Investigation 9. Tracing the Change in Discourse in a Collaborative Dynamic-Geometry Environment: From Visual to More Mathematical

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Abstract

This case study investigated the development of group cognition by tracing the change in mathematical discourse of a team of three middle-school students as they worked on a construction problem within a virtual collaborative dynamic geometry environment. Sfard's commognitive framework was employed to examine how the student team's word choice, use of visual mediators, and adoption of geometric construction routines changed character during an hour-long collaborative problem-solving session. The findings indicated that the team gradually moved from a visual discourse toward a more formal discourse—one that is primarily characterized by a routine of constructing geometric dependencies. This significant shift in mathematical discourse was accomplished in a CSCL setting where tools to support peer collaboration and pedagogy are developed through cycles of design-based research. The analysis of how this discourse development took place at the group level has implications for the theory and practice of computer-supported collaborative mathematical learning. Discussion of which features of the specific setting proved effective and which were problematic suggests revisions in the design of the setting.

Keywords

Mathematical discourse development · Mathematical routines · Group cognition · Collaborative dynamic geometry · Dependencies

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Introduction

Documenting processes by which learning takes place in collaborative settings has been one of the most important research agendas for CSCL researchers. This endeavor is even more challenging in the context of learning geometry, which has been considered a classic example of individual intellectual development (Stahl, 2016). Shifting the focus from individual cognition to group cognition, this study examines the development of a group of students' geometrical thinking in the Virtual Math Teams (VMT) environment (Stahl, 2009). VMT is an open-source, virtual, collaborative learning setting that affords synchronous text-based interaction (chat) with an embedded multi-user dynamic geometry application, GeoGebra (www.GeoGebra.org). VMT is regarded as the first sustained effort supporting a collaborative form of dynamic geometry (Stahl, 2013a).

Learning within a dynamic geometry environment (DGE) is indicated by the ability to construct figures, which marks the transition toward formal mathematics. There is a crucial distinction between *drawing* and *construction* within a DGE. Drawing refers to the juxtaposition of geometrical objects that *look* like some intended figure (Hoyles & Jones, 1998). Construction, however, depends on creating theory-based relationships, in other words *dependencies* (Stahl, 2013a), among the elements of a figure. Once relationships are constructed accordingly, the dynamic figure maintains these theoretical relationships even under dragging.

The transition from visual to formal mathematics is, however, neither straightforward nor easy for students working with dynamic geometry (Jones, 2000; Marrades & Gutierrez, 2000). Students often think that it is possible to construct a geometric figure based on visual cues (Laborde, 2004), although constructing dynamic geometry figures requires defining dependencies. Corresponding to this contrast, one can distinguish between two different *mathematical* discourses (Sfard, 2008) in which students may engage when working within DGEs. Within one of these, students may talk about geometrical figures as if they are merely visually perceptible entities without making any connections between them and the theoretical relationships they signify. When presented with a geometry construction problem, students might adopt a solution *routine* (Sfard, 2008) that is based on visual placement and verification, which produces a *drawing* (Hoyles & Jones, 1998). Taking a more sophisticated mathematical discourse, however, they would frame the problem as *construction*, that is, one that involves establishing dependencies.

Sfard (2008) argues that such a discursive jump to more sophisticated discourses takes place "while participating in the discourse with more experienced interlocutors" (p. 191). However, this study will show that participation within a well-designed collaborative learning setting, such as VMT, can also help students move forward from visual toward more formal ways of dealing with construction problems. That is, interacting with expert interlocutors (e.g., teachers) may not be the only path toward advancing one's mathematical discourse. This process may also take place within a virtual collaborative setting where feedback from dynamic geometry software, collaboration with peers, and guidance from task instructions collectively fulfill a role similar to that of the discourse of experts.

Constructing Dependencies with Dynamic Geometry

In geometry, entering the theoretical domain is challenging given that students need to deal with the double role that diagrams play. On the one hand, diagrams refer to *theoretical* properties of geometrical objects and their relations. On the other hand, they are *spatio-graphical* figures that are immediately accessible through *visual* perception (Laborde, 2004). These two worlds come in close contact in DGEs. When one uses theory to *construct* a geometrical object, theoretical relationships are preserved even when the elements of the construction are visually altered through dragging. That is, spatio-graphical aspects of the construction keep reflecting invariant theoretical properties dynamically. For instance, when one properly constructs two line segments to be perpendicular bisectors of each other, not only will the segments look and measure as though they bisect each other at 90°, but they will remain so even if the points of the construction are dragged into other positions. Within a DGE, in order to construct a perpendicular bisector, one needs to create *dependencies* by defining the theoretical relationships that determine perpendicularity. The counterpart of the classical Euclidean compass-and-straightedge construction within a DGE makes uses of circle and line software tools, which can, for instance, create a rhombus whose diagonals bisect at right angles. In that way, dynamic geometry constructions provide a computer-based context in which the connections between spatio-graphical and theoretical worlds are maintained.

Although dynamic geometry affords unique possibilities for learning geometry, there have been concerns regarding the nature of mathematical truth that students may be deriving when working in DGEs (Chazan, 1993a; Hadas, Hershkowitz, & Schwarz, 2000; Hoyles & Jones, 1998). Some researchers and teachers worry that when students can easily generate empirical evidence, the need and motivation for formal explanations may vanish. More fundamentally, students may not make the transition toward the theoretical aspects of geometry (Marrades & Gutierrez, 2000) and build the connection between spatio-graphical and theoretical worlds that is an essential aspect of meaning in geometry (Laborde, 2004). Learners may become stuck in the transition area between a visually produced solution and the underlying theoretical relationships (Hölzl, 1995).

On the other hand, it can be argued that focusing on constructing dependencies may help students move toward noticing relevant mathematical relationships (Jones, 2000). Dynamic geometry constructions are associated with formal geometry because created dependencies can correspond to elements of a mathematical proof (Stahl, 2013a). One starts with creating dependencies as if listing the givens in a mathematical proof task. These built-in relationships in turn constrain the elements of a figure in certain ways that lead to further relationships, which reflect the ideas underlying a corresponding explanatory proof.

Some researchers stress the differences between Euclidean geometry and dynamic geometry. For instance, Hölzl (1996) argues that dynamic geometry software imposes a hierarchy of dependencies that alters the relational character of geometric objects. He states that a distinction arises between free points (that can be dragged) and restricted points (such as intersections), which may not be geometrical or necessary in a paper-and-pencil environment. This is not surprising given that Euclidean geometry and dynamic geometry rely on "qualitatively different technologies" (Shaffer & Kaput, 1999). Despite the lack of complete congruence between the two, many researchers believe that explicitly stating the steps of a dynamic geometry construction can break down the separation between deduction and construction (Chazan & Yerushalmy, 1998; Hoyles & Jones, 1998; Stahl, 2013a), that is, well-designed DGEs may be able to help students to transition toward formal mathematics.

Constructions are also taken as a form of *mathematization* (Gattegno, 1988; Treffers, 1987; Wheeler, 1982) by Jones (2000), who defined the term for elementary-school geometry using dynamic geometry software. When mathematizing,

> students can be said to be involved in modeling the geometrical situation using the tools available in the software. This involves setting up a construction and seeing if it is appropriate, and quite probably having to adjust the construction to fit the specification of the problem. (p. 62)

Thus, when students move forward from a visual solution toward one that is based on constructing dependencies in a DGE, this is taken as an indication of the development of students' geometric thinking.

Theoretical Framework

In this study, Sfard's (2008) commognitive framework is used to examine students' mathematical discourse. Defining learning as the development of discourses, Sfard frames (mathematical) thinking as an individualized form of communication. Thus, she suggests a developmental unity between the processes of thinking and communicating, which leads to naming her approach "commognitive." Commognitive researchers are interested in mathematical discourses, as this is where one can trace the processes of learning. Sfard distinguishes mathematical discourses in terms of their tools (*words* and *visual means*) and the form and outcomes of their processes—*routines* and *narratives* (Table 9.1). Each of these constructs is explained below, but the focus will be on the notion of routines, which is the most relevant construct for the analysis in this study.

Different mathematical discourses employ certain mathematical *words*, which might signify different things in different discourses, and *visual objects*, such as figures or symbolic artifacts. In addition to using these discourse tools, participants functioning in different discourses produce what Sfard calls *narratives*, that is, sequences of utterances about mathematical objects and relations among them. Narratives are subject to endorsement or rejection under certain substantiation procedures by the community. Endorsed narratives usually take the form of definitions, axioms, theorems, and proofs. In order to produce mathematical narratives, participants engage in mathematical tasks in certain ways. They follow what are called *metarules*, which are different than object-level rules. Rules that express patterns about mathematical objects, say about triangles, are defined as object-level rules (e.g., the sum of interior angles of a triangle is 180°). Metarules, on the other hand, are about actions of participants, and they relate to the production and substantiation of object-level rules. The set of metarules that describe a patterned discursive action are named *routines*, since they are repeated in specific types of situations.

Routines take two forms: the how and the when of a routine. The how of a routine, which may be called *course of action* or *procedure*, refers to a set of metarules describing the course of the patterned discursive action. The when of a routine, on the other hand, is a collection of metarules used by participants to determine the appropriateness of the performance. The researcher might observe the how of a routine more easily when a specific task is assigned. Examining the when of a routine, however, requires extended periods of observation, when participants are asked to solve problems that are more complex. In this study, given that students were provided with a well-defined task, the how of a routine was analyzed.

Sfard (2008) states that metarules and routines are the researcher's construct based on observations of participants' discursive actions. Therefore, they are about the observed past. They are useful constructs for the researcher because "constructed metarules allow us to map thetrajectoryofone's discursive development" (p. 209).

Table 9.1 The four distinguishing aspects of mathematical discourses

Tools of math discourses		Form and outcomes of math discourses	
Words	Visual means	Routines	Narratives
Use of certain keywords that signify different things in different discourses	Visible objects that are operated upon within communication	Set of metarules that describe a patterned discursive action and that relate to the production and substantiation of object-level rules	Sequences of utterances about mathematical objects and relations among them

Method

This is a case study of a team of three eighth-grade students (about 14 years old) who worked on a geometry construction problem collaboratively within the Virtual Math Teams (VMT) environment. These three students were participants in the VMT Project, the larger design-based research (DBR) project that incorporates cycles of data collection and analysis to refine technology, curriculum, and theory for collaborative learning. As part of the VMT project, the participants worked on the tasks of a geometry curriculum for the VMT environment written by Stahl (2013b) for about a semester. Although the participants had very little formal background in geometry, this particular team was able to solve a challenging task (Oner, 2013) in session 5. That brought this team to the attention of the project research team leading to this study to understand the team's mathematical development (see Stahl, 2016 for an analysis of all eight of their sessions).

The study focuses on one of the team's problem-solving sessions, namely, session 3. This session was chosen for analysis as it represented an "extreme case" (Patton, 1990) given that it displayed characteristics from which one could learn the most for the purposes of the larger DBR project. Detailed analyses of such cases could suggest ways of refining the VMT technology, pedagogy, and curriculum to provide better support for future online groups.

The Context and Participants

The team was named the "Cereal Team," because the members selected their online handles to be Cheerios, Cornflakes, and Fruitloops. None of the team members had previously studied geometry; they were taking first-year algebra at the time of data collection. They are all females. Before the session analyzed in this study, they had met within the VMT online environment for 2 h-long sessions, trying basic GeoGebra tools, such as the software tools for creating points, lines, and line segments, or working on the task of equilateral-triangle construction (in sessions 1 and 2).

In session 3, students worked on Topic 3 of the VMT dynamic geometry curriculum (Stahl, 2013b) that involved two tasks:

> *Task* 1: Construct two lines that are perpendicular bisectors of each other. A list of steps is provided so that students can construct the diagonals (AB and CD) of a rhombus (ACBD). A completed construction is provided as an illustration for students (Fig. 9.1a).
>
> *Task* 2: Construct a perpendicular line to a given line through a given point. The expected solution for this task is provided in Fig. 9.1b. Here, one first needs to define the given point H as a midpoint between two points using the circle tool (i.e., drawing the circle at center H with radius AH). Since H is the center of this circle, AH and HB are congruent, which are the radii of this smaller circle. Now one can use points A and B (the intersections of line FG and the small circle) as centers and line segment AB as the radius to construct the two larger circles. As line segments DB, BC, CA, and AD are all radii for these circles (r), they are congruent. Connecting these line segments would create four congruent triangles (by the SSS congruency theorem involving triangles CHB, CHA, DHA, and DHB). This implies that angle CHB is a right angle and line CD is perpendicular to the line FG at H.

Participants work on geometry problems in the VMT software environment within chat rooms created for each session. Figure 9.1c shows the VMT room created for session 3. The screenshot was taken at the very beginning of the session. Note that a completed perpendicular bisector construction is provided for students. In VMT rooms, there is a chat panel on the right hand side and a whiteboard area for multi-user GeoGebra. One can post a chat anytime during the session. However, in order to manipulate objects in the GeoGebra area, one has to click on the "Take Control" button (at the bottom). Thus, only one person at a time can interact with the dynamic geometry section of the room. The GeoGebra view is, however, shared by everyone in the team so they can all observe changes to the figures as they are made.

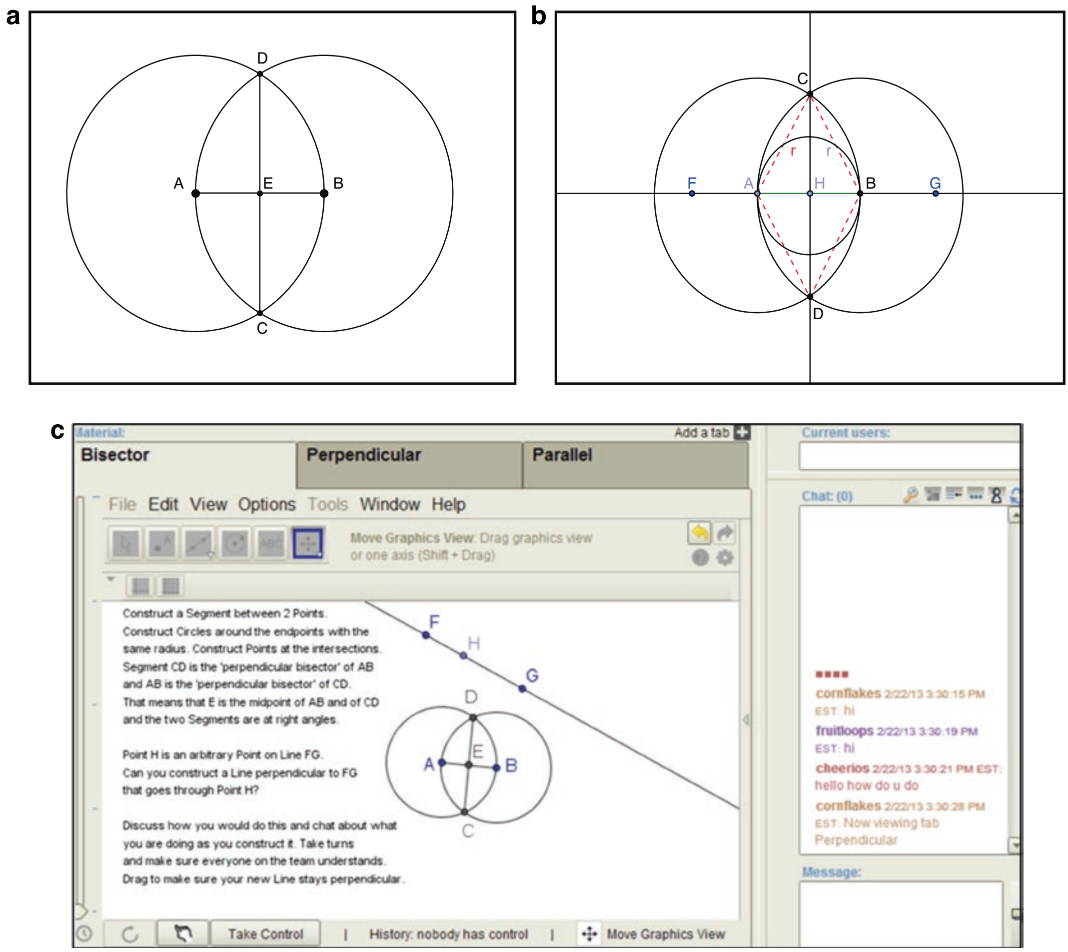

Fig. 9.1 (**a**) Construction of two line segments that are perpendicular bisectors of each other (Task 1). (**b**) Construction of the perpendicular to the line FG through a given point H (solution for Task 2). (**c**) The VMT window at the start of work on session 3. Note the task instructions and example figures. The chat section is in the panel on the right

Data Collection and Analysis

The team's meeting in the VMT environment was part of an after-school club organized by their math teacher in an American public school. The Cereal Team worked on Topic 3 for about an hour. The problem-solving session was recorded as a VMT log file to be replayed later allowing subsequent observation of the team's problem-solving process in micro-detail. All chat postings and GeoGebra actions produced by the team members are automatically logged and digitally recorded.

In order to investigate the changes in participants' discourse, both the chat postings and the actions of the participants recorded in their VMT session were examined through Sfard's (2008) discursive lens. As summarized in Table 9.2, the particular focus was on the changes in (a) the team's use of the word "perpendicular," (b) the visual mediators they acted upon (i.e., the perpendicular bisector construction), and (c) their mathematical routines, since the changes in these features were the most salient aspects of their changing discourse.

Given the nature of the assigned geometry tasks, this study investigated two routines:

Table 9.2 Sfard's (2008) three discourse aspects used in the present analysis

Words	Visual means	Routines
The use of the word "perpendicular"	The perpendicular bisector construction	The production of the perpendicular The verification of perpendicularity

Table 9.3 Characteristics of *visual* vs. *formal* mathematical discourses in session 3

Visual discourse	Formal discourse
Production of the perpendicular is based on visual placement of a perpendicular-looking line (spatio-graphical)	The production of the perpendicular is based on constructing dependencies
Verification of perpendicularity involves visual check (spatio-graphical)	Verification of perpendicularity derives from theoretical relationships
The use of the word perpendicular reflects a visual image of which two lines look perpendicular	The use of the word perpendicular signifies a theoretical relationship between geometrical objects

- *The production of the perpendicular*: This routine involved the use of a set of procedures referring to the repetitive actions in producing a perpendicular line, such as construction (by creating dependencies) or visual placement (drawing).
- *The verification of perpendicularity*: This routine is a set of procedures describing the repetitive actions in substantiating whether a solution (a line produced) is in fact perpendicular to a given line. These procedures could include visual judgment, numerical measurements, or use of theoretical geometry knowledge to justify proposed solutions.

Two discourses are considered different when they are *incommensurable*, that is, when they have different rules for the same type of task (Sinclair & Moss, 2012). One can therefore distinguish between two mathematical discourses when they entail two different ways of solving the tasks in Topic 3 as summarized in Table 9.3. In one discourse, students' production of the perpendicular and verification of perpendicularity are exclusively based on spatio-graphical cues without any concern for theoretical relationships. More specifically, the solution and verification routine is based on visual placement of a perpendicular-looking line (spatio-graphical solution), which produces a drawing (Hoyles & Jones, 1998). Along the same lines, the use of the word "perpendicular" reflects a visual image in which two lines perceptually look perpendicular. Thus, this discourse is categorized as *visual*. In another discourse, which is called *formal*, the production of the perpendicular line involves constructing dependencies— that is, defining relationships using the software tools. The verification routine within this discourse is theoretical deriving from geometrical relationships. The word "perpendicular" within this discourse signifies a theoretical relationship between geometrical objects.

As the first step in the analysis, the chat postings and GeoGebra actions of the Cereal Team were divided into episodes, mainly based on the detected changes in participants' routines of solving the task (i.e., routines of production and verification). In each episode, what is said and done was examined focusing on the three aspects of their mathematical discourse when relevant: their use of the word "perpendicular," the visual means acted upon, and routines of the production of the perpendicular or verification of perpendicularity in each episode. In what follows, an analysis of the most notable moments of these episodes will be presented by providing excerpts from the chat postings and VMT room screenshots.[1]

Analysis

Based on the team's routines of production and verification, the interaction is divided into the following episodes: (1) constructing the perpendicular bisector, (2) drawing a perpendicular-looking line, (3) drawing the perpendicular using the perpendicular bisector construction (PBC) as straightedge, (4) use of circles with no dependencies defined, (5) constructing dependencies, and (6) discussing why the construction worked.

Episode 1: Constructing the Perpendicular Bisector (3:32:15–3:40:20)

As the first task, the team was asked to construct two line segments that are perpendicular bisectors of each other. They were provided the steps to construct a line segment first and then to construct two circles around its endpoints, with the line segment as their radii (see Fig. 9.1a for the expected answer). By constructing the two intersections of the two circles and connecting them, the participants would obtain two line segments perpendicular to each other at their midpoints.

At the start of the first episode, Fruitloops and Cheerios were active with the construction of the two line segments as perpendicular bisectors of each other. The team decided that Fruitloops should take control and tackle the task (Log 9.1, Lines 14–16). However, Fruitloops asked how she could make a line segment after creating two points (I and J). At that moment, the segment tool was not visible; it needed to be pulled down in the toolbar. Cornflakes provided some direction by saying that the segment tool is next to the circle tool (Log 9.1, Line 19). This information was sufficient for Fruitloops, as she was then able to construct a line segment (IJ).

Log 9.1

Line	Post time	User	Message
11	3:31:02.6	fruitloops	who wants to take control
12	3:31:16.1	fruitloops	do you was to delete the instruction
13	3:31:21.5	fruitloops	want*
14	3:32:11.4	fruitloops	want me to start?
15	3:32:13.4	cheerios	take control
16	3:32:16.0	cornflakes	Yes
17	3:33:03.9	fruitloops	how do i make the line segment?
18	3:33:08.0	cheerios	do u need help
19	3:33:26.1	cornflakes	its by the circle thingy
20	3:33:38.1	fruitloops	got it thanks
21	3:34:06.5	cornflakes	no problem
22	3:35:54.1	fruitloops	i did it
23	3:36:02.0	cheerios	good job my peer
24	3:36:14.4	cornflakes	Nice
25	3:36:15.6	fruitloops	someone else want to continue?
26	3:36:23.6	fruitloops	thankyou thankyou
27	3:36:32.5	cheerios	release control
28	3:37:40.4	fruitloops	so now you need to construck points at the intersection
29	3:38:12.1	fruitloops	no you dont make a line you make a line segment
30	3:38:35.1	fruitloops	good!!
31	3:39:20.4	fruitloops	so continue
32	3:39:29.9	cheerios	i just made the intersecting line and point in the middle
33	3:39:40.0	cheerios	it made a perpindicular line

Another problem Fruitloops had difficulty with was constructing circles at the endpoints of the line segment with the same radius, which establishes the dependency crucial for the construction. She created two circles centered at points I and J with radius IK and JL, respectively, which were not congruent but *looked* the same (Fig. 9.2a). To define the radii of the circles centered at points I and J, she used arbitrary points (K and L), not the line segment IJ, that is, her circles *looked* to have the same radius,

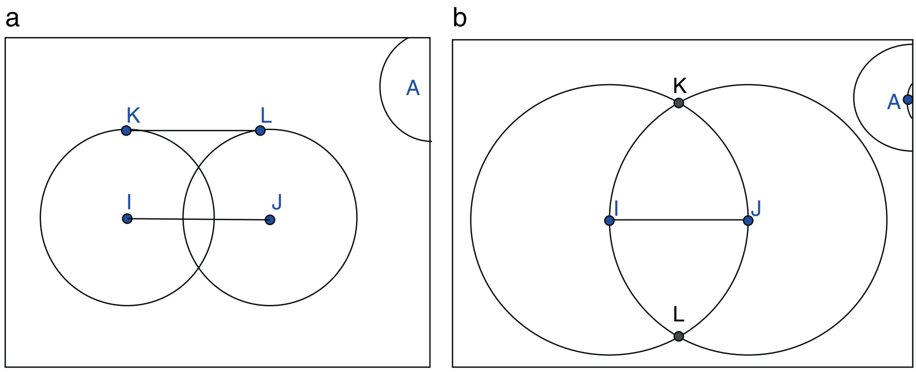

Fig. 9.2 (**a**) Two circles with different radii (IK and JL) centered at I and J. (**b**) Two circles with the same radius IJ centered at I and J

but they were not *constructed* based on an equal-radius relationship. Later, however, after playing with the circle tool for a while, Fruitloops did the construction again and managed to construct two circles around the endpoints (points I and J) with the same radii (IJ) (Fig. 9.2b).

Next, Cheerios took control and continued the work by constructing the intersection points of the two circles (new points K and L) and the line that passed through them. Yet, as the following move, Cheerios removed the line she just constructed. Next, she reconstructed it, and then again deleted it and the intersection points. Finally, she reconstructed the intersections. At this point, Fruitloops drew attention to the instructions, saying they needed to construct a line segment, not a line (Log 9.1, Line 29). This time, Cheerios constructed the line segment KL through the intersections and created point M, the intersection of the two segments (KL and IJ). Cheerios explained her actions by saying "i just made the intersecting line and point in the middle," calling M "the point in the middle." She continued, "it made a perpindicular line" (Log 9.1, Lines 32–33).

In this episode, the routine for solving the first task simply involved following the instructions. Yet, Fruitloops had two difficulties. While one had to do with finding the needed menu item in the software, the other was related to constructing the key dependency, that is, same-radius circles at the endpoints of the line segment. Cheerios also had to pay attention to the wording in the instructions (i.e., the difference between "line" and "line segment"). She used the word "perpendicular" once (Log 9.1, Line 33). At this point, it seems reasonable to argue that the word "perpendicular" was just a revoicing of the task instructions.

Episode 2: Drawing a Perpendicular-Looking Line (3:40:27–3:55:30)

Moving to the second part of the given task, the team now had to work on a more challenging problem, which was constructing a perpendicular to a line through a given point. In this episode, the team's problem-solving discourse took a visual character, which was evidenced by (a) producing a perpendicular-looking line (a drawing), (b) verifying perpendicularity by visual perception, and (c) using the word "perpendicular" to refer to a visual image. One other important aspect of this episode was Cornflakes' bringing the illustrative perpendicular-bisector construction to the team's attention.

On their screen, a line FG and the point H was provided to them (Fig. 9.1c). Initially, however, how to use these givens was not clear to any of the team members. For Cornflakes and Cheerios, the production of the perpendicular first required creating another reference line that was somehow related to the line FG, as they both tried to construct lines that either *looked* parallel to or intersected the line FG. Fruitloops elegantly suggested using the line that was already there (Log 9.2, Line 37). Furthermore, she next uttered the word "perpendicular." She said "perpindicular no intersecting" (Log 9.2, Line 39). This use was different than that of Cheerios in the first episode. Fruitloops used the word to evaluate Cheerios' line, which intersected the line FG. At this stage, this use of "perpendicular" may have just implied a visual image rather than a construct with mathematical properties.

Log 9.2

Line	Post time	User	Message
34	3:40:27.5	fruitloops	okay cornflakes go next
35	3:41:11.5	cornflakes	what are you supposed to do?
36	3:41:42.6	fruitloops	just follow the instructions
37	3:43:48.5	fruitloops	were we supposed to just use the line that was already there?
38	3:44:10.2	cornflakes	i think so
39	3:44:44.2	fruitloops	perpindicular no intersecting
40	3:44:46.1	fruitloops	not*

After this initial stage, Cornflakes took control. She constructed a point N and a line through N and H that looked perpendicular to line FG at H (Fig. 9.3a). Then she removed this line but later reconstructed it in the same manner and deleted it once more. She was just picking a location for point N such that a line NH would visually appear to look perpendicular to line FG.

Next, however, she did something rather unexpected: she started moving the perpendicular-bisector constructions (PBCs) around. She dragged both the one that was given with the topic and the one they had just constructed in Episode 1 changing their shape and location. Not seeing any of the use of the PBC immediately, she repeated her production of a line that seemed (visually) perpendicular to line FG through H, after creating points N and O. While the line looked as if it passed through O, N, and H, it was only passing through O and H (Fig. 9.3b).

After Cornflakes' attempt to provide a solution, Fruitloops took control. She first deleted the line Cornflakes constructed (line OH), the one that appeared to be perpendicular to FG at H (Fig. 9.3b). She played with constructing some other points and line segments, which did not seem relevant. It is reasonable to argue that she was not happy with Cornflakes' seemingly perpendicular line. She then released control and asked in the chat: "can you remake it?" (Log 9.3, Line 43). In response, Cheerios took control and added points O and Q and a line through them that passed through H (Fig. 9.4a). This line again was a visual solution that looked perpendicular to FG through H. Cheerios then added another point (R) on the line placing it in the upper plane. Fruitloops, however, questioned defining extra points (O and Q) (Log 9.3, Line 44) while Cornflakes was fine with them (Log 9.3, Line 45). In response, Cheerios removed point R and then her line OQ. She reconstructed point R and constructed another line through R, which this time did not even look perpendicular to line FG at H (Fig. 9.4b). She then asked if the line was okay (Log 9.3, Line 46). Fruitloops once again evaluated the line Cheerios constructed saying "its not perpinicuklar" (Log 9.3, Line 48). Then Cornflakes deleted this line and constructed a more perpendicular-looking one first through H and S (a new point) and then, deleting line HS, through H and N (Fig. 9.4c). Even though Fruitloops seemed satisfied this time saying, "I think that'sgood," (Log 9.3, Line 49), Cornflakes erased the perpendicular-looking line (line HN) once more

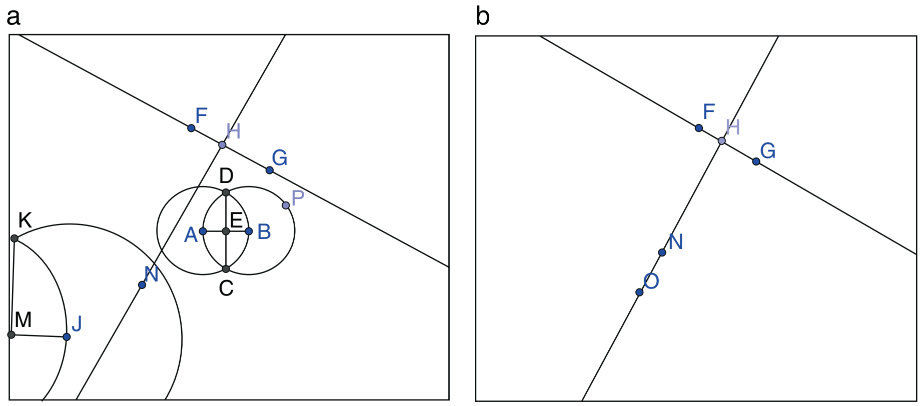

Fig. 9.3 (**a**) Construction of line NH that looks perpendicular to line FG. (**b**) Construction of another line (line OH) that looks perpendicular to line FG

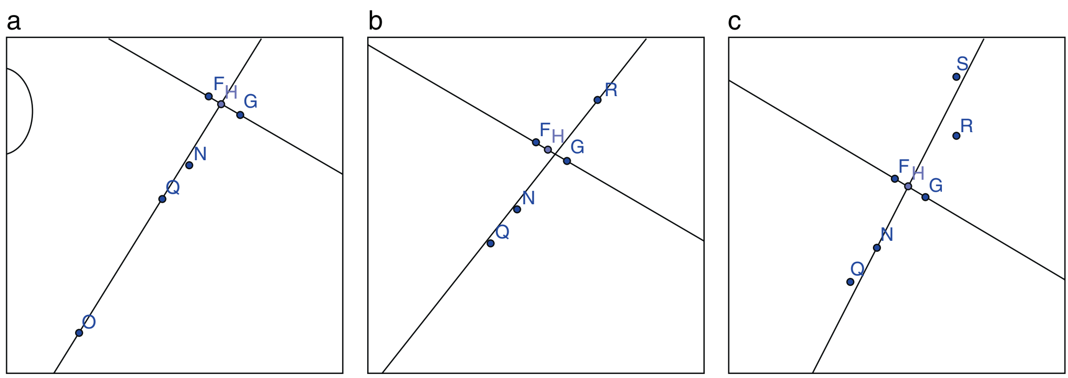

Fig. 9.4 (**a**) Construction of the perpendicular-looking line OQ. (**b**) Construction of a perpendicular-looking line passing point R. (**c**) Construction of the perpendicular-looking line HN

The solution offered by Cornflakes included placing a perpendicular-looking line visually (a spatio-graphical solution), which did not depend on creating dependencies. Cheerios also worked toward producing a line that would look perpendicular to the line FG at point H. However, there was also some level of discomfort with this solution, which was evidenced by deletion actions immediately followed by creating such lines. Fruitloops did not explicitly undertake the same production routine. She used the word "perpendicular," judging Cheerios' line as not fitting her notion of perpendicular. However, she eventually agreed on the line produced by Cornflakes in response (Log 9.3, Line 49). Therefore, at this stage, one can say that all team members' production of the perpendicular routine involved creating a line that was a *drawing*. An important aspect of this episode was Cornflakes' little play with the available PBC. Even though PBC had not been used as a mediator of the production of the perpendicular routine just yet, Cornflakes made its presence known and highlighted it as a potential tool.

Log 9.3

Line	Post time	User	Message
41	3:48:09.7	fruitloops	sorry i did it by accident
42	3:48:23.5	cheerios	its fine :) my dear peer
43	3:48:38.3	fruitloops	can you remake it
44	3:48:52.7	fruitloops	why did you make point o and q
45	3:48:55.0	cornflakes	its alright
46	3:49:09.5	cheerios	is the line ok
47	3:49:16.0	cornflakes	i didn' t make point o and q
48	3:49:23.0	fruitloops	its not perpinicuklar
49	3:50:57.7	fruitloops	i think thats good

As the team did not seem completely satisfied with their (visual) solution, some of their efforts next focused on finding ways to judge perpendicularity. This stage was marked and initiated by Cheerios when she suggested rotating the line FG (she referred to it as FHG) "so it is easier to make it horizontal" (Log 9.4, Line 50). With this statement, she meant dragging the given line FG into a horizontal-looking position so that one can test when a line was perpendicular to it more easily. Presumably, the prototypical visual image of perpendicularity involves a horizontal base line and a vertical perpendicular to it. This statement added a new routine to the problem: verification of perpendicularity along with a production routine.

However, neither Cornflakes nor Fruitloops took up this suggestion. Cornflakes was busy reconstructing another perpendicular-looking line passing through H. Fruitloops also adjusted this line so that it would look more perpendicular. Cheerios first helped Fruitloops by removing some of the extra points on or around that line and adjusting the line. Next, she implemented what she suggested by making the line FG horizontal looking, so that the team could better test the perpendicularity of the line it was to construct (Fig. 9.5a). This would of course be a visual test, not a mathematical one. Seeing the line FG in a horizontal position, Cornflakes asked Cheerios to construct the perpendicular line (Log 9.4, Line 53). Cheerios then constructed another two points (R and O) and a line through them that looked perpendicular to FG, but this did not go through point H. Cheerios deleted her first construction and then cleared the area deleting some extra points. Then she constructed line NH, which looked nearly perpendicular to FG through H (Fig. 9.5b). Cornflakes seemed satisfied with the new line, saying, "that's good" (Log 9.4, Line 54). Fruitloops said, "I think its perpendicular cause they are all 90° angles" (Log 9.4, Line 55).

Log 9.4

Line	Post time	User	Message
50	3:50:59.8	cheerios	turn line fhg so its easier make it horizontal
51	3:52:54.4	fruitloops	Hey
52	3:54:06.9	fruitloops	which point did you move to get the line like that
53	3:54:07.5	cornflakes	now construct the line
54	3:55:10.7	cornflakes	thats good
55	3:55:30.5	fruitloops	i think its perpendicular cause they are all 90° angles

To summarize, Cheerios produced yet another drawing (Line NH, Fig. 9.5b) at this point, and Cornflakes and Fruitloops agreed on that solution (Log 9.4, Lines 54–55). Furthermore, Fruitloops' approval involved the use of the word "perpendicular." She said: "i think its perpendicular cause they are all 90° angles" (Log 9.4, Line 55). With this sentence, it became clearer that she used the word as representing a visual image of perpendicularity as she referred to the measure of the angles without measuring. Thus, all group members were still realizing the perpendicular line as a figure that could be produced perceptually. Moreover, Cheerios felt the need to verify their solution. She suggested

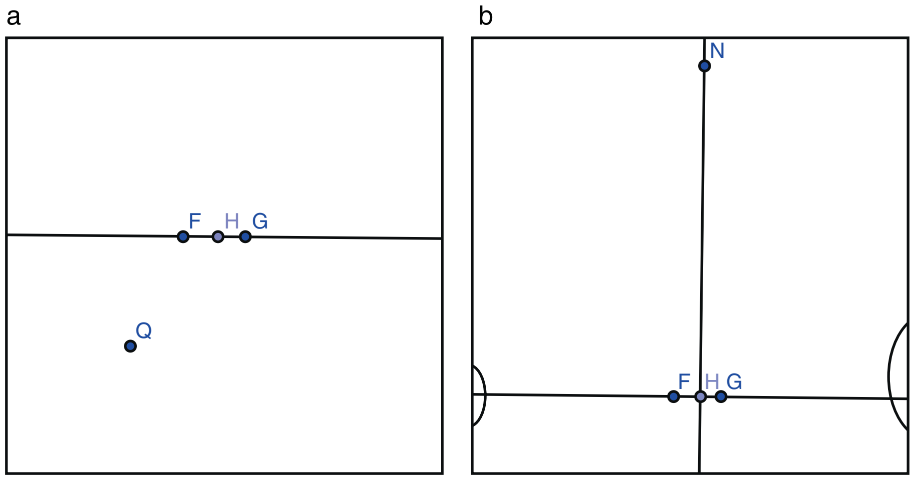

Fig. 9.5 (**a**) Dragging line FG into a horizontal position. (**b**) Construction of line NH that looks perpendicular to line FG

Table 9.4 Summary of Episode 2 in terms of discourse characteristics

Production of the perpendicular routine	Verification of perpendicularity routine	Use of the word *perpendicular*	Use of visual mediators
Creating another reference line in relation to line FG (Cornflakes and Cheerios)		Signifying a visual image of perpendicular to disagree with a spatio-graphical solution (Fruitloops)	
Spatio-graphical solution/ drawing a perpendicular-looking line (Cornflakes)			PBC random dragging (Cornflakes)
Spatio-graphical solution/ drawing a perpendicular-looking line (Cheerios & Cornflakes)		Signifying a visual image of perpendicular to disagree and then agree with a spatio-graphical solution (Fruitloops)	
Spatio-graphical solution (Cornflakes, Fruitloops, Cheerios)	Spatio-graphical verification/ vertical-horizontal alignment of the lines (Cheerios)	Signifying a visual image of perpendicular to agree with a spatio-graphical solution (Fruitloops)	

producing the perpendicular line in a horizontal-vertical arrangement of two lines (the prototypical visual image for perpendicularity), which allowed a visual verification. Therefore, at this stage, a new routine for verifying perpendicularity emerged, although it was also spatio-graphical.

Table 9.4 provides a summary of the analysis presented for Episode 2.

Episode 3: Drawing the Perpendicular Using the PBC as Straightedge (3:55:55–3:58:26)

Something interesting happened next. Cornflakes started moving the PBC around as if she wanted to use it as a protractor—to verify the right angles. She was not able to get the orientation correct. Getting the idea, Fruitloops took control and dragged the PBC (the one they constructed) placing the middle point M on top of H and aligning with the line FG (Fig. 9.6a). Cornflakes was satisfied, as she responded with a "yes" (Log 9.5, Line 56). These moves signaled a new and different verification routine of perpendicularity, one that is based on measurement rather than based on a visual judgment.

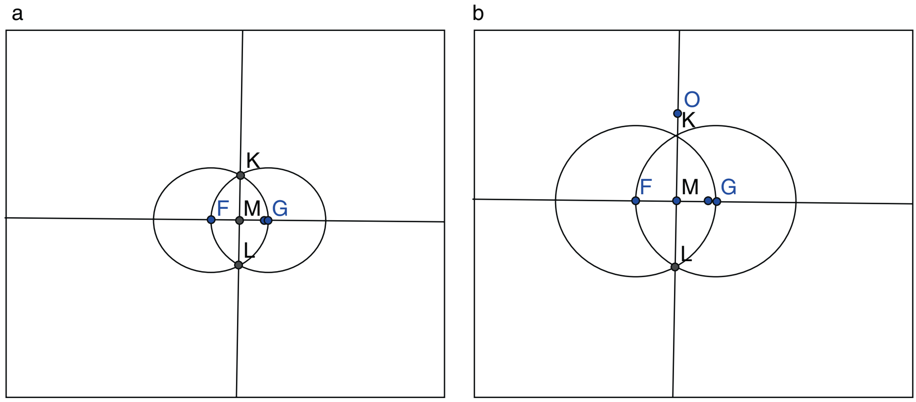

Fig. 9.6 (**a**) Placing PBC on top of H and aligning with line FG. (**b**) Constructing line OH using PBC as a guide

Meanwhile, Fruitloops realized another procedure for producing the perpendicular. Even though she was able to superimpose the two figures well, she deleted the perpendicular-looking line (Line NH). This move suggested that rather than using the PBC as a tool for measuring the angles, she could use it as a straightedge to draw the perpendicular. This still represented a visual production of the perpendicular (a spatio-graphical solution); meanwhile it perhaps marked the point of new possibilities for approaching the problem. Cornflakes was following Fruitloops one step behind saying "so after construting the line we put the circle on top" (Log 9.5, Line 57). She was still seeing the PBC as a tool for checking perpendicularity rather than as a tool for drawing. Fruitloops, on the other hand, constructed another line (line OH) that looked like it concurred with the line segment KL (the segment perpendicular to segment IJ in the PBC construction, Fig. 9.6b). Cornflakes then realized what Fruitloops was trying to do as she typed "so put the line thru the line on the circle" (Log 9.5, Line 58). Fruitloops, however, was not sure how to proceed. She deleted her line (line OH) and even constructed an intersecting line (not a perpendicular). She next deleted that too and finally said "I don't know what I am doing help" (Log 9.5, Line 59).

In this episode, two new routines emerged. First, initiated by Cornflakes, the routine of verification shifted from one that is based on perception to one that is based on measurement by making use of a new visual mediator, the PBC. She wanted to use the PBC, which is known to be perpendicular, to check perpendicularity. She got help from Fruitloops to do that. Secondly, the production of the perpendicular also changed character involving the same visual mediator. While helping Cornflakes, Fruitloops wanted to imitate a paper-pencil routine of drawing the perpendicular using the PBC as a straightedge, yet she left the work unfinished. Cornflakes adopted this new routine as well.

Log 9.5

Line	Post time	User	Message
56	3:56:28.6	cornflakes	Yes
57	3:57:05.2	cornflakes	so after construting the line we put the circle on top
58	3:57:56.8	cornflakes	so put the line thru the line on the circle
59	3:58:18.5	fruitloops	i dont know what i am doing help
60	3:58:24.8	fruitloops	someone else take control

Table 9.5 provides a summary of the analysis presented for Episode 3.

Table 9.5 Summary of Episode 3 in terms of discourse characteristics

Production of the perpendicular routine	Verification of perpendicularity routine	Use of the word *perpendicular*	Use of visual mediators
Spatio-graphical solution/imitation of paper-pencil routine of drawing the perpendicular using PBC as straightedge (Fruitloops & Cornflakes)	Measurement-based verification using PBC (Cornflakes & Fruitloops)		PBC as protractor (Cornflakes) PBC as straightedge (Fruitloops)

Episode 4: Use of Circles with No Dependencies (3:58:27–3:59:52)

Taking control after Fruitloops, Cornflakes first dragged the PBC away. For a while, she seemed to play with the PBC: randomly constructing points on it, dragging them, and moving the labels of the points. Then, Cheerios jumped in, suggesting to "make the line first" (Log 9.6, Line 61). One can infer that Cheerios was still trying to produce the perpendicular line visually. In response, Fruitloops clarified her approach: "i think you need to make the circles first" (Log 9.6, Line 62). This statement signaled a new routine regarding the production of the perpendicular, that is, Fruitloops proposed using the construction of circles to produce the perpendicular just as the team had done with the PBC and the equilateral triangle in the previous topic (Topic 2).

Following her statement, Fruitloops took control and embarked on constructing. At this moment, Cornflakes said "put point m on top of h" (Log 9.6, Line 63), that is, she proposed moving the PBC back on top of point H. This statement suggested that she was not yet following Fruitloops. She wanted either to use the PBC to check perpendicularity or, more plausibly, to use it as a guide to draw the perpendicular. Fruitloops, on the other hand, started the construction by creating two circles with centers at F and G and with radii GQ and FR, respectively (Fig. 9.7). However, although GQ and FR looked the same, they were not constructed as equal. This was, in fact, the same procedure she had initially followed with the PBC construction at the very beginning of their session (Fig. 9.2a). She later constructed another and larger circle with center H and radius HS around these two circles but immediately deleted it. Thus, although she realized that there had to be a construction involving circles, she failed to create the dependency for equal-radius circles. She then released control.

At this stage, Fruitloops suggested a new routine for the production of the perpendicularity, the one that included creating circles. It is quite plausible that this newly emerged routine had been triggered by the presence of the PBC in the problem-solving environment. Although she wanted to follow a procedure that involved constructing circles, she was not able to build the necessary dependencies. Neither Cornflakes nor Cheerios was at this level yet.

Log 9.6

Line	Post time	User	Message
61	3:58:35.8	cheerios	make the line first
62	3:58:51.2	fruitloops	i think you need to make the circles first
63	3:59:19.0	cornflakes	put point m on tp of h

Table 9.6 provides a summary of the analysis presented for Episode 4.

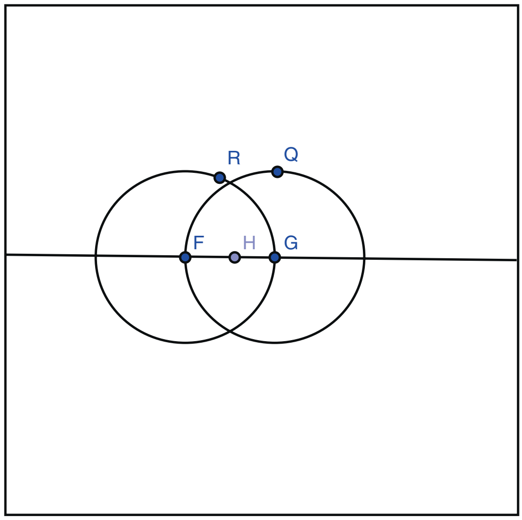

Fig. 9.7 Construction of two circles with different radii (GQ and FR) centered at F and G

Table 9.6 Summary of Episode 4 in terms of discourse characteristics

Production of the perpendicular routine	Verification of perpendicularity routine	Use of the word *perpendicular*	Use of visual mediators
Use of circles with no dependencies defined (Fruitloops)			PBC as image of construction (Fruitloops)

Episode 5: Constructing Dependencies (3:59:53–4:14:15)

Although Fruitloops was not able to complete what she had started immediately, Cheerios eventually took up her new reframing of the problem. After Fruitloops, Cheerios took control. She constructed a line through points T and S (new points) and adjusted it so that line TS would look like it passed through not only H but also the intersections of the circles that Fruitloops constructed (Fig. 9.8a). Cheerios tried several strategies to make the line TS go through the intersections of the circles and point H, such as constructing a point very close to H (point U) and a line through that. However, as Fruitloops observed, the line was not going through H (Log 9.7, Line 64). Thus, although Cheerios was now building on what Fruitloops had started, there were two problems with their attempts to construct the perpendicular. First, H was not defined as the midpoint of a line segment. Secondly, the circles around the endpoints did not have the same radius. In other words, although their production of the perpendicular routine now included the use of circles, no dependencies were constructed.

At this point, Cornflakes provided a definition for bisection, saying, “bisection is a division of something into two equal parts” (Log 9.7, Line 65), which was not given to them with this task. Cheerios then took control and moved point H to the line; however it did not attach to the line. Next, Cornflakes played with the line as well moving it around point H and saw that it was not set to pass through H. Then Fruitloops realized the problem saying, “we didn’t put a point between the circles so the line isn’t perpendicular” (Log 9.7, Line 66) and later adding “the part where the circles intersect” (Log 9.7, Line 69).

Although Fruitloops was not using a formal mathematical language to explain her reasoning, this statement provided a new perspective on the production of the perpendicular as creating certain dependencies (which she demonstrated by actually performing the construction later). In response, Cornflakes dragged line FG and saw that dragging messed up their solution (Fig. 9.8b). Cheerios

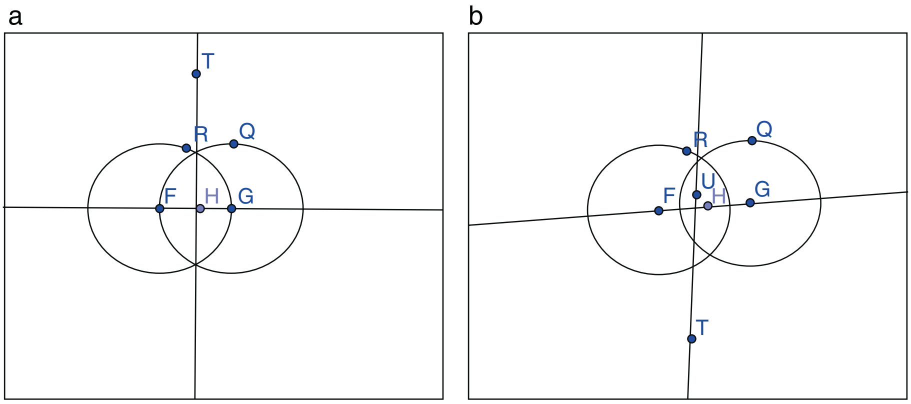

Fig. 9.8 (**a**) Construction of line TS. (**b**) Dragging line FG

agreed with Fruitloops immediately saying "oh I see now" (Log 9.7, Line 68). Cornflakes, however, kept on moving other parts of the figure (such as points H and F) to make intersections and their perpendicular-looking line (TH) concur. Observing that Cornflakes was not convinced, Fruitloops suggested that she look at the examples. Finally, Cornflakes said, "ok I see" (Log 9.7, Line 71).

Now that the team members seemed to be all on the same page, they spent some time discussing who would do the construction. Finally, Fruitloops took control and cleared up the space first by removing some points and their perpendicular-looking line. Then she created two circles at centers F and G with the same radius FG correctly (Fig. 9.9). She also constructed the intersections (points Q and R) and explained what she did: "so i made two circles that intersect and the radius is the same in both circles right?" (Log 9.7, Line 79). Cheerios agreed, "yea they are the same" (Log 9.7, Line 80). Fruitloops highlighted once more that their radii were FG: "and segment fg is the radius" (Log 9.7, Line 81). These statements confirmed that Fruitloops wanted to focus the group's attention on constructing certain relationships. Cornflakes followed with a "yes" (Log 9.7, Line 82). Cheerios said, "now we have to make another line" (Log 9.7, Line 83). However, Fruitloops did not want to continue, saying: "yeah someone else can do that" (Log 9.7, Line 84).

Log 9.7

Line	Post time	User	Message
64	4:02:26.9	fruitloops	the line isn't going through part h
65	4:02:39.5	cornflakes	bisection is a division of something into two equal parts
66	4:04:58.2	fruitloops	we didn't put a point between the circles so the line isn't perpendicular
67	4:05:03.8	fruitloops	line*
68	4:05:19.4	cheerios	oh i see now
69	4:05:20.6	fruitloops	the part where the circles intersect
70	4:05:34.8	fruitloops	look at the examples and you'll see
71	4:05:46.9	cornflakes	ok i see
72	4:05:51.8	cheerios	r u fixing it
73	4:05:54.7	fruitloops	do you want to do it?
74	4:06:02.0	cornflakes	so we have to put a poijt between the circles
75	4:06:19.4	fruitloops	yeah you can do it if you want
76	4:06:43.5	fruitloops	or should i do it?
77	4:06:49.4	cornflakes	you can
78	4:06:49.6	cheerios	yea u should
79	4:08:23.3	fruitloops	so i made two circles that intersect and the radius is the same in both circles right?
80	4:08:41.9	cheerios	yea they are the same

Line	Post time	User	Message
81	4:08:55.1	fruitloops	and segment fg is the radius
82	4:08:58.4	cornflakes	yes
83	4:09:04.1	cheerios	now we have to make another line
84	4:09:14.8	fruitloops	yeah someone else can do that

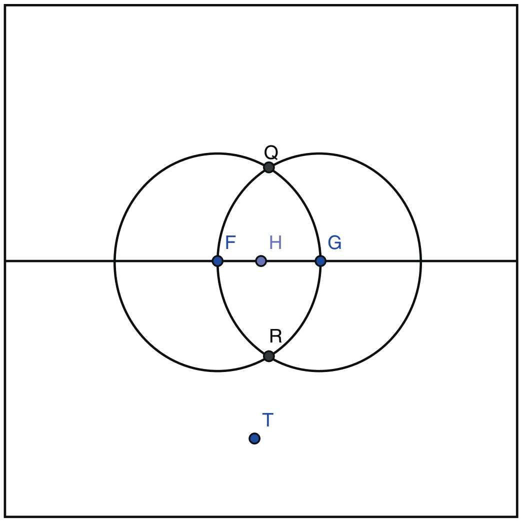

Fig. 9.9 Construction of two circles with the same radius (FG) centered at F and G

In this episode, Fruitloops identified one of the problems with the construction in Line 66 (Log 9.7): the need to create equal-radius circles. Although one can argue that she was not fully aware of the mathematical meaning of this dependency, she must have come to a realization that the way circles are constructed matters. She furthermore carried out the construction and drew attention to the defined relationships (circles with the same radius). The team members agreed upon this procedure. Thus, Fruitloops turned the routine of production of the perpendicular into a construction, one that is based on defining dependencies. Her use of the word "perpendicular" in Line 66 (Log 9.7) also reflected this change in the production routine. Here "perpendicular" was not used to represent a visual image or to evaluate a figure based on that image, as in her previous uses of the word. Rather, the word referred to a mathematical relationship that results from the way the circles were constructed.

There was still one other dependency the team needed to consider. This issue came up when Cornflakes responded to Fruitloops' invitation and constructed a line passing through Q and R (the circle intersections) and U (Fig. 9.10a). Seeing that it did not pass through H, Cornflakes deleted almost half of Fruitloops' construction hoping to solve it, even going back to making the same mistake Fruitloops made (not noticing the role of equal-radius circles at the endpoints of a line segment). However, she eventually repeated the same construction steps and went back to the point where she started. Since H was not defined as the midpoint of the radius, the line through the circles' intersection points was not going through it. At this point, Fruitloops suggested a solution with the problem of H: "you make the points go through qr and then you move h ontop of the line" (Log 9.8, Line 85). Q and R were the intersection points of the circles Cornflakes deleted. Next, Fruitloops took control and she performed what she said; she constructed the intersection points Q and R back again and the line through them and attached H to that line by simply dragging it (Fig. 9.10b). Then she announced that she finally did it (Log 9.8, Line 86).

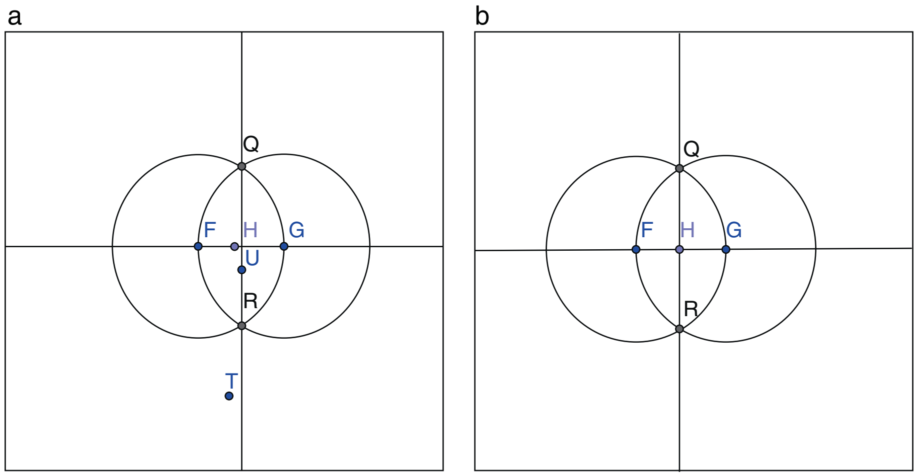

Fig. 9.10 (**a**) Construction of line QR. (**b**) Attaching point H to line QR

Table 9.7 Summary of Episode 5 in terms of discourse characteristics

Production of the perpendicular routine	Verification of perpendicularity routine	Use of the word *perpendicular*	Use of visual mediators
Constructing dependencies/use of equal-radius circles (Fruitloops) Dynamic solution/attaching the arbitrary point H to the line (Fruitloops)		Signifying a mathematical relationship (Fruitloops)	

Although the team seemed to be on the same page regarding one of the dependencies (constructing equal-radius circles), the dependency regarding the point H was overlooked. Fruitloops simply attached the arbitrary point to their perpendicular line, and this procedure seemed to work. Therefore, her routine of constructing a perpendicular through an arbitrary point did not involve taking that arbitrary point as the reference point as the task author intended. Rather, she took advantage of the dynamic geometry by simply dragging the point to the perpendicular.

Log 9.8

Line	Post time	User	Message
85	4:11:09.8	fruitloops	you make the points go through qr and then you move h on top of the line
86	4:13:08.4	fruitloops	i think i did it finallyu
87	4:13:49.1	cornflakes	the klines bisec the circle
88	4:14:15.3	cornflakes	*the lines bisect the circle

Table 9.7 provides a summary of the analysis presented for Episode 5.

Episode 6: Discussing Why the Construction Worked (4:14:29–4:16:17)

Immediately after producing a solution, Fruitloops raised the question, “but how do we know for sure that the line is perpinmdicular” (Log 9.9, Line 89). Cheerios said she was not sure (Log 9.9, Line 90).

Cornflakes first mentioned the spatio-graphical aspect of the figure by saying: “there 90° angles” (Log 9.9, Line 91). However, Fruitloops was looking for another explanation. She said, “but you cant really prove that by looking at it” (Log 9.9, Line 93). In response, Cornflakes participated within this new discourse sensing that the explanation had to do with the circles. She said, “they intersect throught the points that go through the circle” (Log 9.9, Line 94). Fruitloops built on that and said, “it has to do with the perpendicular bisector” (Log 9.9, Line 95). The two continued the discussion with Cornflakes saying “they ‘bisect’ it” (Log 9.9, Line 96). Fruitloops must have thought Cornflakes referred to the line segment by “it” and added, “and the circles” (Log 9.9, Line 97). Cheerios was relatively quiet when Fruitloops and Cornflakes were looking for a deeper understanding. She simply said, “oh I see” (Log 9.9, Line 98) as a response. However, before they moved to the next tab, she was the one who dragged their perpendicular construction extensively, confirming the integrity of the construction as suggested by the final step in the topic instructions.

In this episode, it became clear that Fruitloops was not content with a spatio-graphical verification routine. She completed the task yet also wondered why it worked. This may indicate that she was ready for a formal mathematical explanation. While Cheerios remained silent, Cornflakes participated within this conversation. Fruitloops’ use of the word “perpendicular” in line 89 (Log 9.9) sounded more mathematical as she asked, “how do we know for sure the line is perpendicular?” She further mentioned the PBC as if highlighting its significant role within this problem-solving session.

Log 9.9

Line	Post time	User	Message
89	4:14:29.8	fruitloops	but how do we know for sure that the line is perpinmdicular
90	4:14:39.6	cheerios	im not sure
91	4:14:42.1	cornflakes	there 90° angles
92	4:14:45.4	cheerios	do u cornflakes
93	4:14:59.4	fruitloops	but you cant really prove that by looking at it
94	4:15:06.8	cornflakes	they intersect throught the points that go through the circle
95	4:15:17.7	fruitloops	it has to do with the perpendicular bisector
96	4:15:19.8	cornflakes	they “bisect” it
97	4:15:31.2	fruitloops	and the circles
98	4:15:37.2	cheerios	oh i see

Table 9.8 provides a summary of the analysis presented for Episode 6.

Table 9.8 Summary of Episode 6 in terms of discourse characteristics

Production of the perpendicular routine	Verification of perpendicularity routine	Use of the word *perpendicular*	Use of visual mediators
	-Spatio-graphical (cornflakes) -Looking for a verification routine beyond spatio-graphical evidence (Fruitloops & Cornflakes)	Signifying a mathematical relationship (Fruitloops)	

Discussion

Mathematical experiences at the middle-school level are considered critical for students to develop deductive and formal thinking (Ellis, Lockwood, Williams, Dogan, & Knuth, 2012). Harel and Sowder (1998) note that it would be unreasonable to expect that students will instantly appreciate sophisticated forms of mathematics in high school, where expectations regarding mathematical rigor are higher. Therefore, it is important to provide learning opportunities for middle-school students to advance their geometric thinking. The VMT environment is designed to serve this purpose by affording virtual collaborative problem-solving with a multi-user GeoGebra component. It is important to study the ways in which teams of students using the VMT software and its curriculum are learning geometry and what problems they encounter. Toward this end, Sfard's (2008) discursive lens was employed to investigate the change in mathematical discourse of a team of three middle-school students as they worked on a geometry construction problem in the VMT environment. The analysis focused on how the team's use of the word "perpendicular," its use of the PBC as a visual mediator, and its use of routines (for production of a perpendicular and for verification of perpendicularity) shifted during an hour-long collaborative problem-solving session. The findings indicated that the Cereal Team, whose members had very limited formal geometry background, moved forward from a visual discourse toward a more sophisticated formal mathematical discourse.

To be specific, the team started constructing two line segments as perpendicular bisectors of each other following the instructions of Topic 3 (Episode 1). In this part, Cheerios' use of the word "perpendicular" was copied from the task instructions as if using a foreign language word in a sentence. The team next moved to the second task, which was built on the first one. This presented a challenge as the team needed to figure out how to construct a perpendicular to a line through a given point, which they had not done before.

Table 9.9 summarizes the team's use of the word "perpendicular," their use of visual mediators, their routines of production of a perpendicular, and their verification of perpendicularity in Episodes 2, 3, 4, 5 and 6, where the team worked on the second task in Topic 3.

In the *production of the perpendicular routine column* in the summary Table 9.9, one can see that the team started by producing spatio-graphical solutions including placing the perpendicular line visually and imitating the paper-and-pencil procedure of drawing the perpendicular by using the PBC as a straightedge guide (in Episodes 2 and 3). These routines, however, evolved into first using circles (in Episode 4) and then defining certain relationships with the circles, such as the use of equal-radius circles with the construction allowing the group to successfully complete the task (in Episode 5). The second dependency, however, was bypassed by simply attaching the arbitrary point H to the perpendicular line. Although no dependencies were created here, as Sfard (personal communication, June 2014) observed, this could be considered a legitimate move in GeoGebra. In a dynamic geometry world where everything moves, the point of reference may be redefined as well, as long as the software supports this use.[1]

A parallel progression can also be observed in the *verification of the perpendicularity routine column*. The team first felt the need to verify their solution, which was not explicitly asked in the instructions. Initially, this took a spatio-graphical form, with Cheerios wanting to arrange the lines into a vertical-horizontal position, which represents the prototypical visual image for perpendicularity (in Episode 2). Then Cornflakes, who received help from Fruitloops, wanted to use the PBC as a protractor turning the verification routine into one that is based on measurement (in Episode 3). Eventually, Fruitloops, upon completing the construction, asked how they could be sure if the line was perpen-

[1] The instructions specified that, "point H is an arbitrary point on line FG." In Euclidean geometry, that would mean that even though H can be any point on line FG, it is not something that moves. Thus, although one looks for a solution that would work for any point H, any treatment of H would be static. In dynamic geometry, however, an arbitrary point H is a free point that can be dragged along line FG. Thus, there is some legitimacy to the students' solution. Ultimately, however, the solution fails the drag test of dynamic geometry. If one properly constructs the perpendicular through point H, then one should be able to drag point H along line FG and have the perpendicular to FG move with it so that it always passes through H and remains perpendicular to FG. Cheerios, however, had only dragged their final construction by moving point G.

Table 9.9 The change in discourse in Episodes 2–6 (summary)

Episode	Production of the perpendicular routine	Verification of perpendicularity routine	Use of the word *perpendicular*	Use of visual mediators
2	Creating another reference line in relation to line FG (Cornflakes and Cheerios) Spatio-graphical solution/drawing a perpendicular-looking line (Cornflakes) Spatio-graphical solution/drawing a perpendicular-looking line (Cheerios & Cornflakes)		Signifying a visual image of perpendicular to disagree with a spatio-graphical solution (Fruitloops) Signifying a visual image of perpendicular to disagree and then agree with a spatio-graphical solution (Fruitloops)	PBC random dragging (Cornflakes)
	Spatio-graphical solution (Cornflakes, Fruitloops, Cheerios)	Spatio-graphical verification/vertical-horizontal alignment of the lines (Cheerios)	Signifying a visual image of perpendicular to agree with a spatio-graphical solution (Fruitloops)	
3	Spatio-graphical solution/imitation of paper-pencil routine of drawing the perpendicular using PBC as straightedge (Fruitloops & Cornflakes)	Measurement-based verification using PBC (Cornflakes & Fruitloops)		PBC as protractor (Cornflakes) PBC as straightedge (Fruitloops)
4	Use of circles with no dependencies defined (Fruitloops)			PBC as image of construction (Fruitloops)
5	Constructing dependencies/use of equal-radius circles (Fruitloops) Dynamic solution/attaching the arbitrary point H to the line (Fruitloops)		Signifying a mathematical relationship (Fruitloops)	
6		Spatio-graphical (Cornflakes) Looking for a verification routine beyond spatio-graphical evidence (Fruitloops & Cornflakes)	Signifying a mathematical relationship (Fruitloops)	

dicular (in Episode 6). In this episode, Cornflakes pointed at the visual appearance of the figure to convince Fruitloops. However, Fruitloops seemed to be looking for a verification routine that would go beyond the spatio-graphical. She even used the word "proof"—though not necessarily in a deductive mathematical sense. This situation is quite contrary to the findings in the literature, as students' validation of a mathematical statement often takes the form of testing it against a few examples, even at the more advanced levels (Chazan, 1993b; Coe & Ruthven, 1994). In the case of dynamic geometry, students often think that they can justify a claim by empirically checking the diagram (Laborde, 2004)—that is, by dragging.

This situation and the difficulty the team had with defining point H as the middle point suggest revisions in Topic 3. The group constructed the PBC at the beginning of their session following scripted steps. Completing the task with Fruitloops, Cheerios said, "I just made the intersecting line and point in the middle," continuing, "it made a perpindicular line" (Log 9.1, Lines 32–33). However, there was not much discussion of its mathematical aspects. The group immediately moved to the next task of constructing a perpendicular to a line through a given point. It may be necessary to lead students explicitly to discuss their constructions mathematically when scaffolding the development of higher-level discourses. If participants are genuinely wondering about the relationships and asking questions, as in the present case, additional task instructions could even provide the geometrical theory behind such constructions. Encouraging students to make explicit connections between their deduction and construction knowledge is important; otherwise, as Schoenfeld (1988) cautioned, students may be learning about dynamic constructions merely as a set of procedures to follow.

The word *perpendicular* was first used by Cheerios in the first part of the task (Episode 1). She uttered the word only once, as if to revoice the instructions. Fruitloops, on the other hand, used the word throughout the problem-solving session. Her use of the word also represented a parallel advancement along with the production and verification routines. Initially the word signified a visual image of perpendicularity and was used to evaluate produced visual solutions (in Episodes 2 and 3). Later, however, her use of the word came to refer to a certain relationship between figures (in Episodes 5 and 6).

Finally, it is reasonable to argue that the *PBC*, the already completed construction, functioned as *the key visual mediator* of the session. The PBC figure is derived from Euclid and was presented as a resource in the Topic 3 instructions. The group was also asked to construct the PBC at the beginning of their session following very specific steps. In the second task, Cornflakes brought it to the team's attention when the team seemed to be out of ideas (in Episode 2). Although at first she only played with it randomly, she later figured out a way to use it as a protractor, thus as a tool for verifying perpendicularity (in Episode 3). This use may have led Fruitloops to view it as a straightedge that could be used to draw the perpendicular (in Episode 3). More importantly, however, the PBC became a crucial resource that probably triggered Fruitloops' use of circles, which led to the framing of the problem as a construction task (in Episode 4).

These observations about the PBC are important for at least three reasons: First, when students appear to be stuck with the problem or run out of ideas, they seem to make use of every resource within their problem-solving space. Ryve, Nilsson, and Pettersson (2013) underline the crucial role that visual mediators play in effective communication. However, along with visual mediators, they have also observed that technical terms (i.e., technical mathematical words) were equally important for communication that is effective. In Episode 5, just before Fruitloops framed the task as construction, Cornflakes provided a definition for the term "bisection" (Log 9.7, Line 65). This definition was not given to the team with the task; thus Cornflakes must have found it somewhere else. A little later, when Fruitloops realized the problem with their circles, she was lacking the mathematical terms to express the situation. She said "we didn't put a point between the circles so the line isn't perpendicular" and then "the part where the circles intersect" (Log 9.7, Lines 66, 69). Hence, CSCL task designers should pay considerable attention to the type of resources to be provided to students with the problems. These resources should encompass not only visual mediators but also the technical mathematical words.

Secondly, Cornflakes initially was not able to place the PBC on top of line FG correctly, but Fruitloops completed what Cornflakes had in mind, and Cornflakes responded with a "yes." Afterward, Fruitloops realized another procedure for producing the perpendicular (i.e., use the PBC as a straightedge to draw the perpendicular). All these suggest that in a setting like VMT, "transactive dialogue" (Berkowitz & Gibbs, 1985 as cited in Barron, 2000) can take place through participants' actions using visual mediators on the shared computer screen. This seems more likely when students lack the technical terms to express themselves, as in this case. The "take control" button opens up a "joint problem space" for dynamic manipulations and affords action-based dialogue, in addition to the conversational turns supported by the chat platform. In that way, as Roschelle and Teasley (1995) observed, participants can still interact productively even when they lack the technical vocabulary to talk about the problem.

Third, and most importantly, one could observe that the PBC accompanied the moments of change in mathematical routines: first from the vertical-horizontal alignment of the lines to the use of PBC as a straight-edge guide in Episode 3 and then to the use of circles in producing the perpendicular in Episode 4. Thus, it is reasonable to assume that it played a significant role in the change in mathematical discourse in this problem-solving session.

Along with the PBC, other aspects of the VMT environment also seemed to play a role in the moments of discourse shifts. In Episode 4, Fruitloops introduced a new production routine when she suggested making the circles first (Log 9.6, Line 62) and started constructing the circles. The team constructed circles in the first part of Topic 3 (to construct PBC) and the equilateral triangle in the previous topic in the VMT curriculum (Topic 2), which also required using circles in defining dependencies. Thus, the VMT curriculum, particularly the sequence of the topics in that curriculum, might have also played an important role in supporting students' discourse development.

Initially Fruitloops' circles were not created using the necessary dependencies such as the equal-radius relationship. As no dependencies were defined, the team had problems creating the line that would go through the intersections of the circles and the point H. That is, the dynamic geometry software provided the essential feedback until Fruitloops realized that they needed to construct the circles with certain relationships (in Episode 5). Both Cheerios and Cornflakes played with their construction to see that there was something missing with their solution at that stage. This situation also confirms Roschelle and Teasley's (1995) observation that when students had differing ideas, they were able to experiment with the computer representation. In a dynamic geometry environment, the drag function enables testing the construction if dependencies are correctly defined. Eventually, this experimentation leads the participants to generate new ideas, when they see that their solution is not supported by the software.

This analysis was conducted at the group unit of analysis involving the team discourse rather than the individual cognition of the students.[2] This analysis is not necessarily meant to suggest that the individual team members, including Fruitloops, decisively moved beyond the visual discourse. Nor is the observed discursive jump by the team necessarily an indication of "individualization" (Sfard, 2008) that the team members will henceforth follow more formal mathematical procedures and

[2]In a similar analysis of all eight sessions of the Cereal Team, Stahl (2016) conceptualizes the development of the group's mathematical cognition in terms of the successive adoption of group practices, rather than routines, in order to emphasize that they are being theorized as group-level rather than individual phenomena. As illustrated in the six episodes here, the Cereal Team questions, negotiates, and adopts new practices through their discourse (including shared GeoGebra actions). This meaning-making process creates a shared understanding within the team. Once the team agrees to use a routine, it may become a group practice, which can be used in the future without further discussion.

employ more formal word uses irrespective of the context. One can observe that Cornflakes and Cheerios were mostly attending to the spatio-graphical aspects of their figure, even toward the end of the session. Even Fruitloops was not able to clearly articulate why and how circles worked.

This team of novices succeeded in participating within a collective discourse that gradually took a more mathematical character. Yet, this more formal discourse was, as Baruch Schwarz (personal communication, June 2014) suggested, *rooted* in the spatio-graphical solutions—i.e., solutions that rely on reasoning and recognition of geometric figures with their appearances without any regard to their mathematical properties (Laborde, 2004). Thus, similar to what Sinclair and Moss (2012) noted, the process of discourse change may be better described as oscillating—rather than simply shifting—between the visual and more formal discourse levels.

Sfard's commognitive framework provided an account for the development of geometrical thinking observed within this episode. Rather than talking about fixed-ordered geometrical cognitive levels, as in van Hiele levels (Van Hiele, 1986), Sfard (2008) talks about incommensurable mathematical discourses. Saying that two discourses are incommensurable does not mean that one cannot participate in both of them at the same time. It simply means that "they do not share criteria for deciding whether a given narrative should be endorsed" (Sfard, 2008, p. 257). However, moving toward higher discourse levels requires "student's acceptance and rationalization (individualization) of the discursive ways of the expert interlocutor" (p. 258). Thus, students need to interact with expert others in order to develop sophisticated mathematical discourses. The findings in this study indicate that an environment such as VMT may provide a context in which students can engage in higher-level mathematical discourses with their peers.

Thus, along with instruction by expert mathematicians, well-designed virtual collaborative learning environments can provide a form of interaction that supports significant mathematical discourse development. In that regard, the findings support Sinclair and Moss (2012), who suggested that dynamic geometry software could function as a stand-in or alternative for the discourse of experts. In the present case, multi-user dynamic geometry was a component of the VMT software, which was built to support collaborative learning with a specific geometry curriculum (Stahl, 2013b). Therefore, in addition to the dynamic geometry component, the curriculum and the collaborative interaction aspects of the VMT environment also played crucial roles in supporting students' mathematical discourse development.

There is a tendency in educational research to reduce cases of group cognition to psychological phenomena of individual cognition. Considering the Cereal Team's problem-solving session, one may be inclined to think that Fruitloops was the higher thinker in this session. Not only did she appear to be the one solving the second task, she also wondered why it worked. However, that was not where she started. Initially, her notion of perpendicular referred to a visual image. It evolved into one that represented a mathematical relationship. Similarly, at the beginning, her routine of the production of the perpendicular involved a spatio-graphical solution, the same as for everyone else in the team, which only later became one that was based on defining dependencies. These transformations took place within the context of interacting with her team members, enacting task instructions, and interacting with the VMT software. Furthermore, most of the time, her lead was negotiated with the other team members, as part of the team's coordination of social resources (Oner, 2013). These took the form of the others building on her actions (as in Episode 5) as well as engaging in *transactive dialogue* (Berkowitz & Gibbs, 1985 as cited in Barron, 2000) with Cornflakes (as in Episode 6). She received help from other team members (as in Episode 1). The PBC was brought to her attention by other team members (Cornflakes) as well. Thus, the team's success was the product of group cognition, not simply attributable to one team member (Stahl, 2006).

Would the findings be applicable for other online groups? Qualitative case studies, such as this one, are not usually designed to make grand generalizations concerning the population. They, however, allow making what Stake (1978) calls "naturalistic generalizations," that is, the findings from a case would generalize to another similar case, rather than to the population and then to particular situations. Furthermore, this case study should not be viewed as a summative assessment of the VMT environment but as part of one cycle in an iterative DBR investigation. Accordingly, it was more concerned with documenting learning and how a team of novice students accomplished significant advance in mathematical discourse within the VMT environment in order to guide modifications in technology, pedagogy, and curriculum—so that more student groups might undergo similar mathematical development in future versions of VMT.

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