to\infty} \|u_0 + u_n\| = \|u_0\|$. Denote $v_n := u_0 + u_n$. By Gao et al. [8, Cor.3.6], $u_n \xrightarrow{\text{uo}} 0$ and hence $v_n \xrightarrow{\text{uo}} u_0$. Since E is a σ-BL-space and $\lim_{n\to\infty} \|v_n\| = \|u_0\|$, then $\|v_n - u_0\| \to 0$, which is impossible in view of $\|v_n - u_0\| = \|u_0 + u_n - u_0\| = \|u_n\| = 1$. In this part of the proof, both σ-Dedekind and norm completeness of E were not used.

If the norm in E is not order continuous then, by the Fremlin-Meyer-Nieberg theorem (see e.g. [1, Thm.4.14]), there exist $y \in E_+$ and a disjoint sequence (e_k) in $[0, y]$ such that $\|e_k\| \not\to 0$. Without loss of generality, we may assume $\|e_k\| = 1$ for all $k \in \mathbb{N}$. By σ-Dedekind completeness of E, for any sequence (α_n) in $\mathbb{R}_+$, there exist

$$x_0 = \bigvee_{k=1}^{\infty} e_k, \quad x_n = \alpha_{2n} e_{2n} + \bigvee_{k=1, k\neq n, k\neq 2n}^{\infty} e_k \qquad (\forall n \in \mathbb{N}). \tag{7}$$

Now, we choose $\alpha_{2n} \geqslant 1$ in (7) such that $\|x_n\| = \|x_0\|$ for all $n \in \mathbb{N}$. Clearly, $x_n \xrightarrow{\text{uo}} x_0$. Since E is a σ-BL-space, then $\|x_n - x_0\| \to 0$, violating

$$\|x_n - x_0\| = \|(\alpha_{2n} - 1)e_{2n} - e_n\| = \|(\alpha_{2n} - 1)e_{2n} + e_n\| \geqslant \|e_n\| = 1.$$

The obtained contradiction shows that the norm in E is order continuous.

(3) $\Rightarrow$ (1) If E is not a BL-space, then there exists a net $(x_\alpha)_{\alpha\in A}$ in E such that $x_\alpha \xrightarrow{\text{uo}} x$ and $\|x_\alpha\| \to \|x\|$, but $\|x_\alpha - x\| \not\to 0$. Then $|x_\alpha| \xrightarrow{\text{uo}} |x|$ and $\||x_\alpha|\| \to \||x|\|$.

Note that $\||x_\alpha| - |x|\| \not\to 0$. Indeed, if $\||x_\alpha| - |x|\| \to 0$, then, for any $\varepsilon > 0$, $(|x_\alpha|)_{\alpha\in A}$ is eventually in $[-|x|, |x|] + \varepsilon B_E$. Thus $(|x_\alpha|)_{\alpha\in A}$, and hence $(\mathbb{Re}[x_\alpha])_{\alpha\in A}$ and $(\mathbb{Im}[x_\alpha])_{\alpha\in A}$ are both almost order bounded. Since E is order continuous and $x_\alpha \xrightarrow{\text{uo}} x$, then $\mathbb{Re}[x_\alpha] \xrightarrow{\text{uo}} \mathbb{Re}[x]$ and $\mathbb{Im}[x_\alpha] \xrightarrow{\text{uo}} \mathbb{Im}[x]$. By Gao and Xanthos [7, Pop.3.7.], $\|\mathbb{Re}[x_\alpha - x]\| \to 0$ and $\|\mathbb{Im}[x_\alpha - x]\| \to 0$, and hence $\|x_\alpha - x\| \to 0$, that is impossible. Therefore, without loss of generality, we may assume $x_\alpha \in E_+$ and—by normalizing—$\|x_\alpha\| = \|x\| = 1$ for all α.

Passing to a subnet, denoted by (x_α) again, we may assume

$$\|x_\alpha - x\| > C > 0 \quad (\forall \alpha \in A). \tag{8}$$

Notice that $x \geqslant (x - x_\alpha)^+ = (x_\alpha - x)^- \xrightarrow{\text{uo}} 0$, and hence $(x_\alpha - x)^- \xrightarrow{\text{o}} 0$. Order continuity of the norm in E ensures

$$\|(x_\alpha - x)^-\| \to 0. \tag{9}$$

Denoting $w_\alpha = (x_\alpha - x)^+$ and using (8) and (9), we assume

$$\|w_\alpha\| = \|(x_\alpha - x)^+\| > C \quad (\forall \alpha \in A). \tag{10}$$

In view of (10), we obtain

$$2 = \|x_\alpha\| + \|x\| \geqslant \|(x_\alpha - x)^+\| = \|w_\alpha\| > C \quad (\forall \alpha \in A). \tag{11}$$

Since $w_\alpha \xrightarrow{\text{uo}} (x - x)^+ = 0$, for any fixed $\beta_1, \beta_2, \ldots, \beta_n$,

$$0 \leqslant w_\alpha \wedge (w_{\beta_1} + w_{\beta_2} + \ldots + w_{\beta_n}) \xrightarrow{\text{o}} 0 \quad (\alpha \to \infty). \tag{12}$$

Since $x_\alpha \xrightarrow{\text{uo}} x$, then $x_\alpha \wedge x \xrightarrow{\text{uo}} x \wedge x = x$ and so $x_\alpha \wedge x \xrightarrow{\text{o}} x$. Due to order continuity of the norm in E, there exists an increasing sequence of indices (α_n) in A with

$$\|x - x_\alpha \wedge x\| \leqslant 2^{-n} \quad (\forall \alpha \geqslant \alpha_n).$$

By (12), we also suppose

$$\|w_\alpha \wedge (w_{\alpha_1} + w_{\alpha_2} + \ldots + w_{\alpha_n})\| \leqslant 2^{-n} \quad (\forall \alpha \geqslant \alpha_{n+1}).$$

Since

$$\begin{aligned}\sum_{k=1, k\neq n}^{\infty} \|w_{\alpha_n} \wedge w_{\alpha_k}\| &\leqslant \sum_{k=1}^{n-1} \|w_{\alpha_n} \wedge (w_{\alpha_1} + \ldots + w_{\alpha_{n-1}})\| \\ &\quad + \sum_{k=n+1}^{\infty} \|w_{\alpha_k} \wedge (w_{\alpha_1} + \ldots + w_{\alpha_{k-1}})\| \\ &\leqslant (n-1) \cdot 2^{-n+1} + \sum_{k=n+1}^{\infty} 2^{-k+1} = n2^{-n+1},\end{aligned} \tag{13}$$

the series $\sum_{k=1,k\neq n}^{\infty} w_{\alpha_n} \wedge w_{\alpha_k}$ converges absolutely and hence in norm for any $n \in \mathbb{N}$. Take

$$\omega_{\alpha_n} := \left(w_{\alpha_n} - \sum_{k=1,k\neq n}^{\infty} w_{\alpha_n} \wedge w_{\alpha_k} \right)^+ \quad (\forall n \in \mathbb{N}).$$

First, we show that the sequence $(\omega_{\alpha_n})_{n=1}^{\infty}$ is disjoint. Let $m \neq p$, then

$$\begin{aligned}
\omega_{\alpha_m} \wedge \omega_{\alpha_p} &= \left(w_{\alpha_m} - \sum_{k=1,k\neq m}^{\infty} w_{\alpha_m} \wedge w_{\alpha_k} \right)^+ \wedge \left(w_{\alpha_p} - \sum_{k=1,k\neq p}^{\infty} w_{\alpha_p} \wedge w_{\alpha_k} \right)^+ \\
&\leqslant (w_{\alpha_m} - w_{\alpha_m} \wedge w_{\alpha_p})^+ \wedge (w_{\alpha_p} - w_{\alpha_p} \wedge w_{\alpha_m})^+ \\
&= (w_{\alpha_m} - w_{\alpha_m} \wedge w_{\alpha_p}) \wedge (w_{\alpha_p} - w_{\alpha_m} \wedge w_{\alpha_p}) \\
&= 0
\end{aligned}$$

It follows by (13), that

$$\begin{aligned}
\|w_{\alpha_n} - \omega_{\alpha_n}\| &= \left\| w_{\alpha_n} - \left(w_{\alpha_n} - \sum_{k=1,k\neq n}^{\infty} w_{\alpha_n} \wedge w_{\alpha_k} \right)^+ \right\| \\
&= \left\| w_{\alpha_n} - \left(w_{\alpha_n} - w_{\alpha_n} \wedge \sum_{k=1,k\neq n}^{\infty} w_{\alpha_n} \wedge w_{\alpha_k} \right) \right\| \\
&= \left\| w_{\alpha_n} \wedge \sum_{k=1,k\neq n}^{\infty} w_{\alpha_n} \wedge w_{\alpha_k} \right\| \\
&\leqslant \| \sum_{k=1,k\neq n}^{\infty} w_{\alpha_n} \wedge w_{\alpha_k} \| \\
&\leqslant n2^{-n+1}, \quad (\forall n \in \mathbb{N}). \qquad (14)
\end{aligned}$$

Combining (14) with (11) gives

$$2 \geqslant \|w_{\alpha_n}\| \geqslant \|\omega_{\alpha_n}\| \geqslant C - n2^{-n+1} \quad (\forall n \in \mathbb{N}).$$

Passing to the further increasing sequence of indices, we may assume that

$$\|w_{\alpha_n}\| \to M \in [C, 2] \quad (n \to \infty).$$

Now

$$\begin{aligned}\lim_{n\to\infty}\left\|M^{-1}x+\|\omega_{\alpha_n}\|^{-1}\omega_{\alpha_n}\right\| &= M^{-1}\lim_{n\to\infty}\|x+\omega_{\alpha_n}\| \quad \text{by (14)}\\ &= M^{-1}\lim_{n\to\infty}\|x+w_{\alpha_n}\| \quad \text{by (9)}\\ &= M^{-1}\lim_{n\to\infty}\|x+(x_{\alpha_n}-x)\|\\ &= M^{-1}\lim_{n\to\infty}\|x_{\alpha_n}\|\\ &= M^{-1}\\ &= \|M^{-1}x\|,\end{aligned}$$

violating the pre-BL property for $u_0 = M^{-1}x$ and $u_n = \|\omega_{\alpha_n}\|^{-1}\omega_{\alpha_n}$, $n \geqslant 1$. The obtained contradiction completes the proof. □

A special case of Theorem 4 was proved by Nakano [10, Thm.33.6]. The following result, which follows from Theorem 4, can be considered as a *lemma of Brezis–Lieb type for nets in* L^p.

Proposition 2 *Let* $\mathbf{f}_\alpha \xrightarrow{\text{uo}} \mathbf{f}$ *and* $\|\mathbf{f}_\alpha\|_p \to \|\mathbf{f}\|_p$ *in* $L^p(\mu)$, $1 \leqslant p < \infty$. *Then* $\|\mathbf{f}_\alpha - \mathbf{f}\|_p \to 0$.

It is not clear whether or not implication (2) $\Rightarrow$ (3) of Theorem 4 holds without the assumption that E is σ-Dedekind complete. Since any σ-Brezis–Lieb Banach lattice has the pre-BL property, for dropping σ-Dedekind completeness assumption in Theorem 4, it is sufficient to have the positive answer to the following weaker question.

Question 1 Does the pre-BL property imply order continuity of the norm?

In the end of this section we mention some possible generalizations of Brezis–Lieb spaces and pre-Brezis–Lieb property. To avoid overloading the text, we restrict ourselves to the case of multi-normed Brezis–Lieb spaces.

A multi-normed vector lattice (shortly, MNVL) $E = (E, \mathcal{M})$ (see [4]):

(a) is said to be a *Brezis–Lieb space* if

$$[x_\alpha \xrightarrow{\text{uo}} x_0 \ \ \& \ \ m(x_\alpha) \to m(x_0) \ \ (\forall m \in \mathcal{M})] \Rightarrow [x_\alpha \xrightarrow{\mathcal{M}} x_0];$$

(b) has the *pre-Brezis–Lieb property* if, for any disjoint sequence $(u_n)_{n=1}^\infty$ in E_+ such that (u_n) does not converge in $\mathcal{M}$ to 0 and for any $u_0 > 0$, there exists $m \in \mathcal{M}$ such that $\limsup_{n\to\infty} m(u_0 + u_n) > m(u_0)$.

A σ-Brezis–Lieb MNVL is defined by replacing of nets with sequences.

By using the above definitions one can derive from Theorem 4 the following result.

Corollary 1 *For an MNVL E with a separating order continuous multinorm $\mathcal{M}$, the following conditions are equivalent*:

(1) *E is a BL-space*;
(2) *E is a σ-BL-space*;
(3) *E has the pre-BL property.*

3 Operator Version of the Brezis–Lieb Lemma in Convergent Vector Spaces

In this section we discuss an operator extension of the Brezis–Lieb lemma in convergent vector spaces. Firstly, let us remind some definitions [3]. A *convergence* "$\xrightarrow{c}$" for nets in a set X is defined by the following conditions:

(a) $x_\alpha \equiv x \Rightarrow x_\alpha \xrightarrow{c} x$, and
(b) $x_\alpha \xrightarrow{c} x \Rightarrow x_\beta \xrightarrow{c} x$ for every subnet (x_β) of (x_α).

A mapping f from a *convergence set* (X, c_X) into a convergence set (Y, c_Y) is said to be *$c_X c_Y$-continuous* (or just continuous), if $x_\alpha \xrightarrow{c_X} x$ implies $f(x_\alpha) \xrightarrow{c_Y} f(x)$ for every net (x_α) in X. Under a *convergence vector space* (X, c_X), we understand a vector space X with the convergence c_X such that the linear operations in X are c_X-continuous. (E, c_E) is a *convergence vector lattice* if (E, c_E) is a convergence vector space that is a vector lattice, where the lattice operations are also c_E-continuous. Motivated by the proof of the famous lemma of Brezis and Lieb [2, Thm.2], we present its operator version in convergent spaces.

The following hypotheses will be used in the next theorem.

$(H1)$ Let (X, c_X) be a convergence complex vector space.
$(H2)$ Let (E, c_E) and (F, c_F) be two convergence complex vector lattices, with F is Dedekind complete.
$(H3)$ Let E_0 be an order ideal in $E_+ - E_+$.
$(H4)$ Let $T : E_0 \to F$ be a $c_{E_0} o_F$-continuous positive linear operator, where o_F stands for the order convergence in F.
$(H5)$ Let $J : X \to E$ be a $c_X c_E$-continuous function with $J(0) = 0$.
$(H6)$ For every $\varepsilon > 0$, there exist two $c_X c_E$-continuous mappings $\Phi_\varepsilon, \Psi_\varepsilon : X \to E_+$ satisfying

$$|J(x + y) - Jx| \leqslant \varepsilon \Phi_\varepsilon x + \Psi_\varepsilon y \quad (\forall x, y \in X). \tag{15}$$

Theorem 5 (An Operator Version of the Brezis–Lieb Lemma for Nets) *Suppose hypotheses $(H1)-(H6)$ are satisfied. Let $(g_\alpha)_{\alpha \in A}$ be a net in X satisfying $g_\alpha \xrightarrow{c_X} 0$,* *let $f \in X$ be such that $|Jf|, \Phi_\varepsilon g_\alpha, \Psi_\varepsilon f \in E_0$ for all $\varepsilon > 0, \alpha \in A$, and let some $u \in F_+$ exist with $T\Phi_\varepsilon g_\alpha \leqslant u$ for all $\varepsilon > 0$, $\alpha \in A$. Then*

$$T\Big(|J(f+g_\alpha) - (Jf + Jg_\alpha)|\Big) \xrightarrow{\mathrm{o_F}} 0 \quad (\alpha \to \infty).$$

Proof It follows from (15) that

$$|J(f+g_\alpha) - (Jf+Jg_\alpha)| \leqslant |J(f+g_\alpha) - Jg_\alpha| + |Jf| \leqslant \varepsilon\Phi_\varepsilon g_\alpha + \Psi_\varepsilon f + |Jf|,$$

and hence

$$|J(f+g_\alpha) - (Jf+Jg_\alpha)| - \varepsilon\Phi_\varepsilon g_\alpha \leqslant \Psi_\varepsilon f + |Jf| \quad (\varepsilon > 0, \alpha \in A).$$

Thus

$$0 \leqslant w_{\varepsilon,\alpha} := \Big(|J(f+g_\alpha) - (Jf+Jg_\alpha)| - \varepsilon\Phi_\varepsilon g_\alpha\Big)_+ \leqslant \Psi_\varepsilon f + |Jf| \tag{16}$$

for all $\varepsilon > 0$ and $\alpha \in A$. It follows from (16) and from $c_X c_E$-continuity of J and Φ_ε, that $E_0 \ni w_{\varepsilon,\alpha} \xrightarrow{\mathrm{c_E}} 0$ as $\alpha \to \infty$. Furthermore, (16) implies

$$|J(f+g_\alpha) - (Jf+Jg_\alpha)| \leqslant w_{\varepsilon,\alpha} + \varepsilon\Phi_\varepsilon g_\alpha \quad (\varepsilon > 0, \alpha \in A). \tag{17}$$

Since $T \geqslant 0$ and $T\Phi_\varepsilon g_\alpha \leqslant u$, we get from (17)

$$0 \leqslant T\Big(|J(f+g_\alpha) - (Jf+Jg_\alpha)|\Big) \leqslant Tw_{\varepsilon,\alpha} + \varepsilon T\Phi_\varepsilon g_\alpha \leqslant Tw_{\varepsilon,\alpha} + \varepsilon u \tag{18}$$

for all $\varepsilon > 0$ and $\alpha \in A$. Since F is Dedekind complete and T is $c_{E_0}o_F$-continuous, $Tw_{\varepsilon,\alpha} \xrightarrow{\mathrm{o_F}} 0$, and in view of (18)

$$0 \leqslant (o_F) - \limsup_{\alpha\to\infty} T\Big(|J(f+g_\alpha) - (Jf+Jg_\alpha)|\Big) \leqslant \varepsilon u \quad (\forall \varepsilon > 0).$$

Then $T\Big(|J(f+g_\alpha) - (Jf+Jg_\alpha)|\Big) \xrightarrow{\mathrm{o_F}} 0$. □

We end up by the following remarks on Theorem 5.

1. Replacing nets by sequences one can obtain a sequential version of Theorem 5, whose details are left to the reader.
2. In the case of $F = \mathbb{R}$ and $X = E = L^0(\mu)$ with the almost everywhere convergence, $E_0 = L^1(\mu)$, $Tf = \int f d\mu$, and $J : X \to E$ given by $Jf = j \circ f$,

where $j : \mathbb{C} \to \mathbb{C}$ is continuous with $j(0) = 0$ such that for every $\varepsilon > 0$ there exist two continuous functions $\phi_\varepsilon, \psi_\varepsilon : \mathbb{C} \to \mathbb{R}_+$ satisfying

$$|j(x+y) - j(x)| \leqslant \varepsilon\phi_\varepsilon(x) + \psi_\varepsilon(y) \quad (\forall x, y \in \mathbb{C}),$$

we obtain Theorem 1 from Theorem 5 by letting $\Phi_\varepsilon(f) := \phi_\varepsilon \circ f$ and $\Psi_\varepsilon(f) := \psi_\varepsilon \circ f$.

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On the Commutant of the Generalized Backward Shift Operator in Weighted Spaces of Entire Functions

O. A. Ivanova and S. N. Melikhov

Abstract We study continuous linear operators, which commute with the generalized backward shift operator (a one-dimensional perturbation of the Pommiez operator) in a countable inductive limit E of weighted Banach spaces of entire functions. This space E is isomorphic with the help of the Fourier-Laplace transform to the strong dual of the Fréchet space of all holomorphic functions on a convex domain Q in the complex plane, containing the origin. Necessary and sufficient conditions are obtained for an operator of the mentioned commutant to be a topological isomorphism of E. The problem of factorization of nonzero operators of this commutant is investigated. In the case when the function determining the generalized backward shift operator, has zeros in Q, the commutant is divided into two classes: the first one consists of isomorphisms and surjective operators with finite-dimensional kernels and the second one contains finite-dimensional operators. Using obtained results, we study the generalized Duhamel product in Fréchet space of all holomorphic functions on Q.

Keywords Backward shift operator · Commutant · Weighted space of entire functions · Duhamel product

Mathematical Subject Classification (2010) Primary 46E10, 47B37; Secondary 47A05, 30D15

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1 Introduction

Let Q be a convex domain in $\mathbb{C}$, containing the origin; $H(Q)$ be the Fréchet space of all holomorphic functions on Q; E be a countable inductive limit of weighted Banach spaces which with the help of Fourier-Laplace transform is topologically isomorphic to the strong dual of $H(Q)$. A function $g_0 \in E$ satisfying the condition $g_0(0) = 1$ defines the generalized backward shift operator $D_{0,g_0}(f)(t) = \frac{f(t)-g_0(t)f(0)}{t}$, which is continuous and linear in E. If $g_0 \equiv 1$, then D_{0,g_0} is the usual backward shift operator (Pommiez operator) D_0. In the general case D_{0,g_0} is a one-dimensional perturbation of D_0.

The problem we solve in this article, is to investigate the structure of the set $\mathcal{K}(D_{0,g_0})$ of all continuous linear operators in E commuting with D_{0,g_0} in E. The set $\mathcal{K}(D_{0,g_0})$ has been described in [5]. In the main results we assume that the function g_0 has a finite number of zeros or has no zeros, i.e., $g_0(z) = P(z)e^{\lambda z}$ for some $\lambda \in Q$ and some polynomial P, such that $P(0) = 1$. In Theorems 4.1 and 4.2 it is shown that $\mathcal{K}(D_{0,g_0})$ is divided into two classes. The first one consists of isomorphisms and surjective operators with finite-dimensional kernels and the second one contains finite-dimensional operators. If g_0 has no zeros, i.e., $g_0(z) = e^{\lambda z}$ for some $\lambda \in Q$, then the second class is empty. Previously V.A. Tkachenko [11] investigated properties of the commutant of the operator of generalized integration in a space of analytic functionals. This space is the dual of a countable inductive limit of weighted Banach spaces of entire functions, the growth of which is defined by a ρ-trigonometrically convex function ($\rho > 0$). The operator of generalized integration is the adjoint map of D_{0,g_0}, defined by the function $g_0 = e^{\mathcal{P}}$ for some polynomial $\mathcal{P}$. Such function g_0 has no zeros.

In the dual E' of E shift operators for D_{0,g_0} define a product $\otimes$ by the convolution rule. If we identify the strong dual of E with $H(Q)$ with the help of the adjoint map of the Fourier-Laplace transform, then the operation $\otimes$ is realized in $H(Q)$ as the generalized Duhamel product. In the case of $g_0 \equiv 1$ it coincides with the Duhamel product (with the derivative of the Mikusinski convolution product). The Duhamel product is closely related to the Volterra operator. It was studied quite intensively (see the paper of M.T. Karaev [8]). This multiplication is used in the theory of ordinary differential equations with constant coefficients, in the boundary value problems of mathematical physics (in the sloping beach problem), in the spectral theory of direct sums of operators. Investigations of the Duhamel product in the space of all holomorphic functions on a domain in $\mathbb{C}$ go back to N. Wigley [12]. In this article we prove a criterion for a multiplication operator defined by generalized Duhamel product to be an isomorphism of $H(Q)$.

Note that the situation, when g_0 has zeros, differ significantly from one when g_0 has no zeros. Namely, our proofs use essentially the description of the lattice of proper closed D_{0,g_0}-invariant subspaces of E, obtained in [7]. If g_0 has no zeros, D_{0,g_0} is unicellular. If g_0 has zeros, then this lattice is not linearly ordered, moreover, the family of finite-dimensional closed D_{0,g_0}-invariant subspaces of $H(Q)$ is also not linearly ordered. The mixed structure of this lattice implies the existence of two "extreme" subsets of $\mathcal{K}(D_{0,g_0})$.

2 Preliminary Information

Let Q be a convex domain in $\mathbb{C}$ containing the origin, $(Q_n)_{n\in\mathbb{N}}$ be a sequence of convex compact subsets of Q with $Q_n \subset \operatorname{int} Q_{n+1}$ for all $n \in \mathbb{N}$, and $Q = \bigcup\limits_{n\in\mathbb{N}} Q_n$. The symbol $\operatorname{int} M$ denotes the interior of a set $M \subset \mathbb{C}$ in $\mathbb{C}$. For a bounded set $M \subset \mathbb{C}$ let H_M be the support function of M defined as $H_M(z) := \sup\limits_{t\in\mathbb{C}} \operatorname{Re}(zt)$, $z \in \mathbb{C}$. We set $H_n := H_{Q_n}$, $n \in \mathbb{N}$.

Define weighted Banach spaces

$$E_n := \left\{ f \in H(\mathbb{C}) \mid \|f\|_n := \sup_{z\in\mathbb{C}} \frac{|f(z)|}{\exp(H_n(z))} < +\infty \right\},\ n \in \mathbb{N}.$$

Here $H(\mathbb{C})$ is the space of all entire functions on $\mathbb{C}$. Note that E_n is embedded continuously in E_{n+1} for each $n \in \mathbb{N}$. Put $E := \bigcup\limits_{n\in\mathbb{N}} E_{Q,n}$ and endow E with the topology of the inductive limit of the sequence of Banach spaces E_n, $n \in \mathbb{N}$, with respect to embeddings E_n in E (see [10, Ch. III, § 24]); in symbols, $E := \operatorname{ind}\limits_{n\to} E_n$.

Let $H(Q)$ be the space of all holomorphic functions on Q with the compact convergence topology. For a locally convex space F we denote by F' the dual of F. We put $e_z(t) := e^{zt}$, $z, t \in \mathbb{C}$. The Fourier-Laplace transform $\mathcal{F}(\varphi)(z) := \varphi(e_z)$, $z \in \mathbb{C}$, $\varphi \in H(Q)'$, is a topological isomorphism of the strong dual of $H(Q)$ onto E [3, Theorem 4.5.3].

Fix a function $g_0 \in E$ with $g_0(0) = 1$. The generalized backward shift operator is defined by

$$D_{0,g_0}(f)(t) := \begin{cases} \frac{f(t)-g_0(t)f(0)}{t}, & t \neq 0, \\ f'(0) - g_0'(0)f(0), & t = 0, \end{cases}$$

$f \in E$. Following [1, 2], we introduce *shift operators for the operator* D_{0,g_0} *by putting*

$$T_{z,g_0}(f)(t) := \begin{cases} \frac{tf(t)g_0(z)-zf(z)g_0(t)}{t-z}, & t \neq z, \\ zg_0(z)f'(z) - zf(z)g_0'(z) + f(z)g_0(z), & t = z, \end{cases}$$

$z \in \mathbb{C}$, $f \in E$. Set

$$\widetilde{T}_{z,g_0}(f)(t) := \begin{cases} \frac{f(t)g_0(z) - f(z)g_0(t)}{t-z}, & t \neq z, \\ g_0(z)f'(z) - f(z)g_0'(z), & t = z, \end{cases}$$

$f \in E$, $z \in \mathbb{C}$. For $z \in \mathbb{C}$ the Pommiez operators D_z are defined by

$$D_z(f)(t) := \begin{cases} \frac{f(t)-f(z)}{t-z}, & t \neq z, \\ f'(z), & t = z, \end{cases}$$

$f \in E$. All operators T_{z,g_0}, $\widetilde{T}_{z,g_0}$, D_z, $z \in \mathbb{C}$, are continuous and linear in E.

For an integer $n \geq 0$ by $\mathbb{C}[z]_n$ we denote the space of polynomials of degree at most n. Note that $\operatorname{Ker} D_{0,g_0}^n = g_0\mathbb{C}[z]_{n-1}$ for all $n \in \mathbb{N}$.

With the help of shift operators for D_{0,g_0} in E' one can define a multiplication $\otimes$ by $(\varphi \otimes \psi)(f) = \varphi_z(\psi(T_{z,g_0}(f)))$, $\varphi, \psi \in E'$, $f \in E$. By Ivanova and Melikhov [5, 2.2] the space E' is an associative and commutative algebra with the multiplication $\otimes$.

Let $\mathcal{K}(D_{0,g_0})$ be the set of all continuous linear operators B in E, such that $BD_{0,g_0} = D_{0,g_0}B$ in E. This set an algebra with the operation of composition of operators taken as multiplication. Note that $T_{z,g_0} \in \mathcal{K}(D_{0,g_0})$ for every $z \in \mathbb{C}$. For a functional $\varphi \in E'$ we define the operator $B_\varphi(f)(z) := \varphi\left(T_{z,g_0}(f)\right)$, $z \in \mathbb{C}$, $f \in E$. It is continuous and linear in E.

In [5, Lemma 17] the following result is proved:

Theorem 2.1 *The map $\varphi \mapsto B_\varphi$ is an isomorphism of the algebra $(E', \otimes)$ onto $\mathcal{K}(D_{0,g_0})$.*

From Theorem 2.1 it follows that the algebra $\mathcal{K}(D_{0,g_0})$ is commutative.

From the commutativity of $\otimes$ it follows that for each $\varphi \in E'$ the convolution operator $S_\varphi : E' \to E'$, $\psi \mapsto \varphi \otimes \psi$, is the adjoint of the operator $B_\varphi : E \to E$ with respect to the dual system (E, E').

Remark 2.1 We will use the following well known properties of support functions H_n:

(i) For each $n \in \mathbb{N}$ there is $\varepsilon > 0$, such that

$$\sup_{z \in \mathbb{C}} (H_n(z) + \varepsilon|z| - H_{n+1}(z)) < +\infty.$$

(ii) $\lim\limits_{|t| \to +\infty} \left(\left(\sup\limits_{|\xi - t| \leq \delta} H_n(\xi)\right) - H_{n+1}(t)\right) = -\infty$ for each $n \in \mathbb{N}$ and $\delta > 0$.

Let $\mathcal{F}^t : E' \to H(Q)$ be the adjoint of the operator $\mathcal{F} : H(Q)' \to E$ with respect to dual systems $(H(Q)', H(Q))$ and (E', E). Then $\mathcal{F}^t(\varphi)(z) = \varphi(e_z)$,

$z \in Q$, $\varphi \in E'$. In addition, $\mathcal{F}^t$ is a topological isomorphism of the strong dual of E onto $H(Q)$ (see [6, 3.2]). We will write $\widehat{\varphi} := \mathcal{F}^t(\varphi)$ for $\varphi \in E'$.

By M we will denote the operator of multiplication by the independent variable.

Remark 2.2

(i) By Ivanova and Melikhov [6, Lemma 14] the equality

$$T_{z,g_0}(f) = g_0(z)D_z(M(f)) - M(f)(z)D_z(g_0),\ \ z \in \mathbb{C},\ f \in E,$$

holds.

(ii) For $f \in E\backslash\{0\}$, $h \in H(Q)$ let $\omega_f(z,h)$ be the Leont'ev's interpolating function (see [9]). Using the equality [4, Example 1] $\omega_f(z,h) = \mathcal{F}^{-1}(D_z(f))(h)$, $z \in \mathbb{C}$, we rewrite the equality in (i) for $\varphi \in E'$ as follows:

$$B_\varphi(f)(z) = g_0(z)\omega_{M(f)}(z,\widehat{\varphi}) - M(f)(z)\omega_{g_0}(z,\widehat{\varphi}),\ \ z \in \mathbb{C},\ \ f \in E\backslash\{0\}. \tag{2.1}$$

(iii) Let $f \in E\backslash\{0\}$, $\lambda \in \mathbb{C}$ and $f(\lambda) = 0$. By Leont'ev [9, Lemma 2] for the function $f_1(t) := \frac{f(t)}{t-\lambda}$

$$\omega_f(z,h) = (z-\lambda)\omega_{f_1}(z,h) - \frac{1}{2\pi i}\int\limits_C \gamma_f(t)h(t)dt,\ \ z \in \mathbb{C},\ h \in H(Q). \tag{2.2}$$

Here C is a closed convex curve in Q, which surrounds the conjugate diagram of f, γ_f is the Borel transform of f.

(iv) For each $\varphi \in E'$, $f \in E$ the equality $\varphi(f) = \frac{1}{2\pi i}\int\limits_C \gamma_f(t)\widehat{\varphi}(t)dt$ holds, where C is a closed convex curve in Q, which surrounds the conjugate diagram of f.

The main purpose of this article is to describe operators of $\mathcal{K}(D_{0,g_0})$, which are isomorphisms of E, and to classify operators of $\mathcal{K}(D_{0,g_0})$, which are not isomorphism of E.

3 Auxiliary Results

We put $B_n := \{f \in E_n \mid \|f\|_n \leq 1\}$ and $\|\varphi\|_n^* := \sup\limits_{f \in B_n} |\varphi(f)|$, $\varphi \in E'$, $n \in \mathbb{N}$.

For $\varphi \in E'$ the operator $A_\varphi(f)(z) := \varphi_t\left(t\widetilde{T}_{z,g_0}(f)(t)\right)$, $z \in \mathbb{C}$, $f \in E$, is continuous and linear in E. For each $\varphi \in E'$ the equality $B_\varphi(f) = \varphi(g_0)f + A_\varphi(f)$, $f \in E$, holds. This enables us to study the properties B_φ using the theory of compact operators in Banach spaces. The key tool to this is the following result.

Lemma 3.1 *Let $g_0 \in E_m$ for some $m \in \mathbb{N}$. For each functional $\varphi \in E'$, each $n \geq m$ the operator A_φ is compact in E_n.*

Proof The proof is similar to the one by V.A. Tkachenko [11, Theorem 2]. Since the restriction of φ on each space E_k is continuous on E_k, for all $n \in \mathbb{N}$, $h \in E_{n+2}$ we have

$$|\varphi(h)| \leq \|\varphi\|^*_{n+2}\|h\|_{n+2}.$$

Fix $n \geq m$ and $\varepsilon > 0$, $z \in \mathbb{C}$, $f \in B_n$. For $t \in \mathbb{C}$ such that $|t - z| \geq 1/\varepsilon$ we obtain

$$\frac{|t||f(t)g_0(z) - f(z)g_0(t)|}{|t - z|\exp(H_{n+2}(t))} \leq \varepsilon \left(\frac{|t||f(t)||g_0(z)|}{\exp(H_{n+2}(t))} + \frac{|t||f(z)||g_0(t)|}{\exp(H_{n+2}(t))}\right) \leq$$

$$\varepsilon(C_1|g_0(z)| + C_2|f(z)|) \leq \varepsilon\left(C_1\|g_0\|_n \exp(H_n(z)) + C_2 \exp(H_n(z))\right) =$$

$$\varepsilon(C_1\|g_0\|_n + C_2)\exp(H_n(z)), \tag{3.1}$$

where

$$C_1 = \sup_{h\in B_n}\sup_{t\in\mathbb{C}} \frac{|t||h(t)|}{\exp(H_{n+2}(t))} < +\infty,\ \ C_2 = \sup_{t\in\mathbb{C}} \frac{|t||g_0(t)|}{\exp(H_{n+2}(t))} < +\infty.$$

Let now $|t - z| \leq 1/\varepsilon$. Applying the maximum modulus principle to the holomorphic function $\frac{f(t)g_0(z)-f(z)g_0(t)}{t-z}$, we conclude that there exists $t_0 \in \mathbb{C}$ with $|t_0 - z| = 1/\varepsilon$ and

$$\frac{|t||f(t)g_0(z) - f(z)g_0(t)|}{|t - z|\exp(H_{n+2}(t))} \leq \varepsilon C_3 \frac{|f(t_0)||g_0(z)| + |f(z)||g_0(t_0)|}{\exp(H_{n+1}(t)))} \leq$$

$$2\varepsilon C_3\|g_0\|_n \exp(H_n(t_0) + H_n(z) - H_{n+1}(t)) \leq$$

$$2\varepsilon C_3\|g_0\|_n \exp(H_n(z) + \beta(z)), \tag{3.2}$$

where $C_3 = \sup\limits_{t\in\mathbb{C}}\left(\exp\left(\log(1 + |t|) + H_{n+1}(t) - H_{n+2}(t)\right)\right) < +\infty$ and

$$\beta(z) = \sup_{|\eta-z|\leq 1/\varepsilon}\left(\left(\sup_{|\xi-\eta|\leq 2/\varepsilon} H_n(\xi)\right) - H_{n+1}(\eta)\right).$$

From inequalities (3.1), (3.2) and Remark 2.1 it follows, that

$$\lim_{|z|\to\infty}\sup_{f\in B_n}\frac{|A_\varphi(f)(z)|}{\exp(H_n(z))} = 0.$$

Hence the set $A_\varphi(B_n)$ is relatively compact in E_n. □

Lemma 3.2 *Let $\varphi \in E'$. If the operator $B_\varphi : E \to E$ is injective and $\varphi(g_0) \neq 0$, then it is a topological isomorphism E onto E.*

Proof Let $g_0 \in E_m$ for some $m \in \mathbb{N}$. By Lemma 3.1 the operator A_φ is compact in each Banach space E_n, $n \geq m$. Since the equality $B_\varphi(f) = \varphi(g_0)f + A_\varphi(f)$, $f \in E$, holds and $\varphi(g_0) \neq 0$, by the Fredholm alternative the restriction of B_φ on each space E_n, $n \geq m$, is a topological isomorphism E_n on itself. From this it follows that $B_\varphi : E \to E$ is a topological isomorphism E onto E. □

In the next part of this section let $g_0 = Pe_\lambda$ for some $\lambda \in Q$ and some polynomial P such that $P(0) = 1$. By $\mathcal{D}(P)$ we denote the set of all polynomials q, dividing P and such that $q(0) = 1$.

We will use the characterization of proper closed D_{0,g_0}-invariant subspaces of E, obtained in [5, Corollary 20] and [7, Theorem 2].

Lemma 3.3 ([5, 7]) *For a subspace S of E the following assertions are equivalent:*

(i) S is a proper closed D_{0,g_0}-invariant subspace of E.
(ii) There exists a polynomial $q \in \mathcal{D}(P)$ of degree greater or equal to 1, such that $S = qE$, or there exist a polynomial $q \in \mathcal{D}(P)$ and an integer $n \geq 0$, such that $n \geq deg(P) - deg(q) - 1$ and $S = qe_\lambda \mathbb{C}[z]_n$.

Lemma 3.4 *Let $\varphi \in E'$ and the operator $B_\varphi : E \to E$ be not injective. Then $B_\varphi(g_0) = 0$.*

Proof For $\varphi = 0$ this statement obvious. Let $\varphi \neq 0$ and $S := \operatorname{Ker} B_\varphi$. Then S is a proper closed D_{0,g_0}-invariant subspace of E. We will apply Lemma 3.3.

If there exists a polynomial $q \in \mathcal{D}(P)$ of degree greater or equal to 1 such that $S = qE$, then $g_0 \in S$. We assume now that there are a polynomial $q \in \mathcal{D}(P)$ and an integer $n \geq 0$ with $n \geq \deg(P) - \deg(q) - 1$ for which $S = qe_\lambda \mathbb{C}[z]_n$. If $\deg(q) = \deg(P)$, then $q = P$ and $g_0 = Pe_\lambda \in S$. Consider the case $\deg(q) < \deg(P)$. In this case there exists $\lambda \in \mathbb{C}$, such that $P(\lambda) = 0$ and $q(z)(z-\lambda)$ divides $P(z)$. Note that the degree of the polynomial $P_1(z) = \frac{P(z)}{q(z)(z-\lambda)}$ is equal to $\deg(P) - \deg(q) - 1$. Hence the function $g_1(z) = q(z)e^{\lambda z}P_1(z) = \frac{P(z)}{z-\lambda}e_\lambda$ belongs to S. From $B_\varphi(g_1) = 0$, by (2.1), it follows that

$$g_0(z)\omega_{M(g_1)}(z, \widehat{\varphi}) - M(g_1)\omega_{g_0}(z, \widehat{\varphi}) = 0. \tag{3.3}$$

for all $z \in \mathbb{C}$ and $0 = B_\varphi(g_1)(0) = \varphi(g_1)$. By Remark 2.2, this implies that $\frac{1}{2\pi i}\int\limits_C \gamma_{g_1}(t)\widehat{\varphi}(t)dt = 0$ (a closed convex curve C in Q surrounds the conjugate diagram of g_1). Multiplying (3.3) by $z - \lambda$ and using the equality (2.2), we infer

$$g_0(z)\omega_{M(g)}(z, \widehat{\varphi}) - M(g)\omega_{g_0}(z, \widehat{\varphi}) = 0$$

for all $z \in \mathbb{C}$. Consequently, by (2.1), $B_\varphi(g_0) = 0$. □

Lemma 3.5 *The following assertions are equivalent:*

(i) *The operator* $B_\varphi : E \to E$ *is injective.*
(ii) $\varphi(g_0) \neq 0$.

Proof (i)⇒(ii): From $B_\varphi(g_0) = \varphi(g_0)g_0$ it follows that $\varphi(g_0) \neq 0$.

(ii)⇒(i): Suppose that B_φ is not injective. By Lemma 3.4 $B_\varphi(g_0) = 0$ and, consequently, $0 = B_\varphi(g_0)(0) = \varphi(g_0)$. A contradiction. □

For $\lambda \in \mathbb{C}$ and an integer $k \geq 0$, we introduce the functional $\delta_{\lambda,k}(f) := f^{(k)}(\lambda)$, $f \in E$. All these functionals are continuous and linear on E.

Lemma 3.6 *Let* $\deg(P) \geq 1$, $k(\lambda)$ *be the multiplicity of a zero* λ *of* P. *Then the following assertions hold:*

(i) $\delta_{\lambda,k} \otimes \delta_{\mu,l} = 0$ *for all zeros* λ, μ *of* P *and for all integers* k, l *with* $0 \leq k \leq k(\lambda) - 1$, $0 \leq l \leq k(\mu) - 1$.
(ii) $B_{\delta_{\lambda,k}} B_{\delta_{\mu,l}} = 0$ *for all zeros* λ, μ *of* P *and for all integers* k, l *with* $0 \leq k \leq k(\lambda) - 1$, $0 \leq l \leq k(\mu) - 1$.

Proof The assertion (i) is verified directly (see, for example, [5, the proof of Lemma 6]).

The equality in (ii) follows from $B_\varphi B_\psi = B_{\varphi \otimes \psi}$, $\varphi, \psi \in E'$ (see Theorem 2.1). □

Suppose that $\deg(P) \geq 1$. For a polynomial $q \in \mathcal{D}(P)$ of degree greater or equal to 1 let λ_j, $1 \leq j \leq m$, be all different zeros of q, k_j be the multiplicity of the zero λ_j of q. We define the "canonical" functional, corresponding to q, by

$$\delta(q) := \sum_{j=1}^{m} \sum_{k=0}^{k_j - 1} \delta_{\lambda_j,k}.$$

Lemma 3.7 *Let* $\deg(P) \geq 1$. *For each polynomial* $q \in \mathcal{D}(P)$ *of degree greater or equal to* 1 *the equality* $\mathrm{Ker}\, B_{\delta(q)} = qE$ *holds.*

Proof Employing standard calculations, for $f \in E$ we obtain:

$$B_{\delta(q)}(f)(z) = \delta(q)_t \left(f(t)g_0(z) + z\frac{f(t)g_0(z) - f(z)g_0(t)}{t - z} \right) =$$

$$g_0(z) \sum_{j=1}^{m} \sum_{k=0}^{k_j-1} f^{(k)}(\lambda_j) + zg_0(z) \sum_{j=1}^{m} \sum_{k=0}^{k_j-1} \sum_{s=0}^{k} C_k^s f^{(k-s)}(\lambda_j) \frac{(-1)^s s!}{(\lambda_j - z)^{s+1}} =$$

$$g_0(z) \sum_{j=1}^{m} \sum_{s=1}^{k_j} \frac{1}{(\lambda_j - z)^s} \sum_{l=0}^{k_j - s} \beta_{s,l} f^{(l)}(\lambda_j), \tag{3.4}$$

where all constants $\beta_{l,s}$ are independent of $f \in E$ and $\beta_{s,k_j-s} \neq 0$, $1 \leq s \leq k_j$, $0 \leq l \leq k_j - s$.

Let $B_{\delta(q)}(f) = 0$ for some $f \in E$. From (3.4) it follows that $\sum_{l=0}^{k_j-s} \beta_{s,l} f^{(l)}(\lambda_j) = 0$, $1 \leq s \leq k_j$, $1 \leq j \leq m$. Hence $f^{(l)}(\lambda_j) = 0$, $0 \leq l \leq k_j - 1$, $1 \leq j \leq m$, and, consequently, $f \in qE$. Vice versa, if $f \in qE$, then (3.4) implies $B_{\delta(q)}(f) = 0$. □

4 Main Results

In this section we fix a point $\lambda \in Q$ and a polynomial P with $P(0) = 1$ and put $g_0 := Pe_\lambda$.

Theorem 4.1 *For $\varphi \in E'$ the following assertions are equivalent:*

(i) The operator B_φ is a topological isomorphism E onto E.
(ii) $\varphi(g_0) \neq 0$.

Proof (i)⇒(ii): If B_φ is a topological isomorphism E onto E, then the operator B_φ is injective. Hence $\varphi(g_0) \neq 0$ by Lemma 3.5.

(ii)⇒(i): By Lemma 3.5 B_φ is injective in E. By Lemma 3.2 $B_\varphi : E \to E$ is a topological isomorphism "onto". □

We will prove a result on the factorization of nonzero operators B_φ. Note, that the lattice of proper closed D_{0,g_0}-invariant subspaces of E is not linearly ordered in the case when the function g_0 has zeros [7, Theorem 2]. This significantly affects factorization.

For polynomials $q, r \in \mathcal{D}(P)$ we denote by $(q, r)_1$ the greatest common divisor d of q and r with $d(0) = 1$.

Theorem 4.2 *Let $\varphi \in E'$, $\varphi \neq 0$ and $\varphi(g_0) = 0$. Then either there exist $\psi \in E'$, $n \in \mathbb{N}$, for which B_ψ is a topological isomorphism E onto E and $B_\varphi = D_{0,g_0}^n B_\psi$, or there are a polynomial $q \in \mathcal{D}(P)$ of degree greater or equal to 1, an integer $n \geq 0$, $\psi \in E'$, such that B_ψ is a topological isomorphism E onto E and $B_\varphi = B_{\delta(q)} D_{0,g_0}^n B_\psi$.*

Proof We will exploit Lemma 3.3. First of all, $S = \operatorname{Ker} B_\varphi$ is a proper closed D_{0,g_0}-invariant subspace of $H(Q)$. We suppose that $S = qe_\lambda\mathbb{C}[z]_n$ for some $q \in \mathcal{D}(P)$ and some integer $n \geq 0$ for which $n \geq \deg(P) - \deg(q) - 1$. We will show that $q = P$. Assume that $\operatorname{eg}(q) < \deg(P)$. Since $\varphi(g_0) = 0$, we have $B_\varphi(g_0) = \varphi(g_0)g_0 = 0$. Consequently, $g_0 \in S$ and hence $Pe_\lambda\mathbb{C}[z]_0 \subset S$. Choose the greatest integer $m \geq 0$ such that $Pe_\lambda\mathbb{C}[z]_m \subset S$. Since the space $\operatorname{Ker} D_{0,g_0}^{m+1} = Pe_\lambda\mathbb{C}[z]_m$ is finite-dimensional and the operator $D_{0,g_0}^{m+1} : E \to E$ is surjective, there exists a continuous linear right inverse $R : E \to E$ to D_{0,g_0}^{m+1} [10, Theorem 10.3]. Then $RD_{0,g_0}^{m+1}(f) - f \in \operatorname{Ker} D_{0,g_0}^{m+1}$ for all $f \in E$. Note that $\varphi = 0$ on $\operatorname{Ker} B_\varphi$, since

$\varphi(f) = B_\varphi(f)(0)$ for each $f \in E$. Consequently, $\gamma D_{0,g_0}^{m+1} = \varphi$ for the functional $\gamma := \varphi R \in E'$. For each $z \in \mathbb{C}$, $f \in E$ we obtain:

$$B_\varphi(f)(z) = \varphi(T_{z,g_0}(f)) = \gamma\left(D_{0,g_0}^{m+1}\left(T_{z,g_0}(f)\right)\right) = \gamma\left(T_{z,g_0}\left(D_{0,g_0}^{m+1}(f)\right)\right) =$$

$$B_\gamma\left(D_{0,g_0}^{m+1}(f)\right)(z) = D_{0,g_0}^{m+1} B_\gamma(f)(z),$$

i.e. $B_\varphi = D_{0,g_0}^{m+1} B_\gamma$. In addition, $\gamma(g_0) = 0$. In fact, otherwise B_γ is injective by Lemma 3.5 and $\mathrm{Ker} B_\varphi = \mathrm{Ker}\, D_{0,g_0}^{m+1} = P e_\lambda \mathbb{C}[z]_m$. A contradiction with $\mathrm{Ker}\, B_\varphi = q e_\lambda \mathbb{C}[z]_n$. Hence there exist $s \in \mathbb{N}$ and $\xi \in E'$ for which $B_\gamma = D_{0,g_0}^s B_\xi$, and consequently, $B_\varphi = D_{0,g_0}^{m+s+1} B_\xi$. This is a contradiction with the maximality of m. Thus, $q = P$. By Theorem 4.1 B_ψ is a topological isomorphism E onto E.

Let now $S = qE$ for a polynomial $q \in \mathcal{D}(P)$ of degree greater or equal to 1. By Lemma 3.7 $\mathrm{Ker} B_{\delta(q)} = qE$. Since $\mathrm{Ker}\, B_{\delta(q)}$ has the finite codimension, the image $B_{\delta(q)}(E)$ is finite-dimensional. From this it follows that $B_{\delta(q)}(E)$ is a Fréchet space with the topology induced from E. Consequently, there is a continuous linear right inverse $R_0 : B_{\delta(q)}(E) \to E$ to $B_{\delta(q)} : E \to B_{\delta(q)}(E)$. Define a functional ξ_0 on $B_{\delta(q)}(E)$ as $\xi_0 := \varphi R_0$. Then ξ_0 is continuous and linear on $B_{\delta(q)}(E)$ with the topology, induced from E. By the Hahn-Banach Theorem ξ_0 can be extended to a continuous linear functional ξ on E. Since $R_0 B_{\delta(q)}(f) - f \in \mathrm{Ker}\, B_{\delta(q)} = \mathrm{Ker}\, B_\varphi$ for all $f \in E$, we have $\xi_0 B_{\delta(q)} = \varphi$ and also $\xi B_{\delta(q)} = \varphi$. As in the first case, from this we infer $B_\varphi = B_{\delta(q)} B_\xi$. If $\xi(g_0) \neq 0$, then the lemma is proved (with $\psi = \xi$ and $n = 0$). If $\xi(g_0) = 0$, then we factorize B_ξ in the form $B_\xi = D_{0,g_0}^n B_\psi$, where $n \in \mathbb{N}$, $\psi \in E'$, $\psi(g_0) \neq 0$, or $B_\xi = B_{\delta(r)} B_\tau$, $\tau \in E'$, $r \in \mathcal{D}(P)$. Second decomposition is not valid, since otherwise $B_\varphi = B_{\delta(q)} B_{\delta(r)} B_\tau = 0$ by Lemma 3.6. In addition, B_ψ is a topological isomorphism E onto E by Theorem 4.1. □

Corollary 4.1 *Each nonzero operator from $\mathcal{K}(D_{0,g_0})$, which is not finite-dimensional, is surjective and has a continuous linear right inverse.*

Remark 4.1 Let $\mathrm{Lat}(D_{0,g_0}, E)$ be the lattice of all closed D_{0,g_0}-invariant subspaces of E. From the proof of Theorem 4.2 it follows, that the set of kernels of all operators B_φ, $\varphi \in E'$, coincides with $\mathrm{Lat}(D_{0,g_0}, E)$ if and only if the function $g_0 = P e_\lambda$ has no zeros, i.e. $P \equiv 1$.

Remark 4.2 V.A. Tkachenko [11] investigated properties of the commutant of the operator of generalized integration $\mathcal{I}$ in the strong dual of a countable inductive limit of weighted Banach spaces of entire functions, whose growth is determined with the help of a ρ-trigonometric convex function ($\rho > 0$) with values in $(-\infty, +\infty]$. The operator $\mathcal{I}$ is the dual map (we use notations of this article) to the operator D_{0,g_0} for a function $g_0 = e^{\mathcal{P}}$, where $\mathcal{P}$ is a polynomial. This function has no zeros in $\mathbb{C}$. The operator $\mathcal{I}$ is unicellular. In the unicellular case of our article, if $g_0(z) = e^{\lambda z}$, Theorems 4.1 and 4.2 follow from statements, proved by V.A. Tkachenko [11, § 4, Property d); Theorem 2].

5 The Generalized Duhamel Product

We will apply Theorem 4.1 to a multiplication in $H(Q)$. Let $g_0 = Pe_\lambda$, where $\lambda \in Q$ and P is a polynomial such that $P(0) = 1$. By Ivanova and Melikhov [7, § 4] $\mathcal{F}^t(\varphi \otimes \psi) = \mathcal{F}^t(\varphi) * \mathcal{F}^t(\psi)$ for all $\varphi, \psi \in E'$, where $*$ is an associative and commutative multiplication in $H(Q)$.

For a polynomial $r(z) = \sum\limits_{j=0}^{n} b_j z^j$ we define the differential operator $r(D)(f) := \sum\limits_{j=0}^{n} b_j f^{(j)}$. Note, that $\varphi(Pe_\lambda) = P(D)\left(\mathcal{F}^t(\varphi)\right)(\lambda)$, $\varphi \in E'$.

Let $m := \deg(P) \geq 1$. We introduce polynomials p_j, $0 \leq j \leq m-1$, for which $\sum\limits_{j=0}^{m-1} p_j(t)z^j = \frac{P(t)-P(z)}{t-z}$. Set $\widetilde{p}_j(t) := tp_j(t)$, $0 \leq j \leq m-1$, $t \in \mathbb{C}$. As shown in [7, § 4], for all $f, h \in H(Q)$, $z \in Q$

$$(f * h)(z) =$$

$$h(\lambda)P(D)(f)(z) + \int\limits_{\lambda}^{z} P(D)(f)(\xi)h'(z+\lambda-\xi)d\xi - \sum_{j=0}^{m-1} \widetilde{p}_j(D)(f)(z)h^{(j)}(\lambda),$$

where the integral is taken along the line segment from λ to z. Employing integration by parts and substitution $\eta = z + \lambda - \xi$, for $f, h \in H(Q)$, $z \in Q$, we infer

$$(f * h)(z) =$$

$$P(D)(f)(\lambda)h(z) + \int\limits_{\lambda}^{z} \left(P(D)(f)\right)'(\eta)h'(z+\lambda-\eta)d\xi - \sum_{j=0}^{m-1} \widetilde{p}_j(D)(f)(z)h^{(j)}(\lambda).$$

This expression for the multiplication $*$ emphasizes the significance of the factor $P(D)(f)(\lambda)$. For $P \equiv 1$

$$(f * h)(z) = f(\lambda)h(z) + \int\limits_{\lambda}^{z} f'(\eta)h(z+\lambda-\eta)d\eta, \;\; z \in Q, \;\; f, h \in H(Q).$$

If $P \equiv 1$, $\lambda = 0$, then $f * h$ is the Duhamel product. In the space of all holomorphic functions on a domain in $\mathbb{C}$, star-shaped with respect to the origin, this product was investigated for the first time by Wigley [12].

Define for $f \in H(Q)$ the Duhamel operator $G_f(h) := f * h$, $h \in H(Q)$, which is continuous and linear in $H(Q)$. Note that $\widehat{S_\varphi(\psi)} = G_{\widehat{\varphi}}\left(\widehat{\psi}\right)$ for all $\varphi, \psi \in E'$. Applying standard dual arguments to Theorem 4.1, we get the following result:

Corollary 5.1 *For $f \in H(Q)$ the operator G_f is a topological isomorphism $H(Q)$ onto $H(Q)$ if and only if $P(D)(f)(\lambda) \neq 0$.*

In the case $g_0 \equiv 1$, i.e. $P \equiv 1$, $\lambda = 0$, this statement was proved by Wigley [12] (for a domain Q, which is star-shaped with respect to the origin).

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Global Boundedness of Solutions of Continuous Social Stratification Model

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Abstract We consider the continuous social stratification model in the special case when the influence of negative factors is uniformly spread among the society. We first obtain spatially homogeneous solutions of the problem. Next we employ the comparison theorems for nonlinear parabolic equations to derive sufficient conditions of global boundedness and blow up for the solutions which correspond to spatially inhomogeneous initial data. Finally we perform numerical and analytical study of the strain localization effect in the case of no external influence on the society.

Keywords Continuous social stratification model · Nonlinear diffusion equation · Global boundedness · Blow up

MSC Code 35K55

1 Introduction

Mathematical modelling of social phenomena is actively studied in literature. The methodology of statistical physics is being successfully applied to models describing social dynamics [1, 2], human cooperation [3] and crime hotspots [4]. In the limit of large populations it becomes possible to apply continuous models, based on partial differential equations [4, 5].

In the paper [6], the quantitative assessment of the social strain has been suggested. In the same paper the authors have developed a simple mathematical model for short-range prediction of the background social strain level. This approach has been consequently successfully applied to the modelling of various social phenomena: strike movement in Russia during the end of the nineteenth century

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and the beginning of the nineteenth century, social strain in the post-war USSR [7], the interaction of elite and workers [8] and protest activities [9].

Mathematical models, considered in [6–10] contain a low number of interacting social groups. The complexity of the equations, however, grows as the number of social groups increases, which makes qualitative analysis extremely difficult. In this case it is natural to consider a model of continuous medium. This means that instead of the discrete set of interacting social groups one can consider their continuous distribution on some finite interval and arrive at a model of continuous social stratification

$$\frac{\partial P}{\partial t} = U(x,t) - \gamma P + \mu(x) P^2 + \frac{\partial}{\partial x}\left(C(x)\frac{P}{1-P}\frac{\partial P}{\partial x}\right), \tag{1}$$

where $P = P(x,t) \in [0,1)$ is the normalized background social strain level (see [6]), $x \in (0,1)$ is the stratification variable, $C(x)$ is the stratification diffusion of social strain, $\mu(x)$ is the autoexitation of society, $U(x,t)$ is the external influence on society, $\gamma > 0$ is the strain dissipation coefficient and $t \geq 0$ is dimensionless time. This model may be considered as a spatially distributed generalization of those studied in [6–10]. The term $\frac{P}{1-P}$ here express the assumption that the speed at which the social strain spreads to the “neighbouring” social groups grows with the local level of strain (see [8] for details).

Let us assume that the influence of negative factors is uniformly spread among the society and put $U \equiv U_0 \geq 0$, $C \equiv C_0 > 0$, $\mu \equiv \mu_0 > 0$. Then the Eq. (1) can be written as

$$\frac{\partial P}{\partial t} = U_0 - \gamma P + \mu_0 P^2 + C_0 \frac{\partial}{\partial x}\left(\frac{P}{1-P}\frac{\partial P}{\partial x}\right). \tag{2}$$

It is also assumed, that the ends of the interval $(0,1)$ are supplied with homogeneous Neumann boundary conditions

$$\frac{\partial P}{\partial x}|_{x=0,1} = 0. \tag{3}$$

The main purpose of the present work can be formulated as follows: to find the sufficient conditions of boundedness for the solutions of continuous social stratification model (2) and to start the numerical and analytical investigation of the strain localization effect in the Eq. (2) in the special case of no external influence.

2 The Change of Variables

In the Eq. (2) the normalized background social strain level $P(x,t)$ is defined on the half-closed interval $[0,1)$. Let us introduce the change of variables, which transforms the half-closed interval $[0,1)$ into the half-line $[0,+\infty)$. Let us define a function $T(x,t)$ by

$$P(x,t) = \frac{T(x,t)}{1+T(x,t)}. \tag{4}$$

Then $T(x,t) \xrightarrow{P\to 1} +\infty$, $T(x,t) \xrightarrow{P\to 0} 0$ and the Eq. (2) could be written in the form

$$\frac{\partial T}{\partial t} = U_0(1+T)^2 - \gamma T(1+T) + \mu_0 T^2 + C_0(1+T)^2 \frac{\partial}{\partial x}\left(\frac{T}{(1+T)^2}\frac{\partial T}{\partial x}\right), \tag{5}$$

while Neumann boundary conditions (3) are expressed by the formula

$$\frac{\partial T}{\partial x}|_{x=0,1} = 0. \tag{6}$$

3 Spatially Homogeneous Solutions

By putting $T = T(t)$ in (5) we arrive at the ordinary differential equation

$$\frac{\partial T}{\partial t} = U_0 + (2U_0 - \gamma)T + (\mu_0 + U_0 - \gamma)T^2. \tag{7}$$

The equilibria of (7) could be obtained from the equation

$$(\mu_0 + U_0 - \gamma)T^2 + (2U_0 - \gamma)T + U_0 = 0. \tag{8}$$

Let us define $U^* = \gamma - \mu_0$, then for $U_0 \neq U^*$ the leading coefficient of the polynomial (8) is not equal to zero and there exists a pair of roots

$$T_0^{1,2} = \frac{\gamma - 2U_0 \pm \sqrt{\gamma^2 - 4U_0\mu_0}}{2(\mu_0 + U_0 - \gamma)}, \tag{9}$$

which are real when $U_0 \leq U_{cr}$, $U_{cr} = \frac{\gamma^2}{4\mu_0}$. If $\mu_0 > \gamma$, then $T_0^{1,2} \geq 0$, otherwise the sign of constant solutions of (7) differs with respect to values of μ_0 and γ (see Table 1). If $U_0 = U^*$ then the Eq. (8) becomes a first-order equation. In the case when $\gamma \neq 2\mu_0$ there exists a root $T^* = -\frac{\gamma-\mu_0}{\gamma-2\mu_0}$, otherwise if $\gamma = 2\mu_0$ we get that $U^* = U_{cr}$ and as a result the Eq. (8) has no roots.

Table 1 Steady state solutions of the Eq. (5) when $\mu_0 \le \gamma$

Condition	$0 \le U_0 < U^*$	$U = U^*$	$U^* < U \le U_{cr}$
$\gamma < 2\mu_0$	$T_0^1 < 0$ $T_0^2 \ge 0$	$T^* = -\frac{\gamma-\mu_0}{\gamma-2\mu_0} \ge 0$	$T_0^1 > 0,\ T_0^2 > 0$
$\gamma = 2\mu_0$		$\nexists T^*$	—
$\gamma > 2\mu_0$		$T^* = -\frac{\gamma-\mu_0}{\gamma-2\mu_0} < 0$	$T_0^1 < 0,\ T_0^2 < 0$

Let us find the solutions of (7) which are different from steady states. First assume that $U_0 \neq U^*$ and $U_0 < U_{cr}$. Then the general solution of this equation can be written as

$$T(t) = T_0^1 + \frac{B}{A(1 - C_0 e^{Bt})},\ A = \mu_0 + U_0 - \gamma \neq 0,\ B = \sqrt{\gamma^2 - 4U_0\mu_0}, \tag{10}$$

where $C_0 = 1 - \frac{B}{A(T(0) - T_0^1)}$. If $0 < C_0 < 1$, which corresponds to the case $T(0) > T_0^1$ for $A > 0$ and $T(0) < T_0^1$ for $A < 0$, then $T(t) \to +\infty,\ t \to t^*$, where $t^* = \frac{1}{B} ln(\frac{1}{C_0})$. Otherwise $T(t) \to T_0^1$ when $t \to +\infty$. If $U = U_{cr}$ then the Eq. (8) has one multiple root and the general solution of (7) can be written as

$$T(t) = \frac{\gamma - 2U_0}{2(\mu_0 + U_{cr} - \gamma)} + \frac{1}{A(C_0 - t)}, \tag{11}$$

where $C_0 = \frac{1}{A(T(0) + T_0^1)}$, and the constant A was defined in (10).

In the case $U > U_{cr}$ the general solution of (7) can be written as

$$T(t) = \frac{\gamma - 2U_0}{4A} + \sqrt{G} tg(\frac{A}{\sqrt{G}} t + C_0),\ G = \frac{4U_0\mu_0 - \gamma^2}{4A^2}, \tag{12}$$

where $C_0 = arctg(\frac{T(0) + \frac{2U_0 - \gamma}{4A}}{\sqrt{G}})$ and for any initial data $T(t) \to +\infty$ when $t \to t^*$.

Let us assume that $U_0 = U^*$ and $\gamma \neq 2\mu_0$. Then the general solution of (7) is given by

$$T(t) = \frac{\gamma - \mu_0}{\gamma - 2\mu_0} + C_0 e^{(\gamma - 2\mu_0)t}, \tag{13}$$

where $C_0 = T(0) - \frac{\gamma - \mu_0}{\gamma - 2\mu_0}$. If $\gamma = 2\mu_0$, then we get

$$T(t) = T(0) + \mu_0 t. \tag{14}$$

4 Sufficient Conditions for Global Boundedness

Spatially homogeneous solutions of the Eq. (5) can be used for studying the properties of solutions, which correspond to spatially inhomogeneous initial data. Let us assume that $\mu_0 > \gamma$. If $U_0 \le U_{cr}$ then there exists a pair of non-negative steady states $T_0^{1,2}$ and Propositions 1–6 hold. These propositions follow from existing comparison theorems for nonlinear parabolic partial differential equations (see. [11–13]).

Proposition 1 *Let $\mu_0 > \gamma$ and $U_0 \le \frac{\gamma^2}{4\mu_0}$ and consider a solution $T(x,t)$ of the problem (5)–(6). Then the following assertions hold:*

1. *If $T(x,0) < T_0^1$ then $T(x,t) \to T_0^2$ as $t \to +\infty$.*
2. *If $T(x,0) > T_0^1$ then there exists such $t^* > 0$ that $T(x,t) \to +\infty$ as $t \to t^*$.*
3. *If $H_0 = \int\limits_0^1 \frac{T(x,0)}{1+T(x,0)}dx > \frac{T_0^1}{1+T_0^1}$ then there exists such $t^* > 0$ that $T(x,t) \to +\infty$ as $t \to t^*$.*

Let us consider the case when $U_0 > U_{cr}$. Then $T_0^{1,2} \in \mathbb{C}$ and all solutions of the Eq. (5) blow up.

Proposition 2 *Let $\mu_0 > \gamma$ and $U_0 > \frac{\gamma^2}{4\mu_0}$. Let us consider a solution $T(x,t)$ of the problem (5)–(6). Then there exists such $t^* > 0$ that $T(x,t) \to +\infty$ as $t \to t^*$.*

Let $\mu_0 < \gamma$. Then for $0 \le U_0 < U^*$ solutions of the Eq. (5) are bounded and tend to the steady state T_0^2 as $t \to +\infty$.

Proposition 3 *Let $\mu_0 \le \gamma$ and $U \le U^* = \gamma - \mu$. Let us consider a solution $T(x,t)$ of the problem (5)–(6). Then $T(x,t) \to T_0^2$ as $t \to +\infty$.*

Proposition 4 *Let $\mu_0 \le \gamma$ and $U = U^* = \gamma - \mu$. Consider a solution $T(x,t)$ of the problem (5)–(6). Then the following assertions hold:*

1. *If $\mu_0 \le \gamma < 2\mu_0$ then $T(x,t) \to T^* = -\frac{\gamma-\mu_0}{\gamma-2\mu_0}$.*
2. *If $\gamma \ge 2\mu_0$ then $T(x,t) \to +\infty$ as $t \to +\infty$.*

Let $U^* < U_0 < U_{cr}$. Then the following propositions hold.

Proposition 5 *Let $U^* < U_0 < U_{cr}$ and $\mu_0 \le \gamma \le 2\mu_0$. Consider a solution $T(x,t)$ of the problem (5)–(6). Then the following assertions hold:*

1. *If $T(x,0) < T_0^1$ then $T(x,t) \to T_0^2$ as $t \to +\infty$.*
2. *If $T(x,0) > T_0^1$ then there exists such $t^* > 0$ that $T(x,t) \to +\infty$ as $t \to t^*$.*
3. *If $H_0 = \int\limits_0^1 \frac{T(x,0)}{1+T(x,0)}dx > \frac{T_0^1}{1+T_0^1}$ then there exists such $t^* > 0$ that $T(x,t) \to +\infty$ as $t \to t^*$.*

Proposition 6 *Let* $\gamma \geq 2\mu_0$ *and* $U > U^* = \gamma - \mu$. *Consider a solution* $T(x,t)$ *of the problem* (5)–(6)*. Then there exists such* $t^* > 0$ *that* $T(x,t) \to +\infty$ *as* $t \to t^*$.

5 Social Strain Localization in the Case of No External Influence

The diffusion term in the Eq. (5) can be written as

$$C_0(1+T)^2\frac{\partial}{\partial x}\left(\frac{T}{(1+T)^2}\frac{\partial T}{\partial x}\right) = C_0\frac{\partial}{\partial x}\left(T\frac{\partial T}{\partial x}\right) - 2C_0\frac{T}{1+T}\left(\frac{\partial T}{\partial x}\right)^2. \quad (15)$$

Let us consider the auxiliary equation

$$\frac{\partial T}{\partial t} = (\mu_0 + U_0 - \gamma)T^2 + (2U_0 - \gamma)T + U_0 + C_0\frac{\partial}{\partial x}\left(T\frac{\partial T}{\partial x}\right), \quad (16)$$

then the solutions of (16) are at the same time the upper solutions of the Eq. (5). Let us put $U_0 = 0$. Then the Eq. (16) can be written as

$$\frac{\partial T}{\partial t} = (\mu_0 - \gamma)T^2 - \gamma T + C_0\frac{\partial}{\partial x}\left(T\frac{\partial T}{\partial x}\right). \quad (17)$$

Let us assume that $\mu_0 > \gamma$. Then the Eq. (17) possess a family of automodel solutions (see [14]):

$$T_F(x,t,\phi_0) = \begin{cases} \frac{1}{1-Fe^{\gamma t}}A_T\cos^2(\omega_T(x-\phi_0)), & |\omega_T(x-\phi_0)| \leq \frac{\pi}{2}; \\ 0, & |\omega_T(x-\phi_0)| > \frac{\pi}{2}, \end{cases} \quad (18)$$

where $A_T = \frac{4\gamma}{3(\mu_0-\gamma)}$, $\omega_T = \frac{\sqrt{\mu_0-\gamma}}{2\sqrt{2C_0}}$, $\phi_0 \in [\frac{\pi}{2\omega_T}, 1 - \frac{\pi}{2\omega_T}]$. If $F < 0$ then $T_F(x,t,\phi_0) \to 0$ as $t \to +\infty$. If $F = 0$ then $T_0(x,\phi_0)$ is a spatially inhomogeneous stationary solution of the Eq. (17). If $F \in (0,1)$ then $T_F(x,t,\phi_0) \to +\infty$ as $t \to t^*$.

The solutions $T_F(x, t, \phi_0)$ are localized in space, which means that for every $t > 0$ we have $supp\, T_F(x, t, \phi_0) = supp\, T_F(x, 0, \phi_0)$ and these solutions are generalized solutions of (17) because there are two degenerate points $x = \phi_0 \pm \frac{\pi}{2\omega_T}$. Therefore, for sufficiently small C_0 it is possible to consider combinations of $T_F(x, t, \phi_0)$, which correspond to different shift values ϕ_0. Since $\frac{\partial}{\partial x} T_F(\phi_0, t, \phi_0) = 0$, it is also possible to consider $T_F(x, t, \phi_0)$ which correspond to $\phi_0 = 0, 1$, as these functions also satisfy Neumann boundary conditions (6).

Proposition 7 *Let $\mu_0 > \gamma$. Consider a solution $T(x, t)$ of the problem (5)–(6) such that $T(x, 0) < \sum_{k=1}^{N} T_0(x, \phi_k)$ for some $\phi_1, \ldots, \phi_N$ with $supp(T_0(x, \phi_i)) \cap supp(T_0(x, \phi_j)) = \varnothing\ \forall i \neq j : 1 \leq i, j \leq N$. Then $T(x, t) \to 0$ as $t \to +\infty$.*

We performed numerical experiments to study the behaviour of the solutions of the Eq. (5) in the case when $T(x, 0) > T_0(x, \phi_0)$. The calculations were done in MATLAB, the discretization of infinite-dimensional system was performed by the Method of Lines (MOL) and the numerical integration of ODE system was done by the Dormand–Prince method. The experiments were performed for different values of diffusion coefficient C_0. The value of strain dissipation coefficient γ was taken from the literature (see [8]): $\gamma = 0.1$, while the value of autoexcitation coefficient μ_0 was taken as $\mu_0 = 0.3$. Initial data were considered in the form: $T(x, 0) = \sigma[\sum_{k=1}^{N} T_0(x, \phi_k)]$, where $\sigma > 1$.

Let us consider $C_0 = 0.00012$ and $T(x, 0) = \sigma[T_0(x, 0)+T_0(x, 0.5)+T_0(x, 1)]$. For $1 < \sigma < \sigma_0 \approx 1.08$ we observed the convergence of the solution to zero as $t \to +\infty$; for $\sigma \geq \sigma_0$ we observed a local blow up regime (see Fig. 1).

Next consider $C_0 = 0.0055$ and $T(x, 0) = \sigma T_0(x, 0)$. In that case the behaviour of the system was similar to the previous case. However, the critical value σ_0 was lower: for $1 < \sigma < \sigma_0 = 1.07$ the numerical solution decayed in time; otherwise monotonous growth of the solution was observed (see Fig. 2).

Let us set $T(x, 0) = 10T(x, 0.5)$ and consider the numerical solution of (5) for the different values of C_0. In this case the social strain $T(x, t)$ spreads uniformly over the whole localization region, which is followed by monotonous growth of the solution (see Fig. 3).

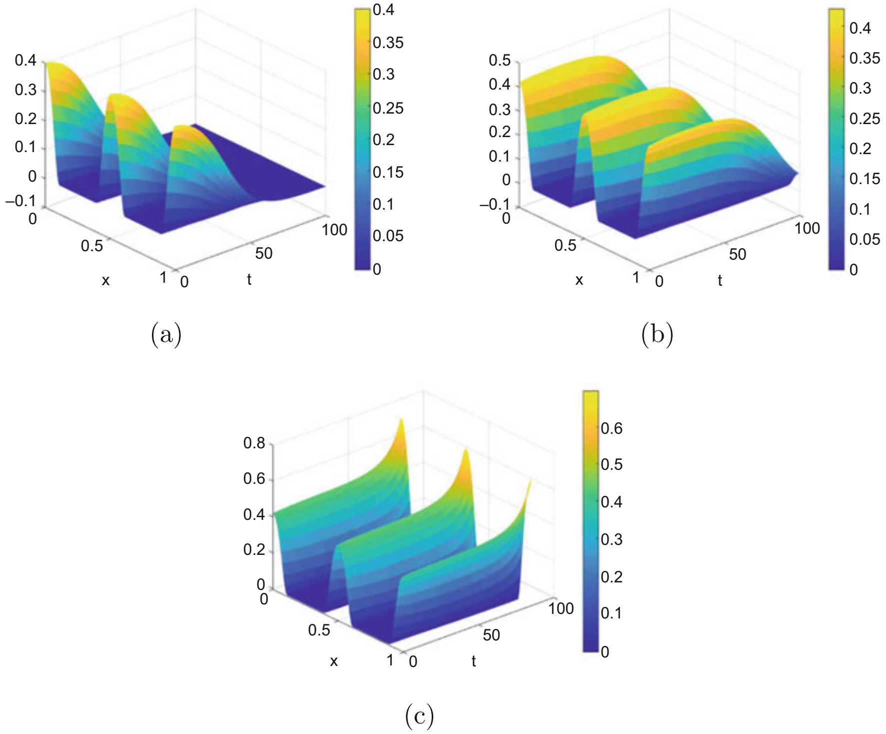

Fig. 1 Social strain level $P(x, t)$ which corresponds to the numerical solution of the Eq. (5) for different values of σ. Initial data: $T(x, 0) = \sigma[T_0(x, 0)+T_0(x, 0.5)+T_0(x, 1)]$. Model parameters: $\mu_0 = 0.3$, $\gamma = 0.1$, $C_0 = 0.00012$. (**a**) $\sigma = 1$, (**b**) $\sigma = 1.08$, (**c**) $\sigma = 1.081$

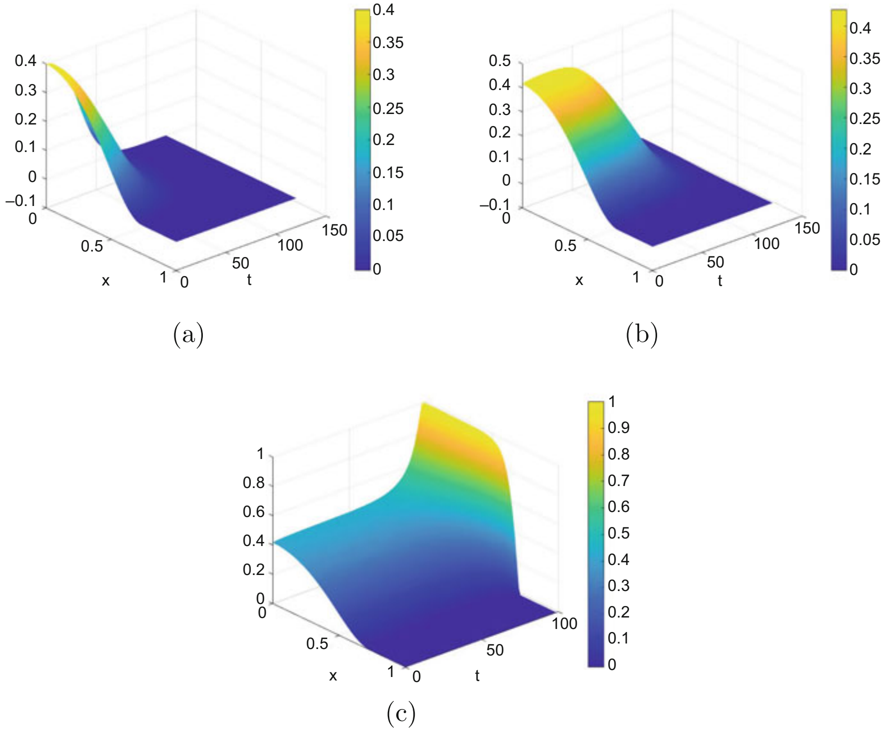

Fig. 2 Social strain level $P(x, t)$ which corresponds to the numerical solution of the Eq. (5) for different values of σ. Initial data: $T(x, 0) = \sigma T_0(x, 0)$. Model parameters: $\mu_0 = 0.3$, $\gamma = 0.1$, $C_0 = 0.0055$. (**a**) $\sigma = 1$. (**b**) $\sigma = 1.07$. (**c**) $\sigma = 1.071$

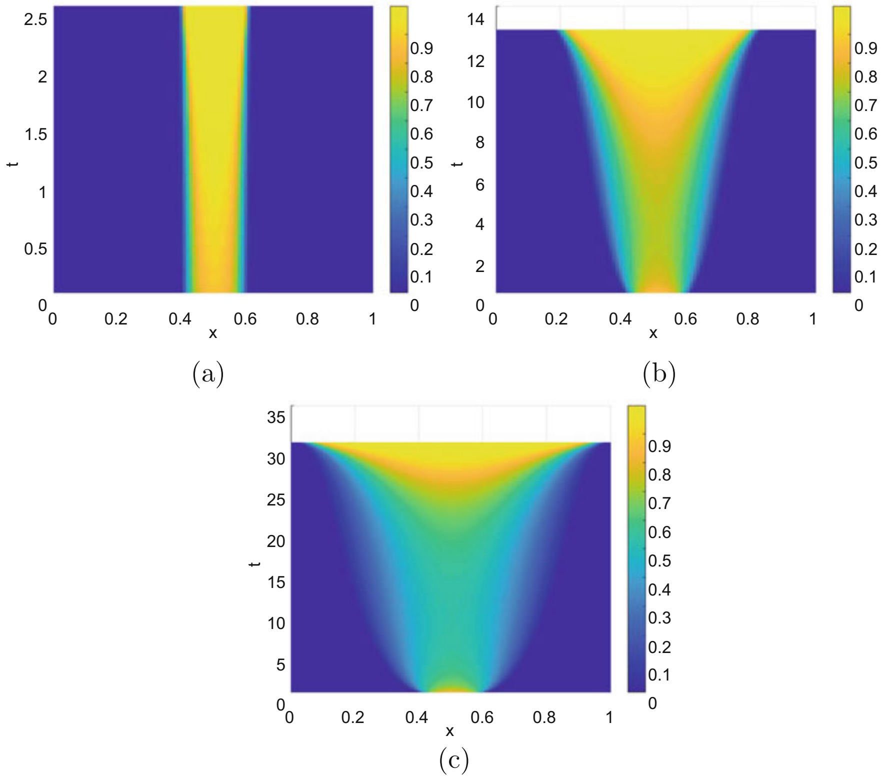

Fig. 3 Social strain level $P(x,t)$ which corresponds to the numerical solution of the Eq. (5) for different values of C_0. Initial data: $T(x,0) = 10T(x,0.5)$. Parameter values: $\mu_0 = 0.3$, $\gamma = 0.1$. (**a**) $C_0 = 0.00012$. (**b**) $C_0 = 0.0012$. (**c**) $C_0 = 0.0025$

6 Concluding Remarks

In the present work we have considered the continuous social stratification model in the special case when the influence of negative factors is uniformly spread among the society. The introduction of auxiliary variable change has enabled us to apply existing comparison theorems for nonlinear parabolic equations and derive the sufficient conditions of global boundedness and blow up for the solutions which correspond to the spatially inhomogeneous initial data.

It was shown that in the case when the autoexcitation coefficient μ_0 is greater than the strain dissipation coefficient γ the model (2) can demonstrate the localization and decay of social strain if there is no external influence on the society and initial strain is localized and bounded, so there exists such constant ϕ_0 that $P(x,0) \le P_0(x,\phi_0) = \frac{T_0(x,\phi_0)}{1+T_0(x,\phi_0)}$. This effect may persist in the model only for small enough values of diffusion coefficient C_0. It was observed from numerical experiments that a localized blow up regime may exist in the system when $P(x, 0) > P_0(x, \phi_0)$. In this case the social strain first spreads uniformly over the localization region, which is followed by monotonous growth of the numerical solution.

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Stochastic Test of a Minimal Surface

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Abstract The paper aims to obtain a stochastic test for minimal surfaces. Such a test is formulated in terms of transition densities of stochastic processes. Two fundamental forms of the surface generate these processes. This work exhausts the problem of stochastic test for regular minimal surfaces.

Keywords Diffusion process · Minimal surface

Mathematical Subject Classification 53A05, 60J25

1 Introduction

A stochastic analogue of the main theorem of surface theory was obtained in [1, 2]. In [1] it was done for the surfaces of positive curvature and in [2] for the surfaces of nonzero mean curvature.

The purpose of the present paper is to establish a stochastic test for surfaces with zero mean curvature (minimal surfaces). The technique from [1, 2] is used.

It is assumed that the reader is familiar with the concepts of a stochastic process and a strictly Markov process, as well as with the basic concepts of the theory of surfaces in three-dimensional Euclidean space.

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2 Some Definitions from the Theory of Random Processes

We denote by $(\Omega, \mathfrak{F}, P)$ some given random space.

Consider a the manifold (phase space) $(E, \mathfrak{B})$, where $\mathfrak{B}$ is a σ-field of Borel's sets in E. One can find a detailed definition of a random process on a manifold in [4].

We introduce some necessary definitions and notation.

Definition 1 ([3, p. 74]) Say that $P(t, x, \Gamma)$ $(t > 0, x \in E, \Gamma \in \mathfrak{B})$ is a transition function if the following conditions are fulfilled:

1. If t and x are fixed, then the function $P(t, x, \Gamma)$ is a measure on the σ-algebra $\mathfrak{B}$.
2. If t and Γ are fixed, then $P(t, x, \Gamma)$ is $\mathfrak{B}$-measurable function of the point x.
3. $P(t, x, \Gamma) \leq 1$.
4. $P(0, x, E \setminus x) = 0$.
5. $P(s + t, x, \Gamma) = \int_E P(s, x, dy)P(t, y, \Gamma)$

Let μ be some measure on the phase space $(E, \mathfrak{B})$.

Definition 2 ([3, p. 75]) Say that $p(t, x, y)$ $(t > 0, x, y \in E)$ is a transition density if the following conditions are fulfilled:

1. $p(t, x, y) \geq 0$.
2. If t is fixed, then $p(t, x, y)$ is $\mathfrak{B} \times \mathfrak{B}$- measurable function in (x, y).
3. $\int_E p(t, x, y)\mu(dy) \leq 1$.
4. $p(s + t, x, z) = \int_E p(s, x, y)p(t, y, z)\mu(dy)$.

It is easy to verify [3, p. 75] that if $p(t, x, y)$ is a transition density then the formula

$$P(t, x, \Gamma) = \int_\Gamma p(t, x, y)dy, t > 0, P(t, x, \Gamma) = \chi_\Gamma, t = 0$$

defines a transition function.

Every transition function generates the contractive semigroup T_t by the following formula [3, p. 80]:

$$T_t f(x) = \int_E P(t, x, dy) f(y),$$

where $f \in B$, and B is a set of bounded measurable functions with natural linear operations and the norm $||f|| = \sup_{x \in E} |f(x)|$.

Definition 3 ([3, p. 214]) An operator A is called an infinitesimal operator of the semigroup T_t (the transition function $P(t, x, \Gamma)$) if

$$Af(x) = \lim_{t \to +0} \frac{T_t f(x) - f(x)}{t}.$$

The domain of the operator A consists of all functions f for which the limit in the right-hand side exists.

Let a phase space be a smooth manifold. If we contract an infinitesimal operator to C^2-functions, then it is called a generator of stochastic process. In the local coordinate (x^i) for a generator we have

$$Af(x) = a^{ij}\partial_i\partial_j f(x) + b^i\partial_i f(x) - Cf(x),$$

where $\partial_i = \dfrac{\partial}{\partial x^i}$, a^{ij} is a positive definite matrix.

The Kolmogorov backward equation takes place [3, p. 238]:

$$\frac{\partial p}{\partial t} = Ap.$$

Here A is the generator, and p is the transition density of a stochastic process.

In [3, Ch. 1, 2] it is shown that every Markov process uniquely defines the contractive semigroup, the transition function and the infinitesimal operator.

3 Main Result

Let $F \in C^3$ be a simply connected two-dimensional surface in the three-dimensional Euclidian space. We assume F is conformal equivalent to a unite disc and has zero mean curvature. We denote by $I = g_{ij}dx^i dx^j$, $II = b_{ij}dx^i dx^j$ and $III = f_{ij}dx^i dx^j$ the first, second and third fundamental form of F respectively. Here x^1, x^2 are the local coordinates on the surface F. Without lost of generality we can assume that the coordinates x^1, x^2 are isothermal, i. e. $I = ds^2 = \lambda(dx^{1^2} + dx^{2^2})$ [5, p. 193].

Consider three stochastic processes on F; namely, X_t, Y_t and Z_t with the generators $A_X = g^{ij}\partial_i\partial_j$, $A_Y = b^{ij}\partial_i\partial_j$ and $A_Z = f^{ij}\partial_i\partial_j$, respectively. We denote by $p^1(t, x, y)$, $p^2(t, x, y)$, $p^3(t, x, y)$ the transition densities of X_t, Y_t and Z_t, respectively.

The next assertions take place.

Theorem 1 *If the transition densities p^1 and p^3 satisfy the relations*

$$\frac{\partial_t p^1}{2\Delta p^1}\Delta \ln \frac{\partial_t p^1}{2\Delta p^1} \leq 0,$$

$$\frac{\Delta p^3}{\partial_t p^3} = -\frac{\partial_t p^1}{2\Delta p^1}\Delta \ln \frac{\partial_t p^1}{2\Delta p^1} \cdot \frac{\Delta p^1}{\partial_t p^1},$$

where Δ is the Laplace operator and ∂_t is a partial time derivative, then the surface F is minimal.

Theorem 2 *Let the conditions of Theorem 1 are satisfied. If the transition density p^1 of the process X_t and a time derivative of the transition function P^3 of Z_t at the moment $t = 0$ satisfy the system*[1]

$$\left(\frac{\Delta p^1}{\partial p^1}\right)^{-2} \times$$
$$\times \left[\frac{1}{4H^2} \left(\begin{array}{l} \left(K\frac{\Delta p^1}{\partial p^1} + \left(\frac{\Delta p^1}{\partial p^1}\right)^2 \int P_0^3\,(t,x,dy)\,\frac{y_1^2}{2}\right)\left(K\frac{\Delta p^1}{\partial p^1} + \left(\frac{\Delta p^1}{\partial p^1}\right)^2 \int P_0^3\,(t,x,dy)\,\frac{y_2^2}{2}\right) - \\ -\left(\left(K\frac{\Delta p^1}{\partial p^1} + \left(\frac{\Delta p^1}{\partial p_t^1}\right)^2 \int P_0^3\,(t,x,dy)\, y_1 y_2\right)\right)^2 \end{array} \right)\right]$$
$$= \frac{\partial_t p^1}{2\Delta p^1}\Delta \ln \frac{\partial_t p^1}{\Delta p^1},$$

$$\frac{\partial}{\partial x^k}\left[\left(\frac{\Delta p^1}{\partial_t p^1}\right)^2 \cdot \left(K\frac{\Delta p^1}{\partial_t p^1} + \int P_0^3\,(t,x,dy)\frac{y_i y_j}{1+\delta_{ij}}\right)\frac{1}{2H}\right] -$$
$$-\Gamma_{ik}^{\alpha}\left(\frac{\Delta p^1}{\partial_t p^1}\right)^2 \cdot \left(K\frac{\Delta p^1}{\partial_t p^1} + \int P_0^3\,(t,x,dy)\frac{y_\alpha y_j}{1+\delta_{\alpha j}}\right)\frac{1}{2H} =$$
$$= \frac{\partial}{\partial x^j}\left[\left(\frac{\Delta p^1}{\partial_t p^1}\right)^2 \cdot \left(K\frac{\Delta p^1}{\partial_t p^1} + \int P_0^3\,(t,x,dy)\frac{y_i y_k}{1+\delta_{ik}}\right)\frac{1}{2H}\right] -$$
$$-\Gamma_{ij}^{\alpha}\left(\frac{\Delta p^1}{\partial_t p^1}\right)^2 \cdot \left(K\frac{\Delta p^1}{\partial_t p^1} + \int P_0^3\,(t,x,dy)\frac{y_\alpha y_k}{1+\delta_{\alpha k}}\right)\frac{1}{2H},$$

where $\Gamma_{ij}^{l} = \frac{1}{2}\frac{\delta^{lk}\partial_t p^1}{\Delta p^1}\left[\frac{\partial\left(\delta_{ik}\frac{\Delta p^1}{\partial_t p^1}\right)}{\partial x^j} + \frac{\partial\left(\delta_{jk}\frac{\Delta p^1}{\partial_t p^1}\right)}{\partial x^i} - \frac{\partial\left(\delta_{ij}\frac{\Delta p^1}{\partial_t p^1}\right)}{\partial x^k}\right]$, $K = \frac{\partial_t p^1}{2\Delta p^1}\Delta \ln \frac{\partial_t p^1}{\Delta p^1}$,
$H^2 = \frac{\frac{\Delta p^1}{\partial p^1}\left[\int P_0^3(t,x,dy)\frac{y_2^2}{2} - \int P_0^3(t,x,dy)\frac{y_1^2}{2}\right]}{4}$, *then the stochastic processes X_t and Z_t uniquely define the minimal surface F.*

[1] We denote by P_0^3 a time derivative $\frac{\partial P^3}{\partial t}(0)$.

3.1 Auxiliary Statements

Lemma 1 ([1]) *The Gauss curvature K of the surface F and the transition density $p^1(t, x, y)$ satisfy the following relation:*

$$K = \frac{\partial_t p^1}{2\Delta p^1} \Delta \ln \frac{\partial_t p^1}{2\Delta p^1},$$

where $\Delta = \frac{\partial^2}{\partial {x^1}^2} + \frac{\partial^2}{\partial {x^2}^2}$, $x = (x^1, x^2)$.

Lemma 2 *If the mean curvature H of the surface F equals zero, then the first and the third fundamental forms of F are proportional.*

Proof The following relation is well-known [6, p. 232]:

$$K \cdot I + H \cdot II + III = 0, \tag{1}$$

In the case under consideration it has the next form:

$$K \cdot I + III = 0.$$

□

Lemma 3 *If on F we have $H \neq 0$, I and III are proportional, then F is a sphere.*

Proof We can rewrite (1) as

$$II = \frac{-K \cdot I - III}{H}.$$

From here we see that I and II are proportional. Hence, F is a sphere [6, p. 212].

□

It is obvious that the Gauss curvature is non-positive if the mean curvature equals zero. From this fact and the previous lemmas, we have the next assertion.

Lemma 4 *If the Gauss curvature of the surface F is non-positive and I and III are proportional then the mean curvature $H = 0$.*

3.2 Proof of Main Result

Let us show that the transition density p^1 of the stochastic process generating by the first fundamental form I of F satisfies to the equation:

$$\frac{\partial p^1}{\partial t} = \frac{1}{\lambda} \Delta p^1,$$

where λ is the isothermal element of F. Similarly, for p^3 we have

$$\frac{\partial p^3}{\partial t} = \frac{1}{\mu} \Delta p^3,$$

where $\mu = -\frac{\partial_t p^1}{2\Delta p^1} \Delta \ln \frac{\partial_t p^1}{2\Delta p^1} \cdot \frac{\Delta p^1}{\partial_t p^1}$ is the coefficient of III.

It follows from the relation

$$\frac{\Delta p^3}{\partial_t p^3} = -\frac{\partial_t p^1}{2\Delta p^1} \Delta \ln \frac{\partial_t p^1}{2\Delta p^1} \cdot \frac{\Delta p^1}{\partial_t p^1},$$

that for p^3 the Kolmogorov backward equation takes place:

$$\partial_t p^3 = \frac{1}{-\frac{\partial_t p^1}{2\Delta p^1} \Delta \ln \frac{\partial_t p^1}{2\Delta p^1} \cdot \frac{\Delta p^1}{\partial_t p^1}} \Delta p^3.$$

From the method of constructing a stochastic process by quadratic form [3, 7], we have

$$III = \mu(dx^{1^2} + dx^{2^2}).$$

We note that the third fundamental form of a surface is nonnegative and defines the stochastic process [7, p. 19].

Thus, we receive that I and III are proportional.

By Lemma 1 the relation

$$\frac{\partial_t p^1}{2\Delta p^1} \Delta \ln \frac{\partial_t p^1}{2\Delta p^1} \leq 0,$$

is equivalent to non-positivity of the Gauss curvature of F. Now the main result follows from Lemma 4 in an obvious way.

From [2] we can deduce a result that is more general.

As above we denote by Z_t the stochastic process with transition density p^3 and by P_0^3 a time derivative of the transition function of Z_t at the moment $t = 0$.

By literately repeating the reasoning from [2], we get

Theorem 3 *Let the assumptions of Theorem 1 be satisfied and let the transition density p^1 of the process X_t and the transition function P^3 of the process Z_t satisfy the system:*

$$\left(\frac{\Delta p^1}{\partial p^1}\right)^{-2}\times$$
$$\times\left[\frac{1}{4H^2}\left(\begin{array}{l}\left(K\frac{\Delta p^1}{\partial p^1}+\left(\frac{\Delta p^1}{\partial p^1}\right)^2\int P_0^3(t,x,dy)\frac{y_1^2}{2}\right)\left(K\frac{\Delta p^1}{\partial p^1}+\left(\frac{\Delta p^1}{\partial p^1}\right)^2\int P_0^3(t,x,dy)\frac{y_2^2}{2}\right)-\\ -\left(\left(K\frac{\Delta p^1}{\partial p^1}+\left(\frac{\Delta p^1}{\partial p_t^1}\right)^2\int P_0^3(t,x,dy)\,y_1y_2\right)\right)^2\end{array}\right)\right]$$
$$=\frac{\partial_t p^1}{2\Delta p^1}\Delta\ln\frac{\partial_t p^1}{\Delta p^1},$$

$$\frac{\partial}{\partial x^k}\left[\left(\frac{\Delta p^1}{\partial_t p^1}\right)^2\cdot\left(K\frac{\Delta p^1}{\partial_t p^1}+\int P_0^3(t,x,dy)\frac{y_iy_j}{1+\delta_{ij}}\right)\frac{1}{2H}\right]-$$
$$-\Gamma_{ik}^{\alpha}\left(\frac{\Delta p^1}{\partial_t p^1}\right)^2\cdot\left(K\frac{\Delta p^1}{\partial_t p^1}+\int P_0^3(t,x,dy)\frac{y_\alpha y_j}{1+\delta_{\alpha j}}\right)\frac{1}{2H}=$$
$$=\frac{\partial}{\partial x^j}\left[\left(\frac{\Delta p^1}{\partial_t p^1}\right)^2\cdot\left(K\frac{\Delta p^1}{\partial_t p^1}+\int P_0^3(t,x,dy)\frac{y_iy_k}{1+\delta_{ik}}\right)\frac{1}{2H}\right]-$$
$$-\Gamma_{ij}^{\alpha}\left(\frac{\Delta p^1}{\partial_t p^1}\right)^2\cdot\left(K\frac{\Delta p^1}{\partial_t p^1}+\int P_0^3(t,x,dy)\frac{y_\alpha y_k}{1+\delta_{\alpha k}}\right)\frac{1}{2H},$$

where $\Gamma_{ij}^{l}=\frac{1}{2}\frac{\delta^{lk}\partial_t p^1}{\Delta p^1}\left[\frac{\partial\left(\delta_{ik}\frac{\Delta p^1}{\partial_t p^1}\right)}{\partial x^j}+\frac{\partial\left(\delta_{jk}\frac{\Delta p^1}{\partial_t p^1}\right)}{\partial x^i}-\frac{\partial\left(\delta_{ij}\frac{\Delta p^1}{\partial_t p^1}\right)}{\partial x^k}\right]$, $K=\frac{\partial_t p^1}{2\Delta p^1}\Delta\ln\frac{\partial_t p^1}{\Delta p^1}$,
$H^2=\frac{\frac{\Delta p^1}{\partial p^1}\left[\int P_0^3(t,x,dy)\frac{y_2^2}{2}-\int P_0^3(t,x,dy)\frac{y_1^2}{2}\right]}{4}$. *Then the stochastic processes X_t and Z_t uniquely define some minimal surface.*

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On Isomorphism of Some Functional Spaces under Action of Two-dimensional Singular Integral Operators

S. B. Klimentov

Abstract We address two-dimensional singular integral equations widely used for constructing and investigating the solutions to the general linear first-order elliptic systems in the 2D domains. Proving the solvability of such integral equations with the use of the Calderón–Zygmund theorem brings us at the statements about the existence of the solution belonging to the spaces of summable functions whose summability exponents have to be close to the value of two, and the increase of regularity of the problem's data does not eliminate this restriction automatically. In this article, we prove a regularity result for the solutions to two-dimensional singular integral equations with the use of the representations of the second kind for the solutions to the first-order general linear elliptic systems discovered by the author in his prior work.

Keywords Two-dimensional singular integral equations

Mathematics Subject Classification 45E99 + 30E20 + 44A15

1 Introduction and Formulation of Results

Two-dimensional singular integral equations are widely used in constructing and investigating of solutions to the general linear first-order elliptic systems in the 2D domains, see, for example, articles [1–6]. Proving the solvability of such integral equations with the use of the Calderón–Zygmund theorem brings us at the statements about the existence of the solution belonging to the spaces of summable functions which the summability exponents have to be close to the value of two, and

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the increase of regularity of the problem's data does not eliminate this restriction automatically. We know of only one result [4] which refines the regularity of the solutions. We'll be discussing this result in Remark 1 below, for now we only note that the constructions used by [4] substantially rely on considering the equations in the special domain which is the unit disk.

In this article, we prove the regularity of solutions to the linear singular integral equations in the bounded simply connected domain provided that the problem's data are sufficiently regular. The proofs employ the representations of the second kind for the solutions to the general first-order linear elliptic systems reported by the author in articles [5, 6].

We now proceed to precise formulations. Let $D = \{\zeta : |\zeta| < 1\}$ be the unit disk in the complex ζ-plane E, $z = x + iy$, $i^2 = -1$; $\Gamma = \partial D$ is the boundary of D; and $\overline{D} = D \cup \Gamma$ and G is the simply connected bounded domain in the complex ζ-plane; $\partial G = \mathcal{L}$; $\overline{G} = G \cup \mathcal{L}$. Throughout the article, we use the following function spaces with standard norms:

- the space of functions summable in $\overline{D}$ with exponent $p \geq 1$ denoted as $L_p(\overline{D})$;
- the Sobolev space of functions having generalized derivatives of order $k = 0, 1, \ldots$ in $\overline{D}$ summable with exponent $p \geq 1$, denoted as $W_p^k(\overline{D})$, at that, we set $W_p^0(\overline{D}) \equiv L_p(\overline{D})$;
- the Hölder space of functions continuously differentiable up to order $k = 0, 1, \ldots$ in $\overline{D}$ which senior derivatives are Hölder-continuous in $\overline{D}$ with exponent $\alpha \in (0, 1)$ denoted as $C_\alpha^k(\overline{D})$ in addition, put $C_\alpha^0(\overline{D}) \equiv C_\alpha(\overline{D})$ and denote by $C_\alpha^k(\Gamma)$ the same spaces but consisting of the functions defined on Γ.

The notation $C_\alpha^k(\overline{G})$, $L_p(\overline{G})$, $W_p^k(\overline{G})$, $C_\alpha^k(\mathcal{L})$ has the same meaning as in the case $G = D$ or $\mathcal{L} = \Gamma$. By the Sobolev–Kondrashov embedding theorem, we always identify the elements of $W_p^k(\overline{G})$ with the continuous functions for $k \geq 1$, $p > 2$. A reader can refer to monograph [2] for more details on the mentioned function spaces. We also employ the space $W_p^{k-\frac{1}{p}}(\mathcal{L})$ of the traces of the functions belonging to space $W_p^k(\overline{G})$ (see [7, Ch. 5], [3, Ch. 6, §1])).

Let $k \geq 1$, $0 < \alpha \leq 1$, $l \geq 2$ and $p > 2$. Let $\mathcal{L}$ denote a simple closed curve. We say that $\mathcal{L} \in C_\alpha^k$ ($\mathcal{L} \in W_p^{l-\frac{1}{p}}$) if there exists a diffemorphism $\mathcal{L} \to \Gamma$ belonging to C_α^k ($W_p^{l-\frac{1}{p}}$).

We define the following integral operators:

$$Tf(\zeta) = T_G f(\zeta) = -\frac{1}{\pi} \iint\limits_G \frac{f(z)}{z - \zeta} dx dy, \quad z = x + iy,$$

$$f(\zeta) \in L_p(\overline{G}), \ p \geq 1, \ \ \partial_{\bar{\zeta}} Tf(\zeta) = f(\zeta), \ \partial_{\bar{\zeta}} = \frac{\partial}{\partial \bar{\zeta}};$$

$$\overline{T} f(\zeta) = \overline{\left(T \overline{f(\zeta)}\right)}, \ \ \partial_\zeta \overline{T} f(\zeta) = f(\zeta), \ \partial_\zeta = \frac{\partial}{\partial \zeta},$$

For more details, see [2]. We set

$$\Pi f(\zeta) = \partial_\zeta T f(\zeta) = -\frac{1}{\pi}\iint\limits_G \frac{f(z)}{(z-\zeta)^2}dxdy,$$

$$\overline{\Pi} f(\zeta) = \partial_{\bar\zeta} \overline{T} f(\zeta) = -\frac{1}{\pi}\iint\limits_G \frac{f(z)}{(\bar z-\bar\zeta)^2}dxdy,$$

where the integral is understood in the sense of the Cauchy principal value.

We define two-dimensional singular integral operators:

$$\begin{aligned} S_1 f(\zeta) \equiv f(\zeta) + q_1(\zeta)\Pi f(\zeta) + q_2(\zeta)\overline{\Pi f(\zeta)} + \\ + A(\zeta) T f(\zeta) + B(\zeta)\overline{T f(\zeta)}; \end{aligned} \tag{1}$$

$$\begin{aligned} S_2 f(\zeta) \equiv f(\zeta) + \Pi[q_1(\zeta) f(\zeta) + q_2(\zeta)\overline{f(\zeta)}] + \\ + \Pi[A(\zeta)\overline{T} f(\zeta) + B(\zeta) T \overline{f(\zeta)}], \end{aligned} \tag{2}$$

where $q_1(\zeta),\ q_2(\zeta),\ A(\zeta),\ B(\zeta)$, are some complex-valued functions.

The main results of the present work read as follows.

Theorem 1 *Let* $q_1(\zeta),\ q_2(\zeta) \in C(\overline{G})$. *Assume that*

$$|q_1(\zeta)| + |q_2(\zeta)| \le q_0 = const < 1,\ \zeta \in \overline{G}, \tag{3}$$

$A(\zeta),\ B(\zeta) \in L_p(\overline{G})$, $p > 2$, $\partial G = \mathcal{L} \in C^1_\alpha$, $0 < \alpha < 1$. *Then the operators* S_1 *and* S_2 *are linear isomorphisms of the real Banach space* $L_p(\overline{G})$.

In what follows, let inequality (3) hold by default.

Theorem 2 *Assume that* $q_1(\zeta),\ q_2(\zeta),\ A(\zeta),\ B(\zeta) \in C^k_\alpha(\overline{G})$, $k \ge 0$, $0 < \alpha < 1$, $\partial G = \mathcal{L} \in C^{k+1}_\alpha$. *Then the operators* S_1 *and* S_2 *are linear isomorphisms of the real Banach space* $C^k_\alpha(\overline{G})$.

Theorem 3 *Assume that* $q_1(\zeta),\ q_2(\zeta),\ A(\zeta),\ B(\zeta) \in W^k_p(\overline{G})$, $k \ge 1$, $p > 2$, $\partial G = \mathcal{L} \in W_p^{k+1-\frac{1}{p}}$. *Then the operators* S_1 *and* S_2 *are linear isomorphisms of the real Banach space* $W^k_p(\overline{G})$.

Remark 1 The result obtained in [4] is a particular case of Theorem 1 which addresses only operator S_1 and $G = D$. In [4], the author did not used operators T and $\Pi = \partial_\zeta T$. Instead, he employed the following operators

$$T_0 f(\zeta) = -\frac{1}{\pi}\iint\limits_D \left[\frac{f(t)}{t-\zeta} + \frac{\zeta\overline{f(t)}}{1-\zeta\bar t}\right]dxdy,\ \operatorname{Re} T_0 f(\zeta)|_{|\zeta|=1} = 0, \quad \Pi_0 = \partial_\zeta T_0.$$

These definitions stick the considerations of the article [4] to the unit disk too firmly which prevents them from being extended to an arbitrary simply connected domain.

2 Auxiliary Statements

We assume that $\partial G = \mathcal{L} \in C^1_\alpha$ by default, and while discussing the spaces $C^k_\alpha(\overline{G})$ ($W^k_p(\overline{G})$, $k \geq 1$), we assume that $\mathcal{L} \in C^{k+1}_\alpha$ ($\mathcal{L} \in W_p^{k+1-\frac{1}{p}}$). We need the following assertions. For their proofs, see [2, Ch. 1, §8] and [6].

Lemma 1 *The singular operator Π acts continuously in the spaces $C^k_\alpha(\overline{G})$, $k \geq 0$, $0 < \alpha < 1$, and $W^k_p(\overline{G})$, $k \geq 0$, $p > 2$. Moreover, $\|\Pi\|_{L_2} = 1$ and for each $q_0 : 0 < q_0 < 1$ there exists $s_0 = s_0(q_0) > 2$ such that $q_0\|\Pi\|_{L_s} < 1$ as $2 < s < s_0$. The same assertions hold true for the operator $\overline{\Pi}$.*

Lemma 2 *The operator T acts $C^k_\alpha(\overline{G}) \to C^{k+1}_\alpha(\overline{G})$ and $W^k_p(\overline{G}) \to W^{k+1}_p(\overline{G})$ continuously for every $k \geq 0$, $0 < \alpha < 1$, $p > 2$, and, hence, compactly $C^k_\alpha(\overline{G}) \to C^k_\alpha(\overline{G})$ and $W^k_p(\overline{G}) \to W^k_p(\overline{G})$. For every $p > 2$ and $f \in L_p(\overline{G})$, function $Tf \in C_\beta(E)$ $\beta = \frac{p-2}{p}$ is holomorphic in the exterior of domain G, and $Tf(\infty) = 0$. The same assertions hold true for the operator $\overline{T}$.*

Also, we need several regularity results proved in [6] regarding the following operator

$$\Omega w(\zeta) \equiv w(\zeta) + T(q_1\partial_\tau w + q_2\partial_{\bar{\tau}}\overline{w} + Aw + B\overline{w})(\zeta). \tag{4}$$

Lemma 3 *Let $p > 2$, $A(\zeta)$, $B(\zeta) \in L_p(\overline{G})$. Let $q_1(\zeta)$, $q_2(\zeta)$ be bounded measurable functions satisfying (3). Then the operator Ω is continuously invertible in the space $W^1_s(\overline{G})$ for some s with $2 < s \leq p$.*

Theorem 4 *Let all the assumptions of Lemma 3 be satisfied. Assume, in addition, that $q_1(\zeta)$, $q_2(\zeta) \in C(\overline{G})$. Then Ω is the linear isomorphism of real Banach space $W^1_p(\overline{G})$.*

Theorem 5 *Let $k \geq 0$ and $0 < \alpha < 1$. Let all the assumptions of Lemma 3 be satisfied. Assume, in addition, that $q_1(\zeta)$, $q_2(\zeta)$, $A(\zeta)$, $B(\zeta) \in C^k_\alpha(\overline{G})$, $\partial G = \mathcal{L} \in C^{k+1}_\alpha$. Then Ω is a linear isomorphism of the real Banach space $C^{k+1}_\alpha(\overline{G})$.*

Theorem 6 *Let $k \geq 1$, $p > 2$. Let all the assumptions of Lemma 3 be satisfied. Assume, in addition, that $q_1(\zeta)$, $q_2(\zeta)$, $A(\zeta)$, $B(\zeta) \in W^k_p(\overline{G})$, $\partial G = \mathcal{L} \in W_p^{k+1-\frac{1}{p}}$. Then Ω is the linear isomorphism of real Banach space $W^{k+1}_p(\overline{G})$.*

Consider now the general linear first-order elliptic system, which we write in the complex form

$$\partial_{\bar{\zeta}} w + q_1(\zeta)\partial_{\zeta} w + q_2(\zeta)\partial_{\bar{\zeta}}\overline{w} + A(\zeta)w + B(\zeta)\overline{w} = R(\zeta). \tag{5}$$

In (5), $\zeta = \xi + i\eta$, $w = w(\zeta) = u(\zeta) + iv(\zeta)$ is an unknown complex-valued function, and $\partial_{\bar{\zeta}} = 1/2(\partial/\partial\xi + i\partial/\partial\eta)$, $\partial_{\zeta} = 1/2(\partial/\partial\xi - i\partial/\partial\eta)$. We are about to formulate the regularity results for solutions to Eq. (5). For this purpose we define the auxiliary transformation

$$\frac{1}{2\pi i}\int\limits_{\mathcal{L}} \frac{w(t)}{t-\zeta}dt \equiv \mathcal{K}w(\zeta) = \mathcal{K}_{\mathcal{L}}w(\zeta). \tag{6}$$

Lemma 4 *Let $s > 2$ and let $w = w(\zeta) \in W_s^1(\overline{G})$ be a solution to Eq. (5), where $q_1(\zeta),\ q_2(\zeta) \in C(\overline{G}),\ A(\zeta),\ B(\zeta),\ R(\zeta) \in L_p(\overline{G})$ for some $p > 2$. Let $\partial G = \mathcal{L} \in C_\alpha^1$ for some $0 < \alpha < 1$. Assume in addition that*

$$\Phi(\zeta) = \mathcal{K}_{\mathcal{L}}w(\zeta) \in W_p^1(\overline{G}). \tag{7}$$

Then $w(\zeta) \in W_p^1(\overline{G})$.

Lemma 5 *Let $s > 2$, $k \geq 0$, $p > 2$ and $0 < \alpha < 1$. Let $w = w(\zeta) \in W_s^1(\overline{G})$ be a solution to Eq. (5), where $q_1(\zeta),\ q_2(\zeta),\ A(\zeta),\ B(\zeta),\ R(\zeta) \in C_\alpha^k(\overline{G})\ (W_p^k(\overline{G}))$. Let $\partial G = \mathcal{L} \in C_\alpha^{k+1}\ (W_p^{k+1-\frac{1}{p}})$). Assume in addition that function Φ defined by equality (7) belongs to $C_\alpha^{k+1}(\overline{G})\ (W_p^{k+1}(\overline{G}))$. Then $w \in C_\alpha^{k+1}(\overline{G})\ (W_p^{k+1}(\overline{G}))$.*

Proof of Lemma 4 By Pompeiu's formula [2, p. 41, 57, 69],

$$w(\zeta) = \frac{1}{2\pi i}\int\limits_{\mathcal{L}} \frac{w(t)}{t-\zeta}dt - \frac{1}{\pi}\iint\limits_{G} \frac{\partial w}{\partial \bar{t}} \cdot \frac{dxdy}{t-\zeta},\ t = x + iy. \tag{8}$$

The use of equality (8) brings us at the integro-differential equation

$$\Omega(w) = TR + \Phi$$

where $TR \in W_p^1(\overline{G})$ by Lemma 2. Hence, $w(\zeta) \in W_p^1(\overline{G})$ by Theorem 4. □

Proof of Lemma 5 repeats the proof of Lemma 4 almost literally but with replacing Theorem 4 by Theorems 5 and 6.

3 Proof of Theorems 1–3

First, we prove all assertions of the three theorems regarding operator S_1 and then regarding operator S_2.

3.1 Operator S_1

Consider equation

$$S_1 f(\zeta) = R(\zeta) \in L_p(\overline{G})\ (C^k_\alpha(\overline{G}),\ W^k_p(\overline{G})). \tag{9}$$

It follows from the results of [1, §4, Sec. 3], [2, Ch. 3, §17, Sec. 4] that Eq. (9) has a unique solution $f(\zeta) \in L_s(\overline{G})$ for some s with $2 < s \leq p$. Moreover, it follows from Lemmas 1 and 2 that S_1 acts continuously in space $L_p(\overline{G})\ (C^k_\alpha(\overline{G}),\ W^k_p(\overline{G}))$. Hence, by the Banach theorem, we have reduced the proof to inspecting whether $f(\zeta)$ belongs to $L_p(\overline{G})\ (C^k_\alpha(\overline{G}),\ W^k_p(\overline{G}))$. To do so, we put $w(\zeta) = Tf(\zeta)$. Given Eq. (9), we see that w is the solution to Eq. (5). At that, $w \in W^1_s(\overline{G})$ by Lemma 2. Finally, $\mathcal{K}w(\zeta) \equiv 0$ since function w is holomorphic in the exterior of domain G and $w(\infty) = 0$. Given the listed observations and Lemma 4 (respectively Lemma 5), we conclude that $w \in W^1_p(\overline{G})\ (C^{k+1}_\alpha(\overline{G}), W^{k+1}_p(\overline{G}))$, and hence

$$f(\zeta) = \partial_\zeta w(\zeta) \in L_p(\overline{G})\ (C^k_\alpha(\overline{G}),\ W^k_p(\overline{G})).$$

Thus, the case of the operator S_1 is completed.

3.2 Operator S_2

Define

$$S^0_2 f(\zeta) = f(\zeta) + \Pi[q_1(\zeta) f(\zeta) + q_2(\zeta)\overline{f(\zeta)}]$$

and

$$Pf(\zeta) = \Pi[A(\zeta)\overline{T} f(\zeta) + B(\zeta) T\overline{f(\zeta)}].$$

Consider the equation

$$S_2 f(\zeta) \equiv S^0_2 f(\zeta) + Pf(\zeta) = R(\zeta) \in L_p(\overline{G})\ (C^k_\alpha(\overline{G}),\ W^k_p(\overline{G})). \tag{10}$$

By Lemma 1, operator S_2^0 acts continuously in space $L_p(\overline{G})$. Moreover, it is continuously invertible in $L_s(\overline{G})$ for some s with $2 < s \leq p$, see [6, proof of Lemma 8].

By Lemmas 1 and 2, the operator P is compact in space $L_p(\overline{G})$. Thus, the proof of existence and uniqueness of solutions to Eq. (10) in the space $L_s(G)$ we have reduced to the problem of uniqueness of the zero solution to the corresponding homogeneous equation

$$S_2 f(\zeta) = 0. \tag{11}$$

Such a reduction is a consequence to the Fredholm theorem, see, e.g., [8, Ch. 13, §5, Theorem 1].

Let $f(\zeta) \in L_s(\overline{G})$ be a solution to the Eq. (11). We set $w(\zeta) = \overline{T} f(\zeta)$, and put $\beta = \frac{s-2}{s}$. By Lemma 2, $w \in C_\beta(\overline{G})$, and w is anti-holomorphic in the exterior of $\overline{G}$ with

$$w(\infty) = 0. \tag{12}$$

Let $D_R = \{\zeta : |\zeta| < R\}$, let $G \subset D_R$. We put $\Gamma_R = \partial D_R$, $\overline{D}_R = D_R \cup \Gamma_R$. We extend the solution f to Eq. (11) and every coefficient of the operator S_2 from domain $\overline{G}$ to the disk D_R with the zero value. Given such an extension, we consider Eq. (11) in D_R. We rewrite equation (11) as

$$\partial_\zeta \Omega w(\zeta) = 0.$$

It follows from the last equation that

$$\Omega w(\zeta) = \Psi(\zeta) \tag{13}$$

where $\Psi(\zeta) \in W_s^1(\overline{D}_R)$ is anti-holomorphic in D_R. It is clear that the values of this function do not depend on R when $\zeta \in G$. Let us show that $\Psi(\zeta) \equiv 0$. Set

$$g(\zeta) = T(q_1 \partial_\zeta w + q_2 \overline{\partial_\zeta w} + Aw + B\overline{w})(\zeta).$$

Without loss of generality, we assume $0 \in G$. By Lemma 2, function $g(\zeta)$ is holomorphic outside of G and $g(\infty) = 0$. Define

$$g^*(\zeta) = g\left(\frac{R^2}{\bar{\zeta}}\right), \ \zeta \in \overline{D}_R.$$

Function g^* delivers the anti-holomorphic extension of function g from the exterior of disk D_R to the interior of it.

We apply the operator $\overline{\mathcal{K}}_{\Gamma_R}$ ($\overline{\mathcal{K}}_{\Gamma_R} f = \overline{\mathcal{K}_{\Gamma_R} \bar{f}}$) to (13). Note that $\Psi(\zeta)$ and $g^*(\zeta)$ are anti-holomorphic in D_R, and $w(\zeta)$ is anti-holomorphic in the exterior of D_R and satisfies (12). Hence, we get

$$\Psi(\zeta) = g^*(\zeta), \ \zeta \in D_R. \tag{14}$$

It follows from Eq. (14) that the absolute values of $\Psi(\zeta)$ in some neighborhood of zero become arbitrary small when $R \to +\infty$. At the same time, the values of function $\Psi(\zeta)$ are independent of R in some neighborhood of zero. Hence $\Psi(\zeta) \equiv 0$. Then $w(\zeta) \equiv 0$ and $f(\zeta) = \partial_\zeta w(z) \equiv 0$ by (13) and Lemma 3.

Thus, Eq. (10) has a unique solution $f(\zeta) \in L_s(\overline{G})$, $s : 2 < s \le p$. From Lemmas 1 and 2, it follows that S_2 acts continuously in the space $L_p(\overline{G})$ ($C^k_\alpha(\overline{G})$, $W^k_p(\overline{G})$). According to Banach theorem, it remains to show that $f(\zeta)$ belongs to $L_p(\overline{G})$ ($C^k_\alpha(\overline{G})$, $W^k_p(\overline{G})$).

Again, we denote $w(\zeta) = \overline{T} f(\zeta)$ and rewrite equation (10) as

$$\partial_\zeta[\Omega w(\zeta) - \overline{T} R(\zeta)] = 0.$$

Then

$$\Omega w(\zeta) - \overline{T} R(\zeta) = \Psi(\zeta), \tag{15}$$

where $\Psi(\zeta) \in W^1_s(\overline{G})$ is anti-holomorphic in G and $\overline{T} R(\zeta)$ is anti-holomorphic outside of G.

By arguing in the way similar to the previous step, we find out that $\Psi(\zeta) \equiv 0$ in Eq. (15). Since $\overline{T} R(\zeta) \in W^1_p(\overline{G})$ ($C^{k+1}_\alpha(\overline{G})$, $W^{k+1}_p(\overline{G})$), we conclude that $w(\zeta) \in W^1_p(\overline{G})$ ($C^{k+1}_\alpha(\overline{G})$, $W^{k+1}_p(\overline{G})$) by Theorem 4 (5, 6). Hence

$$f(\zeta) = \partial_\zeta w(\zeta) \in L_p(\overline{G}) \ (C^k_\alpha(\overline{G}), \ W^k_p(\overline{G})),$$

and this completes the case of operator S_2.

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On One Class of Solutions of the Darboux System

R. C. Kulaev and A. B. Shabat

Abstract We construct a reduction of the three-dimensional Darboux system for the Christoffel symbols, which describes orthogonal curvilinear coordinate systems. We show that the class of solutions of the Darboux system is parametrized by six functions of one variable (two for each of three independent variables). We give an explicit formulas for Darboux system solutions. In addition, we study the linear system associated with the Darboux system. This system reduces to a three-dimensional Goursat problem for a third-order equation with data on coordinate planes. It is shown that the solution to the Goursat problem admits the separation of variables and is determined by its values on the coordinate lines.

Keywords Darboux three-dimensional system · Integrated system · Goursat problem with three variables · Hydrodynamic type system · Hamiltonian system

Mathematics Subject Classification 35L40, 37K15

We consider a system of six differential equations (*Darboux system*)

$$\left(\partial_j + \Gamma_{kj}\right)\Gamma_{ki} = \Gamma_{ki}\Gamma_{ij} + \Gamma_{kj}\Gamma_{ji}, \quad i \neq j \neq k, \quad 1 \leq i, j, k \leq 3, \tag{0.1}$$

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with respect to the functions Γ_{ij} of three variables $(x, y, z) \in \mathbb{R}^3_+$. Here and below we follow the notation $\partial_1 = \partial_x = \frac{\partial}{\partial x}$, $\partial_2 = \partial_y = \frac{\partial}{\partial y}$, $\partial_3 = \partial_z = \frac{\partial}{\partial z}$. We are interested in issues related to the integrability of the system (0.1) and, in particular, constructing its explicit solutions.

The modern interest in the system (0.1) is due to its close connection to differential geometry. It occurs when describing conjugate curvilinear coordinate systems. In the terminology of differential geometry, the unknown functions Γ_{ij} are the Kristoffel symbols $\Gamma_{ij} \equiv \Gamma^i_{ij}$, $i \neq j$, Levi-Civita connections (see, for example [1]). Today, there are several approaches to the problem of constructing conjugate and orthogonal coordinate systems. An approach based on the inverse problem method in the theory of integrable equations was developed in [2–5]. In the works of Dubrovin-Novikov [6], Zakharov [3], Tsarev [1], and Krichever [7], an approach to the construction of such coordinates was proposed, which is based on the methods of the theory of hydrodynamic type systems. In particular, it was found that an arbitrary orthogonal coordinate system corresponds to a family of diagonal Hamiltonian systems of hydrodynamic type. That systems determined from the system

$$\begin{aligned} u_y &= \Gamma_{12}(v-u), \quad v_z = \Gamma_{23}(w-v), \quad w_x = \Gamma_{31}(u-w), \\ u_z &= \Gamma_{13}(w-u), \quad v_x = \Gamma_{21}(u-v), \quad w_y = \Gamma_{32}(v-w). \end{aligned} \tag{0.2}$$

with respect to the functions u, v, w of three variables $(x, y, z) \in \mathbb{R}^3_+$.

Unfortunately, the solution of the systems (0.1), (0.2) is not possible without introducing additional conditions. In the works [8, 9] the system (0.2) corresponding to weakly nonlinear systems of hydrodynamic type was considered. It is characterized by the fact that the functions u, v, and w do not depend on the variables x, y, and z, respectively. This feature makes it possible to obtain explicit formulas for the solutions of the system (0.2):

$$u = \frac{g_2(y) - g_3(z)}{f_2(y) - f_3(z)}, \quad v = \frac{g_3(z) - g_1(x)}{f_3(z) - f_1(x)}, \quad w = \frac{g_1(x) - g_2(y)}{f_1(x) - f_2(y)}, \tag{0.3}$$

where f_i, g_i are arbitrary functions. In [10] a class of solutions of the Darboux system satisfying the reduction condition

$$\partial_j \Gamma_{jk} = \partial_k \Gamma_{kj} = \Gamma_{jk}\Gamma_{kj}, \quad i \neq j \neq k. \tag{0.4}$$

was introduced. Such reduction provided (local) solvability of the system (0.1) on the Christoffel symbols in terms of their values on the coordinate axes, allowed to obtain explicit formulas for solutions of the associated system (0.2). In particular (see the paper [11]), it was shown that the solutions of the system (0.2) are parametrized by nine functions of one variable (three for each variable x, y, z). In addition, the reduction to a weakly nonlinear system (see formulas (0.2)) is a special case of the approach under consideration.

In this paper, we give another approach to finding explicit solutions of the systems of Eqs. (0.1), and (0.2). In Sect. 2, we construct a reduction of the three-dimensional Darboux system (0.1) with one additional algebraic condition on the functions Γ_{ij}. The corresponding class of solutions is characterized by the fact that the function Γ_{ij} does not depend on the k-th variable, where the indices i, j, k are pairwise different. It is shown that the selected class of solutions is parametrized by six functions of one variable (two for each of the three variables). There is also some ideological connection with the formulas (0.3). However, we note at once that the class of solutions of the Darboux system constructed in this article differs from the class of solutions corresponding to the reduction (0.4). It is determined by the associated systems (0.2), which, generally speaking, are not related to the property of weak nonlinearity.

1 Linear System (0.2)

It is well known (see, for example, [1, 3, 6]) that the Darboux system (0.1) occurs when determining the compatibility conditions of the first order system (0.2). Substitution of Eqs. (0.2) in the system of Darboux equations (0.1) reduces equations with quadratic nonlinearity to linear equations of the second order on the function u, v, w.

Lemma 1 *If the functions* Γ_{ij} *and* u, v, w *are related by the Eqs.* (0.2)*, then the functions* Γ_{ij} *satisfy the equation system* (0.1)*, and the functions* u, v, w *satisfy the second-order equation system,*

$$\begin{aligned} w_{xy} + (\Gamma_{32} - \Gamma_{32}\Gamma_{21}/\Gamma_{31})\, w_x + (\Gamma_{31} - \Gamma_{31}\Gamma_{12}/\Gamma_{32})\, w_y &= 0, \\ u_{yz} + (\Gamma_{13} - \Gamma_{13}\Gamma_{32}/\Gamma_{12})\, u_y + (\Gamma_{12} - \Gamma_{12}\Gamma_{23}/\Gamma_{13})\, u_z &= 0, \\ v_{zx} + (\Gamma_{21} - \Gamma_{21}\Gamma_{13}/\Gamma_{23})\, v_z + (\Gamma_{23} - \Gamma_{23}\Gamma_{31}/\Gamma_{21})\, v_x &= 0. \end{aligned} \tag{1.1}$$

Proof Let us consider the pair of Eqs. (0.2):

$$w_y + \Gamma_{32} w = \Gamma_{32} v, \qquad w_x + \Gamma_{31} w = \Gamma_{31} u.$$

Differentiating first equation with respect to y, and second equation with respect to x, we get

$$\begin{aligned} w_{xy} + \Gamma_{32,x} w + \Gamma_{32} w_x - \Gamma_{32,x} v - \Gamma_{32} v_x &= 0, \\ w_{xy} + \Gamma_{31,y} w + \Gamma_{31} w_y - \Gamma_{31,y} u - \Gamma_{31} u_y &= 0. \end{aligned} \tag{1.2}$$

To the first derivatives of the function w, on the left hand side, we attract the equations of this system, and for the derivatives of the functions u, v, on the right hand side, we attract equations from other pairs:

$$\Gamma_{32} w_x = \Gamma_{32}\Gamma_{31}(u - w), \quad \Gamma_{31} w_y = \Gamma_{31}\Gamma_{32}(v - w),$$
$$\Gamma_{32} v_x = \Gamma_{32}\Gamma_{21}(u - v), \quad \Gamma_{31} u_y = \Gamma_{31}\Gamma_{12}(v - u).$$

Then, subtracting the first equation in (1.2) from the second one, we get a pair of equations with respect to the functions Γ_{ij}

$$\Gamma_{32,x} = \Gamma_{31,y}, \quad \Gamma_{32,x} = \Gamma_{32}\Gamma_{21} + \Gamma_{31}\Gamma_{12} - \Gamma_{32}\Gamma_{31}.$$

From the latter in turn the equations for the functions u, v, w are obtained. Indeed, taking the first equality in (1.2), we have

$$\begin{aligned}
& w_{xy} + \Gamma_{32} w_x - \Gamma_{32,x}(v - w) - \Gamma_{32} v_x \\
&= w_{xy} + \Gamma_{32} w_x - \Gamma_{32,x}(v - w) - \Gamma_{32}\Gamma_{21}(u - v) \\
&= w_{xy} + \Gamma_{32} w_x - \frac{\Gamma_{32,x}}{\Gamma_{32}} w_y - \Gamma_{32}\Gamma_{21}(u - w) + \Gamma_{32}\Gamma_{21}(v - w) \\
&= w_{xy} + \left(\Gamma_{32} - \frac{\Gamma_{32}\Gamma_{21}}{\Gamma_{31}}\right) w_x + \left(\Gamma_{31} - \frac{\Gamma_{31}\Gamma_{12}}{\Gamma_{32}}\right) w_y.
\end{aligned}$$

The remaining equations can be obtained similarly.

Lemma 2 *If the functions Γ_{ij} and u, v, w are related by the Eqs. (0.2), then the function u satisfies the second-order system*

$$\begin{aligned}
& u_{xy} + \Gamma_{12} u_x + \left(\Gamma_{21} - \frac{\Gamma_{12,x}}{\Gamma_{12}}\right) u_y = 0, \\
& u_{xz} + \Gamma_{13} u_x + \left(\Gamma_{31} - \frac{\Gamma_{13,x}}{\Gamma_{13}}\right) u_z = 0.
\end{aligned} \tag{1.3}$$

The proof is similar to the proof of Lemma 1. To derive the first equation, it is necessary to consider a pair of equations of the system (0.2) with the functions Γ_{12} and Γ_{21}, whilst the pair of functions Γ_{13} and Γ_{31} should be considered in case of the second equation.

Remark 1 From (1.3) by cyclic permutations

$$x \to y \to z \to x, \quad 1 \to 2 \to 3 \to 1. \tag{1.4}$$

one can obtain two more systems of the same form for the functions v and w. Consequently, each of the functions u, v, w satisfies a system of three second order hyperbolic equations.

Remark 2 Darboux equations (0.1) admit "conformal" transformations:

$$x \to X(x), \quad y \to Y(y), \quad z \to Z(z)$$

which translate the solution Γ_{ij} of the system (0.1) into the following set of functions:

$$\frac{\Gamma_{21}}{X'(x)}, \frac{\Gamma_{31}}{X'(x)}, \quad \frac{\Gamma_{12}}{Y'(y)}, \frac{\Gamma_{32}}{Y'(y)}, \quad \frac{\Gamma_{13}}{Z'(z)}, \frac{\Gamma_{23}}{Z'(z)}.$$

In particular, constant solution of the system (0.1) is transferred to the system of functions of one variable.

2 Reduction of the Darboux System

This section deals with the reduction of the Darboux system. It is determined by the following property: *for $i \neq j \neq k \neq i$, the function Γ_{ij} does not depend on the k-th variable.* It is obvious that in the class of functions Γ_{ij} under consideration, all constant solutions of the system (0.1) are contained. It is easy to verify that in this case the Darboux system (0.1) reduces to a system of three algebraic equations

$$\frac{\Gamma_{21}}{\Gamma_{31}} + \frac{\Gamma_{12}}{\Gamma_{32}} = 1, \quad \frac{\Gamma_{32}}{\Gamma_{12}} + \frac{\Gamma_{23}}{\Gamma_{13}} = 1, \quad \frac{\Gamma_{13}}{\Gamma_{23}} + \frac{\Gamma_{31}}{\Gamma_{21}} = 1. \tag{2.1}$$

In our further considerations, the key role will be played by determinant of the matrix

$$\mathbf{\Gamma} = \begin{pmatrix} 0 & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & 0 & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & 0 \end{pmatrix}, \quad \det \mathbf{\Gamma} = \Gamma_{12}\Gamma_{23}\Gamma_{31} + \Gamma_{13}\Gamma_{32}\Gamma_{21}. \tag{2.2}$$

Therefore, before proceeding to solving the Darboux system, we first establish the relationship of the system (2.1) with the determinant of the matrix $\mathbf{\Gamma}$.

Lemma 3 *The general solution of algebraic equations* (2.1) *depends on four free parameters. Moreover, any solution of the system* (2.1) *satisfies the condition*

$$\det \mathbf{\Gamma} = 0. \tag{2.3}$$

Proof Let Γ_{12}, Γ_{21} and Γ_{13}, Γ_{31} be selected as free parameters. From the first and third equations of the system (2.1) we get

$$\Gamma_{23} = \frac{\Gamma_{13}\Gamma_{21}}{\Gamma_{21} - \Gamma_{31}}, \quad \Gamma_{32} = \frac{\Gamma_{12}\Gamma_{31}}{\Gamma_{31} - \Gamma_{21}}. \tag{2.4}$$

Substituting in the second equation (2.1) instead of Γ_{23}, Γ_{32}, the right-hand sides of the Eqs. (2.4), we obtain the identity. Therefore, the Eqs. (2.4) are equivalent to the three Eqs. (2.1).

To prove the second part of the assertion of the lemma, we note that (2.4) implies the equality (2.4)

$$\frac{\Gamma_{13}\Gamma_{21}}{\Gamma_{23}} = -\frac{\Gamma_{12}\Gamma_{31}}{\Gamma_{32}}.$$

Now condition (2.3) is obvious by virtue of formulas (2.2).

Remark 3 The "conformal" transformations $x \to X(x),\ y \to Y(y),\ z \to Z(z)$ preserve the form of Eqs. (2.1). Moreover, they do not change the value of the determinant of the matrix $\mathbf{\Gamma}$.

Example 1 It is easy to verify that if $\Gamma_{12} = \Gamma_{13} = \Gamma_{21} = 1$, $\Gamma_{31} = 2$, $\Gamma_{23} = s$ and $\Gamma_{32} = -2s$, then the matrix $\mathbf{\Gamma}$ is degenerate for any $s \in \overline{\mathbb{R}}$. At the same time, the substitution of these values in the left parts of the first and third equations in (2.1) gives

$$\frac{1}{2} + \frac{1}{-2s} = \frac{s-1}{2s}, \qquad \frac{1}{s} + \frac{2}{1} = \frac{2s+1}{s}.$$

Note that the Eqs. (2.1) are satisfied only for $s = -1$.

Further, we will show that solutions of the system of three algebraic equations (2.1) are parametrized by six functions of one variable (two for each of the variables x, y, z).

Since for $k \neq i \neq j \neq k$, the function Γ_{ij} does not depend on the k-th variable, then from Lemma 3, we have

$$\begin{aligned}
\frac{\Gamma_{21}\Gamma_{32}}{\Gamma_{12}\Gamma_{23}} = -\frac{\Gamma_{31}}{\Gamma_{13}} \quad &\Rightarrow \quad \frac{\Gamma_{21}}{\Gamma_{12}} = \frac{X'(x)}{Y'(y)}, \quad \frac{\Gamma_{32}}{\Gamma_{23}} = \frac{Y'(y)}{Z'(z)}, \\
\frac{\Gamma_{21}\Gamma_{13}}{\Gamma_{12}\Gamma_{31}} = -\frac{\Gamma_{23}}{\Gamma_{32}} \quad &\Rightarrow \quad \frac{\Gamma_{21}}{\Gamma_{12}} = \frac{X'(x)}{Y'(y)}, \quad \frac{\Gamma_{13}}{\Gamma_{31}} = -\frac{Z'(z)}{X'(x)}.
\end{aligned} \tag{2.5}$$

where X, Y, Z—arbitrary functions of one variable.

Using first equation from (2.1) and (2.5), we get

$$\frac{X'\Gamma_{12}}{Y'\Gamma_{31}}+\frac{Z'\Gamma_{12}}{Y'\Gamma_{23}}=1 \quad \Rightarrow \quad \frac{\Gamma_{12}}{Y'}\left(\frac{X'}{\Gamma_{31}}+\frac{Z'}{\Gamma_{23}}\right)=1 \quad \Rightarrow \quad \frac{\Gamma_{12}}{Y'}\left(-\frac{Z'}{\Gamma_{13}}+\frac{Y'}{\Gamma_{32}}\right)=1,$$

therefore,

$$\frac{X'}{\Gamma_{31}}+\frac{Z'}{\Gamma_{23}}=\frac{Y'}{\Gamma_{12}}, \qquad \frac{Z'}{\Gamma_{13}}+\frac{Y'}{\Gamma_{12}}=\frac{Y'}{\Gamma_{32}}. \tag{2.6}$$

Attracting other equations of system (2.1), we obtain another pair of equations

$$\frac{X'}{\Gamma_{21}}+\frac{Z'}{\Gamma_{13}}=\frac{Z'}{\Gamma_{23}}, \qquad \frac{X'}{\Gamma_{31}}+\frac{Y'}{\Gamma_{32}}=\frac{X'}{\Gamma_{21}}. \tag{2.7}$$

Since each fraction in (2.6), (2.7) depends only on two variables, we have

$$\frac{X'}{\Gamma_{31}}=f_3-f_1, \qquad \frac{Z'}{\Gamma_{23}}=f_2-f_3, \qquad \frac{Y'}{\Gamma_{12}}=f_2-f_1,$$
$$\frac{Z'}{\Gamma_{13}}=f_1-f_3, \qquad \frac{Y'}{\Gamma_{32}}=f_2-f_3, \qquad \frac{X'}{\Gamma_{21}}=f_2-f_1,$$

where f_i are arbitrary functions, each of which depends only on the ith variable from the ordered set $\{x, y, z\}$. From this, we finally obtain the solution of the Darboux system (0.1), which, taking into account the arbitrariness of f_i and remark 1, can be written as

$$\Gamma_{31}=\frac{X'}{f_3(Z)-f_1(X)}, \qquad \Gamma_{23}=\frac{Z'}{f_2(Y)-f_3(Z)}, \qquad \Gamma_{12}=\frac{Y'}{f_2(Y)-f_1(X)},$$
$$\Gamma_{13}=\frac{Z'}{f_1(X)-f_3(Z)}, \qquad \Gamma_{32}=\frac{Y'}{f_2(Y)-f_3(Z)}, \qquad \Gamma_{21}=\frac{X'}{f_2(Y)-f_1(X)}. \tag{2.8}$$

By virtue of Lemma 3 condition (2.3) is a consequence of the algebraic system (2.1), and Example 1 indicates that condition (2.3) and the system (2.1) are not the same. Further, we show that if we supplement system (0.1) with the condition (2.3), then the solutions Γ_{ij} of the resulting system will not depend on the k-th variable, i.e. the system of relations (0.1), (2.3) is reduced to an algebraic system (2.1).

Lemma 4 *The solutions of the Darboux system* (0.1) *satisfy equality* (2.3) *if and only if the condition*

$$\partial_x(\Gamma_{32}\Gamma_{23})=\partial_y(\Gamma_{13}\Gamma_{31})=\partial_z(\Gamma_{12}\Gamma_{21})=0 \tag{2.9}$$

holds.

Proof From Eqs. (0.1), we get

$$\partial_x(\Gamma_{32}\Gamma_{23}) = \Gamma_{32}\partial_x\Gamma_{23} + \Gamma_{23}\partial_x\Gamma_{32}$$
$$= \Gamma_{32}(\Gamma_{23}\Gamma_{31} + \Gamma_{21}\Gamma_{13} - \Gamma_{23}\Gamma_{21}) + \Gamma_{23}(\Gamma_{32}\Gamma_{21} + \Gamma_{31}\Gamma_{12} - \Gamma_{32}\Gamma_{31})$$
$$= \Gamma_{32}\Gamma_{21}\Gamma_{13} + \Gamma_{23}\Gamma_{31}\Gamma_{12}.$$

Consequently, we arrive at the equivalence

$$\partial_x(\Gamma_{32}\Gamma_{23}) = 0 \quad \Leftrightarrow \quad \Gamma_{32}\Gamma_{21}\Gamma_{13} + \Gamma_{23}\Gamma_{31}\Gamma_{12} = 0.$$

The rest of the statement of the lemma can be obtained by cyclic permutations of the indices $\{1, 2, 3\}$ and variables $\{x, y, z\}$.

Let Γ_{ij} be solutions of the Darboux system satisfying (2.3). Then from Lemma 4 it follows that the product $\Gamma_{12}\Gamma_{21}$ does not depend on z. Hence, there are functions $a_{12}(x, y)$, $a_{21}(x, y)$ and $c(z)$ such that

$$\Gamma_{12} = a_{12}c, \quad \Gamma_{21} = \frac{a_{21}}{c}. \tag{2.10}$$

Similarly, one can show that

$$\begin{aligned} \Gamma_{13} &= a_{13}b, \quad \Gamma_{31} = \frac{a_{31}}{b}, \\ \Gamma_{23} &= a_{23}a, \quad \Gamma_{32} = \frac{a_{32}}{a}, \end{aligned} \tag{2.11}$$

where the functions a_{ij} depend only on the i and j variables, and the functions a and b depend only on x and y, respectively. There are two possible options: either $a'(x)b'(y)c'(z) = 0$ or $a'(x)b'(y)c(z) \neq 0$. Consider each of them separately.

Let $a'(x)b'(y)c'(z) = 0$. Since the functions Γ_{ij} satisfy the semi-Hamiltonian conditions $\Gamma_{ij,k} = \Gamma_{ik,j}$ for any triple $i \neq j \neq k \neq i$ (see formula (0.1)), the equality

$$a_{12}c' = a_{13}b' \tag{2.12}$$

holds. It follows from (2.12) that $c(z) = c$, $b(y) = b$, where c and b are nonzero constants. Then, from the equality $\Gamma_{31,y} = \Gamma_{32,x}$, in turn, it follows that the function $a(x)$ does not depend on x. Consequently, the function Γ_{ij} does not depend on the kth variable, $k \neq i \neq j \neq k$. Therefore Γ_{ij} satisfy system 2.1.

Now suppose that $a'(x)b'(y)c'(z) \neq 0$. Then from (2.12) it follows that

$$\frac{a_{12}}{b'} = \frac{a_{13}}{c'}.$$

The left side does not depend on z, and the right side does not depend on y. Consequently

$$a_{12} = a_1(x)b', \quad a_{13} = a_1(x)c'.$$

Hence, Γ_{12} and Γ_{13} admit presentation

$$\Gamma_{12} = a_1(x)b'(y)c(z), \quad \Gamma_{13} = a_1(x)b(y)c'(z).$$

Using similar reasoning we obtain equalities for the remaining four functions Γ_{ij}

$$\Gamma_{23} = b_1(y)a(x)c_2'(z), \quad \Gamma_{21} = b_1(y)a'(x)c_2(z),$$
$$\Gamma_{31} = c_1(z)a_2'(x)b_2(y), \quad \Gamma_{32} = c_1(z)a_2(x)b_2'(y).$$

Comparing the obtained expressions for Γ_{ij} with the formulas (2.10) and (2.11), we obtain that the Darboux system solution (0.1) is parametrized by the procession of functions of one variable and can be represented as

$$\Gamma_{12} = \frac{\widetilde{X}(x)Y'(y)Z(z)}{Y^2(y)}, \quad \Gamma_{23} = \frac{\widetilde{Y}(y)X(x)Z'(z)}{Z^2(z)}, \quad \Gamma_{31} = \frac{\widetilde{Z}(z)X'(x)Y(y)}{X^2(x)},$$
$$\Gamma_{13} = -\frac{\widetilde{X}(x)Z'(z)}{Y(y)}, \quad \Gamma_{21} = -\frac{\widetilde{Y}(y)X'(x)}{Z(z)}, \quad \Gamma_{32} = -\frac{\widetilde{Z}(z)Y'(y)}{X(x)}, \tag{2.13}$$

where

$$X'(x)Y'(y)Z'(z) \neq 0. \tag{2.14}$$

Moreover, in view of the Darboux equations (0.1), all six functions of one variable, defining the parametrization, are interconnected by one functional equation

$$\frac{X\widetilde{Y}}{Z} + \frac{Y\widetilde{Z}}{X} + \frac{Z\widetilde{X}}{Y} = 1, \tag{2.15}$$

which is invariant under cyclic permutations of symbols $\{X, Y, Z\}$.

Let us show that the system of relations (2.14), and (2.15) is incompatible. To do this, we rewrite the Eq. (2.15) as

$$X^2Y\widetilde{Y} + Y^2Z\widetilde{Z} + Z^2X\widetilde{X} = XYZ.$$

Differentiating this equation with respect to x, we get

$$2XX'Y\widetilde{Y} + Z^2(X\widetilde{X})' = X'YZ.$$

Further, differentiating the obtained equality with respect to z, we have

$$2ZZ'(X\widetilde{X})' = X'YZ' \quad \Rightarrow \quad Z'(2Z(X\widetilde{X})' - X'Y) = 0.$$

It follows that $Z(z) = \text{const}$, which contradicts (2.14).

We summarize the results of this section in the form of the following theorem.

Theorem 1 *Let the condition* (2.3) *be satisfied. Then the Darboux system solutions are parametrized by six functions of one variable (two for each, each of the variables* x, y, z*) and represented by formulas* (2.8)*.*

Example 2 It is easy to show that following collection of functions

$$\Gamma_{31} = \frac{1}{e^x + e^{-z}} = -\Gamma_{13}, \quad \Gamma_{23} = \frac{1}{2e^y - e^{-z}} = \Gamma_{32}, \quad \Gamma_{12} = \frac{1}{e^x + 2e^y} = \Gamma_{21}$$

is a solution of the system (0.1). Moreover, the equality (2.3) holds and the reduction condition (0.4) is not satisfied.

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Boolean Valued Analysis: Background and Results

A. G. Kusraev and S. S. Kutateladze

Abstract The paper provides a brief overview of the origins, methods and results of Boolean valued analysis. Boolean valued representations of some mathematical structures and mappings are given in tabular form. A list of some problems arising outside the theory of Boolean valued models, but solved using the Boolean valued approach, is given. The relationship between the Kantorovich's heuristic principle and the Boolean valued transfer principle is also discussed.

Keywords Vector lattice · Kantorovich's principle · Gordon's theorem · Boolean valued analysis · Boolean valued representation

Mathematics Subject Classification (2000) 06F25, 46A40.

1 Introduction

In 1977, Eugene Gordon, a young teacher of Lobachevsky Nizhny Novgorod State University, published the short note [13] which begins with the words:

> This article establishes that the set whose elements are the objects representing reals in a Boolean valued model of set theory $\mathbb{V}^{(\mathbb{B})}$, can be endowed with the structure of a vector

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space and an order relation so that it becomes an extended K-space[1] with base[2] $\mathbb{B}$. It is shown that in some cases this fact can be used to generalize the theorems about reals to extended K-spaces.

His note has become the bridge between various areas of mathematics which helps, in particular, to solve numerous problems of functional analysis in "semiordered vector spaces" [36] by using the techniques of Boolean valued models of set theory [6].

In the same year, at the Symposium on Applications of Sheaf Theory to Logic, Algebra, and Analysis (Durham, July 9–11, 1977), Gaisi Takeuti, a renowned expert in proof theory, observed that if $\mathbb{B}$ is a complete Boolean algebra of orthogonal projections in a Hilbert space H, then the set whose elements represent reals in the Boolean valued model $\mathbb{V}^{(\mathbb{B})}$ can be identified with the vector lattice of selfadjoint operators in H whose spectral resolutions take values in $\mathbb{B}$; see [93].

These two events marked the birth of a new section of functional analysis, which Takeuti designated by the term *Boolean valued analysis*. The history and achievements of Boolean valued analysis are reflected in [56–58].

It should be mentioned that in 1969 Dana Scott foresaw that the new nonstandard models must be of mathematical interest beyond the independence proof, but he was unable to give a really good evidence of this; see [87]. In fact Takeuti found a narrow path whereas Gordon paved a turnpike to the core of mathematics, which justifies the vision of Scott.

Boolean valued analysis signifies the technique of studying the properties of an arbitrary mathematical object by comparison between its representations in two different Boolean valued models of set theory. As the models, we usually take the *von Neumann universe* $\mathbb{V}$ (the mundane embodiment of the classical Cantorian paradise) and the *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ (a specially-trimmed universe whose construction utilizes a complete Boolean algebra $\mathbb{B}$). The principal difference between $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$ is the way of verification of statements: There is a natural way of assigning to each statement ϕ about $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$ the *Boolean truth-value* $[\![\phi(x_1, \dots, x_n)]\!] \in \mathbb{B}$. The sentence $\phi(x_1, \dots, x_n)$ is called true in $\mathbb{V}^{(\mathbb{B})}$ if $[\![\phi(x_1, \dots, x_n)]\!] = \mathbb{1}$. All theorems of Zermelo–Fraenkel set theory with the axiom of choice are true in $\mathbb{V}^{(\mathbb{B})}$ for every complete Boolean algebra $\mathbb{B}$. There is a smooth and powerful mathematical technique for revealing interplay between the interpretations of one and the same fact in the two models $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$. The relevant *ascending-and-descending machinery* rests on the functors of *canonical embedding* $X \mapsto X^\wedge$, *descent* $X \mapsto X\!\downarrow$, and *ascent* $X \mapsto X\!\uparrow$ acting between $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$, see [56, 57]. Everywhere below $\mathbb{B}$ is a complete Boolean algebra and $\mathbb{V}^{(\mathbb{B})}$ the corresponding Boolean valued model of set theory; see [6, 99]. A *partition of unity* in $\mathbb{B}$ is a family $(b_\xi)_{\xi\in\Xi} \subset \mathbb{B}$ such that $\bigvee_{\xi\in\Xi} b_\xi = \mathbb{1}$ and $b_\xi \wedge b_\eta = \mathbb{0}$

[1] A *K-space* or a *Kantorovich space* is a Dedekind complete vector lattice. An *extended K-space* is a universally complete vector lattice, cp. [4] and [104].

[2] The *base* of a vector lattice is the inclusion ordered set of all of its bands (that forms a complete Boolean algebra) [36, 104].

whenever $\xi \neq \eta$. The unexplained terms of vector lattice theory can be found in [4, 70, 71, 85, 104].

2 Kantorovich's Heuristic Principle

Definition 1 A *vector lattice* or a *Riesz space* is a real vector space X equipped with a partial order $\leq$ for which the *join* $x \vee y$ and the *meet* $x \wedge y$ exist for all $x, y \in X$, and such that the *positive cone* $X_+ := \{x \in X : 0 \leq x\}$ is closed under addition and multiplication by positive reals and for any $x, y \in X$ the relations $x \leq y$ and $0 \leq y - x$ are equivalent. A *band* in a vector lattice X is the *disjoint complement* Y' of any set $Y \subset X$ where $Y' := \{x \in X : (\forall y \in Y)\, |x| \wedge |y| = 0\}$. Let $\mathbb{P}(X)$ stand for the complete Boolean algebra of all band projections in X.

Definition 2 A subset $U \subset X$ is *order bounded* if U lies in an *order interval* $[a, b] := \{x \in X : a \leq x \leq b\}$ for some $a, b \in X$. A vector lattice X is *Dedekind complete* (respectively, *laterally complete*) if each nonempty order bounded set (respectively, each nonempty set of pairwise disjoint positive vectors) U in X has a least upper bound $\sup(U) \in X$. The vector lattice that is laterally complete and Dedekind complete simultaneously is referred to as *universally complete*.

Definition 3 An *f-algebra* is a vector lattice X equipped with a distributive multiplication such that if $x, y \in X_+$ then $xy \in X_+$, and if $x \wedge y = 0$ then $(ax) \wedge y = (xa) \wedge y = 0$ for all $a \in X_+$. An f-algebra is *semiprime* provided that $xy = 0$ implies $x \perp y$ for all x and y. A *complex vector lattice* $X_{\mathbb{C}}$ is the complexification $X_{\mathbb{C}} := X \oplus iX$ (with i standing for the imaginary unity) of a real vector lattice X.

Leonid Kantorovich was among the first who studied operators in ordered vector spaces. He distinguished an important instance of ordered vector spaces, a Dedekind complete vector lattice, often called a *Kantorovich space* or a *K-space*. This notion appeared in Kantorovich's first fundamental article [35] on this topic where he wrote:

> In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in such a space) as linear functionals.

Here Kantorovich stated an important methodological principle, the *heuristic transfer principle* for K-spaces, claiming that the elements of a K-space can be considered as generalized reals. Essentially, this principle turned out to be one of those profound ideas that, playing an active and leading role in the formation of a new branch of analysis, led eventually to a deep and elegant theory of K-space rich in various applications. At the very beginning of the development of the theory, attempts were made at formalizing the above heuristic argument. In this direction, there appeared the so-called *identity preservation theorems* which claimed that if some proposition involving finitely many relations is proven for the reals then an analogous fact remains valid automatically for the elements of every K-space (see [36, 71, 104]). The depth and universality of Kantorovich's principle were demonstrated within Boolean valued analysis. See more about the Kantorovich's universal heuristics and innate integrity of his methodology in [67]. The contemporary forms of above mentioned relation preservation theorems, basing on Boolean valued models, may be found in Gordon [15, 18, 21] and Jech [30].

3 Boolean Valued Reals

Boolean valued analysis stems from the fact that each internal field of reals of a Boolean valued model descends into a universally complete vector lattice. Thus, a remarkable opportunity opens up to expand and enrich the mathematical knowledge by translating information about the reals to the language of other branches of functional analysis.

According to the principles of Boolean valued set theory there exists an internal field of reals $\mathscr{R}$ in a model $\mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism. In other words, there exists $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ for which $[\![\, \mathscr{R}$ is a field of reals $]\!] = \mathbb{1}$. Moreover, if $[\![\, \mathscr{R}'$ is a field of reals $]\!] = \mathbb{1}$ for some $\mathscr{R}' \in \mathbb{V}^{(\mathbb{B})}$ then $[\![$ the ordered fields $\mathscr{R}$ and $\mathscr{R}'$ are isomorphic $]\!] = \mathbb{1}$.

By the same reasons there exists an internal field of complex numbers $\mathscr{C} \in \mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism. Moreover, $\mathbb{V}^{(\mathbb{B})} \models \mathscr{C} = \mathscr{R} \oplus i\mathscr{R}$. We call $\mathscr{R}$ and $\mathscr{C}$ the *internal reals* and *internal complexes* in $\mathbb{V}^{(\mathbb{B})}$.

The fundamental result of Boolean valued analysis is *Gordon's Theorem* [13] which reads as follows: *Each universally complete vector lattice is an interpretation of the reals in an appropriate Boolean valued model.* Formally:

Gordon Theorem *Let $\mathbb{B}$ be a complete Boolean algebra, $\mathscr{R}$ be a field of reals within $\mathbb{V}^{(\mathbb{B})}$. Endow $\mathbf{R} := \mathscr{R}{\downarrow}$ with the descended operations and order. Then*

(1) *The algebraic structure $\mathbf{R}$ is a universally complete vector lattice.*
(2) *The field $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ can be chosen so that $[\![\, \mathbb{R}^{\wedge}$ is a dense subfield of $\mathscr{R}\,]\!] = \mathbb{1}$.*
(3) *There is a Boolean isomorphism χ from $\mathbb{B}$ onto $\mathbb{P}(\mathbf{R})$ such that*

$$\chi(b)x = \chi(b)y \Longleftrightarrow b \le [\![\, x = y \,]\!],$$
$$\chi(b)x \le \chi(b)y \Longleftrightarrow b \le [\![\, x \le y \,]\!]$$
$$(x, y \in \mathbf{R};\ \ b \in \mathbb{B}).$$

For a detailed proof of the Gordon Theorem, see [45, 56, 58]. Observe also some additional properties of Boolean valued reals, namely multiplicative structure and complexification:

Corollary 1 *The universally complete vector lattice $\mathscr{R}{\downarrow}$ with the descended multiplication is a semiprime f-algebra with the ring unity $\mathbb{1} := 1^{\wedge}$. Moreover, for every $b \in \mathbb{B}$ the band projection $\chi(b) \in \mathbb{P}(\mathbf{R})$ acts as multiplication by $\chi(b)\mathbb{1}$.*

Corollary 2 *Let $\mathscr{C}$ be the field of complex numbers within $\mathbb{V}^{(\mathbb{B})}$. Then the algebraic system $\mathscr{C}{\downarrow}$ is a universally complete complex f-algebra. Moreover, $\mathscr{C}{\downarrow}$ is the complexification of the universally complete real f-algebra $\mathscr{R}{\downarrow}$; i.e., $\mathscr{C}{\downarrow} = \mathscr{R}{\downarrow} \oplus i\mathscr{R}{\downarrow}$.*

Example 1 Assume that a measure space (Ω, Σ, μ) is semi-finite; i.e., if $A \in \Sigma$ and $\mu(A) = \infty$ then there exists $B \in \Sigma$ with $B \subset A$ and $0 < \mu(B) < \infty$. The vector lattice $L^0(\mu) := L^0(\Omega, \Sigma, \mu)$ (of cosets) of μ-measurable functions on Ω is universally complete if and only if (Ω, Σ, μ) is *localizable* ($\equiv$ *Maharam*). In this event $L^p(\Omega, \Sigma, \mu)$ is Dedekind complete; see [11, 241G]. Observe that $\mathbb{P}(L^0(\Omega, \Sigma, \mu)) \simeq \Sigma/\mu^{-1}(0)$.

Example 2 Given a complete Boolean algebra $\mathbb{B}$ of orthogonal projections in a Hilbert space H, denote by $\langle\mathbb{B}\rangle$ the space of all selfadjoint operators on H whose spectral resolutions are in $\mathbb{B}$; i.e., $A \in \langle\mathbb{B}\rangle$ if and only if $A = \int_{\mathbb{R}} \lambda \, dE_\lambda$ and $E_\lambda \in \mathbb{B}$ for all $\lambda \in \mathbb{R}$. Define the partial order in $\langle\mathbb{B}\rangle$ by putting $A \geq B$ whenever $\langle Ax, x\rangle \geq \langle Bx, x\rangle$ holds for all $x \in \mathscr{D}(A) \cap \mathscr{D}(B)$, where $\mathscr{D}(A) \subset H$ stands for the domain of A. Then $\langle\mathbb{B}\rangle$ is a universally complete vector lattice and $\mathbb{P}(\langle\mathbb{B}\rangle) \simeq \mathbb{B}$.

Example 3 Let $\Lambda = \mathscr{R}{\Downarrow}$ stands for the bounded part of the universally complete vector lattice $\mathscr{R}{\downarrow}$, that is, $\Lambda := \{x \in \mathscr{R}{\downarrow} : |x| \leq C^{\wedge}$ for some $C \in \mathbb{R}\}$. Then Λ is a Dedekind complete vector lattice and $\bar{\Lambda} := \Lambda \oplus i\Lambda$ is a complex Dedekind complete vector lattice. Moreover, Λ can be endowed with a norm $\|x\|_\infty := \inf\{\alpha > 0 : |x| \leq \alpha\mathbb{1}\}$.

If μ is a Maharam measure and $\mathbb{B}$ in the Gordon Theorem is the algebra of all μ-measurable sets modulo μ-negligible sets, then $\mathscr{R}{\downarrow}$ is lattice isomorphic to $L^0(\mu)$; see Example 1. If $\mathbb{B}$ is a complete Boolean algebra of projections in a Hilbert space H then $\mathscr{R}{\downarrow}$ is isomorphic to $\langle\mathbb{B}\rangle$; see Example 2. The two indicated particular cases of Gordon's Theorem were intensively and fruitfully exploited by Takeuti [92–95]. The object $\mathscr{R}{\downarrow}$ for general Boolean algebras was also studied by Jech [30, 31], and [32] who in fact rediscovered Gordon's Theorem. The difference is that in [30] a (complex) universally complete vector lattice with unit is defined by another system of axioms and is referred to as a complete *Stone algebra*. By selecting special $\mathbb{B}$'s, it is possible to obtain some properties of $\mathscr{R}$.

Remark 1 In 1963 P. Cohen discovered his *method of 'forcing'* and also proved the independence of the Continuum Hypothesis. A comprehensive presentation of the Cohen forcing method gave rise to the *Boolean valued models of set theory*, which were first introduced by D. Scott and R. Solovay (see Scott [87]) and P. Vopěnka [103]. A systematic account of the theory of Boolean valued models and its applications to independence proofs can be found in [6, 33, 91, 99].

Remark 2 Gordon came to his theorem, while trying to solve the Solovay's famous problem. Assuming the consistency with ZFC of the existence of inaccessible cardinal, R. Solovay established the following result: *The statement "Every subset of* $\mathbb{R}$ *is Lebesgue measurable" is consistent with* ZF+DC (*Dependent choice*), see [90]. The Solovay's problem asks whether or not this result remains true without assumption of consistency of existence of inaccessible cardinal? Gordon failed to solve this problem but proved the following weaker assertion: *The statement "The Lebesgue measure on* $\mathbb{R}$ *can be extended to a* σ*-additive invariant measure on the* σ*-algebra of sets definable by a countable sequence of ordinals" is consistent with* ZFC,[3] see [13, Theorem 7] and [16]. In order to prove this result he needed to examine a Boolean algebra $\mathscr{B}$ with a measure $\mu : \mathscr{B} \to \mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$ and identify the descent $\mu{\downarrow} : \mathscr{B}{\downarrow} \to \mathscr{R}{\downarrow}$ of μ in $\mathbb{V}$. Thus, he discovered that the algebraic structure of $\mathscr{R}{\downarrow}$ is a well-known object, and it is K-space, which he learned from the book [101].

Remark 3 Many delicate properties of the objects inside $\mathbb{V}^{(\mathbb{B})}$ depend essentially on the structure of the initial Boolean algebra $\mathbb{B}$. The diversity of opportunities together with a great stock of information on particular Boolean algebras ranks Boolean valued models among the most powerful tools of foundational studies, see [6, 33, 99]. Here it is worth mentioning two deep independence results in analysis: The sentences SH[4] (Souslin's Hypothesis) and NDH[5] (No Discontinuous Homomorphisms) are independent of ZFC, see [29, 91] and [10], respectively.

4 Boolean Valued Representation of Structures

Every Boolean valued universe has the collection of mathematical objects in full supply. Available in plenty are all sets with extra structure: groups, rings, algebras, normed spaces, operators etc. Applying the descent functor to these internal algebraic systems of a Boolean valued model, we distinguish some bizarre entities or recognize old acquaintances, which leads to revealing the new facts of their life and structure.

[3] Earlier G. Saks [88] without assumption of existence of inaccessible cardinal proved that the statement "The Lebesgue measure on $\mathbb{R}$ can be extended to the σ-additive invariant measure defined on all subsets of $\mathbb{R}$" is consistent with ZF + DC.

[4] H: Every order complete order dense linearly ordered set having no first or last element is order isomorphic to the ordered set of reals $\mathbb{R}$, provided that every collection of mutually disjoint non-empty open intervals in it is countable.

[5] NDH: For each compact space X, each homomorphism from $C(X, \mathbb{C})$ into a Banach algebra is continuous.

It thus stands to reason to raise the following question: What structures significant for mathematical practice are obtainable by the Boolean values interpretation of the most typical algebraic systems? The answer is given in terms of Boolean sets.

1. A *Boolean set* or, more precisely, a $\mathbb{B}$-*set* is by definition a pair (X, d), where $X \in \mathbb{V}$, $X \neq \varnothing$, and d is a mapping from $X \times X$ to $\mathbb{B}$ satisfying for all $x, y, z \in X$ the conditions: (1) $d(x, y) = \mathbb{O}$ if and only if $x = y$; (2) $d(x, y) = d(y, x)$; (3) $d(x, y) \leq d(x, z) \vee d(z, y)$. Each nonempty subset $\varnothing \neq X \subset \mathbb{V}^{(\mathbb{B})}$ provides an example of a $\mathbb{B}$-set on assuming that $d(x, y) := [\![x \neq y]\!] = [\![x = y]\!]^*$ for all $x, y \in X$. Another example arises if we furnish a nonempty set X with the "discrete $\mathbb{B}$-metric" d; i. e., on letting $d(x, y) = \mathbb{1}$ in case $x \neq y$ and $d(x, y) = \mathbb{O}$ in case $x = y$.

2. For every $\mathbb{B}$-set (X, d) there are a member $\mathscr{X}$ of $\mathbb{V}^{(\mathbb{B})}$ and an injection $\iota : X \to X' := \mathscr{X}{\downarrow}$ such that $d(x, y) = [\![\iota(x) \neq \iota(y)]\!]$ for all $x, y \in X$ and every $x' \in X'$ admits the representation $x' = \operatorname{mix}_{\xi \in \Xi}(b_\xi \iota(x_\xi))$, with $(x_\xi)_{\xi \in \Xi} \subset X$ and $(b_\xi)_{\xi \in \Xi}$ a partition of unity in $\mathbb{B}$. The element $\mathscr{X}$ of $\mathbb{V}^{(\mathbb{B})}$ is said to be the *Boolean valued representation* of the $\mathbb{B}$-set X. If X is a discrete $\mathbb{B}$-set then $\mathscr{X} = X^{\wedge}$ and $\iota(x) = x^{\wedge}$ for all $x \in X$. If $X \subset \mathbb{V}^{(\mathbb{B})}$ then $\iota{\uparrow}$ is an injection from $X{\uparrow}$ to $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$. Say that X is $\mathbb{B}$-*complete* (or $\mathbb{B}$-*cyclic*), whenever $\iota(X) = X'$.

3. A mapping f from a $\mathbb{B}$-set (X, d) to a $\mathbb{B}$-set (X', d') is *contractive* provided that $d'(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$. Assume that X and Y are some $\mathbb{B}$-sets. Assume further that $\mathscr{X}$ and $\mathscr{Y}$ are the Boolean valued representations of X and Y, while $\iota : X \to \mathscr{X}{\downarrow}$ and $\jmath : Y \to \mathscr{Y}{\downarrow}$ are the corresponding injections. If $f : X \to Y$ is a contractive mapping then there is a unique member g of $\mathbb{V}^{(\mathbb{B})}$ such that $[\![g : \mathscr{X} \to \mathscr{Y}]\!] = \mathbb{1}$ and $f = \jmath^{-1} \circ g{\downarrow} \circ \iota$.

4. In case a $\mathbb{B}$-set X has some a priori structure we may try to furnish the Boolean valued representation of X with an analogous structure, so as to apply the technique of ascending and descending to the study of the original structure of X. Consequently, the above questions may be treated as instances of the unique problem of searching a well-qualified Boolean valued representation of a $\mathbb{B}$-set with some additional structure, *algebraic* $\mathbb{B}$-*systems*.

5. Thus an algebraic $\mathbb{B}$-system refers to a $\mathbb{B}$-set endowed with a few contractive operations and $\mathbb{B}$-predicates, the latter meaning $\mathbb{B}$-valued contractive mappings. The Boolean valued representation of an algebraic $\mathbb{B}$-system appears to be a conventional two valued algebraic system of the same type. This means that an appropriate $\mathbb{B}$-completion of each algebraic $\mathbb{B}$-system coincides with the descent of some two valued algebraic system.

6. The following table shows Boolean valued representations of some structures. Of course, all these representation results are applied to the study of their properties by means of Boolean valued analysis. For details, we refer to the sources indicated in the third column of the table (Table 1).

Table 1 Structures

Algebraic structure with order, norm, etc.	Boolean valued representation	Author [·], year
Complete Boolean algebra with a complete subalgebra	Complete Boolean algebra	Solovay and Tennenbaum [91]
Amalgated free product of Boolean algebras over $\mathbb{B}$	Free product of Boolean algebras	Can be extracted from [91]
Universally complete Kantorovich space	Field of reals	Gordon [13]
Boolean extension of a uniform space	Complete uniform space	Gordon and Lyubetskiĭ [22–24]
Rationally complete semiprime abelian ring	Arbitrary field	Gordon [19]
Complete ring of fractions of a semiprime abelian ring	The field of fractions of an integral domain	Gordon [19]
Unital separated injective module	Vector space	Gordon [20]
Continuous geometry[a]	Irreducible CG[b]	Nishimura [73]
Von Neumann algebra	Von Neumann factor	Ozawa [78], Takeuti [96]
Kaplansky–Hilbert module	Hilbert space	Takeuti [96], Ozawa [79, 80]
$\mathbb{B}$-complete C^*-algebra	C^*-algebra	Takeuti [97]
Type I AW^*-algebra	W^*-algebra $\mathrm{End}(H)$ for a Hilbert space H	Ozawa [80]
AW^*-module	Hilbert space	Ozawa [80]
Embeddable AW^*-algebra	Von Neumann algebra	Ozawa [81]
Banach–Kantorovich space	Banach space	Kusraev [41]
Operator caps and faces	Caps and faces of sets of functionals	Kutateladze [64, 65]
$\mathbb{B}$-simple groups and $\mathbb{B}$-simple rings	Simple groups and Simple rings	Takeuti [98]
$\mathbb{B}$-complete Banach space	Banach space	Kusraev [41, 42], Ozawa [84]
$\mathbb{B}$-compactification (or cyclic compactification)	Stone-Čech compactification	Abasov and Kusraev [1]
$\mathbb{B}$-Dedekind domain[b]	Dedekind domain[b]	Nishimura [75]
$\mathbb{B}$-complete Lie algebra over a Stone algebra	Lie algebra	Nishimura [76]
AL^*-algebra[c]	L^*-algebra[c]	Nishimura [77]

$\mathbb{B}$-complete JB-algebra	JB-algebra	Kusraev [43]
$\mathbb{B}$-complete $\mathbb{B}$-dual JB-algebra	Dual JB-algebra	Kusraev [43]
Injective Banach lattice	AL-space (L_1 space)	Kusraev [50, 54][e]
Kaplansky–Hilbert lattice[d]	Hilbert lattice	Kusraev [51]
Ordered preduals to injective Banach lattices	L^1-preduals	Kusraev, Kutateladze [59]

[a]A *continuous geometry* (= CG) is a complete complemented modular lattice L satisfying the axioms: $\sup_{\alpha\in\mathrm{A}}(x_\alpha \wedge z) = (\sup_{\alpha\in\mathrm{A}} x_\alpha) \wedge z$ and $\inf_{\alpha\in\mathrm{A}}(y_\alpha \vee z) = (\inf_{\alpha\in\mathrm{A}} y_\alpha) \vee z$ for all $z \in L$, increasing family $(x_\alpha)_{\alpha\in\mathrm{A}}$, and decreasing family $(y_\alpha)_{\alpha\in\mathrm{A}}$ in L. A continuous geometry with a trivial center is called *irreducible, Neuman [102]*

[b]A $\mathbb{B}$-*Dedekind domain* is a $\mathbb{B}$-integral domain that is $\mathbb{B}$-hereditary. A $\mathbb{B}$-*integral domain* is a $\mathbb{B}$-complete ring R in which every $\mathbb{B}$-ideal of R is $\mathbb{B}$-projective and for all $a, b \in R$ with $ab = 0$ there exist $e, f \in \mathbb{B}$ such that $ef = 0$, $e + f = 1$, $ea = 0$, and $fb = 0$; see [75, p. 69]. A Dedekind domain is an integral domain in which every ideal is projective or, equivalently, each nonzero ideal is a product of prime ideals [7, Chap. 7, § 2]

[c]An AL^*-algebra is an AW^*-module $\mathscr{L}$ over a commutative von Neumann algebra A endowed with an A-bilinear operation $[\cdot,\cdot] : \mathscr{L} \times \mathscr{L} \to \mathscr{L}$ and a unary $*$-operation $(\cdot)^* : \mathscr{L} \to \mathscr{L}$ such that for all $u, v, w \in \mathscr{L}$ we have: (1) $[u, u] = 0$; (2) $[[u, v], w] + [[v, w]u] + [[w, u]v] = 0$; (3) $\langle [u, v], w\rangle = \langle v, [u^*, w]\rangle$; see [77, p. 245]. An L^*-algebra is a complex Lie algebra $\mathscr{L}$ that is simultaneously a Hilbert space endowed with a $*$-operation satisfying $\langle [u, v], w\rangle = \langle v, [u^*, w]\rangle$ for all $u, v, w \in \mathscr{L}$; see [86]

[d]A *Kaplansky–Hilbert lattice* over Λ is a real Banach lattice X such that $X \oplus iX$ is a Kaplansky–Hilbert module over $\bar{\Lambda}$ and $\|x\| := \left\|\sqrt{\langle x, x\rangle}\right\|_\infty$ for all $x \in X$, see Example 3. A Kaplansky–Hilbert lattice over $\Lambda = \mathbb{R}$ is called a *Hilbert lattice,* see [71]. The norm $\|x + iy\| := \sqrt{\|\langle x, x\rangle + \langle y, y\rangle\|_\infty}$ is given incorrectly in [51]

[e]Some related results can be found in [51, 59, 60]

5 Boolean Valued Representation of Operators

1. Let X be a normed space and let E be a vector lattice. Say that a linear operator $T : X \to E$ has an *abstract norm* or is *dominated* if the image $T(B_X)$ of the unit ball B_X of X is order bounded in E. Assume now that X is a multinormed space and E has an order unit $\mathbb{1}$. An operator T is called *piecewise bounded* if there is a partition of unity (π_α) in $\mathbb{P}(E)$ and a family of continuous seminorms (p_α) such that $|\pi_\alpha Tx| \leq \mathbb{1}p_\alpha(x)$ for all α and $x \in X$

2. An operator $T : E \to F$ between two vector lattices is said to be *interval preserving* whenever T is a positive operator and $T[0, x] = [0, Tx]$ holds for each $x \in E_+$. A Maharam operator is an order continuous interval preserving operator. An operator $T : E \to E$ on vector lattice is said to be *band preserving* if $x \perp y$ implies $Tx \perp y$ for all $x, y \in E$ or, equivalently, whenever T keeps all bands of E invariant, i. e., $T(B) \subset B$ holds for each band B of E.

3. Consider a $\mathbb{B}$-complete Banach space Y. Denote by $\mathrm{Prt}_\sigma(\mathbb{B})$ the set of all countable partitions of unity in $\mathbb{B}$. Say that a *sequence* $(y_n)_{n\in\mathbb{N}}$ $\mathbb{B}$-*approximates* $y \in Y$ if, for each $k \in \mathbb{N}$, we have $\inf\{\sup_{n\geqslant k} \|\pi_n(y_n - y)\| : (\pi_n)_{n\geq k} \in \mathrm{Prt}_\sigma(\mathbb{B})\} = 0$. Call a set $K \subset Y$ $\mathbb{B}$-*compact* if K is $\mathbb{B}$-complete and every sequence $(y_n)_{n\in\mathbb{N}} \subset K$ $\mathbb{B}$-approximates some $y \in K$. An operator from a normed space X to Y is called $\mathbb{B}$-*compact* or *cyclically compact* if the image of every norm bounded subset of X lies in some $\mathbb{B}$-compact subset of Y.

4. Suppose E is a Banach lattice. A linear operator $T : E \to Y$ is *cone* $\mathbb{B}$*-summing* if and only if there exists a positive constant C such that for every finite collection $x_1, \dots x_n \in E$ there is a countable partition of unity $(\pi_k)_{k\in\mathbb{N}}$ in $\mathbb{B}$ such that the inequality

$$\sup_{k\in\mathbb{N}} \sum_{i=1}^{n} \|\pi_k T x_i\| \le C \left\| \sum_{i=1}^{n} |x_i| \right\|$$

holds, see [50]. Observe that if $\mathbb{B} = \{0, I_Y\}$ then a cone $\mathbb{B}$-summing operator is a *cone absolutely summing operator*; cp. [85, Ch. 4].

5. Let $\mathbb{P} = \mathbb{R}$ or $\mathbb{P} = \mathbb{C}$. Given an algebra A over the field $\mathbb{P}$, we call a $\mathbb{P}$-linear operator $D : A \to A$ a *derivation* provided that $D(uv) = D(u)v + uD(v)$ for all $u, v \in A$. It can be easily seen that an order bounded derivation of a universally complete f-algebra is trivial (Table 2).

Table 2 Operators

Operator, representation homomorphism, etc.	Boolean valued representation	Author [·], year
Unitary representation of an LCA group	Character of an LCA group	Takeuti [93]
Ordinary differential operator with parameters in $\langle\mathbb{B}\rangle$[a]	Ordinary differential operator	Takeuti [93]
$\langle\mathbb{B}\rangle$-valued Fourier transform on LCA groups	Fourier transform on LCA groups	Takeuti [94]
Linear operator with abstract norm	Norm bounded linear functional	Gordon [14, 17]
Conditional expectation	Lebesgue integral	Gordon [17]
$\mathbb{B}$-Compact operator	Compact operator	Kusraev [39]
Maharam operator	Order continuous positive functional	Kusraev [40]
Piecewise bounded linear operator	Continuous linear functional	Sikorskiĭ [89]
Differential polynomial on $\mathscr{D}'(\mathbb{R}^n, \mathbf{C})$ or $\mathscr{S}'(\mathbb{R}^n, \mathbf{C})$ with coefficients in $\mathbf{C}$[b]	Constant coefficients differential polynomial on $\mathscr{D}'(\mathbb{R}^n)$, $\mathscr{S}'(\mathbb{R}^n)$	Sikorskiĭ [88, 89]
Unitary representation of a locally compact group	Irreducible unitary representation	Nishimura [74]
Band preserving operator	$\mathbb{R}^\wedge$-linear function on Boolean valued reals	Kusraev [46]
Derivation on a universally complete f-algebra over $\mathbb{C}$	Derivation on the complex plane	Kusraev [47]
Cone $\mathbb{B}$-summing operator	Cone absolutely summing operator	Kusraev [49]
Weighted conditional expectation type operator	Weighted conditional expectation operator	Kusraev, Kutateladze [58]

[a] See Example 2 in § 3
[b] $\mathscr{D}'(\mathbb{R}^n, \mathbf{C})$ (resp. $\mathscr{S}'(\mathbb{R}^n, \mathbf{C})$) is the space of all piecewise bounded operators from $\mathscr{D}(\mathbb{R}^n)$ (resp. $\mathscr{S}(\mathbb{R}^n)$ to $\mathbf{C}$), where $\mathbf{C} := \mathscr{C}{\downarrow} = \mathbf{R} \oplus i\mathbf{R}$ is a complex universally complete vector lattice, see Corollary 2

6 Problems and Solutions

Boolean valued analysis sheds new light on some old problems and generates a large number of new ones. We now give a small list of problems that arose independently of the theory of Boolean valued models, but which were solved by means of Boolean valued analysis. Details as well as many other aspects of Boolean valued analysis may be found in the books [10, 21, 56–58, 92] and the survey papers [23, 34, 61] (Table 3).

Table 3 Problems and solutions

Problem	Stems from	Reduced to (by means of BA)	Solved
Intrinsic characterization of subdifferentials	Kutateladze [63]	Weakly compact convex sets of functionals	Kusraev and Kutateladze [55]
General disintegration in Kantorovich spaces	Ioffe, Levin [28]; Neumann [72]	Hahn–Banach and Radon–Nikodým theorems	Kusraev [40]
Kaplansky Problem: Homogeneity of a type I AW^*-algebra	Kaplansky [38]	Homogeneity of $\mathrm{End}(H)$ with H a Hilbert space	Ozawa [80]
The trace problem for finite AW^*-algebra	Kaplansky [37]	The trace problem for a W^*-factor	Ozawa [82, 83]
Wickstead problem: Order boundedness of all band preserving operators	Wickstead [105]	Solvability of Cauchy type functional equations	Gutman [26] and Kusraev [47]
Maharam extension of a positive operator	Luxemburg and Schep [69]	Daniel extension of an elementary integral	Akilov, Kolesnikov, and Kusraev [2, 3]
Goodearl problem 18 in [12]	Goodearl [12]	Theorem 12.16 in [12]	Chupin [9]
$\mathbb{B}$-Atomic decomposition of vector measures (into a sum of spectral measures)	Hoffman-Jørgenson [27]	Hammer–Sobczyc decomposition theorem	Kusraev and Malyugin [62]
Classification of AJW-algebras[a]	Topping [100]	Classification of predual JB-factors (JBW-factors)	Kusraev [52, 53]
Description of operators T with $\lvert T\rvert$ a sum of two lattice homomorphisms	Grothendieck [25]	Description of functionals with the same property	Kutateladze [66]
Classification of injective Banach lattices	Cartwright [8] and Lotz [68]	Classification of AL-spaces (L_1 spaces)	Kusraev [52, 53]

[a] An *AJW-algebra* is a JB-algebra with a Jordan counterpart of Baire condition (= annihilators are generated by projections), see [5]. For some related results, see [44, 48]

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Approximation Properties of Vallée Poussin Means for Special Series of Ultraspherical Jacobi Polynomials

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Abstract It is shown that the approximation rate of continuous functions by Vallée Poussin means $V_{n,m}^{\alpha}(f)$ of special series partial sums (α is a parameter in special series construction) is of the order of the best approximation provided $\frac{1}{2} < \alpha < \frac{3}{2}$, $m \asymp n$.

Keywords Jacobi polynomials · Special (sticking) series of ultraspherical polynomials · Approximation properties · Vallée Poussin means

Mathematics Subject Classification (2010) 33C45; 41A25; 41A10.

1 Main Result

Let $\mu^{\alpha}(x) = (1-x^2)^{\alpha}$ be a weight with $\alpha > 0$ and let $\{\hat{P}_n^{\alpha}(x), n = 0, 1, \ldots\}$ be a system of polynomials (with positive leading coefficients) orthonormal with respect to the inner product

$$\int_{-1}^{1} \mu^{\alpha}(x)\hat{P}_n^{\alpha}(x)\hat{P}_k^{\alpha}(x)dx = \delta_{nk}. \tag{1.1}$$

The polynomials $\hat{P}_n^{\alpha}(x)$ are called the orthonormal Jacobi polynomials. We will also use standardized Jacobi polynomials $P_n^{\alpha}(x)$ that differ from orthonormal ones by constant and are normed by the condition $P_n^{\alpha}(1) = \binom{n+\alpha}{n}$.

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The special series is defined for functions $f(x) \in C[-1,1]$ as follows:

$$f(x) \sim l(f)(x) + (1-x^2)\sum_{k=0}^{\infty} c_k^\alpha(F)\hat{P}_k^\alpha(x), \quad \alpha > 0, \tag{1.2}$$

where $l(f)(x)$ is a straight line connecting points of function $f(x)$ at the segment $[-1,1]$ ends, that is,

$$l(f)(x) = \frac{f(-1)+f(1)}{2} + \frac{f(1)-f(-1)}{2}x,$$

$F(x) = \frac{f(x)-l(f)(x)}{1-x^2}$ and $c_k^\alpha(F)$ is a Fourier–Jacobi coefficient of function F:

$$c_k^\alpha(F) = \int_{-1}^{1}\Big(f(t)-l(f)(t)\Big)(1-t^2)^{\alpha-1}\hat{P}_k^\alpha(t)dt. \tag{1.3}$$

The partial sums of the special series (1.2) we will denote as

$$\sigma_n^\alpha(f,x) = l(f)(x) + (1-x)^2\sum_{k=0}^{n-2} c_k^\alpha(F)\hat{P}_k^\alpha(x). \tag{1.4}$$

Special series was introduced in the article [1] as a generalization of limit ultraspherical series [2]. In [1] approximation properties of partial sums $\sigma_n^\alpha(f)$ were investigated. In particular, it was shown that

$$|f(x)-\sigma_n^\alpha(f,x)| \le c(\alpha)E_n(f)\big(1+\ln(1+n\sqrt{1-x^2})\big),\ \frac{1}{2} \le \alpha < \frac{3}{2},\ f \in C[-1,1],$$

where $E_n(f)$ is the best approximation of a function f by algebraic polynomials p_n of degree n:

$$E_n(f) = \inf_{p_n}\max_{x\in[-1,1]}|f(x)-P_n(x)|.$$

In [3] approximation properties of Vallée Poussin means

$$V_{n,m}^\alpha(f) = V_{n,m}^\alpha(f,x) = \frac{1}{m+1}[\sigma_n^\alpha(f,x)+\cdots+\sigma_{n+m}^\alpha(f,x)] \tag{1.5}$$

for special series were studied in the case when $\alpha = \frac{1}{2}$ and $n \le qm$, where q is an arbitrary positive fixed number, and the following estimate was obtained:

$$|f(x) - V_{n,m}^{\frac{1}{2}}(f,x)| \le c(q)E_n(f),\ \ f \in C[-1,1].$$

Here and elsewhere, $c(\alpha)$ and $c(\beta)$ are positive numbers (or positive constants) the dependence only on the given parameters, which, in general, may be different in different places.

In this paper the above mentioned result is extended to the case $\frac{1}{2} < \alpha < \frac{3}{2}$ provided $n \asymp m$ (i. e. $c_1 m \le n \le c_2 m$).

Theorem 1.1 *The following estimate of the remainder is valid when the functions $f \in C[-1, 1]$ are approximated by Vallée Poussin means $V_{n,m}^{\alpha}(f, x)$:*

$$|f(x) - V_{n,m}^{\alpha}(f, x)| \le c(\alpha) E_n(f), \quad n \asymp m, \quad \frac{1}{2} < \alpha < \frac{3}{2}.$$

2 Proof of the Main Result

The following expression can be obtained for partial sums (1.4) using formula (1.3):

$$\sigma_n^{\alpha}(f, x) = l(f)(x) + (1-x^2) \int_{-1}^{1} \big[f(t) - l(f)(t)\big](1-t^2)^{\alpha-1} \mathcal{K}_{n-2}^{\alpha}(x, t) dt, \qquad (2.1)$$

where

$$\mathcal{K}_n^{\alpha}(x, t) = \sum_{k=0}^{n} \hat{P}_k^{\alpha}(x) \hat{P}_k^{\alpha}(t) dt.$$

From the Vallée Poussin means definition (1.5) and (2.1) we derive equality:

$$V_{n,m}^{\alpha}(f, x) = l(f)(x) + (1 - x^2) \int_{-1}^{1} \Big(f(t) - l(f)(t)\Big)(1 - t^2)^{\alpha-1} K_{n-2,m}^{\alpha}(x, t) dt,$$

where

$$K_{n,m}^{\alpha}(x, t) = \frac{1}{m+1} \sum_{k=n}^{n+m} K_k^{\alpha}(x, t).$$

Let H_n be a space of algebraic polynomials with degree at most n. Note that since the operators $\sigma_n^{\alpha}(f, x)$ leave the polynomials $p_n \in H_n$ invariant [1, p. 1042], the Vallée Poussin means (1.5) have a similar property:

$$V_{n,m}^{\alpha}(p_n, x) = p_n(x), \quad p_n \in H_n.$$

Let $H_n^{\pm}(f)$ be a subset of H_n containing only those polynomials that coincide with the function $f(x)$ at the endpoints of the segment $[-1, 1]$. Let $E_n^{\pm}(f)$ be the best approximation of a function $f(x) \in C[-1, 1]$ by polynomials p_n from $H_n^{\pm}(f)$ [1, p. 1042]. If $p_n^*(x)$ is the best approximation polynomial for a function f by the set $H_n^{\pm}(f)$, then

$$|f(x) - V_{n,m}^{\alpha}(f, x)| \le E_n^{\pm}(f) + |V_{n,m}^{\alpha}(f - p_n^*)| \le E_n^{\pm}(f)\Big(1 + \Lambda_{n,m}^{\alpha}(x)\Big),$$

where

$$\Lambda_{n,m}^{\alpha}(x) = (1 - x^2)\int_{-1}^{1}(1 - t^2)^{\alpha-1}\Big|K_{n-2,m}^{\alpha}(x, t)\Big|dt \tag{2.2}$$

is a Lebesgue function of Vallée Poussin means for partial sums of special series of ultraspherical Jacobi polynomials.

The following will be proved in Sect. 4..

Theorem 2.1 *If $n \asymp m$, then*

$$\Lambda_{n,m}^{\alpha}(x) \le c(\alpha), \quad \frac{1}{2} < \alpha < \frac{3}{2}, \quad -1 \le x \le 1. \tag{2.3}$$

Since $E_n^{\pm}(f) \le 2E_n(f)$ [1, p. 1055], Theorem 1.1 follows from Theorem 2.1. In the next section we give some properties and statements that will be used in the proof of Theorem 2.1.

3 Auxiliary Information

For convenience of reference we will give here some properties of Jacobi polynomials that can be found, for example, in [4].

1. Relation between orthonormal and standardized polynomials:

$$\hat{P}_n^{\alpha,\beta}(x) = \sqrt{\frac{n!(\alpha + \beta + 2n + 1)\Gamma(\alpha + \beta + n + 1)}{2^{\alpha+\beta+1}\Gamma(\alpha + n + 1)\Gamma(\beta + n + 1)}}P_n^{\alpha,\beta}(x). \tag{3.1}$$

In particular, it follows from (3.1) that

$$\hat{P}_n^{\alpha,\beta}(x) \asymp \sqrt{n}P_n^{\alpha,\beta}(x). \tag{3.2}$$

2. Symmetry:

$$\hat{P}_n^{\alpha}(-x) = (-1)^n\hat{P}_n^{\alpha}(x). \tag{3.3}$$

3. Weighted estimate ($-1 < x < 1, \alpha, \beta \geq -1/2$):

$$|\hat{P}_n^{\alpha,\beta}(x)| \leq c(\alpha) \left(\sqrt{1-x}\right)^{-\alpha-\frac{1}{2}} \left(\sqrt{1+x}\right)^{-\beta-\frac{1}{2}}. \tag{3.4}$$

4. Uniform estimate ($-1 \leq x \leq 1, \max\{\alpha, \beta\} \geq -1/2$):

$$|\hat{P}_n^{\alpha,\beta}(x)| \leq n^{\max\{\alpha,\beta\}+\frac{1}{2}}. \tag{3.5}$$

5. The Cristoffel–Darboux formula:

$$K_k^{\alpha,\beta}(x,t) = \sum_{j=0}^{k} \hat{P}_j^{\alpha,\beta}(x)\hat{P}_j^{\alpha,\beta}(t) =$$
$$\sqrt{\lambda_{n+1}} \frac{\hat{P}_{k+1}^{\alpha,\beta}(x)\hat{P}_k^{\alpha,\beta}(t) - \hat{P}_k^{\alpha,\beta}(x)\hat{P}_{k+1}^{\alpha,\beta}(t)}{x-t}, \quad \sqrt{\lambda_{n+1}} \leq 1.$$

Using the recurrence formula for Jacobi polynomials we can derive from the latter the following equality [1, p. 1043]:

$$K_k^{\alpha}(x,t) = \frac{2^{-2\alpha-1}\Gamma(n+2)\Gamma(n+2\alpha+2)}{(n+1)(\Gamma(n+\alpha+1))^2} \times$$
$$\frac{(1-t)P_k^{\alpha+1,\alpha}(t)P_k^{\alpha,\alpha}(x) - (1-x)P_k^{\alpha+1,\alpha}(x)P_k^{\alpha,\alpha}(t)}{x-t}.$$

Applying (3.2) and (3.4) to the last equation and taking into account Gamma-function properties yields the following estimate for $-1 < x, t < 1$ [1, p. 1043]:

$$|K_k^{\alpha}(x,t)| \leq \frac{c(\alpha)}{|x-t|}(1-t^2)^{-\frac{\alpha}{2}-\frac{1}{4}}(1-x^2)^{-\frac{\alpha}{2}-\frac{1}{4}}\Big((1-t^2)^{\frac{1}{2}} + (1-x^2)^{\frac{1}{2}}\Big).$$

Since the right-hand side is independent on k, the same estimate is valid for $K_{n,m}^{\alpha}(x,y)$:

$$|K_{n,m}^{\alpha}(x,t)| \leq \frac{c(\alpha)}{|x-t|}(1-t^2)^{-\frac{\alpha}{2}-\frac{1}{4}}(1-x^2)^{-\frac{\alpha}{2}-\frac{1}{4}}\Big((1-t^2)^{\frac{1}{2}} + (1-x^2)^{\frac{1}{2}}\Big). \tag{3.6}$$

We also need the following statements, the proof of which can be found in [5].

Lemma 3.1 *The representation* $K^{\alpha}_{m,n}(x,t) = \sum_{\nu=0}^{6} S^{\nu}(x,t)$ *holds with*

$$S^1(x,t) = \frac{1}{(x-t)(m+1)} \sum_{k=n}^{n+m} h_k^1 P_{k+1}^{\alpha,\alpha}(x) P_k^{\alpha,\alpha}(t),$$

$$S^2(x,t) = \frac{1}{(x-t)(m+1)} \sum_{k=n}^{n+m} h_k^2 P_k^{\alpha,\alpha}(x) P_{k+1}^{\alpha,\alpha}(t),$$

$$S^3(x,t) = \frac{1}{(x-t)(m+1)} \sum_{k=n}^{n+m} h_k^3 P_k^{\alpha,\alpha}(x) P_k^{\alpha,\alpha}(t),$$

$$S^4(x,t) = \frac{1}{(x-t)(m+1)} \sum_{k=n}^{n+m} h_k^4 P_{k+1}^{\alpha,\alpha}(x) P_{k+1}^{\alpha,\alpha}(t),$$

$$S^5(x,t) = \frac{(1-x)(1+t)}{(x-t)(m+1)} \sum_{k=n}^{n+m} \delta_k P_k^{\alpha+1,\alpha}(x) P_k^{\alpha,\alpha+1}(t),$$

$$S^6(x,t) = -\frac{(n+m+\alpha+2) P_{n+m+1}^{\alpha,\alpha}(x) P_{n+m+1}^{\alpha,\alpha}(t) - (n+\alpha+1) P_n^{\alpha,\alpha}(x) P_n^{\alpha,\alpha}(t)}{2^{2\alpha+1}(m+1)(x-t)},$$

where $h_k^i = O(1)$, $\delta_k = O(1)$, $k \to \infty$.

Lemma 3.2 *Equality* $S^0(x,t) = \sum\limits_{\nu=1}^{5} S_{\nu}^0(x,t)$ *holds, where*

$$S_1^0(x,t) = \frac{(1-x)^2(1+t)}{2^{2\alpha+1}(m+1)(x-t)^2} \times$$
$$[(n+m+2+2\alpha) P_{n+m}^{\alpha+2,\alpha}(x) P_{n+m}^{\alpha,\alpha+1}(t) - (n+1+2\alpha) P_{n-1}^{\alpha+2,\alpha}(x) P_{n-1}^{\alpha,\alpha+1}(t)],$$

$$S_2^0(x,t) = -\frac{(1-x)(1-t)(1+t)}{2^{2\alpha+1}(m+1)(x-t)^2} \times$$
$$[(n+m+2+2\alpha) P_{n+m}^{\alpha+1,\alpha+1}(t) P_{n+m}^{\alpha+1,\alpha}(x) - (n+1+2\alpha) P_{n-1}^{\alpha+1,\alpha+1}(t) P_{n-1}^{\alpha+1,\alpha}(x)],$$

$$S_3^0(x,t) = \frac{(1-x)(1+t)}{2^{2\alpha+1}(m+1)(x-t)^2} \times$$
$$\left[\frac{n+m+2+2\alpha}{2(n+m+1)+2\alpha+1} P_{n+m}^{\alpha+1,\alpha}(x) P_{n+m}^{\alpha,\alpha+1}(t) - \frac{n+1+2\alpha}{2n+2\alpha+1} P_{n-1}^{\alpha+1,\alpha}(x) P_{n-1}^{\alpha,\alpha+1}(t)\right],$$

$$S_4^0(x,t) = \frac{(1-x)(1+t)}{2^{2\alpha}(m+1)(x-t)^2} \sum_{k=n}^{n+m} \frac{(2\alpha+1)(k+\alpha+1)}{(2k+2\alpha+3)(2k+2\alpha+1)} P_k^{\alpha+1,\alpha}(x) P_k^{\alpha,\alpha+1}(t),$$

$$S_5^0(x,t) = \frac{(1-x)(1+t)}{2^{2\alpha+1}(m+1)(x-t)^2}\times$$
$$\sum_{k=n}^{n+m} \frac{\alpha(\alpha+1)}{2k+2\alpha+1}[P_{k-1}^{\alpha+1,\alpha}(x)P_k^{\alpha,\alpha+1}(t) - P_k^{\alpha+1,\alpha}(x)P_{k-1}^{\alpha,\alpha+1}(t)].$$

4 Proof of Theorem 2.1

It is easy to see, that $\Lambda_{n,m}(-x) = \Lambda_{n,m}(x)$ due to the symmetry property (3.3). Therefore it is sufficient to prove the result in case of $0 \le x \le 1$.

Denote

$$J_a^b(x) = (1-x^2)\int_a^b (1-t^2)^{\alpha-1}\left|K_{n,m}^{\alpha}(x,t)\right|dt \tag{4.1}$$

and represent the Lebesgue function (2.2) in the following form:

$$\Lambda_{n+2,m}^{\alpha}(x) = J_{-1}^{-1/2}(x) + J_{-1/2}^{1}(x). \tag{4.2}$$

We show first that

$$J_{-1}^{-1/2}(x) \le c(\alpha)(1-x)^{\frac{3}{4}-\frac{\alpha}{2}} \le c(\alpha). \tag{4.3}$$

Indeed, applying the Christoffel–Darboux formula, we get

$$J_{-1}^{-1/2}(x) \le (1-x^2)\times$$
$$\int_{-1}^{-1/2} \frac{(1-t^2)^{\alpha-1}}{(m+1)|x-t|}\sum_{k=n}^{n+m}\Big(|\hat{P}_{k+1}^{\alpha}(x)||\hat{P}_k^{\alpha}(t)| + |\hat{P}_k^{\alpha}(x)||\hat{P}_{k+1}^{\alpha}(t)|\Big)dt.$$

From here using weighted estimate (3.4) and taking into account that $|x-t| \ge 1/2$ we deduce the inequality

$$J_{-1}^{-1/2}(x) \le c(\alpha)(1-x^2)^{\frac{3}{4}-\frac{\alpha}{2}}\int_{-1}^{-1/2}(1-t^2)^{\frac{\alpha}{2}-\frac{5}{4}}dt,$$

which yields the required estimate (4.3) provided $\frac{1}{2} < \alpha < \frac{3}{2}$.

Further, we show that under our assumptions the following estimate holds:

$$J_{-1/2}^{1}(x) \le c(\alpha). \tag{4.4}$$

To this end consider two cases:

$$0 \le x \le 1 - \frac{c^2}{(m+n)^2}, \tag{4.5}$$

$$1 - \frac{c^2}{(m+n)^2} \le x \le 1, \tag{4.6}$$

where $c > 1$.

4.1 *The Case* $0 \le x \le 1 - \frac{c^2}{(m+n)^2}$

Break the expression $J^1_{-1/2}(x)$ as follows:

$$J^1_{-1/2}(x) = J_{-1/2}^{x-\frac{m+1}{m+n}\sqrt{1-x}}(x) + J_{x-\frac{m+1}{m+n}\sqrt{1-x}}^{x-\frac{1}{m+n}\sqrt{1-x}}(x) + J_{x-\frac{1}{m+n}\sqrt{1-x}}^{x+\frac{1}{m+n}\sqrt{1-x}}(x) + J_{x+\frac{1}{m+n}\sqrt{1-x}}^{x+\frac{m+1}{m+n}\sqrt{1-x}}(x) + J^1_{x+\frac{m+1}{m+n}\sqrt{1-x}}(x). \tag{4.7}$$

It should be noted, that for some values of the parameters in the given partition one or another term may be absent. In particular, we will assume that $J_a^b(x) = 0$ if $a \ge b$ and $J_a^b(x) = J_a^1(x)$ if $b \ge 1$.

The boundedness of the right-hand side terms of the equality (4.7) is proved in the following Sects. 4.1.1–4.1.4.

4.1.1 Estimation of $J_{-1/2}^{x-\frac{m+1}{m+n}\sqrt{1-x}}(x)$

Applying (3.6) and taking into account that $0 \le x < 1$, $-1/2 \le t \le x$, we get

$$J_{-1/2}^{x-\frac{m+1}{m+n}\sqrt{1-x}}(x) \le c(\alpha) \int_{-1/2}^{x-\frac{m+1}{m+n}\sqrt{1-x}} \left[\left(\frac{1-x}{1-t}\right)^{\frac{3}{4}-\frac{\alpha}{2}} + \left(\frac{1-x}{1-t}\right)^{\frac{5}{4}-\frac{\alpha}{2}}\right]\frac{dt}{x-t}.$$

Since $\frac{1-x}{1-t} \le 1$ when $t \le x$, it follows from the obtained inequality that

$$J_{-1/2}^{x-\frac{m+1}{m+n}\sqrt{1-x}}(x) \le c(\alpha) \int_{-1/2}^{x-\frac{m+1}{m+n}\sqrt{1-x}} \left(\frac{1-x}{1-t}\right)^{\frac{3}{4}-\frac{\alpha}{2}} \frac{dt}{x-t} = c(\alpha) I. \tag{4.8}$$

In order to estimate integral I we make variable substitution $\tau = 1 - \frac{1-x}{1-t}$:

$$I = I_{\frac{1}{1+\frac{m+n}{m+1}\sqrt{1-x}}}^{\frac{2x+1}{3}} = \int_{\frac{1}{1+\frac{m+n}{m+1}\sqrt{1-x}}}^{\frac{2x+1}{3}} \frac{d\tau}{\tau(1-\tau)^{\frac{\alpha}{2}+\frac{1}{4}}}.$$

The condition $0 \le x < 1$ implies that the lower bound of integration $a = \frac{1}{1+\frac{m+n}{m+1}\sqrt{1-x}}$ is inside the segment $[0, 1]$ and upper bound is inside $[\frac{1}{3}, 1]$. Hence we can write the following estimate:

$$I \le \begin{cases} I_a^{1/3} + I_{1/3}^{(2x+1)/3}, & a < 1/3, \\ I_{1/3}^{(2x+1)/3}, & a \ge 1/3. \end{cases} \tag{4.9}$$

So, since

$$I_a^{1/3} \le c(\alpha) \int_a^{1/3} \frac{d\tau}{\tau} \le c(\alpha) \ln(1 + \frac{m+n}{m+1}\sqrt{1-x}),$$
$$I_{1/3}^{(2x+1)/3} \le \int_{1/3}^{(2x+1)/3} \frac{d\tau}{(1-\tau)^{\frac{\alpha}{2}+\frac{1}{4}}} \le c(\alpha), \quad \alpha < \frac{3}{2},$$

the inequality $J_{-1/2}^{x-\frac{m+1}{m+n}\sqrt{1-x}}(x) \le c(\alpha)$ follows from (4.8) and (4.9) provided $m \asymp n$.

4.1.2 Estimation of $J^1_{x+\frac{m+1}{m+n}\sqrt{1-x}}(x)$

Reasoning similar to that of the previous section, with taking into account the relations $t \ge x$, $\frac{1-x}{1-t} \ge 1$, lead to the inequality

$$J^1_{x+\frac{m+1}{m+n}\sqrt{1-x}}(x) \le c(\alpha) \int_{x+\frac{m+1}{m+n}\sqrt{1-x}}^{1} \left(\frac{1-x}{1-t}\right)^{\frac{5}{4}-\frac{\alpha}{2}} \frac{dt}{t-x} = c(\alpha) I. \tag{4.10}$$

If we make here integration variable substitution $\tau = 1 - \frac{1-t}{1-x}$ then we get

$$I = \int_{\frac{1}{\frac{m+n}{m+1}\sqrt{1-x}}}^{1} \frac{d\tau}{\tau(1-\tau)^{\frac{5}{4}-\frac{\alpha}{2}}}.$$

Note that the lower bound of integration $a = \frac{1}{\frac{m+n}{m+1}\sqrt{1-x}} < 1$. It follows from the inequality $x + \frac{m+1}{m+n}\sqrt{1-x} < 1$ which we assume to be true, since otherwise the indicated term in the partition (4.7) would be absent, as we have already noted.

Considering as in previous subsection two cases $a < a_0$ and $a \geq a_0$, where $a_0 \in (0, 1)$ is a fixed number, it is easy to show that

$$I \leq c(\alpha)\Big(1 + \ln(1 + \frac{m+n}{m+1}\sqrt{1-x})\Big), \quad \alpha > \frac{1}{2}.$$

Hence, $J^1_{x+\frac{m+1}{m+n}\sqrt{1-x}}(x) \leq c(\alpha)$ provided that $m \asymp n$.

4.1.3 Estimation of $J_{x-\frac{1}{m+n}\sqrt{1-x}}^{x+\frac{1}{m+n}\sqrt{1-x}}(x)$

Using weighted estimate (3.4), from (4.1) we get the relation:

$$J_{x-\frac{1}{m+n}\sqrt{1-x}}^{x+\frac{1}{m+n}\sqrt{1-x}}(x) \leq c(\alpha)(1-x)^{-\frac{\alpha}{2}+\frac{3}{4}}(n+m+1)\int_{x-\frac{1}{m+n}\sqrt{1-x}}^{x+\frac{1}{m+n}\sqrt{1-x}}(1-t)^{\frac{\alpha}{2}-\frac{5}{4}}dt. \tag{4.11}$$

Since under the hypotheses we have $\frac{\alpha}{2} - \frac{5}{4} < 0$, the integrand can be estimated from above by $(1 - x - \frac{\sqrt{1-x}}{m+n})^{\frac{\alpha}{2}-\frac{5}{4}}$. Then from (4.11) we get:

$$J_{x-\frac{1}{m+n}\sqrt{1-x}}^{x+\frac{1}{m+n}\sqrt{1-x}}(x) \leq \frac{c(\alpha)}{\left(1-\frac{1}{(m+n)\sqrt{1-x}}\right)^{\frac{5}{4}-\frac{\alpha}{2}}}. \tag{4.12}$$

But x satisfies condition (4.5), which implies that $(m+n)\sqrt{1-x} \geq c > 1$. Hence, inequality (4.12) yields $J_{x-\frac{1}{m+n}\sqrt{1-x}}^{x+\frac{1}{m+n}\sqrt{1-x}}(x) < c(\alpha)$.

4.1.4 Estimation of $J_{x-\frac{m+1}{m+n}\sqrt{1-x}}^{x-\frac{1}{m+n}\sqrt{1-x}}(x)$ and $J_{x+\frac{1}{m+n}\sqrt{1-x}}^{x+\frac{m+1}{m+n}\sqrt{1-x}}(x)$

Let's begin with an estimation of $J_{x-\frac{m+1}{m+n}\sqrt{1-x}}^{x-\frac{1}{m+n}\sqrt{1-x}}(x)$. We apply Lemma 3.1 and represent the estimated value in the following form:

$$J_{x-\frac{m+1}{m+n}\sqrt{1-x}}^{x-\frac{1}{m+n}\sqrt{1-x}}(x) = (1-x^2)\int_{x-\frac{m+1}{m+n}\sqrt{1-x}}^{x-\frac{1}{m+n}\sqrt{1-x}}(1-t^2)^{\alpha-1}\Big|\sum_{\nu=0}^{6}S^\nu(x,t)\Big|dt \leq \sum_{\nu=0}^{6}a_\nu,$$

where $a_\nu = (1-x^2)\int_{x-\frac{m+1}{m+n}\sqrt{1-x}}^{x-\frac{1}{m+n}\sqrt{1-x}}(1-t^2)^{\alpha-1}|S^\nu(x,t)|dt$.

Using weighted estimate (3.4) with relation (3.2) and inequality $\sum_{k=n}^{n+m} \frac{1}{k} \le \ln(1+\frac{m}{n})$ we obtain for $1 \le \nu \le 4$:

$$a_\nu \le c(\alpha)\frac{1}{\sqrt{1-x}}\frac{\ln(1+\frac{m}{n})}{m+1}\int_{x-\frac{m+1}{m+n}\sqrt{1-x}}^{x-\frac{1}{m+n}\sqrt{1-x}}\left(\frac{1-x}{1-t}\right)^{-\frac{\alpha}{2}+\frac{5}{4}}\frac{dt}{x-t} = c(\alpha)\frac{1}{\sqrt{1-x}}\frac{\ln(1+\frac{m}{n})}{m+1}I. \tag{4.13}$$

After variable substitution $\tau = 1 - \frac{1-x}{1-t}$ the integral I will take a form:

$$I = \int_{\frac{1}{1+(m+n)\sqrt{1-x}}}^{\frac{1}{1+\frac{m+n}{m+1}\sqrt{1-x}}} \frac{d\tau}{\tau(1-\tau)^{\frac{\alpha}{2}-\frac{1}{4}}}.$$

If we introduce the notation

$$I_u^v = \begin{cases} \int_u^v \frac{d\tau}{\tau(1-\tau)^{\frac{\alpha}{2}-\frac{1}{4}}}, & u < v, \\ 0, & u \ge v, \end{cases}$$

we can write:

$$I \le I_{\frac{1}{1+(m+n)\sqrt{1-x}}}^{1/2} + I_{1/2}^{1}. \tag{4.14}$$

The first term can be estimated as follows:

$$I_{\frac{1}{1+(m+n)\sqrt{1-x}}}^{1/2} \le \int_{\frac{1}{1+(m+n)\sqrt{1-x}}}^{1/2} \frac{d\tau}{\tau} \le \ln(1+(m+n)\sqrt{1-x}). \tag{4.15}$$

It can be easily seen that $I_{1/2}^{1} < c(\alpha)$ provided $a < 3/2$, hence from (4.14) and (4.15) we get:

$$I \le c(\alpha)\ln(1+(m+n)\sqrt{1-x}), \quad a < 3/2. \tag{4.16}$$

Using this inequality, the relation $\ln(1+x) < x$ and assuming $m \asymp n$, for $1 \le \nu \le 4$ we deduce from (4.13):

$$a_\nu \le c(\alpha)\frac{1}{\sqrt{1-x}}\frac{\ln(1+\frac{m}{n})}{m+1}\ln(1+(m+n)\sqrt{1-x}) \le c(\alpha)(1+\frac{m}{n}) \le c(\alpha).$$

Similarly we can obtain the following estimate for a_5:

$$a_5 \le c(\alpha)\frac{\ln(1+\frac{m}{n})}{m+1} I \le c(\alpha).$$

To estimate a_6 we use again weighted estimate (3.4) with relation (3.2) and apply inequality (4.16):

$$a_6 \le \frac{c(\alpha)}{(m+1)\sqrt{1-x}}\Big(\frac{n+m+\alpha+2}{n+m+1} + \frac{n+\alpha+1}{n}\Big) I \le \frac{c(\alpha)}{(m+1)\sqrt{1-x}} I \le c(\alpha).$$

We now proceed to the estimation of a_0. To do this we use Lemma 3.2 and represent a_0 as follows:

$$a_0 = (1-x^2)\int_{x-\frac{m+1}{m+n}\sqrt{1-x}}^{x-\frac{1}{m+n}\sqrt{1-x}} (1-t^2)^{\alpha-1}\Big|\sum_{\mu=1}^{5} S_\mu^0(x,t)\Big| dt \le \sum_{\mu=1}^{5} b_\mu, \tag{4.17}$$

where $b_\mu = (1-x^2)\int_{x-\frac{m+1}{m+n}\sqrt{1-x}}^{x-\frac{1}{m+n}\sqrt{1-x}} (1-t^2)^{\alpha-1}|S_\mu^0(x,t)|dt$.

Applying weighted estimate (3.2) to S_μ^0, it can be shown that

$$b_1 \le c(\alpha)\frac{\sqrt{1-x}}{m+1}\int_{x-\frac{m+1}{m+n}\sqrt{1-x}}^{x-\frac{1}{m+n}\sqrt{1-x}} \Big(\frac{1-x}{1-t}\Big)^{5/4-\alpha/2}\frac{dt}{(x-t)^2}, \tag{4.18}$$

$$b_2 \le c(\alpha)\frac{\sqrt{1-x}}{m+1}\int_{x-\frac{m+1}{m+n}\sqrt{1-x}}^{x-\frac{1}{m+n}\sqrt{1-x}} \Big(\frac{1-x}{1-t}\Big)^{3/4-\alpha/2}\frac{dt}{(x-t)^2}, \tag{4.19}$$

$$b_\mu \le \frac{c(\alpha)}{(m+1)n}\int_{x-\frac{m+1}{m+n}\sqrt{1-x}}^{x-\frac{1}{m+n}\sqrt{1-x}} \Big(\frac{1-x}{1-t}\Big)^{5/4-\alpha/2}\frac{dt}{(x-t)^2}, \quad 3 \le \mu \le 5. \tag{4.20}$$

Note that

$$\int_{x-\frac{m+1}{m+n}\sqrt{1-x}}^{x-\frac{1}{m+n}\sqrt{1-x}} \frac{dt}{(x-t)^2} = \frac{1}{x-t}\Big|_{x-\frac{m+1}{m+n}\sqrt{1-x}}^{x-\frac{1}{m+n}\sqrt{1-x}} \le \frac{m+n}{\sqrt{1-x}}. \tag{4.21}$$

Since $\frac{1-x}{1-t} \le 1$ and $\sqrt{1-x} \ge \frac{c}{(m+n)}$ (see (4.5)), inequalities (4.18), (4.19), (4.20), and (4.21) yield the estimations:

$$b_\mu \le c(\alpha)\frac{m+n}{m+1}, \quad 1 \le \mu \le 2, \qquad b_\mu \le c(\alpha)\frac{(m+n)^2}{(m+1)n}, \quad 3 \le \mu \le 5,$$

from which it follows that $b_\mu \le c(\alpha)$, $1 \le \mu \le 5$ provided $m \asymp n$. Hence, we have $a_0 \le c(\alpha)$ due to (4.17).

Thus, we have shown that $a_\nu \le c(\alpha)$, $0 \le \nu \le 6$, and, consequently, $J_{x-\frac{m+1}{m+n}\sqrt{1-x}}^{x-\frac{1}{m+n}\sqrt{1-x}}(x) \le c(\alpha)$.

The estimate $J_{x+\frac{1}{m+n}\sqrt{1-x}}^{x+\frac{m+1}{m+n}\sqrt{1-x}}(x) \le c(\alpha)$ is proved quite similarly.

4.2 *The Case* $1 - \frac{c^2}{(m+n)^2} \le x \le 1$

We represent $J_{-1/2}^{1}(x)$ in the following form:

$$J_{-1/2}^{1}(x) = J_{-1/2}^{1-\frac{2c^2}{(m+n)^2}}(x) + J_{1-\frac{2c^2}{(m+n)^2}}^{1}(x). \tag{4.22}$$

Assuming that $\frac{1-x}{1-t} \le 1$, we get from (4.1) and (3.6).

$$J_{-1/2}^{1-\frac{2c^2}{(m+n)^2}}(x) \le c(\alpha) \int_{-1/2}^{1-\frac{2c^2}{(m+n)^2}} \left[\left(\frac{1-x}{1-t}\right)^{-\frac{\alpha}{2}+\frac{3}{4}} + \left(\frac{1-x}{1-t}\right)^{-\frac{\alpha}{2}+\frac{5}{4}} \right] \frac{dt}{x-t} \le$$

$$c(\alpha) \int_{-1/2}^{1-\frac{2c^2}{(m+n)^2}} \left(\frac{1-x}{1-t}\right)^{-\frac{\alpha}{2}+\frac{3}{4}} \frac{dt}{x-t} = c(\alpha) I.$$

Variable substitution $\tau = 1 - \frac{1-x}{1-t}$ in integral I gives us:

$$I = \int_{1-\frac{(1-x)(m+n)^2}{2c^2}}^{\frac{2x+1}{3}} \frac{d\tau}{\tau(1-\tau)^{\frac{\alpha}{2}+\frac{1}{4}}}.$$

Condition (4.6) implies that the lower bound of the obtained integral is not less than $\frac{1}{2}$. Therefore, assuming that $\alpha < \frac{3}{2}$ we get:

$$I \le 2 \int_{\frac{1}{2}}^{\frac{2x+1}{3}} \frac{d\tau}{(1-\tau)^{\frac{\alpha}{2}+\frac{1}{4}}} \le c(\alpha).$$

Thus, the first term in the right-hand side of (4.22) is bounded. Let's proceed to the second term.

We apply uniform estimate (3.5):

$$J^1_{1-\frac{2c^2}{(m+n)^2}}(x) \le c(\alpha)(1-x)\int_{1-\frac{2c^2}{(m+n)^2}}^1 (1-t)^{\alpha-1}dt\frac{1}{m+1}\sum_{k=n}^{n+m}\sum_{l=0}^{k}(l+1)^{2\alpha+1} \le$$
$$c(\alpha)(1-x)\frac{1}{(n+m)^{2\alpha}}(n+m)^{2\alpha+2} = c(\alpha)(1-x)(n+m)^2. \quad (4.23)$$

It follows from condition (4.6) that $(1-x)(n+m)^2 \le c$. Consequently, inequality (4.23) yields the boundedness of the second term in (4.22).

The proof of Theorem 2.1 is complete.

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Laplace's Integrals and Stability of the Open Flows of Inviscid Incompressible Fluid

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Abstract In this article, we study the spectra of the boundary value problems which arise from linearizing the Euler equations of incompressible hydrodynamics near a stationary solution describing a steady flow through a given domain, in the case when the fluid enters the domain and leave it through some parts of the boundary. It's natural to call such flows as open. The spectra of the open flows are not widely studied compared to the classical case of the fully impermeable boundaries. Moreover, the methods widely used for the latter fail to cover the former. We propose a new reduction of finding the eigenmodes of an open flow to finding 'zeroes' of an entire operator-valued function which is a kind of Laplace's integral. Here, by zeroes, we mean the values of the complex variable which deliver degenerations to the mentioned integral. Correspondingly, studying the flow stability reduces itself to Routh–Hurwitz problem for this integral. For several particular flows, this problem is solved explicitly with the use of the Polia theorem on the zeroes of the Laplace integral. As a result, we prove the stability of spectra for several concrete flows for which were unknown with such proofs before.

Keywords Euler equations · Inviscid incompressible fluid · Stability · Spectra · Entire function · Laplace's integral · Routh-Hurwits problem

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1 Introduction

Let $Q = Q(x)$ be a smooth real vector field defined in some domain of an Euclidian space, which is finite-dimensional, to begin with. Assume there exists an equilibrium $y : \; Q(y) = 0$. The linear equation governing the small perturbations of such an equilibrium reads as $\dot{z} = Q'(y)z$, where $Q'(y)$ stands for the differential of vector field Q at the equilibrium. Each eigenvalue λ of operator $Q'(y)$ gives rise to an eigenmode of the small perturbations that has the form $z(t) = e^{\lambda t}b$, where b denotes the eigenvector of operator $Q'(y)$ which corresponds to eigenvalue λ. The classical linearization principle of Liapunov's stability theory asserts that the equilibrium is asymptotically stable provided that all of its eigenvalues belong to semi-plane $\mathrm{Re}\lambda < 0$, and it is unstable if at least one of the eigenvalues belongs to semi-plane $\mathrm{Re}\lambda > 0$.

The applications lead us to the study of the smooth families of vector fields depending on some physical parameters. Then the equilibria also form families, which, however, can be non-smooth somewhere due to the bifurcation points. Assume, nevertheless, that the family of equilibria is being observed is smooth in the vicinity of certain point in the space of parameters. Consider a smooth path passing through this point in the space of the parameters, and let the parameters change themselves along it together with the equilibrium and with its spectrum. Such an altering can lead to transversal intersecting of the imaginary axis by some branches of the eigenvalues. It is what is called the occurrence of instability. For the 1-parametric families of the vector fields, there are two generic ways of occurring the instability of equilibria. Namely, either two branches of simple complex eigenvalues intersect the imaginary axis at two conjugated points or a single branch of simple real eigenvalues intersects the imaginary axis at the origin. The former instability is called oscillatory while the latter is called monotone.

The occurrences of instability indicate the local bifurcations of the equilibria family, which, in turn, reveal themselves by qualitative changes in the non-linear dynamics governed by equation $\dot{y} = Q(y)$.

If there are no additional degenerations, branching the equilibria family itself accompanies the monotone instability and branching off the limit cycle from the family accompanies the oscillatory instability (Poincare–Andronov–Hopf bifurcation), and more complex bifurcations occur in the case of additional degeneracy, e.g. when the neutral spectrum is multiple. For more information on this subject, a reader could refer to monographs [1, 2, 5].

Considering the evolutional PDEs as infinite-dimensional ODEs is the common practice in mathematical physics and, in particular, in the continuum mechanics. For instance, one can treat a steady fluid flow as an equilibrium of the ODE associated to the Navier–Stokes equation, etc. In many respects, the above assertions remain valid in the infinite dimensions even for the vector fields associated with PDEs [3, 4]. For many important PDEs, one can reduce the study of local bifurcations to the finite dimensions using the central manifold technique [5].

The spectra of the steady flows of an inviscid incompressible fluid stands as the classical subject of studies. A reader can find a summary of the classical results and modern advances in [6, 7] and can also follow the references provided therein. Since the inviscid fluid with impermeable boundary represents a Hamiltonian system [8] and the steady flows stand as the equilibria of it, their spectra is symmetrically relative to the imaginary axis. Also, they include both the continuous and point-wise components. At that, the former is always non-empty and can be unstable while the latter can be empty. The situation is very different upon considering the confined inviscid incompressible flows with pumping and withdrawing the fluid through the flow boundary. We name such call flows open. Although the spectral properties of the open flows are not so well known as those which associated with the case of the empty or fully impermeable boundaries, from what we have learned to the date, it follows that the spectrum of an open flow can be empty or fully point-wise. Also, it can belong to the open left complex semi-plane except for a finite number of the eigenvalues or even in full [9, 10].

In the present article, we discover new classes of inviscid open flows the spectra of which belongs to the left complex semi-plane. Proofs of these spectral properties were unknown earlier despite the simplicity of the flows. Here we mean the proofs in a rigorous sense not relying on certain issues such as the numeric computations and etc. We treat the mentioned examples uniformly using the rather general novel reduction of the spectral problem to finding the 'zeroes' of an 'operator-valued Laplace's integral'. Here, by zeroes, we mean the values of the complex variable which deliver degenerations to the mentioned integral. Correspondingly, examining the stability of eigenmodes brings us at the Routh–Hurwitz problem for the operator-valued Laplace's integral. In concrete examples, this integral often acts as a multiplicative transformation of the Fourier series, and then we arrive at the Routh–Hurwitz problem for the scalar Laplace's integral. Although this problem is solved in general [11], applying the general solution to the concrete problems is often not easy. Fortunately, the sufficient conditions known as Polia theorem [12] is enough for our purposes.

As we have mentioned, the spectra of the inviscid open flows had been studied earlier in [9, 10]. However, the flows examined here are not consistent with boundary conditions imposed there, while the methods employed there fail for the topology of the flow domain and boundary conditions considered here.

2 Formulation of the Initial-Boundary Value Problems

The Euler equations governing the inviscid incompressible homogeneous fluid flows read as

$$\mathbf{v}_t + \boldsymbol{\omega} \times \mathbf{v} = -\nabla H; \quad H = P + \mathbf{v}^2/2 \quad \operatorname{curl} \mathbf{v} = \boldsymbol{\omega}; \quad \operatorname{div} \mathbf{v} = 0. \tag{2.1}$$

Here, the unknowns are the vector field of the flow velocity denoted as $\mathbf{v}$ and the scalar field of pressure denoted as P. They have to be determined at every instant of time at every point of the flow domain that we denote as D. We consider a specified and non-variable domain $D \subset \mathbb{R}^3$ (or $D \subset \mathbb{R}^2$), and we assume it to be at least piece-wisely smooth. The quantity denoted as H is called the Bernoulli function. Note that the pressure as well as the Bernoulli function is defined up to a constant. The equation of motion written in (2.1) is known as the Lamb form of the Euler equation. The equivalent form is

$$\mathbf{v}_t + (\mathbf{v}, \nabla)\mathbf{v} = -\nabla P, \tag{2.2}$$

where$(\mathbf{v}, \nabla)$ stands for the covariant derivative along field $\mathbf{v}$ induced by a standard Riemann metric on $\mathbb{R}^3$ or on $\mathbb{R}^2$. In the Cartesian coordinates x_i, $((\mathbf{v}, \nabla)\mathbf{v})_i = v_j v_{i x_j}$, where repeating subscript indexes presumes summation. The equivalence of Eqs. (2.2) and (2.1) follows from the identity

$$\boldsymbol{\omega} \times \mathbf{v} + \nabla \mathbf{v}^2/2 = (\mathbf{v}, \nabla)\mathbf{v}.$$

Let $S = \partial D$ and let $\mathbf{n}$ be the exterior normal unit on S. For every t there is a partition $S = S^+(t) \cup S^-(t) \cup S^0(t)$ (up to the set of the zero $n-1$-dimensional measure), where $S^+(t) = \{x \in S : \mathbf{v}(x,t) \cdot \mathbf{n} < 0\}$ is the flow inlet, $S^-(t) = \{x \in S : \mathbf{v}(x,t) \cdot \mathbf{n} > 0\}$ is the flow outlet, and $S^0(t) = \{x \in S : \mathbf{v}(x,t) \cdot \mathbf{n} = 0\}$ is impermeable wall. By definition, the inlet and outlet are not empty for every open flow.

Let function γ defined on S be specified, and let

$$\int_S \gamma \, ds = 0.$$

Let us impose boundary condition

$$\mathbf{v} \cdot \mathbf{n} = \gamma(x,t), \ t \geqslant 0. \tag{2.3}$$

Specifying the normal velocity determines the inlet, outlet and impermeable wall. If $\gamma \equiv 0$ and $S^0 = S$ for every t then adding the initial condition leads to the correct initial-boundary value problem, which is a subject of numerous research articles, e.g. see monograph [13] and the references therein.

Setting boundary condition (2.3) where $\gamma \not\equiv 0$ brings us at the open flows. At that, however, a correct formulation of the initial-boundary value problem requires an additional boundary condition. It is not an easy issue. Likely, one should always impose the extra boundary conditions at the inlet, and there are more than one way to do so. For instance, one can specify the tangential vorticity:

$$\mathbf{n} \times (\boldsymbol{\omega} - \boldsymbol{\omega}^+) = 0 \text{ on } S^+, \tag{2.4}$$

The results on this boundary condition trace back to Yudovich [14] who considered the 2D problem. In two dimensions, $\mathbf{v} = u\mathbf{e}_1 + v\mathbf{e}_2$, and $\operatorname{rot}\mathbf{v} = \omega\mathbf{e}_3$, where $\omega = v_{x_2} - u_{x_1}$, and $\mathbf{e}_i,\ i = 1, 2, 3$ is the cartesian frame such that the flow is invariant to translations along $\mathbf{e}_3$. Then condition (2.4) takes the form $\omega - \omega^+ = 0$ on S^+, where ω^+ stands for a given scalar function. For such a boundary condition, Yudovich proved the global existence of the classical solution to the 2D problem provided that the problem data possess certain regularity. In addition to the natural smoothness and consistency of the data, he assumed that the flow inlet, outlet and impermeable wall are unions of connected components of the boundary. Succeeding authors had gradually got rid of these restrictions [15, 16]. Extending the Yudovich boundary condition to the 3D flows is due to Kazhikhov [17]. Also, he discovered an alternative extra boundary condition that reads as

$$\mathbf{n} \times (\mathbf{v} - \mathbf{v}^+) = 0 \text{ on } S^+. \tag{2.5}$$

In other words, in (2.4), the tangential vorticity can be replaced by the tangential velocity [18]. Note in passing, that neither the global solvability nor the collapses are known for the Kazhikhov formulation even in two dimensions.[1]

The Kazhikhov problem arises from the consideration of the vanishing viscosity limit for the Navier–Stokes system provided that the boundary conditions specify the flow velocity both on the inlet and on the outlet [19, 20]. Such boundary conditions describe the pumping/withdrawal of the fluid through the porous walls in the leading approximation [21].

3 The Liapunov Functionals

Let $\mathrm{W} : \mathbb{Y} \to \mathbb{R}$ be a functional on the phase space of some dynamical system $\mathcal{S}_t : \mathbb{Y} \to \mathbb{Y}, t \geqslant 0$. For a fixed $x \in \mathbb{Y}$, define a scalar function $w_x(t) = \mathrm{W}(\mathcal{S}_t x)$. Strictly monotone-decreasing (increasing) function w_x for every x belonging to a sufficiently representative set should be seen as an essential manifestation of the non-conservatism of system $\mathcal{S}_t$. We call in an informal way the functionals possessing such property as the Liapunov functionals (for classical theory, see [22]). Let us consider several simple examples of boundary conditions under which the inviscid open flows possess the Liapunov functionals.

Consider the Yudovich problem for the planar flows. We recall that we identify the values of curl with scalars for the 2D vector fields. Let $\gamma \not\equiv 0$ be defined arbitrarily and $\omega^+ = \Omega \equiv \text{const}$. Let $f = f(r) > 0$ for $r \neq 0$, $f(0) = 0$.

[1]In three dimensions, the same problem arises irrespective of the boundary conditions and persists even for the flows on the smooth closed manifolds.

Then

$$\frac{d}{dt}\int_D f(\omega - \Omega)\,dx = -\int_{S^-} f(\omega - \Omega)\gamma\,ds \leqslant 0.$$

Consequently, in the case of $\omega^+ = \text{const}$, the Yudovich problem admits a great family of the positive decreasing Liapunov functionals

$$\omega \mapsto \int_D f(\omega - \Omega)\,dx, \quad \omega = \text{curl}\,\mathbf{v}, \tag{3.1}$$

where f acts as the family parameter. The expression for the derivative of functional (3.1) follows from the so-called vorticity equation that arises upon applying curl to the Euler equation (2.1). In effect, the Liapunov functionals allied to (3.1) exists under the Yudovich boundary conditions for much wider classes of boundary data [9, 10].

Consider now the Yudovich problem in a multiply connected domain. Assume that the flow inlet includes a connected component of the boundary denoted as c. Define functional

$$\mathbf{v} \mapsto \oint_c \mathbf{v}\cdot\mathbf{dx}. \tag{3.2}$$

Integrating equation (2.1) across c brings us at the following identity

$$\frac{d}{dt}\oint_c \mathbf{v}\cdot\mathbf{dx} = -\int_c \omega^+\gamma\,ds. \tag{3.3}$$

Thus, the system admits the Liapunov functional (3.2), that grows linearly provided that the right-hand side of equality (3.3) does not depend on time and not equal to zero. If $\omega^+ = \Omega \equiv \text{const} \neq 0$ and γ does not depend on time then the system admits both the growing functional (3.2) and the decreasing positive functional (3.1). In the case of such coexistence, the Yudovich problem allows an asymptotic for $t \to +\infty$, the leading term of which describes an accelerating purely circulational flow placed upon the flow with constant vorticity [23].

Now let us pass to the Kazhikhov problem. Let $\boldsymbol{\Omega}$ and $\boldsymbol{U}$ be constant vectors. Define a vector field $\boldsymbol{V} = \boldsymbol{\Omega}\times\mathbf{x} + \boldsymbol{U}$. Let $\mathbf{v}$ be solution to the Kazhikhov problem arising for $\gamma = \boldsymbol{V}\cdot\mathbf{n}$ and $\mathbf{v}^+ = \boldsymbol{V}$ in boundary conditions (2.3) and (2.5). Then

$$\frac{d}{dt}\int_D (\mathbf{v} - \boldsymbol{V})^2\,dx = -\int_{S^-} \gamma\,(\mathbf{v} - \boldsymbol{V})^2\,dS \leqslant 0.$$

Deriving these assertions becomes absolutely transparent upon using the form of Euler equations given in (2.2). Thus, we get the Kazhikhov problem possessing the decreasing Liapunov functional defined as

$$\mathbf{v} \mapsto \int_D (\mathbf{v} - \boldsymbol{V})^2 \, dx. \tag{3.4}$$

For the Yudovich problem, the Liapunov functionals (3.1) under certain additional conditions imply the asymptotic stability of the steady flow with $\mathrm{curl}\mathbf{v} = \Omega = \mathrm{const}$, and this result can be generalized to the wider classes of steady vortex flows, see [9] and references in [10]. Under the Kazhikhov boundary conditions, despite the decreasing positive Liapunov functional (3.4), it is not clear whether the flow $\mathbf{V}$ possesses the asymptotic stability. Note in passing, that this is true for every positive value of the fluid viscosity. Such conclusion follows from the Navier–Stokes equations endowed with boundary condition $\mathbf{v} = \boldsymbol{V}$ on S. However, the decay rate of the perturbations tends to zero together with the fluid viscosity.

4 Elementary and Instructive Example

Let $\Gamma \in \mathbb{R}$, $D = \{(x, y), 0 < x < 1\}$, and $\mathbf{e}_x$ ($\mathbf{e}_y$) denote the unit vector field parallel to Ox–axis (Oy–axis). In domain D, consider the Kazhikhov problem the boundary conditions of which are as follows

$$\mathbf{v}|_{x=0} = \mathbf{e}_x + \Gamma\mathbf{e}_y, \quad \mathbf{v}\cdot\mathbf{e}_x|_{x=1} = 1, \tag{4.1}$$

and all the solutions have to be 2ℓ-periodic in variable y. Thus, $S^+ = \{x = 0\}$, $S^- = \{x = 1\}$, $S^0 = \varnothing$.

Under the boundary conditions formulated in (4.1), there exist a steady flow defined as $\mathbf{V} = \mathbf{e}_x + \Gamma\mathbf{e}_y$, $H = \mathrm{const}$. For this flow, the eigenvalue problem determining the eigenmodes of the small perturbations reads as

$$\lambda u - \Gamma\omega = -h_x, \quad \lambda v + \omega = -h_y \tag{4.2}$$

$$v_x - u_y = \omega, \quad u_x + v_y = 0, \tag{4.3}$$

$$u|_{x=0} = v|_{x=0} = 0, \quad u|_{x=1} = 0. \tag{4.4}$$

Here, u, v, h denote the unknown functions. At that, vector field (u, v) stands for the velocity perturbation while h stands for the perturbation of the Bernoulli function.

Eliminating the unknown h from Eq. (4.2) leads us to the equation

$$\lambda\omega + \omega_x + \Gamma\omega_y = 0.$$

The solution to it has the form $\omega = \omega^+(y - \Gamma x)e^{-\lambda x}$, where function ω^+ is still undefined. Replacing ω by the obtained solution in the second equation in (4.2), equating $x = 0$ and using the boundary condition (4.4) yields the equality $\omega^+ = -h_y(0, y)$. Now let us turn to reconstructing the velocity. For this purpose, we introduce the stream function $\psi = \psi(x, y)$, then we resolve the second equation in (4.3) by setting $u = \psi_y$, $v = -\psi_x$. As a result, the first equation in (4.3) takes the form

$$-\Delta\psi = \omega = -\mathrm{e}^{-\lambda x}\chi_y(y - \Gamma x), \quad (x, y) \in D, \tag{4.5}$$

where $\chi(y) = h(0, y)$ is an unknown $2\ell-$periodic function. We supply Eq. (4.5) with the condition requiring $2\ell-$periodicity in variable y and boundary conditions

$$\psi_x|_{x=0} = 0, \quad \psi|_{x=1} = 0. \tag{4.6}$$

Let $\mathrm{G} : \omega \mapsto \psi$ be the Green operator for the described boundary value problem. Then

$$\psi = -\mathrm{G}e_\lambda S_{\Gamma x}\partial_y\chi$$

where $e_\lambda(z) = \mathrm{e}^{\lambda z}$ and $(S_h f)(z) = f(z + h)$. The boundary conditions (4.6) are equivalent to the second and third boundary conditions in (4.4). Satisfying the first boundary condition in (4.4) brings us at an equation

$$\partial_y|_{x=0}(\mathrm{G}e_\lambda S_{\Gamma x}\partial_y\chi) = 0.$$

We define

$$\mathrm{K}(\lambda) : \chi \mapsto \partial_y|_{x=0}\mathrm{G}e_\lambda S_{\Gamma x}\partial_y\chi.$$

Every $\chi \in \ker \mathrm{K}(\lambda)$ gives rise to an eigenmode

$$u = -\partial_y Ge_\lambda S_{\Gamma x}\partial_y\chi, \quad v = \partial_x \mathrm{G}e_\lambda S_{\Gamma x}\partial_y\chi \quad h(x, y) = \chi(y) + \int_0^x (\Gamma\omega - \lambda u)(y, s)ds.$$

The Fourier decomposition allows us to express $\mathrm{K}(\lambda)$ explicitly. It acts as a multiplicative transform; namely, the multipliers has the form

$$\kappa_\beta(z) = -\beta^2 \int_0^1 g_\beta(0, \xi)e^{-z\xi}\, d\xi,\ \beta \in \alpha\mathbb{Z} \setminus \{0\},\ \alpha = \frac{\pi}{\ell},\ z = \lambda + i\beta\Gamma$$

where g_β stands for the Green kernel of the problem

$$-w''(y) + \beta^2 w = f(y), \ y \in (0,1) \quad w'(0) = w(1) = 0.$$

Kernel g_β reads as

$$g_\beta(x,\xi) = \frac{1}{|\beta| \operatorname{ch} \beta} \begin{cases} \operatorname{sh} |\beta|(1-\xi) \operatorname{ch} \beta x, \ x < \xi \\ \operatorname{sh} |\beta|(1-x) \operatorname{ch} \beta \xi, \ x > \xi. \end{cases}$$

Hence,

$$\kappa_\beta(z) = \frac{|\beta| e^{-\lambda}}{\operatorname{ch} \beta} \int_0^1 \operatorname{sh}(|\beta|\tau) e^{z\tau} \, d\tau, \ \beta \in \alpha\mathbb{Z} \setminus \{0\}, \ \alpha = \frac{\pi}{\ell}. \tag{4.7}$$

Each zero of function κ_β on the plane of the complex variable z corresponds to eigenvalue $\lambda = z - i\beta\Gamma$. Evidently, the real part of every zero of function κ_β on the z-plane is equal to the real part of eigenvalue $\lambda = z - i\beta\Gamma$. However, the former, and, therefore, the latter belong to the open left $z-$semiplane for every $\beta \in \alpha\mathbb{Z} \setminus \{0\}$ by Polia's theorem.

Remark 1 The conclusion we are about to make matches the existence of the Liapunov functional. Note that the latter circumstance itself implies only locating the eigenvalues inside the *closed* left semi-plane.

Remark 2 Let $\widetilde{\mathrm{L}_2}$ denote the space of square-summable $2\ell-$periodic functions vanishing on average. Then correspondence $\lambda \mapsto \mathrm{K}(\lambda) : \widetilde{\mathrm{L}_2} \to \widetilde{\mathrm{L}_2}$ determines the operator-valued entire function on the $\lambda-$plane.

Remark 3 Consider now the Yudovich problem in the same strip D with boundary conditions

$$\mathbf{v} \cdot \mathbf{e}_x|_{x=0} = 1, \quad \mathbf{v} \cdot \mathbf{e}_x|_{x=1} = 1, \quad \omega|_{x=0} = 0.$$

Again, there exists a steady solution $\boldsymbol{V} = \mathbf{e}_x + \Gamma \mathbf{e}_y$, but this time with *arbitrary* $\Gamma \in \mathbb{R}$. Hence, the steady solution is not isolated. For every steady flow of that kind, we have the spectral problem consisting of Eqs. (4.2), (4.3) and boundary conditions

$$u|_{x=0} = 0, \quad u|_{x=1} = 0, \quad v_x - u_y|_{x=0} = 0.$$

Under such boundary conditions, there exists only one eigenvalue $\lambda = 0$, and the corresponding eigenvectors has the form $u = 0$, $v = \text{const}$, $h = 0$.

Henceforth, we focus ourselves on the Kazhikhov problem and extend the analysis of the above example to more general flows.

5 Lagrangian Coordinates: Non-Separated Flows

For describing the motion of the material particles of a fluid flow, consider the Cauchy problem

$$\partial_s X = \mathbf{v}(X, s); \quad X|_{s=t} = x \in D,\ t > 0, \tag{5.1}$$

where $\mathbf{v} = \mathbf{v}(x, t)$ stands for the flow velocity defined in a given domain, which we denote as D. D is. Let $\mathbf{v} \in \mathrm{C}^1(\overline{D \times \{t > 0\}})$. There exist $\tau_1 = \tau_1(x, t) \in (0, t)$ è $\tau_2 = \tau_2(x, t) > t$, such that solution $X = X(s, x, t)$ to problem (5.1) is well-defined for $s \in (\tau_1(x, t), \tau_2(x, t))$. Mapping $s \mapsto X(s, x, t)$ parameterizes the path of the material particle which the fluid flow brings at point x to the time moment t. Every such path is a characteristic of the Euler equations. Hence the inlet and outlet are the non-characteristic parts of the boundary while the impermeable part is the characteristic part.

Consider the evolutional family X induced by a stationary flow $\boldsymbol{V} = \boldsymbol{V}(x) \in \mathrm{C}^1(\bar{D})$.

Definition 1 We call vector field $\boldsymbol{V}$ as non-separated in $D \cup S$ if there exists a function t^+ positive and bounded in $D \cup S$ and such that

$$\forall x \in D \cup S \;\; X(\max(t - t^+(x), 0), x, \max(t, t^+(x))) \in S^+. \tag{5.2}$$

Note that this time the Cauchy problem (5.1) is time-independent. Consequently,

$$a^+(x) \overset{\text{def}}{=} X(\max(t - t_+(x), 0), x, \max(t, t^+(x))) = X(0, x, t^+(x)).$$

For every $t > t_+(x)$, mapping $s \mapsto X(s, x, t)$, $s \in (t - t^+(x), t)$ parameterizes the same segment ℓ_x of the trajectory of the field $\boldsymbol{V}$. This segment connects point x to $a^+(x) \in S^+$. The trajectories of the field $\boldsymbol{V}$ are called streamlines. Mapping $x \mapsto a^+(x)$ is nothing else then the projection of $D \cup S \to S^+$ along the streamlines.

Remark 4 For every non-separated flow,

$$t_* = \sup_D t^+ < \infty.$$

Hence, during time t_*, every such flow renews in full the material particles of which it consists. In particular, for $t > t_*$ the flow consists of the particles which have entered the flow domain afterwards the initial time moment, and they have forced out all the particles initially constituted the flow.

Example 1 Let $D = \{(x, y) : 0 < x < 1\}$. Choose function $U \in \mathrm{C}^1(\mathbb{R})$, and define vector field $\mathbf{V} : (x, y) \mapsto (U(y), 0)$. Field $\mathbf{V}$ is the steady solution to the Euler equations. The corresponding flow is called as shear flow. Let $U(y) > 0 \; \forall\, y \in \mathbb{R}$.

Then $S^+ = \{x = 0\}$, $S^- = \{x = 1\}$, $S^0 = \varnothing$,

$$X(s, x, y, t) = (x - (t - s)U(y), y),\ t^+(x, y) = x/U(y),\ a^+(x, y) = (0, y).$$

The shear flow is non-separated provided that $\inf_{\mathbb{R}} U(y) > 0$. At that, $\sup t^+ = t_* = \sup_{\mathbb{R}} U^{-1}(y)$.

6 Harmonic Flows and Operator-Valued Laplace's Integral

Definition 2 We call vector field $\boldsymbol{V} \in \mathrm{C}^1(D)$ *harmonic* if curl $\boldsymbol{V} = 0$ and div $\boldsymbol{V} = 0$ in D.

In a sufficiently regular domain, the harmonic fields parallel to the boundary constitute the finite-dimensional space, the dimension of which is equal to the 1-dimensional Betty number of the domain. However, definition 1 does not require the harmonic field to be tangential to the boundary.

Every field $\boldsymbol{V}$ harmonic in the domain D satisfies the Euler equations in D with $H = \text{const}$.

Let $D \subset \mathbb{R}^n$, $n = 2, 3$ be a smooth domain, and let $S = \partial D$. Let M_n denote the direct product of the unit segment and $n - 1$ copies of the unit circumference.

Definition 3 We call domain $D \subset \mathbb{R}^2$ *annular* if M_2 is homeomorphic either to $D \cup S$ or to the space of orbits of a discrete subgroup of translations isomorphic to $\mathbb{Z}$, that acts on $D \cup S$.

Definition 4 We call domain $D \subset \mathbb{R}^3$ *annular* if either the manifold with boundary $D \cup S$ is homeomorphic to the direct product of unit segment and the 2D smooth closed manifold or a discrete subgroup of translations isomorphic to $\mathbb{Z}^n$, $n = 1, 2$ acts on $D \cup S$ and the space of its orbits is homeomorphic to M_3.

The set of the annular domains includes the plane annuli, the planar stripes, the gaps between the planes, or the cylinders, or the spheres, or the tori, etc.

If a discrete subgroup of translations acts on an annular domain then we always assume that all the functions considered on such a domain possess the periodicity consistent with this acting.

Let D be an annular domain. Then $S = S_1 \cup S_2$, $S_1 \cap S_2 = \varnothing$ and S_1, S_2 are the connected smooth surfaces homeomorphic one to another. Consider the Kazhikhov problem in an annular domain D and let the boundary data of it allow a harmonic flow such that the inlet coincides with one component of the boundary and the outlet coincides with the other component. Such a harmonic solution is unique in the class of harmonic fields. Indeed, the difference between two harmonic solutions is a harmonic field parallel to the boundary and equal to zero on the inlet. Hence, it has zero circulations across every closed path in D since every path in D is homological to some path on the inlet.

Let D be an annular domain and let $\boldsymbol{V}$ be a harmonic solution to some Kazhikhov's problem in D such that $S_1 = S^+$ è $S_2 = S^-$. The spectrum of the eigenmodes of such a flow obeys the following eigenvalue problem

$$\lambda\mathbf{v} + \boldsymbol{\omega} \times \boldsymbol{V} = -\nabla h; \quad \mathrm{curl}\,\mathbf{v} = \boldsymbol{\omega}; \quad \mathrm{div}\ \mathbf{v} = 0, \tag{6.1}$$

$$\mathbf{v} \times \mathbf{n} = 0 \text{ on } S^+, \quad \mathbf{n} \cdot \mathbf{v} = 0 \text{ on } S. \tag{6.2}$$

We will be using space of square-integrable functions on S^+ vanishing on average denoted as $\tilde{\mathrm{L}}_2(S^+)$ and the space of bounded operators acting in $\tilde{\mathrm{L}}_2(S^+)$ denoted as $\mathcal{B}^+$. Let $\Lambda(\boldsymbol{V})$ denote the point-wise spectrum of problem (6.1)–(6.2) for a specified field $\boldsymbol{V}$.

Theorem 1 *Let D be an annular domain. Assume that vector field $\boldsymbol{V}$ is harmonic in D and such that $S_1 = S^+$ and $S_2 = S^-$. Assume that vector field $\boldsymbol{V}$ generates the non-separated flow. Then there exists an entire function $\mathrm{K}_+ : \lambda \mapsto \mathrm{K}_+(\lambda) \in \mathcal{B}^+$ such that*

$$\Lambda(\boldsymbol{V}) = \{\lambda : \ \ker \mathrm{K}_+(\lambda) \neq \{0\}\}.$$

Function K_+ allows an explicit form of writing itself modulo reconstructing the divergence free field in domain D by its curl *under boundary conditions $\mathbf{u} \cdot \mathbf{n} = 0$ íà S^-, $\mathbf{u} \times \mathbf{n} = 0$ íà S^+.*

Lemma 1 *Let $D \subset \mathbb{R}^3$ be an annular domain, and let $\boldsymbol{\omega}$ be a smooth vector field in D, such that* $\mathrm{div}\ \boldsymbol{\omega} = 0$ *in D and $\boldsymbol{\omega} \cdot \mathbf{n} = 0$ on S_1. Consider a boundary value problem*

$$\mathrm{curl}\mathbf{v} = \boldsymbol{\omega}, \ \mathrm{div}\ \mathbf{v} = 0 \textit{ in } D; \ \mathbf{v} \times \mathbf{n} = 0 \textit{ íà } S_1, \ \mathbf{v} \cdot \mathbf{n} = 0 \textit{ on } S_2. \tag{6.3}$$

There exists unique solution $\mathbf{v}$.

Proof of Lemma 1 Let $\mathbf{u}$ be a solution to the homogeneous problem (6.3). Then the circulation of it across every closed path in D is equal to zero by the boundary condition at S_1. (Here we use the fact that domain D is annular). Hence, there exists scalar function ϕ such that $\mathbf{u} = \nabla\phi$ in D. Then $\Delta\phi = 0$ in D, $d\phi/dn = 0$ on S_2, and $\phi \equiv \mathrm{const}$ on S_1. From these observations, it follows that $\phi \equiv \mathrm{const}$, and $\mathbf{u} \equiv 0$ â D. Consider now constructing the solution. For the sake of definiteness, assume that S_1 is inside S_2. Extend $\boldsymbol{\omega}$ up to some field $\tilde{\boldsymbol{\omega}}$ defined on $\mathbb{R}^3$ by setting $\tilde{\boldsymbol{\omega}} = 0$ inside S_1, and $\tilde{\boldsymbol{\omega}} = \nabla\psi$ outside S_2 where ψ is a solution to the problem $\Delta\psi = 0$, $d\psi/dn = \omega \cdot \mathbf{n}$ on S_2, and $\psi(x) = o(1)$, $|x| \to \infty$. Field $\tilde{\boldsymbol{\omega}}$ is divergence-free in the generalized sense since its normal component is continuous. The decay rate of field $\tilde{\boldsymbol{\omega}}$ on the infinity is suitable for defining a convolution $G_0 * \tilde{\boldsymbol{\omega}}$, where $G_0 = (4\pi|x|)^{-1}$. Consider $\mathbf{b} = G_0 * \tilde{\boldsymbol{\omega}}$. Then $\mathrm{curlcurl}\,\mathbf{b} = \tilde{\boldsymbol{\omega}}$ in $\mathbb{R}^3$ and in D, therefore. Since $\boldsymbol{\omega} \cdot \mathbf{n} = 0$ on S_1, there exists field $\mathbf{h}$ harmonic in D and parallel to S such that field $\mathbf{b}_0 = \mathrm{curl}\,\mathbf{b} + \mathbf{h}$ has zero circulations across every closed path on S_1. Hence, there exists function $\phi_0 : \ (\mathbf{b}_0 - \nabla\phi_0) \times \mathbf{n} = 0$ on S_1. Finally, put

$\mathbf{v} = \mathbf{b}_0 + \nabla\phi$, where $\Delta\phi = 0$ in D, $\phi = -\phi_0$ on S_1, and $d\phi/dn = -\mathbf{b}_0 \cdot \mathbf{n}$ on S_2. This step completes the proof. □

Proof of Theorem 1 By applying curl to Eq. (6.1), we get equation

$$\lambda\boldsymbol{\omega} + [\boldsymbol{\omega}, \boldsymbol{V}] = 0. \tag{6.4}$$

Integration of these equations along the characteristics leads to the following equality

$$\partial_s \left(e^{\lambda s} \left(X'(s, x, t) \right)^{-1} \boldsymbol{\omega}(y) \right) = 0,\ \ y = X(s, x, t),\ \ s \in (t - t^+(x), t), \tag{6.5}$$

where $X'(s, x, t)$ is diffrential of the mapping $x \mapsto X(s, x, t)$ evaluated at point x. By equality (6.5),

$$X'(s, x, t)\boldsymbol{\omega}(x) = e^{\lambda(s-t)}\boldsymbol{\omega}(y), \quad y = X(s, x, t)\ s \in (t - t^+(x), t).$$

When $s \to t - t^+(x)$ this equality reads as

$$\boldsymbol{\omega}(x) = e^{-\lambda t^+(x)}\mathcal{X}^+(x)\boldsymbol{\omega}^+(a^+(x)), \quad x \in D. \tag{6.6}$$

where field $\boldsymbol{\omega}^+$ is the trace of $\boldsymbol{\omega}$ on S^+, and

$$\mathcal{X}^+(x) = X'(t^+(x), a^+(x), 0). \tag{6.7}$$

Since the flow $\boldsymbol{V}$ is non-separated, equality (6.6) defines field $\boldsymbol{\omega}$ everywhere in D correctly provided that one has specified the field $\boldsymbol{\omega}^+$ on the flow inlet. Field $\boldsymbol{\omega}$ arising from this definition generally does not allow to represent itself by curl of some other field. For existing such a representation, two conditions have to be hold. First, div $\boldsymbol{\omega} = 0$. Second, the flux of field $\boldsymbol{\omega}$ through each component of S must be equal to zero. Since S consists of two components by definition of the annular domains, it is enough to check the former condition and to show that one of the fluxes vanishes. From the projection of Eq. (6.1) onto the planes tangential to S^+, it follows that

$$\boldsymbol{\omega}^+ = -\gamma^{-1}\mathbf{n} \times \nabla h,\ \text{ where } \gamma = \mathbf{V} \cdot \mathbf{n} \neq 0. \tag{6.8}$$

By equality (6.8), $\boldsymbol{\omega}^+ \cdot \mathbf{n} \equiv 0$ on S^+, and the flux of it is equal to zero too. The vanishing of the normal component of field $\boldsymbol{\omega}^+$ on S^+ is consistent with boundary condition (6.2). Indeed, $\text{rot}_n\mathbf{v}$ on S^+ does not include normal derivatives of $\mathbf{v}$ and does not depend on the normal component of $\mathbf{v}$. Hence, $\text{rot}_n\mathbf{v} = 0$ on S^+ together with the tangential velocity.

Let us turn to the divergence of $\boldsymbol{\omega}$. Let $\rho = \operatorname{div}\, \boldsymbol{\omega}$. From Eq. (6.4), it follows that

$$\lambda\rho + \boldsymbol{V} \cdot \nabla\rho = 0.$$

Since the flow is non-separated, this equation implies that $\rho \equiv 0$ provided that $\rho \equiv 0$ on S^+. Let ρ^+ denote the trace of function ρ on S^+. Projecting equation (6.4) on the normal to S^+ and keeping in mind equality (6.8) brings us at equality

$$\rho^+ = \gamma^{-1}\mathrm{rot}_n((\gamma^{-1}\mathbf{n} \times \nabla h) \times \boldsymbol{V})\Big|_{S^+} = \gamma^{-1}\mathrm{rot}_n(\nabla h)\,|_{S^+} \equiv 0. \tag{6.9}$$

Actually, equality (6.9) remains true even for every function h and field $\boldsymbol{V}$ provided that $\boldsymbol{V} \cdot \mathbf{n} = \gamma$ on S^+. To see this, consider field

$$(\gamma^{-1}\mathbf{n} \times \nabla h) \times \boldsymbol{V} - \nabla h,$$

and note that it is normal to S^+.

Let G be the operator reconstructing the velocity by vorticity as described by lemma 1. Let χ be a scalar function on S^+ having zero mean value, and let $\lambda \in \mathbb{C}$. Put $h = \chi$ in equality (6.8) and define operator $\mathrm{L}_+(\lambda) : \chi \mapsto \omega$ by equalities (6.6) and (6.8). Define operator

$$\mathrm{K}_+(\lambda) : \chi \mapsto \mathbf{n} \cdot (\mathrm{GL}_+(\lambda)\chi)|_{S^+}\,, \tag{6.10}$$

Note that $\mathbf{n} \cdot (\mathrm{GL}_+(\lambda)\chi) = 0$ on S^- and field $\mathrm{GL}_+(\lambda)\chi$ is divergence-free in D by the definition of operator G. Therefore, the range of $\mathrm{K}_+(\lambda)$ consists of the functions vanishing on average. Thus, every $\chi_\lambda \in \ker \mathrm{K}_+(\lambda)$ gives rise to the eigenmodes defined as $\mathbf{v}_\lambda = \mathrm{GL}_+\chi_\lambda$. This observation completes the proof. □

Remark 5 Adding a constant to χ_λ does not alter the corresponding eigenmodes. Therefore, eliminating the constant functions from the domain of operator K_+ does not lead to the loss of the eigenmodes.

Remark 6 One can treat function χ as the trace of the perturbation of the Bernoulli function on S^+, where it coincides with the pressure perturbation because of the boundary conditions. Hence, vanishing function χ on average matches the fact that the Bernoulli function, as well as the pressure, are defined up to a constant. In particular, zero mean value of function χ is equivalent to preserving the mean pressure on the inlet upon perturbing the basic flow.

Remark 7 We omit the detailed proof of the boundedness of operator $\mathrm{K}(\lambda)$.

7 Flow Through a Gap Between Two Cylinders

Let D be a gap between two coaxial round cylinders the radii of which are equal to 1 and $a > 1$. We set up a cylindrical coordinate system the axis of which coincides with the axis of cylinders. Let r, θ, z stand for the radial, azimuthal and axial coordinates, and $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ stand for the corresponding unit vectors. In domain D, consider the Kazhikhov problem the boundary conditions of which reads as

$$\mathbf{v} \cdot \mathbf{n}|_{r=1} = -1; \quad \mathbf{v} \cdot \mathbf{n}|_{r=a} = 1/a, \tag{7.1}$$

$$\mathbf{n} \times (\mathbf{v} - \Gamma \mathbf{e}_\theta)|_{r=1} = 0. \tag{7.2}$$

From boundary conditions (7.1), it follows that $S^+ = \{r = 1\}$, $S^- = \{r = a\}$. At that, the value of $\Gamma \in \mathbb{R}$ determines the circulation of the flow across the inlet, that is equal to $2\pi\Gamma$. Also, we require $2\ell-$periodicity along $z-$axis. The harmonic solutions to this problem has the form

$$\boldsymbol{V} = r^{-1}(\mathbf{e}_r + \Gamma \mathbf{e}_\theta). \tag{7.3}$$

The field defined in (7.3) is non-separated. If $\Gamma = 0$, the flow admits the single-valued scalar potential—that is, $\boldsymbol{V} = \nabla \Phi$.

Note that vector field (7.3) directs the flow outwards the axis of the cylinders. It is natural to call such a flow diverging and to call the oppositely directed flow converging. The converging flow arises from the following boundary conditions

$$\mathbf{v} \cdot \mathbf{n}|_{r=1} = 1; \quad \mathbf{v} \cdot \mathbf{n}|_{r=a} = -1/a, \tag{7.4}$$

$$\mathbf{n} \times \left(\mathbf{v} - a^{-1}\Gamma \mathbf{e}_\theta\right)\Big|_{r=a} = 0. \tag{7.5}$$

Then $S^- = \{r = 1\}$ è $S^+ = \{r = a\}$, and the velocity field of the converging flow reads as

$$\boldsymbol{V} = r^{-1}(-\mathbf{e}_r + \Gamma \mathbf{e}_\theta).$$

We consider in detail only the diverging flows. For the converging flows, the results are the same as for the converging flows and their proof does not require a new idea.

Consider the eigenvalue problem (6.1)–(6.2) with field $\boldsymbol{V}$ specified in (7.3). Put

$$\mathrm{b}_{n\alpha} = \exp\, i(n\theta + \alpha z),\ n \in \mathbb{Z},\ \alpha \in \alpha_0\mathbb{Z},\ \alpha_0 = \pi/\ell,\ n^2 + \alpha^2 \neq 0.$$

Due to the symmetry of the problem, operator $\mathrm{K}_+(\lambda)$ takes the diagonal form in the basis of Fourier harmonic $\mathrm{b}_{n\alpha}$; namely,

$$\mathrm{K}_+(\lambda) : \mathrm{b}_{n,\alpha} \mapsto \kappa_{n,\alpha}(\Gamma, a, \lambda)\mathrm{b}_{n,\alpha}.$$

Hence, $\Lambda(\boldsymbol{V})$ coincides with the union of null-sets of all the multipliers denoted as $\kappa_{n,\alpha}(\Gamma, a, \lambda)$ on the complex plane λ.

Let us calculate $\mathrm{K}_+(\lambda)$ explicitly. Put $\mathbf{v} = \mathrm{b}_{n,\alpha}\hat{\mathbf{v}}(r)$, $\boldsymbol{\omega} = \mathrm{b}_{n,\alpha}\hat{\boldsymbol{\omega}}(r)$,

$$\hat{\boldsymbol{\omega}} = \xi\mathbf{e}_r + \eta\mathbf{e}_\theta + \zeta\mathbf{e}_z, \quad \hat{\mathbf{v}} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_z;$$

at that boundary conditions (6.2) takes the form

$$u(1) = v(1) = w(1) = 0; \; u(a) = 0. \tag{7.6}$$

First, consider $\mathrm{L}_+(\lambda)$. Relative to the Fourier basis, the system consisting of Eq. (6.4), condition $\operatorname{div}\boldsymbol{\omega} = 0$, and the boundary conditions at S^+ take the form

$$\lambda\xi - inr^{-2}(\eta - \Gamma\xi) - i\alpha r^{-1}\zeta = 0; \quad \xi|_{r=1} = 0; \tag{7.7}$$

$$\lambda\eta - i\alpha\Gamma r^{-1}\zeta + \Big(r^{-1}(\eta - \Gamma\xi)\Big)_r = 0; \quad \eta|_{r=1} = i\alpha; \tag{7.8}$$

$$\lambda\zeta + r^{-1}\zeta_r + i\beta r^{-2}\zeta = 0; \quad \zeta|_{r=1} = -in; \tag{7.9}$$

$$r^{-1}\left((r\xi)_r + in\eta\right) + i\alpha\zeta = 0. \tag{7.10}$$

Here, the boundary conditions on S^+ arise from representing of condition (6.8) using the Fourier basis. Eliminating η and ζ from (7.7) with the use of (7.10) brings us at the system

$$r\lambda\xi + \xi_r + r^{-1}(in\Gamma + 1)\xi = 0, \quad \xi(1) = 0,$$

which implies that $\xi \equiv 0$. Bearing in mind this conclusion, we note that Eqs. (7.8), (7.9), and (7.10) are not independent. Then integrating them gives

$$\zeta = -in\mathcal{E}(r); \; \eta = i\alpha r\mathcal{E}(r); \; \mathcal{E}(r) = \mathcal{E}(r, \lambda, n\Gamma) = r^{-in\Gamma}\mathrm{e}^{-\frac{\lambda(r^2-1)}{2}}. \tag{7.11}$$

We turn to calculating the operator $\mathrm{GL}_+(\lambda)$. Reconstructing the velcoity by vorticity with the use of boundary conditions (7.6) leads to expressions

$$w(r) = \int_1^r (i\alpha u - \eta)\, ds; \; rv(r) = \int_1^r (s\zeta + inu)\, ds.$$

Eliminating the unknowns w and v from equation $\operatorname{div}\mathbf{v} = 0$ yields the equation for finding unknown u. Introducing new unknown function

$$\phi(r) = \int_1^r u(s)\, ds$$

brings us to the boundary value problem

$$r^2\phi_{rr} + r\phi_r - (n^2 + \alpha^2 r^2)\phi = -r^2 F_{n,\alpha};\ \phi(1) = 0,\ \phi_r(a) = 0,\ \text{where} \tag{7.12}$$
$$F_{n,\alpha}(r, \lambda, n\Gamma) = ((n/r)^2 + \alpha^2)\int_1^r \mathcal{E}(s, \lambda, n\Gamma)\, s ds,$$

Given with this problem, we note that $\kappa_{n,\alpha}(\lambda, \Gamma, a) = u(1) = \phi_r(1)$. From this observation, using the Green function of problem (7.12), we get

$$\kappa_{n,\alpha}(\lambda, \Gamma, a) = K^{-1}_{\alpha,n,a}(1)\int_1^a K_{\alpha,n,a}(s) F_{n,\alpha}(s, \lambda, n\Gamma)\, s ds, \tag{7.13}$$

where $K_{\alpha,n,a}$ is a solution to Cauchy problem

$$r^2 K_{rr} + r K_r - (n^2 + r^2\alpha^2)K = 0; \quad K(a) = 1;\ K_r(a) = 0. \tag{7.14}$$

Given with the Cauchy problem (7.14), integrating by parts transforms expression (7.13) into the Laplace's integral that reads as

$$\kappa_{n,\alpha}(\lambda, \beta, a) = -K^{-1}_{\alpha,n,a}(1)\int_1^a r^{-i\beta} \mathrm{e}^{-\lambda(r^2-1)/2} K'_{\alpha,n,a}(r) r^2 dr,\ \beta = n\Gamma, \tag{7.15}$$

and $(\cdot)'$ stands for the derivatives in variable r.

Let $\Lambda_{n,\alpha,\beta,a}$ denote the null set of the Laplace integral (7.15) on the plane of complex variable λ, and let $\Lambda_{\Gamma,a}$ stand for the point-wise spectrum of problem (6.1)–(6.2) in domain $D = \{(r, \theta, z) : 1 < r < a\}$, when field $\boldsymbol{V}$ has the form specified in (7.3).

Lemma 2

$$\Lambda_{\Gamma,a} = \bigcup_{\alpha\in\alpha_0\mathbb{Z},\ \beta=n\Gamma,\ n\in\mathbb{Z},\ n^2+\alpha^2\neq 0} \Lambda_{n,\alpha,\beta,a}$$

Thus, we have reduced the Routh–Hurwitz problem for the spectrum of flow (7.3) to the same problem for scalar Laplace's integrals (7.15).

Lemma 3

$$\Lambda_{n,\alpha,0,a} \subset \{\lambda : \operatorname{Re}\lambda < 0\}\ \forall n \in \mathbb{Z},\ \alpha \in \mathbb{R}.$$

Proof Let us apply Polia's theorem about the nulls of Laplace integrals (see [12]). The integral is to be dealt with reads as

$$\int_1^a \mathrm{e}^{-\lambda(r^2-1)/2}\,(-rK_r)\,r\,dr, \quad K = K_{\alpha,n,a}.$$

We have to check the following assertions:

$$\forall r \in (1, a) \quad \text{(i) } K_r(r) < 0; \quad \text{(ii) } (rK_r(r))_r > 0.$$

By the Cauchy problem (7.14),

$$\int_s^a (\alpha^2 r + n^2/r)K^2 dr = -K(s)K_r(s)s - \int_s^a K_r^2 r\,dr, \ \ \forall s:\ 1 \le s < a.$$

Hence, $K(s)K_r(s) < 0$ for $s \in [1, a)$ and $K(a) = 1$ by the initial condition of problem (7.14). Consequently,

$$K(r) > 0, \ K_r(r) < 0, \forall r \in (1, a).$$

Then Eq. (7.14) implies assertion (ii). It completes the proof.

By lemmas 2 and 3, we get □

Theorem 2 $\Lambda_{\Gamma,a} \cap \{\lambda : \operatorname{Re}\lambda \ge 0\} \ne \varnothing$. *Then* $\beta = n\Gamma \ne 0$.

Remark 8 By Theorem 2, every eigenmode of potential flow is stable. Moreover, an eigenmode is stable provided it possesses the rotational symmetry.

Remark 9 At the inlet, every eigenmode of pressure has the form $\exp(\lambda t + in\theta + i\alpha z)$ as it follows from remark at page 134). At the same time, there exist the small perturbations of flow (7.3) which preserve the pressure on the inlet. Such a perturbation vanishes in a finite time. Indeed, consider a small perturbation $\mathbf{v} = v(r, t)\mathbf{e}_\theta + w(r, t)\mathbf{e}_z$, $p = p(r, t)$ (p denotes the perturbation of the pressure). The system governing such a class of perturbations reads as

$$\tfrac{2\Gamma v}{r^2} = p_r, \ (rv)_t + \tfrac{(rv)_r}{r} = 0, \ w_t + \tfrac{w_r}{r} = 0, \ t > 0, \ r \in (1, a), \quad (7.16)$$

$$v|_{r=1} = 0, \quad w|_{r=1} = 0, \quad p|_{r=1} = 0. \quad (7.17)$$

Integrating this system shows that every solution to it is equal to zero everywhere on interval $r \in (1, a)$ for every $t > (a^2-1)/2$. Hence, the system of spectral projectors of the point-wise spectrum of flow (7.3) is incomplete.

Remark 10 The assertions of Theorem 2 and Remarks 8 and 9 remains true for the converging flows.

Remark 11 Irrespective of the direction of the basic flow, the unstable modes occur upon increasing the circulation of it—that is, upon increasing the parameter denoted as Γ. Moreover, this instability survives upon taking into account the fluid viscosity. For more details, a reader should refer to articles [24–26].

8 Flow Through a Gap Between Two Spheres

Consider the Kazhikhov problem for Euler equations in a gap between two concentric spheres. Then the flow domain is $D = \{1 < |x| < a\}$, $a > 1$, and we assume that the boundary conditions are such that the normal velocities on the smaller and greater sphere are equal either to the values of -1 and a^{-2} correspondingly or to the values of 1 and $-a^{-2}$ correspondingly. The flow obeying the former (the latter) boundary conditions is called as diverging (converging). The inlet of the diverging (converging) flow is the smaller (the greater) sphere, and the other sphere stands as the outlet. We assume that the boundary conditions on the inlet impose zero tangential velocity. The formulated problems are rotationally invariant, and each has the rotationally invariant solution that reads as

$$\boldsymbol{V} = \pm \mathbf{e}_r/r^2,\ H = 0,\ r = |x|,\ \mathbf{e}_r = \nabla r \tag{8.1}$$

where sign of $+$ $(-)$ corresponds to the diverging (converging) flow. Note that this flow admits a scalar potential—that is, $\boldsymbol{V} = \mp\nabla r^{-1}$.

We will be consider in detail the case of diverging flows. Treatment of the converging flows is quite similar and leads to the results identical to the case of the diverging ones. Thus, we will study the eigenmodes of problem (6.1)–(6.2), where $\boldsymbol{V}$ is the diverging flow determined by equality (8.1). We set the spherical coordinates

$$x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta,\ \theta\in(0,\pi),\ \varphi\in(-\pi,\pi),$$

and we denote as $\mathbf{e_r}, \mathbf{e}_\theta, \mathbf{e}_\varphi$ the corresponding unit vectors. Let (ξ,η,ζ) and (u, v, w) be the coordinates of fields $\boldsymbol{\omega}$ and $\mathbf{v}$ relative to frame $\mathbf{e_r}, \mathbf{e}_\theta, \mathbf{e}_\varphi$ correspondingly. Thus,

$$\boldsymbol{\omega} = \xi\mathbf{e_r} + \eta\mathbf{e}_\theta + \zeta\mathbf{e}_\varphi,\ \mathbf{v} = u\mathbf{e_r} + v\mathbf{e}_\theta + w\mathbf{e}_\varphi.$$

In the spherical coordinates, the Cauchy problem determining operator $\mathrm{L}_+(\lambda)$ reads as

$$\lambda\xi - \frac{1}{r^3\sin\theta}\left\{(\eta\sin\theta)_\theta + \zeta_\varphi\right\} = 0,\ \xi|_{r=1} = 0, \tag{8.2}$$

$$\lambda\eta + \frac{1}{r^3}\{r\eta_r - \eta\} = 0,\ \eta|_{r=1} = \frac{\chi_\varphi}{\sin\theta}, \tag{8.3}$$

$$\lambda\zeta + \frac{1}{r^3}\{r\zeta_r - \zeta\} = 0,\ \zeta|_{r=1} = -\chi_\theta. \tag{8.4}$$

Here, we have been setting the boundary conditions in accordance with condition (6.8), and, therefore, $\chi = \chi(\theta, \phi)$ is an unknown function defined on the inlet. Integrating the problem (8.2)–(8.4) gives us the following solution

$$\xi = 0,\ \eta \sin\theta = \mathcal{E}(\lambda, r)\chi_\varphi,\ \zeta = -\mathcal{E}(\lambda, r)\chi_\theta,\ \mathcal{E}(\lambda, r) = re^{-\frac{\lambda(r^3-1)}{3}}. \quad (8.5)$$

Thus, we have an explicit expression for $\mathrm{L}_+(\lambda)$. Let us turn to calculating operator $\mathrm{GL}_+(\lambda)$. Given with $\boldsymbol{\omega}$ specified in (8.5), equation curl $\mathbf{v} = \boldsymbol{\omega}$ reads as

$$(w \sin\theta)_\theta = v_\varphi,\ u_\varphi - (rw)_r \sin\theta = r\mathcal{E}\chi_\varphi,\ (rv)_r - u_\theta = -r\mathcal{E}\chi_\theta. \quad (8.6)$$

Integrating the second and third equations in (8.6) and keeping in mind the boundary conditions at the inlet gives the following equalities

$$wr \sin\theta = \Phi_\varphi - \mathcal{E}_1\chi_\varphi,\ rv = \Phi_\theta - \mathcal{E}_1\chi_\theta,\ \text{where} \quad (8.7)$$

$$\Phi(r, \theta, \varphi) = \int_1^r u\, dr,\ \mathcal{E}_1(\lambda, r) = \int_1^r \mathcal{E}(\lambda, s)\, sds. \quad (8.8)$$

The expressions (8.7) implies the first equation in (8.6). Then equation div $\mathbf{v} = 0$ takes the form

$$\Delta\Phi + \mathcal{E}_1(\lambda, r)r^{-2}\Delta_+\chi = 0, \quad (8.9)$$

where $\Delta_+ f = -\frac{1}{\sin\theta}(f_\theta \sin\theta)_\theta - \frac{1}{\sin^2\theta} f_{\varphi\varphi}$ is the spherical Beltrami operator. Let b_m be the eigenfunction of operator Δ_+ that corresponds to the eigenvalue $\mu_m = m(m+1), m \in \mathbb{N}$. Putting $\chi = \mathrm{b}_m$, $\Phi = \widehat{\Phi}_m(r)\mathrm{b}_m$ brings us at the problem

$$\left(r^2\hat{\Phi}'_m\right)' - \mu_m\,(\widehat{\Phi}_m - \mathcal{E}_1(\lambda, r)) = 0, \quad (8.10)$$

$$\hat{\Phi}'_m(1) = 0,\ \hat{\Phi}_m(1) = 0,\ \hat{\Phi}'_m(a) = 0, \quad (8.11)$$

Hence,

$$\mathrm{K}_+(\lambda)\mathrm{b}_m = u|_{r=1} = \hat{\Phi}'_m(1)\mathrm{b}_m,\ \forall b_m \in \mathrm{H}_m,$$

where H_m stands for the eigen-space of operator Δ_+ corresponding to the eigenvalue $\mu_m = m(m+1), m \in \mathbb{N}$. Thus, the multipliers are functions

$$\kappa_m(\lambda, a) = \hat{\Phi}'_m(1)$$

where $\hat{\Phi}_m$ is the solution to problem (8.10–8.11). Expressing this solution using the Green function leads to the following equalities

$$\kappa_m(\lambda, a) = \frac{\mu_m}{K_{m,a}(1)} \int_1^a K_{m,a}(r)\mathcal{E}_1(\lambda, r)\, dr. \tag{8.12}$$

Here, $K_{m,a}$ is the solution to homogeneous equation (8.10) satisfying the initial conditions $K_{m,a}(a) = 1$, $K'_{m,a}(a) = 0$. Bearing this in mind, we arrive at equality

$$\kappa_m(\lambda, a) = -\frac{1}{K_{m,a}(1)} \int_1^a K'_{m,a}(r)\mathcal{E}(\lambda, r) r^3 dr = 0. \tag{8.13}$$

Simple transformation of this integral enables us to apply Polia's theorem and conclude that the zeroes of every multiplier belong to open left λ−semiplane. Thus we have arrived at

Theorem 3 *Let Λ_a be the point-wise spectrum of the problem (6.1)–(6.2), where* $\mathbf{V}$ *is one of the fields defined in* (8.1) *and* $x \in D = \{x : 1 < |x| < a\} \subset \mathbb{R}^3$. *Then* $\mathrm{Re}\,(\lambda) < 0\ \forall \lambda \in \Lambda_a\ \forall\, a \in (1, \infty)$.

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Spectral Properties of Killing Vector Fields of Constant Length and Bounded Killing Vector Fields

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Abstract This paper is a survey of recent results related to spectral properties of Killing vector fields of constant length and of some their natural generalizations on Riemannian manifolds. One of the main result is the following: If $\mathfrak{g}$ is a Lie algebra of Killing vector fields on a given Riemannian manifold (M, g), and $X \in \mathfrak{g}$ has constant length on (M, g), then the linear operator $\operatorname{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ has a pure imaginary spectrum (Nikonorov, J. Geom. Phys. 145 (2019), 103485). We discuss also more detailed structure results on the corresponding operator $\operatorname{ad}(X)$. Related results for geodesic orbit Riemannian spaces are considered. Finitely, we discuss some generalizations obtained recently by Xu and Nikonorov (Algebraic properties of bounded Killing vector fields. Asian J. Math. 2020 (accepted), see also. arXiv:1904.08710) for bounded Killing vector fields.

Keywords Bounded Killing vector fields · Geodesic orbit space · Homogeneous Riemannian space · Killing vector field of constant length

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1 Introduction

We discuss some general properties of Killing vector fields of constant length and of some their natural generalizations on an arbitrary Riemannian manifold (M, g). A comprehensive survey on classical results in this direction could be found in [5, 6]. Important properties of Killing vector fields of constant length (abbreviated as KVFCL) on compact homogeneous Riemannian spaces are studied in [20]. Some resent results about Killing vector field of constant length on some special

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Riemannian manifolds are obtained in [28, 30]. All manifolds are supposed to be connected.

Let us consider a Riemannian manifold (M, g) and any Lie group G acting effectively on (M, g) by isometries. We will identify the Lie algebra $\mathfrak{g}$ of G with the corresponded Lie algebra of Killing vector field on (M, g) as follows. For any $U \in \mathfrak{g}$ we consider a one-parameter group $\exp(tU) \subset G$ of isometries of (M, g) and define a Killing vector field $\widetilde{U}$ by a usual formula

$$\widetilde{U}(x) = \frac{d}{dt}\exp(tU)(x)\Big|_{t=0}. \tag{1}$$

It is clear that the map $U \to \widetilde{U}$ is linear and injective, but $[\widetilde{U}, \widetilde{V}] = -\widetilde{[U, V]}_{\mathfrak{g}}$, where $[\cdot, \cdot]_{\mathfrak{g}}$ is the Lie bracket in $\mathfrak{g}$ and $[\cdot, \cdot]$ is the Lie bracket of vector fields on M. We will use this identification repeatedly in this paper.

Any $X \in \mathfrak{g}$ determines a linear operator $\operatorname{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ acting by $Y \mapsto [X, Y]$. If we consider X as a Killing vector field on (M, g), then some geometric type assumptions on X imply special properties (in particular, spectral properties) of the corresponding operator $\operatorname{ad}(X)$. In this paper, we study the property of X to be of constant length and the property to be bounded.

Let us fix some notation. For a Lie algebra $\mathfrak{g}$, we denote by $\mathfrak{n}(\mathfrak{g})$ and $\mathfrak{r}(\mathfrak{g})$ the nilradical (the maximal nilpotent ideal) and the radical of $\mathfrak{g}$ respectively. A maximal semisimple subalgebra of $\mathfrak{g}$ is called a Levi factor or a Levi subalgebra. There is a semidirect decomposition $\mathfrak{g} = \mathfrak{r}(\mathfrak{g}) \rtimes \mathfrak{s}$, where $\mathfrak{s}$ is an arbitrary Levi factor. The Malcev–Harish-Chandra theorem states that any two Levi factors of $\mathfrak{g}$ are conjugate by an automorphism $\exp(\operatorname{Ad}(Z))$ of $\mathfrak{g}$, where Z is in the nilradical $\mathfrak{n}(\mathfrak{g})$ of $\mathfrak{g}$. We have $\mathfrak{r}(\mathfrak{g}) = [\mathfrak{s}, \mathfrak{r}(\mathfrak{g})] \oplus C_{\mathfrak{r}(\mathfrak{g})}(\mathfrak{s})$ (a direct sum of linear subspaces), where $C_{\mathfrak{r}(\mathfrak{g})}(\mathfrak{s})$ is the centralizer of $\mathfrak{s}$ in $\mathfrak{r}(\mathfrak{g})$. Recall also that $[\mathfrak{g}, \mathfrak{r}(\mathfrak{g})] \subset \mathfrak{n}(\mathfrak{g})$, therefore, $[\mathfrak{s}, \mathfrak{r}(\mathfrak{g})] \subset [\mathfrak{g}, \mathfrak{r}(\mathfrak{g})] \subset \mathfrak{n}(\mathfrak{g})$. Moreover, $D(\mathfrak{r}(\mathfrak{g})) \subset \mathfrak{n}(\mathfrak{g})$ for every derivation D of $\mathfrak{g}$. For any Levi factor $\mathfrak{s}$, we have $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{r}(\mathfrak{g}) + \mathfrak{s}, \mathfrak{r}(\mathfrak{g}) + \mathfrak{s}] = [\mathfrak{g}, \mathfrak{r}(\mathfrak{g})] \rtimes \mathfrak{s} \subset \mathfrak{n}(\mathfrak{g}) \rtimes \mathfrak{s}$. For a more detailed discussion of the Lie algebra structure we refer to [14].

Recall that a subalgebra $\mathfrak{k}$ of a Lie algebra $\mathfrak{g}$ is said to be *compactly embedded in* $\mathfrak{g}$ if $\mathfrak{g}$ admits an inner product relative to which the operators $\operatorname{ad}(X) : \mathfrak{g} \mapsto \mathfrak{g}$, $X \in \mathfrak{k}$, are skew-symmetric. This condition is equivalent to the following one: the closure of $\operatorname{Ad}_G(\exp(\mathfrak{k}))$ in $\operatorname{Aut}(\mathfrak{g})$ is compact, see e.g. [14]. Note that for a compactly embedded subalgebra $\mathfrak{k}$, every operator $\operatorname{ad}(X) : \mathfrak{g} \to \mathfrak{g}$, $X \in \mathfrak{k}$, is semisimple and the spectrum of $\operatorname{ad}(X)$ lies in $\mathbf{i}\mathbb{R}$, where $\mathbf{i} = \sqrt{-1}$. Recall also that a subalgebra $\mathfrak{k}$ of a Lie algebra $\mathfrak{g}$ is said to be *compact* if it is compactly embedded in itself. It is equivalent to the fact that there is a compact Lie group with a given Lie algebra $\mathfrak{k}$. It is clear that any compactly embedded subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is compact.

The following theorem is one of the main results for our discussion.

Theorem 1 (Theorem 1 in [22]) *For any Killing field of constant length* $X \in \mathfrak{g}$ *on a Riemannian manifold* (M, g)*, the spectrum of the operator* $\operatorname{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ *is pure imaginary, i. e. is in* $\boldsymbol{i}\mathbb{R}$.

This result is a particular case of Theorem 3. It is clear that it is non-trivial only for noncompact Lie algebras $\mathfrak{g}$ and only when X is not a central element of $\mathfrak{g}$. On the other hand, there are examples of Killing fields of constant length $X \in \mathfrak{g}$ for noncompact $\mathfrak{g}$. Moreover, Theorems 4 and 5 give us examples when $X \in \mathfrak{n}(\mathfrak{g})$ and $\operatorname{ad}(X)$ is non-trivial and nilpotent. In particular, $\operatorname{ad}(X)$ is not semisimple in this case.

The paper is organized as follows. In Sect. 2, we consider some important spectral properties of the operator $\operatorname{ad}(X) : \mathfrak{g} \mapsto \mathfrak{g}$ for any Killing vector field of constant length $X \in \mathfrak{g}$ on a given Riemannian manifold (M, g). One of the main results is Theorem 2, that implies non-trivial geometric properties of (M, g) in the case when the Lie algebra $\mathfrak{g}$ could be decomposed as a direct Lie algebra sum. In Sect. 3, we discuss some results on Killing vector fields of constant length on geodesic orbit Riemannian spaces. In particular, Theorems 4 and 5 imply that any Killing vector field in the center of $\mathfrak{n}(\mathfrak{g})$ has constant length on a given geodesic orbit space $(G/H, g)$. This observation provides non-trivial examples of Killing vector field of constant length X such that the operator $\operatorname{ad}(X) : \mathfrak{g} \mapsto \mathfrak{g}$ is nilpotent. In Sect. 4, we discuss general properties of bounded Killing vector fields on a given Riemannian manifold (M, g). In particular, we consider the generalization of Theorem 1 and some other spectral properties for bounded Killing vector fields. We also study the Lie algebra of all bounded vectors in $\mathfrak{g}$ for a Riemannian homogeneous space G/H, on which G acts effectively. This Lie algebra is compact and we completely describe all bounded Killing vector fields for a Riemannian homogeneous space.

2 KVFCL on General Riemannian Manifolds

In what follows, we assume that a Lie group G acts effectively on a Riemannian manifold (M, g) by isometries, $\mathfrak{g}$ is the Lie algebra of G, elements of $\mathfrak{g}$ are identified with Killing vector field on (M, g) according to (1).

The following characterizations of Killing vector fields of constant length on Riemannian manifolds are very useful.

Lemma 1 (Lemma 3 in [6]) *Let X be a non-trivial Killing vector field on a Riemannian manifold (M, g). Then the following conditions are equivalent:*

(1) *X has constant length on M;*
(2) *$\nabla_X X = 0$ on M;*
(3) *every integral curve of the field X is a geodesic in (M, g).*

Lemma 2 (Lemma 2 in [21]) *If a Killing vector field $X \in \mathfrak{g}$ has constant length on (M, g), then for any $Y, Z \in \mathfrak{g}$ the equalities*

$$g([Y, X], X) = 0\,, \tag{2}$$

$$g([Z, [Y, X]], X) + g([Y, X], [Z, X]) = 0 \tag{3}$$

hold at every point of M. If G acts on (M, g) transitively, then condition (2) implies that X has constant length. Moreover, the condition (3) also implies that X has constant length for compact M and transitive G.

Now, we are going to get some more detailed results.

Theorem 2 (Theorem 2 in [22]) *Let $X \in \mathfrak{g}$ be a Killing vector field of constant length on (M, g). Suppose that we have a direct Lie algebra sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_l$, $l \geq 2$. Then for every $i = 1, \dots, l$, there is an ideal $\mathfrak{u}_i$ in $\mathfrak{g}_i$ such that $[X, \mathfrak{g}_i] \subset \mathfrak{u}_i$ and $g(\mathfrak{u}_i, \mathfrak{u}_j) = 0$ on M for every $i \neq j$.*

Proof Since X is of constant length, then $\mathfrak{g}_i \cdot g(X, X) = g([\mathfrak{g}_i, X], X) = 0$ for any i by Lemma 2. If we take $j \neq i$, then

$$0 = \mathfrak{g}_j \cdot g([\mathfrak{g}_i, X], X) = g([\mathfrak{g}_i, X], [\mathfrak{g}_j, X]).$$

Let $\{\mathfrak{u}_i\}$, $i = 1, \dots, l$, be a set of maximal (by inclusion) subspaces $\mathfrak{u}_i \subset \mathfrak{g}_i$, such that $[\mathfrak{g}_i, X] \subset \mathfrak{u}_i$ and $g(\mathfrak{u}_i, \mathfrak{u}_j) = 0$ for every $i \neq j$ (such a set of subspaces should not be unique in general). Since

$$0 = \mathfrak{g}_i \cdot g(\mathfrak{u}_i, \mathfrak{u}_j) = 0 = g([\mathfrak{g}_i, \mathfrak{u}_i], \mathfrak{u}_j),$$

then $[\mathfrak{g}_i, \mathfrak{u}_i] \subset \mathfrak{u}_i$ due to the choice of $\mathfrak{u}_i$. Hence, every $\mathfrak{u}_i$ is an ideal in $\mathfrak{g}_i$ and in $\mathfrak{g}$. ■

Remark 1 If $X = X_1 + X_2 + \cdots + X_l$, where $X_i \in \mathfrak{g}_i$, then $\mathfrak{u}_i \neq 0$ if X_i is not in the center of $\mathfrak{g}_i$. In particular, if $X_i \neq 0$ and $\mathfrak{g}_i$ is simple, then $\mathfrak{u}_i = \mathfrak{g}_i$. Note, that Theorem 2 leads to a more simple proof of Theorem 1 in [20] about properties of Killing vector fields of constant length on compact homogeneous Riemannian manifolds. See also Remark 4 about geodesic orbit spaces.

In what follows, for a Killing vector field of constant length $X \in \mathfrak{g}$ on (M, g), we denote by $L = L(X)$ the linear operator $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$.

The above results lead to the following theorem (see [22] for a detailed proof).

Theorem 3 (Theorem 3 in [22]) *Let $X \in \mathfrak{g}$ be a Killing vector field of constant length on (M, g). Then the following assertions hold.*

(1) We have an L-invariant linear space decomposition $\mathfrak{g} = A_1 \oplus A_2$, where $A_1 =$ $\mathrm{Ker}(L^2)$ *and $A_2 =$* $\mathrm{Im}(L^2)$*. Moreover, A_1 is the root space for L with the eigenvalue* 0 *and L is invertible on A_2.*
(2) If $\mathfrak{o}$ is a two-dimensional L-invariant subspace, corresponding to a complex conjugate pair of eigenvalues $\alpha \pm \beta \boldsymbol{i}$ ($\beta \neq 0$), i. e. $L(U) = \alpha \cdot U - \beta \cdot V$ and $L(V) = \beta \cdot U + \alpha \cdot V$ for some non-trivial $U, V \in \mathfrak{o}$, then $\alpha = 0$, $g(U, V) = 0$, and $g(U, U) = g(V, V)$ on G/H.
(3) All eigenvalues of L have trivial real parts.

Obviously, this result implies Theorem 1.

In what follows, we use the notation $A_1 = \mathrm{Ker}(L^2)$ and $A_2 = \mathrm{Im}(L^2)$ *as in Theorem 3.*

Remark 2 Note that $\mathfrak{g} = A_1 \oplus A_2 = \mathrm{Ker}(L^2) \oplus \mathrm{Im}(L^2)$ is the Fitting decomposition (see e.g. Lemma 5.3.11 in [14]) for the operator L. It should be noted that the decomposition $\mathfrak{g} = \mathrm{Ker}(L) \oplus \mathrm{Im}(L)$ is not valid at least for $X \in C(\mathfrak{n}(\mathfrak{g})) \setminus C(\mathfrak{g})$ (see Theorem 5) since $\mathrm{Im}(L) \subset C(\mathfrak{n}(\mathfrak{g})) \subset \mathrm{Ker}(L)$ in this case.

Remark 3 It is interesting to study KVFCL X with $\mathrm{Ker}(L) \neq A_1$. For such X the operator $L = \mathrm{ad}(X)$ is not semisimple. One class of suitable examples are $X \in C(\mathfrak{n}(\mathfrak{g}))$ for geodesic orbit spaces $(G/H, g)$ as in Theorem 5 (if there is a vector $V \in \mathfrak{g} \setminus \mathfrak{n}(\mathfrak{g})$ such that $[X, V] \neq 0$).

Since always $X \in \mathrm{Ker}(L) = \mathrm{Ker}(\mathrm{ad}(X)) \subset A_1$, we get $A_1 \neq 0$. On the other hand, it is possible that $A_1 = \mathfrak{g}$ and $A_2 = 0$ (see Remark 5 and Proposition 5).

Proposition 1 (Proposition 7 in [22]) *Suppose that* $X \in \mathfrak{g}$ *has constant length on* (M, g) *and* $A_1 = \mathfrak{g}$. *Then* $X \in \mathfrak{n}(\mathfrak{g})$.

Proof If $A_1 = \mathfrak{g}$, then $L^2 = (\mathrm{ad}(X))^2 = 0$ on $\mathfrak{g}$. Elements $X \in \mathfrak{g}$ with this property are called *absolute zero divisors* in $\mathfrak{g}$. Using the Levi decomposition $\mathfrak{g} = \mathfrak{r}(\mathfrak{g}) \rtimes \mathfrak{s}$, one can show that $X \in \mathfrak{r}(\mathfrak{g})$. If $X = X_{\mathfrak{r}(\mathfrak{g})} + X_{\mathfrak{s}}$, where $X_{\mathfrak{r}(\mathfrak{g})} \in \mathfrak{r}(\mathfrak{g})$ and $X_{\mathfrak{s}} \in \mathfrak{s}$, then for any $Y \in \mathfrak{s}$ we have $(\mathrm{ad}(X))^2(Y) = [X_{\mathfrak{s}}, [X_{\mathfrak{s}}, Y]] + Z$, where $Z = [X, [X_{\mathfrak{r}(\mathfrak{g})}, Y]] + [X_{\mathfrak{r}(\mathfrak{g})}, [X_{\mathfrak{s}}, Y]] \in \mathfrak{r}(\mathfrak{g})$. Hence, $(\mathrm{ad}(X))^2 = 0$ imply $[X_{\mathfrak{s}}, [X_{\mathfrak{s}}, Y]] = 0$ for all $Y \in \mathfrak{s}$, i. e. $X_{\mathfrak{s}}$ is an absolute zero divisor in $\mathfrak{s}$, that impossible for $X_{\mathfrak{s}} \neq 0$. Indeed, if $U \in \mathfrak{s}$ is a non-trivial absolute zero divisor, then U is a non-trivial nilpotent element in $\mathfrak{s}$. Hence, there are $V, W \in \mathfrak{s}$ such that $[W, U] = 2U$, $[W, V] = -2V$, and $[U, V] = W$ by the Morozov–Jacobson theorem (see e.g. Theorem 3 in [15]). But this implies $[U, [U, V]] = -2U$ that contradicts to $(\mathrm{ad}(U))^2 = 0$ (see [16] for a more detailed discussion). Therefore, we get $X_{\mathfrak{s}} = 0$ and $X \in \mathfrak{r}(\mathfrak{g})$.

Moreover, it is known that $\mathfrak{n}(\mathfrak{g}) = \{Y \in \mathfrak{r}(\mathfrak{g}) \mid (\mathrm{ad}(Y))^p = 0 \text{ for some } p \in \mathbb{N}\}$, see e.g. Remark 7.4.7 in [14]. Therefore, $X \in \mathfrak{n}(\mathfrak{g})$. ∎

3 KVFCL on Geodesic Orbit Spaces

Let $(M = G/H, g)$ be a homogeneous Riemannian space, where H is a compact subgroup of a Lie group G and g is a G-invariant Riemannian metric. We will suppose that G acts effectively on G/H (otherwise it is possible to factorize by U, the maximal normal subgroup of G in H).

We recall the definition of one important subclass of homogeneous Riemannian spaces.

A Riemannian manifold (M, g) is called *a manifold with homogeneous geodesics or a geodesic orbit manifold* (shortly, *GO-manifold*) if any geodesic γ of M is an orbit of a 1-parameter subgroup of the full isometry group of (M, g). Obviously, any geodesic orbit manifold is homogeneous. A Riemannian homogeneous space ($M = G/H, g$) is called *a space with homogeneous geodesics or a geodesic orbit space* (shortly, *GO-space*) if any geodesic γ of M is an orbit of a 1-parameter subgroup of the group G. Hence, a Riemannian manifold (M, g) is a geodesic orbit Riemannian manifold, if it is a geodesic orbit space with respect to its full connected isometry group. This terminology was introduced in [17] by O. Kowalski and L. Vanhecke, who initiated a systematic study of such spaces. In the same paper, O. Kowalski and L. Vanhecke classified all GO-spaces of dimension ≤6. A detailed exposition on geodesic orbit spaces and some important subclasses could be found also in [4, 11, 21], see also the references therein. In particular, one can find many interesting results about GO-manifolds and their subclasses in [1–3, 6–8, 10, 12, 13, 18, 23–25, 31, 32].

Recall that all symmetric, weakly symmetric, normal homogeneous, naturally reductive, generalized normal homogeneous, and Clifford–Wolf homogeneous Riemannian spaces are geodesic orbit, see [10]. Besides the above examples, every isotropy irreducible Riemannian space is naturally reductive, and hence geodesic orbit, see e.g. [9].

The following simple result is very useful (M_x denotes the tangent space to M at the point $x \in M$).

Lemma 3 ([19], Lemma 5) *Let (M, g) be a Riemannian manifold and $\mathfrak{g}$ be its Lie algebra of Killing vector fields. Then (M, g) is a GO-manifold if and only if for any $x \in M$ and any $v \in M_x$ there is $X \in \mathfrak{g}$ such that $X(x) = v$ and x is a critical point of the function $y \in M \mapsto g_y(X, X)$. If (M, g) is homogeneous, then the latter condition is equivalent to the following one: for any $Y \in \mathfrak{g}$ the equality $g_x([Y, X], X) = 0$ holds.*

Now, we recall the following remarkable result.

Proposition 2 ([19], Theorem 1) *Let (M, g) be a GO-manifold, $\mathfrak{g}$ is its Lie algebra of Killing vector fields. Suppose that $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$. Then any $X \in \mathfrak{a}$ has constant length on (M, g).*

As is noted in [19], Proposition 2 could be generalized for geodesic orbit spaces. For the reader's convenience, we provide also the proof of the corresponding result.

Theorem 4 (Theorem 4 in [22]) *Let $(M = G/H, g)$ be a geodesic orbit Riemannian space. Suppose that $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g} = \mathrm{Lie}(G)$. Then any $X \in \mathfrak{a}$ (as a Killing vector field) has constant length on (M, g). As a corollary, $g(X, Y) \equiv \mathrm{const}$ on M for every $X, Y \in \mathfrak{a}$.*

Proof Let x be any point in M. We will prove that x is a critical point of the function $y \in M \mapsto g_y(X, X)$. Since $(M = G/H, g)$ is homogeneous, then (by Lemma 3) it suffices to prove that $g_x([Y, X], X) = 0$ for every $Y \in \mathfrak{g}$.

Consider any $Y \in \mathfrak{a}$, then $Y \cdot g(X, X) = 2g([Y, X], X) = 0$ on M, since $\mathfrak{a}$ is abelian.

Now, consider $Y \in \mathfrak{g}$ such that $g_x(Y, U) = 0$ for every $U \in \mathfrak{a}$. We will prove that $g_x([Y, X], X) = 0$. By Lemma 3, for the vector $X(x) \in M_x$ there is a Killing field

$Z \in \mathfrak{g}$ such that $Z(x) = X(x)$ and $g_x([V, Z], Z) = 0$ for any $V \in \mathfrak{g}$. In particular, $g_x([Y, Z], Z) = 0$. Now, $W = X - Z$ vanishes at x and we get

$$g_x([Y, X], X) = g_x([Y, Z + W], Z + W) = g_x([Y, Z + W], Z) =$$

$$g_x([Y, Z], Z) + g_x([Y, W], Z) = g_x([Y, W], Z).$$

Note that $g_x([Y, W], Z) = -g_x([W, Y], Z) = g_x(Y, [W, Z]) = 0$ due to $W(x) = 0$ $(0 = W \cdot g(Y, Z)|_x = g_x([W, Y], Z) + g_x(Y, [W, Z]))$ and $[W, Z] = [X, Z] \in \mathfrak{a}$. Therefore, $g_x([Y, X], X) = 0$.

Since every $Y \in \mathfrak{g}$ could be represented as $Y = Y_1 + Y_2$, where $Y_1 \in \mathfrak{a}$ and $g_x(Y_2, \mathfrak{a}) = 0$, then x is a critical point of the function $y \in M \mapsto g_y(X, X)$. Since every $x \in M$ is a critical point of the function $y \in M \mapsto g_y(X, X)$, then X has constant length on (M, g).

The last assertion follows from the equality $2g(X, Y) = g(X + Y, X + Y) - g(X, X) - g(Y, Y)$. ■

Corollary 1 *Every geodesic orbit Riemannian space $(M = G/H, g)$ with non-semisimple group G has non-trivial Killing vector fields of constant length.*

Proof If the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is non semisimple, then it has a non-trivial abelian ideal $\mathfrak{a}$ (for instance, this property has the center of the nilradical $\mathfrak{n}(\mathfrak{g})$ of $\mathfrak{g}$). Now, it suffices to apply Theorem 4. ■

We recall some other important properties of geodesic orbit spaces. Any semisimple Lie algebra $\mathfrak{s}$ is a direct Lie algebra sum of its compact and noncompact parts ($\mathfrak{s} = \mathfrak{s}_c \oplus \mathfrak{s}_{nc}$). The following proposition is asserted in [12], a detailed proof could be found in [13].

Proposition 3 *Let $(G/H, g)$ be a connected geodesic orbit space and let $\mathfrak{s}$ be any Levi factor of G. Then the noncompact part $\mathfrak{s}_{nc}$ of $\mathfrak{s}$ commutes with the radical $\mathfrak{r}(\mathfrak{g})$.*

Remark 4 For a geodesic orbit space $(G/H, g)$ we have a direct Lie algebra sum $\mathfrak{g} = (\mathfrak{r}(\mathfrak{g}) \rtimes \mathfrak{s}_c) \oplus \mathfrak{s}_{nc}$ by Proposition 3. Moreover, we can represent $\mathfrak{s}_{nc}$ as a direct sum of simple noncompact ideals. This decomposition is useful for applying of Theorem 2.

Proposition 4 (C. Gordon, [12]) *Let $(G/H, \rho)$ be a geodesic orbit space. Then the nilradical $\mathfrak{n}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is commutative or two-step nilpotent.*

Theorem 5 (Theorem 5 in [22]) *For a geodesic orbit space $(G/H, g)$, we consider any $X \in \mathfrak{n}(\mathfrak{g})$. Then the following conditions are equivalent:*

(1) X is in the center $C(\mathfrak{n}(\mathfrak{g}))$ of $\mathfrak{n}(\mathfrak{g})$;
(2) X has constant length on $(G/H, g)$.

Proof (1) $\Rightarrow$ (2). Since the center $C(\mathfrak{n}(\mathfrak{g}))$ is an abelian ideal in $\mathfrak{g}$, then X has constant length due to Theorem 4.

(2) $\Rightarrow$ (1). Since $X \in \mathfrak{n}(\mathfrak{g})$ and $\mathfrak{n}(\mathfrak{g})$ is at most two step nilpotent by Proposition 4, then $[Z, [Z, X]] = 0$ for any $Z \in \mathfrak{n}(\mathfrak{g})$. Now, by Lemma 2, we have

$$g([Z, X], [Z, X]) = g([Z, [Z, X]], X) + g([Z, X], [Z, X]) = 0,$$

hence $[Z, X] = 0$ for any $Z \in \mathfrak{n}(\mathfrak{g})$. Consequently, $X \in C(\mathfrak{n}(\mathfrak{g}))$. ■

Corollary 2 *Under conditions of Theorem 5, any abelian ideal $\mathfrak{a}$ in $\mathfrak{g}$ is in $C(\mathfrak{n}(\mathfrak{g}))$. In particular, $C(\mathfrak{n}(\mathfrak{g}))$ is a maximal abelian ideal in $\mathfrak{g}$ by inclusion.*

Proof It is clear that $\mathfrak{a}$ is a nilpotent ideal in $\mathfrak{g}$, hence $\mathfrak{a} \subset \mathfrak{n}(\mathfrak{g})$. By Proposition 4, $\mathfrak{a}$ consists of Killing fields of constant length, hence, $\mathfrak{a} \subset C(\mathfrak{n}(\mathfrak{g}))$ by Theorem 5. ■

Remark 5 It should be recalled that there are many examples of geodesic orbit nilmanifolds [12]. Therefore, Theorems 4 and 5 give non-trivial examples X of KVFCL on $(M = G/H, g)$, where $X \in C(\mathfrak{n}(\mathfrak{g}))$. For any such example, the operator $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is non semisimple, since it is nilpotent. In this case, $A_1 = \mathfrak{g}$ and $L^2 = (\mathrm{ad}(X))^2 = 0$.

Let us recall Problem 2 in [20]: *Classify geodesic orbit Riemannian spaces with nontrivial Killing vector fields of constant length*. Now, this problem is far from being resolved. We have one modest result in this direction.

Proposition 5 (Proposition 7 in [22]) *Let $(G/H, g)$ be a geodesic orbit space and $X \in \mathfrak{g} = \mathrm{Lie}(G)$. Then the following conditions are equivalent:*

(1) X has constant length on $(G/H, g)$ and $A_1 = \mathrm{Ker}(L^2) = \mathfrak{g}$;
(2) X is in the center $C(\mathfrak{n}(\mathfrak{g}))$ of $\mathfrak{n}(\mathfrak{g})$.

Proof (1) $\Rightarrow$ (2). By Proposition 1, we get $X \in \mathfrak{n}(\mathfrak{g})$. Hence, $X \in C(\mathfrak{n}(\mathfrak{g}))$ by Theorem 5.

(2) $\Rightarrow$ (1). By Theorem 5, X has constant length on $(G/H, g)$. Since $\mathfrak{n}(\mathfrak{g})$ is an ideal in $\mathfrak{g}$, then $[X, Y] \in \mathfrak{n}(\mathfrak{g})$ and $L^2(Y) = [X, [X, Y]] \in [C(\mathfrak{n}(\mathfrak{g})), \mathfrak{n}(\mathfrak{g})] = 0$ for all $Y \in \mathfrak{g}$. ■

4 Algebraic Properties of Bounded Killing Vector Fields

In this section, we discuss results obtained recently in [29]. Results from the previous sections lead to the following conjecture.

Conjecture 1 (Conjecture 1 in [22]) If $\mathfrak{g}$ is semisimple, then any Killing field of constant length $X \in \mathfrak{g}$ on (M, g) is a compact vector in $\mathfrak{g}$, i. e. the Lie algebra $\mathbb{R} \cdot X$ is compactly embedded in $\mathfrak{g}$.

Initial motivation of the authors of [29] was to prove this conjecture. For this goal, they applied a special new tool based on the study of *bounded Killing vector fields*.

Recall that a Killing vector field on a Riemannian manifold is called bounded if *its length function with respect to the given metric is a bounded function*. This condition is relatively weak. For example, any Killing vector field on a compact Riemannian manifold is bounded. On the other hand, curvature conditions may provide serious restrictions for bounded Killing vector fields. For instance, on a complete negatively curved Riemannian manifold, bounded Killing vector field must be zero [27]. On a complete non-positively curved Riemannian manifold, a bounded Killing vector field must be parallel [5].

The following result proves Conjecture 1 (in particular).

Theorem 6 (Theorem 1.2 in [29]) *Let M be a connected Riemannian manifold on which a connected semisimple Lie group G acts effectively and isometrically. Assume $X \in \mathfrak{g}$ defines a bounded Killing vector field. Then X is contained in a compact ideal in $\mathfrak{g}$.*

It is natural to further study this spectral property of bounded Killing vector fields when G is not semisimple. For this purpose, one can consider a Levi decomposition $\mathfrak{g} = \mathfrak{r}(\mathfrak{g}) + \mathfrak{s}$ for $\mathfrak{g} = \mathrm{Lie}(G)$ and the corresponding decomposition $X = X_r + X_s$. The following result gives some details on algebraic properties of the bounded Killing vector fields. For a semisimple Lie algebra $\mathfrak{s}$ we consider the direct sum $\mathfrak{s} = \mathfrak{s}_c \oplus \mathfrak{s}_{nc}$ of compact and noncompact ideals.

Theorem 7 (Theorem 1.3 in [29]) *Let M be a connected Riemannian manifold on which the connected Lie group G acts effectively and isometrically. Assume that $X \in \mathfrak{g}$ defines a bounded Killing vector field, and $X = X_r + X_s$ according to the Levi decomposition $\mathfrak{g} = \mathfrak{r}(\mathfrak{g}) + \mathfrak{s}$, where $\mathfrak{s} = \mathfrak{s}_c \oplus \mathfrak{s}_{nc}$. Then we have the following:*

(1) *The vector $X_s \in \mathfrak{s}$ is contained in the compact semisimple ideal $\mathfrak{c}_{\mathfrak{s}_c}(\mathfrak{r}(\mathfrak{g}))$ of $\mathfrak{g}$;*
(2) *The vector $X_r \in \mathfrak{r}$ is contained in the center $\mathfrak{c}(\mathfrak{n})$ of $\mathfrak{n}$.*

Here the centralizer $\mathfrak{c}_{\mathfrak{a}}(\mathfrak{b})$ of the subalgebra $\mathfrak{b} \subset \mathfrak{g}$ in the subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is defined as $\mathfrak{c}_{\mathfrak{a}}(\mathfrak{b}) = \{u \in \mathfrak{a} \,|\, [u, \mathfrak{b}] = 0\}$. In particular, the center $\mathfrak{c}(\mathfrak{a})$ of $\mathfrak{a} \subset \mathfrak{g}$ coincides with $\mathfrak{c}_{\mathfrak{a}}(\mathfrak{a})$.

Theorem 7 helps us find more algebraic properties for bounded Killing vector fields. In particular, $X = X_r + X_s$ is an abstract Jordan decomposition which is irrelevant to the choice of the Levi subalgebra $\mathfrak{s}$, and the eigenvalues of $\mathrm{ad}(X)$ coincide with those of $\mathrm{ad}(X_s)$, which are all imaginary.

Theorem 8 (Theorem 4.1 in [29]) *Let M be a connected Riemannian manifold on which the connected Lie group G acts effectively and isometrically. Assume that $X \in \mathfrak{g}$ defines a bounded Killing vector field. Let X be decomposed as $X = X_r + X_s$ according to any Levi decomposition $\mathfrak{g} = \mathfrak{r}(\mathfrak{g}) + \mathfrak{s}$, then we have the following:*

(1) *The decomposition* $\mathrm{ad}(X) = \mathrm{ad}(X_r) + \mathrm{ad}(X_s)$ *is the unique Jordan–Chevalley decomposition for* $\mathrm{ad}(X)$ *in* $\mathfrak{gl}(\mathfrak{g})$*;*

(2) *The decomposition* $X = X_r + X_s$ *is the abstract Jordan decomposition which is unique in the sense that* X_s *is contained in all Levi subalgebras, i.e. this decomposition is irrelevant to the choice of the Levi subalgebra;*
(3) *The eigenvalues of* $\mathrm{ad}(X)$ *coincide with those of those of* $\mathrm{ad}(X_s)$*, counting multiples.*

As a direct corollary, we obtain the following spectral property (that is an obvious generalization of Theorem 1).

Corollary 3 *Let* M *be a connected Riemannian manifold on which the connected Lie group* G *acts effectively and isometrically. Assume that* $X \in \mathfrak{g}$ *defines a bounded Killing vector field. Then all eigenvalues of* $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ *are imaginary.*

When $M = G/H$ is a Riemannian homogeneous space on which the connected Lie group G acts effectively, we can apply Theorem 7 to prove Theorem 9, which completely determine all bounded vectors in $\mathfrak{g}$ for G/H, or equivalently all bounded Killing vector fields induced by vectors in $\mathfrak{g}$. Let us recall necessary definition.

For any smooth coset space G/H, where H is a closed subgroup of G, $\mathrm{Lie}(G) = \mathfrak{g}$ and $\mathrm{Lie}(H) = \mathfrak{h}$, we denote $\mathrm{pr}_{\mathfrak{g}/\mathfrak{h}}$ the natural linear projection from $\mathfrak{g}$ to $\mathfrak{g}/\mathfrak{h}$. We call any vector $X \in \mathfrak{g}$ a *bounded vector* for G/H, if

$$f(g) = \|\mathrm{pr}_{\mathfrak{g}/\mathfrak{h}}\big(\mathrm{Ad}(g)X\big)\|, \quad \forall\, g \in G, \tag{4}$$

is a bounded function, where $\|\cdot\|$ is any norm on $\mathfrak{g}/\mathfrak{h}$.

Since the space $\mathfrak{g}/\mathfrak{h}$ has a finite dimension, any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on it are equivalent in the sense that

$$c_1\|u\|_1 \le \|u\|_2 \le c_2\|u\|_1, \quad \forall\, u \in \mathfrak{g}/\mathfrak{h},$$

where c_1 and c_2 are some positive constants. So the boundedness of $X \in \mathfrak{g}$ for G/H does not depend on the choice of the norm.

When $\|\cdot\|$ is an $\mathrm{Ad}(H)$-invariant quadratic norm, which defines a G-invariant Riemannian metric on G/H, the function $f(\cdot)$ on G defined in (4) is right H-invariant, so it can be descended to G/H, and coincides with the length function of the Killing vector field induced by X. Hence, we have

Lemma 4 *If* $X \in \mathfrak{g}$ *is a bounded vector for* G/H*, then it defines a bounded Killing vector field for any* G*-invariant Riemannian metric on* G/H*. Conversely, if* G/H *is endowed with a* G*-invariant Riemannian metric and* $X \in \mathfrak{g}$ *induces a bounded Killing vector field, then* X *is a bounded vector for* G/H*.*

We have the following characterization of bounded vectors.

Theorem 9 (Theorem 1.5 in [29]) *Let* G/H *be a Riemannian homogeneous space on which the connected Lie group* G *acts effectively. Let* $\mathfrak{r}(\mathfrak{g})$*,* $\mathfrak{n}(\mathfrak{g})$ *and* $\mathfrak{s} = \mathfrak{s}_c \oplus \mathfrak{s}_{nc}$ *be the radical, the nilradical, and the Levi subalgebra respectively. Then the space of all bounded vectors in* $\mathfrak{g}$ *for* G/H *is a compact subalgebra. Its semisimple part coincides with the ideal* $\mathfrak{c}_{\mathfrak{s}_c}(\mathfrak{r}(\mathfrak{g}))$ *of* $\mathfrak{g}$*, which is independent of the choice of the*

Levi subalgebra $\mathfrak{s}$*, and its abelian part* $\mathfrak{v}$ *is contained in* $\mathfrak{c}(\mathfrak{n}(\mathfrak{g}))$*, which coincides with the sum of* $\mathfrak{c}_{\mathfrak{c}(\mathfrak{r}(\mathfrak{g}))}(\mathfrak{s}_{nc})$ *and all two-dimensional irreducible representations of* $\mathrm{ad}(\mathfrak{r}(\mathfrak{g}))$ *in* $\mathfrak{c}_{\mathfrak{c}(\mathfrak{n}(\mathfrak{g}))}(\mathfrak{s}_{nc})$ *corresponding to nonzero imaginary weights, i.e.* $\mathbb{R}$*-linear functionals* $\lambda : \mathfrak{r} \to \mathfrak{r}/\mathfrak{n} \to \mathbb{R}\sqrt{-1}$.

Note that $\mathfrak{c}_{\mathfrak{s}_c}(\mathfrak{r}(\mathfrak{g}))$ is a compact semisimple summand in the Lie algebra direct sum decomposition of $\mathfrak{g}$, which can be easily determined. On the other hand, for the abelian factor $\mathfrak{v}$, there is a theoretic algorithm which explicitly describes all bounded vectors in $\mathfrak{c}(\mathfrak{n}(\mathfrak{g}))$ (see Section 4 in [29]).

Theorem 9 provides a simple proof of the following theorem.

Theorem 10 (Theorem 1.6 in [29]) *The space of bounded vectors in* $\mathfrak{g}$ *for a Riemannian homogeneous space* G/H *on which the connected Lie group* G *acts effectively is irrelevant to the choice of* H.

Notice that the arguments in [26] indicate that the subset of all bounded isometries in G is irrelevant to the choice of H. So Theorem 10 can also be proved by J. Tits' Theorem 1 in [26], which implies that all bounded isometries in G are generated by bounded vectors in $\mathfrak{g}$.

It should be noted that all lemmas, theorems and corollaries of this section are still valid when M is a *Finsler manifold.*

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The Radon-Nikodým Theorem for Disjointness Preserving Orthogonally Additive Operators

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Abstract In this article we prove the Radon-Nikodým type theorem for positive disjointness preserving orthogonally additive operators defined on a vector lattice E and taking values in a Dedekind complete vector lattice F.

Keywords Orthogonally additive operator · Positive operator · Disjointness preserving operator · Vector lattice

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1 Introduction and Preliminaries

Orthogonally additive operators in vector lattices first were introduced in [10]. Today the theory of these operators is an area of intense research (see for instance [1, 6, 11–15]). The aim of this notes is to continue this line of investigation. We prove a Radon-Nikodým theorem for positive disjointness preserving orthogonally additive operators.

In this section we present some basic facts concerning vector lattices. For the standard information we refer to [3]. All vector lattices below are assumed to be Archimedean.

Let E be a vector lattice. A net $(x_\alpha)_{\alpha\in\Lambda}$ in E *order converges* to an element $x \in E$ (notation $x_\alpha \xrightarrow{\text{o}} x$) if there exists a net $(e_\alpha)_{\alpha\in\Lambda}$ in E_+ such that $e_\alpha \downarrow 0$ and $|x_\alpha - x| \leq e_\alpha$ for all $\alpha \in \Lambda$ satisfying $\alpha \geq \alpha_0$ for some $\alpha_0 \in \Lambda$. For an element

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$x \in E$ the *band generated by* x is the set

$$B_x = \{y \in E \colon |y| \wedge n|x| \uparrow |y|\}.$$

Two elements x, y of the vector lattice E are said to be *disjoint* (notation $x \perp y$), if $|x| \wedge |y| = 0$. For an element $x \in E$ its disjoint complement is defined by $\{x\}^{\perp} = \{y \in E \colon |y| \wedge |x| = 0\}$. The band B_x is then $\{x\}^{\perp\perp}$.

The equality $x = \bigsqcup_{i=1}^{n} x_i$ means that $x = \sum_{i=1}^{n} x_i$ and $x_i \perp x_j$ for all $i \neq j$. If $n = 2$ we use the notation $x = x_1 \sqcup x_2$. An element y of E is called a *fragment* of an element $x \in E$ (or a *component*), if $y \perp (x - y)$. The notation $y \sqsubseteq x$ means that y is a fragment of x. The set of all fragments of the element $x \in E$ is denoted by $\mathcal{F}_x$.

Definition 1.1 Let E be a vector lattice, and let F be a real vector space. An operator $T : E \to F$ is called *orthogonally additive* if $T(x + y) = Tx + Ty$ for every disjoint elements $x, y \in E$.

It is clear that $T(0) = 0$. The set of all orthogonally additive operators is a real vector space with respect to the natural linear operations.

Definition 1.2 Let E and F be vector lattices. An orthogonally additive operator $T : E \to F$ is said to be:

- *positive* if $Tx \geq 0$ holds in F for all $x \in E$;
- *disjointness preserving, if* $x \perp y$ *implies* $Tx \perp Ty$;
- *order bounded*, if it maps order bounded sets in E to order bounded sets in F;
- *laterally-to-order bounded*, if the set $T(\mathcal{F}_x)$ is order bounded in F for every $x \in E$.
- *regular*, if $T = S_1 - S_2$, where S_1 and S_2 are positive orthogonally additive operators from E to F.

The sets of all positive, regular and laterally-to-order bounded orthogonally additive operators from E to F are denoted by $\mathcal{OA}_+(E, F)$, $\mathcal{OA}_r(E, F)$ and $\mathcal{P}(E, F)$, respectively. The order in $\mathcal{P}(E, F)$ is introduced as follows: $S \leq T$ whenever $(T - S) \geq 0$. Then $\mathcal{P}(E, F)$ becomes an ordered vector space. For a Dedekind complete vector lattice F we have the following properties of $\mathcal{OA}_r(E, F)$ and $\mathcal{P}(E, F)$.

Lemma 1.3 ([12, Theorem 3.6]) *Let E and F be a vector lattices with F Dedekind complete. Then* $\mathcal{P}(E, F) = \mathcal{OA}_r(E, F)$ *and* $\mathcal{OA}_r(E, F)$ *is a Dedekind complete vector lattice. Moreover, for every* $S, T \in \mathcal{OA}_r(E, F)$ *and every* $x \in E$ *the following formulas hold*

(1) $(T \vee S)(x) = \sup\{Ty + Sz : x = y \sqcup z\}$;
(2) $(T \wedge S)(x) = \inf\{Ty + Sz : x = y \sqcup z\}$;
(3) $T^+(x) = \sup\{Ty : y \sqsubseteq x\}$;
(4) $T^-(x) = -\inf\{Ty : y \sqsubseteq x\}$;
(5) $|T|(x) = \sup\{Ty - Tz \colon x = y \sqcup z\}$
(6) $|Tx| \leq |T|(x)$.

2 Result

The Radon-Nikodým Theorem is the well-known classical result in the measure theory.

In the theory of integration the calculation of an integral with respect to a measure ν can be reduced to the calculation of the integral with respect to a given measure μ if the former possesses a density with respect to μ (sometimes called *Radon-Nikodým derivative*). The cited Radon-Nikodým Theorem ([9], chapt. XI.2, [2], chapt. 10.6) tells us that this is true for any finite measure ν which is *absolutely continuous* with respect to a σ-finite measure μ, i.e. if $\nu(A) = 0$ whenever $\mu(A) = 0$. We remark that for finite measures ν and μ the first one is absolutely continuous with respect to μ if and only if $\nu \in \{\mu\}^{\perp\perp}$ (see the detailed explanation in [19], (Chap. 14)).

Some generalizations of the Radon-Nikodým Theorem in the framework of the theory of linear operators on Banach spaces and on vector lattices were obtained in [4, 5, 7, 8, 16–18]. In particular, in [7] a positive operator $S: E \to F$ is defined to be *absolutely continuous* with respect to the operator $T: E \to F$ if $Sx \in \{Tx\}^{\perp\perp}$ for any $x \in E_+$, where the vector lattices E, F are Dedekind complete. The equivalence between absolute of a positive linear operator S with respect to a lattice homomorphism T and the condition $S \in \{T\}^{\perp\perp}$ was proved in [5]. We remark that this connection is considered as the operator version of the Radon-Nikodým Theorem. The aim of this note is to prove the following Radon-Nikodým Theorem for (nonlinear) orthogonally additive operators.

Theorem 2.1 *Let E, F be vector lattices with F Dedekind complete, $T \in \mathcal{OA}_+(E, F)$ be a disjointness preserving operator and $S \in \mathcal{OA}_+(E, F)$. Then the following statements are equivalent:*

(1) $S \in \{T\}^{\perp\perp}$;
(2) $Sx \in \{Tx\}^{\perp\perp}$ *for all* $x \in E$.

Now, we prove that like in the linear case, the module, the positive part and the negative part of a disjointness preserving orthogonally additive operator can be evaluated pointwise.

Lemma 2.2 *Let E and F be vector lattices, with F Dedekind complete and $T: E \to F$ be a disjointness preserving operator. Then $T \in \mathcal{OA}_r(E, F)$ and for every $x \in E$ the following conditions hold:*

(1) $|T|x = |Tx|$;
(2) $T^+x = (Tx)^+$;
(3) $T^-x = (Tx)^-$;
(4) $T^+x \wedge T^-x = 0$.

Proof First we prove that $T \in \mathcal{OA}_r(E, F)$. By Lemma 1.3 it is enough to show that $T \in \mathcal{P}(E, F)$. Fix $x \in E$. We claim that the set $T(\mathcal{F}_x)$ is order bounded in F. Indeed, for every $y \in \mathcal{F}_x$ we have $(x - y) \perp y \Rightarrow T(x - y) \perp Ty$. Since T

preserves disjointness[1] there is also

$$|Ty| \leq |T(x-y)| + |Ty| = |T(x-y) + Ty| = |Tx|.$$

Hence T is a laterally-to-order bounded operator and therefore $T \in \mathcal{OA}_r(E, F)$. Now we prove the equality $|T|x = |Tx|$ for all $x \in E$. By Lemma 1.3 (5) we have

$$|T|x = \sup\{Ty - Tz : x = y \sqcup z\} \geq Tx \vee (-Tx) = |Tx|.$$

We need to prove the reverse inequality. Take $y, z \in E$ such that $x = y \sqcup z$. Then $Ty \perp Tz$ and we may write

$$Ty - Tz \leq |Ty - Tz| = |Ty + Tz| = |T(y+z)| = |Tx|.$$

Passing to the supremum on the left-hand side of the above inequality over all fragments y, z of x such that $x = y \sqcup z$ we deduce that $|T|x \leq |Tx|$. Thus $|T|x = |Tx|$ for any $x \in E$. Since

$$T^+ = \frac{1}{2}(|T| + T) \text{ and } T^- = \frac{1}{2}(|T| - T)$$

we may write

$$T^+x = \frac{1}{2}(|T| + T)x = \frac{1}{2}(|T|x + Tx) = \frac{1}{2}(|Tx| + Tx) = (Tx)^+;$$
$$T^-x = \frac{1}{2}(|T| - T)x = \frac{1}{2}(|T|x - Tx) = \frac{1}{2}(|Tx| - Tx) = (Tx)^-.$$

Finally, we get

$$T^+x \wedge T^-x = (Tx)^+ \wedge (Tx)^- = 0.$$

□

For the proof of the main result we need the following auxiliary propositions.

Proposition 2.3 *Let E, F be vector lattices with F Dedekind complete, $S, T \in \mathcal{OA}_+(E, F)$ be disjointness preserving operators. Then $T + S$ is a disjointness preserving operator if and only if $Sx \perp Ty$ for every disjoint elements $x, y \in E$.*

Proof Fix a pair of disjoint elements $x, y \in E$ and assume that $T + S$ is a disjointness preserving operator. Then due to the positivity of S and T we have

$$0 \leq |Sx| \wedge |Ty| = Sx \wedge Ty \leq (S+T)x \wedge (S+T)y = 0.$$

[1] And by using the relation $|a + b| = |a| + |b|$ for $a \perp b,\ a, b, \in E$.

Hence $Sx \perp Ty$. Now assume that $Sx \perp Ty$ for every disjoint elements $x, y \in E$. Note that the assumptions ensure that

$$Sx \wedge Sy = Tx \wedge Ty = Sx \wedge Ty = Sy \wedge Tx = 0.$$

Thus

$$0 \leq (S+T)x \wedge (S+T)y = (Sx + Tx) \wedge (Sy + Ty) \leq$$
$$(Sx \wedge Sy) + (Tx \wedge Sy) + (Sx \wedge Ty) + (Tx \wedge Ty) = 0.$$

□

Proposition 2.4 *Let E, F be vector lattices with F Dedekind complete, $T \in \mathcal{OA}_+(E, F)$ be a disjointness preserving operator and $S \in \mathcal{OA}_+(E, F)$ satisfies $S \in \{T\}^{\perp\perp}$. Then $Sx \in \{Tx\}^{\perp\perp}$ for all $x \in E$.*

Proof Since $0 \leq S \in \{T\}^{\perp\perp}$ we have that the sequence $R_n := S \wedge nT$, $n \in \mathbb{N}$ increases and order converges to S in the vector lattice $\mathcal{OA}_r(E, F)$. Thus $(S \wedge nT)x \uparrow Sx$ for every $x \in E$. Then by using the order closedness of bands

$$0 \leq (S \wedge nT)x \leq nTx \Rightarrow (S \wedge nT)x \in \{Tx\}^{\perp\perp}$$

we deduce that $Sx \in \{Tx\}^{\perp\perp}$. □

Proposition 2.5 *Let E, F be vector lattices with F Dedekind complete, $T \in \mathcal{OA}_+(E, F)$ be a disjointness preserving operator and $S \in \mathcal{OA}_+(E, F)$ satisfies $Sx \in \{Tx\}^{\perp\perp}$ for all $x \in E$. Then S is a disjointness preserving operator.*

Proof If $x, y \in E$, $x \perp y$ then for any $m, n \in \mathbb{N}$ we have

$$0 \leq (Sx \wedge nTx) \wedge (Sy \wedge mTy) \leq (n+m)(Tx \wedge Ty) = 0$$

and hence

$$0 \leq (Sx \wedge nTx) \wedge (Sy \wedge mTy) = 0.$$

Since $Sx \in \{Tx\}^{\perp\perp}$ and $Sy \in \{Ty\}^{\perp\perp}$ we have $Sx \wedge nTx \uparrow Sx$ and $Sy \wedge mTy \uparrow Sy$. Therefore $Sx \wedge Sy = 0$. □

Remark 2.6 We observe that by Proposition 2.4 for a disjointness preserving operator $T \in \mathcal{OA}_+(E, F)$, the inclusion $0 \leq S \in \{T\}^{\perp\perp}$ implies that S is a disjointness preserving operator too.

Proposition 2.7 *Let E, F be vector lattices with F Dedekind complete. Let $T \in \mathcal{OA}_+(E, F)$ be a disjointness preserving operator and $0 \leq S_1, S_2 \in \{T\}^{\perp\perp}$. Then*

$$(S_1 \wedge S_2)x = S_1x \wedge S_2x.$$

Proof Let be $S_1' = S_1 - S_1 \wedge S_2$, $S_2' = S_2 - S_1 \wedge S_2$ and $S = S_1' - S_2'$. Then $S_1', S_2', |S| \in \{T\}^{\perp\perp}$ and $S_1' \wedge S_2' = 0$. It follows that $S^+ = S_1'$ and $S^- = S_2'$. By the Remark 2.6 the operator $|S|$ is disjointness preserving and so, by Lemma 2.2, we get $S_1'x \wedge S_2'x = 0$ for every $x \in E$. This means

$$\big(S_1x - (S_1 \wedge S_2)x\big) \wedge \big(S_2x - (S_1 \wedge S_2)x\big) = 0,$$

and we deduce that

$$(S_1 \wedge S_2)x = S_1x \wedge S_2x.$$

for all $x \in E$. □

Corollary 2.8 *Let E, F be vector lattices with F Dedekind complete, $S_1, S_2 \in \mathcal{OA}_+(E, F)$ and $S_1 + S_2$ be a disjointness preserving operator. Then*

$$S_1 \wedge S_2 = 0 \Leftrightarrow S_1x \wedge S_2x = 0 \quad \textit{for all} \quad x \in E.$$

Proof Due to the positivity of S_1, S_2 we have $S_1, S_2 \leq S_1 + S_2$. Hence applying Proposition 2.7 to $S_1 + S_2$ we finish the proof. □

Now we are ready to prove the main result.

Proof of Theorem 2.1 The implication (1) $\Rightarrow$ (2) is proved in Proposition 2.4.

We prove the implication (2) $\Rightarrow$ (1). By Proposition 2.5 the operator S is disjointness preserving. We claim that $Sx \perp Ty$ for any disjoint elements $x, y \in E$. Actually $Tx \wedge Ty = 0$, hence $\{Tx\}^{\perp\perp} \cap \{Ty\}^{\perp\perp} = \{0\}$. Since $Ty \in \{Ty\}^{\perp\perp}$ and by assumption $Sx \in \{Tx\}^{\perp\perp}$, according to Proposition 2.3, the operator $S + T$ is disjointness preserving. Taking into account the decomposition

$$\mathcal{OA}_r(E, F) = \{T\}^{\perp\perp} \oplus \{T\}^{\perp}$$

there is a representation $S = S_1 + S_2$, with $0 \leq S_1 \in \{T\}^{\perp\perp}$ and $0 \leq S_2 \in \{T\}^{\perp}$. Since $0 \leq S_2 + T \leq S + T$ the operator $S_2 + T$ is also disjointness preserving. By Corollary 2.8 we have that $S_2x \in \{Tx\}^{\perp}$.

On the other hand from $0 \leq S_2x \leq Sx$ and $Sx \in \{Tx\}^{\perp\perp}$ for all $x \in E$ we deduce that $S_2x \in \{Tx\}^{\perp\perp}$, what implies $S_2x = 0$ for all elements $x \in E$ and therefore $S_2 = 0$. Thus $S = S_1 \in \{T\}^{\perp\perp}$ and the proof is finished. □

We remark that the Radon-Nikodým type theorem for order bounded positive disjointness preserving orthogonally additive operators was proved in [1].

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On the Continuous Linear Right Inverse for a Convolution Operator

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Abstract We consider a surjective convolution operator in the Beurling space of ultradifferentiable functions of mean type on the real axis. We obtain necessary and sufficient conditions on the symbol of the operator under which it has a continuous linear right inverse.

Keywords Spaces of ultradifferentiable functions · Convolution operator · Continuous linear right inverse

Mathematical Subject Classification (2010) 44A35, 46E10

1 Introduction

The problem of the existence of a continuous linear right inverse operator (briefly, CLRI) for a convolution one was posed by L. Schwartz (see, e.g., [1]) in the particular case of linear partial differential operators on spaces of C^∞-functions. Later it was studied intensively for spaces of analytic and real analytic functions by many authors (see [2, 3] and references therein). On the other hand, the case of C^∞-functions and especially ultradifferentiable functions which are very important in applications have not been studied much. In this relation, recall that the theory of ultradifferentiable functions (briefly, UDF) and ultradistributions was developed by Roumier [4], Beurling and Björk [5] in early 1960s and later essentially developed by Braun, Meise, and Taylor in [6].

Returning to the problem of the existence of CLRI for convolution operators, note that in [7] Meise and Vogt solved the Schwartz problem for nonquasianalytic Beurling spaces $\mathcal{E}_{(\omega)}(\mathbb{R})$ of UDF having, in a certain sense, a maximal type

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with respect to some weight function ω. Later this problem was studied for nonquasianalytic spaces $\mathcal{E}_{(\omega)}(\Omega)$ on an open set $\Omega \subset \mathbb{R}^N$ [8] and quasianalytic ones $\mathcal{E}_{(\omega)}(\mathbb{R})$ [9] of the same maximal type. Similarly to the growth theory of entire functions, UDF of mean type satisfy sharper estimates than functions of maximal one. This gives us an opportunity to obtain sharper solutions of some problems in spaces of UDF $\mathcal{E}^p_{(\omega)}(\Omega)$ of mean type $p \in (0, \infty)$. For more details see [10] where an analog of Borel's extension theorem for $\mathcal{E}^p_{(\omega)}(\mathbb{R}^N)$ was established, and [11] where the Schwartz problem for the space $\mathcal{E}^1_{(\omega)}(I)$ on a bounded interval $I \subset \mathbb{R}$ was completely solved. It should be noted that in studying spaces of mean type it is enough to consider the case $p = 1$ (see [10]).

This paper is devoted to the Schwartz problem for the Beurling space $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ of UDF of mean type on $\mathbb{R}$. It should be noted that in our previous papers a systematical study of convolution operators T_μ on $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ given by their symbols μ that are the multipliers of the Fourier–Laplace representation of the dual space to $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ was undertaken. In details, in [12] those symbols μ were completely characterised that generate surjective operators T_μ on $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ or, in other words, there were established some conditions on μ under which the convolution equation

$$T_\mu f = g \tag{1}$$

has a solution for every $g \in \mathcal{E}^1_{(\omega)}(\mathbb{R})$. After that in [13] a particular solution of (1) having an explicit form was constructed. At last, in [14] a form of a general solution of the corresponding homogeneous equation $T_\mu f = 0$ has been established. For these reasons, the present paper can be considered as a final point in the study of convolution operators on $\mathcal{E}^1_{(\omega)}(\mathbb{R})$. On the other hand, it is an extension of the paper [7] from the case of the Beurling UDF of maximal type to the ones of mean type. It should be noted that the case $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ has some essential differences from both $\mathcal{E}_{(\omega)}(\mathbb{R})$ and $\mathcal{E}^1_{(\omega)}(I)$. To see this it is enough to observe that an important role in the Schwartz problem for UDF belongs to the structure of the set of entire multipliers of the dual spaces. For $\mathcal{E}_{(\omega)}(\mathbb{R})$ and $\mathcal{E}^1_{(\omega)}(I)$ this set has an inductive and, respectively, a projective structure, while for $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ it has a mixed inductive–projective one. Thus, the case treated here has its own interest and novelty.

Note also (see [14]) that Eq. (1) includes as particular cases differential equations of infinite order with constant coefficients

$$\sum_{k=0}^{\infty} a_k f^{(k)}(x) = 0, \quad a_k \in \mathbb{C};$$

difference-differential ones

$$\sum_{k=0}^{m} \sum_{j=0}^{n} a_{kj} f^{(k)}(x + x_j) = 0, \quad m, n \in \mathbb{N}_0, \quad a_{kj} \in \mathbb{C}, \quad x_j \in \mathbb{R};$$

and integro-differential equations

$$\sum_{k=0}^{m}\sum_{j=0}^{n} a_{kj}\int\limits_{\mathbb{R}} f^{(k)}(x+y)\varphi_j(y)\,d\lambda = 0, \quad m,n\in\mathbb{N}_0, \quad a_{kj}\in\mathbb{C},$$

where φ_j, $0 \le j \le n$, are Lebesgue integrable functions that vanish out of some segment almost everywhere.

The structure of the paper is as follows. Section 2 contains all we need about weight functions, spaces and convolution operators that are considered in the paper. Section 3 contains a functional criterion for a convolution operator on $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ to have a CLRI. In Sects. 4 and 5 necessary and, respectively, sufficient conditions for the existence of a CLRI are formulated in terms of the behaviour of the symbols of convolution operators. The final Sect. 6 contains the main result of the paper and some examples are established.

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2 Preliminaries

In the approach of Beurling-Bjorck, spaces of UDF are determined by a weight function. An increasing continuous function $\omega : [0,\infty) \to [0,\infty)$ is called a *weight function* if it satisfies the following conditions:

(α) $\forall p > 1 \quad \exists C > 0 : \omega(x+y) \le p\big(\omega(x)+\omega(y)\big) + C, \quad x, y \ge 0;$
(α') $\omega(t) = O(t), \quad t \to \infty;$
(γ) $\ln t = o\big(\omega(t)\big), \quad t \to \infty;$
(δ) $\varphi_\omega(x) := \omega(e^x)$ is convex on $[0,\infty)$.

If, in addition, the weight ω has the property

(β) $\int_1^\infty \frac{\omega(t)}{t^2}\,dt < \infty,$

then it is called *nonquasianalytic*; otherwise—*quasianalytic*. The examples of nonquasianalytic weight functions are

$$\omega(t) = \ln^\beta(1+t), \quad \beta > 1; \qquad \omega(t) = \frac{t}{\ln^\beta(e+t)}, \quad \beta > 1;$$

$$\omega(t) = t^{\rho(t)}, \quad \text{where } \rho(t) \to \rho \in (0,1) \text{ is some proximate order;}$$

and the quasianalytic ones are

$$\omega(t) = \frac{t}{\ln^\beta(e+t)}, \quad 0 < \beta \le 1; \qquad \omega(t) = t.$$

We may assume without loss of generality that $\omega(1) = 0$. Put $\omega(z) := \omega(|z|)$, $z \in \mathbb{C}$.

Now let us recall some known properties of weight functions. Obviously,

$$\omega(x+y) \le K\big(\omega(x) + \omega(y) + 1\big), \quad x, y \ge 0, \tag{2}$$

for some $K \ge 1$. Next, there is $A > 0$ such that

$$\omega(t) \le At, \quad t \ge 0. \tag{3}$$

Finally, it is well known that

$$\lim_{r \downarrow 1} \limsup_{t \to \infty} \frac{\omega(rt)}{\omega(t)} = 1. \tag{4}$$

In addition, list some properties of nonquasianalytic weight functions. (see [14] and references therein). Firstly, each nonquasianalytic weight ω satisfies the condition $\omega(t) = o(t)$, $t \to \infty$. Next, as usual, we denote by P_ω the harmonic extension of a nonquasianalytic weight ω to the open upper and lower halfplanes:

$$P_\omega(x+iy) := \begin{cases} \frac{|y|}{\pi} \int\limits_{-\infty}^{\infty} \frac{\omega(t)}{(t-x)^2+y^2}\, dt & \text{for } y \ne 0, \\ \omega(x) & \text{for } y = 0. \end{cases}$$

Then the function P_ω is continuous and subharmonic in $\mathbb{C}$. Moreover, $P_\omega(z) \ge \omega(z)$, $z \in \mathbb{C}$; $P_\omega(iy) = o(y)$, $y \to \infty$.

The Beurling space of UDF of mean type on the real axis is defined as

$$\mathcal{E}^1_{(\omega)}(\mathbb{R}) := \Big\{ f \in C^\infty(\mathbb{R}) \; : \; \forall q \in (0,1), \quad \forall l \in (0,\infty) \quad |f|_{\omega,q,l} := \sup_{j \in \mathbb{N}_0} \sup_{|x| \le l} \frac{|f^{(j)}(x)|}{\exp q\varphi^*_\omega(j/q)} < \infty \Big\}.$$

Here $\varphi^*_\omega(y) = \sup\{xy - \varphi_\omega(x) \, : \, x \ge 0\}$, $y \ge 0$, is the Young conjugate of φ_ω. Functions f from $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ are supposed complex-valued. The space $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ is endowed with the natural topology determined by the family of pre-norms $\big\{ | \cdot |_{\omega,q,l} : q \in (0,1), l \in (0,\infty) \big\}$ and it is an (FS)-space (see [15]). Let $\big(\mathcal{E}^1_{(\omega)}(\mathbb{R})\big)'_\beta$ denote the strong dual space of $\mathcal{E}^1_{(\omega)}(\mathbb{R})$. Then the Fourier-Laplace transform of functionals $F : \varphi \mapsto \widehat{\varphi}(z) := \varphi_x(e^{-ixz})$, $z \in \mathbb{C}$, is a topological isomorphism from

$\left(\mathcal{E}^1_{(\omega)}(\mathbb{R})\right)'_\beta$ onto $H^1_{(\omega)}(\mathbb{C})$, where

$$H^1_{(\omega)}(\mathbb{C}) := \Big\{ f \in H(\mathbb{C}) \mid \exists q \in (0,1), \quad \exists l \in (0,\infty) : \\ \|f\|_{\omega,q,l} := \sup_{z\in\mathbb{C}} \frac{|f(z)|}{\exp\left(q\omega(z) + l|\operatorname{Im} z|\right)} < \infty \Big\}.$$

The space $H^1_{(\omega)}(\mathbb{C})$ endowed with natural inductive topology is a (DFS)-space [15].

As usual, we define a convolution operator in $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ as the conjugate one of a multiplication operator in $H^1_{(\omega)}(\mathbb{C})$. Recall that by Theorem 1 in [12], the set of all multipliers of the space $H^1_{(\omega)}(\mathbb{C})$, i.e. of all entire functions μ with $\mu H^1_{(\omega)}(\mathbb{C}) \subset H^1_{(\omega)}(\mathbb{C})$, coincides with

$$M^1_{(\omega)}(\mathbb{C}) = \left\{ \mu \in H(\mathbb{C}) \mid \forall \varepsilon > 0 \ \exists l > 0 : \ \|\mu\|_{\omega,\varepsilon,l} < \infty \right\}.$$

For each $\mu \in M^1_{(\omega)}(\mathbb{C})$, the multiplication operator $\Lambda_\mu : f \mapsto \mu f$ is a continuous linear operator from $H^1_{(\omega)}(\mathbb{C})$ to $H^1_{(\omega)}(\mathbb{C})$. Then we find the functional $\psi_\mu := F^{-1}(\mu)$ in $\left(\mathcal{E}^1_{(\omega)}(\mathbb{R})\right)'$ and introduce the corresponding *convolution operator* by

$$(T_\mu f)(x) := \langle \psi_\mu, f(x + \cdot) \rangle, \quad x \in \mathbb{R}, \ f \in \mathcal{E}^1_{(\omega)}(\mathbb{R}).$$

Clearly, T_μ acts linearly and continuously in $\mathcal{E}^1_{(\omega)}(\mathbb{R})$. Everywhere below an entire function $\mu \in M^1_{(\omega)}(\mathbb{C})$ generating the operators Λ_μ and T_μ will be called the *symbol* of these operators.

As it was said in the Introduction, some differential operators of infinite order with constant coefficients, difference-differential and integro-differential operators are particular cases of operators T_μ.

The problem of the existence of a CLRI is always studied for surjective operator T_μ. A complete description of surjective convolution operators in $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ was obtained in [12, Theorem 2] (see also [14]). The corresponding result is formulated in terms of the symbol μ. It should be noted that in [12], the case of an arbitrary (quasianalytic or nonquasianalytic) weight function was considered. Also, all multipliers μ from $M^1_{(\omega)}(\mathbb{C})$ were studied. In paper [14] devoted to construction of an absolute basis in $\ker T_\mu$, we only considered multipliers from subclass $\widetilde{M}^1_{(\omega)}(\mathbb{C})$ of $M^1_{(\omega)}(\mathbb{C})$, where

$$\widetilde{M}^1_{(\omega)}(\mathbb{C}) = \left\{ \mu \in H(\mathbb{C}) \mid \exists l_0 > 0 : \ \forall \varepsilon > 0 \quad \|\mu\|_{\omega,\varepsilon,l_0} < \infty \right\}.$$

The necessity of that exchange was discussed in [14]. It was also shown that $\widetilde{M}^1_{(\omega)}(\mathbb{C}) = M^1_{(\omega)}(\mathbb{C})$ for all nonquasianalytic weights as well as weights $\omega(t) = kt$, $k > 0$. Since the present paper is essentially based on paper [14], we also should study symbols μ from $\widetilde{M}^1_{(\omega)}(\mathbb{C})$ only. However, this restriction is not so important because the main result of the paper (Theorem 6) is technically obtained for nonquasianalytic weights.

Note also that, if it is suitable, one can replace $\omega(z)$ by $\omega(\operatorname{Re} z)$ in the definition of spaces $H^1_{(\omega)}(\mathbb{C})$, $M^1_{(\omega)}(\mathbb{C})$, and $\widetilde{M}^1_{(\omega)}(\mathbb{C})$ (see [12, Lemmma 1]).

Now, let ω be an arbitrary (not necessarily nonquasianalytic) weight function and T_μ a surjective convolution operator in $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ with a symbol $\mu \in \widetilde{M}^1_{(\omega)}(\mathbb{C})$. Obviously, only the case of infinite number of zeros of μ is interesting. Indeed, if μ has a finite number of zeros, the kernel $\ker T_\mu$ of the operator T_μ has finite dimension. Thus it is complemented in $\mathcal{E}^1_{(\omega)}(\mathbb{R})$, and T_μ has a CLRI (see Theorem 1 below). Let now $(\lambda_s)_{s=1}^\infty$ be the sequence of all zeros of μ, $|\lambda_s| \uparrow \infty$, and k_s denote the multiplicity of the zero λ_s. In [14], using the surjectivity of an operator T_μ, we constructed a special open covering $(U_j)_{j=1}^\infty$ of the zero set of μ.

Let us recall the properties of that covering which will be used in the sequel. Fix any sequence $\varepsilon_k \downarrow 0$; $\varepsilon_1 < \frac{1}{A}$, where A is the constant from the condition (3). Put $\delta_k = \frac{\varepsilon_k}{64KHel_0}$. Here K is the constant from (2); l_0 is determined by the multiplier μ from $\widetilde{M}^1_{(\omega)}(\mathbb{C})$; $H := 3 + \ln 48$. Each number $k \in \mathbb{N}$ generates in a certain way a number $j_k \in \mathbb{N}$, $j_k \uparrow \infty$ (see [14]). Given sequences (ε_k) and (δ_k), in [14] we constructed the open covering $(U_j)_{j=1}^\infty$ for zeros of μ. In each set U_j it was chosen a certain point z_j by the rule: if U_j ($j_k \le j < j_{k+1}$) contains a point z_j with $|\operatorname{Im} z_j| \le \delta_k \omega(\operatorname{Re} z_j)$, then we fix this z_j; otherwise we fix an arbitrary point z_j from U_j. It was shown that in the first case $\operatorname{diam} U_j \le 12\delta_k\omega(\operatorname{Re} z_j)$; and in the second one $\operatorname{diam} U_j \le 2|\operatorname{Im} z_j|$. Here $\operatorname{diam} U_j = \sup\{\|z-t\| : z, t \in U_j\}$; $\|z\| = \max\{|\operatorname{Re} z|, |\operatorname{Im} z|\}$, $z \in \mathbb{C}$.

Finally, for $j_k \le j < j_{k+1}$, $k \in \mathbb{N}$, put

$$\sigma_j := \min_{z \in \overline{U_j}} \exp\left\{-4\varepsilon_k\omega(\operatorname{Re} z) - 4L_0|\operatorname{Im} z|\right\}. \tag{5}$$

In accordance with [14, Lemma 5], for all $z \in (\partial U_j)(\sigma_j) = \left\{z \in \mathbb{C} : \operatorname{dist}(z, \partial U_j) < \sigma_j\right\}$, the following estimate holds:

$$\ln|\mu(z)| \ge -3\varepsilon_k\omega(\operatorname{Re} z) - 3L_0|\operatorname{Im} z|. \tag{6}$$

Here $L_0 = 112KK_1H(H+1)(H_1+1)el_0$; the constants K, H, and l_0 have been already described above; $H_1 := 3 + \ln\frac{24(1+\beta)}{\beta}$, β is an arbitrary fixed number from $(0, \frac{1}{32}]$ (for example, $\beta = \frac{1}{32}$); the constant $K_1 \ge 1$ is determined by the condition

$$\omega(2s + 8e\eta) \le K_1\big(\omega(s) + \omega(\eta) + 1\big), \ \ s, \eta \ge 0\,.$$

Introduce numbers $m_j := \sum_{\lambda_s \in U_j} k_s$. Then m_j is a number of zeros of μ lying in U_j, $j \in \mathbb{N}$ (with multiplicities).

In conclusion, let us prove the following auxiliary lemma which will be used below.

Lemma 1 *Let ω be an arbitrary weight function, $\mu \in \widetilde{M}^1_{(\omega)}(\mathbb{C})$ a symbol of the surjective convolution operator T_μ in $\mathcal{E}^1_{(\omega)}(\mathbb{R})$, and $(U_j)_{j=1}^{\infty}$ and $(z_j)_{j=1}^{\infty}$ chosen above. If*

$$\lim_{j\to\infty} \frac{|\operatorname{Im} z_j|}{\omega(\operatorname{Re} z_j)} = 0\,, \tag{7}$$

then for any positive numbers q, l, and ε there is $C > 0$ such that, for all $j \in \mathbb{N}$ and $z \in U_j$,

$$q\omega(\operatorname{Re} z_j) + l|\operatorname{Im} z_j| + l|\operatorname{Im} z| \le (q+\varepsilon)\omega(\operatorname{Re} z) + C\,.$$

Proof Fix any q, l, ε in $(0, \infty)$. By the property (4) of the weight ω, we can find $\delta \in \left(0, \frac{1}{A}\right)$ and $C > 0$ so that

$$\left(q + \frac{\varepsilon}{2}\right)\omega\left(\frac{1}{1-\delta A}t\right) \le (q+\varepsilon)\omega(t) + C\,,\ \ t \ge 0\,. \tag{8}$$

Using the above estimates for $\operatorname{diam} U_j$ and condition (7), we conclude that, for some $j_0 \in \mathbb{N}$, $\operatorname{diam} U_j \le \delta\omega(\operatorname{Re} z_j)$, $j \ge j_0$. Then, by (3), $\operatorname{diam} U_j \le \delta A|\operatorname{Re} z_j|$, $j \ge j_0$. Thus, $|\operatorname{Re} z| \ge |\operatorname{Re} z_j| - \operatorname{diam} U_j \ge (1-\delta A)|\operatorname{Re} z_j|$ for all $j \ge j_0$, $z \in U_j$, and, consequently,

$$|\operatorname{Re} z_j| \le \frac{1}{1-\delta A}|\operatorname{Re} z|\,,\ \ j \ge j_0\,,\ \ z \in U_j\,. \tag{9}$$

In [14, Lemma 3] the following estimates for $|\operatorname{Im} z|$, $z \in U_j$, $j \in \mathbb{N}$, were obtained. If the set U_j contains the point z_j with $|\operatorname{Im} z_j| \le \delta_k\omega(\operatorname{Re} z_j)$, then $|\operatorname{Im} z| \le 13\delta_k\omega(\operatorname{Re} z_j)$; otherwise $|\operatorname{Im} z| \le 3|\operatorname{Im} z_j|$. So, in any case,

$$l|\operatorname{Im} z_j| + l|\operatorname{Im} z| \le (l+3)|\operatorname{Im} z_j| + 13l\delta_k\omega(\operatorname{Re} z_j)\,,\ \ z \in U_j\,.$$

Taking into account once again the condition (7) and increasing the number j_0, if it is necessary, we have that

$$l|\operatorname{Im} z_j| + l|\operatorname{Im} z| \le \frac{\varepsilon}{2}\,\omega(\operatorname{Re} z_j)\,,\ \ z \in U_j\,,\ \ j \ge j_0\,. \tag{10}$$

Combining estimates (8)–(10), we finally obtain that, for all $z \in U_j$, $j \ge j_0$,

$$q\omega(\operatorname{Re} z_j) + l|\operatorname{Im} z_j| + l|\operatorname{Im} z| \le \left(q + \frac{\varepsilon}{2}\right)\omega\left(\frac{1}{1-\delta A}\operatorname{Re} z\right) \le (q+\varepsilon)\omega(\operatorname{Re} z) + C\,.$$

This completes the proof. □

3 Functional Criterion

In this section we state a functional criterion for a surjective convolution operator T_μ on $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ to have a CLRI. It is based on the duality theory and is similar to the corresponding one established in [11] for the space $\mathcal{E}^1_{(\omega)}(I)$ on a bounded interval $I \subset \mathbb{R}$. In this relation, we only introduce some necessary notation without details and omit the proof of the criterion.

Denote by J the image $\operatorname{Im} \Lambda_\mu$ of the multiplication operator $\Lambda_\mu : H^1_{(\omega)}(\mathbb{C}) \to H^1_{(\omega)}(\mathbb{C})$, where $\mu \in H^1_{(\omega)}(\mathbb{C})$ is such that the corresponding convolution operator T_μ is surjective from $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ onto $\mathcal{E}^1_{(\omega)}(\mathbb{R})$. Since T_μ is supposed surjective, J is a closed subspace of the space $H^1_{(\omega)}(\mathbb{C})$ (see [12, Theorem 2]). This implies that J and the corresponding quotient space $H^1_{(\omega)}(\mathbb{C})/J$ are (DFS) ones.

In [14, Lemma 7] the space $H^1_{(\omega)}(\mathbb{C})/J$ was isomorphically described as a sequence space k^∞ which is defined as follows. At first, for every $j \in \mathbb{N}$, we introduce the space $H^\infty(U_j)$ of all bounded holomorphic functions in U_j equipped with the norm $\|g\|_{\infty,j} = \sup_{z\in U_j} |g(z)|$. Then, its closed subspace

$$J_j = \left\{ g \in H^\infty(U_j) \ : \ g^{(l)}(\lambda_s) = 0, \quad l = 0, \ldots, k_s - 1, \quad \lambda_s \in U_j \right\}$$

is considered and the corresponding quotient space $X_j := H^\infty(U_j)/J_j$, $j \in \mathbb{N}$. As it was pointed out above, $|\mu(z)|$ is bounded away from zero in some neighbourhood of the boundary ∂U_j of the set U_j. So, any equivalence class $[f]$ in the space X_j that is generated by some function $f \in H^\infty(U_j)$ coincides with the space $\left\{ f + \mu g : g \in H^\infty(U_j) \right\}$. The quotient norm in this space is defined by

$$|||\,[f]\,|||_{\infty,j} := \inf_{f\in[f]} \|f\|_{\infty,j} = \inf_{g\in H^\infty(U_j)} \sup_{z\in U_j} |f(z) + \mu(z)g(z)|.$$

Remark Note that for every $[f] \in X_j$ there exists $f \in [f]$ such that $|||\,[f]\,|||_{\infty,j} = \|f\|_{\infty,j}$. Indeed, given $f_0 \in [f]$, we can find a sequence $(g_n)_{n=1}^\infty$ of functions from J_j so that $\|f_0 + g_n\|_{\infty,j} \to |||\,[f]\,|||_{\infty,j}$ as $n \to \infty$. From this it follows, in particular, that, for some $M > 0$,

$$\|g_n\|_{\infty,j} \leq M \text{ for all } n \geq 1\,,$$

that is, the sequence $(g_n)_{n=1}^\infty$ is bounded in $H^\infty(U_j)$ and, consequently, in $H(U_j)$. Then, by Montel's theorem, there exists a subsequence $(g_{n_k})_{k=1}^\infty$ which converges in $H(U_j)$ to some function g_0. This implies, in particular, that $g_0^{(l)}(\lambda_s) = 0$ ($l = 0, \ldots, k_s - 1$; $\lambda_s \in U_j$). In addition, we have that $\|g_0\|_{\infty,j} \leq M$, that is, $g_0 \in H^\infty(U_j)$. Thus, $g_0 \in J_j$. Putting $f = f_0 + g_0 \in [f]$, we have for every $z \in U_j$

$$|f(z)| = \lim_{k\to\infty} |f_0(z) + g_{n_k}(z)| \leq \lim_{k\to\infty} \|f_0 + g_{n_k}\|_{\infty,j} = |||\,[f]|||_{\infty,j}\,.$$

Hence, $\|f\|_{\infty,j} \leq |||\,[f]|||_{\infty,j}$, which implies that $|||\,[f]|||_{\infty,j} = \|f\|_{\infty,j}$.

Obviously, $\dim X_j = \sum_{\lambda_s \in U_j} k_s = m_j$, $j \in \mathbb{N}$. To this end, put $X := \prod_{j=1}^{\infty} X_j$ and

$$k^{\infty} := \Big\{ \varphi = \big([\varphi_j]\big)_{j=1}^{\infty} \in X \mid \exists q \in (0,1), \quad \exists l \in (0,\infty) : \\ \widetilde{|\varphi|}_{\omega,q,l} = \sup_{j \geq 1} \frac{|||\,[\varphi_j]\,|||_{\infty,j}}{\exp\big(q\omega(\operatorname{Re} z_j) + l|\operatorname{Im} z_j|\big)} < \infty \Big\}.$$

The space k^{∞} is equipped with its natural inductive topology and, with respect to this topology, it is a (DFS) one.

In [14, Lemma 7] it was proved that the map $\rho : [f] \in H^1_{(\omega)}(\mathbb{C})/J \mapsto \big([f|_{U_j}]\big)_{j=1}^{\infty}$ is a topological isomorphism from $H^1_{(\omega)}(\mathbb{C})/J$ onto k^{∞}. Clearly this implies that the map $\widetilde{\rho} : f \in H^1_{(\omega)}(\mathbb{C}) \mapsto \big([f|_{U_j}]\big)_{j=1}^{\infty}$ is a linear continuous surjective operator from $H^1_{(\omega)}(\mathbb{C})$ onto k^{∞}.

Now we are ready to state the functional criterion for T_μ to have a CLRI.

Theorem 1 *Let ω be an arbitrary weight function and T_μ a surjective convolution operator in $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ generated by a symbol $\mu \in \widetilde{M}^1_{(\omega)}(\mathbb{C})$. The following assertions are equivalent:*

(i) *T_μ admits a CLRI;*
(ii) *$\ker T_\mu$ is complemented in $\mathcal{E}^1_{(\omega)}(\mathbb{R})$;*
(iii) *Λ_μ has a continuous linear left inverse operator;*
(iv) *$J = \operatorname{Im} \Lambda_\mu$ is complemented in $H^1_{(\omega)}(\mathbb{C})$;*
(v) *$\widetilde{\rho}$ admits a CLRI.*

Note that, further conditions for the existence of a CLRI for operators T_μ that will be established in the next sections and lead to our main result are based on the equivalence $(i) \Leftrightarrow (v)$.

4 Necessary Conditions

The first step to our main result is to prove necessary conditions in terms of entire functions.

Theorem 2 *Let ω be an arbitrary weight function and T_μ a surjective convolution operator in $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ with a symbol $\mu \in \widetilde{M}^1_{(\omega)}(\mathbb{C})$. If T_μ admits a CLRI, then there is a family of functions $(g_s)_{s=1}^{\infty}$ in $H^1_{(\omega)}(\mathbb{C})$ with $g_s(\lambda_s) = 1$, $s \in \mathbb{N}$, satisfying the condition*

$$(A)\ \ \forall q \in (0,1),\ \forall l \in (0,\infty) \quad \exists \widetilde{q} \in (0,1),\ \exists \widetilde{l} \in (0,\infty),\ \exists C > 0 : \\ \ln|g_s(z)| + q\omega(\operatorname{Re}\lambda_s) + l|\operatorname{Im}\lambda_s| \leq \widetilde{q}\omega(\operatorname{Re} z) + \widetilde{l}|\operatorname{Im} z| + C\,,\ z \in \mathbb{C}\,,\ s \in \mathbb{N}\,. \tag{11}$$

Proof Since T_μ has a CLRI, by Theorem 1, $\widetilde{\rho}$ also has a CLRI $L : k^\infty \to H^1_{(\omega)}(\mathbb{C})$.
Let us consider the following elements in the space k^∞

$$\varphi^j = \big([0], \dots, [0], \underset{j}{[1]}, [0], \dots\big), \;\; j \in \mathbb{N}.$$

Obviously, $|||\,[1]|||_{\infty,j} = 1$, so that for all $q \in (0, 1)$, $l \in (0, \infty)$, and $j \in \mathbb{N}$,

$$\widetilde{|\varphi^j|}_{\omega,q,l} = \exp\big(-q\omega(\operatorname{Re} z_j) - l|\operatorname{Im} z_j|\big).$$

Let $f^j := L(\varphi^j)$, $j \in \mathbb{N}$. Then $f^j \in H^1_{(\omega)}(\mathbb{C})$, $j \in \mathbb{N}$, and $\widetilde{\rho}(f^l) = \big([f^l|_{U_j}]\big)_{j=1}^\infty = \varphi^l$, $l \in \mathbb{N}$. Therefore, $[f^j|_{U_j}] = [1]$, $j \in \mathbb{N}$, so that $f^j(\lambda_s) = 1$, $\lambda_s \in U_j$.

For each $s \in \mathbb{N}$ find $j \in \mathbb{N}$ with $\lambda_s \in U_j$ and put $g_s(z) := f^j(z)$, $z \in \mathbb{C}$. Hence, $g_s(\lambda_s) = 1$, $s \in \mathbb{N}$, and it remains only to check the condition (A).

Fix any $q \in (0, 1)$ and $l \in (0, \infty)$ and choose an arbitrary $q_1 \in (q, 1)$. By [14, Lemma 3], there are $l_1 \in (l, \infty)$ (l_1 can be written explicitly) and $C_1 > 0$ such that

$$q\omega(\operatorname{Re} z) + l|\operatorname{Im} z| \le q_1\omega(\operatorname{Re} z_j) + l_1|\operatorname{Im} z_j| + C_1$$

for all $z \in U_j$, $j \in \mathbb{N}$. Taking there $z = \lambda_s$, we get

$$q\omega(\operatorname{Re} \lambda_s) + l|\operatorname{Im} \lambda_s| \le q_1\omega(\operatorname{Re} z_j) + l_1|\operatorname{Im} z_j| + C_1\,. \tag{12}$$

Next, since k^∞ and $H^1_{(\omega)}(\mathbb{C})$ are (DFS)-spaces and L acts continuously from k^∞ into $H^1_{(\omega)}(\mathbb{C})$, it follows that, for q_1 and l_1, we can find $\widetilde{q} \in (0, 1)$, $\widetilde{l} \in (0, \infty)$, and $C_2 > 0$ such that

$$\|L\varphi\|_{\omega,\widetilde{q},\widetilde{l}} \le C_2 \cdot \widetilde{|\varphi|}_{\omega,q_1,l_1}$$

for all $\varphi \in k^\infty$ with $\widetilde{|\varphi|}_{\omega,q_1,l_1} < \infty$. In particular, $\|L\varphi^j\|_{\omega,\widetilde{q},\widetilde{l}} \le C_2 \cdot \widetilde{|\varphi^j|}_{\omega,q_1,l_1}$, $j \in \mathbb{N}$. Thus,

$$\ln|f^j(z)| \le \widetilde{q}\omega(\operatorname{Re} z) + \widetilde{l}|\operatorname{Im} z| - q_1\omega(\operatorname{Re} z_j) - l_1|\operatorname{Im} z_j| + \ln C_2\,, \;\; z \in \mathbb{C}\,, \;\; j \in \mathbb{N}\,.$$

Taking into account (12), we finally obtain that functions g_s satisfy inequality (11) with $C = C_1 + \ln C_2$. □

The next result contains necessary conditions for the existence of a CLRI in the desired form.

Theorem 3 *Let ω be a nonquasianalytic weight and $(\lambda_s)_{s=1}^\infty$, $|\lambda_s| \uparrow \infty$, the sequence of zeros of a function $\mu \in M^1_{(\omega)}(\mathbb{C})$. If μ generates the surjective* *convolution operator T_μ admitting a CLRI, then*

$$\lim_{s\to\infty} \frac{|\operatorname{Im}\lambda_s|}{\omega(\operatorname{Re}\lambda_s)} = 0\,. \tag{13}$$

Proof By Theorem 2, there is a family of functions $(g_s)_{s=1}^{\infty}$ in $H^1_{(\omega)}(\mathbb{C})$ with $g_s(\lambda_s) = 1$, $s \in \mathbb{N}$, satisfying the condition (A).

Given $\varepsilon > 0$, choose $q \in (1-\varepsilon, 1)$ and $l \in (0, \infty)$. Then we find $q_1 \in (q, 1)$, $l_1 \in (l, \infty)$, and $C_1 > 0$ such that

$$\ln|g_s(z)| + q\omega(\operatorname{Re}\lambda_s) + l|\operatorname{Im}\lambda_s| \le q_1\omega(\operatorname{Re} z) + l_1|\operatorname{Im} z| + C_1 \tag{14}$$

for all $z \in \mathbb{C}$ and $s \in \mathbb{N}$. By the same reason, for q and l_1+2, there are $q_2 \in (q_1, 1)$, $l_2 \in (l_1+2, \infty)$, and $C_2 > 0$ such that

$$\ln|g_s(z)| + q\omega(\operatorname{Re}\lambda_s) + (l_1+2)|\operatorname{Im}\lambda_s| \le q_2\omega(\operatorname{Re} z) + l_2|\operatorname{Im} z| + C_2 \tag{15}$$

for any $z \in \mathbb{C}$, $s \in \mathbb{N}$. Next, from condition (α) on weight ω it follows that for each $q_3 \in (q_2, 1)$ there exists $C_3 > 0$ such that

$$q_2\omega(x+y) \le q_3\big(\omega(x) + \omega(y)\big) + C_3\,,\ \ x, y \ge 0\,. \tag{16}$$

Finally, by the properties of the function P_ω (see Sect. 2),

$$P_\omega(iy) \le |y| + C_4\,,\ \ y \in \mathbb{R}\,, \tag{17}$$

for appropriate $C_4 > 0$.

Applying the Phragmèn-Lindelöf principle [16, 6.5.4], we have for all $u+iv \in \mathbb{C}$ with $v \neq 0$,

$$\ln|g_s(u+iv)| \le \frac{|v|}{\pi}\int_{-\infty}^{\infty} \frac{\ln|g_s(t)|}{(u-t)^2+v^2}\,dt + |v|d_s\,,\ \ s \in \mathbb{N}\,, \tag{18}$$

where $d_s = \limsup_{r\to\infty} \frac{2}{\pi r}\int_0^\pi \ln|g_s(re^{i\theta})|\sin\theta\, d\theta$.

Consider d_s. From (14) it follows that

$$\ln|g_s(re^{i\theta})| \le q_1\omega(r\cos\theta) + l_1 r|\sin\theta| + C_1\,,\ \ \ r > 0\,,\ \ \theta \in [0, 2\pi)\,.$$

Consequently,

$$\frac{2}{\pi r}\int_0^\pi \ln|g_s(re^{i\theta})|\sin\theta\, d\theta \le \frac{4}{\pi r}\big(q_1\omega(r) + C_1\big) + l_1\,,\ \ r > 0\,.$$

Since $\frac{\omega(r)}{r} \to 0$, $r \to \infty$, we get that $d_s \le l_1$, $s \in \mathbb{N}$.

Now estimate the integral term in (18). By (15), we have

$$\ln|g_s(t)| + q\omega(\operatorname{Re}\lambda_s) + (l_1+2)|\operatorname{Im}\lambda_s| \le q_2\omega(t) + C_2\,,\ \ t\in\mathbb{R}\,.$$

Using the last inequality and (16), (17), we finally obtain that

$$\frac{|v|}{\pi}\int_{-\infty}^{\infty}\frac{\ln|g_s(t)|}{(u-t)^2+v^2}\,dt + q\omega(\operatorname{Re}\lambda_s) + (l_1+2)|\operatorname{Im}\lambda_s|$$
$$\le q_2\cdot\frac{|v|}{\pi}\int_{-\infty}^{\infty}\frac{\omega(t)}{(u-t)^2+v^2}\,dt + C_2 =$$
$$= q_2\cdot\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\omega(u+vt)}{t^2+1}\,dt + C_2 \le q_3\cdot\frac{2}{\pi}\int_{0}^{\infty}\frac{\omega(u)+\omega(vt)}{t^2+1}\,dt + C_2 + C_3 =$$
$$= q_3\omega(u) + q_3P_\omega(iv) + C_2 + C_3 \le q_3\omega(u) + |v| + C_5\,,$$

where $C_5 := C_2 + C_3 + C_4$.

Returning to (18), we now get

$$\ln|g_s(u+iv)| \le q_3\omega(u) + (l_1+1)|v| + C_5 - q\omega(\operatorname{Re}\lambda_s) - (l_1+2)|\operatorname{Im}\lambda_s|$$

for each $u+iv\in\mathbb{C}$ with $v\ne 0$ and any $s\in\mathbb{N}$. Since all functions are continuous, this estimate holds in $\mathbb{C}$. Taking $u+iv=\lambda_s$, by $g_s(\lambda_s)=1$, $s\in\mathbb{N}$, we conclude that

$$(q_3-q)\omega(\operatorname{Re}\lambda_s) - |\operatorname{Im}\lambda_s| + C_5 \ge 0\,,\ \ s\in\mathbb{N}\,.$$

Here $q_3 - q < 1 - q < \varepsilon$, so that

$$|\operatorname{Im}\lambda_s| \le \varepsilon\omega(\operatorname{Re}\lambda_s) + C_5\,,\ \ s\in\mathbb{N}\,.$$

Since $\varepsilon>0$ is arbitrary and $|\lambda_s|\uparrow\infty$, this completes the proof. □

5 Sufficient Conditions

Similarly to the previous Sect. 4, sufficient part consists of sufficient conditions in terms of a special family of subharmonic functions and conditions in terms of the symbol μ.

We start with an auxiliary lemma. Recall that $X_j = H^\infty(U_j)/J_j$ is the Banach space of dimension m_j, $j\in\mathbb{N}$. Then, by using the Auerbach lemma [17, 10.5], we can take bases $\{[\varphi_{jp}] : p=1,\dots,m_j\}$ and $\{\nu_{jp} : p=1,\dots,m_j\}$ in the spaces X_j

and X'_j, respectively, having the following properties:

$$||| [\varphi_{jp}] |||_{\infty,j} = 1\,,\ \ ||| \nu_{jp} |||'_{\infty,j} = 1\,,\ \ p = 1, \dots, m_j\,;$$

$$< [\varphi_{jp}], \nu_{jm} >= \delta_{pm} = \begin{cases} 1\,, & p = m\,, \\ 0\,, & p \neq m\,. \end{cases}$$

Put

$$E_{jp} = \big([0], \dots, [0], \underset{j}{[\varphi_{jp}]}, [0], \dots\big)\,,\ \ p = 1, \dots, m_j\,,\ \ j \in \mathbb{N}\,.$$

Evidently, $E_{jp} \in k^\infty$ and

$$\widetilde{|E_{jp}|}_{\omega,q,l} = \exp\big\{-q\omega(\operatorname{Re} z_j) - l|\operatorname{Im} z_j|\big\}\,,\ \ q \in (0,1)\,,\ \ l \in (0,\infty)\,. \tag{19}$$

Lemma 2 *System $\{E_{jp} : p = 1, \dots, m_j\,;\ j \in \mathbb{N}\}$ forms an absolute basis in k^∞. For each $x = \big([x_j]\big)_{j=1}^\infty \in k^\infty$, its expansion by this system has the form*

$$x = \sum_{j=1}^{\infty} \sum_{p=1}^{m_j} < [x_j], \nu_{jp} > E_{jp}\,. \tag{20}$$

Proof Let $x = \big([x_j]\big)_{j=1}^\infty$ be an arbitrary element in k^∞. Then in X_j

$$[x_j] = \sum_{p=1}^{m_j} < [x_j], \nu_{jp} > \varphi_{jp}\,,\ \ j \in \mathbb{N}\,.$$

So it remains only to check that the series (20) converges absolutely in k^∞. Find $q \in (0,1)$ and $l \in (0,\infty)$ such that $\widetilde{|x|}_{\omega,q,l} < \infty$. Then

$$||| [x_j] |||_{\infty,j} \le \widetilde{|x|}_{\omega,q,l} \cdot \exp\big\{q\omega(\operatorname{Re} z_j) + l|\operatorname{Im} z_j|\big\}$$

and

$$| < [x_j], \nu_{jp} > | \le \widetilde{|x|}_{\omega,q,l} \cdot \exp\big\{q\omega(\operatorname{Re} z_j) + l|\operatorname{Im} z_j|\big\}\,. \tag{21}$$

Fix $\varepsilon > 0$ such that $q + \varepsilon < 1$. Using (19) with $q + \varepsilon$ instead of q and $l + \varepsilon$ instead of l, we get

$$\sum_{j=1}^{\infty} \sum_{p=1}^{m_j} | < [x_j], \nu_{jp} > | \cdot \widetilde{|E_{jp}|}_{\omega,q+\varepsilon,l+\varepsilon} \le \widetilde{|x|}_{\omega,q,l} \sum_{j=1}^{\infty} \frac{m_j}{\exp\big\{\varepsilon\omega(\operatorname{Re} z_j) + \varepsilon|\operatorname{Im} z_j|\big\}}\,.$$

By [14, Corollary 1 from Lemma 4], the last series converges. This completes the proof. □

Theorem 4 *Let ω be an arbitrary weight function and T_μ a surjective convolution operator in $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ with a symbol $\mu \in \widetilde{M}^1_{(\omega)}(\mathbb{C})$. Suppose that there is a family $(u_j)_{j=1}^\infty$ of subharmonic functions in $\mathbb{C}$ with $u_j\big|_{U_j} \ge 0$, $j \in \mathbb{N}$, satisfying the condition*

$$(B)\ \ \forall q \in (0,1)\,,\ \forall l \in (0,\infty) \quad \exists \widetilde{q} \in (0,1)\,,\ \exists \widetilde{l} \in (0,\infty)\,,\ \exists C > 0\ :$$
$$u_j(z) + q\omega(z_j) + l|\operatorname{Im} z_j| \le \widetilde{q}\omega(z) + \widetilde{l}|\operatorname{Im} z| + C\,,\ z \in \mathbb{C}\,,\ j \in \mathbb{N}\,.$$

Then T_μ admits a CLRI.

Proof By Theorem 1, it is sufficient to construct a continuous linear operator L from k^∞ into $H^1_{(\omega)}(\mathbb{C})$ which is a right inverse for $\widetilde{\rho}$. To do this, we first determine (see item (a) below) elements $f_{jp} = LE_{jp}$, where $\{E_{jp} : p = 1, \ldots, m_j\,;\ j \in \mathbb{N}\}$ is the absolute basis in k^∞ from Lemma 2. Then (item (b) in this proof) we continue operator L on the whole space k^∞ in the natural way. Recall that L acts continuously from k^∞ into $H^1_{(\omega)}(\mathbb{C})$ if and only if for each $q \in (0,1)$ and $l \in (0,\infty)$ there are $\widetilde{q} \in (0,1), \widetilde{l} \in (0,\infty)$, and $C > 0$ such that $\|L\varphi\|_{\omega,\widetilde{q},\widetilde{l}} \le C \cdot \widetilde{|\varphi|}_{\omega,q,l}$ for all $\varphi \in k^\infty$ with $\widetilde{|\varphi|}_{\omega,q,l} < \infty$.

(a) Let us now construct functions f_{jp} in $H^1_{(\omega)}(\mathbb{C})$ with $\widetilde{\rho}(f_{jp}) = E_{jp}$, $p = 1, \ldots, m_j$, $j \in \mathbb{N}$, which norms in $H^1_{(\omega)}(\mathbb{C})$ satisfy some estimates we need in what follows.

(1) By the Remark in Sect. 3, in each class $[\varphi_{jp}]$ there exists a function $\varphi_{jp} \in H^\infty(U_j)$ with $\|\varphi_{jp}\|_{\infty,j} = |||\,[\varphi_{jp}]\,|||_{\infty,j} = 1$. Then

$$|\varphi_{jp}(z)| \le 1\,,\ z \in U_j\,. \tag{22}$$

Let $k \in \mathbb{N}$, $j_k \le j < j_{k+1}$. Put $V_j = \{z \in U_j : \operatorname{dist}(z, \partial U_j) \ge \sigma_j\}$. Recall that σ_j are determined by (5). Thus, for all $z \in U_j \setminus V_j$, $j \in \mathbb{N}$, we have the estimate (6). Take an infinitely differentiable function g in $\mathbb{R}^2$ (see, e.g., [18, Theorem 1.4.1 and estimate (1.4.2)]) such that

$$g(z) \equiv 1 \text{ on } \bigcup_j V_j\,; \quad \operatorname{supp} g \subset \bigcup_j U_j\,; \quad 0 \le g(z) \le 1 \text{ in } \mathbb{C}\,;$$

$$\left|\frac{\partial g}{\partial \overline{z}}(z)\right| \le \frac{C_0}{\sigma_j}\,,\ z \in U_j \setminus V_j\,,$$

where C_0 is a constant independent of $z \in U_j \setminus V_j$ and j. Hence,

$$\left|\frac{\partial g}{\partial \overline{z}}(z)\right| \le C_0 \exp\left\{4\varepsilon_k \omega(\operatorname{Re} z) + 4L_0|\operatorname{Im} z|\right\}, \quad z \in U_j \setminus V_j. \tag{23}$$

(2) Fix $j \in \mathbb{N}$ and $p : 1 \le p \le m_j$. Put

$$\Phi_{jp}(z) := \begin{cases} \varphi_{jp}(z), & z \in U_j, \\ 0, & z \in \mathbb{C} \setminus U_j; \end{cases} \qquad h_{jp}(z) := -\frac{\Phi_{jp}(z)}{\mu(z)} \cdot \frac{\partial g}{\partial \overline{z}}(z), \quad z \in \mathbb{C}.$$

It is clear that $h_{jp}(z) = 0$ for all $z \notin U_j \setminus V_j$. Since μ has no zeros in $U_j \setminus V_j$, the function h_{jp} is infinitely differentiable in $\mathbb{R}^2$. Using (22), (6), and (23), for $z \in U_j \setminus V_j$, we get

$$|h_{jp}(z)| \le C_0 \exp \max_{z \in \overline{U_j}} \left\{7\varepsilon_k \omega(\operatorname{Re} z) + 7L_0 |\operatorname{Im} z|\right\} = C_0 \exp\left\{7\varepsilon_k \omega(\operatorname{Re} \widetilde{z}_j) + 7L_0 |\operatorname{Im} \widetilde{z}_j|\right\}. \tag{24}$$

Here $\widetilde{z}_j$ is some point in $\overline{U_j}$.

Now obtain a special integral estimate for the function h_{jp}. Let $q \in (0, 1)$ and $l \in (2A + 21L_0, \infty)$ be arbitrary (A is the constant from condition (3)). Put

$$\psi_j(z) := u_j(z) + q\omega(z_j) + l |\operatorname{Im} z_j|, \ z \in \mathbb{C}, \ j \in \mathbb{N},$$

where $(u_j)_{j=1}^{\infty}$ is a family of subharmonic functions from the assumptions of the Theorem. Then ψ_j are subharmonic in $\mathbb{C}$ and

$$\psi_j(z) \le \widetilde{q}\omega(z) + \widetilde{l} |\operatorname{Im} z| + \ln C_1, \ z \in \mathbb{C}, \ j \in \mathbb{N}, \tag{25}$$

for some $\widetilde{q} \in (q, 1)$, $\widetilde{l} \in (l, \infty)$, and $C_1 > 1$. Using (24) and $u_j\big|_{U_j} \ge 0$, $j \in \mathbb{N}$, we have

$$\int_{\mathbb{C}} |h_{jp}(z)|^2 \exp\left\{-2\psi_j(z) - 2\ln(1 + |z|^2)\right\} d\lambda_z \le C_0^2 M^2 D_j^2,$$

where

$$M^2 := \int_{\mathbb{C}} \frac{d\lambda_z}{(1 + |z|^2)^2}, \ D_j^2 := \exp\left\{14\varepsilon_k \omega(\operatorname{Re} \widetilde{z}_j) + 14L_0 |\operatorname{Im} \widetilde{z}_j| - 2q\omega(z_j) - 2l |\operatorname{Im} z_j|\right\}.$$

Applying [19, Theorem 4.4.2], we solve the $\overline{\partial}$-problem $\frac{\partial v}{\partial \overline{z}} = h_{jp}$ and find its solution $v_{jp} \in C^{\infty}(\mathbb{R}^2)$ such that

$$\int_{\mathbb{C}} |v_{jp}(z)|^2 \exp\left\{-2\psi_j(z) - 4\ln(1 + |z|^2)\right\} d\lambda_z \le \frac{C_0^2 M^2 D_j^2}{2}. \tag{26}$$

(3) Consider the functions $f_{jp}(z) := v_{jp}(z)\mu(z) + \Phi_{jp}(z)g(z)$, $z \in \mathbb{C}$. From the proof of Lemma 7 in [14] it follows that $f_{jp} \in H^1_{(\omega)}(\mathbb{C})$ and $\widetilde{\rho}(f_{jp}) = E_{jp}$, $p = 1, \dots, m_j$, $j \in \mathbb{N}$. Now we improve the estimates for norms of f_{jp} in $H^1_{(\omega)}(\mathbb{C})$ obtained in [14].

Obviously,

$$|f_{jp}(z)|^2 \le 2\big(|v_{jp}(z)|^2|\mu(z)|^2 + |\Phi_{jp}(z)|^2\big), \quad z \in \mathbb{C}. \tag{27}$$

Fix any $\varepsilon \in \big(0, \min\big\{\frac{1-\widetilde{q}}{2}, q\big\}\big)$. Since $\mu \in \widetilde{M}^1_{(\omega)}(\mathbb{C})$, there is $C_2 > 0$ such that

$$|\mu(z)| \le C_2 \exp\big\{\varepsilon\omega(z) + l_0|\operatorname{Im} z|\big\}, \ z \in \mathbb{C}. \tag{28}$$

Next, put $\widetilde{\psi}(z) := (\widetilde{q} + \varepsilon)\omega(z) + (\widetilde{l} + l_0)|\operatorname{Im} z| + 2\ln(1 + |z|^2)$, $z \in \mathbb{C}$. By (28), (25), and (26), we get

$$\int_{\mathbb{C}} |v_{jp}(z)|^2|\mu(z)|^2 \exp\big\{-2\widetilde{\psi}(z)\big\}\, d\lambda_z \le$$
$$\le C_2^2 \int_{\mathbb{C}} |v_{jp}(z)|^2 \exp\big\{-2\widetilde{q}\omega(z) - 2\widetilde{l}|\operatorname{Im} z| - 4\ln(1+|z|^2)\big\}\, d\lambda_z \le$$
$$\le C_1^2C_2^2 \int_{\mathbb{C}} |v_{jp}(z)|^2 \exp\big\{-2\psi_j(z) - 4\ln(1+|z|^2\big\}\, d\lambda_z \le \frac{C_0^2C_1^2C_2^2M^2}{2} D_j^2\,.$$

Note now that $\Phi_{jp} = 0$ for $z \notin U_j$ and, by (22), $|\Phi_{jp}(z)| \le 1$ for $z \in U_j$, $p = 1, \dots, m_j$, $j \in \mathbb{N}$. Applying Lemma 3 in [14], we find $C_3 > 1$ such that

$$(\widetilde{q} + \varepsilon)\omega(z) + (\widetilde{l} + l_0)|\operatorname{Im} z| \ge q\omega(z_j) + l|\operatorname{Im} z_j| - \frac{\ln C_3}{2}\,, \ z \in U_j\,, \ \ j \in \mathbb{N}\,.$$

Then

$$\int_{\mathbb{C}} |\Phi_{jp}(z)|^2 \exp\big\{-2\widetilde{\psi}(z)\big\}\, d\lambda_z \le C_3^2M^2 \exp\big\{-2q\omega(z_j) - 2l|\operatorname{Im} z_j|\big\} \le C_3M^2D_j^2\,.$$

Using this and the above integral estimate for $|v_{jp}(z)|^2|\mu(z)|^2$ in (27), we get

$$\int_{\mathbb{C}} |f_{jp}(z)|^2 \exp\big\{-2\widetilde{\psi}(z)\big\}\, d\lambda_z \le C_4^2D_j^2\,,$$

where $C_4^2 := (C_0^2C_1^2C_2^2 + 2C_3^2)M^2$. Since the function $|f_{jp}|^2$ is subharmonic, we finally have

$$|f_{jp}(z)|^2 \le \frac{1}{\pi} \int_{|t-z|\le 1} |f_{jp}(t)|^2 \exp\big\{-2\widetilde{\psi}(t)\big\}\, d\lambda_t \cdot \exp \sup_{|t-z|\le 1} 2\widetilde{\psi}(t) \le$$
$$\le C_4^2D_j^2 \exp\big\{2(\widetilde{q} + \varepsilon)\omega\big(|z|+1\big) + 2(\widetilde{l} + l_0)\big(|\operatorname{Im} z| + 1\big) + 4\ln\big(1 + (|z|+1)^2\big)\big\}\,.$$

By the properties of the weight ω, this implies that

$$|f_{jp}(z)|^2 \le C_5^2 D_j^2 \exp\left\{2(\widetilde{q}+2\varepsilon)\omega(z)+2(\widetilde{l}+l_0)|\operatorname{Im} z|\right\}, \quad z \in \mathbb{C}, \tag{29}$$

for all $p = 1, \ldots, m_j$, $j \in \mathbb{N}$, and some $C_5 \ge C_4$ (C_5 is independent of j, p and z).

Let us now consider D_j. Take $\widetilde{k} \in \mathbb{N}$ such that $14\varepsilon_k < \varepsilon$, $k \ge \widetilde{k}$. Using again Lemma 3 from [14] with $q = \varepsilon, l = 14L_0$, and ε, we have that, for $L_1 = 2A + 21L_0$ and some $C_6 > 1$, the following estimate holds:

$$\varepsilon\omega(\operatorname{Re} z) + 14L_0|\operatorname{Im} z| \le 2\varepsilon\omega(\operatorname{Re} z_j) + 2L_1|\operatorname{Im} z_j| + \ln C_6\,, \quad z \in U_j\,, \quad j \in \mathbb{N}\,.$$

Consequently,

$$14\varepsilon_k\omega(\operatorname{Re}\widetilde{z}_j) + 14L_0|\operatorname{Im}\widetilde{z}_j| \le 2\varepsilon\omega(\operatorname{Re} z_j) + 2L_1|\operatorname{Im} z_j| + \ln C_6\,, \quad j \ge j_{\widetilde{k}}\,.$$

This implies that

$$D_j^2 \le C_6^2 \exp\left\{-2(q-\varepsilon)\omega(\operatorname{Re} z_j) - 2(l-L_1)|\operatorname{Im} z_j|\right\}, \quad z \in \mathbb{C}\,, \quad j \ge j_{\widetilde{k}}\,.$$

Note that $l > 2A + 21L_0 = L_1$, so $l - L_1 > 0$.

From (29) it now follows that, for all $z \in \mathbb{C}$ and $j \ge j_{\widetilde{k}}$,

$$\frac{|f_{jp}(z)|}{\exp\left\{(\widetilde{q}+2\varepsilon)\omega(z)+(\widetilde{l}+l_0)|\operatorname{Im} z|\right\}} \le C_5C_6 \exp\left\{-(q-\varepsilon)\omega(\operatorname{Re} z_j) - (l-L_1)|\operatorname{Im} z_j|\right\}.$$

Therefore, $\|f_{jp}\|_{\omega,\widetilde{q}+2\varepsilon,\widetilde{l}+l_0} \le C_5C_6\widetilde{|E_{jp}|}_{\omega,q-\varepsilon,l-L_1}$, $p = 1, \ldots, m_j$, $j \ge j_{\widetilde{k}}$. Recall that C_5 and C_6 are independent of j and p; l_0 and L_1 are determined by ω and μ; $\varepsilon > 0$ is arbitrary small.

In fact, we have shown that for each $q \in (0, 1)$ and $l \in (0, \infty)$ there exist $\widetilde{q} \in (0, 1)$, $\widetilde{l} \in (0, \infty)$, and $C > 0$ such that

$$\|f_{jp}\|_{\omega,\widetilde{q},\widetilde{l}} \le C\widetilde{|E_{jp}|}_{\omega,q,l}\,, \quad p = 1, \ldots, m_j\,, \quad j \ge j_{\widetilde{k}}\,. \tag{30}$$

These are the desired estimates for the norms of f_{jp}.

To complete the item (a) of the proof, put $L(E_{jp}) := f_{jp}$.

(b) Continue now operator L onto the whole space k^∞. Let $x = \big([x_j]\big)_{j=1}^\infty \in k^\infty$ be an arbitrary element in k^∞. Then its expansion by the basis $\{E_{jp} : p = 1, \ldots, m_j, \ j \in \mathbb{N}\}$ has the form (20). Put

$$Lx := \sum_{j=1}^{\infty}\sum_{p=1}^{m_j} < \nu_j, [x_{jp}] > f_{jp}\,. \tag{31}$$

We claim that this operator is continuous from k^∞ to $H^1_{(\omega)}(\mathbb{C})$. Then it is a CLRI for the operator $\widetilde{\rho}$ which completes the proof.

It remains to check our claim. Fix $q \in (0,1)$ and $l \in (0,\infty)$. If $\widetilde{|x|}_{\omega,q,l} < \infty$, then the coefficients of the series (31) satisfy (21). Take any $\varepsilon \in \left(0, \frac{1-q}{2}\right)$. By (30), for $q+\varepsilon$ instead of q and $l+\varepsilon$ instead of l, there are $\widetilde{q} \in (0,1)$, $\widetilde{l} \in (0,\infty)$, and $C > 0$ such that $\| f_{jp}\|_{\omega,\widetilde{q},\widetilde{l}} \le C\widetilde{|E_{jp}|}_{\omega,q+\varepsilon,l+\varepsilon}$. Then

$$\sum_{j=1}^{\infty}\sum_{p=1}^{m_j} \Big| < \nu_j, [x_{jp}] > \Big| \cdot \| f_{jp}\|_{\omega,\widetilde{q},\widetilde{l}} \le C\widetilde{|x|}_{\omega,q,l} \sum_{j=1}^{\infty} \frac{m_j}{\exp\left\{\varepsilon\omega(\operatorname{Re} z_j) + \varepsilon|\operatorname{Im} z_j|\right\}}\,.$$

As above, the last series converges. Clearly this implies that $L : k^\infty \to H^1_{(\omega)}(\mathbb{C})$ is continuous. □

Theorem 5 *Let ω be a nonquasianalytic weight function and T_μ a surjective convolution operator in $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ with a symbol $\mu \in M^1_{(\omega)}(\mathbb{C})$. If the sequence $(z_j)_{j=1}^\infty$ generated by μ satisfies condition (7), then there is a family $(u_j)_{j=1}^\infty$ of continuous subharmonic functions in $\mathbb{C}$ with $u_j\big|_{U_j} \ge 0$ for all $j \in \mathbb{N}$ and the following property*

$$(\widetilde{B})\ \ \forall q \in (0,1)\,,\ \forall l \in (0,\infty)\,,\ \forall \widetilde{q} \in (q,1)\quad \exists C > 0\ :$$
$$u_j(z) + q\omega(z_j) + l|\operatorname{Im} z_j| \le \widetilde{q}\omega(z) + |\operatorname{Im} z| + C\,,\ z \in \mathbb{C}\,,\ j \in \mathbb{N}\,.$$

Proof Using [11, Lemma 1], take a continuous subharmonic function v such that $v(0) = 0$ and

$$v(z) \le |\operatorname{Im} z| - q\omega(z) + C\,,\ z \in \mathbb{C}\,,$$

for each $q \in (0,1)$ and some $C = C(q) > 0$. Let $u_j(z) := \sup_{t \in U_j} v(z-t)$, $z \in \mathbb{C}$, $j \in \mathbb{N}$. Then u_j are continuous and subharmonic in $\mathbb{C}$. Moreover, $u_j(z) \ge v(0) = 0$ for $z \in U_j$.

Let us show that the family $(u_j)_{j=1}^\infty$ satisfies the condition $(\widetilde{B})$. Fix $q \in (0,1)$ and $l \in (0,\infty)$ and choose q_1, q_2: $q < q_1 < q_2 < 1$. Given q_2, find $C_2 > 0$ such that

$$v(z) \le |\operatorname{Im} z| - q_2\omega(z) + C_2 \le |\operatorname{Im} z| - q_2\omega(\operatorname{Re} z) + C_2\,,\ z \in \mathbb{C}\,.$$

Next, by condition (α) on the weight ω,

$$q_1\omega(x+y) \le q_2\big(\omega(x) + \omega(y)\big) + C_1\,,\ x, y \ge 0\,,$$

for some $C_1 > 0$. This implies that

$$q_2\omega(x-y) \ge q_1\omega(x) - q_2\omega(y) - C_1\,,\ x, y \ge 0\,.$$

Thus, for all $z, t \in \mathbb{C}$,

$$v(z-t) \le |\operatorname{Im} z - \operatorname{Im} t| - q_2\omega(\operatorname{Re} t - \operatorname{Re} z) + C_2 \le$$
$$\le |\operatorname{Im} z| + |\operatorname{Im} t| - q_1\omega(\operatorname{Re} t) + q_2\omega(\operatorname{Re} z) + C_1 + C_2 \,. \tag{32}$$

By Lemma 1, there exists $C_3 > 0$ such that

$$q\omega(\operatorname{Re} z_j) + l|\operatorname{Im} z_j| + |\operatorname{Im} t| \le q_1\omega(\operatorname{Re} t) + C_3\,,\ \ t \in U_j\,,\ \ j \in \mathbb{N}\,.$$

Hence, the estimate (32) can be continued as follows

$$v(z-t) \le q_2\omega(\operatorname{Re} z) + |\operatorname{Im} z| - \big(q\omega(\operatorname{Re} z_j) - l|\operatorname{Im} z_j|\big) + C\,,$$

Evidently, this implies that

$$u_j(z) + q\omega(\operatorname{Re} z_j) + l|\operatorname{Im} z_j| \le q_2\omega(\operatorname{Re} z) + |\operatorname{Im} z| + C\,,\ \ z \in \mathbb{C}\,,\ \ j \in \mathbb{N}\,.$$

□

Corollary *Under the assumptions of Theorem 5, operator T_μ admits a CLRI.*

6 The Main Result

In this Section we give a complete description for symbols of convolution operators T_μ in $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ admitting a CLRI. To do this, prove the following lemma.

Lemma 3 *Let ω be an arbitrary weight function and $\mu \in \widetilde{M}^1_{(\omega)}(\mathbb{C})$ a symbol of a surjective convolution operator in $\mathcal{E}^1_{(\omega)}(\mathbb{R})$. Then conditions (7) and (13) are equivalent.*

Proof (1) Let (7) hold. Then we can find $j_0 \in \mathbb{N}$ such that $|\operatorname{Im} z_j| \le \frac{1}{8A}\,\omega(\operatorname{Re} z_j)$, $j \ge j_0$. Next, it is clear that $\omega(t) \le K_0\omega\big(\frac{t}{4}\big)$, $t \ge t_0$, for some $K_0 \ge 1$ and some $t_0 > 0$. Choose $\widetilde{j} \ge j_0$ such that $|\operatorname{Re} z_j| \ge t_0$ for all $j \ge \widetilde{j}$.

Fix $k \in \mathbb{N}$ with $j_k \ge \widetilde{j}$. Consider any $j \in \mathbb{N}$: $j_k \le j < j_{k+1}$. Two situations are possible. If U_j contains a point z_j with $|\operatorname{Im} z_j| \le \delta_k\omega(\operatorname{Re} z_j)$, then, by [14, Lemma 2],

$$(1 - 12\delta_k A)|\operatorname{Re} z_j| \le |\operatorname{Re}\lambda_s|\,,\quad |\operatorname{Im}\lambda_s| \le 13\delta_k\omega(\operatorname{Re} z_j)\,.$$

Hence,

$$|\operatorname{Im}\lambda_s| \le 13\delta_k\omega\Big(\frac{1}{1-12\delta_k A}\ \operatorname{Re}\lambda_s\Big)\,.$$

Obviously, we may assume that $l_0 \geq 1$. Then $\delta_1 < \frac{\varepsilon_1}{64} < \frac{1}{64A}$ and $1 - 12\delta_k A > \frac{1}{2}$. This gives that

$$|\operatorname{Im}\lambda_s| \leq 13\delta_k\omega(2\operatorname{Re}\lambda_s) \leq 26\delta_k K_0\omega(\operatorname{Re}\lambda_s).$$

Let now $|\operatorname{Im} z| > \delta_k\omega(\operatorname{Re} z)$ for all $z \in U_j$. From Lemma 2 in [14], it then follows that

$$|\operatorname{Re}\lambda_s| \geq |\operatorname{Re} z_j| - 2|\operatorname{Im}\lambda_s|, \quad |\operatorname{Im}\lambda_s| \leq 3|\operatorname{Im} z_j|. \tag{33}$$

By (3), $|\operatorname{Im} z_j| \leq \frac{1}{8A}\omega(\operatorname{Re} z_j) \leq \frac{1}{8}|\operatorname{Re} z_j|$. Using the second inequality from (33), we get that $|\operatorname{Im}\lambda_s| \leq \frac{3}{8}|\operatorname{Re} z_j|$. Consequently, by the first inequality in (33), $|\operatorname{Re}\lambda_s| \geq \frac{1}{4}|\operatorname{Re} z_j|$. Therefore,

$$\frac{|\operatorname{Im}\lambda_s|}{\omega(\operatorname{Re}\lambda_s)} \leq \frac{3|\operatorname{Im} z_j|}{\omega(\frac{1}{4}\operatorname{Re} z_j)} \leq 3K_0\frac{|\operatorname{Im} z_j|}{\omega(\operatorname{Re} z_j)}.$$

Thus, in both situations we have that

$$\frac{|\operatorname{Im}\lambda_s|}{\omega(\operatorname{Re}\lambda_s)} \leq \max\left\{26\delta_k K_0,\, 3K_0\frac{|\operatorname{Im} z_j|}{\omega(\operatorname{Re} z_j)}\right\}.$$

Since the right-hand side tends to 0 as $j \to \infty$ (or $k \to \infty$, or $s \to \infty$), it follows that $\frac{|\operatorname{Im}\lambda_s|}{\omega(\operatorname{Re}\lambda_s)} \to 0$, $s \to \infty$.

(2) Let now (13) hold. Obviously, it is sufficient to prove condition (7) in case when $|\operatorname{Im} z| > \delta_k\omega(\operatorname{Re} z)$ for all $z \in U_j$. Applying again Lemma 2 from [14], we get

$$|\operatorname{Re} z_j| \geq |\operatorname{Re}\lambda_s| - 2|\operatorname{Im} z_j|, \quad |\operatorname{Im} z_j| \leq 3|\operatorname{Im}\lambda_s|,$$

i.e., we swap λ_s and z_j in (33). Arguing further in the same manner as in the previous item (1), we have

$$\frac{|\operatorname{Im} z_j|}{\omega(\operatorname{Re} z_j)} \leq 3K_0\frac{|\operatorname{Im}\lambda_s|}{\omega(\operatorname{Re}\lambda_s)}, \quad j \geq \widetilde{j}.$$

This completes the proof. □

Applying Theorem 3, Corollary from Theorem 5, and Lemma 3, we get the main result of the paper.

Theorem 6 *Let ω be a nonquasianalytic weight, $\mu \in M^1_{(\omega)}(\mathbb{C})$ a symbol of the surjective convolution operator T_μ in $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ and $(\lambda_s)_{s=1}^\infty$, $|\lambda_s| \uparrow \infty$, the sequence of zeros of μ. The following assertions are equivalent:*

(i) *T_μ admits a CLRI;*

(ii) $\lim\limits_{s\to\infty} \dfrac{|\operatorname{Im}\lambda_s|}{\omega(\operatorname{Re}\lambda_s)} = 0\,.$

In conclusion let us consider some examples.

Example 1 Let ω be a nonquasianalytic weight function such that $\ln^2 t = o\big(\omega(t)\big)$, $t \to \infty$, and $(\lambda_n)_{n=1}^\infty$ be an increasing lacunary sequence of positive numbers. This means that $\liminf_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} > 1$. Put

$$\mu(z) = \prod_{n=1}^\infty \Big(1 - \frac{z^2}{\lambda_n^2}\Big), \quad z \in \mathbb{C}.$$

As it follows from [14, Example 1], μ generates in $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ a surjective convolution operator T_μ which is a differential operator of infinite order with constant coefficients. More exactly, if $\mu(z)$ is represented by the Taylor series $\mu(z) = \sum_{k=0}^\infty a_k(-i)^k z^k$, then $T_\mu f = \sum_{k=0}^\infty a_k f^{(k)}$, $f \in \mathcal{E}^1_{(\omega)}(\mathbb{R})$. Since all zeros of μ are real, T_μ has a CLRI. In other words, differential equation $\sum_{k=0}^\infty a_k f^{(k)} = g$ has always a solution $f \in \mathcal{E}^1_{(\omega)}(\mathbb{R})$ which depends linearly and continuously on the right-hand side $g \in \mathcal{E}^1_{(\omega)}(\mathbb{R})$ of the equation.

Example 2 (see [14, Example 2]) Let ω be an arbitrary nonquasianalytic weight and $\mu(z) = \sin z$, $z \in \mathbb{C}$. Then μ generates in $\mathcal{E}^1_{(\omega)}(\mathbb{R})$ a surjective difference operator $(T_\mu f)(x) = \frac{1}{2i}\big(f(x+1) - f(x-1)\big)$. Obviously, condition (13) holds, so T_μ has a CLRI. This means that for each $g \in \mathcal{E}^1_{(\omega)}(\mathbb{R})$, difference equation $f(x+1) - f(x-1) = g(x)$ has a solution $f \in \mathcal{E}^1_{(\omega)}(\mathbb{R})$ depending linearly and continuously on g.

Example 3 Let $\omega(t) = t^{\rho(t)}$, where $\rho(t) \to \rho \in (0,1)$ is some proximate order. Suppose that $(\lambda_s)_{s=1}^\infty$, $|\lambda_s| \uparrow \infty$, is a sequence of complex numbers satisfying the condition $\lim_{s\to\infty} \frac{s}{|\lambda_s|^{\rho(|\lambda_s|)}} = 0$ and the condition (C') (see [20, Section II, §1, p. 95]), that is, the points λ_s lie inside angles with a common vertex at the origin and no other points in common; moreover, if we arrange the points of (λ_s) inside any of these angles in the order of increasing of their moduli, then for the points that are inside the same angle we have $|\lambda_{k+1}| - |\lambda_k| > d|\lambda_k|^{1-\rho(|\lambda_k|)}$ for some $d > 0$. In [14, Remark 4] it was shown that the function $\mu(z) = \prod_{s=1}^\infty \Big(1 - \frac{z}{\lambda_s}\Big)$ is a symbol of the surjective convolution operator T_μ in $\mathcal{E}^1_{(\omega)}(\mathbb{R})$. Note, that, as in Example 1, T_μ is a differential operator of infinite order with constant coefficients. It is clear that T_μ has no a CLRI, if there are α, β such that $0 < \alpha < \beta < \pi$ or $\pi < \alpha < \beta < 2\pi$ and such that the angle $\alpha \le \arg z \le \beta$ contains the infinite set of zeros λ_s of μ. In particular, T_μ has no a CLRI when all points λ_s are imaginary, i.e. $\lambda_s = \pm i|\lambda_s|$.

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Long Wavelength Asymptotics of Self-Oscillations of Viscous Incompressible Fluid

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Abstract We obtain the long wavelength asymptotics of a secondary regime formed at stability loss of a stationary spatially periodic shear flow with non-zero average as one of the spatial periods tends to infinity (the wave number vanishes). It is known that if certain non-degeneracy conditions are satisfied, then from the basic solution a self-oscillatory regime branches. Recurrence formulas for kth term of the asymptotics of this secondary solution are obtained. To study the bifurcations of basic flow we obtain the scheme of Lyapunov-Schmidt method proposed by V.I. Yudovich. At each step of the Lyapunov-Schmidt method series expansion in the small parameter α is applied.

Keywords Stability of two-dimensional viscous flows · Kolmogorov flow · Long wavelength asymptotics

Mathematical Subject Classification 35Q30, 35P20, 35B35

1 Introduction

Spatial-periodic flows of a viscous fluid are widely used in mathematical modelling of various physical processes. On the one hand, the condition of the periodicity of the flow velocity allows one to obtain explicit analytical representations of space-time structures. On the other hand, such flows can be realized in physical experiments.

In this paper we consider the two-dimensional ($\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$) viscous incompressible flow driven by an external forces field $\boldsymbol{F}(\boldsymbol{x}, t)$ that is periodic in x_1

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and x_2 with periods ℓ_1 and ℓ_2, respectively. The flow is described by the Navier-Stokes equations:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}, \nabla)\boldsymbol{v} - \nu \Delta \boldsymbol{v} = -\nabla p + \boldsymbol{F}(\boldsymbol{x}, t), \qquad \operatorname{div} \boldsymbol{v} = 0, \tag{1.1}$$

where $\nu = 1/Re$ is the kinematic viscosity and Re is the Reynolds number. The period $\ell_1 = 2\pi$, and the ratio of the periods is characterized by the wave number α: $\ell_2 = 2\pi/\alpha$, $\alpha \to 0$. Let $\langle f \rangle$ denote the average with respect to x_1, while $\langle\langle f \rangle\rangle$ denote the average over the period rectangle $\Omega = [0, \ell_1] \times [0, \ell_2]$:

$$\langle f \rangle = \frac{1}{\ell_1} \int_0^{\ell_1} f(\boldsymbol{x}, t)\, dx_1, \quad \langle\langle f \rangle\rangle(t) = \frac{1}{|\Omega|} \int_\Omega f(\boldsymbol{x}, t)\, dx_1\, dx_2,$$

The spatial average velocity is assumed to be given:

$$\langle\langle \boldsymbol{v} \rangle\rangle = \boldsymbol{q}. \tag{1.2}$$

Assume that the velocity field $\boldsymbol{v}$ is periodic in x_1 and x_2 with the same periods ℓ_1 and ℓ_2 as the external force field.

If $\boldsymbol{F} = (0, \nu F(x_1))$ and $\boldsymbol{q} = (0, q)$, the unique stationary shear flow $\boldsymbol{V} = (0, V(x_1))$ can be found as the solution of the problem

$$-V'' = F(x_1) - \langle F \rangle, \quad \langle V \rangle = q.$$

The class of such solutions includes the Kolmogorov flow [1]:

$$\boldsymbol{V} = (0, \gamma \sin(x_1)). \tag{1.3}$$

In [2] the stability of Kolmogorov flow was analyzed using continuous fractions. In [3] it was proved that, for $\alpha \geqslant 1$, the Kolmogorov flow is a globally stable and unique, while, for any $\alpha < 1$ and sufficiently small ν, new stationary solutions bifurcate from solution (1.3). In [4] the uniqueness and stability results for short-wave perturbations of the Kolmogorov flow were extended to an unbounded periodic channel with rigid walls. It was proved in [5] that, for $\alpha \geqslant 1$, the Kolmogorov flow in a channel with rigid walls remains stable for all Reynolds numbers.

Many modern studies are devoted to the Kolmogorov flow and its various generalizations. An overview of some of them is given in [6].

In this paper, we are interested in bifurcations of a stationary solution of system (1.1)–(1.2) of a general form

$$\boldsymbol{V} = (0, V(x_1)) \quad \langle V \rangle \neq 0, \tag{1.4}$$

which is called the basic flow.

It is known that for sufficiently large viscosity values (small Reynolds numbers), the stability spectrum lies strictly in the left half-plane of the complex plane and the basic flow is stable. The loss of stability of steady flow (1.4) is characterized by the fact that, as a parameter ν varies, the eigenvalues of linear spectral problem pass from the left to the right half of the complex plane. Critical value is the value of the parameter $\nu = \nu_c$, in which one or more eigenvalues of the linear spectral problem cross the imaginary axis. There are two types of stability loss. In the absence of additional degeneracies at the critical value of the parameter, either a pair of purely imaginary complex conjugate eigenvalues appears, or the eigenvalue passes through zero. In the first case the stability loss is oscillatory. In the second case the stability loss is monotone.

For the first time, a long-wavelength asymptotic expansion for the problem of stability of general flow (1.4) was constructed in [7]. It was shown in [8] that, for $\langle V \rangle \neq 0$, a self-oscillation mode bifurcates from (1.4) and the first terms of asymptotics were found in terms of the stream function. The linear stability of three-dimensional flows was considered in [9]. In the case when the Reynolds number passes the critical value found in [9], the leading terms of the asymptotics in a self-oscillation mode bifurcating from the basic flow were explicitly constructed in [10].

In [11] the leading terms of the asymptotics of secondary modes for basic flow of the form

$$\boldsymbol{V} = (\alpha V_1, V_2)(\boldsymbol{x}) \tag{1.5}$$

were found but general rules in coefficients expressions were not obtained.

In [12] recurrence formulas for finding the kth term of the long wavelength asymptotics for the stability of steady shear flows were derived in the case of nonzero average corresponding to oscillatory loss of stability. The coefficients of the expansions are explicitly expressed in terms of some Wronskians, as well as integral operators of Volterra type. It is shown that the eigenvalues of the linear spectral problem are odd functions of the parameter α, and the critical viscosity is an even function. In the particular case, when the deviation of the velocity from its mean value $V(x) - \langle V \rangle$ is an odd function of x, the coefficients of expansion of the eigenvalues in series in powers of α, starting from the third, are zero and the eigenvalues can be found exactly: $\sigma_{1,2} = \pm im\langle V \rangle\alpha$, $m \neq 0$.

Long-wave asymptotics of linear adjoint problem in two-dimensional case is under consideration in [13]. Recurrence formulas for kth term of the velocity and pressure asymptotics are obtained. The relations between coefficients of linear adjoint problem and linear spectral problem are obtained.

In [14], recurrence formulas for finding the kth term of the long-wave asymptotics for the linear stability problem of two-dimensional basic shear flows of a viscous incompressible fluid with zero average are derived. It was proved that, if $\langle V \rangle = 0$ and the basic velocity profile is odd, as in the case of Kolmogorov flow, then the loss of stability is monotone.

In [15] the results [12] related to shear flows (1.4) are generalized to the case of flows close to shear (1.4) with $\langle V \rangle \neq 0$. In [15] the first terms of the long-wavelength asymptotics are found. The coefficients of the asymptotic expansions are explicitly expressed in terms of some Wronskians and integral operators of Volterra type, as in the case of shear basic flow. The structure of eigenvalues and eigenfunctions for the first terms of asymptotics is identified, a comparison with the case of shear flow is made.

The aim of this paper is to obtain the recurrence formulas for finding the kth term of the self-oscillations bifurcating from the basic flow (1.4). The results obtained can be used to derive the kth term of the secondary regime branching from shear flow with zero average, which generalizes Kolmogorov flow, and from almost-periodic shear flow.

Let S_2 be the closure in $L_2(\Omega)$ of the set of smooth solenoidal vector functions periodic in the spatial variables x_1, x_2 with periods ℓ_1 and ℓ_2 respectively, Π is the orthogonal projector in $L_2(\Omega)$ onto the subspace S_2 (hydrodynamic projector). Linearizing the Navier-Stokes equations on the main flow (1.4), we obtain a linear spectral problem in S_2:

$$A(\nu_c)\boldsymbol{\varphi} + i\omega_0\boldsymbol{\varphi} = 0, \quad A(\nu)\boldsymbol{\varphi} = -\nu\Pi\Delta\boldsymbol{\varphi} + \Pi\left[\varphi_1 V'(x_1)\boldsymbol{e_2} + V(x_1)\frac{\partial\boldsymbol{\varphi}}{\partial x_2}\right], \tag{1.6}$$

here $\boldsymbol{e_1}$, $\boldsymbol{e_2}$ are coordinate unit vectors.

To study the bifurcations of the basic flow, we apply the scheme of the Lyapunov-Schmidt method proposed by V. I. Yudovich [16, 17]. First we consider the linear spectral problem (1.6), at the second step we find the eigenvectors of the linear adjoint problem

$$A^*(\nu_c)\boldsymbol{\Phi} - i\omega_0\boldsymbol{\Phi} = 0, \quad A^*(\nu)\boldsymbol{\Phi} = -\Pi\left[\nu\Delta\boldsymbol{\Phi} - V(x_1)\sum_{j=1}^{2}\left(\frac{\partial\Phi_j}{\partial x_2} + \frac{\partial\Phi_2}{\partial x_j}\right)\boldsymbol{e_j}\right]. \tag{1.7}$$

where A^* is the Hilbert conjugate to the operator A in S_2. In this paper, at each step of the Lyapunov-Schmidt method, series expansion in the small parameter α is applied.

Assuming for any solution of the Navier-Stokes equations (1.1)

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{V},$$

we arrive at the nonlinear perturbation equation in the space S_2:

$$\frac{d\boldsymbol{u}}{dt} + A(\nu)\boldsymbol{u} = K(\boldsymbol{u}, \boldsymbol{u}),$$

here $K(\boldsymbol{u}, \boldsymbol{u}) = -\Pi(\boldsymbol{u}, \nabla)\boldsymbol{u}$. We denote supercriticality by $\varepsilon^2 = \nu_c - \nu$, in the perturbation equation we replace $\tau = \omega t$, where ω is the unknown cyclic frequency, velocity and pressure perturbations $(\boldsymbol{u}, P)$ and the frequency ω we will look for in the form of series in powers of the parameter ε:

$$\boldsymbol{u} = \sum_{k=1}^{\infty} \varepsilon^k \boldsymbol{u}_k, \quad P = \sum_{k=1}^{\infty} \varepsilon^k P_k, \quad \omega = \sum_{k=1}^{\infty} \varepsilon^k \omega_k \tag{1.8}$$

Then the velocity perturbation in the first order in ε has the form

$$\boldsymbol{u}_1 = \eta(\boldsymbol{\varphi} e^{i\tau} + \overline{\boldsymbol{\varphi}} e^{-i\tau}), \tag{1.9}$$

where overline denotes complex conjugate, $\boldsymbol{\varphi}$ is a solution to the linear spectral problem and is found in [12], the amplitude η is determined from the solvability condition of the perturbation equation at ε^3. The velocity perturbation at ε^2 has the following structure

$$\boldsymbol{u}_2 = \eta^2(\boldsymbol{w} + \boldsymbol{v} e^{i2\tau} + \overline{\boldsymbol{v}} e^{-i2\tau}), \tag{1.10}$$

where $\boldsymbol{w}$ and $\boldsymbol{v}$ are solutions of linear inhomogeneous equations in S_2

$$A\boldsymbol{w} = K(\boldsymbol{\varphi}, \overline{\boldsymbol{\varphi}}) + K(\overline{\boldsymbol{\varphi}}, \boldsymbol{\varphi}), \quad 2i\omega_0 \boldsymbol{v} + A\boldsymbol{v} = K(\boldsymbol{\varphi}, \boldsymbol{\varphi}) \tag{1.11}$$

Let H denote the subspace of functions $f \in L_2(0, \ell_1)$ that are orthogonal to unity: $\langle f \rangle = 0$. The operator $I : H \to H$ is the inverse of the differentiation operator and is completely continuous:

$$If = \int_0^x f(s)ds - \left\langle \int_0^x f(s)ds \right\rangle \tag{1.12}$$

Let $W_x(f, g)$ and $W_z(f, g)$ denote the Wronskians of functions $f(x, z)$ and $g(x, z)$ in x and z respectively:

$$W_x(f, g) = f\frac{\partial g}{\partial x} - g\frac{\partial f}{\partial x}, \quad W_z(f, g) = f\frac{\partial g}{\partial z} - g\frac{\partial f}{\partial z},$$

the auxiliary function θ characterizes the deviation of the second component of velocity from its average value:

$$\frac{d^2\theta}{dx^2} = V - \langle V \rangle, \quad \langle \theta \rangle = 0.$$

The deviation of a periodic function from its period-average value is denoted by curly brackets:

$$\{F\} = F(x) - \langle F \rangle.$$

We present some results [12] related to the linear spectral problem (1.6) that will be used in this paper.

The components of solution of (1.6) $\boldsymbol{\varphi}$ and $P(x, z)$ are sought in the form of asymptotics series in powers of parameter α:

$$\boldsymbol{\varphi} = \sum_{k=0}^{\infty} \boldsymbol{\varphi}^k \alpha^k, \quad P = \sum_{k=0}^{\infty} P^k \alpha^k. \tag{1.13}$$

The eigenvalues $\sigma = i\omega_0$ and critical viscosity ν_c are also represented in the form of series

$$\sigma(\alpha) = \sum_{k=0}^{\infty} \sigma_k \alpha^k, \quad \nu_c = \nu_* + \sum_{k=1}^{\infty} \nu_k \alpha^k. \tag{1.14}$$

Then

$$\varphi_1^0 = e^{-imz}, \quad \varphi_2^0 = \frac{1}{\nu_*}\varphi_1^0(z)\frac{d\theta}{dx} = \frac{1}{\nu_*}\varphi_1^0(z)a_0(x), \quad a_0(x) = \frac{d\theta}{dx}, \tag{1.15}$$

$\sigma_0 = 0$ and $\sigma_1 = im\langle V \rangle$, where $m \neq 0$ is the wave number.

Substituting expansions (1.13) and (1.14) into (1.6) and equating the coefficients of α^k, $k \geqslant 1$, yields a system for finding kth term of the asymptotics of linear spectral problem:

$$\nu_* \frac{\partial^2 \varphi_1^k}{\partial x^2} = \frac{\partial P^k}{\partial x} + \sum_{j=1}^{k} \sigma_j \varphi_1^{k-j} + V \frac{\partial \varphi_1^{k-1}}{\partial z} - \sum_{j=1}^{k-1} \nu_j \frac{\partial^2 \varphi_1^{k-j}}{\partial x^2} - \sum_{j=0}^{k-2} \nu_j \frac{\partial^2 \varphi_1^{k-2-j}}{\partial z^2}, \tag{1.16}$$

$$\nu_* \frac{\partial^2 \varphi_2^k}{\partial x^2} = \frac{\partial P^{k-1}}{\partial z} + \sum_{j=1}^{k} \sigma_j \varphi_2^{k-j} + V \frac{\partial \varphi_2^{k-1}}{\partial z} - \sum_{j=1}^{k} \nu_j \frac{\partial^2 \varphi_2^{k-j}}{\partial x^2} -$$

$$- \sum_{j=0}^{k-2} \nu_j \frac{\partial^2 \varphi_2^{k-2-j}}{\partial z^2} + \varphi_1^k \frac{dV}{dx}, \tag{1.17}$$

$$\frac{\partial \varphi_1^k}{\partial x} + \frac{\partial \varphi_2^{k-1}}{\partial z} = 0, \quad \langle \varphi_2^k \rangle = 0, \quad \int_0^{2\pi} \varphi_1^k \, dz = 0. \tag{1.18}$$

Here, the sum is assumed to extend over those values of j for which the upper boundary in the sum is not smaller then the lower one.

As a result, we obtain recurrence formulas [12].

Theorem *For shear basic flow with non-zero average* (1.4) *the critical eigenvalues are odd functions of wave number* α*, while the critical values of viscosity are even functions and*

$$\sigma_{2j+1} = (-1)^j \frac{im^{2j+1}}{\nu_*^{2j+2}} \langle \theta' a_{2j-1} \rangle, \quad \nu_{2j} = (-1)^j \frac{m^{2j}}{\nu_*^{2j+1}} \langle \theta' a_{2j} \rangle,$$

here a_k *are the coefficients of expansions of* φ^k*:*

$$\varphi_1^k = -\frac{1}{\nu_*^k} \frac{d^k \varphi_1^0}{dz^k} I(a_{k-1}(\theta)) - \frac{\nu_{k-1}}{\nu_*} \varphi_1^1 \tag{1.19}$$

$$P^k = \frac{1}{\nu_*^{k-1}} \frac{d^k \varphi_1^0}{dz^k} q_k(\theta) - \frac{\nu_{k-2}}{\nu_*} \{P^2\} + \langle P^k \rangle \tag{1.20}$$

$$\varphi_2^k = \frac{1}{\nu_*^{k+1}} \frac{d^k \varphi_1^0}{dz^k} a_k(\theta) - \frac{\nu_k}{\nu_*} \varphi_2^0 \tag{1.21}$$

where a_k*,* q_k *are expressed in terms of* a_j *and* q_j *for* $j \leqslant k-1$*. The term with* ν_{k-2} *in the expression for* P^k *appears for even* $k \geqslant 4$*.*

The explicit formulas for coefficients a_k and q_k were found in [12].

2 Recurrence Formulas for Asymptotics of $\boldsymbol{w}$

The first Eq. (1.11) for finding $\boldsymbol{w}$ has a form:

$$-\nu \left(\frac{\partial^2 w_1}{\partial x^2} + \alpha^2 \frac{\partial^2 w_1}{\partial z^2} \right) + \alpha V \frac{\partial w_1}{\partial z} = -\frac{\partial Q}{\partial x} +$$
$$+\alpha \left(W_z(\varphi_1, \overline{\varphi_2}) + \overline{W_z(\varphi_1, \overline{\varphi_2})} \right), \tag{2.1}$$

$$-\nu \left(\frac{w^2 \varphi_2}{\partial x^2} + \alpha^2 \frac{\partial^2 w_2}{\partial z^2} \right) + \alpha V \frac{\partial w_2}{\partial z} + w_1 \frac{\partial V}{\partial x} = -\alpha \frac{\partial Q}{\partial z} -$$
$$- \left(W_x(\varphi_1, \overline{\varphi_2}) + \overline{W_x(\varphi_1, \overline{\varphi_2})} \right), \tag{2.2}$$

$$\frac{\partial w_1}{\partial x} + \alpha \frac{\partial w_2}{\partial z} = 0, \tag{2.3}$$

$$\langle w_2 \rangle = 0, \quad \int_0^{2\pi} w_1 \, dz = 0. \tag{2.4}$$

We will seek the unknown components of the velocity and the pressure in the form of series of powers of the parameter α:

$$\boldsymbol{w} = \sum_{k=0}^{\infty} \boldsymbol{w}^k \alpha^k, \quad Q = \sum_{k=0}^{\infty} Q^k \alpha^k. \tag{2.5}$$

2.1 First Terms of the Asymptotic Expansion of w

We substitute (2.5) in (2.1)–(2.4) and equate the coefficients with the same powers of α. Up to α^0 from the continuity equation (2.3) we deduce that w_1^0 is a function of only z: $w_1^0 = w_1^0(z)$. Then from (2.1) $Q^0 = Q^0(z)$.

Substituting the explicit expressions for φ_1^0 and φ_2^0 into (2.2), we obtain

$$\nu_* \frac{\partial^2 w_2^0}{\partial x^2} = w_1^0 \frac{\partial V}{\partial x} + \frac{2}{\nu_*}\theta''(x)$$

Using the integral operator I, we find the solution of this equation:

$$w_2^0(x, z) = \frac{1}{\nu_*} w_1^0(z) a_0(x) + \frac{2}{\nu_*^2} A_0(x); \quad A_0(x) = \theta(x). \tag{2.6}$$

Averaging Eq. (2.1) and equating the coefficients up to α^1, we obtain the equation:

$$\langle V \rangle \frac{d w_1^0}{dz} = 0. \tag{2.7}$$

From (2.1) and the assumption $\langle V \rangle \neq 0$ we find $w_1^0(z) = const$. Taking into account (2.4) we find

$$w_1^0(z) = 0. \tag{2.8}$$

After substitution (2.8) into (2.6) we conclude that w_2^0 is a function of only x:

$$w_2^0(x) = \frac{2}{\nu_*^2}\theta(x) = \frac{2}{\nu_*^2} A_0(x); \quad A_0(x) = \theta(x). \tag{2.9}$$

Hence from continuity equation (2.1) we obtain that $w_1^1 = w_1^1(z)$ and its mean value over period is zero.

In the particular case when $\theta(x)$ is an odd function (as for the Kolmogorov flow), $w_2^0(x)$ is also an odd function.

For $k = 1$ Eq. (2.1) takes a form:

$$\frac{\partial Q^1}{\partial x} = 0,$$

hence $Q^1 = Q^1(z)$.

From (2.2) and (2.4) we obtain the problem for finding w_2^1:

$$-\nu_* \frac{\partial^2 w_2^1}{\partial x^2} = -w_1^1 \frac{\partial V}{\partial x} - \frac{dQ^0}{\partial z}; \quad \langle w_2^1 \rangle = 0. \tag{2.10}$$

From the solvability condition of this equation it follows that $Q^0 = const$, and w_2^1 is found by the formula

$$w_2^1 = \frac{1}{\nu_*} w_1^1(z) \frac{d\theta}{dx}. \tag{2.11}$$

2.2 kth Term of w

For α^k and odd k from (2.1)–(2.4) we derive the equations:

$$-\nu_* \frac{\partial^2 w_1^k}{\partial x^2} + V(x) \frac{dw_1^{k-1}}{dz} = -\frac{\partial Q^k}{\partial x} + \sum_{j=0}^{k-1} [W_z(\varphi_1^j, \overline{\varphi_2^{k-1-j}}) + \overline{W_z(\varphi_1^j, \overline{\varphi_2^{k-1-j}})}], \tag{2.12}$$

$$-\nu_* \frac{\partial^2 w_2^k}{\partial x^2} + w_1^k(z) \frac{dV}{dx} = -\frac{\partial \langle Q^{k-1} \rangle}{\partial z} - \sum_{j=0}^{k} [W(\varphi_1^j, \overline{\varphi_2^{k-j}}) + \overline{W(\varphi_1^j, \overline{\varphi_2^{k-j}})}], \tag{2.13}$$

$$\frac{\partial w_1^k}{\partial x} + \frac{\partial w_2^{k-1}}{\partial z} = 0, \tag{2.14}$$

$$\langle w_2^k \rangle = 0, \quad \int_0^{2\pi} w_1^k \, dz = 0. \tag{2.15}$$

Since the Wronskians on the right-hand sides of the Eqs. (2.12)–(2.15) contain an odd number of derivatives with respect to z, they are purely imaginary and, therefore, together with their complex conjugates give zero. In the expression Q^{k-1}, only the average depends on z. Therefore, Eqs. (2.12)–(2.13) take the form:

$$-\nu_* \frac{\partial^2 w_1^k}{\partial x^2} + V(x) \frac{dw_1^{k-1}}{dz} = -\frac{\partial Q^k}{\partial x}, \tag{2.16}$$

$$-\nu_* \frac{\partial^2 w_2^k}{\partial x^2} + w_1^k(z) \frac{dV}{dx} = -\frac{d\langle Q^{k-1} \rangle}{dz} \tag{2.17}$$

Averaging the first equation of the system (2.16) over the variable x, we arrive at the equality:

$$\langle V\rangle \frac{dw_1^{k-1}}{dz} = 0. \tag{2.18}$$

Hence, from $\langle V\rangle \neq 0$, it follows that $w_1^{k-1}(z) = const$. Given the fact that the average of this constant is equal to zero, we arrive at the equality,

$$w_1^{k-1}(z) = 0. \tag{2.19}$$

Substituting (2.19) into (2.14), we obtain that w_2^{k-1} depends only on x:

$$w_2^{k-1} = -\sum_{j=2}^{k-1} \frac{\nu_j}{\nu_*} w_2^{k-1-j} +$$

$$+\frac{1}{\nu_*}\sum_{j=0}^{k-1} I^2 \left[\{W(\varphi_1^j, \overline{\varphi_2^{k-1-j}})\} + \overline{\{W(\varphi_1^j, \overline{\varphi_2^{k-1-j}})\}}\right]. \tag{2.20}$$

Then from the continuity equation (2.14) and Eq. (2.15) it follows that $w_1^k = w_1^k(z)$ and has zero period average. This implies, in particular, that Eq. (2.16) takes the form

$$\frac{\partial Q^k}{\partial x} = 0$$

and $Q^k = Q^k(z)$.

In the particular case of an odd velocity profile ($\theta(x)$ is odd), it follows from the formula (2.20) that w_2^{k-1} is also odd. Indeed, in the previous steps it was shown that the first term is odd. In the second term, both arguments of each Wronskian have the same parity (since $k-1$ is even), therefore, all Wronskians are odd in x. Since the integral operator I is applied to Wronskian twice, the result is that the oddness will be preserved.

Equation (2.17) takes the form:

$$\nu_* \frac{\partial^2 w_2^k}{\partial x^2} = w_1^k \frac{dV}{dx} + \frac{d\langle Q^{k-1}\rangle}{dz}, \quad \langle w_2^k\rangle = 0. \tag{2.21}$$

From the solvability condition of (2.21) it follows that $\langle Q^{k-1}\rangle = 0$. Therefore, for odd k the second component of the stationary solution w_2^k has the form:

$$w_2^k = \frac{1}{\nu_*} w_1^k(z) \frac{d\theta}{dx}. \tag{2.22}$$

For α^k and even k we derive the equalities:

$$-\nu_* \frac{\partial^2 w_1^k}{\partial x^2} + V(x)\frac{dw_1^{k-1}}{dz} = -\frac{\partial Q^k}{\partial x} + 2W_z(\varphi_1^0, \overline{\varphi_2^{k-1}}) +$$

$$+2W_z(\varphi_1^{\frac{k}{2}}, \overline{\varphi_2^{\frac{k}{2}-1}}) + 2\sum_{j=1}^{\frac{k}{2}-1}[W_z(\varphi_1^j, \overline{\varphi_2^{k-1-j}}) + W_z(\varphi_1^{k-j}, \overline{\varphi_2^{j-1}})] \quad (2.23)$$

$$-\nu_* \frac{\partial^2 w_2^k}{\partial x^2} + V(x)\frac{\partial w_2^{k-1}}{\partial z} + w_1^k(z)\frac{dV}{dx} = -\frac{dQ^{k-1}}{dz} - \sum_{j=2}^{k}\nu_j \frac{d^2 w_2^{k-j}}{dx^2} -$$

$$-2W(\varphi_1^0, \overline{\varphi_2^k}) - 2\sum_{j=1}^{\frac{k}{2}}[W(\varphi_1^j, \overline{\varphi_2^{k-j}}) + W(\varphi_1^{k-j+1}, \overline{\varphi_2^{j-1}})]. \quad (2.24)$$

As in the case of odd k, we arrive at the equalities $w_1^{k-1}(z) = 0$, $w_2^{k-1} = 0$. Then it follows from the continuity equation that $w_1^k = w_1^k(z)$. From (2.23)we find Q^k:

$$Q^k = 2I\{W_z(\varphi_1^0, \overline{\varphi_2^{k-1}})\} + 2I\{W_z(\varphi_1^{\frac{k}{2}}, \overline{\varphi_2^{\frac{k}{2}-1}})\} +$$

$$+2\sum_{j=1}^{\frac{k}{2}-1} I\{W_z(\varphi_1^j, \overline{\varphi_2^{k-1-j}}) + W_z(\varphi_1^{k-j}, \overline{\varphi_2^{j-1}})\} + \langle Q^k\rangle. \quad (2.25)$$

In the particular case of an odd velocity profile ($\theta(x)$ is odd), it follows from the formula (2.25) that Q^k is even. Indeed, since $k-1$ is odd, and the Wronskians on the right-hand side of the equality (2.25) are taken over the variable z, all Wronskians are odd in x. A single application of the integral operator I changes the parity.

From the second equation of the system we find w_2^k: From here we find w_2^k:

$$w_2^k = \frac{1}{\nu_*}w_1^k(z)\frac{d\theta}{dx} + \sum_{j=2}^{k}\frac{\nu_j}{\nu_*}w_2^{k-j} + \frac{2}{\nu_*}I^2\{W(\varphi_1^0, \overline{\varphi_2^k})\} +$$

$$+\frac{2}{\nu_*}\sum_{j=1}^{\frac{k}{2}} I^2\{W(\varphi_1^j, \overline{\varphi_2^{k-j}}) + W(\varphi_1^{k-j+1}, \overline{\varphi_2^{j-1}})\}, \quad (2.26)$$

Thus, the kth term of the nonlinear correction of the velocity perturbation is found that satisfies the first Eq. (1.11).

3 Recurrence Formulas for Asymptotics of $\boldsymbol{v}$

The second Eq. (1.11) for finding $\boldsymbol{v}$ has a form:

$$2\sigma v_1 - \nu\left(\frac{\partial^2 v_1}{\partial x^2} + \alpha^2\frac{\partial^2 v_1}{\partial z^2}\right) + \alpha V\frac{\partial v_1}{\partial z} = -\frac{\partial P}{\partial x} + \alpha W_z(\varphi_1, \varphi_2), \quad (3.1)$$

$$2\sigma v_2 - \nu\left(\frac{\partial^2 v_2}{\partial x^2} + \alpha^2\frac{\partial^2 v_2}{\partial z^2}\right) + \alpha V\frac{\partial v_2}{\partial z} + v_1\frac{\partial V}{\partial x} = -\alpha\frac{\partial P}{\partial z} - W_x(\varphi_1, \varphi_2), \quad (3.2)$$

$$\frac{\partial v_1}{\partial x} + \alpha\frac{\partial v_2}{\partial z} = 0, \quad (3.3)$$

$$\langle v_2\rangle = 0, \quad \int_0^{2\pi} v_1\, dz = 0. \quad (3.4)$$

We will seek the unknown components of the velocity and the pressure in the form of a series of powers of the parameter α:

$$\boldsymbol{v} = \sum_{k=0}^{\infty}\boldsymbol{v}^k\alpha^k, \quad P = \sum_{k=0}^{\infty}P^k\alpha^k. \quad (3.5)$$

3.1 First Terms of the Asymptotic Expansion of $\boldsymbol{v}$

Up to α^0 from the continuity equation (3.3) we deduce that v_1^0 is a function of only z: $v_1^0 = v_1^0(z)$. Then from (3.1) $P^0 = P^0(z)$.

Substituting the explicit expressions for φ_1^0 and φ_2^0 into (3.2), we obtain

$$\nu_*\frac{\partial^2 v_2^0}{\partial x^2} = v_1^0\frac{dV}{dx} + W(\varphi_1^0, \varphi_2^0)$$

Since the average of the right-hand side of this equation is zero, the solvability condition is satisfied. We find a solution

$$v_2^0(x,z) = \frac{1}{\nu_*}v_1^0(z)a_0(x) + \frac{1}{\nu_*^2}(\varphi_1^0(z))^2A_0(x), \quad A_0(x) = Ia_0 = \theta(x). \quad (3.6)$$

In the particular case when $\theta(x)$ is odd, the first term containing $a_0(x)$ is even in x, and the second term containing $A_0(x)$ is odd in x. Thus, the function $v_2^0(x,z)$ is not even or odd in x, but in z, as will be shown below, both terms behave identically. The first term is a solution to a homogeneous equation, and the second is a heterogeneous one.

We also note that both terms on the right-hand side of Eq. (3.6) have the same structure as φ_2^0 in solving the linear spectral problem.

The following notation is used below. If a function f is expressed as a linear combination of φ_1^0 and its derivatives with coefficients $a_k(x)$ depending on x, then $f(\langle\varphi_1^k\rangle)$ (or $f(v_1^0(z))$) denotes the expression for f with φ_1^0 replaced by $\langle\varphi_1^k\rangle$ (or $v_1^0(z)$). Similarly, $\widetilde{f}(\langle\varphi_1^k\rangle)$ (or $\widetilde{f}((\varphi_2^0)^2)$) denotes the expression for f with the same property and, in addition, $a_k(x)$ replaced by $A_k(x)$ and

$$\widetilde{f} = \frac{1}{\nu_*} f.$$

Using this notation, we rewrite Eq. (3.6):

$$v_2^0(x,z) = \varphi_2^0(v_1^0(z)) + \widetilde{\varphi_2^0}((\varphi_1^0(z))^2). \tag{3.7}$$

We average the first Eq. (3.1):

$$2\sigma\langle v_1\rangle - \nu\alpha^2 \frac{d^2}{dz^2}\langle v_1\rangle + \alpha\frac{d}{dz}\langle V(x)v_1\rangle = \alpha\langle W_z(\varphi_1,\varphi_2)\rangle. \tag{3.8}$$

From the equality to zero of the Wronskian average the equation for α^1 has the form:

$$2\sigma_1 v_1^0(z) + \frac{d}{dz}\langle V(x)v_1^0(z)\rangle = 0. \tag{3.9}$$

or

$$\langle V\rangle\left[2imv_1^0(z) + \frac{dv_1^0}{dz}\right] = 0. \tag{3.10}$$

Hence

$$v_1^0(z) = C_1^0 e^{-2imz} = C_1^0(\varphi_1^0(z))^2. \tag{3.11}$$

If $\langle\theta' a_1\rangle \neq 0$, then we find C_1^0 from the solvability condition of the averaged Eq. (3.8) for α^3

$$C_1^0 = -\frac{4\langle\theta' A_1\rangle}{3\nu_*\langle\theta' a_1\rangle} \tag{3.12}$$

If $\langle\theta' a_1\rangle = 0$, then expansion in powers of α is needed, starting from degree -1. This case is not considered in this paper.

Similarly, the averaged equation for α^2 has the form:

$$\langle V\rangle\left[2im\langle v_1^1\rangle+\frac{d\langle v_1^1\rangle}{dz}\right]=\nu_*\frac{d^2v_1^0}{dz^2}-\frac{d^2}{dz^2}\langle\theta' v_2^0\rangle. \tag{3.13}$$

Since the right side of this equation is zero, then

$$\langle v_1^1\rangle=C_1^1e^{-2imz}=C_1^1(\varphi_1^0(z))^2. \tag{3.14}$$

If $\langle\theta' a_1\rangle\neq 0$, then we find C_1^1 from the solvability condition of the averaged Eq. (3.8) for α^4:

$$C_1^1=\frac{i}{3\langle\theta' a_1\rangle}\left(\frac{6\langle\theta' a_2\rangle C_1^0m}{\nu_*}+\frac{8\langle\theta' A_2\rangle m}{\nu_*^2}\right) \tag{3.15}$$

Note, that the constant C_1^0 is real and the constant C_1^1 is purely imaginary.

3.2 kth Term of v

Up to α^k, $k\geqslant 1$, from (3.1)–(3.4) we derive the following system of equations:

$$2\sum_{j=1}^{k}\sigma_jv_1^{k-j}-\sum_{j=0}^{k-1}\nu_j\frac{\partial^2v_1^{k-j}}{\partial x^2}-\sum_{j=0}^{k-2}\nu_j\frac{\partial^2v_1^{k-2-j}}{\partial z^2}+$$

$$+V(x)\frac{\partial v_1^{k-1}}{\partial z}=-\frac{\partial P^k}{\partial x}, \tag{3.16}$$

$$2\sum_{j=1}^{k}\sigma_jv_2^{k-j}-\sum_{j=0}^{k}\nu_j\frac{\partial^2v_2^{k-j}}{\partial x^2}-\sum_{j=0}^{k-2}\nu_j\frac{\partial^2v_2^{k-2-j}}{\partial z^2}+$$

$$+V(x)\frac{\partial v_2^{k-1}}{\partial z}+v_1^k\frac{dV}{dx}=-\frac{\partial P^{k-1}}{\partial z}-\sum_{j=0}^{k}W_x(\varphi_1^j,\varphi_2^{k-j}), \tag{3.17}$$

$$\frac{\partial v_1^k}{\partial x}+\frac{\partial v_2^{k-1}}{\partial z}=0, \tag{3.18}$$

$$\langle v_2^k\rangle=0,\quad \int\limits_0^{2\pi}v_1^k\,dz=0. \tag{3.19}$$

Taking into account the equalities $\sigma_1 = im\langle V\rangle$ and $\sigma_2 = 0$, we average Eq. (3.16):

$$\langle V\rangle\left[2im\langle v_1^{k-1}\rangle + \frac{d\langle v_1^{k-1}\rangle}{dz}\right] = -2\sum_{j=3}^{k}\sigma_j\langle v_1^{k-j}\rangle -$$
$$-\frac{d^2}{dz^2}\langle\theta' v_2^{k-2}\rangle + \sum_{j=0}^{k-2}\nu_j\frac{d^2\langle v_1^{k-2-j}\rangle}{dz^2} \tag{3.20}$$

Carrying out the proofs as shown above, we obtain that, for $k \leqslant 6$ the coefficients of the expansions in powers of α have the following structure:

$$v_1^k = \varphi_1^k(v_1^0(z)) + \widetilde{\varphi_1^k}((\varphi_1^0)^2) + \sum_{j=1}^{k-1}\varphi_1^j(\langle v_1^{k-j}\rangle) + \langle v_1^k\rangle \tag{3.21}$$

$$P^k = P^k(v_1^0(z)) + \widetilde{P^k}((\varphi_1^0)^2) + \sum_{j=1}^{k-1}P^j(\langle v_1^{k-j}\rangle) + \langle P^k\rangle \tag{3.22}$$

$$v_2^k = \varphi_2^k(v_1^0(z)) + \widetilde{\varphi_2^k}((\varphi_1^0)^2) + \sum_{j=0}^{k-1}\varphi_2^j(\langle v_1^{k-j}\rangle) \tag{3.23}$$

Here

$$\varphi_1^k(v_1^0(z)) = -\frac{1}{\nu_*^k}\frac{d^k v_1^0}{dz^k}I(\widetilde{a_{k-1}}) - \frac{\nu_{k-1}}{\nu_*}\varphi_1^1(v_1^0(z)) = \widehat{\varphi_1^k} - \frac{\nu_{k-1}}{\nu_*}\varphi_1^1 \tag{3.24}$$

$$P^k(v_1^0(z)) = \frac{1}{\nu_*^{k-1}}\frac{d^k v_1^0}{dz^k}\widetilde{q}_k(\theta) - \frac{\nu_{k-2}}{\nu_*}\{P^2\} + \langle P^k\rangle \tag{3.25}$$

$$\varphi_2^k(v_1^0(z)) = \frac{1}{\nu_*^{k+1}}\frac{d^k v_1^0}{dz^k}\widetilde{a}_k(\theta) - \frac{\nu_k}{\nu_*}\varphi_2^0(v_1^0(z)) = \widehat{\varphi_2^k} - \frac{\nu_k}{\nu_*}\varphi_2^0 \tag{3.26}$$

The formulas (3.24)–(3.26) have the same structure as in solving the linear spectral problem (1.19)–(1.21), but the equations for finding the coefficients $\widetilde{a}_k$ and $\widetilde{q}_k$, in contrast to a_k and q_k, include $2\sigma_j$ instead of σ_j. At the same time, for $0 \leqslant j \leqslant 2$ we have the equalities

$$\widetilde{a}_j = a_j, \quad \widetilde{q}_j = q_j. \tag{3.27}$$

Expressions $\varphi_m^j(\langle v_1^{k-j}\rangle)$ and $P^j(\langle v_1^{k-j}\rangle)$ in (3.21)–(3.23) have the same form as (3.24)–(3.26), but in (3.24)–(3.26) it is necessary to substitute $\langle v_1^{k-j}\rangle$ instead of $v_1^0(z)$.

Similarly,

$$\widetilde{\varphi_1^k}((\varphi_1^0)^2) = -\frac{1}{\nu_*^k}\frac{d^k((\varphi_1^0)^2)}{dz^k}I(A_{k-1}) - \frac{\nu_{k-1}}{\nu_*}\widetilde{\varphi_1^1}((\varphi_1^0)^2) \tag{3.28}$$

$$\widetilde{P^k}((\varphi_1^0)^2) = \frac{1}{\nu_*^{k-1}}\frac{d^k((\varphi_1^0)^2)}{dz^k}Q_k(\theta) - \frac{\nu_{k-2}}{\nu_*}\{P^2\} + \langle\widetilde{P^k}\rangle \tag{3.29}$$

$$\widetilde{\varphi_2^k}((\varphi_1^0)^2) = \frac{1}{\nu_*^{k+1}}\frac{d^k((\varphi_1^0)^2)}{dz^k}A_k(\theta) - \frac{\nu_k}{\nu_*}\widetilde{\varphi_2^0}((\varphi_1^0)^2) = \widehat{\widetilde{\varphi_2^k}} - \frac{\nu_k}{\nu_*}\widetilde{\varphi_2^0} \tag{3.30}$$

Below there are more detailed expressions for the coefficients of the expansion of the eigenfunctions in powers of α for even k, which were derived for of $k \leqslant 6$:

$$\widetilde{\varphi_1^k}((\varphi_1^0)^2) = -\frac{1}{\nu_*^k}\frac{d^k(\varphi_1^0)^2}{dz^k}I(B_{k-1}(\theta)) - \sum_{j=2}^{k-4}\frac{\nu_j}{\nu_*}\widetilde{\varphi_1^{k-j}} - \frac{2\nu_{k-2}}{\nu_*}\widetilde{\varphi_1^2} -$$
$$-\sum_{j=2}^{k-4}\frac{\nu_j}{\nu_*}I^2\frac{\partial^2\widetilde{\varphi_1^{k-2-j}}}{\partial z^2} + 2\sum_{j=3}^{k-1}\frac{\sigma_j}{\nu_*}I^2(\widetilde{\varphi_1^{k-j}}) \tag{3.31}$$

$$\widetilde{\varphi_2^k}((\varphi_1^0)^2) = -\frac{1}{\nu_*^{k+1}}\frac{d^k(\varphi_1^0)^2}{dz^k}B_k(\theta) - \sum_{j=2}^{k}\frac{\nu_j}{\nu_*}\widetilde{\varphi_2^{k-j}} -$$
$$-\sum_{j=2}^{k-4}\frac{\nu_j}{\nu_*}I^2\frac{\partial^2\widetilde{\varphi_2^{k-2-j}}}{\partial z^2} + 2\sum_{j=3}^{k-1}\frac{\sigma_j}{\nu_*}I^2(\widetilde{\varphi_2^{k-j}}) \tag{3.32}$$

Here, the coefficients $B_m(\theta)$ for even m are given by the formula:

$$B_m(\theta) = I^2[\{W(\theta'', IA_{m-1})\} + \nu_*^2(Q_{m-1}(\theta) - A_{m-2}(\theta))] + D_m. \tag{3.33}$$

For odd m, this expression contains an additional term:

$$B_m(\theta) = I^2[\{W(\theta'', IA_{m-1})\} + \nu_*^2(Q_{m-1}(\theta) - A_{m-2}(\theta)) -$$
$$-\frac{1}{2}\langle\theta' A_{m-3}\rangle Q_2(\theta)] + D_m. \tag{3.34}$$

Function $D_m(x)$ is a solution of following equation

$$\frac{1}{\nu_*^{m+2}}\frac{d^m((\varphi_1^0)^2)}{dz^m}\frac{d^2 D_m}{dx^2} = \sum_{j=0}^{m}\{W_x(\varphi_1^j, \varphi_2^{m-j})\} + 2\frac{\nu_{m-1}}{\nu_*}\{W_x(\varphi_1^1, \varphi_2^0)\} \tag{3.35}$$

The coefficients $Q_m(\theta)$ (through which pressure is expressed) are given by the formula

$$Q_m(\theta) = I\{\theta'' I A_{m-2}\} - B_{m-1}(\theta) - \nu_*^2 I^2(A_{m-3}) \tag{3.36}$$

for any m.

For odd k, detailed expressions for the coefficients of the eigenfunctions expanded in powers of α are given by

$$\widetilde{\varphi_1^k}((\varphi_1^0)^2) = -\frac{1}{\nu_*^k}\frac{d^k\varphi_1^0}{dz^k} I(b_{k-1}(\theta)) - \sum_{j=2}^{k-1}\frac{\nu_j}{\nu_*}\widetilde{\varphi_1^{k-j}} -$$
$$-\sum_{j=2}^{k-5}\frac{\nu_j}{\nu_*} I^2 \frac{\partial^2 \widetilde{\varphi_1^{k-2-j}}}{\partial z^2} + 2\sum_{j=3}^{k-2}\frac{\sigma_j}{\nu_*} I^2(\widetilde{\varphi_1^{k-j}}) \tag{3.37}$$

$$\widetilde{\varphi_2^k}((\varphi_1^0)^2) = -\frac{1}{\nu_*^{k+1}}\frac{d^k\varphi_1^0}{dz^k} b_k(\theta) - \sum_{j=2}^{k-3}\frac{\nu_j}{\nu_*}\widetilde{\varphi_2^{k-j}} - \frac{2\nu_{k-1}}{\nu_*}\widetilde{\varphi_2^1} -$$
$$-\sum_{j=2}^{k-3}\frac{\nu_j}{\nu_*} I^2 \frac{\partial^2 \widetilde{\varphi_2^{k-2-j}}}{\partial z^2} + 2\sum_{j=3}^{k}\frac{\sigma_j}{\nu_*} I^2(\widetilde{\varphi_2^{k-j}}), \tag{3.38}$$

where $B_m(\theta)$ and $Q_m(\theta)$ are determined in (3.12)–(3.14).

The proof of the formulas (3.21)–(3.38) is by induction.

The kth terms of the asymptotic expansion are found by applying the following scheme. Given v_2^{k-1}, we first find v_1^k from continuity equation (3.18):

$$v_1^k = -I\left(\frac{\partial v_2^{k-1}}{\partial z}\right) + \langle v_1^{k+1}\rangle. \tag{3.39}$$

In the second step, taking into account the averaging Eq. (3.20), we determine P_k from (3.16) up to the average value. In the third step, averaging Eq. (3.17), we find

$$\frac{d\langle P^{k-1}\rangle}{dz} = -\langle W_x(\{v_1^k\}, V)\rangle - \sum_{j=0}^{k}\langle W_x(\varphi_1^j, \varphi_2^{k-j})\rangle \tag{3.40}$$

Representing the right-hand side of Eq. (3.17) for finding v_2^k in divergent form, we obtain v_2^k. In the forth step, by using the quantities found at the kth step, we can write the condition for the solvability of (3.17) for the $(k+2)$th term of the asymptotic expansion. This condition is given by (3.20) with k replaced by $k+2$, and it is used to find average value of $\langle v_1^{k+1}\rangle$:

$$\langle v_1^{k+1}\rangle = C_1^{k+1}e^{-2imz}. \tag{3.41}$$

From the solvability condition for Eq. (3.20) with k replaced by $k+2$ we find C_1^{k-1}. Next, the process is repeated.

4 Conclusions

We have constructed the recurrence formula for the long-wavelength asymptotics of self-oscillations bifurcating from the basic shear flow in general form with non-zero average.

We have considered two problems (1.11). The solution $\boldsymbol{w}$ of the first problem is stationary part of non-linear term of secondary flow, while the solution $\boldsymbol{v}$ of the second problem belongs to the non-stationary part. Both these solutions have been represented as a series in powers of wave number α, $\alpha \to 0$.

The first component of $\boldsymbol{w}$ is zero

$$w_k^1 = 0$$

for any k and the expansion coefficients of the second component of velocity and coefficients of pressure vanish

$$w_2^k = 0, \quad Q^k = 0$$

for odd k.

If k is even then coefficients Q^k of pressure are given by (2.25) with $\langle Q^k\rangle = 0$ and coefficients w_2^k of velocity are given by (2.26) with $w_k^1 = 0$.

If the deviation $\{\mathrm{V}\}$ of velocity from its average value is an odd function, then $\theta(x)$ is odd as well. Then the formulas obtained for $\boldsymbol{w}$ and Q imply that $\boldsymbol{w}$ is odd and Q is even. Note, that if $\theta(x)$ is odd, then $a_1 = I\{W(\theta', \theta)\}$ is odd. Therefore, $\langle\theta' a_1\rangle = 0$.

For the second problem, a solution $\boldsymbol{v}$ was obtained under the assumption that $\langle\theta' a_1\rangle \neq 0$. We obtain kth term of the asymptotics by formulas (3.21)–(3.38).

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The Convergence of the Fourier–Jacobi Series in Weighted Variable Exponent Lebesgue Spaces

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Abstract The article focuses on the problem of basis property for the Jacobi polynomial system $P_n^{\alpha,\beta}(x)$ in the weighted variable exponent Lebesgue space $L_\mu^{p(x)}([-1, 1])$. The sufficient, and in a certain sense, necessary conditions on the variable exponent $p = p(x) > 1$ ensuring the uniform boundedness of Fourier-Jacobi sums $S_n^{\alpha,\beta}(f)$ $(n = 0, 1, \ldots)$ with $-1 < \alpha, \beta < -1/2$ in $L_\mu^{p(x)}([-1, 1])$ are obtained.

Keywords Jacobi polynomials · Fourier-Jacobi sums · Weighted Lebesgue space with variable exponent

Mathematics Subject Classification (2010) 42C10, 46E30

1 Introduction

Let E be an arbitrary set on which the Lebesgue measure m is given and let $p = p(x)$ be nonnegative m-measurable function defined on E. We denote by $L_m^{p(x)}(E)$ the space of m-measurable functions $f = f(x)$ defined on E for which the Lebesgue integral $\int_E |f(x)|^{p(x)}m(dx)$ is finite. If $p = p(x)$ is essentially bounded on E then, as shown in [1], $L_m^{p(x)}(E)$ is a linear topological space. If an additional condition $1 \le p(x) \le \overline{p} < \infty$ holds then $L_m^{p(x)}(E)$ is a Banach space with the norm

$$\|f\|_{p(\cdot)}(E) = \inf\{\alpha > 0 : \int_E \left|\frac{f(x)}{\alpha}\right|^{p(x)} m(dx) \le 1\}. \tag{1.1}$$

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If $E \in \mathbb{R}^n$, $m(dx) = w(x)dx$ then we will use notation $L_w^{p(x)}(E)$ instead of $L_m^{p(x)}(E)$ and call $L_w^{p(x)}(E)$ weighted Lebesgue space with variable exponent $p(x)$ and weight $w = w(x)$. If in addition $w(x) \equiv 1$ then we will use notation $L^{p(x)}(E)$ instead of $L_1^{p(x)}(E)$. If $1 < p(x) < \infty$, then we define new variable exponent $q(x) = p(x)/(p(x) - 1)$ and the corresponding variable exponent space $L_m^{q(x)}(E)$. As shown in the paper [1], if $1 < p(x) \le \overline{p}(E) < \infty$ then the space $(L_m^{p(x)}(E))'$ of linear continuous functionals F defined on $L_m^{p(x)}(E)$ coincides with the linear span of the space $L_m^{q(x)}(E)$: an arbitrary element $F \in (L_m^{p(x)}(E))'$ can be represented as

$$F(f) = \int_E f(x)g(x)m(dx), \quad \frac{g}{\alpha} \in L_m^{q(x)}(E), \quad \alpha \neq 0.$$

In particular, if $1 < \underline{p}(E) \le p(x) \le \overline{p}(E) \le \infty$, then q also has this property, and therefore $(L_m^{p(x)}(E))' = L_m^{q(x)}(E)$ as a consequence, the space $L_m^{p(x)}(E)$ is reflexive. This result allows us to introduce in $L_m^{p(x)}(E)$ other norms equivalent to the original one $\|f\|_{p(\cdot)}(E)$ (see (1.1)). Thus, let $1 < \underline{p}(E) \le p(x) \le \overline{p}(E) < \infty$, then the conjugate exponent $q(x) = p(x)/(p(x)-1)$ has, as has already been noted, the same properties i.e. $1 < \underline{q}(E) \le q(x) \le \overline{q}(E) < \infty$, so we can introduce the following norm of $f \in L_m^{p(x)}(E)$:

$$\|f\|^*_{p(\cdot)}(E) = \sup_{\substack{g \in L_m^{q(x)}(E), \\ \|g\|_{q(\cdot)}(E) \le 1}} \int_E f(x)g(x)m(dx), \tag{1.2}$$

for which (see [2]) inequality holds

$$\|f\|^*_{p(\cdot)}(E) \le 2\|f\|_{p(\cdot)}(E). \tag{1.3}$$

Let p and q be two variable exponents defined on E for which $1 \le p(x) \le q(x) \le \overline{q}(E) < \infty$. Then for $f \in L_m^{q(x)}(E)$ the following inequality holds (see [2–4])

$$\|f\|_{p(\cdot)}(E) \le c(p,q)\|f\|_{q(\cdot)}(E), \tag{1.4}$$

where here and everywhere in the sequel $c, c(p), c(p,q), \ldots$ denote positive numbers that depend only on these parameters.

Next, let E_1 and E_2 be two measurable sets with corresponding Lebesgue measures m_1 and m_2, with $m_i(E_i) < \infty$, $i = 1, 2$. Let $f = f(x,t)$ be a measurable function defined on the Cartesian product $E_1 \times E_2$. Then the following inequality

$$\left\| \int_{E_2} f(\cdot, t)m_2(dt) \right\|_{p(\cdot)} (E_1) \le 2 \int_{E_2} \|f(\cdot, t)\|_{p(\cdot)}(E_1)m_2(dt)$$

holds (see [2–4]).

We denote by $\mathcal{P}(-1,1)$ the space of variable exponents p, defined on $[-1,1]$ and satisfying the following conditions:

(A) the Dini–Lipschitz condition

$$|p(x)-p(y)| \le \frac{d}{-\ln|x-y|}, \quad |x-y| \le \frac{1}{2};$$

(B) $\underline{p}([-1,1]) = \min_{x\in[-1,1]} p(x) > 1$;

(C) for p there exist positive (arbitrarily small) numbers $\delta_i = \delta_i(p)$ $(i=1,2)$ such that $p(x) = p(-1)$ for $x \in [-1,-1+\delta_1]$ and $p(x) = p(1)$ for $x \in [1-\delta_2, 1]$.

Recently, the approximation theory of Lebesgue spaces with variable exponent has been intensively developed by many authors (see [4–8] and the references therein). We note that the most important results obtained in these works are related to the Dini-Lipschitz condition. In particular, it was shown in [6] that the system of Legendre polynomials form the basis in $L^{p(x)}(-1,1)$ if the variable exponent $p(x)$ satisfies the Dini–Lipschitz condition (A) and the additional conditions (B), (C).

In this paper we consider the problem of basis property for the Jacobi polynomial system $P_n^{\alpha,\beta}(x)$ in the weighted variable exponent Lebesgue space $L_\mu^{p(x)}([-1,1])$ with weight $\mu = \mu(x) = (1-x)^\alpha(1+x)^\beta$. In the case of $-1 < \alpha, \beta < -1/2$ it will be shown that if the variable exponent $p = p(x)$ satisfies the conditions (A)–(C) then the orthonormal system of Jacobi polynomials $p_n^{\alpha,\beta}(x) = (h_n^{\alpha,\beta})^{-\frac{1}{2}} P_n^{\alpha,\beta}(x)$ $(n = 0, 1, \ldots)$ is a basis in $L_\mu^{p(x)}([-1,1])$ provided that $1 < p(1), p(-1) < \infty$.

In the paper [9] a similar problem was considered in the variable exponent Lebesgue space $L_\mu^{p(x)}([-1,1])$ when $\alpha, \beta > -1/2$. Moreover, we note that this problem was solved by I.I. Sharapudinov in [10] in the case when $\alpha = \beta$.

2 The Hilbert Transform in $L^{p(x)}(\mathbb{R})$

Let $p > 1$, $f \in L^p(\mathbb{R})$. Then we can define the Hilbert transform

$$Hf = H(f) = H(f)(x) = \int_{\mathbb{R}} \frac{f(t)dt}{t-x}, \tag{2.1}$$

where the integral in (2.1) is understood in the sense of the Cauchy principal value, i.e. $H(f)(x) = \lim_{\varepsilon\to 0} H_\varepsilon(f)(x)$, $\quad H_\varepsilon(f)(x) = \int_{|t-x|>\varepsilon} \frac{f(t)dt}{t-x}$. It is well known that the function $Hf(x)$ is finite for almost all $x \in \mathbb{R}$. Moreover, if $f \in L^p(\mathbb{R})$, where $p = const > 1$, then it follows from the well-known Riesz theorem that $\|H(f)\|_p(\mathbb{R}) \le c(p)\|f\|_p(\mathbb{R})$. As shown in [11, 12], this estimate can be generalized to the case when $f \in L^{p(x)}(\mathbb{R})$ if the variable exponent $p : \mathbb{R} \to [0,\infty)$

satisfies the conditions (A) and

$$|p(x)-p(y)|\le \frac{c}{\ln(e+|x|^n)},\quad x,y\in\Omega,\quad |y|>|x|, \tag{2.2}$$

where $|x|=(x_1^2+\ldots+x_n^2)^{1/2}$ for $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$, Ω is an open set in $\mathbb{R}^n$. In other words, then we have the estimate

$$\|H(f)\|_{p(\cdot)}(\mathbb{R})\le c(p)\|f\|_{p(\cdot)}(\mathbb{R}). \tag{2.3}$$

Consider the Hilbert transform of the following form

$$H^{a,b}f=H^{a,b}f(x)=\int_a^b \frac{f(t)dt}{t-x}. \tag{2.4}$$

Let the variable exponent p satisfies the condition (A) on $[a,b]$. Then we can extend $p=p(x)$ to all $\mathbb{R}$ in the following way. Let $A=1+\max\{|a|,|b|\}$, $p(x)=3/2$ for $|x|\ge A$, and extend $p(x)$ linearly and continuously to the segments $[-A,a]$ and $[b,A]$. Then it is not difficult to verify that the extended function p satisfies the conditions (A) and (2.2) on the entire $\mathbb{R}$. On the other hand, if the function $f\in L^{p(x)}([a,b])$, then it can be extended to all $\mathbb{R}$, assuming $f(x)\equiv 0$ for $x\notin[a,b]$ and we can consider the Hilbert transform

$$Hf=Hf(x)=\int_{\mathbb{R}}\frac{f(t)dt}{t-x}=\int_a^b\frac{f(t)dt}{t-x}=H^{a,b}f(x).$$

Thus from (2.3) we deduce

$$\|H^{a,b}(f)\|_{p(\cdot)}(\mathbb{R})=\|H(f)\|_{p(\cdot)}(\mathbb{R})\le c(p)\|f\|_{p(\cdot)}(\mathbb{R})=c(p)\|f\|_{p(\cdot)}([a,b]). \tag{2.5}$$

3 Some Information About the Jacobi Polynomials

For arbitrary real α and β the Jacobi polynomials $P_n^{\alpha,\beta}(x)$ can be defined [13] using the Rodrigues formula

$$P_n^{\alpha,\beta}(x)=\frac{(-1)^n}{2^n n!}\frac{1}{\mu(x)}\frac{d^n}{dx^n}\left\{\mu(x)\sigma^n(x)\right\},$$

where $\mu(x)=\mu(x;\alpha,\beta)=(1-x)^\alpha(1+x)^\beta$, $\sigma(x)=1-x^2$. If $\alpha,\beta>-1$, Jacobi polynomials form an orthogonal system with weight $\mu(x)$, i.e.

$$\int_{-1}^1 P_n^{\alpha,\beta}(x)P_m^{\alpha,\beta}(x)\mu(x)dx=h_n^{\alpha,\beta}\delta_{nm},$$

where

$$h_n^{\alpha,\beta} = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)2^{\alpha+\beta+1}}{n!\Gamma(n+\alpha+\beta+1)(2n+\alpha+\beta+1)}. \tag{3.1}$$

For $-1 < \alpha, \beta$ and $x \in [-1, 1]$ the following estimate holds

$$\sqrt{n}|P_n^{\alpha,\beta}(x)| \leq c(\alpha, \beta)\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}}\left(\sqrt{1+x}+\frac{1}{n}\right)^{-\beta-\frac{1}{2}}. \tag{3.2}$$

This estimate together with the Christoffel-Darboux formula

$$K_n^{\alpha,\beta}(x, y) = \sum_{k=0}^{n} \frac{P_k^{\alpha,\beta}(x)P_k^{\alpha,\beta}(y)}{h_k^{\alpha,\beta}} = \frac{2^{-\alpha-\beta}}{2n+\alpha+\beta+2}\frac{\Gamma(n+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}$$

$$\times \frac{P_{n+1}^{\alpha,\beta}(x)P_n^{\alpha,\beta}(y) - P_n^{\alpha,\beta}(x)P_{n+1}^{\alpha,\beta}(y)}{x-y} \tag{3.3}$$

play a fundamental role in the study of the approximative properties of Fourier–Jacobi sums. We also note the following properties of the Jacobi polynomials:

$$P_n^{\alpha,\beta}(-x) = (-1)^n P_n^{\beta,\alpha}(x), \tag{3.4}$$

$$(1-x)P_n^{\alpha+1,\beta}(x) =$$

$$\frac{2}{2n+\alpha+\beta+2}\left[(n+\alpha+1)P_n^{\alpha,\beta}(x) - (n+1)P_{n+1}^{\alpha,\beta}(x)\right]. \tag{3.5}$$

4 Main Result

Let $\alpha, \beta > -1$, $\mu = \mu(x) = \mu(x; \alpha, \beta) = (1-x)^\alpha(1+x)^\beta$, $L_\mu^{p(x)}([-1, 1])$ be the Lebesgue space with variable exponent $p = p(x)$ and weight μ. If $p \geq 1$ and $f \in L_\mu^{p(x)}([-1, 1])$, then we can define the Fourier–Jacobi coefficients

$$f_k^{\alpha,\beta} = \frac{1}{h_k^{\alpha,\beta}}\int_{-1}^{1} f(t)P_k^{\alpha,\beta}(t)\mu(t)dt \tag{4.1}$$

and the Fourier–Jacobi sum

$$S_n^{\alpha,\beta}(f) = S_n^{\alpha,\beta}(f, x) = \sum_{k=0}^{n} f_k^{\alpha,\beta}P_k^{\alpha,\beta}(x). \tag{4.2}$$

The main goal of this paper is to find conditions on the variable exponent $p = p(x)$, that guarantee the convergence of Fourier sums $S_n^{\alpha,\beta}(f)$ on the Jacobi polynomials $P_k^{\alpha,\beta}(x)$ to the function $f \in L_\mu^{p(x)}([-1, 1])$ with respect to the norm of the space $L_\mu^{p(x)}([-1, 1])$ for $n \to \infty$. In other words, we consider the problem of the basis property in the Banach space $L_\mu^{p(x)}([-1, 1])$ of the orthonormal Jacobi polynomial system $p_k^{\alpha,\beta}(x) = (h_k^{\alpha,\beta})^{-1/2} P_k^{\alpha,\beta}(x)$ $(k = 0, 1, \ldots)$, restricting ourselves to the case $-1 < \alpha, \beta < -1/2$. To this end, we introduce a class of variable exponents $\mathcal{P}(-1, 1)$ satisfying the conditions (A), (B), and (C). It is shown that the conditions (A), (B), and (C) taken together provide the uniform boundedness in $L_\mu^{p(x)}([-1, 1])$ of the sequence of linear operators $S_n^{\alpha,\beta}(f)$ $(n = 0, 1, \ldots)$ under the additional condition (4.10). At the same time, we note that (A) and (B) are also necessary conditions for the uniform boundedness in $L_\mu^{p(x)}([-1, 1])$ of the sequence of linear operators $S_n^{\alpha,\beta}(f)$ $(n = 0, 1, \ldots)$.

We need certain transformations for Fourier–Jacobi sums $S_n^{\alpha,\beta}(f, x)$. To this end, we use the Christoffel-Darboux formula (3.3). Then (4.2) can be rewritten as

$$S_n^{\alpha,\beta}(f, x) = \int_{-1}^{1} K_n^{\alpha,\beta}(x, y) f(y)\mu(y)dy. \tag{4.3}$$

Lemma 4.1 *If $x \neq y$, then the Kristoffel-Darboux kernel $K_n^{\alpha,\beta}(x, y)$ admits the following representation*

$$K_n^{\alpha,\beta}(x, y) = K_{n1}^{\alpha,\beta}(x, y) + K_{n2}^{\alpha,\beta}(x, y),$$

where

$$K_{n1}^{\alpha,\beta}(x, y) = -\gamma_n(\alpha, \beta)\frac{(1 - x)P_n^{\alpha+1,\beta}(x)P_n^{\alpha,\beta}(y)}{x - y},$$

$$K_{n2}^{\alpha,\beta}(x, y) = \gamma_n(\alpha, \beta)\frac{(1 - y)P_n^{\alpha+1,\beta}(y)P_n^{\alpha,\beta}(x)}{x - y},$$

$\gamma_n(\alpha, \beta) = O(n)$ $(n \to \infty)$.

The proof of lemma 4.1 follows immediately from (3.3) and (3.5)

Taking into account Lemma 4.1, the equality (4.3) can be rewritten as follows

$$S_n^{\alpha,\beta}(f, x) = \gamma_n(\alpha, \beta)P_n^{\alpha,\beta}(x) \int_{-1}^{1} \frac{(1 - y)P_n^{\alpha+1,\beta}(y)f(y)\mu(y)dy}{x - y}$$

$$- \gamma_n(\alpha, \beta)(1 - x)P_n^{\alpha+1,\beta}(x) \int_{-1}^{1} \frac{P_n^{\alpha,\beta}(y)f(y)\mu(y)dy}{x - y}, \tag{4.4}$$

where the integrals are understood in the sense of the Cauchy principal values. We consider the case when $-1 < \alpha, \beta < -1/2$, $\mu(x) = (1-x)^{\alpha}(1-x)^{\beta}$, $x \in [0, 1)$. Since the function $f(x)(1-x)^{\alpha}(1-x)^{\beta}$ is integrable on $(-1, 1)$, then in this case, from (4.4) taking into account the weighted estimate (3.2) we derive for almost all $x \in [0, 1]$

$$(1-x)^{\frac{\alpha}{p(x)}}|S_n^{\alpha,\beta}(f,x)| \le$$

$$\gamma_n(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}}\left|P_n^{\alpha,\beta}(x)\int_{-1}^{1}\frac{(1-y)P_n^{\alpha+1,\beta}(y)f(y)\mu(y)dy}{x-y}\right|$$

$$+\gamma_n(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}}(1-x)\left|P_n^{\alpha+1,\beta}(x)\int_{-1}^{1}\frac{P_n^{\alpha,\beta}(y)f(y)\mu(y)dy}{x-y}\right| \le$$

$$c(\alpha,\beta)\sqrt{n}(1-x)^{\frac{\alpha}{p(x)}}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}}\left|\int_{-1}^{1}\frac{(1-y)P_n^{\alpha+1,\beta}(y)f(y)\mu(y)dy}{x-y}\right|$$

$$+c(\alpha,\beta)\sqrt{n}(1-x)^{\frac{\alpha}{p(x)}+1}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{3}{2}}\left|\int_{-1}^{1}\frac{P_n^{\alpha,\beta}(y)f(y)\mu(y)dy}{x-y}\right| \le$$

$$\sigma_1(f,x)+\sigma_2(f,x)+\sigma_3(f,x)+\sigma_4(f,x), \tag{4.5}$$

where

$$\sigma_1(f,x) = c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}} \times\left|\int_{-1}^{-\frac{1}{2}}\frac{(1-y)\sqrt{n}P_n^{\alpha+1,\beta}(y)f(y)\mu(y)dy}{x-y}\right| \tag{4.6}$$

$$\sigma_2(f,x) = c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}} \times\left|\int_{-\frac{1}{2}}^{1}\frac{(1-y)\sqrt{n}P_n^{\alpha+1,\beta}(y)f(y)\mu(y)dy}{x-y}\right| \tag{4.7}$$

$$\sigma_3(f,x) = c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}+1}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{3}{2}} \times \left|\int_{-1}^{-\frac{1}{2}} \frac{\sqrt{n}P_n^{\alpha,\beta}(y)f(y)\mu(y)dy}{x-y}\right| \tag{4.8}$$

$$\sigma_4(f,x) = c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}+1}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{3}{2}} \times \left|\int_{-\frac{1}{2}}^{1} \frac{\sqrt{n}P_n^{\alpha,\beta}(y)f(y)\mu(y)dy}{x-y}\right| \tag{4.9}$$

The following statement is fundamental.

Lemma 4.2 *Let* $-1 < \alpha, \beta < -1/2$, $p \in \mathcal{P}(-1,1)$, $\mu = \mu(x) = (1-x)^{\alpha}(1-x)^{\beta}$, $f \in L_{\mu}^{p(x)}([-1,1])$, $\|f\|_{p(\cdot)}([-1,1]) \le 1$,

$$1 < p(1) < \infty, \quad 1 < p(-1) < \infty. \tag{4.10}$$

Then

$$\|\sigma_i(f)\|_{p(\cdot)}([0,1]) \le c(\alpha,\beta,p) \quad (i = 1,2,3,4). \tag{4.11}$$

5 The Proof of Lemma 4.2

We need the following statement (see [9])

Lemma 5.1 *Let* $-1 < a, b < 1$, $0 < \gamma < 1$. *Then*

$$\frac{1}{|a-b|}\left|\left(\frac{1-b}{1-a}\right)^{\gamma} - 1\right| \le \frac{2}{(1-a)^{\gamma}((1-a)^{1-\gamma}+(1-b)^{1-\gamma})}. \tag{5.1}$$

We suppose

$$\|f\|_{p(\cdot),\mu} \le 1. \tag{5.2}$$

Consider the case $i = 1$. From (4.6), (3.2) and (5.2) we have

$$\sigma_1(f,x) \le c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}} \left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}} \times \int_{-1}^{-\frac{1}{2}} \frac{\mu(y)\left(\sqrt{1+y}+\frac{1}{n}\right)^{-\beta-\frac{1}{2}}|f(y)|}{x-y}dy$$

$$\le c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}} \left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}} \int_{-1}^{-\frac{1}{2}} \mu(y)|f(y)|dy$$

$$\le c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}} \left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}}. \quad (5.3)$$

Let $1-x \le \frac{c}{n^2} < \delta_2$. Then from (5.3) we get

$$(1-x)^{\frac{\alpha}{p(x)}} \left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}} = (1-x)^{\frac{\alpha}{p(1)}} \left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}}$$

$$\le c(\alpha)(1-x)^{\frac{\alpha}{p(1)}} n^{\alpha+\frac{1}{2}}. \quad (5.4)$$

For the case $\frac{c}{n^2} < 1-x \le 1$ we have

$$(1-x)^{\frac{\alpha}{p(x)}} \left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}} = (1-x)^{\frac{\alpha}{p(1)}} \left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}}$$

$$\le c(\alpha)(1-x)^{\frac{\alpha}{p(1)}} (1-x)^{-\frac{\alpha}{2}-\frac{1}{4}} = c(\alpha)(1-x)^{\frac{\alpha}{p(1)}-\frac{\alpha}{2}-\frac{1}{4}}. \quad (5.5)$$

From (5.4) and (5.5) we obtain

$$\sigma_1(f,x) \le c(p,\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}-\frac{\alpha}{2}-\frac{1}{4}} \; (0 \le x \le 1). \quad (5.6)$$

From (5.6) we have

$$\int_0^1 (\sigma_1(f,x))^{p(x)}dx \le c(\alpha,\beta,p)\int_0^1 (1-x)^{\alpha-\frac{\alpha p(x)}{2}-\frac{p(x)}{4}}dx \le c(\alpha,\beta,p), \quad (5.7)$$

since $\alpha - \frac{\alpha p(x)}{2} - \frac{p(x)}{4} > -1$ as $-1 < \alpha < -1/2$ and $p(1) > 1$. Hence from (5.7) we derive $\|\sigma_1(f)\|_{p(\cdot)}([0,1]) \le c(\alpha, \beta, p)$.

In a similar way we estimate $\|\sigma_3(f)\|_{p(\cdot)}([0,1])$. From (4.8), (3.2) and (5.2) we get

$$\sigma_3(f,x) \le c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}+1}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{3}{2}} \times \left|\int_{-1}^{-\frac{1}{2}} \frac{\sqrt{n}P_n^{\alpha,\beta}(y)f(y)\mu(y)dy}{x-y}\right|$$

$$\le c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}+1}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{3}{2}} \le c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}-\frac{\alpha}{2}+\frac{1}{4}}. \tag{5.8}$$

Since $\alpha - \frac{\alpha p(x)}{2} + \frac{p(x)}{4}$ from (5.8) we derive

$$\int_0^1 (\sigma_3(f,x))^{p(x)}dx \le c(\alpha,\beta,p)\int_0^1 (1-x)^{\alpha-\frac{\alpha p(x)}{2}+\frac{p(x)}{4}}dx \le c(\alpha,\beta,p). \tag{5.9}$$

We proceed to the case $i = 2$. We set

$$A_n(y) = \sqrt{n}P_n^{\alpha+1,\beta}(y)\left(\sqrt{1-y}+\frac{1}{n}\right)^{\alpha+\frac{3}{2}}\left(\sqrt{1+y}+\frac{1}{n}\right)^{\beta+\frac{1}{2}}(1+y)^{\beta}. \tag{5.10}$$

Then

$$\sigma_2(f,x) = c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}} \times \left|\int_{-\frac{1}{2}}^{1} \frac{(1-y)\sqrt{n}P_n^{\alpha+1,\beta}(y)f(y)\mu(y)dy}{x-y}\right|$$

$$= c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}}$$

$$\times \left|\int_{-\frac{1}{2}}^{1} \frac{A_n(y)(1-y)^{\alpha+1}\left(\sqrt{1-y}+\frac{1}{n}\right)^{-\alpha-\frac{3}{2}}\left(\sqrt{1+y}+\frac{1}{n}\right)^{-\beta-\frac{1}{2}} f(y)dy}{x-y}\right|$$

$$\leq c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}}$$
$$\times\left|\int_{-\frac{1}{2}}^{1}\frac{A_n(y)(1-y)^{\alpha+1}\left(\sqrt{1-y}+\frac{1}{n}\right)^{-\alpha-\frac{3}{2}}f(y)dy}{x-y}\right|$$
$$\leq \sigma_{21}(f,x)+\sigma_{22}(f,x), \tag{5.11}$$

where

$$\sigma_{21}(f,x)=c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}}$$
$$\times\left|\int_{-\frac{1}{2}}^{1-\delta_2}\frac{A_n(y)(1-y)^{\alpha+1}\left(\sqrt{1-y}+\frac{1}{n}\right)^{-\alpha-\frac{3}{2}}f(y)dy}{x-y}\right|,$$

$$\sigma_{22}(f,x)=c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}}$$
$$\times\left|\int_{1-\delta_2}^{1}\frac{A_n(y)(1-y)^{\alpha+1}\left(\sqrt{1-y}+\frac{1}{n}\right)^{-\alpha-\frac{3}{2}}f(y)dy}{x-y}\right|.$$

We assume $\varphi_n(y)=A_n(y)(1-y)^{\alpha+1}\left(\sqrt{1-y}+\frac{1}{n}\right)^{-\alpha-\frac{3}{2}}$ and note that by virtue of (3.2) there exists a constant $c(\alpha,\beta,p)$ for which

$$|\varphi_n(y)|\leq c(\alpha,\beta,p),\ \left(-\frac{1}{2}\leq y\leq 1-\frac{\delta_2}{2}\right). \tag{5.12}$$

We write $\sigma_{21}(f,x)$ in the form

$$\sigma_{21}(f,x)=c(\alpha,\beta,p)(1-x)^{\frac{\alpha}{p(x)}}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}}\left|\int_{1-\delta_2}^{1}\frac{\varphi_n(y)f(y)dy}{x-y}\right| \tag{5.13}$$

and consider two cases. If $0 \le x \le 1-\delta_2/2$ then from (2.4) and (5.13) we have

$$\sigma_{21}(f,x) \le c(\alpha,\beta,p)\left|\int_{-\frac{1}{2}}^{1-\delta_2} \frac{\varphi_n(y)f(y)dy}{x-y}\right| = c(\alpha,\beta,p)|H^{-1/2,1-\delta_2}g(x)|, \tag{5.14}$$

where $g(y)=\varphi_n(y)f(y)$. We now turn to the estimate (2.5), which, taking into account (5.12), gives

$$\|H^{-\frac{1}{2},1-\delta_2}g\|_{p(\cdot)}(\mathbb{R}) \le c(\alpha,\beta,p)\|g\|_{p(\cdot)}([-1/2,1-\delta_2]) \le$$
$$c(\alpha,\beta,p)\|f\|_{p(\cdot)}([-1/2,1-\delta_2]) \le c(\alpha,\beta,p)\|f\|_{p(\cdot),\mu}([-1/2,1-\delta_2]) \le$$
$$c(\alpha,\beta,p)\|f\|_{p(\cdot),\mu}([-1,1]) \le c(\alpha,\beta,p). \tag{5.15}$$

From (5.14) and (5.15) we have

$$\|\sigma_{21}(f)\|_{p(\cdot)}([0,1-\delta_2/2]) \le c(\alpha,\beta,p). \tag{5.16}$$

If $1-\delta_2/2 < x < 1$, then

$$\sigma_{21}(f,x) = c(\alpha,\beta,p)(1-x)^{\frac{\alpha}{p(1)}}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}}\int_{-\frac{1}{2}}^{1-\delta_2}|\varphi_n(y)f(y)|dy. \tag{5.17}$$

On the other hand, from (1.4) and (5.12) we have

$$\int_{-\frac{1}{2}}^{1-\delta_2}|g(y)|dy \le c(p)\|g\|_{p(\cdot)}([-1/2,1-\delta_2]) \le$$
$$c(\alpha,\beta,p)\|f\|_{p(\cdot),\mu}([-1/2,1-\delta_2]) \le c(\alpha,\beta,p). \tag{5.18}$$

From (5.17) and (5.18) we find (here $x_n = 1-c/n^2 \ge 1-\delta_2$)

$$\int_{1-\delta_2/2}^{1}(\sigma_{21}(f,x))^{p(x)}dx = \int_{1-\delta_2/2}^{1}(\sigma_{21}(f,x))^{p(1)}dx \le$$
$$c(\alpha,\beta,p)\left(\int_{1-\delta_2/2}^{x_n}(1-x)^{\frac{\alpha}{p(1)}-\frac{\alpha}{2}-\frac{1}{4}}dx + n^{\frac{\alpha}{2}+\frac{1}{4}}\int_{x_n}^{1}(1-x)^{\frac{\alpha}{p(1)}}dx\right) \le c(\alpha,\beta,p). \tag{5.19}$$

Combining the estimates (5.16) and (5.16), we can write

$$\|\sigma_{21}(f)\|_{p(\cdot)}([0,1]) \leq c(\alpha,\beta,p). \tag{5.20}$$

We now turn to an estimate of the norm of the function $\sigma_{22}(f)$. We consider two cases. If $0 \leq x \leq 1-\delta_2$, then from (5.11) we have

$$\sigma_{22}(f,x) = c(\alpha,\beta,p)\left|\int\limits_{1-\delta_2}^{1} \frac{A_n(y)(1-y)^{\alpha+1}\left(\sqrt{1-y}+\frac{1}{n}\right)^{-\alpha-\frac{3}{2}} f(y)dy}{x-y}\right|.$$

$$\leq c(\alpha,\beta,p)\left|\int\limits_{1-\delta_2}^{1-\frac{\delta_2}{2}}[\dots]\right| + c(\alpha,\beta,p)\left|\int\limits_{1-\frac{\delta_2}{2}}^{1}[\dots]\right| = \sigma_{221}(f,x)+\sigma_{222}(f,x). \tag{5.21}$$

We apply the Gilbert transform (2.4) to the function $g(y) = \varphi_n(y)f(y)$, then from (5.21) we have $\sigma_{221}(f,x) = c(\alpha,\beta,p)|H^{1-\delta_2,1-\delta_2/2}g(x)|$, therefore, by virtue of (2.5) and (5.12) we can write

$$\|\sigma_{221}(f)\|_{p(\cdot)}([1-\delta_2,1-\delta_2/2]) \leq c(\alpha,\beta,p)\|g\|_{p(\cdot)}([1-\delta_2,1-\delta_2/2]) \leq$$

$$c(\alpha,\beta,p)\|f\|_{p(\cdot),\mu}([1-\delta_2,1-\delta_2/2]) \leq$$

$$c(\alpha,\beta,p)\|f\|_{p(\cdot),\mu}([-1,1]) \leq c(\alpha,\beta,p). \tag{5.22}$$

Next, from (3.2) and (5.10) it follows that

$$|A_n(y)| \leq c(\alpha,\beta) \quad (-1/2 \leq y < 1), \tag{5.23}$$

so from (5.21) we have using Holder inequality

$$\sigma_{222}(f,x) = c(\alpha,\beta,p)\left|\int\limits_{1-\delta_2/2}^{1}(1-y)^{\alpha+1}\left(\sqrt{1-y}+\frac{1}{n}\right)^{-\alpha-\frac{3}{2}} f(y)dy\right|$$

$$\leq c(\alpha,\beta,p)\left|\int\limits_{1-\delta_2/2}^{1}(1-y)^{\frac{\alpha}{2}+\frac{1}{4}-\frac{\alpha}{p(1)}}\mu^{\frac{\alpha}{p(1)}}(y)f(y)dy\right|$$

$$\leq c(\alpha,\beta,p)\|f\|_{p(\cdot),\mu}([1-\delta_2/2,1])\left(\int\limits_{1-\delta_2/2}^{1}(1-y)^{\left(\frac{\alpha}{2}+\frac{1}{4}-\frac{\alpha}{p(1)}\right)q(1)}dy\right)^{\frac{1}{q(1)}}, \tag{5.24}$$

where $q(1)=p(1)/(p(1)-1)$. The last integral is finite, since by virtue of (4.10) $\left(\frac{\alpha}{2}+\frac{1}{4}-\frac{\alpha}{p(1)}\right)q(1)>-1$. Thus, from (5.24) we derive $(0\leq x\leq 1-\delta_2)$

$$\sigma_{222}(f,x)\leq c(\alpha,\beta,p)\|f\|_{p(\cdot),\mu}([1-\delta_2/2,1])$$

$$\leq c(\alpha,\beta,p)\|f\|_{p(\cdot),\mu}([-1,1])\leq c(\alpha,\beta,p),$$

from which, in turn, we find

$$\|\sigma_{222}(f)\|_{p(\cdot)}([0,1-\delta_2])\leq c(\alpha,\beta,p). \tag{5.25}$$

The estimates (5.21), (5.22) and (5.25) taken together give

$$\|\sigma_{22}(f)|_{p(\cdot)}([0,1-\delta_2])\leq c(\alpha,\beta,p). \tag{5.26}$$

Consider the case $1-\delta_2\leq x<1$ and note that in this case $p(x)=p(1)$, so we can obtain for $\sigma_{22}(f,x)$ the following estimate

$$\sigma_{22}(f,x)\leq c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(1)}-\frac{\alpha}{2}-\frac{1}{4}}\left|\int\limits_{1-\delta_2}^{1}\frac{A_n(y)(1-y)^{\frac{\alpha}{2}+\frac{1}{4}}f(y)dy}{x-y}\right|$$

$$=c(\alpha,\beta)\left|\int\limits_{1-\delta_2}^{1}\frac{A_n(y)(1-y)^{\frac{\alpha}{p(x)}}f(y)}{x-y}\left(\frac{1-y}{1-x}\right)^{\frac{\alpha}{2}+\frac{1}{4}-\frac{\alpha}{p(x)}}dy\right|$$

$$\leq U(f,x)+V(f,x), \tag{5.27}$$

where

$$U(f,x)=c(\alpha,\beta)\left|\int\limits_{1-\delta_2}^{1}\frac{A_n(y)(1-y)^{\frac{\alpha}{p(1)}}f(y)}{x-y}dy\right|, \tag{5.28}$$

$$V(f,x) = c(\alpha,\beta)\int\limits_{1-\delta_2}^{1} \frac{|A_n(y)|(1-y)^{\frac{\alpha}{p(1)}}|f(y)|}{|x-y|}\left|\left(\frac{1-y}{1-x}\right)^{\frac{\alpha}{2}+\frac{1}{4}-\frac{\alpha}{p(1)}}-1\right|dy. \tag{5.29}$$

We note that

$$0 < \frac{\alpha}{2}+\frac{1}{4}-\frac{\alpha}{p(1)} < 1 \text{ if } 1 < p(1) < \frac{4\alpha}{2\alpha+1}, \tag{5.30}$$

$$\frac{\alpha}{2}+\frac{1}{4}-\frac{\alpha}{p(1)} = 0 \text{ if } p(1) = \frac{4\alpha}{2\alpha+1}, \tag{5.31}$$

$$0 < \frac{\alpha}{p(1)}-\frac{\alpha}{2}-\frac{1}{4} < 1 \text{ if } \frac{4\alpha}{2\alpha+1} < p(1) < \infty. \tag{5.32}$$

First we estimate $U(f,x)$. We set $\psi(y) = A_n(y)(1-y)^{\frac{\alpha}{p(1)}}f(y)$ and use the definition of (2.4), then from (5.28) we have $U(f,x) = c(\alpha,\beta)|H^{1-\delta_2,1}\psi(x)|$, therefore, from the estimates (2.5) and (5.23) we find

$$\|U(f)\|_{p(\cdot)}([1-\delta_2,1]) \le c(\alpha,\beta,p)\|\psi\|_{p(\cdot)}([1-\delta_2,1]) \le$$
$$c(\alpha,\beta,p)\|f\|_{p(\cdot),\mu}([1-\delta_2,1]) \le c(\alpha,\beta,p). \tag{5.33}$$

We proceed to estimate $\|V(f)\|_{p(\cdot)}([1-\delta_2,1])$ in the case of (5.32). Under condition (5.30), the value of $\|V(f)\|_{p(\cdot)}([1-\delta_2,1])$ is estimated similarly and $\|V(f)\|_{p(\cdot)}([1-\delta_2,1]) = 0$ in the case of (5.31).

We turn to the relations (2.3), (1.3) and (5.23), from which, assuming $\gamma = \alpha/p(1)-\alpha/2-1/4$,

$$K_\gamma(x,y) = \frac{1}{|x-y|}\left|\left(\frac{1-y}{1-x}\right)^{\gamma}-1\right|,$$

we find

$$\|V(f)\|_{p(\cdot)}([1-\delta_2,1]) \le c(\alpha,\beta)\sup_g \int\limits_{1-\delta_2}^{1}\int\limits_{1-\delta_2}^{1} |g(x)\psi(y)|K_\gamma(x,y)dxdy, \tag{5.34}$$

where the supremum is taken over all $g \in L^{q(1)}([1-\delta_2, 1])$, for which $\|g\|_{p(\cdot)}([1-\delta_2, 1]) \leq 1$. We estimate the double integral I from (5.34). We have

$$I = \int_{1-\delta_2}^{1} \int_{1-\delta_2}^{1} |\psi(y)| K_\gamma^{\frac{1}{p(1)}}(x, y) \left(\frac{1-x}{1-y}\right)^{-\frac{1}{p(1)q(1)}} \times$$

$$|g(x)| K_\gamma^{\frac{1}{q(1)}}(x, y) \left(\frac{1-x}{1-y}\right)^{\frac{1}{p(1)q(1)}} dx dy \leq$$

$$\left(\int_{1-\delta_2}^{1} |g(x)|^{q(1)} F_1(x) dx\right)^{\frac{1}{q(1)}} \left(\int_{1-\delta_2}^{1} |\psi(y)|^{p(1)} F_2(y) dy\right)^{\frac{1}{p(1)}}, \tag{5.35}$$

where

$$F_1(x) = \int\limits_{1-\delta_2}^{1} K_\gamma(x, y) \left(\frac{1-x}{1-y}\right)^{\frac{1}{p(1)}} dy,$$

$$F_2(y) = \int\limits_{1-\delta_2}^{1} K_\gamma(x, y) \left(\frac{1-y}{1-x}\right)^{\frac{1}{q(1)}} dx.$$

Let us show that functions F_1 and F_2 are bounded on $[1-\delta_2, 1]$. To this end we note that $0 < \gamma < \frac{1}{4}$ when $-1 < \alpha < -1/2$ so we can use Lemma 5.1, from which we have

$$F_1(x) \leq 2 \int_{1-\delta_2}^{1} \frac{(1-x)^{\frac{1}{p(1)}} dy}{(1-y)^{\frac{1}{p(1)}+\gamma}((1-y)^{1-\gamma} + (1-x)^{1-\gamma})} \leq$$

$$2 \int_{1-\delta_2}^{x} \frac{(1-x)^{\frac{1}{p(1)}} dy}{(1-y)^{\frac{1}{p(1)}+1}} + 2 \int_{x}^{1} \frac{(1-x)^{\frac{1}{p(1)}} dy}{(1-y)^{\frac{1}{p(1)}+\gamma}(1-x)^{1-\gamma}} \leq$$

$$2(1-x)^{\frac{1}{p(1)}} p(1)(1-x)^{-\frac{1}{p(1)}} + 2(1-x)^{\frac{1}{p(1)}-1+\gamma} \frac{(1-x)^{1-\frac{1}{p(1)}-\gamma}}{1-\frac{1}{p(1)}-\gamma} \leq c(\alpha, p), \tag{5.36}$$

$$F_2(x) \leq 2 \int_{1-\delta_2}^{1} \frac{(1-y)^{\frac{1}{q(1)}} dx}{(1-x)^{\gamma+\frac{1}{q(1)}}((1-y)^{1-\gamma} + (1-x)^{1-\gamma})} \leq$$

$$2 \int_{1-\delta_2}^{y} \frac{(1-y)^{\frac{1}{q(1)}} dx}{(1-x)^{\frac{1}{q(1)}+1}} + 2 \int_{y}^{1} \frac{(1-y)^{\frac{1}{q(1)}-1+\gamma} dx}{(1-x)^{\frac{1}{q(1)}+\gamma}} \leq$$

$$2(1-y)^{\frac{1}{q(1)}}\frac{(1-y)^{-\frac{1}{q(1)}}}{1/q(1)}+2(1-y)^{\frac{1}{q(1)}-1+\gamma}\frac{(1-y)^{1-\frac{1}{q(1)}-\gamma}}{1-\frac{1}{q(1)}-\gamma}\le c(\alpha,p). \tag{5.37}$$

It follows from (5.23), (5.35), (5.36) and (5.37) that

$$I\le c(\alpha,p)\|g\|_{q(1)}([1-\delta_2,1])\|\psi\|_{p(1)}([1-\delta_2,1])\le$$
$$c(\alpha,\beta,p)\|A_n\mu^{\frac{1}{p(1)}}f\|_{p(1)}([1-\delta_2,1])\le c(\alpha,\beta,p)\|f\|_{p(\cdot),\mu}([1-\delta_2,1])\le$$
$$c(\alpha,\beta,p)\|f\|_{p(\cdot),\mu}([-1,1]), \tag{5.38}$$

Putting together the estimates (5.34) and (5.38) yields

$$\|V(f)\|_{p(\cdot)}([1-\delta_2,1])\le c(\alpha,\beta,p). \tag{5.39}$$

From the inequality (5.27), (5.33) and (5.39) we derive

$$\|\sigma_{22}(f)\|_{p(\cdot)}([1-\delta_2,1])\le c(\alpha,\beta,p), \tag{5.40}$$

and from (5.26), (5.40) we get

$$\|\sigma_{22}(f)\|_{p(\cdot)}([0,1])\le c(\alpha,\beta,p). \tag{5.41}$$

Comparing (5.20) and (5.41) with (5.11) we see that the assertion of Lemma 4.2 holds for $i=2$. We proceed to proof the assertion of Lemma 4.2 for $i=4$. We set

$$B_n(y)=\sqrt{n}(1+y)^{\beta}(\sqrt{1-y}+\frac{1}{n})^{\alpha+\frac{1}{2}}P_n^{\alpha,\beta}(y), \tag{5.42}$$

$$\Phi_n(y)=B_n(y)(1-y)^{\alpha}(\sqrt{1-y}+\frac{1}{n})^{-\alpha-\frac{1}{2}} \tag{5.43}$$

and note, that by virtue of (3.2) we have

$$|B_n(y)|\le c(\alpha,\beta)\quad(-1/2\le y<1), \tag{5.44}$$

$$|\Phi_n(y)|\le c(\alpha,\beta,p)\quad(-1/2\le y<1-\delta_2/2). \tag{5.45}$$

From (4.9), (5.42) and (5.43) we have

$$\sigma_4(f,x)=c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}+1}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{3}{2}}\left|\int_{-\frac{1}{2}}^{1}\frac{\Phi_n(y)f(y)dy}{x-y}\right|$$
$$\le\sigma_{41}(f,x)+\sigma_{42}(f,x), \tag{5.46}$$

where

$$\sigma_{41}(f,x)=c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}+1}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{3}{2}}\left|\int_{-\frac{1}{2}}^{1-\delta_2}\frac{\Phi_n(y)f(y)dy}{x-y}\right|, \tag{5.47}$$

$$\sigma_{42}(f,x)=c(\alpha,\beta)(1-x)^{\frac{\alpha}{p(x)}+1}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{3}{2}}\left|\int_{1-\delta_2}^{1}\frac{\Phi_n(y)f(y)dy}{x-y}\right|. \tag{5.48}$$

Let us estimate $|\sigma_{41}(f,x)|$. To this end we consider two cases: (1) $0\le x\le 1-\delta_2/2$; (2) $1-\delta_2/2\le x<1$. If $0\le x\le 1-\delta_2/2$ then it follows from (2.4) and (5.47) that

$$\sigma_{41}(f,x)\le c(\alpha,\beta,p)\left|\int_{-\frac{1}{2}}^{1-\delta_2}\frac{\Phi_n(y)f(y)dy}{x-y}\right|=c(\alpha,\beta,p)\left|H^{-\frac{1}{2},1-\delta_2}g(x)\right|, \tag{5.49}$$

where $g(y)=\Phi_n(y)f(y)$, and for $1-\delta_2/2\le x<1$ we have

$$\sigma_{41}(f,x)\le c(\alpha,\beta,p)(1-x)^{\frac{\alpha}{p(1)}+1}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{3}{2}}\int_{-\frac{1}{2}}^{1-\delta_2}|\Phi_n(y)f(y)|dy. \tag{5.50}$$

If we turn to the estimate (2.5), then, setting $g(y)=\Phi_n(y)f(y)$ and taking into account (5.42)–(5.45), we can write

$$\|H^{-\frac{1}{2},1-\delta_2}g\|_{p(\cdot)}(\mathbb{R})\le c(\alpha,\beta,p)\|g\|_{p(\cdot)}([-1/2,1-\delta_2])\le$$
$$c(\alpha,\beta,p)\|f\|_{p(\cdot)}([-1/2,1-\delta_2])\le c(\alpha,\beta,p)\|f\|_{p(\cdot),\mu}([-1/2,1-\delta_2]). \tag{5.51}$$

From (5.49) and (5.51) we derive

$$\|\sigma_{41}(f)\|_{p(\cdot)}([0,1-\delta_2])\le c(\alpha,\beta,p)\|f\|_{p(\cdot),\mu}([-1,1])\le c(\alpha,\beta,p). \tag{5.52}$$

On the other hand, by virtue of (1.4) and (5.45)

$$\int_{-\frac{1}{2}}^{1-\delta_2}|g(y)|dy\le c(p)\|g\|_{p(\cdot)}([-\frac{1}{2},1-\delta_2])\le c(\alpha,\beta,p)\|f\|_{p(\cdot),\mu}([-\frac{1}{2},1-\delta_2])\le c(\alpha,\beta,p),$$

therefore, for $p \in \mathcal{P}(-1, 1)$ from (5.50) we have

$$\int\limits_{1-\delta_2}^{1} (\sigma_{41}(f, x))^{p(x)} dx \leq$$

$$c(\alpha, \beta, p) \int\limits_{1-\delta_2}^{1} \left[(1-x)^{\frac{\alpha}{p(1)}+1} \left(\sqrt{1-x} + \frac{1}{n} \right)^{-\alpha-\frac{3}{2}} \right]^{p(1)} dx \leq$$

$$c(\alpha, \beta, p) \int\limits_{1-\delta_2}^{1} (1-x)^{\alpha+(\frac{1}{4}-\frac{\alpha}{2})p(1)} dx \leq$$

$$\frac{c(\alpha, \beta, p)\delta_2^{\alpha+(\frac{1}{4}-\frac{\alpha}{2})p(1)}}{\alpha + (\frac{1}{4} - \frac{\alpha}{2})p(1) + 1} \leq c(\alpha, \beta, p). \tag{5.53}$$

From (5.52) and (5) we find

$$\|\sigma_{41}(f)\|_{p(\cdot)}([0, 1]) \leq c(\alpha, \beta, p). \tag{5.54}$$

We proceed to estimate $|\sigma_{42}(f, x)|$. To this end, we consider two cases: 1) $0 \leq x \leq 1 - \delta_2$; 2) $1 - \delta_2 \leq x < 1$. If $0 \leq x \leq 1 - \delta_2$ then according to (5.48) we have

$$\sigma_{42}(f, x) = c(\alpha)(1-x)^{\frac{\alpha}{p(1)}+1} \left(\sqrt{1-x} + \frac{1}{n} \right)^{-\alpha-\frac{3}{2}}$$
$$\times \left| \left(\int_{-\frac{1}{2}}^{1-\delta_2} + \int_{1-\delta_2}^{1} \right) \frac{\Phi_n(y) f(y) dy}{x-y} \right| \leq \sigma_{421}(f, x) + \sigma_{422}(f, x), \tag{5.55}$$

where

$$\sigma_{421}(f, x) = c(\alpha, \beta, p) \left| \int_{1-\delta_2}^{1-\delta_2/2} \frac{\Phi_n(y) f(y) dy}{x-y} \right|, \tag{5.56}$$

$$\sigma_{422}(f, x) = c(\alpha, \beta, p) \left| \int_{1-\delta_2/2}^{1} \frac{\Phi_n(y) f(y) dy}{x-y} \right|. \tag{5.57}$$

Let us set $r_n = r_n(y) = \Phi_n(y) f(y)$ and use the definition (2.4), then we can rewrite (5.56) as $\sigma_{421}(f, x) = c(\alpha, \beta, p) \left| H^{1-\delta_2, 1-\delta_2/2} r_n(x) \right|$, therefore, by virtue of (2.5) and (5.45)

$$\|\sigma_{421}(f)\|_{p(\cdot)}([0,1-\delta_2]) \le c(\alpha,\beta,p)\|r_n\|_{p(\cdot)}([1-\delta_2,1-\delta_2/2]) \le$$
$$c(\alpha,\beta,p)\|f\|_{p(\cdot)}([1-\delta_2,1-\delta_2/2]) \le$$
$$c(\alpha,\beta,p)\|f\|_{p(\cdot),\mu}([-1,1]) \le c(\alpha,\beta,p). \tag{5.58}$$

Next, by virtue of (5.44), (5.57) and taking into account that (5.2) we can write

$$\sigma_{422}(f,x) \le c(\alpha,\beta,p)\int_{1-\delta_2/2}^{1}(1-y)^{\alpha}(\sqrt{1-y}+\frac{1}{n})^{-\alpha-\frac{1}{2}}|f(y)|dy \le$$
$$c(\alpha,\beta,p)\int_{1-\delta_2/2}^{1}(1-y)^{\frac{\alpha}{q(1)}}(\sqrt{1-y}+\frac{1}{n})^{-\alpha-\frac{1}{2}}|f(y)|\mu^{\frac{\alpha}{p(1)}}(y)dy \le$$
$$c(\alpha,\beta,p)\|f\|_{p(\cdot),\mu}([-1,1])\left(\int_{1-\delta_2/2}^{1}(1-y)^{\alpha}(\sqrt{1-y}+\frac{1}{n})^{-(\alpha+\frac{1}{2})q(1)}dy\right)^{\frac{1}{q(1)}}.$$

Thus (see (5.19)) $\sigma_{422}(f,x) \le c(\alpha,\beta,p)$ and

$$\|\sigma_{422}(f)\|_{p(\cdot)}([0,1-\delta_2]) \le c(\alpha,\beta,p). \tag{5.59}$$

The estimates (5.55), (5.58) and (5.59), taken together, give

$$\|\sigma_{42}(f)\|_{p(\cdot)}([0,1-\delta_2]) \le c(\alpha,\beta,p). \tag{5.60}$$

We consider now the case $1-\delta_2 \le x < 1$. Let $y_n = 1-c/n^2 > 1-\delta_2$, then

$$\sigma_{42}(f,x) \le c(\alpha)(1-x)^{\frac{\alpha}{p(1)}-\frac{\alpha}{2}+\frac{1}{4}}$$
$$\times\left|\int_{1-\delta_2}^{1-y_n}\frac{B_n(y)(1-y)^{\alpha}(\sqrt{1-y}+\frac{1}{n})^{-\alpha-\frac{1}{2}}f(y)dy}{x-y}\right|$$
$$+c(\alpha)(1-x)^{\frac{\alpha}{p(1)}-\frac{\alpha}{2}+\frac{1}{4}}\left|\int_{1-y_n}^{1}\frac{B_n(y)(1-y)^{\alpha}(\sqrt{1-y}+\frac{1}{n})^{-\alpha-\frac{1}{2}}f(y)dy}{x-y}\right|$$
$$= I_1 + I_2. \tag{5.61}$$

We estimate I_1. To this end we write

$$I_1 \le G(f,x) + Q(f,x), \tag{5.62}$$

where

$$G(f,x) = c(\alpha)\left|\int_{1-\delta_2}^{1-y_n} \frac{B_n(y)(1-y)^{\frac{\alpha}{p(1)}} f(y)dy}{x-y}\right|, \tag{5.63}$$

$$Q(f,x) = c(\alpha)\left|\int_{1-\delta_2}^{1-y_n} \frac{B_n(y)(1-y)^{\frac{\alpha}{p(1)}} f(y)dy}{x-y}\left[\left(\frac{1-x}{1-y}\right)^{\frac{\alpha}{p(1)}-\frac{\alpha}{2}+\frac{1}{4}} - 1\right]\right|. \tag{5.64}$$

Value $G(f,x)$ can be estimated in a similar way as $U(f,x)$ therefore we have

$$\|G(f)\|_{p(\cdot)}([1-\delta_2,1]) \le c(\alpha,\beta,p)\|f\|_{p(\cdot)}([-1,1]) \le c(\alpha,\beta,p). \tag{5.65}$$

Next, we note

$$0 < -\frac{\alpha}{2} + \frac{1}{4} + \frac{\alpha}{p(1)} < 1. \tag{5.66}$$

and set

$$K_\gamma(x,y) = \frac{1}{|x-y|}\left|\left(\frac{1-x}{1-y}\right)^\gamma - 1\right|,\ \gamma = -\frac{\alpha}{2} + \frac{1}{4} + \frac{\alpha}{p(1)}. \tag{5.67}$$

Then, taking into account (5.64), we can write

$$\|Q(f)\|_{p(\cdot)}([1-\delta_2,1]) \le c(\alpha)\sup_g \int_{1-\delta_2}^{1}\int_{1-\delta_2}^{1} |g(x)Z(y)|K_\gamma(x,y)dxdy, \tag{5.68}$$

where $Z(y) = B_n(y)(1-y)^{\frac{\alpha}{p(1)}}$, and the supremum is taken over all $g \in L^{q(1)}([1-\delta_2,1])$, for which $\|g\|_{p(\cdot)}([1-\delta_2,1]) \le 1$. The double integral J from (5.68) can be estimated in a similar way as I. Thus

$$J \le c(\alpha,\beta,p).$$

Comparing this estimate with (5.68), we derive

$$\|Q(f)\|_{p(\cdot)}([1-\delta_2,1]) \le c(\alpha,\beta,p). \tag{5.69}$$

From estimates (5.62), (5.65) and (5.69)

$$I_1 \le c(\alpha,\beta,p). \tag{5.70}$$

We proceed to I_2 estimation. From (5.44) assuming $(1-y)^\alpha f(y) = g(y)$ we get

$$I_2 \le c(\alpha) n^{\alpha+\frac{1}{2}} (1-x)^{\frac{\alpha}{p(1)}-\frac{\alpha}{2}+\frac{1}{4}} \left| \int_{1-y_n}^{1} \frac{(1-y)^\alpha f(y) dy}{x-y} \right|$$
$$\le c(\alpha,\beta) n^{\alpha+\frac{1}{2}} (1-x)^{\frac{\alpha}{p(1)}-\frac{\alpha}{2}+\frac{1}{4}} H^{1-y_n,1} g(x)$$
$$\le c(\alpha,\beta,p) n^{\alpha+\frac{1}{2}} (1-x)^{\frac{\alpha}{p(1)}-\frac{\alpha}{2}+\frac{1}{4}} \tag{5.71}$$

The estimates (5.61), (5.70) and (5.71) give

$$\|\sigma_{42}(f)\|_{p(\cdot)}([1-\delta_2, 1]) \le c(\alpha,\beta,p), \tag{5.72}$$

and from (5.60) and (5.72) we get

$$\|\sigma_{42}(f)\|_{p(\cdot)}([0, 1]) \le c(\alpha,\beta,p). \tag{5.73}$$

Comparing (5.54) and (5.73) with (5.46), we see that the assertion of Lemma 4.2 holds for $i = 4$. Thus, Lemma 4.2 is completely proved.

Theorem 5.2 *Let $-1 < \alpha, \beta < -1/2$, $\mu = \mu(x) = (1-x)^\alpha (1+x)^\beta$, $p \in \mathcal{P}(-1, 1)$, $1 < p(1) < \infty$, $1 < p(-1) < \infty$. Then*

$$\|S_n^{\alpha,\beta}(f)\|_{p(\cdot),\mu}([-1, 1]) \le c(\alpha,\beta,p) \|f\|_{p(\cdot),\mu}([-1, 1]) \tag{5.74}$$

holds for any function $f \in L_\mu^{p(x)}([-1, 1])$.

The proof of this assertion is similar to the Theorem 5.1 from [9].

Corollary 5.3 *If Theorem 5.2 conditions hold, then the system of orthonormal Jacobi polynomials $\{p_n^{\alpha,\beta}(x)\}_{n=0}^\infty$ with $-1 < \alpha, \beta < -1/2$ is a basis of the space $L_\mu^{p(x)}([-1, 1])$, where $\mu = \mu(x) = (1-x)^\alpha (1+x)^\beta$ and consequently*

$$\|f - S_n^{\alpha,\beta}(f)\|_{p(\cdot),\mu}([-1, 1]) \to 0$$

as $n \to \infty$ for arbitrary function $f \in L_\mu^{p(x)}([-1, 1])$.

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Hyperbolic B-Potentials: Properties and Inversion

E. L. Shishkina

Abstract The paper is devoted to the study of the fractional integral operator which is a negative real power of the singular wave operator generated by Bessel operator, its properties and its inverse using weighted distributions.

Keywords Hyperbolic Riesz B-potential · Fractional power of singular hyperbolic operator · Lorentz distance · Singular Bessel differential operator · Generalized translation · Multidimensional Hankel transform

MSC Classification 31B15, 31B10, 26A33, 46E30

1 Introduction

In recent years, the interest to the Fractional Calculus has been increasing due to its applications in many fields. As for multidimensional case the most developed type of fractional integrals are Riesz potentials which are generalized both Newton potential to the fractional case and Riemann-Liouville fractional integral to the multidimensional case.

Marcel Riesz was a Hungarian mathematician who first established the fractional powers of the Laplace and D'Alembert operators (see [1] and [2]). Such potentials are called the **Riesz potentials** now and have the forms

$$I^{\alpha}_{\Delta} f(P) = \frac{1}{\gamma_n(\alpha)} \int\limits_{\mathbb{R}^n} f(Q) r^{\alpha-n} dQ$$

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and

$$I_{\square}^{\alpha} f(P) = \frac{1}{H_n(\alpha)} \int\limits_D f(Q) r_{PQ}^{\alpha-n} dQ,$$

where $P = (x_1, \ldots, x_n)$, $Q = (\xi_1, \ldots, \xi_n)$, $\gamma_n(\alpha)$, $H_n(\alpha)$ is normalizing constant,

$$r=\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + \ldots + (x_n - \xi_n)^2}$$

is the Euclidean distance,

$$r_{PQ}=\sqrt{(x_1 - \xi_1)^2 - (x_2 - \xi_2)^2 - \ldots - (x_n - \xi_n)^2}$$

is the Lorentz distance, $D = \{x : x_1^2 \geq x_2^2 + \ldots + x_n^2\}$ is the positive cone.

In [2] was shown that

$$\Delta I_{\Delta}^{\alpha+2} f(P) = -I_{\Delta}^{\alpha} f(P)$$

and

$$\square I_{\square}^{\alpha+2} f(P) = I_{\square}^{\alpha} f(P).$$

For further properties such as conditions of existence, semigroup property and inversion see [2–6]. The theory of hyperbolic potentials introduced in [5] was developed in the articles [7, 8]. As for classical Riesz potentials with Lorentz distance we refer to [6, 9].

The theory of fractional powers of elliptic operators with Bessel operator

$$B_{\nu}=D^2 + \frac{\nu}{x} D, \qquad D = \frac{d}{dx}$$

acting instead of all or some second derivatives in Δ is well developed (see [10–16]).

Fractional powers of hyperbolic operators, with Bessel operators instead of all or some second derivatives were studied in [17–21]. Such operators have wide areas of application such as singular differential equations, differential geometry and random walks.

In this article we study real powers of

$$\square_{\gamma} = B_{\gamma_1} - B_{\gamma_2} - \ldots - B_{\gamma_n}, \qquad B_{\gamma_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}, \qquad i = 1, \ldots, n.$$

Composition method (see [22]) was used for construction of $(\square_{\gamma})^{-\frac{\alpha}{2}}$, $\alpha > 0$.

2 Basic Definitions

In this section we derive some definitions that we use later in this article.

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space,

$$\mathbb{R}^n_+=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n,\quad x_1>0,\ldots,x_n>0\},$$

$\gamma=(\gamma_1,\ldots,\gamma_n)$ is a multi-index consisting of positive fixed real numbers γ_i, $i=1,\ldots,n$, and $|\gamma|=\gamma_1+\ldots+\gamma_n$.

Let Ω be finite or infinite open set in $\mathbb{R}^n$ symmetric with respect to each hyperplane $x_i=0$, $i=1,\ldots,n$, $\Omega_+=\Omega\cap\mathbb{R}^n_+$ and $\overline{\Omega}_+=\Omega\cap\overline{\mathbb{R}}^n_+$ where $\overline{\mathbb{R}}^n_+=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n, x_1\geq0,\ldots,x_n\geq0\}$. We deal with the class $C^m(\Omega_+)$ consisting of m times differentiable on Ω_+ functions and denote by $C^m(\overline{\Omega}_+)$ the subset of functions from $C^m(\Omega_+)$ such that all derivatives of these functions with respect to x_i for any $i=1,\ldots,n$ are continuous up to $x_i=0$. Class $C^m_{ev}(\overline{\Omega}_+)$ consists of all functions from $C^m(\overline{\Omega}_+)$ such that $\left.\frac{\partial^{2k+1}f}{\partial x_i^{2k+1}}\right|_{x=0}=0$ for all non-negative integer $k\leq\frac{m-1}{2}$ (see [23] and [24, p. 21]). In the following we will denote $C^m_{ev}(\overline{\mathbb{R}}^n_+)$ by C^m_{ev}. We set

$$C^\infty_{ev}(\overline{\Omega}_+)=\bigcap C^m_{ev}(\overline{\Omega}_+)$$

with intersection taken for all finite m and $C^\infty_{ev}(\overline{\mathbb{R}}_+)=C^\infty_{ev}$.

As the space of test functions we will use the subspace of the space of rapidly decreasing functions:

$$S_{ev}=\left\{f\in C^\infty_{ev}:\sup_{x\in\mathbb{R}^n_+}\left|x^\alpha D^\beta f(x)\right|<\infty\quad\forall\alpha,\beta\in\mathbb{Z}^n_+\right\},$$

where $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\beta=(\beta_1,\ldots,\beta_n)$, $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n$ are integer non-negative numbers, $x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$, $D^\beta=D^{\beta_1}_{x_1}\ldots D^{\beta_n}_{x_n}$, $D_{x_j}=\frac{\partial}{\partial x_j}$.

Let $L^\gamma_p(\mathbb{R}^n_+)=L^\gamma_p$, $1\leq p<\infty$, be the space of all measurable in $\mathbb{R}^n_+$ functions even with respect to each variable x_i, $i=1,\ldots,n$ such that

$$\int\limits_{\mathbb{R}^n_+}|f(x)|^p x^\gamma dx<\infty,$$

here and further

$$x^\gamma=\prod_{i=1}^n x_i^{\gamma_i}.$$

For a real number $p \geq 1$, the L_p^γ-norm of f is defined by

$$||f||_{L_p^\gamma(\mathbb{R}_+^n)} = ||f||_{p,\gamma} = \left(\int\limits_{\mathbb{R}_+^n} |f(x)|^p x^\gamma dx \right)^{1/p}.$$

Weighted measure of Ω_+ is denoted by $\text{mes}_\gamma(\Omega_+)$ and is defined by formula

$$\text{mes}_\gamma(\Omega_+) = \int\limits_{\Omega_+} x^\gamma dx.$$

For every measurable function $f(x)$ defined on $\mathbb{R}_+^n$ we consider

$$\mu_\gamma(f, t) = \text{mes}_\gamma\{x \in \mathbb{R}_+^n : |f(x)| > t\} = \int\limits_{\{x:\, |f(x)|>t\}^+} x^\gamma dx$$

where $\{x{:}|f(x)|{>}t\}^+{=}\{x{\in}\mathbb{R}_+^n{:}|f(x)|{>}t\}$. We will call the function $\mu_\gamma = \mu_\gamma(f, t)$ a *weighted distribution function* $|f(x)|$.

Space $L_\infty^\gamma(\mathbb{R}_+^n){=}L_\infty^\gamma$ is the space of all measurable in $\mathbb{R}_+^n$ functions even with respect to each variable x_i, $i = 1, \dots, n$ for which the norm

$$||f||_{L_\infty^\gamma(\mathbb{R}_+^n)} = ||f||_{\infty,\gamma} = \underset{x\in\mathbb{R}_+^n}{\text{ess sup}_\gamma} |f(x)| = \inf_{a\in\mathbb{R}}\{\mu_\gamma(f, a) = 0\}$$

is finite.

For $1 \leq p \leq \infty$ the $L_{p,loc}^\gamma(\mathbb{R}_+^n) = L_{p,loc}^\gamma$ is the set of functions $u(x)$ defined almost everywhere in $\mathbb{R}_+^n$ such that $uf \in L_p^\gamma$ for any $f \in S_{ev}$.

Definition 1 The **space of weighted distributions** $S'_{ev}(\mathbb{R}_+^n) = S'_{ev}$ is a class of continuous linear functionals that map a set of test functions $f \in S_{ev}$ into the set of real numbers. Each function $u(x) \in L_{1,loc}^\gamma$ will be identified with the functional $u \in S'_{ev}(\mathbb{R}_+^n) = S'_{ev}$ acting according to the formula

$$(u, f)_\gamma = \int\limits_{\mathbb{R}_+^n} u(x)\, f(x)\, x^\gamma\, dx, \qquad f \in S_{ev}. \tag{1}$$

Functionals $u \in S'_{ev}$ acting by the formula (1) will be called **regular weighted functionals**. All other continuous linear functionals $u \in S'_{ev}$ will be called **singular weighted functionals**.

We consider regular distributions

$$(\mathcal{P}_\gamma^\lambda, \varphi)_\gamma = \int\limits_{\mathbb{R}_+^n} \mathcal{P}^\lambda(x)\varphi(x)x^\gamma dx, \qquad x^\gamma = \prod_{i=1}^n x_i^{\gamma_i}, \tag{2}$$

where $\mathcal{P}(x) = \alpha_1 x_1^2 + \ldots + \alpha_n x_n^2$ is quadratic form with complex coefficients, φ is an appropriate test function. Let $P = x_1^2 - x_2^2 - \ldots - x_n^2$, and $P' = \varepsilon(x_1^2 + \ldots + x_n^2)$, $\varepsilon > 0$. Weighted distributions $(P \pm i0)_\gamma^\lambda$ are defined by

$$(P \pm i0)_\gamma^\lambda = \lim_{\varepsilon \to 0} (P \pm i P')_\gamma^\lambda$$

in which we are passing to the limit under the integral sign in (2).

Generalized function δ_γ is defined by the equality

$$(\delta_\gamma, \varphi)_\gamma = \varphi(0), \quad \varphi(x) \in S_{ev}.$$

Definition 2 The **multidimensional generalized translation** is defined by the equality

$$({}^\gamma\mathbf{T}_x^y f)(x) = {}^\gamma\mathbf{T}_x^y f(x) = ({}^{\gamma_1}T_{x_1}^{y_1} \ldots {}^{\gamma_n}T_{x_n}^{y_n} f)(x), \tag{3}$$

where each of one-dimensional generalized translation ${}^{\gamma_i}T_{x_i}^{y_i}$ acts for $i = 1, \ldots, n$ according to

$$({}^{\gamma_i}T_{x_i}^{y_i} f)(x) = \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma_i}{2}\right)} \times$$

$$\times \int\limits_0^\pi f(x_1, \ldots, x_{i-1}, \sqrt{x_i^2 + \tau_i^2 - 2x_i y_i \cos\varphi_i}, x_{i+1}, \ldots, x_n)\ \sin^{\gamma_i - 1}\varphi_i\, d\varphi_i.$$

We will use the generalized convolution product defined by the formula

$$(f * g)_\gamma(x) = \int\limits_{\mathbb{R}_+^n} f(y)({}^\gamma\mathbf{T}_x^y g)(x) y^\gamma dy, \qquad f, g \in S_{ev}$$

where ${}^\gamma\mathbf{T}_x^y$ is multidimensional generalized translation (3).

The generalized convolution $(u * f)_\gamma$ of a weighted distribution $u \in S'_{ev}$ and a function $f \in S_{ev}$ is defined by

$$(u * f)_\gamma(x) = (u, {}^\gamma\mathbf{T}_x^{\cdot} f)_\gamma$$

where the right-hand side denotes u acting on ${}^\gamma\mathbf{T}_x^y f$ as a function of y.

We will deal with the **singular Bessel differential operator** B_ν (see, for example, [24, p. 5]):

$$(B_\nu)_t = \frac{\partial^2}{\partial t^2} + \frac{\nu}{t}\frac{\partial}{\partial t} = \frac{1}{t^\nu}\frac{\partial}{\partial t}t^\nu\frac{\partial}{\partial t}, \qquad t>0$$

and the elliptical singular operator or the Laplace-Bessel operator $\triangle_\gamma$:

$$\triangle_\gamma = (\triangle_\gamma)_x = \sum_{i=1}^n (B_{\gamma_i})_{x_i} = \sum_{i=1}^n \left(\frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x}\right) = \sum_{i=1}^n \frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial}{\partial x_i}. \tag{4}$$

The operator (4) belongs to the class of B-elliptic operators by I. A. Kipriyanovs' classification (see [24]).

The natural method for the study of operators associated with the Bessel differential operator is the use of the multidimensional Hankel transform instead of the Fourier transform.

Definition 3 The **Hankel transform** of a function $f \in L_1^\gamma(\mathbb{R}_+^n)$ is expressed as

$$\mathbf{F}_\gamma[f](\xi) = \mathbf{F}_\gamma[f(x)](\xi) = \widehat{f}(\xi) = \int\limits_{\mathbb{R}_+^n} f(x)\,\mathbf{j}_\gamma(x;\xi)x^\gamma dx,$$

where

$$\mathbf{j}_\gamma(x;\xi) = \prod_{i=1}^n j_{\frac{\gamma_i-1}{2}}(x_i\xi_i), \qquad \gamma_1>0,\ldots,\gamma_n>0,$$

the symbol j_ν is used for the normalized Bessel function:

$$j_\nu(r) = \frac{2^\nu\Gamma(\nu+1)}{r^\nu}J_\nu(r) \tag{5}$$

and $J_\nu(r)$ is the Bessel function of the first kind of order ν.

For $f \in S_{ev}$ its inverse Hankel transform is defined by

$$\mathbf{F}_\gamma^{-1}[\widehat{f}(\xi)](x) = f(x) = \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}_+^n} \mathbf{j}_\gamma(x,\xi)\widehat{f}(\xi)\xi^\gamma\, d\xi.$$

If $g \in S'_{ev}$ then equality

$$(\mathbf{F}_\gamma g, \varphi)_\gamma = (g, \mathbf{F}_\gamma\varphi)_\gamma, \qquad \varphi \in S_{ev} \tag{6}$$

defines Hankel transform of functional $g \in S'_{ev}$.

In [25] the space Ψ_V consisting of functions vanishing on a given closed set V of measure zero was considered. The Lizorkin–Samko space Ψ_V is dual to Φ_V in the sense of Fourier transforms. We introduce the space Ψ_V^γ of functions S_{ev} vanishing with all their derivatives on a given closed set V:

$$\Psi_V^\gamma = \{\psi \in S_{ev}(\mathbb{R}_+^n) : (D^k\psi)(x) = 0, x \in V, |k| = 0, 1, 2, \dots\}.$$

Space Ψ_V^γ is dual to Φ_V^γ in the sense of Hankel transforms:

$$\Phi_V^\gamma = \{\varphi : \mathbf{F}_\gamma\varphi \in \Psi_V^\gamma\}. \tag{7}$$

Appell hypergeometric function $F_4(a, b, c_1, c_2; x, y)$ (see [26, p. 658]) for $|x|^{1/2} + |y|^{1/2} < 1$ has the form

$$F_4(a, b, c_1, c_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c_1)_m(c_2)_n\, m!\, n!}\, x^m y^n. \tag{8}$$

For $|x|^{1/2} + |y|^{1/2} \geq 1$ function $F_4(a, b; c_1, c_2; x, y)$ is understood as an analytical continuation, which is determined by the formulas from [27].

3 Definition of the Hyperbolic B-Potentials

Here we consider fractional powers of the hyperbolic expression with Bessel operators

$$\Box_\gamma = B_{\gamma_1} - B_{\gamma_2} - \ldots - B_{\gamma_n}, \qquad B_{\gamma_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \qquad i = 1, \dots, n$$

in S_{ev} and L_p^γ. Negative real powers of $\Box_\gamma$ will be called **hyperbolic B-potentials**.

Definition 4 Hyperbolic B-potentials $I_{P\pm i0,\gamma}^\alpha$ for $\alpha > n + |\gamma| - 2$ are defined by formulas

$$(I_{P\pm i0,\gamma}^\alpha f)(x) = \frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha)} \int_{\mathbb{R}_+^n} (P \pm i0)_\gamma^{\frac{\alpha-n-|\gamma|}{2}} ({}^\gamma\mathbf{T}_x^y f)(x) y^\gamma dy$$

$$= \frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha)} \left((P \pm i0)_\gamma^{\frac{\alpha-n-|\gamma|}{2}} * f\right)_\gamma (x), \tag{9}$$

where $\gamma'=(\gamma_2,\ldots,\gamma_n)$, $|\gamma'|=\gamma_2+\ldots+\gamma_n$,

$$H_{n,\gamma}(\alpha)=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}{2^{n-\alpha}\Gamma\left(\frac{n+|\gamma|-\alpha}{2}\right)}.$$

It is well known (see for example [24]) that generalized convolution of a weighted distribution and a regular function is again a regular function.

Using property of weighted distributions $(P\pm i0)^{\lambda}_{\gamma}$ from [28] we can rewrite formulas (9) as

$$(I^{\alpha}_{P\pm i0,\gamma}f)(x)=\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha)}\left[\int\limits_{K^+}r^{\alpha-n-|\gamma|}(y)(^{\gamma}\mathbf{T}^y_x f)(x)y^{\gamma}dy+\right.$$

$$\left.+e^{\pm\frac{\alpha-n-|\gamma|}{2}\pi i}\int\limits_{K^-}|r(y)|^{\alpha-n-|\gamma|}(^{\gamma}\mathbf{T}^y_x f)(x)y^{\gamma}dy\right],\qquad(10)$$

where

$$K^+=\{x:x\in\mathbb{R}^n_+:P(x)\geq 0\},\qquad K^-=\{x:x\in\mathbb{R}^n_+:P(x)\leq 0\},$$

$$r(y)=\sqrt{P(y)}=\sqrt{y_1^2-y_2^2-\ldots-y_n^2}.$$

Function $r(y)$ is a Lorentz distance and K^+ is a part of the light cone.

Introducing the notations

$$(I^{\alpha}_{P_+,\gamma}f)(x)=\int\limits_{K^+}r^{\alpha-n-|\gamma|}(y)(^{\gamma}\mathbf{T}^y_x f)(x)y^{\gamma}dy,\qquad(11)$$

$$(I^{\alpha}_{P_-,\gamma}f)(x)=\int\limits_{K^-}|r(y)|^{\alpha-n-|\gamma|}(^{\gamma}\mathbf{T}^y_x f)(x)y^{\gamma}dy,\qquad(12)$$

we can write

$$(I^{\alpha}_{P\pm i0,\gamma}f)(x)=\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha)}\left[(I^{\alpha}_{P_+,\gamma}f)(x)+e^{\pm\frac{\alpha-n-|\gamma|}{2}\pi i}(I^{\alpha}_{P_-,\gamma}f)(x)\right].\qquad(13)$$

Remark Let $y' = (y_2, \ldots, y_n)$, $|y'| = \sqrt{y_2^2 + \ldots + y_n^2}$, $(y')^{\gamma'} = y_2^{\gamma_2} \ldots y_n^{\gamma_n}$. For $n \geq 3$ we have

$$(I^{\alpha}_{P_+,\gamma} f)(x) = \int\limits_0^{\infty} y_1^{\gamma_1} dy_1 \int\limits_{\{|y'|<y_1\}^+} (y_1^2 - |y'|^2)^{\frac{\alpha-n-|\gamma|}{2}} ({}^{\gamma}\mathbf{T}_x^y f)(x)(y')^{\gamma'} dy', \tag{14}$$

$$(I^{\alpha}_{P_-,\gamma} f)(x) = \int\limits_0^{\infty} y_1^{\gamma_1} dy_1 \int\limits_{\{|y'|>y_1\}^+} (|y'|^2 - y_1^2)^{\frac{\alpha-n-|\gamma|}{2}} ({}^{\gamma}\mathbf{T}_x^y f)(x)(y')^{\gamma'} dy', \tag{15}$$

where $\{|y'| < y_1\}^+ = \{y \in \mathbb{R}^n_+ : |y'| < y_1\}$, $\{|y'| > y_1\}^+ = \{y \in \mathbb{R}^n_+ : |y'| > y_1\}$.
For $n = 2$ we have

$$(I^{\alpha}_{P_+,\gamma} f)(x) = \int\limits_0^{\infty} y_1^{\gamma_1} dy_1 \int\limits_0^{y_1} (y_1^2 - y_2^2)^{\frac{\alpha-2-|\gamma|}{2}} ({}^{\gamma}\mathbf{T}_x^y f)(x) y_2^{\gamma_2} dy_2,$$

$$(I^{\alpha}_{P_-,\gamma} f)(x) = \int\limits_0^{\infty} y_1^{\gamma_1} dy_1 \int\limits_{y_1}^{\infty} (y_2^2 - y_1^2)^{\frac{\alpha-2-|\gamma|}{2}} ({}^{\gamma}\mathbf{T}_x^y f)(x) y_2^{\gamma_2} dy_2.$$

Theorem 1 *Let $f \in S_{ev}$ and $\alpha > n + |\gamma| - 2$. Then integrals $(I^{\alpha}_{P\pm i0,\gamma} f)(x)$ converge absolutely for $x \in \mathbb{R}^n_+$.*

Proof Let us prove absolute convergence of each term in (10). Passing in (10) to spherical coordinates $y=\rho\sigma$, $\rho=|y|$, $\sigma'=(\sigma_2, \ldots, \sigma_n)$ we obtain

$$\int\limits_{K^+} r^{\alpha-n-|\gamma|}(y)({}^{\gamma}\mathbf{T}_x^y f)(x) y^{\gamma} dy =$$

$$= \int\limits_0^{\infty} \rho^{\alpha-1} d\rho \int\limits_{\{S_1^+(n), |\sigma'|<\sigma_1\}} (\sigma_1^2 - |\sigma'|^2)^{\frac{\alpha-n-|\gamma|}{2}} ({}^{\gamma}\mathbf{T}^{\rho\sigma} f)(x) \sigma^{\gamma} dS,$$

where

$$\{S_1^+(n), |\sigma'| < \sigma_1\} = \{\sigma' \in \mathbb{R}^{n-1}_+ : \sigma_1^2 + |\sigma'|^2 = 1,\ |\sigma'| < \sigma_1\}.$$

Using formula ${}^\gamma\mathbf{T}_x^y f(x) = {}^\gamma\mathbf{T}_y^x f(y)$, inequality $|{}^\gamma\mathbf{T}_x^y f(x)| \le \sup\limits_{\mathbb{R}_+^n} |f(x)|$ (see [29, p. 124]) and considering that $f \in S_{ev}$ we get

$$\left| \int\limits_{K^+} r^{\alpha-n-|\gamma|}(y)({}^\gamma\mathbf{T}_x^y f)(x) y^\gamma dy \right| \le$$

$$\le C \int\limits_0^\infty \frac{\rho^{\alpha-1}}{(1+\rho^2)^{\frac{\alpha+1}{2}}} d\rho \int\limits_{S_1^+(n), |\sigma'|<\sigma_1} (\sigma_1^2 - |\sigma'|^2)^{\frac{\alpha-n-|\gamma|}{2}} \sigma^\gamma dS < \infty,$$

for $\alpha > n + |\gamma| - 2$. Similarly, we get that (12) converges absolutely for $\alpha > n + |\gamma| - 2$. So for $\alpha > n + |\gamma| - 2$ integrals $(I^\alpha_{P\pm i0,\gamma} f)(x)$ converge absolutely. □

Next we show how hyperbolic B-potentials are connected with the operator $(\Box_\gamma)^k$, $k \in \mathbb{N}$. We will use this connection in order to define hyperbolic B-potentials for $0 \le \alpha \le n + |\gamma| - 2$.

Theorem 2 *If $f \in S_{ev}$, $n + |\gamma| - 2 < \alpha$ and $k \in \mathbb{N}$ then*

$$(\Box_\gamma)^k I^{\alpha+2k}_{P\pm i0,\gamma} f = I^\alpha_{P\pm i0,\gamma} f, \tag{16}$$

where $\Box_\gamma = B_{\gamma_1} - \sum\limits_{i=2}^n B_{\gamma_i}$.

Proof Using representation (9) and the property ${}^{\gamma_i}T_{x_i}^{y_i}(B_{\gamma_i})_{x_i} = (B_{\gamma_i})_{x_i}^{\gamma_i} T_{x_i}^{y_i}$ (see formula 1.8.3 from [24]) we obtain

$$(\Box_\gamma)^k (I^{\alpha+2k}_{P\pm i0,\gamma} f)(x) =$$

$$= \frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha+2k)} (\Box_\gamma)^k \int\limits_{\mathbb{R}_+^n} (P \pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}} ({}^\gamma\mathbf{T}_x^y f)(x) y^\gamma dy =$$

$$= \frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha+2k)} \int\limits_{\mathbb{R}_+^n} \left({}^\gamma\mathbf{T}_x^y (\Box_\gamma)^k (P \pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}} \right) f(y) y^\gamma dy.$$

For function $(P \pm i0)_\gamma^\lambda$ the next equalities are true (see [28])

$$(\Box_\gamma)^k (P \pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}} = 2^{2k} \frac{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+k+1\right)}{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+1\right)} \frac{\Gamma\left(\frac{\alpha}{2}+k\right)}{\Gamma\left(\frac{\alpha}{2}\right)} (P \pm i0)_\gamma^{\frac{\alpha-n-|\gamma|}{2}}. \tag{17}$$

Since

$$2^{2k}\frac{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+k+1\right)}{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+1\right)}\frac{\Gamma\left(\frac{\alpha}{2}+k\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\cdot\frac{1}{H_{n,\gamma}(\alpha+2k)}=\frac{1}{H_{n,\gamma}(\alpha)},$$

then using (17) we get

$$(\Box_\gamma)^k(I^{\alpha+k}_{P\pm i0,\gamma}f)(x)=$$

$$=\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha)}\int\limits_{\mathbb{R}^n_+}\left({}^\gamma\mathbf{T}^y_x(P\pm i0)_\gamma^{\frac{\alpha-n-|\gamma|}{2}}\right)f(y)y^\gamma dy=(I^\alpha_{P\pm i0,\gamma}f)(x)$$

and the proof is complete. □

Taking into account (16) let define hyperbolic B-potentials $I^\alpha_{P\pm i0,\gamma}$ for $0\leq\alpha\leq n+|\gamma|-2$ as

$$(I^\alpha_{P\pm i0,\gamma}f)(x)=(\Box_\gamma)^k(I^{\alpha+2k}_{P\pm i0,\gamma}f)(x)=\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha+2k)}(\Box_\gamma)^k$$

$$\times\int\limits_{\mathbb{R}^n_+}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}({}^\gamma\mathbf{T}^y_x f)(x)y^\gamma dy, \tag{18}$$

where $k=\left[\frac{n+|\gamma|-\alpha}{2}\right]$.

The boundedness of $I^\alpha_{P\pm i0,\gamma}$ from L^γ_p to L^γ_q can be extracted from [17].

Theorem 3 ([30]) *Let $n+|\gamma|-2<\alpha<n+|\gamma|$, $1\leq p<\frac{n+|\gamma|}{\alpha}$. For the next estimate*

$$||I^\alpha_{P\pm i0,\gamma}f||_{q,\gamma}\leq C_{n,\gamma,p}||f||_{p,\gamma},\qquad f(x)\in S_{ev} \tag{19}$$

to be valid it is necessary and sufficient that $q=\frac{(n+|\gamma|)p}{n+|\gamma|-\alpha p}$. Constant $C_{n,\gamma,p}$ does not depend on f.

Further operators $I^\alpha_{P_\pm,\gamma}$ on function class L^γ_p we will define as continuations of operators (9) with preservation of boundedness. If integral (9) converges absolutely for $f\in L^\gamma_p$ then these continuations are representable as

$$(I^\alpha_{P\pm i0,\gamma}f)(x)=\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{\gamma_{n,\gamma}(\alpha)}\int\limits_{\mathbb{R}^n_+}(P\pm i0)_\gamma^{\frac{\alpha-n-|\gamma|}{2}}({}^\gamma\mathbf{T}^y_x f)(x)y^\gamma dy,\quad y^\gamma=\prod_{i=1}^n y_i^{\gamma_i}.$$

Theorem 4 ([30]) *The mixed hyperbolic Riesz B-potentials satisfy the following semigroup property for* $f \in S_{ev}$*:*

$$I^{\beta}_{P\pm i0,\gamma} I^{\alpha}_{P\pm i0,\gamma} f = I^{\alpha+\beta}_{P\pm i0,\gamma} f. \tag{20}$$

By virtue of the density S_{ev} in L_p^γ equalities (16) and (20) spread on function from L_p^γ for $1 < p < \frac{n+|\gamma|}{\alpha}$ when integrals $I^{\alpha}_{P\pm i0,\gamma} f$ converge absolutely for $f \in L_p^\gamma$.

Theorem 5 ([30]) *For* $f \in \Phi_V$, $V = \{x \in \mathbb{R}^n_+ : P(x) = 0\}$ *Hankel transform of the Riesz hyperbolic B-potential*

$$\mathbf{F}_\gamma[(I^{\alpha}_{P\pm i0,\gamma} f)(x)](\xi) = (P \mp i0)^{-\frac{\alpha}{2}} \mathbf{F}_\gamma[f](\xi). \tag{21}$$

4 Method of Approximative Inverse Operators and General Poisson Kernel

Here we describe one approach for inverting potential type operators, based on the idea of approximative inverse operators developed in [9, 31].

Equation (9) is a convolution operator, where generalized convolution is used. The problem to invert this or that convolution operator $Af = a * f$ reduces to multiplication of some convenient integral transform of a function f by the reciprocal $\frac{1}{\hat{a}}$ of chosen integral transform of the kernel:

$$Af = a * f, \qquad \widehat{Af} = \hat{a} \cdot \hat{f}, \qquad \widehat{A^{-1}f} = \frac{1}{\hat{a}} \cdot \hat{f}.$$

Indeed we have

$$g = Af, \qquad \widehat{A^{-1}g} = \frac{1}{\hat{a}} \cdot \hat{a} \cdot \hat{f} = \hat{f}.$$

However, in the case of potentials, the multiplier $\frac{1}{\hat{a}}$, is unbounded at infinity and, maybe, on some sets. In this case we use the multiplier m_ε, which is dependent on ε such that $\frac{m_\varepsilon}{\hat{a}}$ vanishes at those sets on which it is necessary and $\lim\limits_{\varepsilon\to 0} m_\varepsilon = 1$. So we can construct $\widehat{A_\varepsilon^{-1} f} = \frac{m_\varepsilon}{\hat{a}} \cdot \hat{f}$. Applying the inverse integral transform and passing to the limit $\varepsilon \to 0$ we obtain A^{-1}. Next it is necessary to prove that the resulting operator will be inverse to the operator A in some appropriate space. Therefore, the factor m_ε should be chosen so that inverse integral transform of $\frac{m_\varepsilon}{\hat{a}} \cdot \hat{f}$ provides a fairly good class of functions.

In our case, we take the Hankel transform. Considering that

$$\mathbf{F}_\gamma I^\alpha_{P\pm i0,\gamma} f = (P \mp i0)_\gamma^{-\frac{\alpha}{2}} \mathbf{F}_\gamma f,$$

where $f \in \Phi_V^\gamma$, $V = \{x \in \mathbb{R}^n_+ : P(x) = 0\}$ we take

$$M_{\varepsilon,\delta} = \frac{(P \mp i0)^m e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m}.$$

So we should prove that left inverse operators to $I^\alpha_{P\pm i0,\gamma}$ are

$$(I^\alpha_{P\pm i0,\gamma})^{-1} f) = \overset{L_p^\gamma}{\lim_{\delta\to 0}} \overset{L_2^\gamma}{\lim_{\varepsilon\to 0}} \left(\left(\mathbf{F}_\gamma^{-1} \frac{(P \mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m}\right)(x) * f(x)\right)_\gamma .$$

We denote

$$(I^\alpha_{P\pm i0,\gamma})^{-1}_{\varepsilon,\delta} f = \left(\left(\mathbf{F}_\gamma^{-1} \frac{(P \mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m}\right)(x) * f(x)\right)_\gamma =$$

$$= \int\limits_{\mathbb{R}^n_+} {}^{\mp}g^\alpha_{\varepsilon,\delta}(y)({}^\gamma\mathbf{T}^y_x f(x)) y^\gamma dy,$$

where

$${}^{\mp}g^\alpha_{\varepsilon,\delta}(x) = \left(\mathbf{F}_\gamma^{-1} \frac{(P \mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m}\right)(x) =$$

$$= \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \int\limits_{\mathbb{R}^n_+} \frac{(P \mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m} \mathbf{j}_\gamma(x,\xi)\xi^\gamma \, d\xi,$$

$m \geq n + |\gamma| - \frac{\alpha}{2}$, $n + |\gamma| - 2 < \alpha < n + |\gamma|$.

Next we consider a certain function used for solving the problem of inverting a hyperbolic B-potential. Based on the type and properties of this function, we will call it the general Poisson kernel.

We first prove an auxiliary lemma.

Lemma 1 *Hankel transform of the* $e^{-\delta|x|}$ *is*

$$\mathbf{F}_\gamma[e^{-\delta|x|}](\xi) = \frac{2^{|\gamma|}\delta \prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right) \Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2 + |\xi|^2)^{\frac{n+|\gamma|+1}{2}}}. \tag{22}$$

Proof We have

$$\mathbf{F}_\gamma[e^{-\delta|x|}](\xi)=\int\limits_{\mathbb{R}^n_+} e^{-\delta|x|}\mathbf{j}_\gamma(x;\xi)x^\gamma dx=\{x=\rho\sigma\}=$$

$$=\int\limits_0^\infty e^{-\delta\rho}\rho^{n+|\gamma|-1}d\rho\int\limits_{S_1^+(n)}\mathbf{j}_\gamma(\rho\sigma;\xi)\sigma^\gamma dS.$$

Applying the formula (5) we obtain

$$\mathbf{F}_\gamma[e^{-\delta|x|}](\xi)=\frac{\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^\infty e^{-\delta\rho}j_{\frac{n+|\gamma|}{2}-1}(\rho|\xi|)\rho^{n+|\gamma|-1}d\rho=$$

$$=\frac{\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{\frac{n-|\gamma|}{2}}|\xi|^{\frac{n+|\gamma|}{2}-1}}\int\limits_0^\infty e^{-\delta\rho}J_{\frac{n+|\gamma|}{2}-1}(\rho|\xi|)\rho^{\frac{n+|\gamma|}{2}}d\rho.$$

Applying the formula 2.12.8.4 from [26, p. 164] of the form

$$\int\limits_0^\infty x^{\nu+2}e^{-px}J_\nu(cx)dx=\frac{2p(2c)^\nu\Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{\pi}(p^2+c^2)^{\nu+\frac{3}{2}}},\qquad \mathrm{Re}\,\nu>-1$$

we get

$$\int\limits_0^\infty e^{-\delta\rho}J_{\frac{n+|\gamma|}{2}-1}(\rho|\xi|)\rho^{\frac{n+|\gamma|}{2}}d\rho=\frac{2\delta(2|\xi|)^{\frac{n+|\gamma|}{2}-1}\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2+|\xi|^2)^{\frac{n+|\gamma|+1}{2}}}$$

and therefore

$$\mathbf{F}_\gamma[e^{-\delta|x|}](\xi)=\frac{\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{\frac{n-|\gamma|}{2}}|\xi|^{\frac{n+|\gamma|}{2}-1}}\,\frac{2\delta(2|\xi|)^{\frac{n+|\gamma|}{2}-1}\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2+|\xi|^2)^{\frac{n+|\gamma|+1}{2}}}=$$

$$=\frac{2^{|\gamma|}\delta\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2+|\xi|^2)^{\frac{n+|\gamma|+1}{2}}}.$$

□

We give the formula from [32] that will be used further

$$\int\limits_{S_1^+(n)} \mathcal{P}_\xi^\gamma f(\langle \xi, x\rangle) x^\gamma d\omega_x = \frac{\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}2^{n-1}\Gamma\left(\frac{|\gamma|+n-1}{2}\right)} \int\limits_{-1}^{1} f(|\xi|p)(1-p^2)^{\frac{n+|\gamma|-3}{2}}\, dp, \tag{23}$$

where $f(t)(1-t^2)^{\frac{n+|\gamma|-3}{2}} \in L_1(-1,1)$.

Definition 5 Function

$$P_\gamma(x,\delta) = \frac{2^n \Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^n \Gamma\left(\frac{\gamma_j+1}{2}\right)} \delta\,(\delta^2+|x|^2)^{-\frac{n+|\gamma|+1}{2}}, \qquad \delta>0 \tag{24}$$

is called the **general Poisson kernel**.

Lemma 2 *For $P_\gamma(x,\delta)$ the following properties are valid*

1. $\mathbf{F}_\gamma[P_\gamma(x,\delta)](\xi) = e^{-\delta|\xi|}$,
2. $\int\limits_{\mathbb{R}_+^n} P_\gamma(x,\delta)x^\gamma dx = \int\limits_{\mathbb{R}_+^n} P_\gamma(x,1)x^\gamma dx = 1$,
3. $P_\gamma(x,\delta) \in L_p^\gamma$, $1 \le p \le \infty$.

Proof

1. From Lemma 1 we get

$$\mathbf{F}_\gamma^{-1}[e^{-\delta|x|}](\xi) = \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \mathbf{F}_\gamma[e^{-\delta|x|}](\xi) =$$

$$= \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \frac{2^{|\gamma|}\delta \prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2+|\xi|^2)^{\frac{n+|\gamma|+1}{2}}} =$$

$$= \frac{2^n\delta\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^n \Gamma\left(\frac{\gamma_j+1}{2}\right)} \frac{1}{(\delta^2+|\xi|^2)^{\frac{n+|\gamma|+1}{2}}} = P_\gamma(x,\delta).$$

Hence we obtain $\mathbf{F}_\gamma[P_\gamma(x,\delta)](\xi) = e^{-\delta|\xi|}$.

2. Consider the integral $\int\limits_{\mathbb{R}^n_+} P_\gamma(x,\delta)x^\gamma dx$. We have

$$\int\limits_{\mathbb{R}^n_+} P_\gamma(x,\delta)x^\gamma dx = \frac{2^n\delta\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+}\frac{x^\gamma\, dx}{(\delta^2+|x|^2)^{\frac{n+|\gamma|+1}{2}}} = \{x=\delta y\} =$$

$$= \frac{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+}\frac{y^\gamma\, dy}{(1+|y|^2)^{\frac{n+|\gamma|+1}{2}}} = \int\limits_{\mathbb{R}^n_+} P_\gamma(x,1)x^\gamma dx.$$

Let us show now that $\int\limits_{\mathbb{R}^n_+} P_\gamma(x,1)x^\gamma dx = 1$. Going over to spherical coordinates we obtain

$$\int\limits_{\mathbb{R}^n_+}\frac{y^\gamma\, dy}{(1+|y|^2)^{\frac{n+|\gamma|+1}{2}}} = \{y=\rho\sigma\} = \int\limits_0^\infty \frac{\rho^{n+|\gamma|-1}\, d\rho}{(1+\rho^2)^{\frac{n+|\gamma|+1}{2}}}\int\limits_{S_1^+(n)}\sigma^\gamma\, dS =$$

$$= \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^\infty \frac{\rho^{n+|\gamma|-1}\, d\rho}{(1+\rho^2)^{\frac{n+|\gamma|+1}{2}}} = \{\rho^2=r\} =$$

$$= \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^\infty \frac{r^{\frac{n+|\gamma|}{2}-1}}{(1+r)^{\frac{n+|\gamma|+1}{2}}}dr.$$

Using the formula 2.2.5.24 from [33, p. 239] of the form

$$\int\limits_0^\infty \frac{x^{\alpha-1}}{(x+z)^\beta}dx = z^{\alpha-\beta}B(\alpha,\beta-\alpha), \qquad 0<\mathrm{Re}\,\alpha<\mathrm{Re}\,\beta,$$

we obtain

$$2\int\limits_0^\infty \frac{\rho^{n+|\gamma|-1}\, d\rho}{(1+\rho^2)^{\frac{n+|\gamma|+1}{2}}} = \int\limits_0^\infty \frac{r^{\frac{n+|\gamma|}{2}-1}}{(1+r)^{\frac{n+|\gamma|+1}{2}}}dr = \frac{\sqrt{\pi}\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\Gamma\left(\frac{n+|\gamma|+1}{2}\right)} \tag{25}$$

and

$$\int\limits_{\mathbb{R}^n_+} \frac{y^\gamma\,dy}{(1+|y|^2)^{\frac{n+|\gamma|+1}{2}}} = \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma|}{2}\right)} \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\Gamma\left(\frac{n+|\gamma|+1}{2}\right)} = \frac{\sqrt{\pi}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}.$$

Finally,

$$\int\limits_{\mathbb{R}^n_+} P_\gamma(x,1)x^\gamma dx = \frac{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)} \int\limits_{\mathbb{R}^n_+} \frac{y^\gamma\,dy}{(1+|y|^2)^{\frac{n+|\gamma|+1}{2}}} =$$

$$= \frac{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)} \frac{\sqrt{\pi}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)} = 1.$$

3. We finally prove that $P_\gamma(x,\delta) \in L_p^\gamma$, $1 \le p \le \infty$. We have

$$\int\limits_{\mathbb{R}^n_+} \frac{x^\gamma dx}{(\delta^2+|x|^2)^{p\,\frac{n+|\gamma|+1}{2}}} = \delta^{(n+|\gamma|)(1-p)-p} \int\limits_{\mathbb{R}^n_+} \frac{x^\gamma dx}{(|x|^2+1)^{p\,\frac{n+|\gamma|+1}{2}}} =$$

$$= \{x=\rho\sigma, |x|=\rho\} = \delta^{(n+|\gamma|)(1-p)-p} \int\limits_0^\infty \frac{\rho^{n+|\gamma|-1}d\rho}{(\rho^2+1)^{p\,\frac{n+|\gamma|+1}{2}}} \int\limits_{S_1^+(n)} \sigma^\gamma dS.$$

Applying (25) for $1 \le p < \infty$ we get

$$||P_\gamma(x,\delta)||_{p,\gamma} = \left(\delta^{(n+|\gamma|)(1-p)-p} \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{2\Gamma\left(\frac{n+|\gamma|+1}{2}\right)} \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\right)^{\frac{1}{p}} =$$

$$= \left(\delta^{(n+|\gamma|)(1-p)-p} \frac{\sqrt{\pi}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}\right)^{\frac{1}{p}} < \infty.$$

For $p=\infty$ we get inequality $||P_\gamma(x,\delta)||_{\infty,\gamma} < \infty$ tending to the limit.

□

Following [34] (see Theorem 1.18, p. 17) we prove that a generalized convolution of a function with the Poisson kernel tends to a function in L_p^γ.

Let

$$(\mathbf{P}_{\gamma,\delta}f)(x)=(f(x)*P_\gamma(x,\delta))_\gamma. \tag{26}$$

Lemma 3 *If* $f\in L_p^\gamma$, $1\leq p\leq\infty$ *or* $f\in C_0\subset L_\infty^\gamma$ *then*

$$||(\mathbf{P}_{\gamma,\delta}f)(x)-f(x)||_{p,\gamma}\to 0 \qquad with \qquad \delta\to 0.$$

Proof Considering the property 2 from Lemma 2 we can write

$$(f(x)*P_\gamma(x,\delta))_\gamma-f(x)=\int\limits_{\mathbb{R}_+^n}[{}^\gamma\mathbf{T}_x^y f(x)-f(y)]P_\gamma(y,\delta)y^\gamma dy.$$

Hence, applying the generalized Minkowski inequality, we obtain

$$||(f(x)*P_\gamma(x,\delta))_\gamma-f(x)||_{p,\gamma}\leq$$

$$\leq\int\limits_{\mathbb{R}_+^n}\left(\int\limits_{\mathbb{R}_+^n}[{}^\gamma\mathbf{T}_x^y f(x)-f(x)]^p x^\gamma dx\right)^{\frac{1}{p}}|P_\gamma(y,\delta)|y^\gamma dy=\{y=\delta t\}=$$

$$=\int\limits_{\mathbb{R}_+^n}\left(\int\limits_{\mathbb{R}_+^n}[{}^\gamma\mathbf{T}_x^{\delta t} f(x)-f(x)]^p x^\gamma dx\right)^{\frac{1}{p}}|P_\gamma(t,1)|t^\gamma dt. \tag{27}$$

From [35] (see Lemma 3.6, p. 166) it follows that for $f\in L_p^\gamma$

$$||{}^\gamma\mathbf{T}_x^{\delta t}f(x)-f(x)||_{p,\gamma}\leq c||f(x)||_{p,\gamma},$$

and from [36] (see the preposition 4.1, p. 182) follows that

$$\lim_{\delta\to 0}\left(\int\limits_{\mathbb{R}_+^n}[{}^\gamma\mathbf{T}_x^{\delta t} f(x)-f(x)]^p x^\gamma dx\right)^{\frac{1}{p}}=0.$$

Then, by the Lebesgue theorem on dominated convergence, the integral (27) tends to zero when $\delta\to 0$, since the integrand is majorized by the integrable function $c||f||_{p,\gamma}|P_\gamma(t,1)|t^\gamma$. □

5 Representation of the Kernel ${}^{\mp}g^{\alpha}_{\varepsilon,\delta}$

In this section we get the integral kernel representation ${}^{\mp}g^{\alpha}_{\varepsilon,\delta}$.

Theorem 6 *Function ${}^{\mp}g^{\alpha}_{\varepsilon,\delta}$ can be presented in the form*

$$
{}^{\mp}g^{\alpha}_{\varepsilon,\delta}(x)=\frac{2^{2-|\gamma|}}{\delta^{n+|\gamma|+\alpha}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\frac{\Gamma(n+|\gamma|+\alpha)}{\Gamma\left(\frac{\gamma_1+1}{2}\right)\Gamma\left(\frac{n+|\gamma'|-1}{2}\right)}\times
$$

$$
\times\int\limits_0^{\infty}r^{n+|\gamma'|-2}\frac{(1-r^2\mp i0)^{m+\frac{\alpha}{2}}}{(1+r^2)^{\frac{n+|\gamma|+\alpha}{2}}(1-r^2+i\varepsilon(1+r^2))^m}\times
$$

$$
\times F_4\left(\frac{\beta}{2},\frac{\beta+1}{2};\frac{\gamma_1+1}{2},\frac{n+|\gamma'|-1}{2};-\frac{x_1^2}{\delta^2(1+r^2)},-\frac{(r|x'|)^2}{\delta^2(1+r^2)}\right)dr.
$$

where $\beta = n+|\gamma|+\alpha$ $F_4(a,b,c_1,c_2;x,y)$ is the Appell hypergeometric function (8).

Proof We represent the function ${}^{\mp}g^{\alpha}_{\varepsilon,\delta}(t)$ as the sum

$$
{}^{\mp}g^{\alpha}_{\varepsilon,\delta}(x)=\mathbf{F}_{\gamma}^{-1}\frac{(P\mp i0)^{m+\frac{\alpha}{2}}e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m}=
$$

$$
=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+}\frac{(P\mp i0)^{m+\frac{\alpha}{2}}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\mathbf{j}_{\gamma}(x,\xi)\xi^{\gamma}\,d\xi=
$$

$$
=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\left[\int\limits_{\{P(\xi)>0\}^+}\frac{P^{m+\frac{\alpha}{2}}(\xi)e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\mathbf{j}_{\gamma}(x,\xi)\xi^{\gamma}\,d\xi+\right.
$$

$$
\left.+e^{\mp(m+\frac{\alpha}{2})\pi i}\int\limits_{\{P(\xi)<0\}^+}\frac{|P(\xi)|^{m+\frac{\alpha}{2}}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\mathbf{j}_{\gamma}(x,\xi)\xi^{\gamma}\,d\xi\right].
$$

Let

$$
J_1=\int\limits_{\{P(\xi)>0\}^+}\frac{P^{m+\frac{\alpha}{2}}(\xi)e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\mathbf{j}_{\gamma}(x,\xi)\xi^{\gamma}\,d\xi,
$$

$$
J_2=\int\limits_{\{P(\xi)<0\}^+}\frac{|P(\xi)|^{m+\frac{\alpha}{2}}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\mathbf{j}_{\gamma}(x,\xi)\xi^{\gamma}\,d\xi.
$$

Going in J_1 over to spherical coordinates $\xi'=\rho\sigma$, $\sigma\in\mathbb{R}^{n-1}_+$, $\rho=|\xi'|$ we obtain

$$J_1=\int\limits_0^\infty j_{\frac{\gamma_1-1}{2}}(x_1\xi_1)\xi_1^{\gamma_1}\,d\xi_1\times$$

$$\times\int\limits_{|\xi'|^2<\xi_1^2}\frac{(\xi_1^2-|\xi'|^2)^{m+\frac{\alpha}{2}}e^{-\delta\sqrt{\xi_1^2+|\xi'|^2}}}{(\xi_1^2-|\xi'|^2+i\varepsilon(\xi_1^2+|\xi'|^2))^m}\mathbf{j}_\gamma(x',\xi')(\xi')^{\gamma'}\,d\xi'=$$

$$=\int\limits_0^\infty j_{\frac{\gamma_1-1}{2}}(x_1\xi_1)\xi_1^{\gamma_1}\,d\xi_1\int\limits_0^{\xi_1}\rho^{n+|\gamma'|-2}\frac{(\xi_1^2-\rho^2)^{m+\frac{\alpha}{2}}e^{-\delta\sqrt{\xi_1^2+\rho^2}}}{(\xi_1^2-\rho^2+i\varepsilon(\xi_1^2+\rho^2))^m}d\rho\times$$

$$\times\int\limits_{S_1^+(n-1)}\mathbf{j}_\gamma(x',\rho\sigma)(\sigma)^{\gamma'}\,dS.$$

The next formula

$$\int\limits_{S_1^+(n-1)}\mathbf{j}_\gamma(x',\rho\sigma)(\sigma)^{\gamma'}\,dS=\frac{\prod\limits_{i=2}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)}\,j_{\frac{n-1+|\gamma'|}{2}-1}(\rho|x'|),$$

is valid (see [37]), therefore

$$J_1=\frac{\prod\limits_{i=2}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)}\int\limits_0^\infty j_{\frac{\gamma_1-1}{2}}(x_1\xi_1)\xi_1^{\gamma_1}\,d\xi_1\times$$

$$\times\int\limits_0^{\xi_1}\rho^{n+|\gamma'|-2}j_{\frac{n-1+|\gamma'|}{2}-1}(\rho|x'|)\frac{(\xi_1^2-\rho^2)^{m+\frac{\alpha}{2}}e^{-\delta\sqrt{\xi_1^2+\rho^2}}}{(\xi_1^2-\rho^2+i\varepsilon(\xi_1^2+\rho^2))^m}d\rho=\{\rho=\xi_1 r\}=$$

$$=\frac{\prod\limits_{i=2}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)}\int\limits_0^\infty j_{\frac{\gamma_1-1}{2}}(x_1\xi_1)\xi_1^{n+|\gamma|-1+\alpha}\,d\xi_1\times$$

$$\times\int\limits_0^1 r^{n+|\gamma'|-2}j_{\frac{n-1+|\gamma'|}{2}-1}(r\xi_1|x'|)\frac{(1-r^2)^{m+\frac{\alpha}{2}}e^{-\delta\xi_1\sqrt{1+r^2}}}{(1-r^2+i\varepsilon(1+r^2))^m}dr=$$

$$= \frac{\prod\limits_{i=2}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)} \frac{2^{\frac{\gamma_1-1}{2}}\Gamma\left(\frac{\gamma_1+1}{2}\right)}{x_1^{\frac{\gamma_1-1}{2}}} \frac{2^{\frac{n-1+|\gamma'|}{2}-1}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)}{|x'|^{\frac{n-1+|\gamma'|}{2}-1}}\times$$

$$\times\int\limits_0^1 r^{\frac{n+|\gamma'|-1}{2}}\frac{(1-r^2)^{m+\frac{\alpha}{2}}}{(1-r^2+i\varepsilon(1+r^2))^m}dr\times$$

$$\times\int\limits_0^\infty \xi_1^{\frac{n+|\gamma|}{2}+\alpha+1}e^{-\delta\xi_1\sqrt{1+r^2}}J_{\frac{\gamma_1-1}{2}}(x_1\xi_1)J_{\frac{n-1+|\gamma'|}{2}-1}(r\xi_1|x'|)\,d\xi_1 =$$

$$= \frac{2^{\frac{|\gamma|-n}{2}}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{x_1^{\frac{\gamma_1-1}{2}}|x'|^{\frac{n-1+|\gamma'|}{2}-1}}\int\limits_0^1 r^{\frac{n+|\gamma'|-1}{2}}\frac{(1-r^2)^{m+\frac{\alpha}{2}}}{(1-r^2+i\varepsilon(1+r^2))^m}dr\times$$

$$\times\int\limits_0^\infty \xi_1^{\frac{n+|\gamma|}{2}+\alpha+1}e^{-\delta\xi_1\sqrt{1+r^2}}J_{\frac{\gamma_1-1}{2}}(x_1\xi_1)J_{\frac{n-1+|\gamma'|}{2}-1}(r\xi_1|x'|)\,d\xi_1.$$

To calculate the internal integral, apply the formula 2.12.38.2 from [26, p. 194] of the form

$$\int\limits_0^\infty x^{a-1}\,e^{-px}\,J_\mu(bx)\,J_\nu(cx)dx = \frac{b^\mu c^\nu}{2^{\mu+\nu}p^{a+\mu+\nu}}\frac{\Gamma(a+\mu+\nu)}{\Gamma(\mu+1)\Gamma(\nu+1)}\times$$

$$\times F_4\left(\frac{a+\mu+\nu}{2},\frac{a+\mu+\nu+1}{2};\mu+1,\nu+1;-\frac{b^2}{p^2},-\frac{c^2}{p^2}\right),$$

$$\mathrm{Re}\,(a+\mu+\nu)>0;\ \mathrm{Re}\,p>0.$$

We have

$$a=\frac{n+|\gamma|}{2}+\alpha+2,\ \ p=\delta\sqrt{1+r^2},\ \ \mu=\frac{\gamma_1-1}{2},$$

$$\nu=\frac{n-1+|\gamma'|}{2}-1,\ \ b=x_1,\ \ c=r|x'|$$

and

$$\int\limits_0^\infty \xi_1^{\frac{n+|\gamma|}{2}+\alpha+1} e^{-\delta\xi_1\sqrt{1+r^2}} J_{\frac{\gamma_1-1}{2}}(x_1\xi_1) J_{\frac{n-1+|\gamma'|}{2}-1}(r\xi_1|x'|)\, d\xi_1 =$$

$$= \frac{x_1^{\frac{\gamma_1-1}{2}}(r|x'|)^{\frac{n+|\gamma'|-3}{2}}}{2^{\frac{n+|\gamma|}{2}-2}(\delta\sqrt{1+r^2})^{n+|\gamma|+\alpha}} \frac{\Gamma(n+|\gamma|+\alpha)}{\Gamma\left(\frac{\gamma_1+1}{2}\right)\Gamma\left(\frac{n+|\gamma'|-1}{2}\right)}\times$$

$$\times F_4\left(\frac{\beta}{2},\frac{\beta+1}{2};\frac{\gamma_1+1}{2},\frac{n+|\gamma'|-1}{2};-\frac{x_1^2}{\delta^2(1+r^2)},-\frac{(r|x'|)^2}{\delta^2(1+r^2)}\right),$$

where $\beta = n+|\gamma|+\alpha$. Then

$$J_1 = \frac{2^{\frac{|\gamma|-n}{2}}\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{x_1^{\frac{\gamma_1-1}{2}}|x'|^{\frac{n-1+|\gamma'|}{2}-1}} \int\limits_0^1 r^{\frac{n+|\gamma'|-1}{2}} \frac{(1-r^2)^{m+\frac{\alpha}{2}}}{(1-r^2+i\varepsilon(1+r^2))^m} dr\times$$

$$\times \frac{x_1^{\frac{\gamma_1-1}{2}}(r|x'|)^{\frac{n+|\gamma'|-3}{2}}}{2^{\frac{n+|\gamma|}{2}-2}(\delta\sqrt{1+r^2})^{n+|\gamma|+\alpha}} \frac{\Gamma(n+|\gamma|+\alpha)}{\Gamma\left(\frac{\gamma_1+1}{2}\right)\Gamma\left(\frac{n+|\gamma'|-1}{2}\right)}\times$$

$$\times F_4\left(\frac{\beta}{2},\frac{\beta+1}{2};\frac{\gamma_1+1}{2},\frac{n+|\gamma'|-1}{2};-\frac{x_1^2}{\delta^2(1+r^2)},-\frac{(r|x'|)^2}{\delta^2(1+r^2)}\right).$$

$$= \frac{\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\delta^\beta} \frac{\Gamma(n+|\gamma|+\alpha)}{\Gamma\left(\frac{\gamma_1+1}{2}\right)\Gamma\left(\frac{n+|\gamma'|-1}{2}\right)}\times$$

$$\times \int\limits_0^1 r^{n+|\gamma'|-2} \frac{(1-r^2)^{m+\frac{\alpha}{2}}}{(1+r^2)^{\frac{n+|\gamma|+\alpha}{2}}(1-r^2+i\varepsilon(1+r^2))^m}\times$$

$$\times F_4\left(\frac{\beta}{2},\frac{\beta+1}{2};\frac{\gamma_1+1}{2},\frac{n+|\gamma'|-1}{2};-\frac{x_1^2}{\delta^2(1+r^2)},-\frac{(r|x'|)^2}{\delta^2(1+r^2)}\right) dr.$$

Similarly, we find

$$J_2 = \frac{\prod_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\delta^{\beta}} \frac{\Gamma(\beta)}{\Gamma\left(\frac{\gamma_1+1}{2}\right)\Gamma\left(\frac{n+|\gamma'|-1}{2}\right)} \times$$

$$\times \int_1^{\infty} r^{n+|\gamma'|-2} \frac{(1-r^2)^{m+\frac{\alpha}{2}}}{(1+r^2)^{\frac{n+|\gamma|+\alpha}{2}}(1-r^2+i\varepsilon(1+r^2))^m} \times$$

$$\times F_4\left(\frac{\beta}{2}, \frac{\beta+1}{2}; \frac{\gamma_1+1}{2}, \frac{n+|\gamma'|-1}{2}; -\frac{x_1^2}{\delta^2(1+r^2)}, -\frac{(r|x'|)^2}{\delta^2(1+r^2)}\right) dr.$$

Multiplying by the corresponding constants, adding $J_1(x)$ with $J_2(x)$ and taking into account that

$$(1-r^2 \mp i0)^{m+\frac{\alpha}{2}} = (1-r^2)_+^{m+\frac{\alpha}{2}} + e^{\mp(m+\frac{\alpha}{2})\pi i}(1-r^2)_-^{m+\frac{\alpha}{2}}$$

we obtain the statement of the proved theorem. □

6 Inversion of the Hyperbolic B-Potentials

Consider a convolution operator

$$Af = (T * f)_\gamma, \qquad f \in S_{ev}. \tag{28}$$

In the images of Hankel transform we can write

$$\mathbf{F}_\gamma[Af] = \mathbf{F}_\gamma[T] \cdot \mathbf{F}_\gamma[f].$$

Definition 6 Let $M \in S'_{ev}$. The weighted distribution is called **B-multiplier in** L_p^γ, if for all $f \in S_{ev}$ the generalized convolution $(\mathbf{F}_\gamma^{-1} M * f)_\gamma$ belongs to L_p^γ and the supremum

$$\sup_{||f||_{p,\gamma}=1} ||(\mathbf{F}_\gamma^{-1} M * f)_\gamma||_{p,\gamma} \tag{29}$$

is finite. Linear space of all such M is denoting by the $M_{p,\gamma} = M_{p,\gamma}(\mathbb{R}_+^n)$. Norm in $M_{p,\gamma}$ is the supremum (29).

Consider a singular differential operator

$$(D_B)^{\beta_i}_{x_i} = \begin{cases} B_{\gamma_i}^{\frac{\beta_i}{2}} & , \beta = 0, 2, 4, \dots , \\ D_{x_i} B_{\gamma_i}^{\frac{\beta_i - 1}{2}} & , \beta = 1, 3, 5, \dots , \end{cases}$$

where $B_{\gamma_i} = \frac{\partial^2}{\partial x_i} + \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}$.

In the article [38] was proved the following criterion of B-multiplier of the type of Mikhlin criterion.

Theorem 7 *Let $M(\xi) \in C^k_{ev}(\mathbb{R}^n_+) \backslash \{0\}$, where k is even number grater then $\frac{n+|\gamma|}{2}$ and there is a constant A which does not depend on $\beta = (\beta_1, \dots, \beta_m)$, $|\beta| < k$, such that for $\xi \neq 0$, $\xi \in \overline{\mathbb{R}}^n_+$ the condition*

$$\left| \xi^\beta \, (D_B)^\beta_\xi \; M(\xi) \right| \leq A$$

holds. Then $M(\xi)$ is B-multiplier for $1 < p < \infty$.

Lemma 4 *Let $\varepsilon, \delta > 0$ are fixed numbers and $m \geq n + |\gamma| - \frac{\alpha}{2}$. Function*

$$M^{\mp}_{\alpha,\varepsilon,\delta}(\xi) = \begin{cases} \frac{(P \mp i0)^{m + \frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m}, & P(\xi) \neq 0; \\ 0, & P(\xi) = 0 \end{cases}$$

is B-multiplier for $1 < p < \infty$.

Proof We prove the estimate

$$\left| \xi_1^{\beta_1} \dots \xi_n^{\beta_n} (D_B)^{\beta_1}_{\xi_1} \dots (D_B)^{\beta_n}_{\xi_n} \, M^{\mp}_{\alpha,\varepsilon,\delta}(\xi) \right| \leq C(\varepsilon, \delta). \tag{30}$$

For $\xi \notin V = \{\xi \in \mathbb{R}^n_+ : P(\xi) = 0\}$ we have

$$|(D_B)^j_\xi (P \mp i0)^{m + \frac{\alpha}{2}}| \leq C_1 |\xi^j| \cdot |P(\xi)|^{m + \frac{\alpha}{2} - |j|},$$

$$|(D_B)^k_\xi (P(\xi) + i\varepsilon|\xi|^2)^{-m}| \leq C_2 |\xi^k| \cdot |P^2(\xi) + \varepsilon^2 |\xi|^4|^{-\frac{m+|k|}{2}},$$

$$|(D_B)^r_\xi e^{-\delta|\xi|}| \leq C_3 |\xi^r| \cdot \frac{e^{-\delta|\xi|}}{|\xi|^{2r-1}}.$$

Using these estimates and the formula of the type of Leibniz formula for B-differentiation of the following form:

$$B^l_i(u\,v) = \sum_{k=0}^{2l} C^k_{2l} \left(D^{2l-k}_{B_i} u \right) \left(D^k_{B_i} v \right) + \sum_{m=1}^{2l-2} \frac{1}{x_i^m} \mathbf{P}_{2l-m} \left(D_{B_i} v; D_{B_i} u \right) ,$$

where

$$\mathbf{P}_{2l-m}\left(D_{B_i}v;\,D_{B_i}u\right)=\sum_{j=1}^{2l-v-1}a_{2l-m-j,j}(\gamma_j)\left(D_{B_i}^{2l-m-j}u\right)\left(D_{B_i}^{j}v\right),$$

we get the required estimate (30).

If $\xi\in V$ then the estimate (30) follows from the continuity of the function $M_{\alpha,\varepsilon,\delta}^{\mp}(\xi)$ and its derivatives on V. □

Lemma 5 *Function* ${}^{\mp}g_{\varepsilon,\delta}^{\alpha}(x)$ *belongs to space* L_p^{γ}, $1<p<\infty$.

Proof Since the function

$${}^{\mp}g_{\varepsilon,\delta}^{\alpha}(t)=\mathbf{F}_{\gamma}^{-1}\frac{(P\mp i0)^{m+\frac{\alpha}{2}}e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m}$$

is representable by an operator generated by the B-multiplier $M_{\alpha,\varepsilon,\delta}^{\mp}(\xi)$ in L_p^{γ} then ${}^{\mp}g_{\varepsilon,\delta}^{\alpha}\in L_p^{\gamma}$. □

Lemma 6 *Let* $f\in S_{ev}$. *The operator*

$$(I_{P\pm i0,\gamma}^{\alpha})_{\varepsilon,\delta}^{-1}f(x)=\int\limits_{\mathbb{R}_+^n}{}^{\mp}g_{\varepsilon,\delta}^{\alpha}(t)({}^{\gamma}\mathbf{T}_x^t f(x))t^{\gamma}dt$$

is bounded in L_p^{γ}, $1<p<\infty$.

Proof By definition of the operator

$$(I_{P\pm i0,\gamma}^{\alpha})_{\varepsilon,\delta}^{-1}f=\left(\left(\mathbf{F}_{\gamma}^{-1}\frac{(P\mp i0)^{m+\frac{\alpha}{2}}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\right)(x)*f(x)\right)_{\gamma}$$

it is a generalized convolution $(\mathbf{F}_{\gamma}^{-1}M_{\alpha,\varepsilon,\delta}^{\mp}*f)_{\gamma}$ with the B-multiplier $M_{\alpha,\varepsilon,\delta}^{\mp}(\xi)$ therefore belongs to L_p^{γ}. □

Lemma 7 *Let* $f\in\Phi_V^{\gamma}$, $V=\{\xi\in\mathbb{R}_+^n:P(\xi)=0\}$ *then*

$$((I_{P\pm i0,\gamma}^{\alpha})_{\varepsilon,\delta}^{-1}I_{P\pm i0,\gamma}^{\alpha}f)(x)=(\mathbf{P}_{\gamma,\delta}f)(x)+\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\sum_{k=0}^{m}C_m^k(-i\varepsilon)^k(\mathbf{A}_k^{\gamma,\delta,\varepsilon}f)(x),$$

where $(\mathbf{P}_{\gamma,\delta}f)(x)$ *is a generalized convolution with the Poisson kernel* (26)

$$(\mathbf{A}_k^{\gamma,\delta,\varepsilon}f)(x)=(A_k^{\gamma,\delta,\varepsilon}(x)*f(x))_{\gamma},\qquad A_k^{\gamma,\delta,\varepsilon}(x)=\int\limits_{\mathbb{R}_+^n}\frac{|\xi|^{2k}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^k}\mathbf{j}_{\gamma}(x,\xi)\xi^{\gamma}d\xi.$$

Proof Let $I^{\alpha}_{P\pm i0,\gamma} f = g$. We have

$$\mathbf{F}_\gamma ((I^{\alpha}_{P\pm i0,\gamma})^{-1}_{\varepsilon,\delta} g)(x) = \mathbf{F}_\gamma \left(\left(\mathbf{F}^{-1}_\gamma \frac{(P\mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m} \right)(x) * g(x) \right)_\gamma =$$

$$= \frac{(P\mp i0)^{m+\frac{\alpha}{2}}_\gamma e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m} \cdot \mathbf{F}_\gamma g = \frac{(P\mp i0)^{m+\frac{\alpha}{2}}_\gamma e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m} \cdot (P\mp i0)^{-\frac{\alpha}{2}}_\gamma \mathbf{F}_\gamma f =$$

$$= \frac{(P\mp i0)^{m}_\gamma e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m} \cdot \mathbf{F}_\gamma f.$$

Then

$$((I^{\alpha}_{P\pm i0,\gamma})^{-1}_{\varepsilon,\delta} I^{\alpha}_{P\pm i0,\gamma} f)(x) = \mathbf{F}^{-1}_\gamma \left(\frac{(P\mp i0)^{m}_\gamma e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m} \cdot \mathbf{F}_\gamma f \right) =$$

$$= \left(\mathbf{F}^{-1}_\gamma \frac{(P\mp i0)^{m}_\gamma e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m} * f \right)_\gamma . \tag{31}$$

Applying to $\mathbf{F}^{-1}_\gamma \frac{(P\mp i0)^{m}_\gamma e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m}$ the Newton's binomial formula we obtain

$$\mathbf{F}^{-1}_\gamma \frac{(P\mp i0)^{m}_\gamma e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m} = \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \left[\int\limits_{\{\xi_1>|\xi'|\}^+} \frac{(\xi_1^2-|\xi'|^2)^m e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m} \mathbf{j}_\gamma(x,\xi)\xi^\gamma\, d\xi + \right.$$

$$\left. + e^{\mp m\pi i} \int\limits_{\{\xi_1<|\xi'|\}^+} \frac{(|\xi'|^2-\xi_1^2)^m e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m} \mathbf{j}_\gamma(x,\xi)\xi^\gamma\, d\xi \right] =$$

$$= \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \left[\int\limits_{\{\xi_1>|\xi'|\}^+} \left((1 - \frac{i\varepsilon|\xi|^2}{P(\xi)+i\varepsilon|\xi|^2} \right)^m e^{-\delta|\xi|} \mathbf{j}_\gamma(x,\xi)\xi^\gamma\, d\xi + \right.$$

$$\left. + e^{\mp m\pi i}(-1)^m \int\limits_{\{\xi_1<|\xi'|\}^+} \left((1 - \frac{i\varepsilon|\xi|^2}{P(\xi)+i\varepsilon|\xi|^2} \right)^m e^{-\delta|\xi|} \mathbf{j}_\gamma(x,\xi)\xi^\gamma\, d\xi \right] =$$

$$= \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \sum_{k=0}^{m} C_m^k (-i\varepsilon)^k \left[\int\limits_{\{\xi_1>|\xi'|\}^+} \frac{|\xi|^{2k} e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^k} \mathbf{j}_\gamma(x,\xi)\xi^\gamma\, d\xi + \right.$$

$$\left. + \int\limits_{\{\xi_1<|\xi'|\}^+} \frac{|\xi|^{2k} e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^k} \mathbf{j}_\gamma(x,\xi)\xi^\gamma\, d\xi \right] =$$

$$= \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \sum_{k=0}^{m} C_m^k (-i\varepsilon)^k \int\limits_{\mathbb{R}_+^n} \frac{|\xi|^{2k} e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^k} \mathbf{j}_\gamma(x,\xi)\xi^\gamma\, d\xi.$$

For $m=0$ applying of (22) gives

$$(\mathbf{F}_\gamma^{-1} e^{-\delta|\xi|})(x) = \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \frac{2^{|\gamma|}\delta \prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2+|x|^2)^{\frac{n+|\gamma|+1}{2}}} =$$

$$= \frac{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)} \delta\,(\delta^2+|x|^2)^{-\frac{n+|\gamma|+1}{2}} = P_\gamma(x,\delta). \tag{32}$$

Here $P_\gamma(x,\delta)$ is general Poisson kernel (24). By Lemma 2 $P_\gamma(x,\delta)\in L_p^\gamma$.

Introducing the notation

$$A_k^{\gamma,\delta,\varepsilon}(x) = \int\limits_{\mathbb{R}_+^n} \frac{|\xi|^{2k} e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^k} \mathbf{j}_\gamma(x,\xi)\xi^\gamma\, d\xi = \mathbf{F}_\gamma \frac{|x|^{2k} e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^k}$$

for $m>0$ we get

$$\mathbf{F}_\gamma^{-1} \frac{(P\mp i0)_\gamma^m e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m} = \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \sum_{k=0}^{m} C_m^k (-i\varepsilon)^k A_k^{\gamma,\delta,\varepsilon}(x). \tag{33}$$

Substituting (32) and (33) in (31) we obtain the statement of the theorem for $f\in\Phi_V^\gamma$. □

Theorem 8 *Let* $f\in\Phi_V^\gamma$, $V=\{\xi\in\mathbb{R}_+^n : P(\xi)=0\}$, $1<p<\frac{n+|\gamma|}{\alpha}$, $p\le 2$, $n+|\gamma|-2<\alpha<n+|\gamma|$, *then*

$$((I_{P\pm i0,\gamma}^\alpha)^{-1} I_{P\pm i0,\gamma}^\alpha f)(x) = f(x),$$

where

$$(I^{\alpha}_{P\pm i0,\gamma})^{-1}f)=\overset{L_p^\gamma}{\lim_{\delta\to 0}}\overset{L_2^\gamma}{\lim_{\varepsilon\to 0}}\left(\left(\mathbf{F}^{-1}_{\gamma}\frac{(P\mp i0)^{m+\frac{\alpha}{2}}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\right)(x)*f(x)\right)_{\gamma}$$

here the limit by ε *is understood by norm in* L_2^γ *and the limit by* δ *is understood by norm in* L_p^γ*.*

Proof From Lemma 7 follows that it is enough to show

$$\overset{L_p^\gamma}{\lim_{\delta\to 0}}\overset{L_2^\gamma}{\lim_{\varepsilon\to 0}}\left[(\mathbf{P}_{\gamma,\delta}f)(x)+\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\sum_{k=0}^{m}C_m^k(-i\varepsilon)^k(\mathbf{A}_k^{\gamma,\delta,\varepsilon}f)(x)\right]=f(x).$$

Find the limit for ε in L_2^γ. We have

$$(\mathbf{A}_k^{\gamma,\delta,\varepsilon}f)(x)=(A_k^{\gamma,\delta,\varepsilon}(x)*f(x))_\gamma=$$

$$=\int\limits_{\mathbb{R}^n_+}\mathbf{F}_\gamma\left[\frac{|x|^{2k}e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^k}\right](y)({}^\gamma\mathbf{T}_x^y f)(x)y^\gamma\,dy=$$

$$=\int\limits_{\mathbb{R}^n_+}\mathbf{F}_\gamma\left[\frac{|x|^{2k}e^{-\frac{\delta}{2}|x|}}{(P(x)+i\varepsilon|x|^2)^k}\,e^{-\frac{\delta}{2}|x|}\right](y)({}^\gamma\mathbf{T}_x^y f)(x)y^\gamma\,dy=$$

$$=\int\limits_{\mathbb{R}^n_+}\mathbf{F}_\gamma\left[\frac{|x|^{2k}e^{-\frac{\delta}{2}|x|}}{(P(x)+i\varepsilon|x|^2)^k}\mathbf{F}_\gamma\left[P_\gamma\left(z,\frac{\delta}{2}\right)\right](x)\right](y)({}^\gamma\mathbf{T}_x^y f)(x)y^\gamma\,dy.$$

Using Parseval Equation to Hankel transform (see [24, p. 20]) we obtain

$$||(-i\varepsilon)^k(\mathbf{A}_k^{\gamma,\delta,\varepsilon}f)(x)||^2_{2,\gamma}=||(A_k^{\gamma,\delta,\varepsilon}(x)*f(x))_\gamma||^2_{2,\gamma}=||\mathbf{F}_\gamma A_k^{\gamma,\delta,\varepsilon}(x)\cdot\mathbf{F}_\gamma f(x))_\gamma||^2_{2,\gamma}=$$

$$=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+}\left|\frac{(-i\varepsilon)^k|x|^{2k}e^{-\frac{\delta}{2}|x|}}{(P(x)+i\varepsilon|x|^2)^k}\mathbf{F}_\gamma\left[P_\gamma\left(x,\frac{\delta}{2}\right)\right]\mathbf{F}_\gamma f(x)\right|^2x^\gamma dx=$$

$$=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+}\left|\frac{(-i\varepsilon)^k|x|^{2k}e^{-\frac{\delta}{2}|x|}}{(P(x)+i\varepsilon|x|^2)^k}\mathbf{F}_\gamma\left[(\mathbf{P}_{\gamma,\delta}f)(x)\right]\right|^2x^\gamma dx.$$

Considering that

$$\left|\frac{(-i\varepsilon)^k |x|^{2k} e^{-\frac{\delta}{2}|x|}}{(P(x)+i\varepsilon|x|^2)^k}\mathbf{F}_\gamma\left[(\mathbf{P}_{\gamma,\delta}f)(x)\right]\right|^2 \leq e^{-\delta|x|}\left|\mathbf{F}_\gamma\left[(\mathbf{P}_{\gamma,\delta}f)(x)\right]\right|^2$$

and $e^{-\delta|x|}\left|\mathbf{F}_\gamma\left[P_\gamma\left(x,\frac{\delta}{2}\right)\right]\right|^2 \in L_1^\gamma$ on the basis of the Lebesgue dominated convergence theorem, we obtain that

$$(-i\varepsilon)^k(\mathbf{A}_k^{\gamma,\delta,\varepsilon}f)(x)\to 0 \qquad \text{for} \qquad \varepsilon\to 0 \qquad \text{in} \qquad L_2^\gamma.$$

The fact that

$$||(\mathbf{P}_{\gamma,\delta}f)(x)-f(x)||_{p,\gamma}\to 0 \qquad \text{for} \qquad \delta\to 0$$

was proved in Lemma 3. Thus, the theorem is proved. □

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Some Properties of Sobolev Orthogonal Polynomials Associated with Chebyshev Polynomials of the Second Kind

M. S. Sultanakhmedov

Abstract This paper considers the polynomials $U_{r,n}(x)$ ($r = 1, 2, \ldots; n = 0, 1, 2, \ldots$) which are orthonormal with respect to Sobolev-type inner product on the segment $[-1, 1]$ with the weight function $\mu(x) = \sqrt{1 - x^2}$ and associated with classical Chebyshev polynomials of the second kind $U_n(x)$. We obtain explicit formulas for $U_{r,n}(x)$ as well as recurrence relations for two special cases $r = 1$ and $r = 2$, which are important for applications. Additionally, the asymptotic properties of the polynomials $U_{r,k}(x)$ are studied.

Keywords Sobolev orthogonal polynomials · Chebyshev polynomials · Asymptotic properties · Recurrence relations · Explicit formulas · Function approximation

Mathematics Subject Classification 26C05, 42C10, 65Q30

1 Introduction

Following the established notation, we denote by $L^p_\omega(a, b)$ the space of functions $f(x)$ measurable on (a, b) for which $\int\limits_a^b |f(x)|^p \omega(x) dx < \infty$, where $\omega = \omega(x)$ is a weight function.

Let $\{\varphi_n\}_{n=0}^\infty$ be a system of functions orthonormal in $L^2_\omega(a, b)$. In other words,

$$\langle \varphi_n, \varphi_m \rangle_{L^2_\omega} = \int_a^b \varphi_n(t)\varphi_m(t)\omega(t)dt = \delta_{n,m},$$

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where $\delta_{n,m}$ is the Kronecker symbol.

By $W^r_{L^2_\omega(a,b)}$ we denote the Sobolev space which consists of functions $f(x)$, continuously differentiable $r-1$-times on $[a,b]$ and such that $f^{(r-1)}(x)$ is absolutely continuous on $[a,b]$ and $f^{(r)}(x) \in L^2_\omega(a,b)$. The inner product in $W^r_{L^2_\omega(a,b)}$ is defined using the equality

$$\langle f, g\rangle = \sum_{\nu=0}^{r-1} f^{(\nu)}(a)g^{(\nu)}(a) + \int_a^b f^{(r)}(t)g^{(r)}(t)\omega(t)dt. \tag{1.1}$$

Inner products of this kind are called Sobolev-type inner products. From the system $\{\varphi_k(x)\}$, we can generate functions orthogonal with respect to inner product (1.1) using the following equations:

$$\varphi_{r,k}(x) = \frac{(x-a)^k}{k!}, \quad k = 0, 1, \ldots, r-1, \tag{1.2}$$

$$\varphi_{r,r+k}(x) = \frac{1}{(r-1)!}\int_a^x (x-t)^{r-1}\varphi_k(t)dt, \quad k = 0, 1, \ldots. \tag{1.3}$$

More precisely, the following statement was proved in [1].

Theorem A *Suppose that the functions* $\varphi_k(x)$ $(k = 0, 1, \ldots)$ *form a complete orthonormal system in* $L^2_\omega(a,b)$*. Then the system* $\{\varphi_{r,k}(x)\}_{k=0}^{\infty}$*, generated by the system* $\{\varphi_k(x)\}_{k=0}^{\infty}$ *with the help of the equalities* (1.2)–(1.3)*, is complete in* $W^r_{L^2_\omega(a,b)}$ *and orthonormal with respect to the inner product* (1.1)*.*

I.I. Sharapudinov in [1] considered the polynomials associated with the classical Chebyshev polynomials of the first kind $T_n(x) = \cos(n \arccos x)$ and orthogonal on $[-1, 1]$ with respect to the Sobolev inner product (1.1) when the weight function has a form $\omega(x) = \frac{1}{\sqrt{1-x^2}}$.

The distinctive feature of inner products of the Sobolev-type of this kind is the presence of special points in the neighborhood of which the behavior of Sobolev orthogonal functions can be "controlled". Due to this feature, it is possible to construct the Fourier series by the Sobolev orthogonal polynomials which partial sums coincide with the approximated function at the ends of the orthogonality segment. Such series proved to be a convenient tool for different applied tasks such as representing solutions of the Cauchy problem for differential equations.

Additionally, in [2, 3] we studied the approximative properties of special wavelets based on classical Chebyshev polynomials of the second kind

$$U_n(x) = \frac{\sin{(n+1)\arccos x}}{\sin \arccos x}, \quad (n = 0, 1, 2, \ldots),$$

which also have the property of coincidence with the approximated function at the end points of the orthogonality segment (at the points ± 1).

In this article, we consider polynomials $U_{r,n}(x)$ $(r = 1, 2, \ldots; n = 0, 1, 2, \ldots)$ which are Sobolev orthonormal on the segment $[-1, 1]$ with the weight function $\omega = \mu(x) = \sqrt{1-x^2}$ and associated with Chebyshev polynomials of the second kind. We obtained the explicit formulas for $U_{r,n}(x)$, as well as recurrence relations for two important special cases $r = 1$ and $r = 2$. Also, we studied the asymptotic properties of the polynomials $U_{r,k}(x)$.

2 Some Properties of Jacobi and Chebyshev Polynomials

For arbitrary real α and β the Jacobi polynomials $P_n^{\alpha,\beta}(x)$ can be determined using the Rodrigues formula (see [4]):

$$P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}\frac{1}{\rho(x)}\frac{d^n}{dx^n}\left\{\rho(x)\sigma^n(x)\right\}, \tag{2.1}$$

where $\rho(x) = \rho(x; \alpha, \beta) = (1-x)^\alpha(1+x)^\beta,\ \sigma(x) = 1 - x^2$. If $\alpha, \beta > -1$, then Jacobi polynomials form a complete orthogonal system in $L^2_\rho(-1, 1)$, i.e.

$$\int_{-1}^{1} P_n^{\alpha,\beta}(x)P_m^{\alpha,\beta}(x)\rho(x)dx = h_n^{\alpha,\beta}\delta_{nm}, \tag{2.2}$$

where

$$h_n^{\alpha,\beta} = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)2^{\alpha+\beta+1}}{n!\Gamma(n+\alpha+\beta+1)(2n+\alpha+\beta+1)}. \tag{2.3}$$

We will need the following properties of Jacobi polynomials (see [4, 5]):

$$P_n^{\alpha,\beta}(-x) = (-1)^n P_n^{\beta,\alpha}(x),$$

$$P_n^{\alpha,\beta}(-1) = (-1)^n\binom{n+\beta}{n}, \quad P_n^{\alpha,\beta}(1) = \binom{n+\alpha}{n}, \tag{2.4}$$

$$\frac{d}{dx}P_n^{\alpha,\beta}(x) = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{\alpha+1,\beta+1}(x), \tag{2.5}$$

$$\frac{d^\nu}{dx^\nu}P_n^{\alpha,\beta}(x) = \frac{(n+\alpha+\beta+1)_\nu}{2^\nu}P_{n-\nu}^{\alpha+\nu,\beta+\nu}(x), \tag{2.6}$$

where $(a)_0 = 1$, $(a)_\nu = a(a+1)\dots(a+\nu-1)$,

$$\binom{n}{l} P_n^{\alpha,-l}(x) = \binom{n+\alpha}{l} \left(\frac{x+1}{2}\right)^l P_{n-l}^{\alpha,l}(x), \quad 1 \le l \le n, \tag{2.7}$$

$$P_n^{\alpha,\beta}(x) = \binom{n+\alpha}{n} \sum_{k=0}^{n} \frac{(-n)_k (n+\alpha+\beta+1)_k}{k!(\alpha+1)_k} \left(\frac{1-x}{2}\right)^k, \tag{2.8}$$

$$(1-x)^\alpha (1+x)^\beta P_n^{\alpha,\beta}(x) = \frac{(-1)^m}{2^m n^{[m]}} \frac{d^m}{dx^m} \left\{(1-x)^{m+\alpha}(1+x)^{m+\beta} P_{n-m}^{m+\alpha,m+\beta}(x)\right\}, \tag{2.9}$$

where $k^{[0]} = 1$, $k^{[r]} = k(k-1)\dots(k-r+1)$.

Let us also consider the expression

$$\frac{P_n^{a,a}(x)}{P_n^{a,a}(1)} = \sum_{j=0}^{[n/2]} \frac{n!(\alpha+1)_{n-2j}(n+2a+1)_{n-2j}(1/2)_j(a-\alpha)_j}{(n-2j)!(2j)!(a+1)_{n-2j}(n-2j+2\alpha+1)_{n-2j}} \times$$
$$\times \frac{1}{(n-2j+a+1)_j(n-2j+\alpha+3/2)_j} \frac{P_{n-2j}^{\alpha,\alpha}(x)}{P_{n-2j}^{\alpha,\alpha}(1)}, \tag{2.10}$$

where $[b]$ is the integer part of number b. Taking into account that $P_{k+r}^{\alpha-r,\alpha-r}(x)$ and $P_{k+r-2j}^{\alpha,\alpha}(x)$ are analytical functions with respect to α, the next statement follows from equality (2).

Lemma 1 *Let $\alpha > -1$ be real and $r \ge 1$, $k \ge r+1$ be integers. Then*

$$P_{k+r}^{\alpha-r,\alpha-r}(x) = \sum_{j=0}^{r} \lambda_j^\alpha P_{k+r-2j}^{\alpha,\alpha}(x),$$

where

$$\lambda_j^\alpha = \lambda_j^\alpha(r,k) = \frac{(-1)^j (k-r+2\alpha+1)_{k+r-2j}(1/2)_j r^{[j]}(\alpha+k)^{[j]}}{(k+r-2j+2\alpha+1)_{k+r-2j}(k+r-2j+\alpha+3/2)_j(2j)!}.$$

The Chebyshev polynomials of the first and second kinds are well-known special cases of Jacobi polynomials. The Chebyshev polynomials of the first kind

$$T_n(x) = \cos n \arccos x, \quad n = 0, 1, 2, \dots,$$

form a complete orthogonal system in $L^2_\kappa(-1,1)$ with the weight function $\kappa(x) = (1-x^2)^{-\frac{1}{2}}$, while the Chebyshev polynomials of the second kind

$$U_n(x) = \frac{\sin{(n+1)\arccos x}}{\sin\arccos x}, \quad n = 0, 1, 2, \ldots,$$

form a complete orthogonal system in $L^2_\mu(-1,1)$ where $\mu(x) = \sqrt{1-x^2}$.

There are several relations between these two kinds of polynomials. We will use this one:

$$2T_n(x) = U_n(x) - U_{n-2}(x), \quad n \geq 2. \tag{2.11}$$

The relation between $U_n(x)$ and the standardized Jacobi polynomials is shown by the next equation:

$$U_n(x) = \frac{4^n n!(n+1)!}{(2n+1)!} P_n^{\frac{1}{2},\frac{1}{2}}(x). \tag{2.12}$$

Using Lemma 1 for the case $\alpha = \frac{1}{2}$, we can derive the following statement.

Lemma 2 *Let k, r be integers, $r \geq 1$, $k \geq r+1$. Then*

$$P_{k+r}^{\frac{1}{2}-r,\frac{1}{2}-r}(x) = \frac{(2k+1)!}{(k!)^2 4^{k+r}} \sum_{j=0}^{r} \frac{(-1)^j}{j!} \frac{(k+r-2j+1)r^{[j]}k^{[r-1]}}{(k+r-j+1)^{[r+1]}} U_{k+r-2j}(x).$$

Finally, we also give here an asymptotic formula for the Jacobi polynomials that will be needed in the study of asymptotic properties of polynomials $U_{r,n}(x)$. Let α and β be arbitrary real numbers,

$$s(\theta) = \pi^{-\frac{1}{2}} \left(\sin\frac{\theta}{2}\right)^{-\alpha-\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}},$$

$$\lambda_n = n + \frac{\alpha+\beta+1}{2}, \quad \gamma = -\left(\alpha+\frac{1}{2}\right)\frac{\pi}{2}.$$

Then for $0 < \theta < \pi$ the next asymptotic formula holds

$$P_n^{\alpha,\beta}(\cos\theta) = n^{-\frac{1}{2}} s(\theta) \left(\cos(\lambda_n\theta + \gamma) + \frac{v_n(\theta)}{n\sin\theta}\right), \tag{2.13}$$

where for $v_n(\theta) = v_n(\theta;\alpha,\beta)$ the estimate

$$|v_n(\theta)| \leq c(\alpha,\beta,\delta) \quad \left(0 < \frac{\delta}{n} \leq \theta \leq \pi - \frac{\delta}{n}\right) \tag{2.14}$$

takes place. In particular, for Chebyshev polynomials of the first kind the asymptotic formula takes the following form

$$T_n(\cos\theta) = \frac{4^n (n!)^2}{(2n)!\sqrt{\pi n}} \left(\cos n\theta + \frac{\hat{v}_n(\theta)}{n \sin\theta} \right), \tag{2.15}$$

where

$$|\hat{v}_n(\theta)| \le c(\delta), \quad 0 < \frac{\delta}{n} \le \theta \le \pi - \frac{\delta}{n}.$$

3 The Chebyshev–Sobolev Polynomials of the Second Kind

It is easy to show that

$$\int_{-1}^{1} U_n(x) U_m(x) \mu(x) dx = \frac{\pi \delta_{nm}}{2}. \tag{3.1}$$

Hence the polynomials $\hat{U}_n(x) = \sqrt{2/\pi}\, U_n(x)$ are orthonormal in $L^2_\mu(-1, 1)$.

Then we consider the polynomials $U_{r,k}(x)$ $(r = 1, 2, \ldots)$ defined on $[-1, 1]$ by the equalities

$$U_{r,k}(x) = \frac{(x+1)^k}{k!}, \quad k = 0, 1, \ldots, r-1, \tag{3.2}$$

$$U_{r,r+n}(x) = \frac{\sqrt{2/\pi}}{(r-1)!} \int_{-1}^{x} (x-t)^{r-1} U_n(t) dt, \quad n = 0, 1, \ldots. \tag{3.3}$$

The next statement directly follows from Theorem A.

Corollary A *For any integer $r > 0$, the system of polynomials $\{U_{r,k}(x)\}_{k=0}^{\infty}$ generated by equalitites (3.2)–(3.3) is complete in $W^r_{L^2_\mu(-1,1)}$ and is orthonormal with respect to the Sobolev-type inner product.*

To proceed with the study of the asymptotic properties of polynomials $U_{r,k}(x)$, first we need to obtain some representations for these polynomials.

Theorem 1 *For an arbitrary integer $r > 0$ and $n \ge 0$ the following equality holds*

$$U_{r,r+n}(x) = (-1)^n \sqrt{\frac{2}{\pi}} \sum_{k=0}^{n} \frac{(-2)^k}{(k+r)^{[r]}} \binom{n+k+1}{n-k} (1+x)^{k+r}. \tag{3.4}$$

Proof We use (2.8) and write

$$P_n^{\frac{1}{2},\frac{1}{2}}(x) = (-1)^n P_n^{\frac{1}{2},\frac{1}{2}}(-x) = (-1)^n \binom{n+\frac{1}{2}}{n} \sum_{k=0}^{n} \frac{(-n)_k (n+2)_k}{k!(\frac{3}{2})_k} \left(\frac{1+x}{2}\right)^k =$$

$$= (-1)^n \frac{(2n+1)!}{4^n (n!)^2} \sum_{k=0}^{n} 4^k \frac{(-n)_k (n+2)_k}{(2k+1)!} \left(\frac{1+x}{2}\right)^k.$$

From (2.12) we get

$$U_n(x) = \frac{4^n n!(n+1)!}{(2n+1)!} P_n^{\frac{1}{2},\frac{1}{2}}(x) = (-1)^n \sum_{k=0}^{n} 4^k \frac{(-n)_k (n+1)_{k+1}}{(2k+1)!} \left(\frac{1+x}{2}\right)^k.$$

Next, from the definition (3.3) we have:

$$U_{r,r+n}(x) = \frac{\sqrt{2/\pi}}{(r-1)!} \int_{-1}^{x} (x-t)^{r-1} U_n(t) dt =$$

$$= (-1)^n \sqrt{\frac{2}{\pi}} \sum_{k=0}^{n} \frac{4^k (-n)_k (n+1)_{k+1}}{(2k+1)!} \frac{1}{(r-1)!} \int_{-1}^{x} (x-t)^{r-1} \left(\frac{1+t}{2}\right)^k dt.$$

Additionally, from the Taylor formula

$$\left(\frac{1+x}{2}\right)^{k+r} = \frac{(k+r)^{[r]}}{2^r (r-1)!} \int_{-1}^{x} (x-t)^{r-1} \left(\frac{1+t}{2}\right)^k dt, \tag{3.5}$$

follows

$$U_{r,r+n}(x) = (-1)^n \sqrt{\frac{2}{\pi}} \sum_{k=0}^{n} \frac{4^k (-n)_k (n+1)_{k+1}}{(2k+1)!} \frac{2^r}{(k+r)^{[r]}} \left(\frac{1+x}{2}\right)^{k+r} =$$

$$= (-1)^n \sqrt{\frac{2}{\pi}} \sum_{k=0}^{n} \frac{(-2)^k}{(k+r)^{[r]}} \binom{n+k+1}{n-k} (1+x)^{k+r}.$$

The established equality (3.4) will be used to study the asymptotic properties of the polynomials $U_{r,n+r}(x)$ in a neighborhood of the point $x = -1$. However, for $-1+\varepsilon \le x$ the formula (3.4) becomes inappropriate for studying the asymptotic behavior of the polynomials $U_{r,n+r}(x)$ as $n \to \infty$. Therefore, it is necessary to find other representations for these polynomials that could be used to study their behavior as $n \to \infty$ in the case when x is not in close proximity to -1.

First, for $n > \max(0, r-2)$ it is obvious that $(n+1)^{[r]} \neq 0$ and we can use equality (2.6):

$$P_n^{\frac{1}{2},\frac{1}{2}}(x) = \frac{2^r}{(n+1)^{[r]}} \frac{d^r}{dt^r} P_{n+r}^{\frac{1}{2}-r,\frac{1}{2}-r}(x),$$

whence

$$U_n(x) = \frac{4^n n!(n+1)!}{(2n+1)!} P_n^{\frac{1}{2},\frac{1}{2}}(x) = 2^{2n+r} \frac{n!(n-r+1)!}{(2n+1)!} \frac{d^r}{dt^r} P_{n+r}^{\frac{1}{2}-r,\frac{1}{2}-r}(x).$$

Then let us consider the expression

$$U_{r,r+n}(x) = \frac{\sqrt{2/\pi}}{(r-1)!} \int_{-1}^{x} (x-t)^{r-1} U_n(t)\,dt =$$

$$= 2^{2n+r} \frac{n!(n-r+1)!}{(2n+1)!} \frac{\sqrt{2/\pi}}{(r-1)!} \int_{-1}^{x} (x-t)^{r-1} \frac{d^r}{dt^r} P_{n+r}^{\frac{1}{2}-r,\frac{1}{2}-r}(t)\,dt =$$

$$= \frac{2^{2n+r} n!(n-r+1)!}{(2n+1)!\sqrt{\pi/2}} \left[P_{n+r}^{\frac{1}{2}-r,\frac{1}{2}-r}(x) - \sum_{\nu=0}^{r-1} \frac{(1+x)^\nu}{\nu!} \left\{ P_{n+r}^{\frac{1}{2}-r,\frac{1}{2}-r}(t) \right\}_{t=-1}^{(\nu)} \right]. \tag{3.6}$$

Due to (2.6) we have

$$\left\{ P_{n+r}^{\frac{1}{2}-r,\frac{1}{2}-r}(t) \right\}^{(\nu)} = \frac{(n+2-r)_\nu}{2^\nu} P_{n+r-\nu}^{\frac{1}{2}+\nu-r,\frac{1}{2}+\nu-r}(t), \tag{3.7}$$

and from (2.4) follows

$$P_{n+r-\nu}^{\frac{1}{2}+\nu-r,\frac{1}{2}+\nu-r}(-1) = (-1)^{n+r-\nu} \binom{n+\frac{1}{2}}{n+r-\nu} =$$

$$= \frac{(-1)^{n+r-\nu}\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(\nu-r+\frac{3}{2}\right)(n+r-\nu)!}. \tag{3.8}$$

From (3.7) and (3.8) we get

$$\left\{ P_{n+r}^{\frac{1}{2}-r,\frac{1}{2}-r}(t) \right\}_{t=-1}^{(\nu)} = \frac{(-1)^{n+r-\nu}\Gamma\left(n+\frac{3}{2}\right)(n+2-r)_\nu}{2^\nu\Gamma\left(\nu-r+\frac{3}{2}\right)(n+r-\nu)!} = A_{\nu,n,r}. \tag{3.9}$$

Comparing (3.6) and (3.9) we can then write

$$U_{r,r+n}(x) = 2^{2n+r}\frac{n!(n-r+1)!}{(2n+1)!}\sqrt{\frac{2}{\pi}}\left[P_{n+r}^{\frac{1}{2}-r,\frac{1}{2}-r}(x) - \sum_{\nu=0}^{r-1}\frac{A_{\nu,n,r}}{\nu!}(1+x)^{\nu}\right]. \tag{3.10}$$

Next, due to the equalities

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad \Gamma(z+1/2) = \frac{\sqrt{\pi}\,\Gamma(2z)}{2^{2z-1}\Gamma(z)}$$

for $n \geq r+1$ we get

$$A_{\nu,n,r} = \frac{(-1)^{n+r-\nu}\Gamma\left(n+\frac{3}{2}\right)(n+2-r)_{\nu}}{2^{\nu}\Gamma\left(\nu-r+\frac{3}{2}\right)(n+r-\nu)!} =$$

$$= \frac{(-1)^{n+r-\nu}\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(r-\nu-\frac{1}{2}\right)(n+2-r)_{\nu}}{2^{\nu}\Gamma\left(\nu-r+\frac{3}{2}\right)\Gamma\left(r-\nu-\frac{1}{2}\right)(n+r-\nu)!} =$$

$$= \frac{(-1)^{n+r-\nu}\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(r-\nu-\frac{1}{2}\right)(n+2-r)_{\nu}\sin\pi(\nu-r+3/2)}{2^{\nu}\pi\,(n+r-\nu)!} =$$

$$= (-1)^{n+1}\frac{(2n+2)!\sqrt{\pi}}{4^{n+1}(n+1)!}\frac{(2(r-\nu-1))!\sqrt{\pi}}{4^{r-\nu-1}(r-\nu-1)!}\frac{(n+2-r)_{\nu}}{2^{\nu}\pi\,(n+r-\nu)!} =$$

$$= (-1)^{n+1}\frac{2^{\nu+1}\,(2n+1)!}{4^{n+r}n!}\frac{(2(r-\nu-1))!}{(r-\nu-1)!}\frac{(n+2-r)_{\nu}}{(n+r-\nu)!}.$$

Now we refer to Lemma 2, from which we deduce

$$\frac{2^{2n+2r}(n!)^2}{(2n+1)!n^{[r-1]}}P_{n+r}^{\frac{1}{2}-r,\frac{1}{2}-r}(x) = \sum_{j=0}^{r}\frac{(-1)^j}{j!}\frac{(n+r-2j+1)r^{[j]}}{(n+r-j+1)^{[r+1]}}U_{n+r-2j}(x),$$

$$\frac{2^{2n+r}n!(n-r+1)!}{(2n+1)!}P_{n+r}^{\frac{1}{2}-r,\frac{1}{2}-r}(x) = \sum_{j=0}^{r}(-1)^j\binom{r}{j}\frac{(n+r-2j+1)\,U_{n+r-2j}(x)}{2^r(n+r-j+1)^{[r+1]}}.$$

Then, returning to (3.10), we derive the following result.

Theorem 2 *If* $n \geq r+1$, *then*

$$U_{r,r+n}(x) = \sqrt{\frac{2}{\pi}} \left[\sum_{j=0}^{r} (-1)^j \binom{r}{j} \frac{(n+r-2j+1)\, U_{n+r-2j}(x)}{2^r (n+r-j+1)^{[r+1]}} - \right.$$
$$\left. - \frac{2^{2n+r}(n!)^2}{(2n+1)! n^{[r-1]}} \sum_{\nu=0}^{r-1} \frac{A_{\nu,n,r}}{\nu!} (1+x)^{\nu} \right]. \tag{3.11}$$

4 Recurrence Relations for Polynomials $U_{1,n}(x)$ and $U_{2,n}(x)$

As it was already mentioned above, Sobolev orthogonal polynomials proved to be an effective tool for the approximate solution of the Cauchy problem for ODEs. Regarding this, two special cases are of particular importance: $r = 1$ and $r = 2$. We will consider them here apart from others.

1. First, let $r = 1$.

$$A_{0,n,1} = \frac{(-1)^{n+1}}{(n+1)!} \frac{(2n+1)!}{2^{2n+1} n!}, \quad n = 2, 3, \ldots.$$

From here and from (3.11) we deduce

$$U_{1,n+1}(x) = \sqrt{\frac{2}{\pi}} \left[\frac{(n+2)\, U_{n+1}(x)}{2(n+2)^{[2]}} - \frac{n\, U_{n-1}(x)}{2(n+1)^{[2]}} - \frac{(-1)^{n+1}}{(n+1)} \right] =$$
$$= \frac{1}{n+1} \sqrt{\frac{2}{\pi}} \left[\frac{U_{n+1}(x) - U_{n-1}(x)}{2} - (-1)^{n+1} \right] =$$
$$= \frac{1}{n+1} \sqrt{\frac{2}{\pi}} \left[T_{n+1}(x) - (-1)^{n+1} \right].$$

Using (3.2)–(3.3), we calculate $U_{1,k}$ for $k = 0, 1, 2$ and finally get

Corollary 1 *The following equalities hold*

$$U_{1,0}(x) = 1, \quad U_{1,1}(x) = \sqrt{\frac{2}{\pi}}(x+1), \quad U_{1,2}(x) = \sqrt{\frac{2}{\pi}} \left(x^2 - 1 \right),$$
$$U_{1,n}(x) = \sqrt{\frac{2}{\pi}} \frac{T_n(x) - (-1)^n}{n}, \quad n \geq 3.$$

2. For $r = 2$ and $n \geq 3$ we have

$$A_{0,n,2} = \frac{(-1)^{n+1}\,(2n+1)!}{4^{n+1} n!\,(n+2)!}, \quad A_{1,n,2} = \frac{(-1)^{n+1}\,(2n+1)!}{4^{n+1}(n-1)!(n+1)!},$$

$$U_{2,n+2}(x) = \sqrt{\frac{2}{\pi}} \left[\sum_{j=0}^{2} (-1)^j \binom{2}{j} \frac{(n-2j+3)\,U_{n-2j+2}(x)}{4\,(n-j+3)^{[3]}} - \right.$$

$$\left. - \frac{4^{n+1}(n!)^2}{(2n+1)!n} \sum_{\nu=0}^{1} \frac{A_{\nu,n,2}}{\nu!}(1+x)^\nu \right] =$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{U_{n+2}(x)}{4\,(n+1)(n+2)} - \frac{U_n(x)}{2\,n(n+2)} + \frac{U_{n-2}(x)}{4\,n(n+1)} - \right.$$

$$\left. - \frac{(-1)^{n+1}(n-1)!}{(n+2)!} - \frac{(-1)^{n+1}n!}{(n+1)!}\,(1+x) \right] =$$

$$= \frac{1}{n+1}\sqrt{\frac{2}{\pi}} \left[\frac{1}{2\,(n+2)} \left(T_{n+2}(x) - \frac{n+2}{2n} U_n(x) \right) + \right.$$

$$\left. + \frac{U_{n-2}(x)}{4\,n} + (-1)^n \left(x + 1 + \frac{1}{n(n+2)} \right) \right] =$$

$$= \frac{1}{n+1}\sqrt{\frac{2}{\pi}} \left[\frac{T_{n+2}(x)}{2\,(n+2)} - \frac{T_n(x)}{2n} + (-1)^n \left(x + 1 + \frac{1}{n(n+2)} \right) \right].$$

Using (3.2)–(3.3), we calculate $U_{2,k}$ for $k = 0, 1, 2, 3, 4$ and finally get

Corollary 2 *The following equalities hold*

$$U_{2,0}(x) = 1, \quad U_{2,1}(x) = x + 1, \quad U_{2,2}(x) = \frac{(x+1)^2}{\sqrt{2\pi}},$$

$$U_{2,3}(x) = \frac{(x-2)(x+1)^2}{3\sqrt{\pi/2}}, \quad U_{2,4}(x) = \frac{2x^4 - 3x^2 + 2x + 3}{3\sqrt{2\pi}},$$

$$U_{2,n}(x) = \frac{1}{n-1}\sqrt{\frac{2}{\pi}} \left[\frac{T_n(x)}{2\,n} - \frac{T_{n-2}(x)}{2\,(n-2)} + (-1)^n \left(x + 1 + \frac{1}{n(n-2)} \right) \right], \quad n \geq 5.$$

Remark See [6] for more information on recurrence formulas for Sobolev orthogonal polynomials associated with classic orthogonal polynomials.

5 Asymptotic Properties of the Polynomials $U_{r,n}(x)$

The results obtained in the previous sections are applied here to the study of the behavior of the polynomials $U_{r,n}(x)$ as $n \to \infty$. Let us start with the case when x is in close proximity to -1.

Theorem 3 *For real $a > 0$ the following asymptotic formula holds*

$$U_{r,n+r}(x) = \frac{(-1)^n(n+1)}{\sqrt{\pi/2}}(1+x)^r\left(\frac{1}{r!} + v_{r,n}(x)\right), \tag{5.1}$$

where for the remainder $v_{r,n}(x)$ an estimate

$$|v_{r,n}(x)| \le c(a,r)n^2|1+x|$$

holds, when $|1+x| \le \frac{a}{n^2}$.

Proof Using Theorem 3, we rewrite equality (3.4) as follows

$$U_{r,r+n}(x) = \frac{(-1)^n(n+1)}{\sqrt{\pi/2}}(1+x)^r\left[\frac{1}{r!} + \sum_{k=1}^{n}\frac{(-2)^k}{(k+r)^{[r]}}\binom{n+k+1}{n-k}\frac{(1+x)^k}{n+1}\right].$$

Denote by

$$v_{r,n}(x) = \sum_{k=1}^{n}\frac{(-2)^k}{(k+r)^{[r]}}\binom{n+k+1}{n-k}\frac{(1+x)^k}{n+1}$$

and estimate this value when $|1+x| \le \frac{a}{n^2}$:

$$|v_{r,n}(x)| \le \sum_{k=1}^{n}\frac{2^k}{(k+r)^{[r]}}\frac{(n+k+1)!}{(n-k)!(2k+1)!}\frac{|1+x|^k}{n+1} \le$$

$$\le \sum_{k=1}^{n}\frac{2^k}{(2k+1)!(k+r)^{[r]}}\frac{(n-k+1)_{2k+1}}{(n+1)}|1+x|^k \le$$

$$\le c\,n^2\,|1+x|\sum_{k=1}^{\infty}\frac{(2a)^{k-1}}{(2k+1)!(k+r)^{[r]}} \le c(a,r)\,n^2\,|1+x|\,.$$

The theorem is proved.

Next we consider the case $-1+\frac{a}{n^2} \le x \le 1-\frac{a}{n^2}$, where $a > 0$. Here we consider only two particular values of r, namely: $r = 1$ and $r = 2$, which are of particular practical interest as was mentioned above.

First, from Corollary 1 and asymptotic formula (2.15) we have

$$U_{1,n}(x) = \frac{1}{n}\sqrt{\frac{2}{\pi}}\left[\frac{4^n (n!)^2}{(2n)!\sqrt{\pi n}}\left(\cos n\theta + \frac{v_n(\theta)}{n\sin\theta}\right) - (-1)^n\right] =$$
$$= \frac{4^n (n!)^2\sqrt{2}}{(2n)!\pi n\sqrt{n}}\left(\cos n\theta + \frac{v_n(\theta)}{n\sin\theta}\right) - \frac{(-1)^n\sqrt{2}}{n\sqrt{\pi}}.$$

Theorem 4 *For* $0 < \theta < \pi$ *the following asymptotic formula holds*

$$U_{1,n}(x) = \frac{4^n (n!)^2\sqrt{2}}{(2n)!\pi n\sqrt{n}}\left(\cos n\theta + \eta_n(\theta)\right) + \gamma_n,$$

where

$$\eta_n(\theta) = \frac{v_n(\theta)}{n\sin\theta}, \quad \gamma_n = (-1)^{n+1}\frac{\sqrt{2}}{n\sqrt{\pi}},$$

and for $v_n(\theta)$ *the estimate*

$$|v_n(\theta)| \le c(\delta)$$

takes place when $0 < \frac{\delta}{n} \le \theta \le \pi - \frac{\delta}{n}$.

Then, from (3.10) for $r = 2$ we have

$$U_{2,n+2}(x) = \frac{4^{n+1} n!(n-1)!}{(2n+1)!}\sqrt{\frac{2}{\pi}}\left[P_{n+2}^{-\frac{3}{2},-\frac{3}{2}}(x) - A_{0,n,2} - (1+x)A_{1,n,2}\right]. \tag{5.2}$$

Additionally, asymptotic formula (2.13) for $\alpha = \beta = -\frac{3}{2}$ gives us

$$P_n^{-\frac{3}{2},-\frac{3}{2}}(\cos\theta) = \frac{1}{2\sqrt{\pi n}}\left(\frac{\cos n\theta - \cos(n-2)\theta}{2} + \frac{v_n(\theta)}{n}\right), \tag{5.3}$$

where

$$|v_n(\theta)| \le c(\delta), \quad 0 < \frac{\delta}{n} \le \theta \le \pi - \frac{\delta}{n}.$$

Then using (5.2)–(5.3) we get

$$U_{2,n}(\cos\theta) = \frac{4^{n-1}}{\pi\sqrt{2n}}\frac{(n-2)!(n-3)!}{(2n-3)!}\left(\frac{\cos n\theta - \cos(n-2)\theta}{2} + \frac{\upsilon_n(\theta)}{n}\right) + \\ + \frac{(-1)^n}{n-1}\sqrt{\frac{2}{\pi}}\left(1 + \cos\theta + \frac{1}{n\,(n-2)}\right).$$

Theorem 5 *For* $0 < \theta < \pi$ *the following asymptotic formula holds*

$$U_{2,n}(\cos\theta) = \hat{\sigma}_n\left(\frac{\cos n\theta - \cos(n-2)\theta}{2} + \frac{\upsilon_n(\theta)}{n}\right) + \hat{\nu}_n(\theta),$$

where

$$\hat{\sigma}_n = \frac{4^{n-1}}{\pi\sqrt{2n}}\frac{(n-2)!(n-3)!}{(2n-3)!}, \quad \hat{\nu}_n(\theta) = \frac{(-1)^n}{n-1}\sqrt{\frac{2}{\pi}}\left(1 + \cos\theta + \frac{1}{n\,(n-2)}\right),$$

and for $\upsilon_n(\theta)$ *the estimate*

$$|\upsilon_n(\theta)| \le c(\delta)$$

takes place when $0 < \frac{\delta}{n} \le \theta \le \pi - \frac{\delta}{n}$.

Finally we consider the case $|1 - x| \le \frac{a}{n^2}$. We will need the following two lemmas.

Lemma 3 *For real* $a > 0$ *and integer* $r \ge 1$*, the following asymptotic formula holds*

$$P_{n+r}^{\frac{1}{2}-r,\frac{1}{2}-r}(x) = \binom{n+\frac{1}{2}}{n+r}\left(1 + q_{r,n}(x)\right) =$$

$$= \frac{\Gamma(n+\frac{3}{2})}{(n+r)!\sqrt{\pi}}\frac{(2r-3)!!}{(-2)^{r-1}}\left(1 + q_{r,n}(x)\right), \tag{5.4}$$

where for the remainder $q_{r,n}(x)$ *estimate*

$$|q_{r,n}(x)| \le c(a,r)n^2|1-x|, \tag{5.5}$$

takes place, when $|1 - x| \le \frac{a}{n^2}$.

Proof From (2.8) we get

$$P_{n+r}^{\frac{1}{2}-r,\frac{1}{2}-r}(x) = \binom{n+\frac{1}{2}}{n+r}\left[1 + \sum_{k=1}^{n+r}\frac{(-n-r)_k(n-r+2)_k}{k!(\frac{3}{2})_k}\left(\frac{1-x}{2}\right)^k\right].$$

For the remainder term in this equality we can use the similar reasoning to the one used in the proof of Theorem 5 to obtain the estimate (5.5).

Lemma 4 *For real $a > 0$ the next asymptotic formula holds*

$$\sum_{\nu=0}^{r-1} \frac{A_{\nu,n,r}}{\nu!}(1+x)^{\nu} = (-1)^{n+1} \frac{\Gamma\left(n+\frac{3}{2}\right)}{(n+1)!\sqrt{\pi}} \frac{(1+x)^{r-1}}{(r-1)!}(1+h_{r,n}(x)), \tag{5.6}$$

where for the remainder $h_{r,n}(x)$ estimate

$$|h_{r,n}(x)| \le c(a,r)n^{-2},$$

takes place, when $|1-x| \le \frac{a}{n^2}$.

Proof This statement directly follows from (3.9).

Finally, from (3.10) and Lemmas 3 and 4 we deduce the following result.

Theorem 6 *For $n \ge r-1$ the next asymptotic formula holds*

$$U_{r,r+n}(x) = \frac{2^r}{(n+1)^{[r+1]}} \frac{1}{\sqrt{2\pi}} \left[\frac{(2r-3)!!}{(-2)^{r-1}(n+2)_{r-1}}(1+q_{r,n}(x)) + \right.$$

$$\left. +(-1)^n \frac{(1+x)^{r-1}}{(r-1)!}(1+h_{r,n}(x)) \right],$$

where for the remainders $q_{r,n}(x)$ and $h_{r,n}(x)$ the estimates

$$|q_{r,n}(x)| \le c(a,r)n^2|1-x|, \quad |h_{r,n}(x)| \le c(a,r)n^{-2},$$

take place, when $|1-x| \le \frac{a}{n^2}$.

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Structure of Essential Spectrum and Discrete Spectrum of the Energy Operator of Five-Electron Systems in the Hubbard Model—Doublet States

S. M. Tashpulatov

Abstract We consider a five-electron system in the Hubbard model with a coupling between nearest-neighbors. The structure of essential spectrum and discrete spectrum of the systems in the first and second doublet states in a ν-dimensional lattice are investigated.

Keywords Essential spectrum · Discrete spectrum · Five-electron system · Bound state · Anti-bound state · Hubbard model · Doublet state · Sextet state · Quartet state

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1 Introduction

In the early 1970s, in these three papers [1–3], a simple metal model was proposed, having become a fundamental model in the theory of strongly correlated electron systems, that appeared almost simultaneously and independently. In that model, a single nondegenerate electron band with a local Coulomb interaction is considered. The model Hamiltonian contains only two parameters: the matrix element t of electron hopping from a lattice site to a neighboring site and the parameter U of the on-site Coulomb repulsion of two electrons. In the secondary quantization representation, the Hamiltonian can be written as

$$H = t \sum_{m,\gamma} a^{+}_{m,\gamma} a_{m,\gamma} + U \sum_{m} a^{+}_{m,\uparrow} a_{m,\uparrow} a^{+}_{m,\downarrow} a_{m,\downarrow}, \tag{1}$$

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where $a_{m,\gamma}^{+}$ and $a_{m,\gamma}$ denote Fermi operators of creation and annihilation of an electron with spin γ on a site m and the summation over τ means summation over the nearest neighbors on the lattice.

The model proposed in [1–3] was called the Hubbard model, after John Hubbard, who had made a fundamental contribution to the study of the statistical mechanics of that system, although the local form of Coulomb interaction had been first introduced for an impurity model in a metal by Anderson [4]. We also recall that the Hubbard model is a particular case of the Shubin-Wonsowsky polaron model [5], which had appeared 30 years before [1–3]. In the Shubin-Wonsowsky model, along with the on-site Coulomb interaction, the interaction of electrons on neighboring sites is also taken into account. The Hubbard model is an approximation used in solid state physics to describe the transition between conducting and insulating states. It is the simplest model describing particle interaction on a lattice. Its Hamiltonian contains only two terms: the kinetic term corresponding to the tunnelling (hopping) of particles between lattice sites and a term corresponding to the on-site interaction. Particles can be fermions, as in Hubbard's original work, and also bosons. The simplicity and sufficiency of Hamiltonian (1) have made the Hubbard model very popular and effective for describing strongly correlated electron systems.

The Hubbard model well describes the behaviour of particles in a periodic potential at sufficiently low temperatures, therefore, all particles that are in the lower Bloch band and involved in long-range interactions can be neglected. If the interaction between particles on different sites is taken into account, then the model is often called the extended Hubbard model. It was proposed to describe electrons in solids, that remains to be of particular interest for high-temperature superconductivity studies. Later, the extended Hubbard model also found applications in describing the behaviour of ultra-cold atoms in optical lattices. In considering electrons in solids, the Hubbard model can be regarded as a sophisticated version of the model of strongly bound electrons, involving only the electron hopping term in the Hamiltonian. In the case of strong interactions, these two models can give essentially different results. The Hubbard model predicts exactly the existence of so-called Mott insulators, where conductance is absent due to strong repulsion between particles. The Hubbard model is based on the approximation of strongly coupled electrons. In the strong coupling approximation, electrons initially occupy orbitals in atoms (lattice sites) and then hop over to other atoms, thus conducting the current. Mathematically, this is represented by the so-called hopping integral. This process can be considered as the physical phenomenon underlying the occurrence of electron bands in crystal materials. But the interaction between electrons is not considered in more general band theories. In addition to the hopping integral, which explains the conductance of the material, the Hubbard model contains the so-called on-site repulsion, corresponding to the Coulomb repulsion between electrons. This leads to a competition between the hopping integral, which depends on the mutual position of lattice sites, and the on-site repulsion, which is independent from the atom positions. As a result, the Hubbard model explains the metal-insulator transition in oxides of some transition metals. When such a material is heated, the distance between nearest-neighbour sites increases, the hopping integral decreases, and on-site repulsion becomes dominant.

The Hubbard model is currently one of the most extensively studied multielectron models of metals [6–10]. But little is known about exact results for the spectrum and wave functions of the crystal described by the Hubbard model, and obtaining the corresponding statements is therefore of great interest. The spectrum and wave functions of the system of two electrons in a crystal described by the Hubbard Hamiltonian were studied in [6]. It is known that two-electron systems can be in two states, in the triplet and singlet ones[6–10]. It was proved in [6] that the spectrum of the system Hamiltonian H^t in the triplet state is purely continuous and coincides with a segment $[m, M]$, and the operator H^s of the system in the singlet state, in addition to the continuous spectrum $[m, M]$, has a unique antibound state for some values of the quasimomentum. For the antibound state, correlated motion of the electrons is realized and a large contribution of binary states takes place. Due to the closeness of the system, the energy must remain constant and large. This prevents the electrons from being separated by long distances. Next, an essential point is that bound states (sometimes called scattering-type states) do not form below the continuous spectrum. This can be easily understood because the interaction is repulsive. We note that a converse situation is realized for $U < 0$: below the continuous spectrum, there is a bound state (antibound states are absent) because the electrons are then attracted to one another.

For the first band, the spectrum is independent of the parameter U of the two-electron on-site Coulomb interaction and corresponds to the energy of two noninteracting electrons, being exactly equal to the triplet band. The second band is determined by Coulomb interaction to a much greater degree: both the amplitudes and the energy of two electrons depend on U, and the band itself disappears as $U \to 0$ and increases without bound as $U \to \infty$. The second band largely corresponds to a one-particle state, namely, the motion of the doublet, i.e., two-electron bound states.

The spectrum and wave functions of the three-electron system in a crystal described by the Hubbard Hamiltonian were studied in [11].

The spectrum of the energy operator of the four-electron system in a crystal described by the Hubbard Hamiltonian in the triplet state were studied in [12]. In the four-electron systems are exists quintet state, and three type triplet states, and two type singlet states. The spectrum of the energy operator of four-electron systems in the Hubbard model in the quintet, and singlet states were studied in [13].

Here, we consider the energy operator of five-electron systems in the Hubbard model and describe the structure of the essential spectra and discrete spectrum of the system for first and second doublet states.

The Hamiltonian of the chosen model has the form

$$H = A \sum_{m,\gamma} a^{+}_{m,\gamma} a_{m,\gamma} + B \sum_{m,\tau,\gamma} a^{+}_{m+\tau,\gamma} a_{m,\gamma} + U \sum_{m} a^{+}_{m,\uparrow} a_{m,\uparrow} a^{+}_{m,\downarrow} a_{m,\downarrow}. \tag{2}$$

Here, A is the electron energy at a lattice site, B is the transfer integral between neighboring sites (we assume that $B > 0$ for convenience), $\tau = \pm e_j$, $j =$

$1, 2, \ldots, \nu$, where e_j are mutually orthogonal unit vectors, which means that the summation is taken over the nearest neighbors, U is the parameter of the on-site Coulomb interaction of two electrons, γ is the spin index, $\gamma = \uparrow$ or $\gamma = \downarrow$, $\uparrow$ and $\downarrow$ denote the spin values $\frac{1}{2}$ and $-\frac{1}{2}$, and $a^+_{m,\gamma}$ and $a_{m,\gamma}$ are the respective electron creation and annihilation operators at a site $m \in Z^\nu$.

There exist a sextet state, four type quartet states and five type doublet states in the five-electron systems.

The energy of the system depends on its total spin S. Along with the Hamiltonian, the N_e electron system is characterized by the total spin S, $S = S_{max}, S_{max} - 1, \ldots, S_{min}$, $S_{max} = \frac{N_e}{2}$, $S_{min} = 0, \frac{1}{2}$.

Hamiltonian (2) commutes with all components of the total spin operator $S = (S^+, S^-, S^z)$, and the structure of eigenfunctions and eigenvalues of the system therefore depends on S. The Hamiltonian H acts in the antisymmetric Fo'ck space $\mathcal{H}_{as}$.

Let $\mathcal{H}$ be a Hilbert space and denote by $\mathcal{H}^n$ the n-fold tensor product $\mathcal{H}^n = \mathcal{H} \bigotimes \mathcal{H} \bigotimes \ldots \bigotimes \mathcal{H}$. We set $\mathcal{H}^0 = C$ and define $\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^n$. The $\mathcal{F}(\mathcal{H})$ is called the Fock space over $\mathcal{H}$; it will be separable, if $\mathcal{H}$ is. For example, if $\mathcal{H} = L_2(R)$, then an element $\psi \in \mathcal{F}(\mathcal{H})$ is a sequence of functions $\psi = \{\psi_0, \psi_1(x_1), \psi_2(x_1, x_2), \psi_3(x_1, x_2, x_3), \ldots\}$, so that

$$|\psi_0|^2 + \sum_{n=1}^{\infty} \int_{R^n} |\psi_n(x_1, x_2, \ldots, x_n)|^2 dx_1 dx_2 \ldots dx_n < \infty.$$

Actually, it is not $\mathcal{F}(\mathcal{H})$ itself, but two of its subspaces which are used most frequently in quantum field theory. These two subspaces are constructed as follows: Let $\mathcal{P}_n$ be the permutation group on n elements, and let $\{\psi_n\}$ be a basis for space $\mathcal{H}$. For each $\sigma \in \mathcal{P}_n$, we define an operator (which we also denote by σ) on basis elements $\mathcal{H}^n$ by $\sigma(\varphi_{k_1} \bigotimes \varphi_{k_2} \bigotimes \ldots \bigotimes \varphi_{k_n}) = \varphi_{k_{\sigma(1)}} \bigotimes \varphi_{k_{\sigma(2)}} \bigotimes \ldots \bigotimes \varphi_{k_{\sigma(n)}}$. The operator σ extends by linearity to a bounded operator (of norm one) on space $\mathcal{H}^n$, so we can define $S_n = \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} \sigma$. It is an easy exercise to show that, the operator S_n is the operator of orthogonal projection: $S_n^2 = S_n$, and $S_n^* = S_n$. The range of S_n is called n-fold symmetric tensor product of $\mathcal{H}$. In the case, where $\mathcal{H} = L_2(R)$ and $\mathcal{H}^n = L_2(R) \bigotimes L_2(R) \bigotimes \ldots \bigotimes L_2(R) = L_2(R^n)$, $S_n \mathcal{H}^n$ is just the subspace of $L_2(R^n)$, of all functions, left invariant under any permutation of the variables. We now define $\mathcal{F}_s(\mathcal{H}) = \bigoplus_{n=0}^{\infty} S_n \mathcal{H}^n$. The space $\mathcal{F}_s(\mathcal{H})$ is called the symmetric Fock space over $\mathcal{H}$, or Boson Fock space over $\mathcal{H}$.

Let $\varepsilon(.)$ be function from $\mathcal{P}_n$ to $\{1, -1\}$, which is one on even permutations and minus one on odd permutations. Define $A_n = \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} \varepsilon(\sigma)\sigma$; then A_n is an orthogonal projector on $\mathcal{H}^n$. $A_n \mathcal{H}^n$ is called the n-fold antisymmetric tensor product of $\mathcal{H}$. In the case where $\mathcal{H} = L_2(R)$, $A_n \mathcal{H}^n$ is just the subspace of $L^2(R^n)$ consisting of those functions odd under interchange of two coordinates. The subspace $\mathcal{F}_a(\mathcal{H}) = \bigoplus_{n=0}^{\infty} A_n \mathcal{H}^n$ is called the antisymmetric Fock space over $\mathcal{H}$, or the Fermion Fock space over $\mathcal{H}$.

2 First Doublet State

There exist five type doublet states in the system. The doublet state corresponds to the basis functions ${}^1d^{1/2}_{m,n,r,t,l\in Z^\nu} = a^+_{m,\downarrow}a^+_{n,\downarrow}a^+_{r,\uparrow}a^+_{t,\uparrow}a^+_{l,\uparrow}\varphi_0$ and ${}^2d^{1/2}_{m,n,r,t,l\in Z^\nu} = a^+_{m,\downarrow}a^+_{n,\uparrow}a^+_{r,\downarrow}a^+_{t,\uparrow}a^+_{l,\uparrow}\varphi_0$ and ${}^3d^{1/2}_{m,n,r,t,l\in Z^\nu} = a^+_{m,\downarrow}a^+_{n,\uparrow}a^+_{r,\uparrow}a^+_{t,\downarrow}a^+_{l,\uparrow}\varphi_0$ and ${}^4d^{1/2}_{m,n,r,t,l\in Z^\nu} = a^+_{m,\downarrow}a^+_{n,\uparrow}a^+_{r,\uparrow}a^+_{t,\uparrow}a^+_{l,\downarrow}\varphi_0$ and ${}^5d^{1/2}_{m,n,r,t,l\in Z^\nu} = a^+_{m,\uparrow}a^+_{n,\downarrow}a^+_{r,\downarrow}a^+_{t,\uparrow}a^+_{l,\uparrow}\varphi_0$.

The subspace ${}^1\mathcal{H}^d_{1/2}$, corresponding to the first five-electron doublet state is the set of all vectors of the forms ${}^1\psi^d_{1/2} = \sum_{m,n,r,t,l\in Z^\nu} f(m,n,r,t,l){}^1d^{1/2}_{m,n,r,t,l\in Z^\nu}$, $f \in l^{as}_2$, where l^{as}_2 is the subspace of antisymmetric functions in the space $l_2((Z^\nu)^5)$.

The restriction ${}^1H^d_{1/2}$ of H to the subspace ${}^1\mathcal{H}^d_{1/2}$, is called the five-electron first doublet state operator.

Theorem 1 *The subspace* ${}^1\mathcal{H}^d_{1/2}$ *is invariant under the operator* H, *and the operator* ${}^1H^d_{1/2}$ *is a bounded self-adjoint operator. It generates a bounded self-adjoint operator* ${}^1\overline{H}^d_{1/2}$, *acting in the space* l^{as}_2 *as*

$$({}^1\overline{H}^d_{1/2}f)(m,n,r,t,l) = 5Af(m,n,r,t,l) + B\sum_\tau[f(m+\tau,n,r,t,l)+$$
$$+f(m,n+\tau,r,t,l) + f(m,n,r+\tau,t,l) + f(m,n,r,t+\tau,l)+$$
$$+f(m,n,r,t,l+\tau)] + U[\delta_{m,r}+\delta_{n,r}+\delta_{m,t}+\delta_{n,t}+\delta_{m,l}+\delta_{n,l}]f(m,n,r,t,l), \quad (3)$$

where $\delta_{k,j}$ *is the Kronecker symbol. The operator* ${}^1H^d_{1/2}$ *acts on a vector* ${}^1\psi^d_{1/2} \in {}^1\mathcal{H}^d_{1/2}$ *as*

$${}^1H^d_{1/2}\,{}^1\psi^d_{1/2} = \sum_{m,n,r,t,l\in Z^\nu} ({}^1\overline{H}^d_{1/2}f)(m,n,r,t,l){}^1d^{1/2}_{m,n,r,t,l\in Z^\nu}. \quad (4)$$

Proof We act with the Hamiltonian H on vectors $\psi \in {}^1\mathcal{H}^d_{1/2}$ using the standard anticommutation relations between electron creation and annihilation operators at lattice sites, $\{a_{m,\gamma}, a^+_{n,\beta}\} = \delta_{m,n}\delta_{\gamma,\beta}$, $\{a_{m,\gamma}, a_{n,\beta}\} = \{a^+_{m,\gamma}, a^+_{n,\beta}\} = \theta$, and also take into account that $a_{m,\gamma}\varphi_0 = \theta$, where θ is the zero element of ${}^1\mathcal{H}^d_{1/2}$. This yields the statement of the theorem.

Lemma 1 *The spectrum of the operators* ${}^1H^d_{1/2}$ *and* ${}^1\overline{H}^d_{1/2}$ *coincide.*

Proof Because ${}^1H^d_{1/2}$ and ${}^1\overline{H}^d_{1/2}$ are bounded self-adjoint operators, it follows from the Weyl criterion (see, for example, [14, Ch. VII, §14]) that there exists a sequence of vectors ψ_n such that

$\psi_n = \sum_{p,q,r,t,s} f_n(p,q,r,t,s)a^+_{p,\downarrow}a^+_{q,\downarrow}a^+_{r,\uparrow}a^+_{t,\uparrow}a^+_{s,\uparrow}\varphi_0$, $||\psi_n|| = 1$, and

$$\lim_{n\to\infty} ||{}^1H^d_{1/2}\psi_n - \lambda\psi_n|| = 0, \tag{5}$$

where $\lambda \in \sigma({}^1H^d_{1/2})$. On the other hand,

$$||{}^1H^d_{1/2}\psi_n - \lambda\psi_n||^2 = ({}^1H^d_{1/2}\psi_n - \lambda\psi_n, {}^1H^d_{1/2}\psi_n - \lambda\psi_n) = \sum_{p,q,r,t,s} ||({}^1\overline{H}^d_{1/2}f_n(p,q,r,t,s)-$$

$$-\lambda f_n(p,q,r,t,s)||^2(a^+_{p,\downarrow}a^+_{q,\downarrow}a^+_{r,\uparrow}a^+_{t,\uparrow}a^+_{s,\uparrow}\varphi_0, a^+_{p,\downarrow}a^+_{q,\downarrow}a^+_{r,\uparrow}a^+_{t,\uparrow}a^+_{s,\uparrow}\varphi_0) = ||{}^1\overline{H}^d_{1/2}F_n - \lambda F_n||^2\times$$

$$\times(a_{p,\downarrow}a_{q,\downarrow}a_{r,\uparrow}a_{t,\uparrow}a_{s,\uparrow}a^+_{p,\downarrow}a^+_{q,\downarrow}a^+_{r,\uparrow}a^+_{t,\uparrow}a^+_{s,\uparrow}\varphi_0,\varphi_0) = ||({}^1\overline{H}^d_{1/2} - \lambda)F_n||^2(\varphi_0,\varphi_0) =$$

$$= ||({}^1\overline{H}^d_{1/2} - \lambda)F_n||^2 \to 0,$$

$n \to \infty$. Here $F_n = (f_n(p,q,r,t,s))_{p,q,r,t,s\in Z^\nu}$ and $||F_n||^2 = \sum_{p,q,r,t,s} |f_n(p,q,r,t,s)|^2 = ||\psi_n||^2 = 1$. It follows that $\lambda \in \sigma({}^1\overline{H}^d_{1/2})$. Consequently, $\sigma({}^1H^d_{1/2}) \subset \sigma({}^1\overline{H}^d_{1/2})$. Conversely, let $\overline{\lambda} \in \sigma({}^1\overline{H}^d_{1/2})$. Again by the Weyl criterion, there then exists a sequence F_n such that $||F_n|| = \sqrt{\sum_{p,q,r,t,s}|f_n(p,q,r,t,s)|^2} = 1$ and

$$||({}^1\overline{H}^d_{1/2}F_n - \overline{\lambda}F_n|| \to 0, \tag{6}$$

as $n \to \infty$.

Setting $\psi_n = \sum_{p,q,r,t,s} f_n(p,q,r,t,s)a^+_{p,\downarrow}a^+_{q,\downarrow}a^+_{r,\uparrow}a^+_{t,\uparrow}a^+_{s,\uparrow}\varphi_0$, we have $||\psi_n|| = ||F_n|| = 1$ and $||{}^1\overline{H}^d_{1/2}F_n - \overline{\lambda}F_n|| = ||{}^1H^d_{1/2}\psi_n - \overline{\lambda}\psi_n||$. This, together with formula (6) and the Weyl criterion, implies that $\overline{\lambda} \in \sigma({}^1H^d_{1/2})$, and hence $\sigma({}^1\overline{H}^d_{1/2}) \subset \sigma({}^1H^d_{1/2})$. These two relations imply that $\sigma({}^1\overline{H}^d_{1/2}) = \sigma({}^1H^d_{1/2})$

We let $\mathcal{F}$ denote the Fourier transform: $\mathcal{F} : l_2((Z^\nu)^5) \to L_2((T^\nu)^5) \equiv {}^1\widetilde{\mathcal{H}}^d_{1/2}$, where T^ν is a ν-dimensional torus with the normalized Lebesgue measure $d\lambda$: $\lambda(T^\nu) = 1$.

We set ${}^1\widetilde{H}^d_{1/2} = \mathcal{F}\,{}^1\overline{H}^d_{1/2}\mathcal{F}^{-1}$. In the quasimomentum representation, the operator ${}^1\widetilde{H}^d_{1/2}$ acts in the Hilbert space $L^{as}_2((T^\nu)^5)$ as

$${}^1\widetilde{H}^d_{1/2}\ {}^1\psi^d_{1/2} = \{5A + 2B\sum_{i=1}^{\nu}[cos\lambda_i + cos\mu_i + cos\gamma_i + cos\theta_i + cos\eta_i]\}f(\lambda,\mu,\gamma,\theta,\eta)+ \tag{7}$$

$$+U\int_{T^\nu}[f(s,\mu,\lambda+\gamma-s,\theta,\eta)+f(s,\mu,\gamma,\lambda+\theta-s,\eta)+f(s,\mu,\gamma,\theta,\lambda+\eta-s)+$$
$$+f(\lambda,s,\mu+\gamma-s,\theta,\eta)+f(\lambda,s,\gamma,\mu+\theta-s,\eta)+f(\lambda,s,\gamma,\theta,\mu+\eta-s)]ds,$$

where $L_2^{as}((T^\nu)^5)$ is the subspace of antisymmetric functions in $L_2((T^\nu)^5)$.

Using tensor products of Hilbert spaces and tensor products of operators in Hilbert spaces [15], we can verify that the operator ${}^1\widetilde{H}_{1/2}^d$ can be represented in the form

$$ {}^1\widetilde{H}_{1/2}^d\,{}^1\psi_{1/2}^d=\widetilde{H}_2^1(\lambda,\gamma)\bigotimes I\bigotimes I+I\bigotimes\widetilde{H}_2^2(\mu,\theta)\bigotimes I+I\bigotimes I\bigotimes\widetilde{H}_2^3(\lambda,\eta) \tag{8}$$

where

$$(\widetilde{H}_2^1f)(\lambda,\gamma)=-\{2A+2B\sum_{i=1}^{\nu}(\cos\lambda_i+\cos\gamma_i)\}f(\lambda,\gamma)-2U\int_{T^\nu}f(s,\lambda+\gamma-s)ds,$$

$$(\widetilde{H}_2^2f)(\mu,\theta)=-\{2A+2B\sum_{i=1}^{\nu}(\cos\mu_i+\cos\theta_i)\}f(\mu,\theta)+2U\int_{T^\nu}f(s,\mu+\theta-s)ds,$$

$$(\widetilde{H}_2^3f)(\lambda,\eta)=\{A+2B\sum_{i=1}^{\nu}\cos\eta_i)\}f(\lambda,\eta),$$

and I is the unit operator.

The spectrum of the operator $A\bigotimes I+I\bigotimes B$, where A and B are densely defined bounded linear operators, was studied in [16–18]. Explicit formulas were given there that express the essential spectrum $\sigma_{ess}(A\bigotimes I+I\bigotimes B)$ and discrete spectrum $\sigma_{disc}(A\bigotimes I+I\bigotimes B)$ of operator $A\bigotimes I+I\bigotimes B$ in terms of the spectrum $\sigma(A)$ and the discrete spectrum $\sigma_{disc}(A)$ of A and in terms of the spectrum $\sigma(B)$ and the discrete spectrum $\sigma_{disc}(B)$ of B :

$$\sigma_{disc}(A\bigotimes I+I\bigotimes B)=\{\sigma(A)\backslash\sigma_{ess}(A)+\sigma(B)\backslash\sigma_{ess}(B)\}\backslash\{(\sigma_{ess}(A)+$$
$$+\sigma(B))\cup(\sigma(A)+\sigma_{ess}(B))\}, \tag{9}$$

$$\sigma_{ess}(A\bigotimes I+I\bigotimes B)=(\sigma_{ess}(A)+\sigma(B))\cup(\sigma(A)+\sigma_{ess}(B)). \tag{10}$$

It is clear that $\sigma(A\bigotimes I+I\bigotimes B)=\{\lambda+\mu:\lambda\in\sigma(A),\mu\in\sigma(B)\}$.

Therefore, we must investigate the spectrum of the operators $\widetilde{H}_2^1$, $\widetilde{H}_2^2$, and $\widetilde{H}_2^3$.

Let $\Lambda_1=\lambda+\gamma$, $\Lambda_2=\lambda+\theta$, $\Lambda_3=\lambda+\eta$, $\Lambda_4=\mu+\gamma$, $\Lambda_5=\mu+\theta$, and $\Lambda_6=\mu+\eta$.

Let the total quasimomentum of the two-electron system $\lambda+\gamma=\Lambda_1$ be fixed. We let $L_2(\Gamma_{\Lambda_1})$ denote the space of functions that are square integrable on the manifold $\Gamma_{\Lambda_1}=\{(\lambda,\gamma):\lambda+\gamma=\Lambda_1\}$. It is known ([19], chapter II, pp. 63–84, and [15], chapter XIII, paragraph 16, pp. 303–341) that the operator $\widetilde{H}_2^1$ and the space

$\widetilde{\mathcal{H}}_2^1 \equiv L_2((T^\nu)^2)$ can be decomposed into a direct integral $\widetilde{H}_2^1 = \bigoplus\int_{T^\nu} \widetilde{H}_{2\Lambda_1}^1 d\Lambda_1$, $\widetilde{\mathcal{H}}_2^1 = \bigoplus\int_{T^\nu} \widetilde{\mathcal{H}}_{2\Lambda_1}^1 d\Lambda_1$ of operators $\widetilde{H}_{2\Lambda_1}^1$ and spaces $\widetilde{\mathcal{H}}_{2\Lambda_1}^1 = L_2(\Gamma_{\Lambda_1})$ such that $\widetilde{\mathcal{H}}_{2\Lambda_1}^1$ are invariant under $\widetilde{H}_{2\Lambda_1}^1$ and $\widetilde{H}_{2\Lambda_1}^1$ act in $\widetilde{\mathcal{H}}_{2\Lambda_1}^1$ according to the formula $(\widetilde{H}_{2\Lambda_1}^1 f_{\Lambda_1})(\lambda) = -\{2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos(\frac{\Lambda_1^i}{2}-\lambda_i)\}f_{\Lambda_1}(\lambda)-2U\int_{T^\nu} f_{\Lambda_1}(s)ds$, where $f_{\Lambda_1}(x) = f(x, \Lambda_1 - x)$.

It is known that the continuous spectrum of $\widetilde{H}_{2\Lambda_1}^1$ is independent of the parameter U and consists of the intervals $\sigma_{cont}(\widetilde{H}_{2\Lambda_1}^1) = G_{\Lambda_1}^\nu = [-2A - 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}, -2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}]$.

Definition 1 The eigenfunction $\varphi_{\Lambda_1} \in L_2(T^\nu \times T^\nu)$ of the operator $\widetilde{H}_{2\Lambda_1}^1$ corresponding to an eigenvalue $z_{\Lambda_1} \notin G_{\Lambda_1}^\nu$ is called a bound state (BS)(antibound state (ABS)) of $\widetilde{H}_2^1$ with the quasi momentum Λ_1, and the quantity z_{Λ_1} is called the energy of this state.

We consider the operator $K_{\Lambda_1}(z)$ acting in the space $\widetilde{\mathcal{H}}_{2\Lambda_1}^1$ according to the formula

$$(K_{\Lambda_1}(z)f_{\Lambda_1})(x) = \int_{T^\nu} \frac{2U}{-\{2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos(\frac{\Lambda_1^i}{2}-t_i)\}-z} f_{\Lambda_1}(t)dt.$$

It is a completely continuous operator in $\widetilde{\mathcal{H}}_{2\Lambda_1}^1$ for $z \notin G_{\Lambda_1}^\nu$.

We set $D_{\Lambda_1}^\nu(z) = 1 - 2U\int_{T^\nu}\frac{ds_1ds_2...ds_\nu}{-2A-4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos(\frac{\Lambda_1^i}{2}-s_i)-z}$.

Lemma 2 *A number $z = z_0 \notin G_{\Lambda_1}^\nu$ is an eigenvalue of the operator $\widetilde{H}_{2\Lambda_1}^1$ if and only if it is a zero of the function $D_{\Lambda_1}^\nu(z)$, i.e., $D_{\Lambda_1}^\nu(z_0) = 0$.*

Proof Let the number $z = z_0 \notin G_{\Lambda_1}^\nu$ be an eigenvalue of the operator $\widetilde{H}_{2\Lambda_1}^1$, and $\varphi_{\Lambda_1}(x)$ be the corresponding eigenfunction, i.e.,

$$-\{2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos(\frac{\Lambda_1^i}{2}-x_i)\}\varphi_{\Lambda_1}(x) - 2U\int_{T^\nu}\varphi_{\Lambda_1}(s)ds = z_0\varphi_{\Lambda_1}(x).$$

Let $\psi_{\Lambda_1}(x) = [-\{2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos(\frac{\Lambda_1^i}{2}-x_i)\}-z]\varphi_{\Lambda_1}(x)$. Then

$$\psi_{\Lambda_1}(x) - \int_{T^\nu}\frac{2U}{-\{2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos(\frac{\Lambda_1^i}{2}-x_i)\}-z}\psi_{\Lambda_1}(t)dt = 0,$$

i.e., the number $\mu = 1$ is a eigenvalue of the operator $K_{\Lambda_1}(z)$. It then follows that $D_{\Lambda_1}^\nu(z_0) = 0$.

Now let $z = z_0$ be a zero of the function $D^{\nu}_{\Lambda_1}(z)$, i.e., $D^{\nu}_{\Lambda_1}(z_0) = 0$. It follows from the Fredholm theorem that the homogeneous equation

$$\psi_{\Lambda_1}(x) - 2U \int_{T^\nu} \frac{\psi_{\Lambda_1}(s)ds_1 \dots ds_\nu}{-2A - 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_1^i}{2} \cos(\frac{\Lambda_1^i}{2} - s_i) - z} ds = 0$$

has a nontrivial solution. This means that the number $z = z_0$ is an eigenvalue of the operator $\widetilde{H}^1_{2\Lambda_1}$.

We consider the one-dimensional case.

Theorem 2

a) At $\nu = 1$ and $U < 0$, and all values of parameters of the Hamiltonian, the operator $\widetilde{H}^1_{2\Lambda_1}$ has a unique eigenvalue $z_1 = -2A + 2\sqrt{U^2 + 4B^2 \cos^2 \frac{\Lambda_1}{2}}$, that is above the continuous spectrum of $\widetilde{H}^1_{2\Lambda_1}$, i.e., $z_1 > M^1_{\Lambda_1}$.

b) At $\nu = 1$ and $U > 0$, and all values of parameters of the Hamiltonian, the operator $\widetilde{H}^1_{2\Lambda_1}$ has a unique eigenvalue $\widetilde{z}_1 = -2A - 2\sqrt{U^2 + 4B^2 \cos^2 \frac{\Lambda_1}{2}}$, that is below the continuous spectrum of $\widetilde{H}^1_{2\Lambda_1}$, i.e., $\widetilde{z}_1 < m^1_{\Lambda_1}$.

Proof If $U < 0$, then in the one-dimensional case, the function $D^{\nu}_{\Lambda_1}(z)$ increases monotonically outside the continuous spectrum domain of the operator $\widetilde{H}^1_{2\Lambda_1}$. For $z < m^1_{\Lambda_1}$ the function $D^{\nu}_{\Lambda_1}(z)$ increases from 1 to $+\infty$, $D^{\nu}_{\Lambda_1}(z) \to 1$ as $z \to -\infty$, $D^{\nu}_{\Lambda_1}(z) \to +\infty$ as $z \to m^1_{\Lambda_1} - 0$. Therefore, below the value $m^1_{\Lambda_1}$, the function $D^{\nu}_{\Lambda_1}(z)$ cannot vanish. For $z > M^1_{\Lambda_1}$, and $U < 0$, the function $D^{\nu}_{\Lambda_1}(z)$ increases from $-\infty$ to 1, $D^{\nu}_{\Lambda_1}(z) \to -\infty$ as $z \to M^1_{\Lambda_1} + 0$, $D^{\nu}_{\Lambda_1}(z) \to 1$ as $z \to +\infty$. Therefore, above the value $M^1_{\Lambda_1}$, the function $D^{\nu}_{\Lambda_1}(z)$ has a single zero at the point $z = z_1 = -2A + 2\sqrt{U^2 + 4B^2 \cos^2 \frac{\Lambda_1}{2}} > M^1_{\Lambda_1}$. If $U > 0$, then the function $D^{\nu}_{\Lambda_1}(z)$ decreases monotonically outside the continuous spectrum domain of the operator $\widetilde{H}^1_{2\Lambda_1}$. For $z < m^1_{\Lambda_1}$ the function $D^{\nu}_{\Lambda_1}(z)$ decreases from 1 to $-\infty$, $D^{\nu}_{\Lambda_1}(z) \to 1$ as $z \to -\infty$, $D^{\nu}_{\Lambda_1}(z) \to -\infty$ as $z \to m^1_{\Lambda_1} - 0$. Therefore, below the value $m^1_{\Lambda_1}$, the function $D^{\nu}_{\Lambda_1}(z)$ has a single zero at the point $\widetilde{z}_1 = -2A - 2\sqrt{U^2 + 4B^2 \cos^2 \frac{\Lambda_1}{2}} < m^1_{\Lambda_1}$.

For $z > M^1_{\Lambda_1}$, and $U > 0$, the function $D^{\nu}_{\Lambda_1}(z)$ decreases from $+\infty$ to 1, $D^{\nu}_{\Lambda_1}(z) \to +\infty$ as $z \to M^1_{\Lambda_1} + 0$, $D^{\nu}_{\Lambda_1}(z) \to 1$ as $z \to +\infty$. Therefore, above the value $M^1_{\Lambda_1}$, the function $D^{\nu}_{\Lambda_1}(z)$ cannot vanish.

In the two-dimensional case, we have similar results.

If $U > 0$, then the function $D^2_{\Lambda_1}(z)$ decreases monotonically outside the continuous spectrum domain of the operator $\widetilde{H}^1_{2\Lambda_1}$. For $z < m^2_{\Lambda_1}$ the function $D^{\nu}_{\Lambda_1}(z)$ decreases from 1 to $-\infty$, $D^{\nu}_{\Lambda_1}(z) \to 1$ as $z \to -\infty$, $D^{\nu}_{\Lambda_1}(z) \to -\infty$

as $z \to m^2_{\Lambda_1} - 0$. Therefore, below the value $m^2_{\Lambda_1}$, the function $D^{\nu}_{\Lambda_1}(z)$ has a single zero at the point $z_1 < m^2_{\Lambda_1}$.

If $U < 0$, for $z > M^2_{\Lambda_1}$, then the function $D^2_{\Lambda_1}(z)$ increases from $-\infty$ to 1, $D^{\nu}_{\Lambda_1}(z) \to 1$ as $z \to +\infty$, $D^{\nu}_{\Lambda_1}(z) \to -\infty$ as $z \to M^2_{\Lambda_1} + 0$. Therefore, above the value $M^2_{\Lambda_1}$, the function $D^{\nu}_{\Lambda_1}(z)$ has a single zero at the point $\widetilde{z}_1 > M^2_{\Lambda_1}$.

We consider three-dimensional case. Denote $m = \int_{T^3} \frac{ds_1 ds_2 ds_3}{\sum_{i=1}^{3} \cos\frac{\Lambda_1^i}{2}(1+\cos(\frac{\Lambda_1^i}{2}-s_i))}$.

For $U < 0$, and $U < -\frac{2B}{m}$, above the continuous spectrum of the operator $\widetilde{H}^1_{2\Lambda_1}$, the function $D^3_{\Lambda_1}(z)$ has a single zero at the point $z = z_1 > M^3_{\Lambda_1}$. For $U < 0$, and $-\frac{2B}{m} \leq U < 0$, above of the continuous spectrum of the operator $\widetilde{H}^1_{2\Lambda_1}$, the function $D^3_{\Lambda_1}(z)$ cannot vanish.

Denote $M = \int_{T^3} \frac{ds_1 ds_2 ds_3}{\sum_{i=1}^{3} \cos\frac{\Lambda_1^i}{2}(1-\cos(\frac{\Lambda_1^i}{2}-s_i))}$.

For $U > 0$, and $U > \frac{2B}{M}$, below the continuous spectrum of the operator $\widetilde{H}^1_{2\Lambda_1}$, the function $D^3_{\Lambda_1}(z)$ has a single zero at the point $\widetilde{z}_1 > M^3_{\Lambda_1}$. For $0 < U \leq \frac{2B}{M}$, above of the continuous spectrum of the operator $\widetilde{H}^1_{2\Lambda_1}$, the function $D^3_{\Lambda_1}(z)$ cannot vanish.

We have now studied the spectrum of the operator $\widetilde{H}^2_{2\Lambda_2}$, i.e., the operator

$$(\widetilde{H}^2_{2\Lambda_2} f_{\Lambda_2})(\lambda) = -\{2A + 4B\sum_{i=1}^{\nu} \cos\frac{\Lambda_2^i}{2}\cos(\frac{\Lambda_2^i}{2} - \lambda_i)\} f_{\Lambda_2}(\lambda) + 2U\int_{T^{\nu}} f_{\Lambda_2}(s) ds.$$

It is known that the continuous spectrum of the operator $\widetilde{H}^2_{2\Lambda_2}$ is independent of U and coincides with the segment $\sigma_{cont}(\widetilde{H}^2_{2\Lambda_2}) = [-2A - 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_2^i}{2}, -2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_2^i}{2}] = G^{\nu}_{\Lambda_2}$.

We let $L_2(\Gamma_{\Lambda_2})$ denote the space of functions that are square integrable on the manifold $\Gamma_{\Lambda_2} = \{(\lambda, \theta) : \lambda + \theta = \Lambda_2\}$. That the operator $\widetilde{H}^2_2$ and the space $\widetilde{\mathcal{H}}^2_2 \equiv L_2((T^{\nu})^2)$ can be decomposed into a direct integral $\widetilde{H}^2_2 = \bigoplus\int_{T^{\nu}} \widetilde{H}^2_{2\Lambda_2} d\Lambda_2$, $\widetilde{\mathcal{H}}^2_2 = \bigoplus\int_{T^{\nu}} \widetilde{\mathcal{H}}^2_{2\Lambda_2} d\Lambda_2$. Each operator $\widetilde{H}^2_{2\Lambda_2}$ acts in $\widetilde{\mathcal{H}}^2_{2\Lambda_2} = L_2(\Gamma_{\Lambda_2})$ as $(\widetilde{H}^2_{2\Lambda_2} f_{\Lambda_2})(\lambda) = -\{2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_2^i}{2}\cos(\frac{\Lambda_1^i}{2} - \lambda_i)\} f_{\Lambda_2}(\lambda) + 2U\int_{T^{\nu}} f_{\Lambda_2}(s) ds$, where $f_{\Lambda_2}(x) = f(x, \Lambda_2 - x)$.

We set $D^{\nu}_{\Lambda_2}(z) = 1 + 2U\int_{T^{\nu}} \frac{ds_1 ds_2 \ldots ds_{\nu}}{-2A - 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_2^i}{2}\cos(\frac{\Lambda_2^i}{2} - s_i) - z}$.

The analogue of the Lemma 2 holds for the in this case. We consider the one-dimensional case.

Theorem 3

a) At $\nu = 1$ *and* $U < 0$ *and for all values of the parameter of the Hamiltonian, the operator* $\widetilde{H}^2_{2\Lambda_2}$ *has a unique eigenvalue* $z = z_2 = -2A - 2\sqrt{U^2 + 4B^2\cos^2\frac{\Lambda_2}{2}}$*, that is below the continuous spectrum of operator* $\widetilde{H}^2_{2\Lambda_2}$*, i.e.,* $z_2 < m^1_{\Lambda_2}$.

b) At $\nu = 1$ *and* $U > 0$ *and for all values of the parameter of the Hamiltonian, the operator* $\widetilde{H}^2_{2\Lambda_2}$ *has a unique eigenvalue* $z = \widetilde{z}_2 = -2A + 2\sqrt{U^2 + 4B^2\cos^2\frac{\Lambda_2}{2}}$*, that is above the continuous spectrum of operator* $\widetilde{H}^2_{2\Lambda_2}$*, i.e.,* $\widetilde{z}_2 > M^1_{\Lambda_2}$.

Theorem 3 is proved totally similarly to Theorem 2.

In the two-dimensional case, we have analogously results.

If $U < 0$, then the equation $D^2_{\Lambda_2}(z) = 0$ has a unique solution $z_2 < m^2_{\Lambda_2}$, below the continuous spectrum of the operator $\widetilde{H}^2_{2\Lambda_2}$. If $U > 0$, then the equation $D^2_{\Lambda_2}(z) = 0$ has a unique solution $\widetilde{z}_2 > M^2_{\Lambda_2}$, above the continuous spectrum of the operator $\widetilde{H}^2_{2\Lambda_2}$.

We consider three-dimensional case. Denote $M = \int_{T^3} \frac{ds_1 ds_2 ds_3}{\sum_{i=1}^3 \cos\frac{\Lambda_2^i}{2}(1-\cos(\frac{\Lambda_2^i}{2}-s_i))}$.

For $U < 0$, and $U < -\frac{2B}{M}$, below of the continuous spectrum of the operator $\widetilde{H}^2_{2\Lambda_2}$, the function $D^3_{\Lambda_2}(z)$ has a single zero at the point $z = z_2 < m^3_{\Lambda_2}$. For $U < 0$, and $-\frac{2B}{M} \le U < 0$, below of the continuous spectrum of the operator $\widetilde{H}^2_{2\Lambda_2}$, the function $D^3_{\Lambda_2}(z)$ cannot vanish. Denote $m = \int_{T^3} \frac{ds_1 ds_2 ds_3}{\sum_{i=1}^3 \cos\frac{\Lambda_2^i}{2}(1+\cos(\frac{\Lambda_2^i}{2}-s_i))}$.

For $U > 0$, and $U > \frac{2B}{m}$, above of the continuous spectrum of the operator $\widetilde{H}^2_{2\Lambda_2}$, the function $D^3_{\Lambda_2}(z)$ has a single zero at the point $z = \widetilde{z}_2 > M^3_{\Lambda_2}$. For $U > 0$, and $0 < U \le \frac{2B}{m}$, above of the continuous spectrum of the operator $\widetilde{H}^2_{2\Lambda_2}$, the function $D^3_{\Lambda_2}(z)$ cannot vanish.

Let $\Lambda_3 = \lambda + \eta$. Now we investigated the spectrum of the operator $\widetilde{H}^3_{\Lambda_3}$.

$$(\widetilde{H}^3_{\Lambda_3} f_{\Lambda_3})(\lambda) = \{A + 2B\sum_{i=1}^{\nu}\cos(\Lambda_3 - \lambda_i)\} f_{\Lambda_3}(\lambda).$$

It is obvious that the spectrum of operator $\widetilde{H}^3_{\Lambda_3}$ is purely continuous and coincides with the value set of the function $h_{\Lambda_3}(\lambda) = A + 2B\sum_{i=1}^{\nu}\cos(\Lambda_3 - \lambda_i)$, i.e., $\sigma(\widetilde{H}^3_{\Lambda_3}) = \sigma_{cont}(\widetilde{H}^3_{\Lambda_3}) = [A - 2B\nu, A + 2B\nu]$.

Now, using the obtained results and representation (8), we describe the structure of essential spectrum and the discrete spectrum of the operator ${}^1\widetilde{H}^d_{1/2}$ of first five-electron doublet state:

Theorem 4 *At* $\nu = 1$ *and* $U < 0$ *the essential spectrum of the system of first five-electron doublet state operator* ${}^1\widetilde{H}^d_{1/2}$ *is exactly the union of four segments:*

$$\sigma_{ess}({}^1\widetilde{H}^d_{1/2}) = [a+c+e, b+d+f] \cup [a+e+z_2, b+f+z_2] \cup [c+e+z_1, d+f+z_1] \cup [e+z_1+z_2, f+z_1+z_2].$$

The discrete spectrum of the operator ${}^1\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^1\widetilde{H}^q_{3/2}) = \emptyset$.

Here and hereafter $a = -2A - 4B\cos\frac{\Lambda_1}{2}$, $b = -2A + 4B\cos\frac{\Lambda_1}{2}$, $c = -2A - 4B\cos\frac{\Lambda_2}{2}$, $d = -2A + 4B\cos\frac{\Lambda_2}{2}$, $e = A - 2B$, $f = A + 2B$, $z_1 = -2A + 2\sqrt{U^2 + 4B^2\cos^2\frac{\Lambda_1}{2}}$, $z_2 = -2A - 2\sqrt{U^2 + 4B^2\cos^2\frac{\Lambda_2}{2}}$.

Theorem 5 *At* $\nu = 1$ *and* $U > 0$ *the essential spectrum of the system of first five-electron doublet state operator* ${}^1\widetilde{H}^d_{1/2}$ *is exactly the union of four segments:*

$$\sigma_{ess}({}^1\widetilde{H}^d_{1/2}) = [a+c+e, b+d+f]\cup[a+e+\widetilde{z}_2, b+f+\widetilde{z}_2]\cup[c+e+\widetilde{z}_1, d+f+\widetilde{z}_1]\cup[e+\widetilde{z}_1+\widetilde{z}_2, f+\widetilde{z}_1+\widetilde{z}_2].$$

The discrete spectrum of the operator ${}^1\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^1\widetilde{H}^q_{3/2}) = \emptyset$.

Here $\widetilde{z}_1 = -2A - 2\sqrt{U^2 + 4B^2\cos^2\frac{\Lambda_1}{2}}$, $\widetilde{z}_2 = -2A + 2\sqrt{U^2 + 4B^2\cos^2\frac{\Lambda_2}{2}}$.

Proof It follows from representation (8) that $\sigma({}^1\widetilde{H}^d_{1/2}) = \{\lambda + \mu + \theta : \lambda \in \sigma(\widetilde{H}^1_{2\Lambda_1}), \mu \in \sigma(\widetilde{H}^2_{2\Lambda_2}), \theta \in \sigma(\widetilde{H}^3_{2\Lambda_3})\}$, and in the one-dimensional case, the continuous spectrum of operator $\widetilde{H}^1_{2\Lambda_1}$ is $\sigma_{cont}(\widetilde{H}^1_{2\Lambda_1}) = [-2A - 4B\cos\frac{\Lambda_1}{2}, -2A + 4B\cos\frac{\Lambda_1}{2}]$, and the discrete spectrum of $\widetilde{H}^1_{2\Lambda_1}$ consists of a single point z_1. The continuous spectrum of operator $\widetilde{H}^2_{2\Lambda_2}$ is $\sigma_{cont}(\widetilde{H}^2_{2\Lambda_2}) = [-2A - 4B\cos\frac{\Lambda_2}{2}, -2A + 4B\cos\frac{\Lambda_2}{2}]$, and the discrete spectrum of $\widetilde{H}^2_{2\Lambda_2}$ consists of a single point z_2. The spectra of the operator $\widetilde{H}^3_{\Lambda_3}$ is a purely continuous and consists of the segment $[A - 2B, A + 2B]$. Therefore, the essential spectrum of the system of first doublet-state operator ${}^1\widetilde{H}^d_{1/2}$ is the union of four segments, and the first doublet-state operator ${}^1\widetilde{H}^d_{1/2}$ has no eigenvalues.

Theorem 5 is proved totally similarly to Theorem 4.
In the two-dimensional case the similar results occur.
We now consider the three-dimensional case.

Theorem 6 *The following statements hold:*

a) Let $\nu = 3$ *and* $U < 0$, $U < -\frac{2B}{m}$, $M > m$, *or* $U < 0$, $U < -\frac{2B}{M}$, $M < m$. *Then the essential spectrum of the system first five-electron doublet state operator* ${}^1\widetilde{H}^d_{1/2}$ *is the union of four segments:*

$$\sigma_{ess}({}^1\widetilde{H}^q_{3/2}) = [a + c + e, b + d + f] \cup [a + e + z_2, b + f + z_2] \cup \times \cup [c + e + z_1, d + f + z_1] \cup [e + z_1 + z_2, f + z_1 + z_2].$$

The discrete spectrum of the operator ${}^1\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^1\widetilde{H}^d_{1/2}) = \emptyset$.

Here, $a = -2A - 4B\sum_{i=1}^{3}\cos\frac{\Lambda_1^i}{2}$, $b = -2A + 4B\sum_{i=1}^{3}\cos\frac{\Lambda_1^i}{2}$, $c = -2A - 4B\sum_{i=1}^{3}\cos\frac{\Lambda_2^i}{2}$, $d = -2A + 4B\sum_{i=1}^{3}\cos\frac{\Lambda_2^i}{2}$, $e = A - 6B$, $f = A + 6B$, *and* z_1, z_2 *are the eigenvalues of the operators* $\widetilde{H}_{2\Lambda_1}^1$, $\widetilde{H}_{2\Lambda_2}^2$, *correspondingly.*

b) *Let* $\nu = 3$ *and* $U < 0$, $-\frac{2B}{m} \le U < -\frac{2B}{M}$, *and* $M > m$, *or* $U < 0$, $-\frac{2B}{M} \le U < -\frac{2B}{m}$, *and* $M < m$. *Then the essential spectrum of the system first five-electron doublet state operator* ${}^1\widetilde{H}_{1/2}^d$ *is the union of two segments:* $\sigma_{ess}({}^1\widetilde{H}_{1/2}^d) = [a + c + e, b + d + f] \cup [a + e + z_2, b + f + z_2]$ *or* $\sigma_{ess}({}^1\widetilde{H}_{1/2}^d) = [a + c + e, b + d + f] \cup [a + e + z_1, b + f + z_1]$. *The discrete spectrum of the operator* ${}^1\widetilde{H}_{1/2}^d$ *is empty:* $\sigma_{disc}({}^1\widetilde{H}_{1/2}^d) = \emptyset$.

c) *Let* $\nu = 3$ *and* $U < 0$, $-\frac{2B}{m} \le U < 0$, *and* $M > m$, *or* $U < 0$, $-\frac{2B}{M} \le U < 0$, *and* $M < m$. *Then the essential spectrum of the system first five-electron doublet state operator* ${}^1\widetilde{H}_{1/2}^d$ *is consists of a single segment:* $\sigma_{ess}({}^1\widetilde{H}_{1/2}^d) = [a + c + e, b + d + f]$, *and the discrete spectrum of the system first singlet-state operator* ${}^1\widetilde{H}_{1/2}^d$ *is empty:* $\sigma_{disc}({}^1\widetilde{H}_{1/2}^d) = \emptyset$.

Theorem 7 *The following statements hold:*

a) *Let* $\nu = 3$ *and* $U > 0$, $U > \frac{2B}{m}$, $M > m$, *or* $U > 0$, $U > \frac{2B}{M}$, $M < m$. *Then the essential spectrum of the system first five-electron doublet state operator* ${}^1\widetilde{H}_{1/2}^d$ *is the union of four segments:*

$$\sigma_{ess}({}^1\widetilde{H}_{3/2}^q) = [a + c + e, b + d + f] \cup [a + e + \widetilde{z}_2, b + f + \widetilde{z}_2] \\ \times \cup [c + e + \widetilde{z}_1, d + f + \widetilde{z}_1] \cup [e + \widetilde{z}_1 + \widetilde{z}_2, f + \widetilde{z}_1 + \widetilde{z}_2].$$

The discrete spectrum of the operator ${}^1\widetilde{H}_{1/2}^d$ *is empty:* $\sigma_{disc}({}^1\widetilde{H}_{1/2}^d) = \emptyset$.

Here, $a = -2A - 4B\sum_{i=1}^{3}\cos\frac{\Lambda_1^i}{2}$, $b = -2A + 4B\sum_{i=1}^{3}\cos\frac{\Lambda_1^i}{2}$, $c = -2A - 4B\sum_{i=1}^{3}\cos\frac{\Lambda_2^i}{2}$, $d = -2A + 4B\sum_{i=1}^{3}\cos\frac{\Lambda_2^i}{2}$, $e = A - 6B$, $f = A + 6B$, *and* $\widetilde{z}_1$ *is an eigenvalue of the operator* $\widetilde{H}_{2\Lambda_1}^1$, *and* $\widetilde{z}_2$ *is an eigenvalue of the operator* $\widetilde{H}_{2\Lambda_2}^2$.

b) *Let* $\nu = 3$ *and* $U > 0$, $\frac{2B}{m} \le U < \frac{2B}{M}$, *and* $M < m$, *or* $U > 0$, $\frac{2B}{M} \le U < \frac{2B}{m}$, *and* $M > m$. *Then the essential spectrum of the system first five-electron doublet state operator* ${}^1\widetilde{H}_{1/2}^d$ *is the union of two segments:* $\sigma_{ess}({}^1\widetilde{H}_{1/2}^d) = [a+c+e, b+d+f] \cup [a+e+\widetilde{z}_2, b+f+\widetilde{z}_2]$ *or* $\sigma_{ess}({}^1\widetilde{H}_{1/2}^d) = [a+c+e, b+d+f] \cup [a+e+\widetilde{z}_1, b+f+\widetilde{z}_1]$, *and the discrete spectrum of the system first doublet-state operator* ${}^1\widetilde{H}_{1/2}^d$ *is empty:* $\sigma_{disc}({}^1\widetilde{H}_{1/2}^d) = \emptyset$.

c) *Let* $\nu = 3$ *and* $U > 0$, $0 < U \le \frac{2B}{m}$, *and* $M < m$, *or* $U > 0$, $0 < U \le \frac{2B}{M}$, *and* $M > m$. *Then the essential spectrum of the system first five-electron doublet state operator* ${}^1\widetilde{H}_{1/2}^d$ *is consists of a single segment:* $\sigma_{ess}({}^1\widetilde{H}_{1/2}^d) = [a + c + e, b + d + f]$, *and the discrete spectrum of the system first five-electron doublet state operator* ${}^1\widetilde{H}_{1/2}^d$ *is empty:* $\sigma_{disc}({}^1\widetilde{H}_{1/2}^d) = \emptyset$.

We now consider the three-dimensional case. Let $\nu = 3$, and $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$, and $\Lambda_2 = (\Lambda_2^0, \Lambda_2^0, \Lambda_2^0)$.

Then the continuous spectrum of the operator $\widetilde{H}^1_{2\Lambda_1}$ is consists of the segment $\sigma_{cont}(\widetilde{H}^1_{2\Lambda_1}) = G^3_{\Lambda_1} = [-2A - 12B\cos\frac{\Lambda_1^0}{2}, -2A + 12B\cos\frac{\Lambda_1^0}{2}]$.

We consider the Watson integral $W = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{3dxdydz}{3-cosx-cosy-cosz} \approx 1516$ (see [20]). Because the measure ν is normalized, $\int_{T^3}\frac{ds_1ds_2ds_3}{3-\sum_{i=1}^3\cos(\Lambda_3^i - s_i)} = \frac{W}{3}$.

Theorem 8 *At $\nu = 3$ and $U < 0$ and the total quasimomentum Λ_1 of the system have the form $\Lambda_1 = (\Lambda_1^1, \Lambda_1^2, \Lambda_1^3) = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$. Then the operator $\widetilde{H}^1_{2\Lambda_1}$ has a unique eigenvalue $\widetilde{z}_1$ if $U < -\frac{6B\cos\frac{\Lambda_1^0}{2}}{W}$, that is above of continuous spectrum of operator $\widetilde{H}^1_{2\Lambda_1}$. Otherwise, the operator $\widetilde{H}^1_{2\Lambda_1}$ has no eigenvalue, that is above of continuous spectrum of operator $\widetilde{H}^1_{2\Lambda_1}$.*

Theorem 9 *At $\nu = 3$ and $U > 0$ and the total quasimomentum Λ_1 of the system have the form $\Lambda_1 = (\Lambda_1^1, \Lambda_1^2, \Lambda_1^3) = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$. Then the operator $\widetilde{H}^1_{2\Lambda_1}$ has a unique eigenvalue z_1 if $U > \frac{6B\cos\frac{\Lambda_1^0}{2}}{W}$, that is below of continuous spectrum of operator $\widetilde{H}^1_{2\Lambda_1}$. Otherwise, the operator $\widetilde{H}^1_{2\Lambda_1}$ has no eigenvalue, that is below of continuous spectrum of operator $\widetilde{H}^1_{2\Lambda_1}$.*

In this case the continuous spectrum of the operator $\widetilde{H}^2_{2\Lambda_2}$ is consists of the segment
$\sigma_{cont}(\widetilde{H}^2_{2\Lambda_2}) = G^3_{\Lambda_2} = [-2A - 12B\cos\frac{\Lambda_2^0}{2}, -2A + 12B\cos\frac{\Lambda_2^0}{2}]$.

Theorem 10 *At $\nu = 3$ and $U < 0$ and the total quasimomentum Λ_2 of the system have the form $\Lambda_2 = (\Lambda_2^1, \Lambda_2^2, \Lambda_2^3) = (\Lambda_2^0, \Lambda_2^0, \Lambda_2^0)$. Then the operator $\widetilde{H}^2_{2\Lambda_2}$ has a unique eigenvalue z_2 if $U < -\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, that is below of continuous spectrum of operator $\widetilde{H}^2_{2\Lambda_2}$. Otherwise, the operator $\widetilde{H}^2_{2\Lambda_2}$ has no eigenvalue, that is below of continuous spectrum of operator $\widetilde{H}^2_{2\Lambda_2}$.*

Theorem 11 *At $\nu = 3$ and $U > 0$ and the total quasimomentum Λ_2 of the system have the form $\Lambda_2 = (\Lambda_2^1, \Lambda_2^2, \Lambda_2^3) = (\Lambda_2^0, \Lambda_2^0, \Lambda_2^0)$. Then the operator $\widetilde{H}^2_{2\Lambda_2}$ has a unique eigenvalue $\widetilde{z}_2$ if $U > \frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, that is above of continuous spectrum of operator $\widetilde{H}^2_{2\Lambda_2}$. Otherwise, the operator $\widetilde{H}^2_{2\Lambda_2}$ has no eigenvalue, that is above of continuous spectrum of operator $\widetilde{H}^2_{2\Lambda_2}$.*

We now using the obtaining results and representation (8), we can describe the structure of essential spectrum and discrete spectrum of the operator of first five-electron quartet state:

Let $\nu = 3$ and $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$, and $\Lambda_2 = (\Lambda_2^0, \Lambda_2^0, \Lambda_2^0)$.

Theorem 12 *The following statements hold:*

a) *Let* $U < 0$, *and* $U < -\frac{6B\cos\frac{\Lambda_1^0}{2}}{W}, \cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *or* $U < 0$, $U < -\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}, \cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$. *Then the essential spectrum of the system first five-electron doublet state operator* ${}^1\widetilde{H}^d_{1/2}$ *is the union of four segments:* $\sigma_{ess}({}^1\widetilde{H}^d_{1/2}) = [a_1+c_1+e_1, b_1+d_1+f_1] \cup [a_1+e_1+z_2, b_1+f_1+z_2] \cup [c_1+e_1+z_1, d_1+f_1+z_1] \cup [e_1+z_1+z_2, f_1+z_1+z_2]$. *The discrete spectrum of the operator* ${}^1\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^1\widetilde{H}^q_{3/2}) = \emptyset$.

Here and hereafter $a_1 = -2A - 12B\cos\frac{\Lambda_1^0}{2}$, $b = -2A + 12B\cos\frac{\Lambda_1^0}{2}$, $c_1 = -2A - 12B\cos\frac{\Lambda_2^0}{2}$, $d_1 = -2A + 12B\cos\frac{\Lambda_2^0}{2}$, $e_1 = A - 6B$, $f_1 = A + 6B$, and z_1 is an eigenvalue of the operator $\widetilde{H}^1_{2\Lambda_1}$, *and* z_2 is an eigenvalue of the operator $\widetilde{H}^2_{2\Lambda_2}$.

b) *Let* $U < 0$, *and* $-\frac{6B\cos\frac{\Lambda_1^0}{2}}{W} \le U < -\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *or* $-\frac{6B\cos\frac{\Lambda_2^0}{2}}{W} \le U < -\frac{6B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$. *Then the essential spectrum of the system first five-electron doublet state operator* ${}^1\widetilde{H}^d_{1/2}$ *is the union of two segments:* $\sigma_{ess}({}^1\widetilde{H}^d_{1/2}) = [a_1+c_1+e_1, b_1+d_1+f_1] \cup [a_1+e_1+z_2, b_1+f_1+z_2]$, *or* $\sigma_{ess}({}^1\widetilde{H}^d_{1/2}) = [a_1+c_1+e_1, b_1+d_1+f_1] \cup [a_1+e_1+z_1, b_1+f_1+z_1]$. *The discrete spectrum of the operator* ${}^1\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^1\widetilde{H}^d_{1/2}) = \emptyset$.

c) *Let* $U < 0$, $-\frac{6B\cos\frac{\Lambda_1^0}{2}}{W} \le U < 0, \cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ *or* $-\frac{6B\cos\frac{\Lambda_2^0}{2}}{W} \le U < 0, \cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$. *Then the essential spectrum of the system first five-electron doublet state operator* ${}^1\widetilde{H}^d_{1/2}$ *is consists of a single segment:* $\sigma_{ess}({}^1\widetilde{H}^d_{1/2}) = [a_1+c_1+e_1, b_1+d_1+f_1]$, *and discrete spectrum of the system first five-electron doublet state operator* ${}^1\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^1\widetilde{H}^d_{1/2}) = \emptyset$.

Theorem 13 *The following statements hold:*

a) *Let* $U > 0$, *and* $U > \frac{6B\cos\frac{\Lambda_1^0}{2}}{W}, \cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *or* $U > 0$, $U > \frac{6B\cos\frac{\Lambda_2^0}{2}}{W}, \cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$. *Then the essential spectrum of the system first five-electron doublet state operator* ${}^1\widetilde{H}^d_{1/2}$ *is the union of four segments:* $\sigma_{ess}({}^1\widetilde{H}^d_{1/2}) = [a_1+c_1+e_1, b_1+d_1+f_1] \cup [a_1+e_1+\widetilde{z}_2, b_1+f_1+\widetilde{z}_2] \cup [c_1+e_1+\widetilde{z}_1, d_1+f_1+\widetilde{z}_1] \cup [e_1+\widetilde{z}_1+\widetilde{z}_2, f_1+\widetilde{z}_1+\widetilde{z}_2]$. *The discrete spectrum of the operator* ${}^1\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^1\widetilde{H}^q_{3/2}) = \emptyset$.

Here and hereafter $a_1 = -2A - 12B\cos\frac{\Lambda_1^0}{2}$, $b_1 = -2A + 12B\cos\frac{\Lambda_1^0}{2}$, $c_1 = -2A - 12B\cos\frac{\Lambda_2^0}{2}$, $d_1 = -2A + 12B\cos\frac{\Lambda_2^0}{2}$, $e_1 = A - 6B$, $f_1 = A + 6B$, and $\widetilde{z}_1$ is an eigenvalue of the operator $\widetilde{H}^1_{2\Lambda_1}$, and $\widetilde{z}_2$ is an eigenvalue of the operator $\widetilde{H}^2_{2\Lambda_2}$.

b) *Let* $U > 0$, *and* $\frac{6B\cos\frac{\Lambda_1^0}{2}}{W} \le U < \frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, *or* $\frac{6B\cos\frac{\Lambda_2^0}{2}}{W} \le U < \frac{6B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$. *Then the essential spectrum of the system first five-electron doublet state operator* $^1\widetilde{H}^d_{1/2}$ *is the union of two segments:* $\sigma_{ess}(^1\widetilde{H}^d_{1/2}) = [a_1+c_1+e_1, b_1+d_1+f_1] \cup [a_1+e_1+\widetilde{z}_2, b_1+f_1+\widetilde{z}_2]$, *or* $\sigma_{ess}(^1\widetilde{H}^d_{1/2}) = [a_1+c_1+e_1, b_1+d_1+f_1] \cup [a_1+e_1+\widetilde{z}_1, b_1+f_1+\widetilde{z}_1]$. *The discrete spectrum of the operator* $^1\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}(^1\widetilde{H}^d_{1/2}) = \emptyset$.

c) *Let* $U > 0$, *and* $0 < U \le \frac{6B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ *or* $0 < U \le \frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$. *Then the essential spectrum of the system first five-electron doublet state operator* $^1\widetilde{H}^d_{1/2}$ *is consists of a single segment:* $\sigma_{ess}(^1\widetilde{H}^d_{1/2}) = [a_1+c_1+e_1, b_1+d_1+f_1]$, *and the discrete spectrum of the system first five-electron doublet state operator* $^1\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}(^1\widetilde{H}^d_{1/2}) = \emptyset$.

Consequently, the essential spectrum of the system first five-electron doublet state operator $^1\widetilde{H}^d_{1/2}$ is the union of no more than four segments, and discrete spectrum of the operator $^1\widetilde{H}^d_{1/2}$ is empty.

3 Second Doublet State

The second doublet state corresponds to the basis functions $^2d^{1/2}_{m,n,r,t,l\in Z^\nu} = a^+_{m,\downarrow}a^+_{n,\uparrow}a^+_{r,\downarrow}a^+_{t,\uparrow}a^+_{l,\uparrow}\varphi_0$. The subspace $^2\mathcal{H}^d_{1/2}$, corresponding to the second five-electron doublet state is the set of all vectors of the form $^2\psi^d_{1/2} = \sum_{m,n,r,t,l\in Z^\nu} f(m,n,r,t,l)^2d^{1/2}_{m,n,r,t,l\in Z^\nu}$, $f \in l_2^{as}$, where l_2^{as} is the subspace of antisymmetric functions in the space $l_2((Z^\nu)^5)$.

The restriction $^2H^d_{1/2}$ of H to the subspace $^2\mathcal{H}^d_{1/2}$, is called the five-electron second doublet state operator.

Theorem 14 *The subspace* $^2\mathcal{H}^d_{1/2}$ *is invariant under the operator* H, *and the operator* $^2H^d_{1/2}$ *is a bounded self-adjoint operator. It generates a bounded self-adjoint operator* $^2\overline{H}^d_{1/2}$ *acting in the space* l_2^{as} *as*

$$(^2\overline{H}^d_{1/2}f)(m,n,r,t,l) = 5Af(m,n,r,t,l) + B\sum_\tau [f(m+\tau,n,r,t,l) + f(m,n+\tau,r,t,l) + f(m,n,r+\tau,t,l) + f(m,n,r,t+\tau,l) + f(m,n,r,t,l+\tau)] + U[\delta_{m,n}+\delta_{m,t} + \delta_{m,l}+\delta_{n,r}+\delta_{r,t}+\delta_{r,l}]f(m,n,r,t,l), \quad (11)$$

where $\delta_{k,j}$ is the Kronecker symbol. The operator ${}^2H^d_{1/2}$ acts on a vector ${}^2\psi^d_{1/2} \in {}^2\mathcal{H}^d_{1/2}$ as

$$
{}^2H^d_{1/2}\,{}^2\psi^d_{1/2} = \sum_{m,n,r,t,l\in Z^\nu} ({}^2\overline{H}^d_{1/2}f)(m,n,r,t,l)^2 d^{1/2}_{m,n,r,t,l\in Z^\nu}. \tag{12}
$$

Proof The proof of the theorem can be obtained from the explicit form of the action of H on vectors $\psi \in {}^2\mathcal{H}^d_{1/2}$ using the standard anticommutation relations between electron creation and annihilation operators at lattice sites, $\{a_{m,\gamma}, a^+_{n,\beta}\} = \delta_{m,n}\delta_{\gamma,\beta}$, $\{a_{m,\gamma}, a_{n,\beta}\} = \{a^+_{m,\gamma}, a^+_{n,\beta}\} = \theta$, and also take into account that $a_{m,\gamma}\varphi_0 = \theta$, where θ is the zero element of ${}^2\mathcal{H}^d_{1/2}$. This yields the statement of the theorem.

We set ${}^2\widetilde{H}^d_{1/2} = \mathcal{F}\,{}^2\overline{H}^d_{1/2}\mathcal{F}^{-1}$. In the quasimomentum representation, the operator ${}^2\widetilde{H}^d_{1/2}$ acts in the Hilbert space $L^{as}_2((T^\nu)^5)$ as

$$
{}^2\widetilde{H}^d_{1/2}\,{}^2\psi^d_{1/2} = \{5A + 2B\sum_{i=1}^{\nu}[cos\lambda_i + cos\mu_i + cos\gamma_i + cos\theta_i + cos\eta_i]\}f(\lambda,\mu,\gamma,\theta,\eta)+ \tag{13}
$$

$$
+U\int_{T^\nu}[f(s,\lambda+\mu-s,\gamma,\theta,\eta) + f(\lambda,s,\mu+\gamma-s,\theta,\eta) + f(\lambda,\mu,s,\gamma+\theta-s,\eta)+
$$

$$
+f(\lambda,\mu,s,\theta,\gamma+\eta-s) + f(s,\mu,\gamma,\lambda+\theta-s,\eta) + f(s,\mu,\gamma,\theta,\lambda+\eta-s)]ds,
$$

where $L^{as}_2((T^\nu)^5)$ is the subspace of antisymmetric functions in $L_2((T^\nu)^5)$.

Taking into account that the function $f(\lambda,\mu,\gamma,\theta,\eta)$ is antisymmetric, we can rewrite formula (13) as

$$
{}^2\widetilde{H}^d_{1/2} = \widetilde{H}^4_2 \bigotimes I \bigotimes I + I \bigotimes \widetilde{H}^5_2 \bigotimes I + I \bigotimes I \bigotimes \widetilde{H}^6_2, \tag{14}
$$

where

$$
(\widetilde{H}^4_2 f)(\lambda,\mu) = \{2A + 2B\sum_{i=1}^{\nu}[\cos\lambda_i + \cos\mu_i]\}f(\lambda,\mu) + U\int_{T^\nu} f(s,\lambda+\mu-s)ds
$$

$$
+U\int_{T^\nu} f(s,\lambda+\theta-s)ds,
$$

$$
(\widetilde{H}^5_2 f)(\gamma,\theta) = \{2A + 2B\sum_{i=1}^{\nu}[\cos\gamma_i + \cos\theta_i]\}f(\gamma,\theta) + U\int_{T^\nu} f(s,\gamma+\theta-s)ds
$$

$$
+U\int_{T^\nu} f(s,\mu+\gamma-s)ds,
$$

$$(\widetilde{H}_2^6 f)(\lambda,\eta) = \{A + 2B\sum_{i=1}^{\nu}\cos\eta_i\} f(\lambda,\eta) - U\int_{T^\nu} f(s,\gamma+\eta-s)ds$$
$$-U\int_{T^\nu} f(s,\lambda+\eta-s)ds,$$

and I is the unit operator.

Taking into account that $\lambda,\mu,\gamma,\theta,\eta \in T^\nu$, we can express the action of operators $\widetilde{H}_2^4$, $\widetilde{H}_2^5$, $\widetilde{H}_2^6$ in the form

$$(\widetilde{H}_2^4 f)(\lambda,\mu) = \{2A + 2B\sum_{i=1}^{\nu}[\cos\lambda_i + \cos\mu_i]\} f(\lambda,\mu) + 2U\int_{T^\nu} f(s,\lambda+\mu-s)ds,$$

$$(\widetilde{H}_2^5 f)(\gamma,\theta) = \{2A + 2B\sum_{i=1}^{\nu}[\cos\gamma_i + \cos\theta_i]\} f(\gamma,\theta) + 2U\int_{T^\nu} f(s,\gamma+\theta-s)ds,$$

$$(\widetilde{H}_2^6 f)(\lambda,\eta) = \{A + 2B\sum_{i=1}^{\nu}\cos\eta_i\} f(\lambda,\eta) - 2U\int_{T^\nu} f(s,\lambda+\eta-s)ds.$$

We must therefore study the spectrum and bound states (antibound states) of the operators $\widetilde{H}_2^4$, $\widetilde{H}_2^5$ and $\widetilde{H}_2^6$. Let the total quasimomentum of the two-electron systems be fixed: $\Lambda_1 = \lambda + \mu$, $\Lambda_2 = \gamma + \theta$, $\Lambda_3 = \lambda + \eta$. We let $L_2(\Gamma_{\Lambda_1})$ (correspondingly, $L_2(\Gamma_{\Lambda_2})$ and $L_2(\Gamma_{\Lambda_3})$) denote the space of functions that are square integrable on the manifold $\Gamma_{\Lambda_1} = \{(\lambda,\mu) : \lambda + \mu = \Lambda_1\}$ (correspondingly, $\Gamma_{\Lambda_2} = \{(\gamma,\theta) : \gamma + \theta = \Lambda_2\}$ and $\Gamma_{\Lambda_3} = \{(\lambda,\eta) : \lambda + \eta = \Lambda_3\}$). That the operator $\widetilde{H}_2^4$, $\widetilde{H}_2^5$ and $\widetilde{H}_2^6$ and the space $\widetilde{\mathcal{H}}_2 \equiv L_2((T^\nu)^2)$ can be decomposed into a direct integral $\widetilde{H}_2^4 = \bigoplus\int_{T^\nu} \widetilde{H}_{2\Lambda_1}^4 d\Lambda_1$ (correspondingly, $\widetilde{H}_2^5 = \bigoplus\int_{T^\nu} \widetilde{H}_{2\Lambda_2}^5 d\Lambda_2$ and $\widetilde{H}_2^6 = \bigoplus\int_{T^\nu} \widetilde{H}_{2\Lambda_3}^6 d\Lambda_3$), $\widetilde{\mathcal{H}}_2 = \bigoplus\int_{T^\nu} \widetilde{\mathcal{H}}_{2\Lambda_1} d\Lambda_1$ (correspondingly, $\widetilde{\mathcal{H}}_2 = \bigoplus\int_{T^\nu} \widetilde{\mathcal{H}}_{2\Lambda_2} d\Lambda_2$ and $\widetilde{\mathcal{H}}_2 = \bigoplus\int_{T^\nu} \widetilde{\mathcal{H}}_{2\Lambda_3} d\Lambda_3$). Each operator $\widetilde{H}_{2\Lambda_1}^4$ (correspondingly, $\widetilde{H}_{2\Lambda_2}^5$ and $\widetilde{H}_{2\Lambda_3}^6$) acts in $\widetilde{\mathcal{H}}_{2\Lambda_1}$ (correspondingly, $\widetilde{\mathcal{H}}_{2\Lambda_2}$ and $\widetilde{\mathcal{H}}_{2\Lambda_3}$) as

$$(\widetilde{H}_{2\Lambda_1}^4 f_{\Lambda_1})(\lambda) = \{2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos(\frac{\Lambda_1^i}{2} - \lambda_i)\} f_{\Lambda_1}(\lambda) + 2U\int_{T^\nu} f_{\Lambda_1}(s)ds,$$

(correspondingly,

$$(\widetilde{H}_{2\Lambda_2}^5 f_{\Lambda_2})(\gamma) = \{2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_2^i}{2}\cos(\frac{\Lambda_2^i}{2} - \gamma_i)\} f_{\Lambda_2}(\gamma) + 2U\int_{T^\nu} f_{\Lambda_2}(s)ds,$$

$$(\widetilde{H}_{2\Lambda_3}^6 f_{\Lambda_3})(\lambda) = \{A + 2B\sum_{i=1}^{\nu}\cos(\Lambda_3^i - \lambda_i)\} f_{\Lambda_3}(\lambda) - 2U\int_{T^\nu} f_{\Lambda_3}(s)ds),$$

where $f_{\Lambda_1}(\lambda) = f(\lambda, \Lambda_1 - \lambda)$ (correspondingly, $f_{\Lambda_2}(\gamma) = f(\gamma, \Lambda_2 - \gamma)$ and $f_{\Lambda_3}(\lambda) = f(\lambda, \Lambda_3 - \lambda)$).

The continuous spectrum of operator $\widetilde{H}^4_{2\Lambda_1}$ (correspondingly, $\widetilde{H}^5_{2\Lambda_2}$ and $\widetilde{H}^6_{2\Lambda_3}$) does not depend on the parameter U and consists of the intervals $G^\nu_{\Lambda_1} = [2A - 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}, 2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}]$ (correspondingly, $G^\nu_{\Lambda_2} = [2A - 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_2^i}{2}, 2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_2^i}{2}]$ and $G^\nu_{\Lambda_3} = [A - 2B\nu, A + 2B\nu]$).

We set $D^\nu_{\Lambda_1}(z) = 1 + 2U\int_{T^\nu}\frac{ds_1 ds_2 \ldots ds_\nu}{2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos(\frac{\Lambda_1^i}{2}-s_i)-z}$.

The analogue of Lemma 2 holds for this case. We consider the one-dimensional case.

Theorem 15

a) At $\nu = 1$ and $U < 0$ and for all values of the parameter of the Hamiltonian, the operator $\widetilde{H}^4_{2\Lambda_1}$ has a unique eigenvalue $z_1 = 2A - 2\sqrt{U^2 + 4B^2\cos^2\frac{\Lambda_1}{2}}$ that is below the continuous spectrum of $\widetilde{H}^4_{2\Lambda_1}$, i.e., $z_1 < m^1_{\Lambda_1}$.

b) At $\nu = 1$ and $U > 0$ and for all values of the parameter of the Hamiltonian, the operator $\widetilde{H}^4_{2\Lambda_1}$ has a unique eigenvalue $\widetilde{z}_1 = 2A + 2\sqrt{U^2 + 4B^2\cos^2\frac{\Lambda_1}{2}}$ that is above the continuous spectrum of $\widetilde{H}^4_{2\Lambda_1}$, i.e., $\widetilde{z}_1 > M^1_{\Lambda_1}$.

Proof If $U < 0$, then in the one-dimensional case, the function $D^\nu_{\Lambda_1}(z)$ decreases monotonically outside the continuous spectrum domain of the operator $\widetilde{H}^4_{2\Lambda_1}$. For $z < m^1_{\Lambda_1}$ the function $D^\nu_{\Lambda_1}(z)$ decreases from 1 to $-\infty$, $D^\nu_{\Lambda_1}(z) \to 1$ as $z \to -\infty$, $D^\nu_{\Lambda_1}(z) \to -\infty$ as $z \to m^1_{\Lambda_1} - 0$. Therefore, below the value $m^1_{\Lambda_1}$, the function $D^\nu_{\Lambda_1}(z)$ has a single zero at the point $z = z_1 = 2A - 2\sqrt{U^2 + 4B^2\cos^2\frac{\Lambda_1}{2}}$. For $z > M^1_{\Lambda_1}$, and $U < 0$, the function $D^\nu_{\Lambda_1}(z)$ decreases from $+\infty$ to 1, $D^\nu_{\Lambda_1}(z) \to +\infty$ as $z \to M^1_{\Lambda_1} + 0$, $D^\nu_{\Lambda_1}(z) \to 1$ as $z \to +\infty$. Therefore, above the value $M^1_{\Lambda_1}$, the function $D^\nu_{\Lambda_1}(z)$ cannot vanish. If $U > 0$, then the function $D^\nu_{\Lambda_1}(z)$ increases monotonically outside the continuous spectrum domain of the operator $\widetilde{H}^1_{2\Lambda_1}$. For $z < m^1_{\Lambda_1}$ the function $D^\nu_{\Lambda_1}(z)$ increases from 1 to $+\infty$, $D^\nu_{\Lambda_1}(z) \to 1$ as $z \to -\infty$, $D^\nu_{\Lambda_1}(z) \to +\infty$ as $z \to m^1_{\Lambda_1} - 0$. Therefore, below the value $m^1_{\Lambda_1}$, the function $D^\nu_{\Lambda_1}(z)$ cannot vanish. For $z > M^1_{\Lambda_1}$, and $U > 0$, the function $D^\nu_{\Lambda_1}(z)$ increases from $-\infty$ to 1, $D^\nu_{\Lambda_1}(z) \to -\infty$ as $z \to M^1_{\Lambda_1} + 0$, $D^\nu_{\Lambda_1}(z) \to 1$ as $z \to +\infty$. Therefore, above the value $M^1_{\Lambda_1}$, $D^\nu_{\Lambda_1}(z)$ has a single zero at the point $\widetilde{z}_1 = 2A + 2\sqrt{U^2 + 4B^2\cos^2\frac{\Lambda_1}{2}}$.

In the two-dimensional case, we have similar results.

We consider three-dimensional case. Denote $m = \int_{T^3}\frac{ds_1 ds_2 ds_3}{\sum_{i=1}^{3}\cos\frac{\Lambda_1^i}{2}(1+\cos(\frac{\Lambda_1^i}{2}-s_i))}$.

For $U < 0$, and $U < -\frac{2B}{m}$, below of the continuous spectrum of the operator $\widetilde{H}^4_{2\Lambda_1}$ the function $D^3_{\Lambda_1}(z)$ has a single zero at the point $z_1 < m^3_{\Lambda_1}$. For $U < 0$, and

$-\frac{2B}{m} \leq U < 0$, below of the continuous spectrum of the operator $\widetilde{H}^4_{2\Lambda_1}$ the function $D^3_{\Lambda_1}(z) = 0$ cannot vanish.

Denote $M = \int_{T^3} \frac{ds_1 ds_2 ds_3}{\sum_{i=1}^3 \cos\frac{\Lambda_1^i}{2}(1-\cos(\frac{\Lambda_1^i}{2}-s_i))}$.

For $U > 0$ and $U > \frac{2B}{M}$, above of the continuous spectrum of the operator $\widetilde{H}^4_{2\Lambda_1}$ the function $D^3_{\Lambda_1}(z)$ has a single zero at the point $\widetilde{z}_1 > M^3_{\Lambda_1}$. For $0 < U \leq \frac{2B}{M}$, above of the continuous spectrum of the operator $\widetilde{H}^4_{2\Lambda_1}$ the function $D^3_{\Lambda_1}(z)$ cannot vanish.

We now study the spectrum of the operator $\widetilde{H}^5_{2\Lambda_2}$. The operators $\widetilde{H}^4_{2\Lambda_1}$ and $\widetilde{H}^5_{2\Lambda_2}$, is the equal operators. Therefore, the spectrum of these operators is coincide, also.

We now study the spectrum of the operator

$$(\widetilde{H}^6_{2\Lambda_3} f_{\Lambda_3})(\lambda) = \{A + 2B\sum_{i=1}^{\nu}\cos(\Lambda_3^i - \lambda_i)\} f_{\Lambda_3}(\lambda) - 2U\int_{T^\nu} f_{\Lambda_3}(s)ds.$$

It is known that the continuous spectrum of $\widetilde{H}^6_{2\Lambda_3}$ is independent on U and coincides with the segment $\sigma_{cont}(\widetilde{H}^6_{2\Lambda_3}) = [A - 2B\nu, A + 2B\nu]$.

We set $D^\nu_{\Lambda_3}(z) = 1 - 2U\int_{T^\nu} \frac{ds_1 ds_2 \ldots ds_\nu}{A + 2B\sum_{i=1}^{\nu}\cos(\Lambda_3^i - s_i) - z}$.

The analogue of Lemma 2 holds for the in this case. We consider the one-dimensional case.

It is clear that the at $U < 0$ ($U > 0$) exists only one solution of the equation $D^\nu_{\Lambda_3}(z) = 0$, that is above (below) the continuous spectrum of the operator $\widetilde{H}^6_{2\Lambda_3}$.

Theorem 16

a) *At $\nu = 1$ and $U < 0$ and for all values of the parameter of the Hamiltonian, the operator $\widetilde{H}^6_{2\Lambda_3}$ has a unique eigenvalue $z_3 = 2A + 2\sqrt{U^2 + B^2}$, that is above the continuous spectrum of operator $\widetilde{H}^6_{2\Lambda_3}$, i.e., $z_3 > M^1_{\Lambda_3}$.*

b) *At $\nu = 1$ and $U > 0$ and for all values of the parameter of the Hamiltonian, the operator $\widetilde{H}^6_{2\Lambda_3}$ has a unique eigenvalue $z = \widetilde{z}_3 = 2A - 2\sqrt{U^2 + B^2}$, that is below the continuous spectrum of operator $\widetilde{H}^6_{2\Lambda_3}$, i.e., $\widetilde{z}_3 < m^1_{\Lambda_3}$.*

In the two-dimensional case the similar situation is to occur. For $U < 0$, the function $D^2_{\Lambda_3}(z)$ above the continuous spectrum of the operator $\widetilde{H}^6_{2\Lambda_3}$ has a single zero at the point $z_2 > M^2_{\Lambda_3}$. For $U > 0$, the function $D^2_{\Lambda_3}(z)$ below the continuous spectrum of the operator $\widetilde{H}^6_{2\Lambda_3}$ has a single zero at the point $\widetilde{z}_2 < m^2_{\Lambda_3}$.

We now consider three-dimensional case. For $U > 0$, and $U > \frac{3B}{W}$, the function $D^3_{\Lambda_3}(z)$ below the continuous spectrum of the operator $\widetilde{H}^6_{2\Lambda_3}$ has a single zero at the point $z_3 < m^3_{\Lambda_3}$. For $0 < U \leq \frac{3B}{W}$, the function $D^3_{\Lambda_3}(z)$ below the continuous spectrum of the operator $\widetilde{H}^6_{2\Lambda_3}$ cannot vanish. For $U < 0$, and $U < -\frac{3B}{W}$, the function $D^3_{\Lambda_3}(z)$ above the continuous spectrum of the operator $\widetilde{H}^6_{2\Lambda_3}$ has a single zero at the point $\widetilde{z}_3 > M^3_{\Lambda_3}$. For $-\frac{3B}{W} \leq U < 0$, the function $D^3_{\Lambda_3}(z)$ above the continuous spectrum of the operator $\widetilde{H}^6_{2\Lambda_3}$ cannot vanish.

Now, using the obtained results and representation (14), we describe the structure of the essential spectrum and the discrete spectrum of the operator ${}^2\widetilde{H}^d_{1/2}$ second five-electron doublet state:

Theorem 17 *At $\nu = 1$ and $U < 0$ the essential spectrum of the system of second five-electron doublet state operator ${}^2\widetilde{H}^d_{1/2}$ is exactly the union of seven segments:*

$$\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a+c+e, b+d+f] \cup [a+e+z_2, b+f+z_2] \cup$$
$$\cup [c+e+z_1, d+f+z_1] \cup [e+z_1+z_3, f+z_1+z_3] \cup$$
$$\cup [a+c+z_3, b+d+z_3] \cup [a+z_2+z_3, b+z_2+z_3] \cup [e+z_1+z_2, f+z_1+z_2].$$

The discrete spectrum of the operator ${}^2\widetilde{H}^d_{1/2}$ is consists of no more than one point: $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \{z_1+z_2+z_3\}$, or $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

Here and hereafter $a = 2A - 4B\cos\frac{\Lambda_1}{2}$, $b = 2A + 4B\cos\frac{\Lambda_1}{2}$, $c = 2A - 4B\cos\frac{\Lambda_2}{2}$, $d = 2A + 4B\cos\frac{\Lambda_2}{2}$, $e = A - 2B$, $f = A + 2B$, $z_1 = 2A - 2\sqrt{U^2 + 4B^2\cos^2\frac{\Lambda_1}{2}}$, $z_2 = 2A - 2\sqrt{U^2 + 4B^2\cos^2\frac{\Lambda_2}{2}}$, $z_3 = A + 2\sqrt{U^2 + B^2}$.

Theorem 18 *At $\nu = 1$ and $U > 0$ the essential spectrum of the system of second five-electron doublet state operator ${}^2\widetilde{H}^d_{1/2}$ is exactly the union of seven segments:*

$$\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a+c+e, b+d+f] \cup [a+e+\widetilde{z}_2, b+f+\widetilde{z}_2] \cup$$
$$\cup [c+e+\widetilde{z}_1, d+f+\widetilde{z}_1] \cup [e+\widetilde{z}_1+\widetilde{z}_3, f+\widetilde{z}_1+\widetilde{z}_3] \cup$$
$$\cup [a+c+\widetilde{z}_3, b+d+\widetilde{z}_3] \cup [a+\widetilde{z}_2+\widetilde{z}_3, b+\widetilde{z}_2+\widetilde{z}_3] \cup [e+\widetilde{z}_1+\widetilde{z}_2, f+\widetilde{z}_1+\widetilde{z}_2].$$

The discrete spectrum of the operator ${}^2\widetilde{H}^d_{1/2}$ is consists of no more than one point: $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \{\widetilde{z}_1+\widetilde{z}_2+\widetilde{z}_3\}$, or $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

Here $\widetilde{z}_1 = 2A + 2\sqrt{U^2 + 4B^2\cos^2\frac{\Lambda_1}{2}}$, $\widetilde{z}_2 = 2A + 2\sqrt{U^2 + 4B^2\cos^2\frac{\Lambda_2}{2}}$, $\widetilde{z}_3 = A - 2\sqrt{U^2 + B^2}$.

In the two-dimensional case the similar results are to occur.

We now consider the three-dimensional case. Let $\nu = 3$.

Theorem 19 *The following statements hold:*

a) Let $U < 0$, and $U < -\frac{2B}{m}, m < \frac{2}{3}W$, or $U < 0$, $U < -\frac{3B}{W}, m > \frac{2}{3}W$. Then the essential spectrum of the system second five-electron quartet state operator

${}^2\widetilde{H}^d_{1/2}$ *is the union of seven segments:*

$$\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a+c+e, b+d+f] \cup [a+c+\widetilde{z}_3, b+d+\widetilde{z}_3] \cup$$
$$\cup [a+e+z_2, b+f+z_2] \cup [a+z_2+\widetilde{z}_3, b+z_2+\widetilde{z}_3] \cup$$
$$\cup [c+e+z_1, d+f+z_1] \cup [c+z_1+\widetilde{z}_3, d+z_1+\widetilde{z}_3] \cup [e+z_1+z_2, f+z_1+z_2].$$

The discrete spectrum of the operator ${}^2\widetilde{H}^d_{1/2}$ *is consists of no more than one point:* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \{z_1+z_2+\widetilde{z}_3\}$, *or* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

Here, $a = 2A - 4B\sum_{i=1}^3 \cos\frac{\Lambda_1^i}{2}$, $b = 2A + 4B\sum_{i=1}^3 \cos\frac{\Lambda_1^i}{2}$, $c = 2A - 4B\sum_{i=1}^3 \cos\frac{\Lambda_2^i}{2}$, $d = 2A + 4B\sum_{i=1}^3 \cos\frac{\Lambda_2^i}{2}$, $e = A - 6B$, $f = A + 6B$, *and* $z_1$is an eigenvalue of the operator $\widetilde{H}^4_{2\Lambda_1}$,and z_2 is an eigenvalue of the operator $\widetilde{H}^5_{2\Lambda_2}$, and $\widetilde{z}_3$ is an eigenvalue of the operator $\widetilde{H}^6_{\Lambda_3}$.

b) *Let* $U < 0$, $-\frac{3B}{W} \leq U < -\frac{2B}{m}$, *and* $m > \frac{2}{3}W$. *Then the essential spectrum of the system second five-electron quartet state operator* ${}^2\widetilde{H}^d_{1/2}$ *is the union of four segments:*

$$\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a+c+e, b+d+f] \cup [a+e+z_2, b+f+z_2] \cup$$
$$\cup [c+e+z_1, d+f+z_1] \cup [e+z_1+z_2, f+z_1+z_2].$$

The discrete spectrum of the operator ${}^2\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

c) *Let* $U < 0$, $-\frac{2B}{m} \leq U < -\frac{3B}{W}$, *and* $m < \frac{2}{3}W$. *Then the essential spectrum of the system second five-electron quartet state operator* ${}^2\widetilde{H}^d_{1/2}$ *is the union of two segments:*

$$\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a+c+e, b+d+f] \cup [a+c+\widetilde{z}_3, b+d+\widetilde{z}_3].$$

The discrete spectrum of the operator ${}^2\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

d) *Let* $U < 0$, $-\frac{2B}{m} \leq U < 0$ *and* $m > \frac{2}{3}W$, *or* $-\frac{3B}{m} \leq U < 0$, *and* $m < \frac{2}{3}W$. *Then the essential spectrum of the system second five-electron quartet state operator* ${}^2\widetilde{H}^d_{1/2}$ *is consists of a single segment:* $\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a+c+e, b+d+f]$, *and the discrete spectrum of the operator* ${}^2\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

Let $\nu = 3$.

Theorem 20 *The following statements hold:*

a) *Let* $U > 0$, *and* $U > \frac{2B}{M}$, $M < \frac{2}{3}W$, *or* $U > 0$, *and* $U > \frac{3B}{W}$, $M > \frac{2}{3}W$. *Then the essential spectrum of the system second five-electron quartet state operator* ${}^2\widetilde{H}^d_{1/2}$ *is the union of seven segments:*

$$\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a+c+e, b+d+f] \cup [a+c+z_3, b+d+z_3] \cup$$
$$\cup [a+e+\widetilde{z}_2, b+f+\widetilde{z}_2] \cup [a+\widetilde{z}_2+z_3, b+\widetilde{z}_2+z_3] \cup$$
$$\cup [c+e+\widetilde{z}_1, d+f+\widetilde{z}_1] \cup [c+\widetilde{z}_1+z_3, d+\widetilde{z}_1+z_3] \cup [e+\widetilde{z}_1+\widetilde{z}_2, f+\widetilde{z}_1+\widetilde{z}_2].$$

The discrete spectrum of the operator ${}^2\widetilde{H}^q_{1/2}$ *is consists of no more than one point:* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \{\widetilde{z}_1+\widetilde{z}_2+z_3\}$, *or* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

Here, $\widetilde{z}_1$ is an eigenvalue of the operator $\widetilde{H}^4_{2\Lambda_1}$, and $\widetilde{z}_2$ is an eigenvalue of the operator $\widetilde{H}^5_{2\Lambda_2}$, and z_3 is an eigenvalue of the operator $\widetilde{H}^6_{\Lambda_3}$.

b) *Let* $U > 0$, $\frac{3B}{W} \le U < \frac{2B}{M}$, *and* $M < \frac{2}{3}W$. *Then the essential spectrum of the system second five-electron quartet state operator* ${}^2\widetilde{H}^d_{1/2}$ *is the union of four segments:*

$$\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a+c+e, b+d+f] \cup [a+e+\widetilde{z}_2, b+f+\widetilde{z}_2] \cup$$
$$\cup [c+e+\widetilde{z}_1, d+f+\widetilde{z}_1] \cup [e+\widetilde{z}_1+\widetilde{z}_2, f+\widetilde{z}_1+\widetilde{z}_2].$$

The discrete spectrum of the operator ${}^2\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

c) *Let* $U > 0$, $\frac{2B}{M} \le U < \frac{3B}{W}$, *and* $M > \frac{2}{3}W$. *Then the essential spectrum of the system second five-electron quartet state operator* ${}^2\widetilde{H}^d_{1/2}$ *is the union of two segments:*

$$\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a+c+e, b+d+f] \cup [a+c+z_3, b+d+z_3].$$

The discrete spectrum of the operator ${}^2\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

d) *Let* $U > 0$, $0 < U \le \frac{3B}{W}$, *and* $M < \frac{2}{3}W$, *or* $U > 0$, $0 < U \le \frac{2B}{M}$, *and* $M > \frac{2}{3}W$. *Then the essential spectrum of the system second five-electron quartet state operator* ${}^2\widetilde{H}^d_{1/2}$ *is consists of a single segment:* $\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a+c+e, b+d+f]$, *and the discrete spectrum of the operator* ${}^2\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

Let $\nu = 3$ and $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$, and $\Lambda_2 = (\Lambda_2^0, \Lambda_2^0, \Lambda_2^0)$.

It is known that the continuous spectrum of $\widetilde{H}^4_{2\Lambda_1}$ is independent of U and coincides with the segment $\sigma_{cont}(\widetilde{H}^4_{2\Lambda_1}) = [2A - 12B\cos\frac{\Lambda_1^0}{2}, 2A + 12B\cos\frac{\Lambda_1^0}{2}]$.

Theorem 21 *At* $\nu = 3$ *and* $U < 0$ *and the total quasimomentum* Λ_1 *of the system have the form* $\Lambda_1 = (\Lambda_1^1, \Lambda_1^2, \Lambda_1^3) = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$. *Then the operator* $\widetilde{H}^4_{2\Lambda_1}$ *has a unique eigenvalue* z_1 *if* $U < -\frac{6B\cos\frac{\Lambda_1^0}{2}}{W}$, *that is below the continuous spectrum of* *operator $\widetilde{H}^4_{2\Lambda_1}$. Otherwise, the operator $\widetilde{H}^4_{2\Lambda_1}$ has no eigenvalue, that is below the continuous spectrum of operator $\widetilde{H}^4_{2\Lambda_1}$*

Theorem 22 *At $\nu = 3$ and $U > 0$ and the total quasimomentum Λ_1 of the system have the form $\Lambda_1 = (\Lambda_1^1, \Lambda_1^2, \Lambda_1^3) = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$. Then the operator $\widetilde{H}^4_{2\Lambda_1}$ has a unique eigenvalue $\widetilde{z}_1$ if $U > \frac{6B\cos\frac{\Lambda_1^0}{2}}{W}$, that is above the continuous spectrum of operator $\widetilde{H}^4_{2\Lambda_1}$. Otherwise, the operator $\widetilde{H}^4_{2\Lambda_1}$ has no eigenvalue, that is above the continuous spectrum of operator $\widetilde{H}^4_{2\Lambda_1}$.*

It is known that the continuous spectrum of $\widetilde{H}^5_{2\Lambda_2}$ is independent of U and coincides with the segment $\sigma_{cont}(\widetilde{H}^5_{2\Lambda_2}) = [2A - 12B\cos\frac{\Lambda_2^0}{2}, 2A + 12B\cos\frac{\Lambda_2^0}{2}]$.

Theorem 23 *At $\nu = 3$ and $U < 0$ and the total quasimomentum Λ_2 of the system have the form $\Lambda_2 = (\Lambda_2^1, \Lambda_2^2, \Lambda_2^3) = (\Lambda_2^0, \Lambda_2^0, \Lambda_2^0)$. Then the operator $\widetilde{H}^5_{2\Lambda_2}$ has a unique eigenvalue z_2 if $U < -\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, that is below the continuous spectrum of operator $\widetilde{H}^5_{2\Lambda_2}$. Otherwise, the operator $\widetilde{H}^5_{2\Lambda_2}$ has no eigenvalue, that is below the continuous spectrum of operator $\widetilde{H}^5_{2\Lambda_2}$.*

Theorem 24 *At $\nu = 3$ and $U > 0$ and the total quasimomentum Λ_2 of the system have the form $\Lambda_2 = (\Lambda_2^1, \Lambda_2^2, \Lambda_2^3) = (\Lambda_2^0, \Lambda_2^0, \Lambda_2^0)$. Then the operator $\widetilde{H}^5_{2\Lambda_2}$ has a unique eigenvalue $\widetilde{z}_2$ if $U > \frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, that is above the continuous spectrum of operator $\widetilde{H}^5_{2\Lambda_2}$. Otherwise, the operator $\widetilde{H}^5_{2\Lambda_2}$ has no eigenvalue, that is above the continuous spectrum of operator $\widetilde{H}^5_{2\Lambda_2}$.*

Now, using the obtained results and representation (14), we describe the structure of the essential spectrum and the discrete spectrum of the system second five-electron quartet state operator ${}^2\widetilde{H}^d_{1/2}$:

Let $\nu = 3$ and $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$, and $\Lambda_2 = (\Lambda_2^0, \Lambda_2^0, \Lambda_2^0)$.

Theorem 25 *The following statements hold:*

a) Let $U < 0$, $U < -\frac{6B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, and $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}$, or $U < 0$, $U < -\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ and $\cos\frac{\Lambda_2^0}{2} > \frac{1}{2}$. Then the essential spectrum of the system second five-electron doublet state operator ${}^2\widetilde{H}^d_{1/2}$ is consists of the union of seven segments: $\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a_1+c_1+e_1, b_1+d_1+f_1]\cup[a_1+e_1+z_2, b_1+f_1+z_2]\cup[c_1+e_1+z_1, d_1+f_1+z_1]\cup[e_1+z_1+z_2, f_1+z_1+z_2]\cup[a_1+c_1+\widetilde{z}_3, b_1+d_1+\widetilde{z}_3]\cup[a_1+z_2+\widetilde{z}_3, b_1+z_2+\widetilde{z}_3]\cup[c_1+z_1+\widetilde{z}_3, d_1+z_1+\widetilde{z}_3]$. The discrete spectrum of the operator ${}^2\widetilde{H}^d_{1/2}$ is consists of no more than one point: $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \{z_1 + z_2 + \widetilde{z}_3\}$, or $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

Here and hereafter $a_1 = 2A - 12B\cos\frac{\Lambda_1^0}{2}$, $b_1 = 2A + 12B\cos\frac{\Lambda_1^0}{2}$, $c_1 = 2A - 12B\cos\frac{\Lambda_2^0}{2}$, $d_1 = 2A + 12B\cos\frac{\Lambda_2^0}{2}$, $e_1 = A - 6B$, $f_1 = A + 6B$, and z_1 is an eigenvalue of the operator $\widetilde{H}^4_{2\Lambda_1}$, and z_2 is an eigenvalue of the operator $\widetilde{H}^5_{2\Lambda_2}$, and $\widetilde{z}_3$is an eigenvalue of the operator $\widetilde{H}^6_{\Lambda_3}$.

b) Let $U < 0$, $-\frac{6B\cos\frac{\Lambda_1^0}{2}}{W} \le U < -\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_2^0}{2} > \frac{1}{2}$, *or* $-\frac{6B\cos\frac{\Lambda_2^0}{2}}{W} \le U < -\frac{6B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}$. *Then the essential spectrum of the system second five-electron doublet state operator* ${}^2\widetilde{H}^d_{1/2}$ *is consists of the union of four segments:* $\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + e_1 + z_2, b_1 + f_1 + z_2] \cup [a_1 + c_1 + \widetilde{z}_3, b_1 + d_1 + \widetilde{z}_3] \cup [a_1 + z_2 + \widetilde{z}_3, b_1 + z_2 + \widetilde{z}_3]$, *or* $\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + e_1 + z_1, b_1 + f_1 + z_1] \cup [a_1 + c_1 + \widetilde{z}_3, b_1 + d_1 + \widetilde{z}_3] \cup [c_1 + z_1 + \widetilde{z}_3, d_1 + z_1 + \widetilde{z}_3]$. *The discrete spectrum of the operator* ${}^2\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

c) Let $U < 0$, $-\frac{6B\cos\frac{\Lambda_1^0}{2}}{W} \le U < -\frac{3B}{W}$, $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}$, *or* $-\frac{6B\cos\frac{\Lambda_2^0}{2}}{W} \le U < -\frac{3B}{W}$, $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_2^0}{2} > \frac{1}{2}$. *Then the essential spectrum of the system second five-electron doublet state operator* ${}^2\widetilde{H}^d_{1/2}$ *is consists of the union of two segments:* $\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + \widetilde{z}_3, b_1 + d_1 + \widetilde{z}_3]$. *The discrete spectrum of the operator* ${}^2\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

d) Let $U < 0$, $-\frac{6B\cos\frac{\Lambda_1^0}{2}}{W} \le U < 0, \cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ *and* $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}$, *or* $-\frac{6B\cos\frac{\Lambda_2^0}{2}}{W} \le U < 0, \cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_2^0}{2} < \frac{1}{2}$. *Then the essential spectrum of the system second five-electron doublet state operator* ${}^2\widetilde{H}^d_{1/2}$ *is consists of a single segments:* $\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1]$, *and discrete spectrum of the operator* ${}^2\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

Theorem 26 *The following statements hold:*

a) Let $U > 0$, $U > \frac{6B\cos\frac{\Lambda_1^0}{2}}{W}, \cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}$, *or* $U > 0$, $U > \frac{6B\cos\frac{\Lambda_2^0}{2}}{W}, \cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_2^0}{2} > \frac{1}{2}$. *Then the essential spectrum of the system second five-electron doublet state operator* ${}^2\widetilde{H}^d_{1/2}$ *is consists of the union of seven segments:* $\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a_1+c_1+e_1, b_1+d_1+f_1] \cup [a_1+e_1+\widetilde{z}_2, b_1+f_1+\widetilde{z}_2] \cup [c_1+e_1+\widetilde{z}_1, d_1+f_1+\widetilde{z}_1] \cup [e_1+\widetilde{z}_1+\widetilde{z}_2, f_1+\widetilde{z}_1+\widetilde{z}_2] \cup [a_1+c_1+z_3, b_1+d_1+z_3] \cup [a_1+\widetilde{z}_2+z_3, b_1+\widetilde{z}_2+z_3] \cup [c_1+\widetilde{z}_1+z_3, d_1+\widetilde{z}_1+z_3]$. *The discrete spectrum of the operator* ${}^2\widetilde{H}^d_{1/2}$ *is consists of no more than one point:* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \{\widetilde{z}_1 + \widetilde{z}_2 + z_3\}$, *or* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

Here and hereafter, $\widetilde{z}_1$ is an eigenvalue of the operator $\widetilde{H}^4_{2\Lambda_1}$, and $\widetilde{z}_2$ is an eigenvalue of the operator $\widetilde{H}^5_{2\Lambda_2}$, and z_3 is an eigenvalue of the operator $\widetilde{H}^6_{\Lambda_3}$.

b) Let $U > 0$, $\frac{6B\cos\frac{\Lambda_1^0}{2}}{W} \leq U < \frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}$, *or* $\frac{6B\cos\frac{\Lambda_2^0}{2}}{W} \leq U < \frac{6B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_2^0}{2} > \frac{1}{2}$. *Then the essential spectrum of the system second five-electron doublet state operator* ${}^2\widetilde{H}^d_{1/2}$ *is consists of the union of four segments:* $\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + e_1 + \widetilde{z}_2, b_1 + f_1 + \widetilde{z}_2] \cup [a_1 + c_1 + z_3, b_1 + d_1 + z_3] \cup [a_1 + \widetilde{z}_2 + z_3, b_1 + \widetilde{z}_2 + z_3]$, *or* $\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + e_1 + \widetilde{z}_1, b_1 + f_1 + \widetilde{z}_1] \cup [a_1 + c_1 + z_3, b_1 + d_1 + z_3] \cup [c_1 + \widetilde{z}_1 + z_3, d_1 + \widetilde{z}_1 + z_3]$. *The discrete spectrum of the operator* ${}^2\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

c) Let $U > 0$, $\frac{6B\cos\frac{\Lambda_1^0}{2}}{W} \leq U < \frac{3B}{W}$, $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_2^0}{2} > \frac{1}{2}$, *or* $\frac{6B\cos\frac{\Lambda_2^0}{2}}{W} \leq U < \frac{3B}{W}$, $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}$. *Then the essential spectrum of the system second five-electron doublet state operator* ${}^2\widetilde{H}^d_{1/2}$ *is consists of the union of two segments:* $\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + \widetilde{z}_1, b_1 + d_1 + \widetilde{z}_1]$, *or* $\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + \widetilde{z}_2, b_1 + d_1 + \widetilde{z}_2]$. *The discrete spectrum of the operator* ${}^2\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

d) Let $U > 0$, $0 < U \leq \frac{3B}{W}$, $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, $(\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2})$, *and* $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}$, *or* $0 < U \leq \frac{6B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ *and* $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}$, *or* $0 < U \leq \frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_2^0}{2} < \frac{1}{2}$. *Then the essential spectrum of the system second five-electron doublet state operator* ${}^2\widetilde{H}^d_{1/2}$ *is consists of a single segments:* $\sigma_{ess}({}^2\widetilde{H}^d_{1/2}) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1]$, *and discrete spectrum of the operator* ${}^2\widetilde{H}^d_{1/2}$ *is empty:* $\sigma_{disc}({}^2\widetilde{H}^d_{1/2}) = \emptyset$.

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On Coefficient Inverse Problems of Heat Conduction for Functionally Graded Materials

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Abstract The inverse heat conduction problem on the identification of thermophysical characteristics of the functionally graded layer of a two-layer hollow long cylinder is considered. The input information is the measured temperature on the upper surface of the cylinder. The direct problem after the Laplace transform is solved using the Galerkin's method and the inversion of transformants, based on the theory of residues. Two approaches are used to solve the inverse problem. The first approach is the development of a previously developed iterative approach, at each step of which the Fredholm integral equation of the 1st kind is solved. The second approach is based on the algebraization of solution of the direct problem. Specific examples of reconstruction of thermal conductivity laws of change and heat capacity of the cylinder are considered. Two approaches are compared.

Keywords Heat conductivity coefficient · Heat capacity · Long hollow cylinder · Functionally graded materials · Galerkin's method · Inverse coefficient problem · Identification · Iterative scheme · Algebraization

MSC 80A20, 80A23

1 Introduction

The study of heat distribution in cylindrical layered bodies is of great importance in metallurgy, aviation and space technology. For quantitative calculations of heat distribution processes, knowledge of the thermophysical properties of materials is

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required. Typically, layered structures are made from uniform materials. Recently, however, to specify the design of the required properties, one of the layers is increasingly being made from functionally graded materials (FGM). FGM are composites with variable physical properties [1]. But the production of FGM is a complex technological process. Due to the multi-stage technological operations, deviations from the established norms may be present in the final product. The thermophysical characteristics of the FGM can be determined only by solving the coefficient inverse problem (CIP) of heat conduction. The number of works devoted to the reconstruction of the thermophysical properties of materials is quite large [2–17].

Two types of formulation of inverse heat conduction problems are widespread. For the first type, additional information is considered known at the internal points of the body at some point in time [2], for the second type—only on a part of the boundary and at a certain time interval [3]. The second formulation of the problem is experimentally the most feasible in practice. But in the case of the second formulation, CIP is a nonlinear problem. The common approach to solving nonlinear CIP is based on the application of well-established iterative algorithms [2–12]. For this, the residual functional is composed, to minimize which, as a rule, gradient methods [2–10] or genetic algorithms [11] are used. Alternative non-iterative solution methods are also used: the quasi-inversion method [14], the method for the treatment of difference scheme [15], etc. In [12, 16–19] for the solution of inverse heat conduction problems for a rod, a layer and a single-layer cylinder the iterative process was constructed, at each step of which the linearized Fredholm integral equations of the 1st kind were solved. However, in these works the functions characterizing heterogeneity had either piecewise constant or continuous character. At the same time, the question of the identification of thermophysical characteristics as functions having breakpoints of the 1st kind at the boundary of contact of the layers remained unexplored. In this paper, the formulation of the CIP of heat conduction for a two-layer cylinder is given. After the Laplace transform, the direct problem is solved on the basis of the Galerkin's projection method. To solve the inverse problem, two approaches are used. The first approach is the development of the previously developed iterative approach, at each step of which the Fredholm integral equations of the 1st kind are solved. The second approach is based on the algebraization of the direct problem. Computational experiments on the reconstruction of various laws of changes in thermophysical characteristics are carried out.

2 Statement of the Problem

Consider the problem of heat propagation in a two-layer infinite hollow cylinder thickness h, the inner layer of which is thickness h_1 that is assumed to be made of FGM with characteristics $k_1(r)$, $c_1(r)$, and the outer layer thickness $h_2 = h - h_1$ made of a homogeneous material with constant characteristics k_2, c_2. Zero temperature is maintained on the inner surface of the cylinder $r = a$, and heat flow $q = q_0 H(t)$ acts on the outer surface $r = b$. The initial-boundary-value problem has the form [16]:

$$\frac{1}{r}\frac{\partial}{\partial r}(k(r)r\frac{\partial T}{\partial r}) = c(r)\frac{\partial T}{\partial t}, \quad a \le r \le b, \quad t > 0, \tag{1}$$

$$T(a,t) = 0, \quad -k(b)\frac{\partial T}{\partial r}(b,t) = q_0 H(t), \tag{2}$$

$$T(r,0) = 0. \tag{3}$$

Go to (1)–(3) to dimensionless parameters and functions, denoting:

$$z = \frac{r-a}{b-a}, \quad z_0 = \frac{a}{b-a}, \quad \bar{k}(z) = \frac{k(r)}{k_2}, \quad \bar{c}(z) = \frac{c(r)}{c_2},$$
$$\tau = \frac{k_2 t}{c_2(b-a)^2}, \quad W(z,\tau) = \frac{k_2 T}{q_0(b-a)}, \quad H_1 = \frac{h_1}{b-a}. \tag{4}$$

After dimensioning, the initial-boundary-value problem (1)–(3) takes the form:

$$\frac{1}{z+z_0}\frac{\partial}{\partial z}(\bar{k}(z)(z+z_0)\frac{\partial W}{\partial z}) = \bar{c}(z)\frac{\partial W}{\partial \tau}, \quad 0 \le z \le 1, \tag{5}$$

$$W(0,\tau) = 0, \quad -\bar{k}(1)\frac{\partial W}{\partial z}(1,\tau) = H(\tau) \tag{6}$$

$$W(z,0) = 0. \tag{7}$$

The direct problem of heat conduction is to determine the function of (5)–(7) at known thermophysical characteristics $\bar{c}(z)$, $\bar{k}(z)$.

In the inverse problem, it is required from (5)–(7) to determine one of the thermophysical characteristics of the functionally graded layer of the cylinder.

As additional information, is the temperature, measured on the outer surface of the cylinder $z = 1$ on the informative time interval $[b_1, b_2]$:

$$W(1,\tau) = f(\tau), \quad \tau \in [b_1, b_2]. \tag{8}$$

We have two settings of the CIP of heat conduction in dimensionless form:

1. from (5)–(7) according to information (8) to find $\bar{c}_1(z)$ under the known law $\bar{k}_1(z)$;
2. from (5)–(7) according to information (8) to find $\bar{k}_1(z)$ under the known law $\bar{c}_1(z)$.

3 The Solution of the Direct Problem of Heat Conduction

The solution of the inverse problem depends on the accuracy of the solution of the direct problem. In the case of arbitrary laws of heterogeneity, sufficient accuracy is provided by approximate analytical methods, for example, the Galerkin's projection method [16, 17].

To do this, we apply to (5), (6) the Laplace transform and, taking into account the initial condition (7), we obtain:

$$\frac{1}{z+z_0}\frac{d}{dz}(\bar{k}(z)(z+z_0)\frac{d\tilde{W}}{dz}) = p\bar{c}(z)\tilde{W}(z,p), \tag{9}$$

$$\tilde{W}(0,p)=0, \quad -\bar{k}(1)\frac{d\tilde{W}}{dz}(1,p)=\frac{1}{p}. \tag{10}$$

Here it is indicated: p is the Laplace transform parameter, $\tilde{W}(z,p)$ is the transformant of dimensionless temperature.

In accordance with the Galerkin's method, we present an approximate solution (9), (10) in the form of an expansion in a system of basis functions [16]:

$$\tilde{W}_N(z,p)=\phi_0(z,p)+\sum_{i=1}^{N}\tilde{a}_i(p)\phi_i(z). \tag{11}$$

Here $\phi_0(z,p) = -\frac{z}{p}$, is a function, satisfying inhomogeneous boundary conditions (10), $\phi_i(z) = \sin\frac{(2i-1)\pi z}{2}$ are the orthogonal functions satisfying homogeneous boundary conditions.

Requiring orthogonality of the residual
$\varepsilon_N = \frac{1}{z+z_0}\frac{d}{dz}(\bar{k}(z)(z+z_0)\frac{d\tilde{W}_N}{dz}) - p\bar{c}(z)\tilde{W}_N(z,p)$ to the basis functions ϕ_i, we obtain a system of algebraic equations with respect to $\tilde{a}_i(p)$:

$$\sum_{j=1}^{n}G_{ji}\tilde{a}_i(p)=D_j, \tag{12}$$

where $G_{ji} = \int\limits_0^1 \frac{1}{z+z_0}\bar{k}(z)\frac{d\phi_i}{dz}\phi_j dz - \int\limits_0^1 \bar{k}(z)\frac{d\phi_i}{dz}\frac{d\phi_j}{dz}dz - p\int\limits_0^1 \bar{c}(z)\phi_i\phi_j dz + \bar{k}(1)\phi_j(1)\frac{d\phi_i}{dz}(1)$, $D_j = \int\limits_0^1 \frac{1}{z+z_0}\bar{k}(z)\frac{d\phi_0}{dz}\phi_j dz - \int\limits_0^1 \bar{k}(z)\frac{d\phi_0}{dz}\frac{d\phi_j}{dz}dz - p\int\limits_0^1 \bar{c}(z)\phi_0\phi_j dz + \bar{k}(1)\phi_j(1)\frac{d\phi_0}{dz}(1)$.

Since $\bar{k}(z)$ and $\bar{c}(z)$ are piecewise continuous functions, in the work the integration interval [0, 1] was divided into sections, corresponding to the individual layers of the cylinder.

After solving system (12), we compose an expression for the temperature transform on the outer surface of the cylinder:

$$\tilde{W}_N(1, p) = \phi_0(1, p) + \sum_{i=1}^{N} \tilde{a}_i(p)\phi_i(1). \tag{13}$$

To find the originals of the functions by their transformants, we need to apply the inverse Laplace transform, i.e. calculate the contour Riemann-Mellin's integral by the formula:

$$F(\tau) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \tilde{F}(p)e^{p\tau} dp, \tag{14}$$

where the integral is taken along the line $p = a > 0$. In the case when the integral (14) is an integral of a meromorphic function, it can be transformed using the residue theory, which allows us to obtain an expression for the original by expanding it into series in exponential functions:

$$F(\tau) = \sum_{j=1}^{\infty} res(\tilde{F}(p_j)e^{p_j \tau}), \quad \tau > 0. \tag{15}$$

Calculations in the Maple showed that function (14) is a fractional-rational function of p and does not have other singular points but real negative poles. Residue theory was used to find the originals of temperature.

The proposed method for solving problem (5)–(8) was tested on the example of a two-layer cylinder with characteristics $\bar{c_1} = 0.5$ $(0 \leq z \leq 0.5)$, $\bar{c_2} = 1$ $(0.5 < z \leq 1)$, $\bar{k_1} = 0.25$ $(0 \leq z \leq 0.5)$, $\bar{k_2} = 1$ $(0.5 < z \leq< 1)$. It was found that if in the Galerkin's method we restrict ourselves to three coordinate functions $N = 3$, then the greatest error arises at short times $\tau \in [0, 0.09]$. At the same time $\tau > 0.09$, the error of the Galerkin's method does not exceed 2%. Figure 1 shows a graph of the change in dimensionless temperature from time on the outer surface of the cylinder, obtained analytically (solid line) and by the Galerkin's method at $N = 3$ (points).

4 The Solution of the Inverse Heat Conduction Problem Based on an Iterative Approach

The inverse problem (5)–(8) is a nonlinear problem. To solve this problem, two approaches are proposed: (1) the iterative process previously developed in [16]; (2) the method of algebraization.

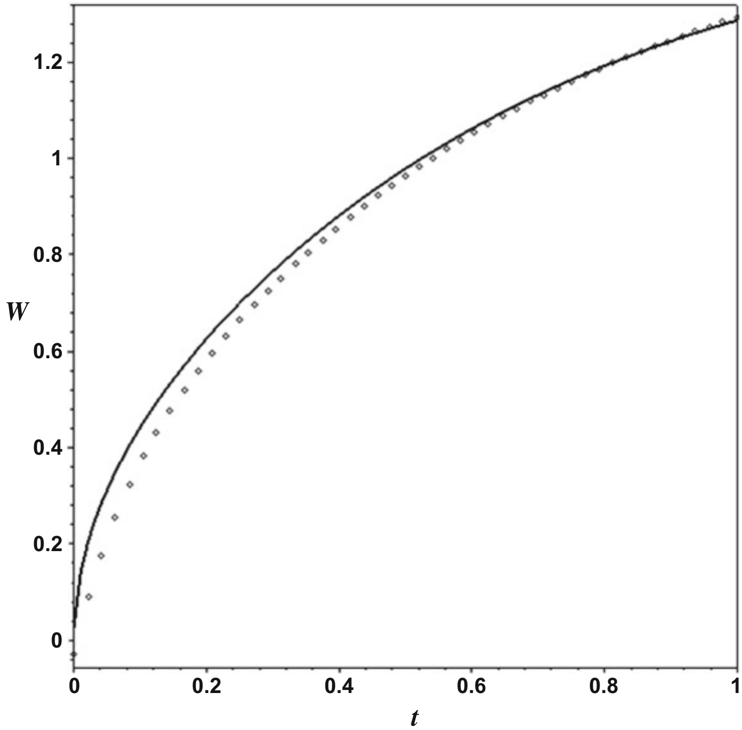

Fig. 1 Temperature change on the outer surface of the cylinder over time

Let us consider in detail the construction of the iterative process of identification of thermophysical characteristics.

The dimensionless thermophysical characteristics of the functionally graded layer of the cylinder $\bar{a}(z)$ ($\bar{c}_1(z)$, $\bar{k}_1(z)$) were restored in two stages.

At the first stage, the initial approximation in the class of positive bounded linear functions $\bar{a}^{(0)}(z) = a_1 z + a_2$ was determined on the basis of minimizing the residual functional, which has the form:

$$J = \int_{b_1}^{b_2} (f(\tau) - W^{(n-1)}(1,\tau))^2 d\tau, \tag{16}$$

At the second stage, based on the solution of operator equations of the 1st kind, the corrections of the reconstructed functions $\delta\bar{a}^{(n-1)}$ were found, and the iterative process of their refinement was constructed according to the scheme: $\bar{a}^{(n)}(z) = \bar{a}^{(n-1)}(z) + \delta\bar{a}^{(n-1)}(z)$.

The operator equations for finding corrections have the form of Fredholm integral equations of the 1st kind [16]:

$$\int_0^1 \delta\bar{k}^{(n-1)} R_1(z,\tau)dz = f(\tau) - W^{(n-1)}(1,\tau), \quad \tau \in [b_1, b_2], \tag{17}$$

$$\int_0^1 \delta\bar{c}^{(n-1)} R_2(z,\tau)dz = f(\tau) - W^{(n-1)}(1,\tau), \quad \tau \in [b_1, b_2]. \tag{18}$$

Here, the kernels of integral equations (17), (18) have the form:

$$R_1(z,\tau) = \int_0^\tau \frac{\partial^2 W^{(n-1)}(z,\tau_1)}{\partial z \partial \tau_1}\left(\frac{\partial W^{(n-1)}(z,\tau_1-\tau)}{\partial z} - \frac{W^{(n-1)}(z,\tau_1-\tau)}{z+z_0}\right)d\tau_1,$$

$$R_2(z,\tau) = \int_0^\tau \frac{\partial W^{(n-1)}(z,\tau_1)}{\partial \tau_1}\frac{\partial W^{(n-1)}(z,\tau_1-\tau)}{\partial \tau_1}d\tau_1.$$

The solution of integral equations (17), (18) is an ill-posed problem; for its regularization, the Tikhonov's method was used [20].

The exit from the iterative process was carried out when the residual functional (16) reached a threshold value equal to 10^{-6}.

5 The Solution of the Inverse Problem by the Algebraization Method

The application of the method of algebraization will be considered in detail on the example of finding the law of change of specific heat capacity $\bar{c}_1(z)$.

We will seek an approximate solution of the CIP in the form of an expansion:

$$\bar{c}_1(z) = \sum_{j=1}^m g_j \psi_j(z), \quad \psi_i(z) = z^{j-1}, \quad j = 1..m. \tag{19}$$

We substitute (19) in the formulas for the coefficients of the system of equations (12) and solve it analytically in the Maple. Further, by the formula (13) we find the expression for the transform of temperature on the outer surface of the cylinder, which has the form:

$$\tilde{W}_N(1, p, g_1, g_2, \ldots, g_m) = \frac{A_N(p, g_1, g_2, \ldots, g_m)}{B_N(p, g_1, g_2, \ldots, g_m)}. \tag{20}$$

The temperature on the outer surface of the cylinder monotonically increases from zero and eventually reaches a certain limiting value. Therefore, we can approximate additional information (8), measured at time instants $\tau_i = \tau_1 + (i -$

$1)\Delta\tau$, $i = 1..2m+1$, on a time interval $[b_1, b_2]$, in the form of a linear combination of exponential functions:

$$f(\tau) \approx s_0 + s_1 e^{p_1 \tau} + s_2 e^{p_2 \tau} + \ldots + s_m e^{p_m \tau}, \tag{21}$$

where $p_1, p_2, \ldots, p_m, s_1, s_2, \ldots, s_m$ are negative numbers, s_0 is a positive number.

We find the indices of exponents $p_1, p_2, \ldots, p_m$ in the expansion (21) by the method of Prony [21]. We introduce the quantities $v_i = f(\tau_i) - f(\tau_{i+1})$. Then the indices of exponents $p_1, p_2, \ldots, p_m$ can be found by the formula:

$$p_j = \frac{1}{\Delta\tau} \ln u_j, \quad j = 1..m, \tag{22}$$

where u_j are the roots of the characteristic equation

$$\alpha_m z^m + \ldots + \alpha_2 z^2 + \alpha_1 z + 1 = 0. \tag{23}$$

To find the coefficients α_j, $j = 1..m$ of the characteristic polynomial (23), it is necessary to solve the system of recursive difference equations. So, for example, when $m = 3$ this system has the form:

$$\begin{aligned} v_1 + \alpha_1 v_2 + \alpha_2 v_3 + \alpha_3 v_4 = 0, \\ v_2 + \alpha_1 v_3 + \alpha_2 v_4 + \alpha_3 v_5 = 0, \\ v_3 + \alpha_1 v_4 + \alpha_2 v_5 + \alpha_3 v_6 = 0. \end{aligned} \tag{24}$$

The numerical values of the parameter p will be the poles of function (20) if they vanish the expression $B_N(p, g_1, g_2, \ldots, g_m)$. Therefore, to find the coefficients of decomposition g_j, $j = 1..m$, we assume $B_N(p, g_1, g_2, \ldots, g_m) = 0$, and as the values of the parameter p, we use the approximate values of the m first indices of the exponents found by the formula (22). As a result of the substitutions we obtain a system of m nonlinear algebraic equations, the solution of which are m sets of numbers $(g_1, g_2, \ldots, g_m)$. Further, for each set of numbers $(g_1, g_2, \ldots, g_m)$, according to formula (19), the function $\bar{c}_1(z)$ was determined. As the first criterion for the selection of a suitable set $(g_1, g_2, \ldots, g_m)$, we introduce the boundedness of the specific heat capacity of the functionally graded part $0 < c_- < \bar{c}_1(z) < c_+$. From further consideration, we remove those sets of numbers $(g_1, g_2, \ldots, g_m)$ that do not satisfy this criterion. The minimum value of the residual functional (16) acts as the second criterion for skipping a suitable set $(g_1, g_2, \ldots, g_m)$.

Similarly, based on the algebraization method, the thermal conductivity coefficient $\bar{k}_1(z)$ was determined.

Comment By the formula (22), no more than three indices of exponents can be stably determined, therefore, the algebraization method is limited by the possibility of reconstruction in the form of linear and quadratic functions. In addition, stable solutions to the inverse problem are obtained if no more than three coordinate functions are used to solve the direct problem by the Galerkin's method. However, in this case, due to the large calculation error at small times, it is necessary to measure additional information at $\tau > 0.09$.

6 Results of Computational Experiments

Computational experiments to reconstruction the thermophysical characteristics of the functionally graded layer of the cylinder are carried out. Only one of the thermophysical characteristics was restored. The second characteristic was assumed to be known, having the same law of heterogeneity as the first. Calculations were made at $z_0 = 1$.

In the first series of experiments, the first approach was used to reconstruction the thermophysical characteristics. The success of the reconstruction depended strongly on the choice of the time interval for the measure of additional information. The most informative is the interval taken in the time zone near the beginning of observation, when the temperature changes most rapidly. It is found that the temperature measurement on the outer surface of the cylinder is most informative in the interval $[b_1, b_2] = [0, 0.8]$. When solving the inverse problem on the basis of the iterative approach, the corrections of the reconstructed coefficients are found from the solution of Fredholm integral equations of the 1st kind, the discretization of which yields a poorly conditioned system of linear algebraic equations. It is enough to know additional information on an informative time interval of 5–8 points in time, since the Tikhonov's method is used to regularize the system of equations [20]. The Tikhonov's method allows solving both underdetermined and overdetermined systems of equations. If we take a large number of points in time, then the volume of calculations increases, but the accuracy of the solution practically does not increase. To satisfy the exit condition, no more than 8 iterations were required. The maximum error in the restoration of monotonic functions did not exceed 4%, and nonmonotonic −7%.

In the second series of experiments, a second approach was used to reconstruct the thermophysical characteristics. The reconstruction took place in the class of quadratic functions ($m = 3$). Additional information was measured in points $\tau_i = 0.1i, i = 1..7$. The maximum error of identification of monotonic functions did not exceed 5%, and nonmonotonic −14%.

Figures 2, 3, and 4 show the result of reconstruction of thermophysical characteristics at $z \in [0, 0.5]$. In this case, the solid line shows the exact law, the dots are restored on the basis of the first approach, the dash-dotted lines are restored on the basis of the second approach.

Figure 2 shows the result of reconstruction of an increasing function $\bar{c}_1(z) = 1.2 + 0.8z^2$. In the case of applying the first approach, the maximum error in the recovery of specific heat capacity (6%) arose in the vicinity of $z = 0$, which is due to the features of the kernel of the integral equation (18). In the case of the second

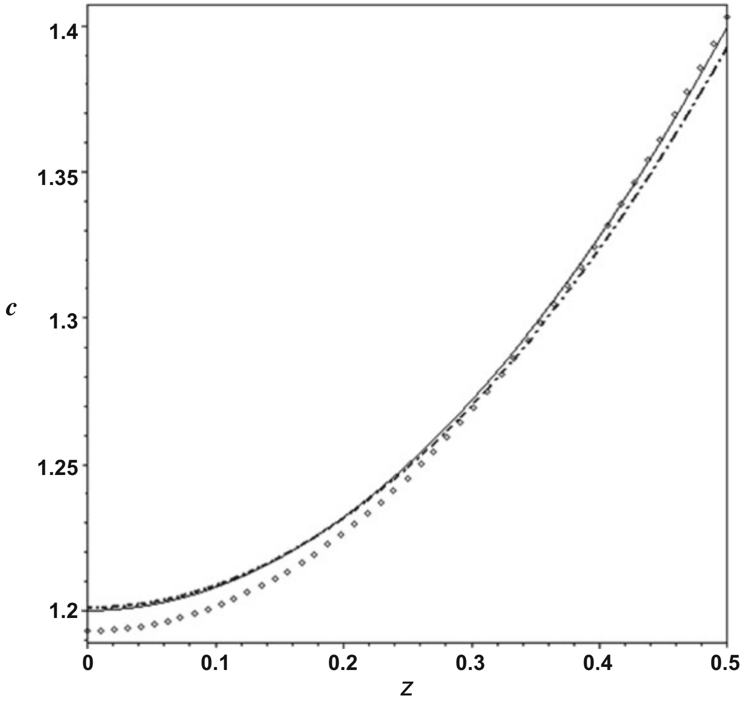

Fig. 2 Result of reconstruction of increasing function $\bar{c}_1(z) = 1.2 + 0.8z^2$

approach, the error of reconstruction did not exceed 4%. Moreover, in comparison with the first approach, the largest error of reconstruction did not occur in the neighborhood of $z = 0$.

Figure 3 shows the result of restoring the decreasing function $\bar{k}_1(z) = 0.6 + e^{-z}$. From Fig. 3 it is seen that in the case of restoration of the coefficient of thermal conductivity as a monotone function, the errors of reconstruction for the first and second approaches differ slightly and do not exceed 4%.

Figure 4 shows the result of reconstruction of a nonmonotonic function $\bar{k}_1(z) = 1 + \sin(2\pi z)$. From Fig. 4 it is seen that the errors of reconstruction of the thermal conductivity as a nonmonotonic function are very different. In the case of applying the first approach, the maximum error of recovery did not exceed 3%. In the case of the second approach, the reconstruction error increased to 13%. For a more successful reconstruction in the case of the second approach, it is necessary to take a larger number of coordinate functions, for example, $m = 4$. However, this is not always possible due to the instability of determining the indices of exponents by the Prony method.

Using the functions $k_1(z) = 0.6 + e^z$ as an example, we studied the effect of the thickness of the functionally graded layer of the cylinder on the results of reconstruction. It was found that when using both methods of solving CIP, the

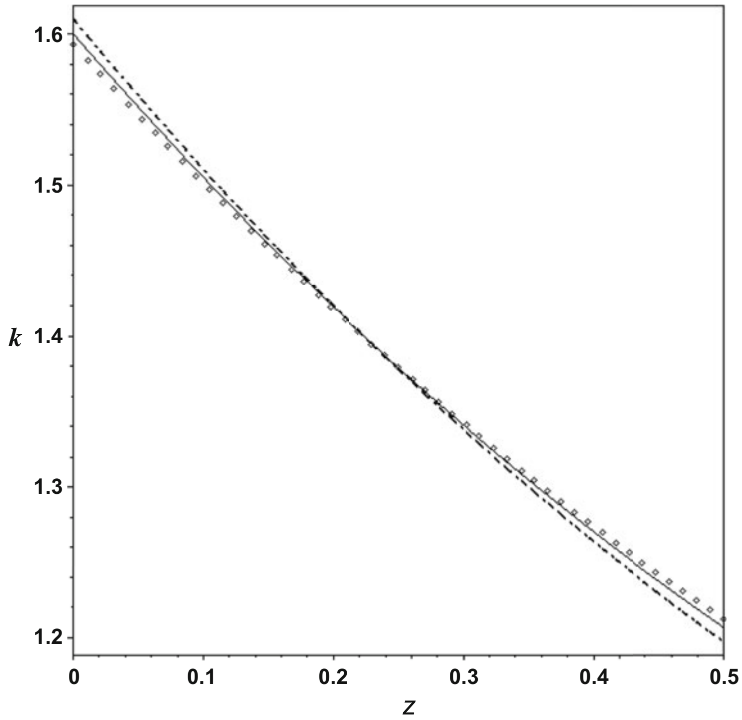

Fig. 3 The result of the reconstruction of a decreasing function $\bar{k}_1(z) = 0.6 + e^{-z}$

reconstruction error increases with decreasing thickness of the functionally graded layer. So, for $H_1 = 0.5$, the reconstruction error when using both approaches does not exceed 4%, however, for $H_1 = 0.1$ it increases to 15%, and for $H_1 < 0.03$ reconstruction becomes impossible. This is due to the fact that the solution of the inverse problem strongly depends on the sensitivity of the input information. With a decrease in the layer thickness, where the functions change most strongly with respect to the total thickness, the sensitivity of the input information decreases and, as a result, the reconstruction results deteriorate.

Using the reconstruction of the function $\bar{k}_1(z) = 0.6 + e^{-z}$ for $H_1 = 0.5$ as an example, we discussed the influence of noise of the input information on accuracy of reconstruction.

The noise of input information was modelled using the ratio:

$$f_\beta(\tau) = f(\tau)(1 + \beta\gamma), \tag{25}$$

where β is the noise value, γ is the random variable with a uniform distribution law on the segment $[-1, 1]$.

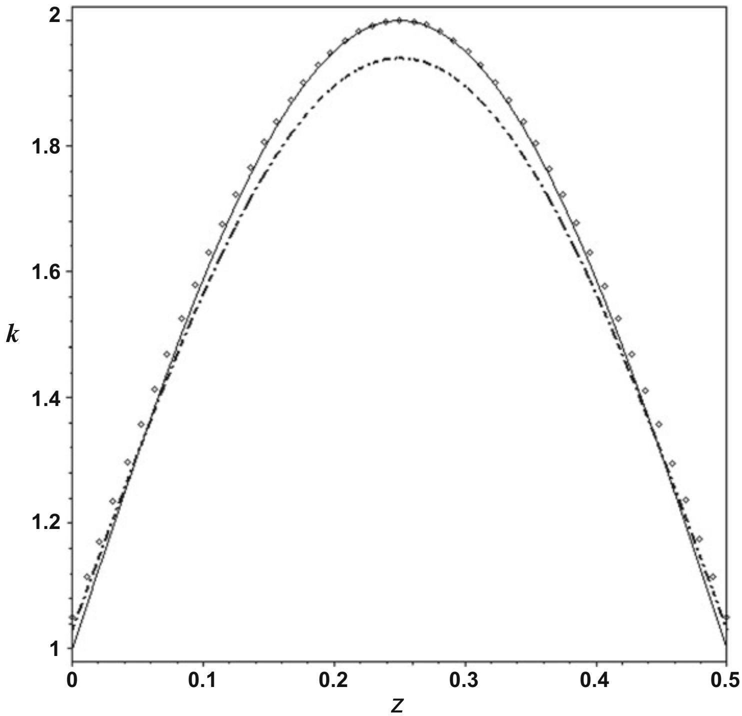

Fig. 4 The result of the reconstruction of nonmonotonic function $\bar{k}_1(z) = 1 + \sin(2\pi z)$

In the presence of noise, the error of reconstruction increased with growth β, and at 2% noise modelling the measurement error, it reached 7% in the case of the first approach, and 23% in the case of the second approach. Therefore, in comparison with the second approach, the reconstruction procedure based on the first approach was much more resistant to noise of input information.

7 Conclusions

The coefficient inverse heat conduction problem for a two-layer cylinder with a functionally graded layer is studied. The direct problem of thermal conductivity is solved on the basis of the Galerkin projection method and residue theory. To solve the inverse problem, two approaches are used. The first approach is the development of the previously developed iterative approach, at each step of which the Fredholm integral equations of the 1st kind are solved. The second approach is based on the algebraization of the direct problem.

It was found that the iterative approach provides high accuracy in the case of the reconstruction of both monotonic and nonmonotonic functions. The method of algebraization allows identification with high accuracy only for monotonic functions, but with much less machine time than the iterative approach. For nonmonotonic functions, the solution obtained by the algebraization method can serve as an initial approximation in the iterative process. Using the iterative approach, the maximum error in the reconstruction of specific heat occurred in the vicinity of $z = 0$, which is associated with the features of the kernel of the integral equation. The algebraization method is deprived of this drawback. However, the reconstruction procedure using the algebraization method turned out to be more sensitive to the noise of input information than using the iterative approach. It was found that with a decrease in the thickness of the functional gradient layer when using both approaches, the error in the reconstruction of thermophysical characteristics increases.

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A Study of the Waves Processes in Inhomogeneous Cylindrical Waveguides

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Abstract A number of problems on oscillations of a cylindrical waveguide, inhomogeneous in radial direction, in the absence and presence of annular delamination are investigated. A canonical system of the first-order differential equations with respect to the four functions is formulated. In the absence of the defect, a technique for calculating fields based on a combination of the Fourier integral transform and the analysis of the auxiliary Cauchy problems is presented. For a waveguide in the presence of the annular delamination, a scheme for constructing a system of integral equations with difference kernels with respect to the jumps of displacement vector components is presented by means of the analysis of a number of other auxiliary Cauchy problems. The structure of hypersingular kernels is studied, and an approach to solving the system of integral equations based on the boundary element method is proposed. The solution is analyzed depending on the number of elements used. The inverse problem on reconstructing the inhomogeneity laws for piecewise gradient functions is solved.

Keywords Cylindrical waveguide · Inhomogeneity · Delamination · Inverse problem · Fourier integral transform · System of integral equations · Boundary element method

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1 Introduction

Cylindrical structures that are non-uniform in the radial direction have been increasingly used in modelling of real objects (e.g., pipelines, vessels, etc.). In order to describe wave propagation in such structures at high frequencies in a proper way, one needs models taking into account laws of variation of elastic modules inside a body, as far as calculations for structures with averaged characteristics do not often provide correct information. In addition, determination of variation laws for inhomogeneity functions based on the wave fields that are measured at the boundary is a problem of certain importance.

Mathematical aspects of wave propagation in inhomogeneous waveguides require the study of the operator sheaf with two spectral parameters. For homogeneous cylindrical waveguides, dispersion relations are constructed in an analytical form through cylindrical functions and were studied in sufficient detail. One may only an asymptotic or numerical analysis for operators with variable properties. In [1], the authors present a brief review of dispersion characteristics of normal modes in an elastic slab and a cylinder. For 125-year history the key topics of the problem and its modern reflection in the global information space are elucidated. The problem on a thick radially inhomogeneous cylinder is solved in [2]. In [3], the oscillations of a cylinder which is inhomogeneous in the circumferential and radial directions are studied. A numerical method for calculating wave propagation in an infinite laminated cylinder is presented in [4]. In [5], the authors describe an approach to the analysis of forced oscillations of an inhomogeneous waveguide. The waveguide inhomogeneity is associated both with the variability of the elastic moduli and with the presence of variable fields of residual stresses of different structures. The results of these studies indicate that in the low-frequency region, where only one mode propagates (close to the rod's one), the velocities and wave fields are determined only by the average values of the elastic moduli in the cross section.

In problems associated with modelling of waves in bodies containing cracks, two approaches are most often used. The first one is historically related to the reduction of mixed problem to a system of integral equations for displacement jumps (usually with hypersingular kernels) and construction of their solutions either the systems of the second kind Fredholm integral equations with subsequent discretization, or using the direct boundary element method [6]. The second approach, which has been actively used in recent years, is based on the conjugation of analytical solutions for an infinite waveguide and finite element methods to study the vibrations of finite regions containing a defect [7, 8]; note that within such an approach, semi-infinite parts of the waveguide are usually assumed to be homogeneous. The advantages of this approach include the significant arbitrariness of the inhomogeneity and geometry of the finite part, however, the emerging problems of interfacing with analytical solutions can introduce a significant error into the solution.

An important aspect of the problems of inhomogeneous bodies oscillation is inverse identification of the inhomogeneity laws. The works [9, 10] present a study of finite-dimensional inverse problems by means of parametrizing the desired laws of inhomogeneity by a finite number of parameters, based on the use of the popular technique for minimizing the residual functional. An approach to identifying coefficients for elliptic boundary value problems is described, in particular when identifying variable Lame coefficients, the minimization problem is studied in a Banach space, and some stability estimates are given. For numerical research, finite element discretization in combination with regularizing approaches was used. The ability to evaluate rapidly changing or even discontinuous coefficients is demonstrated. The proposed method allows to determine efficiently the Lame parameters in a linear isotropic elastic body.

A plenty of inverse problems were solved by minimizing the residual functional, including those that determined constant characteristics [11, 12], where the elastic modules of an orthotropic composite were determined from additional data on the displacement field of the surface.

The work consists of three parts. The first part is devoted to the study of wave fields of inhomogeneous waveguide with annular cross-section in the case of an arbitrary law of moduli change. In the second part, wave fields at the boundary in the presence of annular delamination in a bounded region are studied, the system of integral equations for displacement jumps and its approximate solution are presented. The third part is concerned with simple inverse problems of inhomogeneity laws identification.

2 Problem for Inhomogeneous Waveguide

Let us consider waves in a cylindrical waveguide with annular cross section $a \leq r \leq b$, inhomogeneous in radial direction. The internal boundary of the waveguide is free from stresses; on the external boundary of the waveguide $r = b$, there is a concentrated normal load $\mu_0 \delta(z) \exp(-i\omega t)$ periodic in time with the frequency ω. We consider the steady-state mode of oscillations in the axisymmetric case, assuming that the components of the physical fields do not depend on ϕ.

Let us introduce the following ratios of dimensional and dimensionless parameters and functions: $u_r = bX_1$, $u_z = ibX_2$ are displacement vector components, $\sigma_r = \mu_0 X_3$, $\sigma_{rz} = i\mu_0 X_4$ are components of the Cauchy stress vector, $\lambda = \mu_0 g_1$, $\mu = \mu_0 g_2$ are the Lame parameters that depend on the radial coordinate, $g_1 + 2g_2 = G$ is auxiliary value, $\kappa = \sqrt{\rho_0 \omega^2 b^2 / \mu_0}$ is dimensionless frequency parameter, μ_0, ρ_0 are characteristic values of shear modulus and density.

Now let us apply the Fourier integral transform along the axial coordinate to the boundary value problem describing steady-state oscillations, entering dimensionless coordinates $r = xb$, $z = yb$.

$$\tilde{X}_n(x, \alpha) = \int_{-\infty}^{\infty} X_n(x, y) \exp(i\alpha y)\, dy, \quad n = 1..4 \tag{1}$$

Then with respect to the transforms we have an operator sheaf with two spectral parameters (κ, α) of the following form

$$\tilde{\mathbf{X}}' = \left(\mathbf{A}_0 - \kappa^2\mathbf{A}_{01} - \alpha\mathbf{A}_1 + \alpha^2\mathbf{A}_2\right)\tilde{\mathbf{X}}, \text{ where } \tilde{\mathbf{X}} = \left(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4\right) \tag{2}$$

where the sheaf matrices have the following nonzero components

$\mathbf{A}_0$: $a_{11}^0 = \dfrac{-g_1}{xG}$, $a_{13}^0 = \dfrac{1}{G}$ $a_{24}^0 = \dfrac{1}{g_2}$, $a_{31}^0 = \dfrac{G}{x^2} - \dfrac{g_1^2}{x^2G}$, $a_{33}^0 = -\dfrac{2g_2}{xG}$, $a_{44}^0 = -\dfrac{1}{x}$.
$\mathbf{A}_{01}$: $a_{31}^{01} = a_{42}^{01} = 1$.
$\mathbf{A}_1$: $a_{12}^1 = \dfrac{g_1}{G}$, $a_{21}^1 = -\dfrac{g_2}{g_2}$, $a_{32}^1 = -\dfrac{2g_1g_2}{xG}$, $a_{34}^1 = 1$, $a_{41}^1 = -\dfrac{2g_1g_2}{xG}$, $a_{43}^1 = -\dfrac{g_1}{G}$.
$\mathbf{A}_2$: $a_{42}^2 = G - \dfrac{g_1^2}{G}$.

The boundary conditions will have the form

$$\tilde{X}_k(\xi_0) = 0, \quad \tilde{X}_k(1) = -\delta_{3k} \tag{3}$$

where $\xi_0 = a/b$, $k = 3, 4$.

The problem (2)–(3) has a solution for any (κ, α), except for the points of the dispersion set, and can be solved by the shooting method. Note that construction of analytical solution in the general case is impossible due to significant variability of the matrix components $\mathbf{A}_0$, $\mathbf{A}_1$, $\mathbf{A}_2$. In Fig. 1, the first 4 branches of the real part of the dispersion set are shown for two sets of parameters:

$$1{:}\ g_1 = \begin{cases} 1.4155, & x < \xi_1 \\ 0.44, & x \geq \xi_1 \end{cases}, \quad g_2 = \begin{cases} 1, & x < \xi_1 \\ 0.44, & x \geq \xi_1 \end{cases}, \quad \xi_1 = 0.95;$$

$$2{:}\ g_1 = \begin{cases} 1.4155, & x < \xi_1 \\ 1.4155 - 0.9755\,(x - \xi_1)\,(1 - \xi_1)^{-1}, & x \geq \xi_1 \end{cases},$$

$$g_2 = \begin{cases} 1, & x < \xi_1 \\ 1 - 0.56\,(x - \xi_1)\,(1 - \xi_1)^{-1}, & x \geq \xi_1 \end{cases}, \quad \xi_1 = 0.9.$$

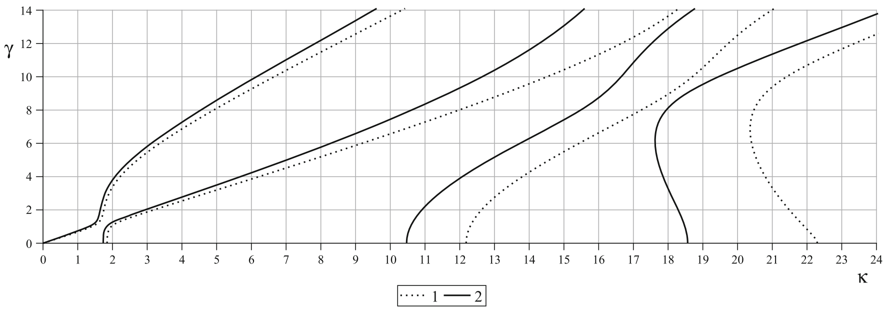

Fig. 1 Dispersion curves for different laws of inhomogeneity

Note that the inhomogeneity laws are chosen in such a way that their averaged values in cross-section coincide; dispersion curves show a weak sensibility to the inhomogeneity law in the low-frequency range and rather strong sensibility in the high-frequency range.

To study wave fields, it only remains to find the inverse Fourier transform by the formula

$$X_j(x, y) = \frac{1}{2\pi} \int\limits_{\Gamma} \tilde{X}_j(x, \alpha) \exp(-i\alpha y)\, d\alpha, \quad j = 1, 2, 3, 4 \tag{4}$$

where the integration is carried out along the contour Γ, which coincides with the real axis everywhere except for the real poles of the transform and envelopes them in accordance with the limit absorption principle; in the normal dispersion case (positive group velocity $\frac{\partial\kappa}{\partial\alpha} > 0$), positive poles are enveloped below, and the negative ones from above; in the abnormal dispersion case $\frac{\partial\kappa}{\partial\alpha} < 0$, the poles are enveloped in the opposite way.

We calculate the inverse Fourier integral transform for all the components of the wave field assuming $y > 0$. The integrands $\tilde{X}_j(x, \alpha)$ are meromorphic, possessing singularities of the real and complex poles type, which are determined by the dispersion equation, and can be found numerically with the aid of the scheme described above. We also note that with the growth of α, the solution of the boundary value problem (2)–(3) begins to possess a boundary layer structure, and the error of the solution found by the Runge-Kutta method grows.

To find the originals (4), let us consider the following contour integral

$$\begin{aligned} &\frac{1}{2\pi} \int\limits_{L} \tilde{X}_j(x, \alpha) \exp(-i\alpha y)\, d\alpha = \\ &= \frac{1}{2\pi} \int\limits_{L_0} \tilde{X}_j(x, \alpha) \exp(-i\alpha y)\, d\alpha + \frac{1}{2\pi} \int\limits_{L_R} \tilde{X}_j(x, \alpha) \exp(-i\alpha y)\, d\alpha \end{aligned} \tag{5}$$

Here, L is a closed contour in the complex plane of the parameter α consisting of two parts $L = L_0 \cup L_R$, where L_0 is the arc of the semicircle $\operatorname{Im}\alpha \le 0$ with the radius R and the centre at $\alpha = 0$, L_R is a part of the contour Γ, which is placed inside a circle of the radius R.

Applying the theory of residues, on the boundary $x = 1$ of the waveguide we obtain

$$\begin{aligned} &\frac{1}{2\pi} \int\limits_{L} \tilde{X}_j(1, \alpha) \exp(-i\alpha y)\, d\alpha = -i \sum_{n=1}^{m} \operatorname{Res}\Big(\tilde{X}_j(1, \alpha_n) \exp(-i\alpha_n y)\Big), \\ &j = 1, 2, 3, 4, \end{aligned} \tag{6}$$

where α_n are poles of the first order inside the contour L, and m is their number.

In accordance with the Jordan lemma, as $R \to \infty$, the integral along the contour L_0 tends to zero, the integral in L_R tends to the integral (4) on the contour Γ, $m \to \infty$. Thus, in accordance with (6), in the regular case it is necessary to sum the residues over all negative real poles and over all complex poles with negative imaginary part.

$$\tilde{X}_1^{(1)}(\xi_0) = 1,\ \tilde{X}_2^{(1)}(\xi_0) = 0,\ \tilde{X}_3^{(1)}(\xi_0) = \tilde{X}_4^{(1)}(\xi_0) = 0 \tag{7}$$

$$\tilde{X}_1^{(2)}(\xi_0) = 0, \tilde{X}_2^{(2)}(\xi_0) = 1,\ \tilde{X}_3^{(2)}(\xi_0) = \tilde{X}_4^{(2)}(\xi_0) = 0 \tag{8}$$

To find the residue at the pole α_n we use the following scheme. While searching for the solution of the problem by the shooting method in the form $\tilde{\mathbf{X}} = c_1\tilde{\mathbf{X}}^{(1)} + c_2\tilde{\mathbf{X}}^{(2)}$ where $\tilde{\mathbf{X}}^{(1)}$, $\tilde{\mathbf{X}}^{(2)}$ are solutions (2)–(7) and (2)–(8), and satisfying the boundary conditions for $x = 1$, we obtain

$$\mathbf{B}(1,\alpha)\,\mathbf{C}(\alpha) = \mathbf{F} \tag{9}$$

where $\mathbf{B}(1,\alpha) = \begin{pmatrix} \tilde{X}_3^{(1)}(1,\alpha) & \tilde{X}_3^{(2)}(1,\alpha) \\ \tilde{X}_4^{(1)}(1,\alpha) & \tilde{X}_4^{(2)}(1,\alpha) \end{pmatrix}$, $\mathbf{C}(\alpha) = \begin{pmatrix} c_1(\alpha) \\ c_2(\alpha) \end{pmatrix}$, $\mathbf{F} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Let us decompose the vector function $\mathbf{C}(\alpha)$ and the matrix-valued function $\mathbf{B}(1,\alpha)$ into the Laurent series in the neighborhood of the pole α_n

$$\mathbf{C}(\alpha) = \mathbf{C}_{-1}\frac{1}{\alpha - \alpha_n} + \mathbf{C}_0 + \ldots$$
$$\mathbf{B}(1,\alpha) = \mathbf{B}_0(1,\alpha_n) + \mathbf{B}_1(1,\alpha_n)(\alpha - \alpha_n) + \ldots,\ \text{where } \mathbf{B}_1(1,\alpha_n) = \left.\frac{\partial \mathbf{B}}{\partial \alpha}\right|_{\alpha=\alpha_n} \tag{10}$$

In accordance with the introduced expansions, the solution of the problem (2)–(3) can also be represented as a Laurent series. The coefficient of the Laurent series at $(\alpha - \alpha_n)^{-1}$: $\left[c_{-1}^1 \tilde{X}_j^{(1)}(1,\alpha_n) + c_{-1}^2 \tilde{X}_j^{(2)}(1,\alpha_n)\right]$ defines the residue at the first-order pole and allows one to switch from (6) to the following formula:

$$X_j(1,y) = -i\sum_{n=1}^{\infty}\left[c_{-1}^1(\alpha_n)\,\tilde{X}_j^{(1)}(1,\alpha_n) + c_{-1}^2(\alpha_n)\,\tilde{X}_j^{(2)}(1,\alpha_n)\right]\exp(-i\alpha_n y),$$
$$j = 1, 2, 3, 4 \tag{11}$$

To find the coefficients used in (11), we apply the scheme described in [5].

3 Problem for Inhomogeneous Waveguide with Annular Delamination

Assume that the waveguide under consideration contains annular delamination located on a cylindrical surface $x = \xi_1 \in (\xi_0, 1)$, $y \in [-l_0, l_0]$. The delamination edges do not interact with each other and therefore are free of stresses; hence, the displacements take a jump. The external normal load causing the propagation of waves in the waveguide is applied in the region $x = 1$, $y \in [l_1, l_2]$, the rest of the external boundary is free of stresses.

We perform the Fourier integral transform along the longitudinal coordinate in regions $S_1 = \{(x, y), \xi_0 \le x \le \xi_1, -\infty < y < \infty\}$, $S_2 = \{(x, y), \xi_1 \le x \le 1, -\infty < y < \infty\}$, provided that the displacement field at the conditional boundary $x = \xi_1$ is known, and each region presents a cylindrical waveguide with an annular cross section. The problem, similar the previous one, is reduced to the vector equation (2). We introduce a number of auxiliary Cauchy problems for the system (2), which differ in initial conditions, allowing to determine the following vectors $\tilde{\mathbf{X}}^{(1)}, \tilde{\mathbf{X}}^{(2)}, \tilde{\mathbf{X}}^{(3)}, \tilde{\mathbf{X}}^{(4)}$.

$$\tilde{X}_1^{(1)}(\xi_0) = 1, \tilde{X}_2^{(1)}(\xi_0) = 0, \tilde{X}_3^{(1)}(\xi_0) = \tilde{X}_4^{(1)}(\xi_0) = 0 \tag{12}$$

$$\tilde{X}_1^{(2)}(\xi_0) = 0, \tilde{X}_2^{(2)}(\xi_0) = 1, \tilde{X}_3^{(2)}(\xi_0) = \tilde{X}_4^{(2)}(\xi_0) = 0 \tag{13}$$

$$\tilde{X}_1^{(3)}(1) = 1, \tilde{X}_2^{(3)}(1) = 0, \tilde{X}_3^{(3)}(1) = \tilde{X}_4^{(3)}(1) = 0 \tag{14}$$

$$\tilde{X}_1^{(4)}(1) = 0, \tilde{X}_2^{(4)}(1) = 1, \tilde{X}_3^{(4)}(1) = \tilde{X}_4^{(4)}(1) = 0 \tag{15}$$

The vectors $\tilde{\mathbf{X}}^{(1)}, \tilde{\mathbf{X}}^{(2)}, \tilde{\mathbf{X}}^{(3)}, \tilde{\mathbf{X}}^{(4)}$ are the solutions of Eq. (2) with the conditions (12), (13), (14) and (15), respectively. For their numerical determination, the Runge-Kutta schemes of the 4 order were used.

We find the vectors $\tilde{\mathbf{X}}^{(1)}, \tilde{\mathbf{X}}^{(2)}$ in the region $S = S_1 \cup S_2$, compose their linear combination $\tilde{\mathbf{Z}}^{(0)} = q_1\tilde{\mathbf{X}}^{(1)} + q_2\tilde{\mathbf{X}}^{(2)}$. Then $\tilde{\mathbf{Z}}^{(0)}$ automatically satisfies the boundary conditions at the boundary $x = \xi_0$. We ensure that the following conditions are met at the border $x = 1$: $\tilde{Z}_3^{(0)}(1) = 1, \tilde{Z}_4^{(0)}(1) = 0$. To do this, we solve the system

$$\begin{aligned} q_1\tilde{X}_3^{(1)}(1) + q_2\tilde{X}_3^{(2)}(1) = 1 \\ q_1\tilde{X}_4^{(1)}(1) + q_2\tilde{X}_4^{(2)}(1) = 0 \end{aligned} \tag{16}$$

We compose the following linear combinations $\tilde{\mathbf{Z}}^{(1)} = p_1\tilde{\mathbf{X}}^{(1)} + p_2\tilde{\mathbf{X}}^{(2)} + p_3\tilde{\mathbf{X}}^{(3)} + p_4\tilde{\mathbf{X}}^{(4)}$, $\tilde{\mathbf{Z}}^{(2)} = d_1\tilde{\mathbf{X}}^{(1)} + d_2\tilde{\mathbf{X}}^{(2)} + d_3\tilde{\mathbf{X}}^{(3)} + d_4\tilde{\mathbf{X}}^{(4)}$, and assume that $\tilde{\mathbf{X}}^{(1)} = \tilde{\mathbf{X}}^{(2)} \equiv 0$ in S_2/S_1 and $\tilde{\mathbf{X}}^{(3)} = \tilde{\mathbf{X}}^{(4)} \equiv 0$ in S_1/S_2 . Then $\tilde{\mathbf{Z}}^{(1)}$ and $\tilde{\mathbf{Z}}^{(2)}$ satisfy the boundary conditions in the absence of stresses at $x = \xi_0$ and $x = 1$.

We provide a single jump in radial $\tilde{Z}_1^{(1)}$ and longitudinal $\tilde{Z}_2^{(2)}$ displacements at the boundary of regions S_1 and S_2 for both solutions $\tilde{\mathbf{Z}}^{(1)}$ and $\tilde{\mathbf{Z}}^{(2)}$, respectively. For this, we solve the following algebraic systems

$$\begin{cases} p_1 \tilde{X}_1^{(1)}(\xi_1) + p_2 \tilde{X}_1^{(2)}(\xi_1) - p_3 \tilde{X}_1^{(3)}(\xi_1) - p_4 \tilde{X}_1^{(4)}(\xi_1) = -1 \\ p_1 \tilde{X}_2^{(1)}(\xi_1) + p_2 \tilde{X}_2^{(2)}(\xi_1) - p_3 \tilde{X}_2^{(3)}(\xi_1) - p_4 \tilde{X}_2^{(4)}(\xi_1) = 0 \\ p_1 \tilde{X}_3^{(1)}(\xi_1) + p_2 \tilde{X}_3^{(2)}(\xi_1) - p_3 \tilde{X}_3^{(3)}(\xi_1) - p_4 \tilde{X}_3^{(4)}(\xi_1) = 0 \\ p_1 \tilde{X}_4^{(1)}(\xi_1) + p_2 \tilde{X}_4^{(2)}(\xi_1) - p_3 \tilde{X}_4^{(3)}(\xi_1) - p_4 \tilde{X}_4^{(4)}(\xi_1) = 0 \end{cases} \tag{17}$$

$$\begin{cases} d_1 \tilde{X}_1^{(1)}(\xi_1) + d_2 \tilde{X}_1^{(2)}(\xi_1) - d_3 \tilde{X}_1^{(3)}(\xi_1) - d_4 \tilde{X}_1^{(4)}(\xi_1) = 0 \\ d_1 \tilde{X}_2^{(1)}(\xi_1) + d_2 \tilde{X}_2^{(2)}(\xi_1) - d_3 \tilde{X}_2^{(3)}(\xi_1) - d_4 \tilde{X}_2^{(4)}(\xi_1) = -1 \\ d_1 \tilde{X}_3^{(1)}(\xi_1) + d_2 \tilde{X}_3^{(2)}(\xi_1) - d_3 \tilde{X}_3^{(3)}(\xi_1) - d_4 \tilde{X}_3^{(4)}(\xi_1) = 0 \\ d_1 \tilde{X}_4^{(1)}(\xi_1) + d_2 \tilde{X}_4^{(2)}(\xi_1) - d_3 \tilde{X}_4^{(3)}(\xi_1) - d_4 \tilde{X}_4^{(4)}(\xi_1) = 0 \end{cases} \tag{18}$$

The linear combinations formed above $\tilde{\mathbf{Z}}^{(0)}(x, \alpha)$, $\tilde{\mathbf{Z}}^{(1)}(x, \alpha)$, $\tilde{\mathbf{Z}}^{(2)}(x, \alpha)$ make it possible to build a solution of the problem in the space of transforms for the waveguide with delamination under the action of an external load of the following form

$$\tilde{\mathbf{Z}}(x, \alpha) = Q(\alpha)\, \tilde{\mathbf{Z}}^{(0)}(x, \alpha) + \tilde{\chi}_1(\alpha)\, \tilde{\mathbf{Z}}^{(1)}(x, \alpha) + \tilde{\chi}_2(\alpha)\, \tilde{\mathbf{Z}}^{(2)}(x, \alpha) \tag{19}$$

where $Q(\alpha) = \int\limits_{l_1}^{l_2} q(y)\, e^{i\alpha y} dy$ is the transform of the external normal load, $\tilde{\chi}_1(\alpha)$ and $\tilde{\chi}_2(\alpha)$ are the Fourier transforms from unknown jumps of radial and longitudinal displacements. A simple analysis showed that $\tilde{Z}_3^{(1)}(\alpha)$, $\tilde{Z}_4^{(2)}(\alpha)$ are even by α, $\tilde{Z}_4^{(1)}(\alpha)$, $\tilde{Z}_3^{(2)}(\alpha)$ are odd by α.

Further, to fulfill the conditions of zero stress vector on the delamination edges, it is necessary to find the fields in the actual space, carrying out the Fourier inverse transform

$$\mathbf{Z}(x, y) = \frac{1}{2\pi} \int\limits_{\Gamma} Q(\alpha)\, \tilde{\mathbf{Z}}^{(0)}(x, \alpha)\, e^{-i\alpha y} d\alpha + \frac{1}{2\pi} \int\limits_{\Gamma} \tilde{\chi}_1(\alpha)\, \tilde{\mathbf{Z}}^{(1)}(x, \alpha)\, e^{-i\alpha y} d\alpha + $$
$$+ \frac{1}{2\pi} \int\limits_{\Gamma} \tilde{\chi}_2(\alpha)\, \tilde{\mathbf{Z}}^{(2)}(x, \alpha)\, e^{-i\alpha y} d\alpha \tag{20}$$

$$\text{where } \tilde{\chi}_j(\alpha) = \int\limits_{-l_0}^{l_0} \chi_j(y)\, e^{i\alpha y} dy, \quad j = 1, 2 \tag{21}$$

In accordance with the boundary conditions, the components Z_3, Z_4 of the solution (20) vanish on the delamination edges. This condition is used further to formulate the operator equations allowing to find the unknown expansion functions $\chi_1(y)$, $\chi_2(y)$.

Taking into account the representation (21), we change the integration order in the second and third terms in (20) and obtain the following system of integral equations for the jumps

$$\int_{-l_0}^{l_0} \chi_j(\eta)\, k_{js}(\eta - y)\, d\eta = f_s(y),\ s = 3, 4,\ j = 1, 2,\ \ y \in [-l_0, l_0] \tag{22}$$

where $f_s(y) = -\frac{1}{2\pi} \int\limits_{\Gamma} Q(\alpha)\, \tilde{Z}_s^{(0)}(1, \alpha)\, e^{-i\alpha y} d\alpha$

We study the components of the introduced vector $k_{js}(t) = \frac{1}{2\pi} \int\limits_{\Gamma} \tilde{Z}_s^{(j)}(\xi_1, \alpha)$ $e^{i\alpha t} d\alpha$, $j = 1, 2$, $s = 3, 4$. Due to the fact that $\tilde{Z}_s^{(j)}(\xi_1, \alpha)$, $j = 1, 2$, $s = 3, 4$ are non-decreasing functions with the asymptotics $\tilde{Z}_s^{(j)}(\xi_1, \alpha) = C_{js}^{+} |\alpha| + C_{js}^{-} \alpha + O(1)$ for $\alpha \to \infty$, the integrals in the representation of the kernels are divergent and need to be given a meaning using the theory of generalized functions [13]. By isolating the principal components corresponding to the limiting values of the functions at infinity, it can be shown that the kernels are hypersingular, and the corresponding integrals are understood in the sense of the final Hadamard value [13].

The system of integral equations (22) for finding the expansion functions can be solved by the boundary element method similar to the scheme described in [14]. We divide the integrals over the interval $[-l_0, l_0]$ into the sum of the integrals over the elements $[-l_0, l_0] = \bigcup\limits_{p=1}^{N} \Delta_p$, where $\Delta_p = [-l_0 + (p-1)h, -l_0 + ph]$, $h = 2l_0 N^{-1}$; we also introduce the coordinates of the elements ends $\eta_p = -l_0 + (p-1)h$, and the collocation points $y_q = -l_0 + (q - 1/2)h$, taking $p = 1..N$, $s = 1..N$. We assume that the functions $\chi_1(\xi)$, $\chi_2(\xi)$ are constant on each element and introduce the notation $\chi_j\big|_{\Delta_p} = \chi_{jp}$. Considering that equations (22) are satisfied in a set of points $\{y_q\}$, we get the following relations

$$\int_{-l_0}^{l_0} \chi_j(\xi)\, k_{js}(\eta - y_q)\, d\eta = \sum_{p=1}^{N} \chi_{jp} \int\limits_{\Delta_p} k_{js}(\eta - y_q)\, d\eta = f_s(y_q) \tag{23}$$

which can be interpreted as an algebraic system with respect to the nodal values χ_{jp} of the expansion functions

$$\sum_{p=1}^{N} \chi_{jp} H_{pq}^{(js)} = f_{sq},\ j = 1, 2,\ s = 3, 4,\ p = 1..N,\ q = 1..2N \tag{24}$$

where the following notation is introduced for the coefficients of the system

$$H_{pq}^{(js)} = \frac{1}{2\pi}\left(\int\limits_{\Gamma} \frac{\tilde{Z}_s^{(j)}(\alpha)}{i\alpha} e^{i\alpha(\eta - y_q)} d\alpha\right)\Bigg|_{\eta_k}^{\eta_{k+1}} = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\tilde{Z}_s^{(j)}(\alpha)}{\alpha} E_{pq}(\alpha)\, d\alpha,$$

$$E_{pq}(\alpha) = \left[e^{i\alpha(\eta_{p+1} - y_q)} - e^{i\alpha(\eta_p - y_q)}\right],\ j = 1, 2,\ s = 3, 4.$$

The main difficulty in solving system (24) is to calculate its coefficients. To calculate them, we will single out the main components corresponding to the limit values of the symbols of the kernels at infinity, the remaining components are found numerically based on quadrature formulas. The solution of system (24) allows one to find the nodal values of the disclosure functions and eventually the displacement fields on the surface of the waveguide.

In numerical calculations, the following values of the parameters and variables were selected: $\kappa = 0.5$, $y_0 = 0.3$, $y_1 = -1$, $y_2 = -0.5$, $\xi_0 = 0.75$, $\xi_1 = 0.9$, $g_1 = 1.5 g_2 = 1.5 \begin{cases} 1, & x < \xi_1 \\ 10, & x \geq \xi_1 \end{cases}$. The selected frequency range corresponds to one propagating wave.

In Fig. 2, the expansion functions are given: the real part of the function $\chi_1(y)$ on the left and the imaginary part $\chi_2(y)$ on the right ($N = 600$). Under the study of the system solutions with increasing number of elements, numerical calculations showed the presence of internal convergence of the discretization procedure. Computational experiments conducted for different number of elements ($N = 30, 60, 90, 270$) indicate the convergence of the algorithm at $N \to \infty$. The results showed that the functions $\mathrm{Im}\,\chi_1(y)$, $\mathrm{Re}\,\chi_2(y)$ represent several orders of

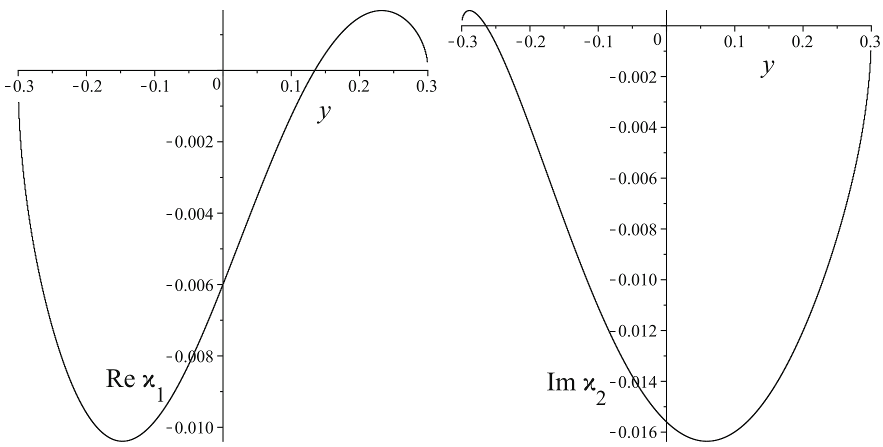

Fig. 2 $\chi_1(y)$, $\chi_2(y)$ for $N = 600$

magnitude smaller than $\mathrm{Re}\,\chi_1(y)$, $\mathrm{Im}\,\chi_2(y)$, and therefore their graphs are not given. Both functions change the sign once in the selected frequency range. The condition of zero opening functions at the delamination edges is quite accurate, despite a fairly simple approximation of the disclosure functions. According to the results of computational experiments, it is established that the laws of change in the opening functions depend on the external load application area. In particular, for the load located symmetrically relative to the centre of the delamination, the functions $\chi_1(y)$, $\chi_2(y)$ also possess a certain symmetry.

4 Inverse Problem

Now we state the inverse problem when it is required to find the complete vector of the unknowns (X_1, X_2, X_3, X_4), the transforms of which satisfy the boundary value problem (2)–(3) and functions $g_1(x)$, $g_2(x)$ according to the information on the radial displacement field at the outer boundary of the waveguide in a bounded region $y \in [y_1, y_2]$ far from the oscillation source

$$X_1(1, y) = \Omega(y), \quad y \in [y_1, y_2], \quad y_1 > 0 \tag{25}$$

We introduce the residual functional

$$J = \max|X_1(1, y) - \Omega(y)|, \quad y \in [y_1, y_2] \tag{26}$$

where in order to calculate $X_1(1, y)$ we use formula (11)

As additional information $\Omega(y)$, we calculate the field of radial displacements for functions describing a homogeneous inner layer and an inhomogeneous outer one

$$g_1 = \begin{cases} 1.5, & x < \xi_1 \\ 1.5 + 1.5(x - \xi_1)(1 - \xi_1)^{-1}, & x \geq \xi_1 \end{cases},$$

$$g_2 = \begin{cases} 1, & x < \xi_1 \\ 1 + 2(x - \xi_1)(1 - \xi_1)^{-1}, & x \geq \xi_1 \end{cases}, \xi_1 = 0.975, \xi_0 = 0.75$$

Let us minimize the residual functional in a fairly simple class of piecewise constant functions in the framework of the following parameterization containing two parameters F_1, F_2:

$$g_1 = \begin{cases} 1.5, & x < \xi_1 \\ \mathrm{F}_1, & x \geq \xi_1 \end{cases}, g_2 = \begin{cases} 1, & x < \xi_1 \\ \mathrm{F}_2, & x \geq \xi_1 \end{cases}.$$

These functions describe the laws of inhomogeneity variation in a two-layer waveguide: the inner and outer layers are considered to be homogeneous.

As a result of minimization of the functional J, one minimum was found that corresponds to the following parameter values $F_1 = 2.65$, $F_2 = 2.08$.

In the calculations, the following parameter values $c = 0.1$, $d = 1$, $\kappa = 15$ were used. The selected frequency parameter corresponds to three propagating modes.

5 Conclusion

A problem of steady-state forced oscillations for a hollow cylindrical inhomogeneous waveguide in the radial direction is solved on the basis of the integral Fourier transform and numerical analysis of auxiliary boundary value problems using the shooting method. The comparative calculations of the dispersion curves for a variety of different inhomogeneity laws of a waveguide are conducted. For a waveguide with an annular delamination, a system of integral equations for radial and longitudinal displacement jumps is formulated. The system is reduced to a linear algebraic system on the basis of ideology of the boundary element method. The opening displacement functions are constructed; the accuracy of their determination depending on the number of boundary elements used is analyzed. A simple inverse problem on reconstruction of two parameters characterizing laws of material inhomogeneity in a two-layer waveguide is solved.

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Solution of a Class of First-Order Quasilinear Partial Differential Equations

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Abstract A method for constructing a solution for some systems of first-order quasilinear partial differential equations is presented. The type of equations can either be hyperbolic or elliptic. The method is based on the application of the generalized hodograph method, which allows us to write the solution in an implicit form. There is a system of first-order linear partial differential equations that is used for commuting flows in the generalized hodograph method. We discover an analogy between the commuting flows and divided differences for the Hermite polynomial. This analogy allows us to obtain an explicit representation for commuting flows. The introduction of new (Lagrangian) variables, which are conserved on the characteristics of the original system, suggests a way to transform the solution of the Cauchy problem for first-order quasilinear partial differential equations to the solution of the Cauchy problem for ordinary differential equations. Numerical, and in some cases analytical, integration of the Cauchy problem makes it possible to construct explicit solutions of the problem on the level lines (isochrons) of the implicit solution. The method proposed is significantly different from the grid method, finite element method, finite volume method, and, in fact, is more precise. The error of the solution can arise only at the last stage in the numerical integration of the Cauchy problem for ordinary differential equations. Moreover, the method allows us to obtain multivalued solutions, in particular, to study the process of

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wave breaking in hyperbolic systems. Particular cases of the equations considered here describe diffusion-free approximation in a wide range of mass transport processes in multicomponent mixtures, such as electrophoresis, chromatography, centrifugation. As a simple example, the solution of the electrophoresis problem (separation multicomponent mixture to individual component) is presented.

Keywords Generalized hodograph method · Commuting flows · Divided difference

Mathematical Subject Classification (2000) 35L45, 35L40, 35L65

1 Introduction

PhD thesis [1], in particular, developed the theory of solutions of the Cauchy problem for ε-systems, some class of the hydrodynamic type systems (first-order quasilinear equations). For such systems the implicit solution is constructed with the help of the generalized hodograph method (see, for example, [2]). In particular, the class of ε-systems includes equations written in the Riemann invariants (diagonalized) which have the form

$$\partial_t R^k + \lambda^k(\boldsymbol{R})\partial_x R^k = 0, \quad \boldsymbol{R} = (R^1, \dots, R^n), \quad k = 1, \dots, n.$$

$$\lambda^k = R^k \left(\prod_{i=1}^{n} R^i \right)^{\varepsilon}.$$

Here, R^k are the Riemann invariants (the superscript corresponds to the number of the invariant, not to its degree), λ^k are the characteristic directions.

The generalized hodograph method allows us to write the solution in an implicit form

$$x - \lambda^i(\boldsymbol{R})t = w^i(\boldsymbol{R}), \quad i = 1, \dots, n,$$

where $w^i(\boldsymbol{R})$ are the commuting flows which satisfy to some system of the linear partial differential equations [2]

$$\frac{\partial_i w^k}{w^i - w^k} = \frac{\partial_j \lambda^k}{\lambda^j - \lambda^k}, \quad i \neq j \neq k, \quad \partial_i = \frac{\partial}{\partial R^i}.$$

In general case, the solution of this system is impossible to obtain. However, if the relations

$$\frac{\partial_k \lambda^i}{\lambda^k - \lambda^i} = \frac{\varepsilon}{R^i - R^k}, \quad i \neq k,$$

which are the definition of ε-system, are satisfied, then in the case of integers $\varepsilon = 1, 2 \ldots$ the commuting flows w^k have the form

$$w^k = \lambda^k \frac{\partial H}{\partial R^k},$$

$$H(\boldsymbol{R}) = \frac{1}{(m-1)!} \sum_{j=1}^{n} \frac{\partial^{m-1}}{\partial (R^j)^{m-1}} \frac{A^j(R^j)}{\prod\limits_{s=1, s\neq j}^{n} (R^j - R^s)^m}, \quad m = \varepsilon.$$

where $A^j(R^j)$ are the arbitrary functions defined by initial conditions of the original equations.

The main objective of the paper is to obtain the explicit formulas for functions $A^j(R^j)$ in the case $m \geqslant 1$. This result generalizes the relations obtained in [3, 4] when $m = 1$. In addition, a method for solving the Cauchy problem which allows us to construct an explicit solution, at least for isolines of the implicit solution, is presented. For the electrophoresis problem ($m = 1$) this algorithm was proposed in [4]. Note that systems of quasilinear equations can be of both hyperbolic and elliptic types. For the elliptic type equations the method of constructing solutions for the case $m = 2$, $m = 1$ is presented in [5].

The paper is organized as follows. In Sect. 2 the Cauchy problem for the quasilinear equations is formulated. In Sect. 3 for constructing the function $H(\boldsymbol{R})$ we use an analogy between function $H(\boldsymbol{R})$ and divided difference for some Hermite polynomial. On the basis of initial data for Cauchy problem we obtain functions $A^k(R^k)$. In fact, in Sect. 3 an implicit solution of the original Cauchy problem is presented. In Sect. 4 we introduce the Lagrangian variables and present the analytical-numerical method for constructing explicit solutions of the original problem. Section 5 contains an example of the solution for the problem of the mass transfer by an electric field.

2 Cauchy Problem

We consider the Cauchy problem for a system of quasilinear equations

$$\partial_t R^k + \lambda^k(\boldsymbol{R}) \partial_x R^k = 0, \tag{1}$$

$$\lambda^k = R^k \left(\prod_{j=1}^{n} R^j \right)^m, \quad k = 1, \ldots, n, \quad m = 1, 2, \ldots$$

$$R^k(x, 0) = R_0^k(x), \quad k = 1, \ldots, n, \tag{2}$$

where $R_0^k(x)$ are given functions.

Implicit solution of the system (1) can be written as [2]

$$x - \lambda^i(\boldsymbol{R})t = w^i(\boldsymbol{R}), \quad i = 1, \dots, n, \tag{3}$$

where $w^i(\boldsymbol{R})$ are commuting flows which have the form [1]

$$w^k = \lambda^k \frac{\partial H}{\partial R^k}, \tag{4}$$

$$H(\boldsymbol{R}) = \frac{1}{(m-1)!} \sum_{j=1}^{n} \frac{\partial^{m-1}}{\partial (R^j)^{m-1}} \frac{A^j(R^j)}{\prod\limits_{\substack{s=1\\ s\neq j}}^{n} (R^j - R^s)^m}. \tag{5}$$

Here, $A^j(R^j)$ are the arbitrary functions.

To determine the functions $A^k(r^k)$ we assume $t = 0$ in (3). Taking into account (4), (5) we obtain the first-order linear partial differential equation system

$$\frac{x}{r^k \prod\limits_{s=1}^{n} (r^s)^m} = \frac{\partial H}{\partial r^k}, \quad k = 1, \dots, n, \tag{6}$$

$$H = \frac{1}{(m-1)!} \sum_{k=1}^{n} \frac{\partial^{m-1}}{\partial (r^k)^{m-1}} \frac{A^k(r^k)}{\prod\limits_{\substack{s=1\\ s\neq k}}^{n} (r^k - r^s)^m}.$$

Here,

$$r^k = R_0^k(x), \quad \frac{\partial}{\partial r^k} = \left.\frac{\partial}{\partial R^k}\right|_{t=0}.$$

Multiplying each equation (6) by $dr^k(x)/dx$, summing and integrating we get

$$\sum_{k=1}^{n} \frac{x}{r^k \prod\limits_{s=1}^{n} (r^s)^m} \frac{dr^k}{dx} = \frac{dH}{dx}, \tag{7}$$

$$H = -\frac{x}{m \prod\limits_{s=1}^{n} (r^s(x))^m} + \frac{1}{m} \int\limits_0^x \frac{d\tau}{\prod\limits_{s=1}^{n} (r^s(\tau))^m}.$$

3 Analogy Between Solution and Divided Difference

To solve the system (6) we use the analogy between the function H and divided differences for Hermite polynomial with multiple points. We recall the formula for divided difference (see, e.g., [6, formula (57)])

$$F(\underbrace{z_1,\ldots,z_1}_{\alpha_1},\ldots,\underbrace{z_k,\ldots,z_k}_{\alpha_k},\ldots,\underbrace{z_n,\ldots,z_n}_{\alpha_n}) = \tag{8}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{\alpha_i-1}\frac{1}{(\alpha_i-1)!}\frac{d^{\alpha_j-1}}{dz^{\alpha_j-1}}\left[\frac{F(z)}{\prod\limits_{s\neq i}(z-z_s)^m}\right]_{z=z_i}.$$

We construct the Hermite interpolation polynomial for function $F(r)$ using m-fold points r^k, $k=1,\ldots,n$, and 1-fold point $r^{n+1}=0$. We assume that at each point r^k the values of the function $rA(r)$ and its derivatives are given as

$$\left.\frac{d^iF(r)}{d(r)^i}\right|_{r=r^k} = \left.\frac{d^i(rA(r))}{d(r)^i}\right|_{r=r^k}, \quad k=1,\ldots,n, \quad i=1,\ldots,m. \tag{9}$$

Then the divided difference constructed on all points r^k, $k=1,\ldots,n+1$ can be written in the following form

$$F(r^1,\ldots,r^1,\ldots,r^n,\ldots,r^n,r^{n+1}) = \tag{10}$$

$$= \frac{m}{(m-1)!}\sum_{k=1}^{n}\frac{\partial^{m-1}}{\partial(r^k)^{m-1}}\frac{A^k(r^k)}{\prod\limits_{s\neq i}(r^k-r^s)^m} + (-1)^n\frac{F(0)}{\prod\limits_{s=1}^{n}(r^s)^m}.$$

The first summand in the right part of (10) coincides with the function H defined by the formula (6). If we select $F(0)$ as

$$F(0) = (-1)^n x, \tag{11}$$

then the divided difference (10) coincides with (7).

Hence, we have

$$F(r^1,\ldots,r^1,\ldots,r^n,\ldots,r^n,r^{n+1}) = \int_0^x \frac{d\tau}{\prod\limits_{s=1}^{n}(r^s(\tau))^m}. \tag{12}$$

Next, we assume that the function $F(z)$ is a polynomial which naturally coincides with the interpolation Hermite polynomial

$$F(z) = \cdots + F(r^1, \ldots, r^1, \ldots, r^n, \ldots, r^n, r^{n+1}) \prod_{s=1}^{n} (z - r^s)^m =$$
$$= \sum_{k=0}^{nm} (-1)^{k+n} F_k(x) z^k.$$

Here F_k are the polynomial coefficients.

In particular, taking into account (12) we get

$$F_{nm}(x) = \int_0^x \frac{d\tau}{\prod_{s=1}^{n} (r^s(\tau))^m}, \qquad F_0 = F(0) = (-1)^n x.$$

All the other polynomial coefficients can be written with the help of the formula (8) (see for example [4], where the coefficients are given in the case $m = 1$).

Thus, there is a polynomial $F(z)$, which due to the choice of its values in the interpolation points (9) is associated with system (6) for the determination of $A^k(r^k)$. The relations (6) can be interpreted as the definition of the function $x = x(r^1, \ldots, r^n)$.

To obtain the functions $A^k(r^k)$ we use formula (9)

$$r^k A^k(r^k) = F(r^k) = \sum_{i=0}^{nm} (-1)^{i+n} F_i(x(r^1, \ldots, r^n))(r^k)^i.$$

However, A^k depends only on r^k. In other words, the explicit dependence on the variable x for functions $A^k(r^k)$ must be absent, i.e. *partial derivative* of x must be zero

$$\frac{\partial}{\partial x} \left(r^k A^k(r^k) \right) = \sum_{i=0}^{nm} (-1)^{i+n} \frac{dF_i(x)}{dx} (r^k(x))^i \equiv 0, \quad k = 1, \ldots, n. \tag{13}$$

If we interpret r^k as polynomial roots, then the relation (13) is valid only in the case when the polynomial coefficients are its invariants which depends on roots of the polynomial

$$\frac{dF_k(x)}{dx} = I_k(x), \quad k = 0, \ldots, nm,$$

where

$$I_0(x) = 1, \quad I_1(x) = m \sum_{k=1}^{n} \frac{1}{r^k(x)}, \dots$$

$$I_{nm-1}(x) = \frac{m \sum\limits_{k=1}^{n} r^k(x)}{\prod\limits_{s=1}^{n} (r^s(x))^m}, \quad I_{nm}(x) = \frac{1}{\prod\limits_{s=1}^{n} (r^s(x))^m}.$$

Note that for the convenience of computing polynomial invariants with m-fold roots one can use 'splitting' of the roots r^k, $r^k + \theta, \dots r^k + (m-1)\theta$, compute the usual invariants of the polynomial, and then pass to the limit at $\theta \to 0$.

Finally, the relations for $A^k(r^k)$ have the following form

$$A^k(r^k(x)) = \frac{1}{r^k(x)} \sum_{s=0}^{nm} (-1)^{s+n} (r^k(x))^s \int\limits_0^x I_s(\tau)\, d\tau. \tag{14}$$

We obtain the solution of the Cauchy problem (1), (2) in the implicit form (3), (4), (5) using the changes of variables $r^k(x) = R_0^k(x)$, $x = (r^k(x))^{-1}$ in the formulas (14).

$$A^k(R_0^k(x)) = \frac{1}{R_0^k(x)} \sum_{s=0}^{nm} (-1)^{s+n} (R_0^k(x))^s \int\limits_0^{(R_0^k(x))^{-1}} I_s(\tau)\, d\tau.$$

Here, $(R_0^k(x))^{-1}$ is the inverse function.

Certainly, the presence of the inverse function imposes significant restrictions on the initial data of the Cauchy problem.

4 Lagrangian Variables

To remove restrictions on the initial data (the existence of inverse functions) we look for an explicit solution of the Cauchy problem (1), (2) in the following form (see [3])

$$R^k = R_0^k(a^k) \equiv r^k \equiv r^k(a^k), \quad k = 1, \dots, n,$$

where a^k are the Lagrangian variables which satisfy the equations

$$a_t^k + \lambda^k(\boldsymbol{a}) a_x^k = 0, \quad \boldsymbol{a} = (a^1, \dots, a^n), \quad a^k(x, 0) = x, \quad k = 1, \dots, n.$$

For new variables the characteristic directions λ^k and the hodograph method relations (3) have the following form

$$\lambda^k(\boldsymbol{a}) = \lambda^k(\boldsymbol{r}(\boldsymbol{a})), \quad x - \lambda^k(\boldsymbol{r}(\boldsymbol{a}))t = w^k(\boldsymbol{r}(\boldsymbol{a})), \quad k = 1, \dots, n. \tag{15}$$

Taking into account (15), (4), (5) we construct functions $t_k(\boldsymbol{a})$

$$t_k(\boldsymbol{a}) \equiv \frac{w^{k+1} - w^k}{\lambda^k - \lambda^{k+1}}, \quad k = 1, \dots, n-1.$$

We parametrize the level lines of functions $t_k(\boldsymbol{a})$ (isochron $t_k(\boldsymbol{a}) = t_*$). Assuming that $a^k = a^k(\mu)$ holds on these lines we obtain the equation system

$$t_* = t_k(\boldsymbol{a}(\mu)), \quad k = 1, \dots, N-1.$$

Differentiation of these equations by the parameter μ allows us to obtain the Cauchy problem for the system of ordinary differential equations

$$\frac{da^i}{d\mu} = \varphi^i(\boldsymbol{a}), \quad a^i(\mu_*) = a^i_*, \quad i = 1, \dots, n, \tag{16}$$

where a_i^* is the value of a_i at some point of isochron $\mu = \mu_*$, $\varphi^i(\boldsymbol{a})$ is the eigenvector of the matrix $\partial t_k/\partial a^i$ (an arbitrary multiplier that arises in the determination of eigenvectors can be included in the parameter μ).

Integrating the Cauchy problem (16) allows us to get the solution of the problem on isochron

$$R^k(x, t_*) = R^k_0(a^k(\mu)), \quad k = 1, \dots, n.$$

Integrating the Cauchy problem we choose the function $A^k(r^k)$ in the form (cf. with (14), see also [3])

$$A^k(r^k(a^k)) = \frac{1}{r^k(a^k)} \sum_{s=0}^{nm} (-1)^{s+n} (r^k(a^k))^s \int\limits_0^{a^k} I_s(\tau)\, d\tau.$$

Thus, the introduction of Lagrangian variables allows us to remove restrictions for initial data of the Cauchy problem. In particular, one can construct solutions described by the multivalued functions.

5 Mass Transport by an Electric Field

An example of an applied problem, for which the theory developed in the work is applicable, is the description of mass transport by an electric field, in particular, by electrophoresis. The system of the first-order quasilinear hyperbolic equations (hyperbolic conservation laws), describing diffusion-less approximation of mass transport in multicomponent mixtures, such as electrophoresis, chromatography, centrifugation, sedimentation, in dimensionless variables in the spatially one-dimensional case has the form (see, e.g. [5, 7])

$$\partial_t u^k + \partial_x(\mu^k u^k E) = 0, \quad k = 1, \dots, n, \tag{17}$$

where $u^k(x,t)$ are the concentrations of mixture components, μ^k are the component velocities, $E(u^1, \dots, u^n)$ is the intensity of the external field.

A characteristic feature of the system (17) is the dependence of the external field intensity E on the 'collective' interaction component

$$E = \frac{1}{1+s}, \quad s = \sum_{k=1}^{n} u^k. \tag{18}$$

System (17), (18) can by written in the Riemann invariants R^k

$$\partial_t R^k + \lambda^k(\boldsymbol{R})\partial_x R^k = 0, \quad \boldsymbol{R} = (R^1, \dots, R^n), \tag{19}$$

$$\lambda^k = R^k \prod_{i=1}^{n} R^i, \quad k = 1, \dots, n.$$

The dependence $R^k(u^1, \dots, u^n)$ and the inverse dependence $u^k(R^1, \dots, R^n)$ are defined by the roots of the polynomial $L(R)$ and relations [5, 7, 8]

$$L(R) \equiv \prod_{k=1}^{n}(\mu^k - R) - R\sum_{j=1}^{n} u^j \prod_{\substack{k=1\\k\neq j}}^{n}(\mu^k - R),$$

$$u^s = \frac{\prod\limits_{k=1}^{n} \mu^k \prod\limits_{k=1}^{n}(\mu^s - R^k)}{\mu^s \prod\limits_{k=1}^{n} R^k \prod\limits_{\substack{k=1\\k\neq s}}^{n}(\mu^s - \mu^k)}.$$

It is obvious that system (19) coincides with (1) at $m = 1$. For a detailed study of this case with arbitrary n, see in [4]. At $n = 2$ quasilinear system of equations (17), (18) (including the case of elliptic type equations) are studied in [5].

To illustrate the method proposed we consider the simplest version of the separation problem for a binary mixture. To calculate the function $t_1(\boldsymbol{a})$ and commuting flows w^1, w^2 we have the following relations

$$t_1(a^1, a^2) = \frac{w^1(\boldsymbol{r}) - w^2(\boldsymbol{r})}{(r^2 - r^1)r^1 r^2}, \quad \boldsymbol{r} = (r^1, r^2).$$

For convenience we introduce the notation

$$P_1 = \frac{1}{r^2 - r^1}, \quad P_2 = \frac{1}{r^1 - r^2}.$$

$$F_{k0}(a^k) = \int_{a_*^k}^{a^k} \left(\frac{1}{R_0^1(z)} + \frac{1}{R_0^2(z)} \right) dz, \quad k = 1, 2.$$

$$F_{k1}(a^k) = \int_{a_*^k}^{a^k} \frac{dz}{R_0^1(z) R_0^2(z)}, \quad k = 1, 2.$$

Then, the functions A^1, A^2 and their derivatives can be written as

$$A^k(a^k) = -\frac{a^k}{r^k} + F_{k0} - r^k F_{k1}, \quad k = 1, 2,$$

$$\frac{dA^k(a^k)}{dr^k} = \frac{a^k}{(r^k)^2} - F_{k1}, \quad k = 1, 2.$$

Calculating the derivatives of the function H we obtain

$$\frac{\partial H}{\partial r^1} = P_1 \left(\frac{dA^1}{dr^1} + \frac{A^1}{r^2 - r^1} \right) - \frac{A^2}{r^1 - r^2} P_2,$$

$$\frac{\partial H}{\partial r^2} = P_2 \left(\frac{dA_2}{dr^2} + \frac{A^2}{r^1 - r^2} \right) - \frac{A^1}{r^2 - r^1} P_1.$$

Commuting flows have the following form

$$w^k = r^k r^1 r^2 \frac{\partial H}{\partial r^k}, \quad k = 1, 2.$$

In case $n = 2$ the Cauchy problem (16) can be written as

$$\frac{da^1}{d\mu} = \frac{\partial t_1}{\partial a^2}, \quad \frac{da^2}{d\mu} = -\frac{\partial t_1}{\partial a^1}, \qquad a^1(\mu_*) = a_*^1, \qquad a^2(\mu_*) = a_*^2.$$

Calculating functions $a^1(\mu)$, $a^2(\mu)$ we get an explicit solution of the original problem on isochron

$$R^1(x, t_*) = R_0^1(a^1(\mu)), \quad R^2(x, t_*) = R_0^2(a^2(\mu)).$$

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