



Nonexistence Certificates for Ovals in a Projective Plane of Order Ten

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Abstract. In 1983, a computer search was performed for ovals in a projective plane of order ten. The search was exhaustive and negative, implying that such ovals do not exist. However, no nonexistence certificates were produced by this search, and to the best of our knowledge the search has never been independently verified. In this paper, we rerun the search for ovals in a projective plane of order ten and produce a collection of nonexistence certificates that, when taken together, imply that such ovals do not exist. Our search program uses the cube-and-conquer paradigm from the field of satisfiability (SAT) checking, coupled with a programmatic SAT solver and the nauty symbolic computation library for removing symmetries from the search.

Keywords: Combinatorial search · Satisfiability checking · Symbolic computation

1 Introduction

Projective geometry—a generalization of the familiar Euclidean geometry where parallel lines do not exist—has been extensively studied since the 1600s. A special case of projective geometry occurs when only a finite number of points exist. A two-dimensional projective geometry with a finite number of points is known as a *finite projective plane*.

Despite a huge amount of study some basic questions about finite projective planes are still open—for example, how many points can a finite projective plane contain? It is well-known [14] that a finite projective plane must contain $n^2 + n + 1$ points for some integer n (known as the plane's *order*) and finite projective planes can be explicitly constructed in all orders that are prime powers. The order six case is excluded by a theoretical result of Bruck and Ryser [8] making ten the first uncertain order.

In the 1970s and 1980s, a significant amount of mathematical ingenuity and computer searches successfully eliminated the possibility of a projective plane of order ten [27]. Today, this remains one of the most prominent achievements of computational combinatorial classification [23]. The search was made feasible due to results of MacWilliams, Sloane, and Thompson [32] concerning the error-correcting code generated by a hypothetical projective plane of order ten. They showed that the weight distribution of this code depends on just two unknown parameters. One of these parameters is the number of ovals that exist in the projective plane of order ten—here an *oval* being a set of twelve points, no three of which are collinear.

In 1983, Lam, Thiel, Swiercz, and McKay [30] showed the nonexistence of ovals in a projective plane of order ten via a computer search. The search space is of a significant size and required about 4,400 h of computation time on the super-mini computer VAX 11/780 (clock speed 5 MHz) to search exhaustively. Because of the nature of the search, Lam et al. specifically encouraged an independent verification:

Since the existence of ovals is an important question, we hope that someone will do an independent search to verify the result.

Despite this hope, there has been little published work independently verifying the search for ovals or their subsequent searches [28, 29] that culminated in the proof that projective planes of order ten do not exist. In his 2011 master’s thesis, Roy [37] performed a verification of the nonexistence of a projective plane of order ten using about 35,000 h on a cluster of desktop machines. However, he did not specifically run a search for the ovals as it was nonessential to his ultimate goal. To the best of our knowledge, there has been no published work specifically replicating the search for ovals.

In this paper, we report our results on verifying the nonexistence of ovals in a projective plane of order ten. Our method relies on a satisfiability (SAT) solver and produces certificates that a third party can use to verify that our search completed successfully. In total, our search used about 1,850 core hours on the supercomputer Graham at the University of Waterloo (clock speed 2.1 GHz) and produced SAT proofs that when compressed use about 3 terabytes of storage.

In addition to using a SAT solver our method also takes advantage of the nauty symbolic computation library [34] to reduce the size of the search space by eliminating redundant symmetries. We present the necessary background on projective geometry, satisfiability checking, and symbolic computation in Sect. 2, describe our SAT encoding in Sect. 3, give details on our implementation and results in Sect. 4, and finally discuss future work in Sect. 5.

2 Preliminaries

The main background necessary to understand our results are some familiarity with projective geometry (see Sect. 2.1), satisfiability checking (see Sect. 2.2), and symbolic computation (see Sect. 2.3).

2.1 Projective Geometry

A *finite projective plane* of order n is a collection of $n^2 + n + 1$ points and $n^2 + n + 1$ lines and an incidence relationship between points and lines where any two points are incident with a unique line and any two lines are incident with a unique point. Furthermore, every line is incident with $n + 1$ points and every point is incident with $n + 1$ lines.

An *oval* of a projective plane of even order n is a set of $n + 2$ points (or $n + 1$ points when n is odd) with no three points collinear (incident with the same line). It can be shown that it is not possible to find a larger set of points with no three points collinear [15], but no characterization of ovals in general projective planes is known. In particular, prior to the search of Lam et al. [30] it was not known if a projective plane of order ten could contain ovals or not.

From a computational perspective, a convenient way of representing a finite projective plane of order n is by a square $\{0, 1\}$ incidence matrix whose (i, j) th entry contains a 1 exactly when the i th line is incident to the j th point. We say that two $\{0, 1\}$ -vectors *intersect* when they share a 1 in the same location and the *weight* of a $\{0, 1\}$ -vector is the number of nonzero entries it contains. In this framework, a projective plane of order n is a $\{0, 1\}$ -matrix with $n^2 + n + 1$ rows that each have weight $n + 1$ and pairwise intersect exactly once (and similarly for the columns). In other words, a $\{0, 1\}$ -matrix A with $n^2 + n + 1$ rows represents a projective plane exactly when it satisfies $AA^T = A^T A = nI + J$ where I denotes the identity matrix and J denotes the matrix consisting of all 1s. Two projective planes that are identical up to row or column permutations are called *isomorphic* and we call a submatrix of a projective plane a *partial projective plane*.

Suppose that A is a projective plane of order ten that contains an oval. Without loss of generality we assume that the first twelve points of the plane consist of an oval. By definition, each pair of points in the oval must define a unique line, and therefore there are $\binom{12}{2} = 66$ lines incident to the oval. Without loss of generality, we assume these lines are ordered in lexicographically increasing order. In other words, the first 66 rows of A have the form

$$B = \left[\begin{array}{c|c} 110000000000 & \\ 101000000000 & \\ \vdots & \\ 000000000011 & \end{array} \right] B'.$$

The first twelve columns contain two 1s on each row, so B' must contain nine 1s in each row. Furthermore, by definition of a projective plane each column in B' must intersect each of the first twelve columns. Each 1 in B' induces an intersection with two of the first twelve columns, so each column in B' contains exactly six 1s.

Without loss of generality, we assume the columns of B' are sorted in lexicographic order. This implies the first nine columns of B' will be incident with the first line (the line through the first and second points). As noted by [26], this also means the i th column of B' (for $1 \leq i \leq 9$) will be incident with the line through the third and $(3 + i)$ th points. We call the nine columns of B' that are

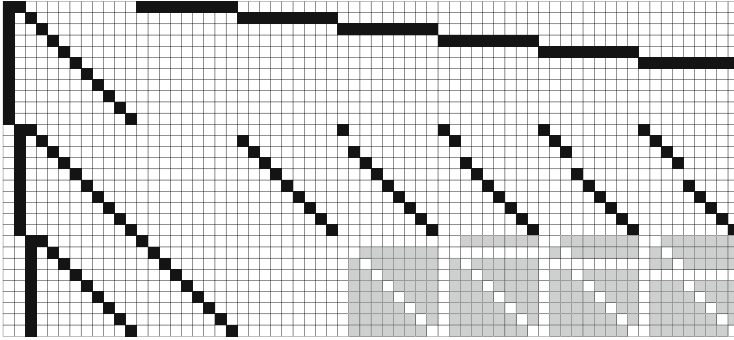


Fig. 1. The upper-left 30×66 submatrix of B under the assumption that the rows are lexicographically ordered and the columns outside the oval are lexicographically ordered. Black entries denote 1s, white entries denote 0s, and gray entries are unknown.

incident with the i th row the i th *block*. In general, all blocks' columns may be ordered similarly to those in the first block [26], and this fixes the first two 1s in each column of B' . Figure 1 contains a visual depiction of the first 30 rows of B up to the sixth block.

Some entries of B' are still undetermined (shown as gray in Fig. 1). At this stage, it is still uncertain if they can be completed in a consistent way to make B' a partial projective plane—since the above description assumes that an oval exists in A . Thus, a proof that there is no way of completing the unknown entries of B' in a consistent way would also imply the nonexistence of ovals in a projective plane of order ten.

The *symmetry group* of a matrix is the group of row and column permutations that fix the entries of the matrix. For example, consider the symmetry group S of the first twelve columns of B . Each row of this submatrix is completely specified by the two columns incident to it, so any column permutation completely specifies a row permutation that undoes the permutation. It follows that S is isomorphic to S_{12} , the symmetric group on twelve elements.

The group S acts on the entries of B' as follows: Given a permutation $\varphi \in S$ the row permutations from φ are applied to the entries of B' , then column permutations are applied to reorder its columns in lexicographic order. The result $\varphi(B')$ is a partial projective plane that is isomorphic to B' . To avoid duplication of work, any search for B' should ideally avoid exploring parts of the search space that are isomorphic under S . Exploiting this leads to a huge reduction in the size of the search space, since S contains about 479 million permutations.

2.2 Satisfiability Checking

Given a formula of Boolean logic, satisfiability (SAT) checking is to determine whether or not the formula is *satisfiable*—that is, is there a way of assigning true

and false to its variables that results in the whole formula becoming true? A *SAT solver* is a program that performs SAT checking on a given formula. Modern SAT solvers require their input to be given in conjunctive normal form or CNF: if x is a Boolean variable then x and $\neg x$ are known as *literals*, expressions of the form $l_1 \vee \cdots \vee l_n$ for literals l_i are known as *clauses*, and expressions of the form $c_1 \wedge \cdots \wedge c_m$ for clauses c_i are in *CNF*. The literal x is satisfied when x is assigned true, $\neg x$ is satisfied when x is assigned false, $l_i \vee \cdots \vee l_n$ is satisfied when at least one l_i is satisfied, and $c_1 \wedge \cdots \wedge c_m$ is satisfied when every c_i is satisfied.

In order to reduce the search for ovals in a projective plane of order ten to a SAT problem we use the incidence structure described in Sect. 2.1 that was based on the assumption that ovals exist. The SAT instance will have a solution when there is a completion of the unknown entries of the matrix B to a partial projective plane—so showing the instance has no solution implies that no ovals exist.

SAT solvers are effective as combinatorial search tools—for example, they were used to resolve the first open case of the Erdős discrepancy conjecture [25]. The *cube-and-conquer* SAT solving paradigm has been particularly effective at solving very large combinatorial search problems [21]. First developed by Heule, Kullmann, Wieringa, and Biere for computing van der Waerden numbers [22], the cube-and-conquer method has since been used to resolve the Boolean Pythagorean triples problem [20] and determine the value of the fifth Schur number [16].

A *cube* is a formula of the form $l_1 \wedge \cdots \wedge l_n$ where l_i are literals. In the cube-and-conquer paradigm a SAT instance is split into a number of distinct subinstances specified by cubes. Each subinstance contains a single cube and the cube is assumed to be true for the purposes of solving the subinstance. The cubes are typically generated by running a “cubing solver” on the SAT instance which attempts to find a set of cubes which split the instance into subinstances of approximately equal difficulty. After the cubes have been generated a “conquering solver” solves the subinstances either in sequence or in parallel. Ideally, the literals in each cube are added to the solver as incremental assumptions. In this case, after each cube is solved the assumptions are removed and the literals from the next cube are added without restarting the SAT solver.

2.3 Symbolic Computation and SAT+CAS

Symbolic computation is a branch computer science devoted to manipulating and simplifying mathematical expressions. Many computer algebra systems (CASs) are available today that contain extensive symbolic computation functionality from a huge number of mathematical domains. However, although CASs contain many sophisticated algorithms, they have not typically been optimized to perform searches in the way that SAT solvers have [1, 4].

For problems that need *both* mathematical sophistication and finely-tuned search it can be useful to combine computer algebra and SAT solvers [10]. Recently SAT+CAS methods have been used in a number of various problems—for example, they have been used to verify the correctness of Boolean arithmetic

circuits [24], improve the best known result in the Hadwiger–Nelson plane-colouring problem [17], find many new algorithms for multiplying 3×3 matrices [19], and improve the best known result in the Ruskey–Savage hypercube conjecture [40].

In addition to a SAT solver we use the nauty symbolic computation library [34] in order to show the nonexistence of ovals in a projective plane of order ten. We call nauty from within the callback function of a “programmatic” SAT solver. A solver is called *programmatic* if it allows learning clauses on-the-fly through a piece of code supplied to the SAT solver. A programmatic SAT solver will run the supplied code from time to time as it is performing its search. The code will examine the current assignment to the variables and test whether the current assignment may be discarded (possibly using knowledge queried from a CAS). If the assignment can be discarded a clause is added to the SAT instance on-the-fly that blocks the current assignment (and ideally other similar assignments). Programmatic SAT solvers were introduced by Ganesh et al. [11] in order to solve an RNA folding problem. They have since been used to search for various combinatorial objects such as Williamson matrices [5], best matrices [7], and complex Golay sequences [6].

3 Satisfiability Encoding

We now describe the encoding that we use to search for ovals in a projective plane of order ten. As described in Sect. 2.1, we may assume a number of entries of this projective plane have been fixed in advance, including all entries in the first twelve columns and all entries in the first 21 rows (see Fig. 1). Specifying these entries removes a substantial amount of symmetry from the search space, however, as described in Sect. 2.1, the remaining search space is still symmetric under the action of the group S generated by permuting the twelve points of the oval. In Sect. 3.1, we give our basic encoding without removing symmetries from S . In Sect. 3.2, we provide a programmatic SAT method of removing symmetries from the group S .

3.1 Basic SAT Encoding

Following Sect. 2.1, let B be the first 66 rows of the incidence matrix of a partial projective plane of order ten whose first twelve points form an oval. As previously outlined, up to isomorphism some points of B can be assumed in advance, but most points remain unspecified. For each unspecified point we define a Boolean variable $b_{i,j}$ that will be true exactly when the i th line is incident to the j th point, i.e., the (i, j) th entry of B is 1.

We now give properties that necessarily hold in B as Boolean constraints in conjunctive normal form. In particular, we encode the two facts that (1) columns of B intersect at most once and (2) each column of B intersects a column in the oval at least once. A similar encoding has been previously used to verify MacWilliams et al.’s result that vectors of weight 15 do not exist in the row space of any projective plane of order ten [3].

Columns Intersect at Most Once. Let i and j be arbitrary column indices of B , so $i, j \in \{1, \dots, 111\}$. By definition of a projective plane these columns cannot intersect twice, so we know there do not exist rows k and l mutually incident to columns i and j . In Boolean logic we write this constraint as

$$\bigwedge_{1 \leq k < l \leq 66} (\neg b_{k,i} \vee \neg b_{k,j} \vee \neg b_{l,i} \vee \neg b_{l,j}).$$

Each Column Intersects a Column in the Oval. Let i be an arbitrary column in the oval, i.e., $i \in \{1, \dots, 12\}$ and let j be an arbitrary column not in the oval, i.e., $j \in \{13, \dots, 111\}$. By definition of a projective plane columns i and j must intersect somewhere in the plane. Since all rows incident with column i occur in the first 66 rows, the intersection of columns i and j must occur in B . In Boolean logic we write this constraint as $\bigvee_{k: B[k,i]=1} b_{k,j}$.

Abbreviated Constraints. For the first set of constraints there are $\binom{111}{2} \cdot \binom{66}{2} \approx 13$ million clauses of the first form and $12 \cdot 99 = 1188$ clauses of the second form. After removing variables whose values are already fixed there are 2696 undetermined variables in these clauses.

SAT solvers often perform better if the number of constraints can be significantly decreased. In our case, we found that it was only necessary to consider a submatrix of B before reaching a contradiction. In particular, our primary searches only used the variables in the blocks 2 to 6 (columns 22 to 66). The first block was skipped since its columns did not intersect the columns of any other block in the known entries (see Fig. 1). This increases the efficiency of the search because contradictions are generally easier to derive from two already-intersecting columns. Using columns 22 to 66 meant there were $\binom{57}{2} \cdot \binom{66}{2} \approx 3.4$ million clauses of the first form, $12 \cdot 45 = 540$ clauses of the second form, and 1199 unknown variables.

Known Row Intersections. We included one further set of constraints that, while not strictly necessary, improved the performance of the SAT solver by enforcing row intersections that must occur. In particular, note that rows 2–6 must intersect rows 22–66 in B and all 1s in row $i \in \{2, \dots, 6\}$ outside the oval occur in the columns $B_i := \{4 + 9i, \dots, 12 + 9i\}$. Thus, we also included clauses of the form $\bigvee_{k \in B_i} b_{j,k}$ for rows $i \in \{2, \dots, 6\}$ and $j \in \{22, \dots, 66\}$ that do not intersect in the oval.

3.2 Symmetry Breaking

The encoding described in Sect. 3.1 could in theory be used to show there is no way of completing the unknown entries of B subject to the given constraints. However, as discussed in Sect. 2.1 the search space is symmetric under the action of relabelling the twelve points of the oval (while appropriately reordering the

rows and remaining columns to preserve our lexicographic presentation of the search space). Since this is an enormous group of symmetries it is worthwhile developing a method that will reduce or “break” these symmetries. Using an “orderly generation” algorithm [36,38] is one way to avoid generating isomorphic partial solutions at each stage of the search. Our approach is similar, though it will only avoid isomorphic partial solutions violating a property that we can show the entries of B satisfy (up to isomorphism).

Mathon [33] provided a characterization of an oval in a projective plane of order ten in terms of K_{12} , the complete graph on vertices $\{1, \dots, 12\}$. Note that a *1-factor* of a graph is a perfect matching of its edges and a *1-factorization* of a graph is a decomposition of its edges into 1-factors. If rows denote edges and columns denote points then the first twelve columns of B are precisely the incidence matrix of K_{12} . Every column of B outside the oval contains six 1s on rows that will not be adjacent (as edges of K_{12}). Therefore, each column of B outside the oval forms a 1-factor of K_{12} .

Furthermore, consider the set of columns in the first block of B . These rows are all incident to the row through points 1 and 2. The other five 1s in each column must each occur on distinct rows and will cover the remaining $\binom{10}{2} = 45$ rows through the points $\{3, \dots, 12\}$. Therefore, the first block of B forms a 1-factorization of $K_{12} \setminus \{1, 2\} \cong K_{10}$ and in general the i th block forms a 1-factorization of $K_{12} \setminus \{1, i + 1\}$. Gelling [12] determined that there are exactly 396 nonisomorphic 1-factorizations of K_{10} and we assume that each has been given a distinct label in the set $\{1, \dots, 396\}$.

Note that the symmetry group S generated by permuting the columns of the oval acts transitively on the set of blocks: there is a permutation in S that will send any block to any other block. Suppose we tag each completed block of B with the label (as described above) of the 1-factorization that it is isomorphic with. We may assume that block 2 of B has the minimal label amongst the blocks of B —if it didn’t, we could send the block with minimal label to block 2 by an appropriate permutation of S . Our symmetry breaking method will enforce the condition that block 2 has the minimal label amongst the other blocks of B for which we are searching (blocks 2–6). However, it is not very easy to concisely express this constraint as clauses in Boolean logic. Therefore, we make use of the programmatic SAT paradigm in order to enforce this constraint on-the-fly.

Programmatic Symmetry Breaking. A programmatic SAT solver is compiled with a “callback” function that often examines the solver’s current assignment (the mapping from variables to truth values). When the callback function determines the current state should be discarded it will add clauses to the SAT instance that block the current assignment.

If all the variables in the i th block have been assigned and p is one of these variables then we let $B_i \models p$ denote that variable p has been assigned true. Suppose all the variables in block 2 and block $i \in \{3, \dots, 6\}$ have been assigned. If the label of block 2 is strictly larger than the label of block i we want to block this configuration from the search space. In such a case we want to add the Boolean constraint

$$\bigwedge_{B_2 \models p} p \rightarrow \left(\neg \bigwedge_{B_i \models p} p \right) \quad \text{or equivalently} \quad \bigvee_{B_2 \models p} \neg p \vee \bigvee_{B_i \models p} \neg p$$

which says that the i th block cannot be assigned the way it currently is while the second block is assigned the way it currently is.

4 Implementation and Results

Our SAT encoding is implemented as a part of the MathCheck project; our scripts are open source and available online at uwaterloo.ca/mathcheck. The search proceeds in three main parts: First, we verify the result of Gelling [12] that there are exactly 396 nonisomorphic 1-factorizations of K_{10} . Second, we generate 396 separate SAT instances (one for each nonisomorphic way of filling in block 2 of B). The cube-and-conquer method is used in parallel to solve each SAT instance. A cubing solver generates a set of cubes from each SAT instance and a programmatic conquering solver is used to show that (up to the symmetry breaking method of Sect. 3.2) there are 58 ways of completing the blocks 2–6. Finally, we generate a new SAT instance for each of the 58 solutions and verify that there are no consistent ways of extending these completions to block 7. Additionally, to increase the confidence that the SAT instances were successfully solved the SAT solvers produced DRUP (delete reverse unit propagation) certificates [18] which were subsequently verified. A flowchart of these steps is available in this paper’s appendix at uwaterloo.ca/mathcheck.

4.1 Generating the SAT Instances

The SAT instances are generated by a Python script that writes the clauses described in Sect. 3.1 to a file in DIMACS (Discrete Mathematics and Theoretical Computer Science) CNF format. The script accepts as a parameter the columns to include in the SAT instance and by default uses the columns in blocks 2–6 (those used in our primary search).

4.2 Generating the Nonisomorphic 1-Factorizations

The 396 nonisomorphic 1-factorizations of K_{10} as reported by Gelling [12] can be quickly generated using a straightforward search, but we used a SAT approach as that was convenient for our purpose. The SAT instance only uses the variables in block 2, the columns of this block corresponding with 1-factors of $K_{12} \setminus \{1, 3\}$. As noted by Gelling, up to isomorphism the entries of the first 1-factor can be completely assumed. By the lexicographic ordering assumption the first 1-factor includes the edge (2, 4) and by permuting the columns $\{5, \dots, 12\}$ of the oval we can assume the first 1-factor contains the edges (5, 6), (7, 8), (9, 10), and (11, 12). Gelling also noted that after fixing the first 1-factor there are exactly two ways (up to isomorphism) of fixing the second 1-factor and the union of the first two 1-factors either form a 4-cycle and a 6-cycle or a 10-cycle.

The entries that can be fixed are given to the SAT solver as unit clauses and a programmatic implementation finds all nonisomorphic 1-factorizations. Whenever a completion of block 2 is found, the program `Traces` from the `nauty` graph isomorphism library determines if the completion is new or isomorphic to a previously found completion. (The graph provided to `Traces` is the *incidence graph* representation [13] of the first 12 columns of B and the columns of block 2.) A new completion is recorded for later use and a clause that blocks the completion (i.e., $\bigvee_{B_2 \models p} \neg p$) is added to the SAT instance until all possible completions have been examined. A programmatic implementation of `MapleSAT` [31] confirms the result of Gelling that 396 nonisomorphic 1-factorizations of K_{10} exist in about 8 s.

4.3 Solving the SAT Instances: Cubing

We now generate 396 distinct SAT instances, one for each of the 396 nonisomorphic ways of filling in block 2. Variables from blocks 2–6 are used in each SAT instance, with the variables in block 2 completely determined by the specific nonisomorphic 1-factorization chosen in each case. We simplify these instances with the preprocessor of the SAT solver `Lingeling` [2] which produces proofs of simplification without renaming variables. After simplification, these instances each contained 912 unknown variables and on average contained 22,883 clauses. Simplifying all 396 SAT instances requires about 15 min in total.

Next, we apply the cubing solver `March_cu` [22] on each of the 396 individual SAT instances. The conquering solver (see Sect. 4.4) typically performs better when the variables in the cubes are not split across blocks. Thus, we modified `March_cu` so that it only produces cubes using variables occurring in the same block as the first variable in the cubes. We controlled the cubing cutoff using the `-n` parameter of `March_cu` which stops cubing once the number of free variables falls below the given bound. Each block contains 228 unknown variables and we stop cubing once the subproblems specified by each cube contain at least 228 fewer free variables than the original instance. On average, `March_cu` produced about 180,000 cubes per SAT instance and spent about 17.5 total hours in this step.

4.4 Solving the SAT Instances: Conquering

The majority of the search work was done by the conquering solver. A programmatic version of `MapleSAT` [31] was used to complete this step. Each of the 396 SAT instances along with the cubes previously computed for each instance were given to separate instances of `MapleSAT` and solved in parallel. The literals in each cube were specified as incremental assumptions [35], so that it was not necessary to restart the SAT solver after solving each cube.

The programmatic encoding from Sect. 3.2 was used to ignore any completions of blocks 3–6 whose label was strictly smaller than the label of block 2 (which was fixed in each SAT instance). The label of each block completion can

be computed by calling `nauty` on the incidence graph representation of the block (as described in Sect. 4.2). However, these incidence graphs contain 87 vertices (from 66 rows and 21 columns) and we found there was significant overhead from calling `nauty` in this way.

Our final implementation makes use of a simpler check based on Gelling’s observation that up to isomorphism each pair of columns in a block are of two types (either a 4-cycle and 6-cycle or a 10-cycle). Given a complete block, we check all $\binom{9}{2} = 36$ pairs of columns and generate the *cycle pattern* for each block up to isomorphism—for example, one cycle pattern consists of the case when all pairs of columns form 10-cycles. In general, a cycle pattern graph on 9 vertices is constructed where two vertices are adjacent exactly when their associated two columns form a 10-cycle. Using `nauty` we determined that the 396 distinct block types gave rise to 359 distinct cycle patterns and in our programmatic implementation we used the cycle pattern as a proxy for determining the block label. In most cases the cycle pattern could be used to uniquely identify the block label, but otherwise the block label was assigned to the largest possible label consistent with the given cycle pattern (i.e., the most pessimistic choice in terms of symmetry breaking).

Following [29], block labels were chosen for the blocks by sorting the blocks in ascending order by the size of their stabilizer groups. Additionally, blocks with identical cycle patterns were given adjacent labels when possible in order to minimize the impact of the above pessimistic choice.

In total, this step required about 1,832 core hours on a cluster of Intel E5-2683 CPUs running at 2.1 GHz. The search produced 58 valid completions of the blocks 2–6 (see uwaterloo.ca/mathcheck for one explicit completion). Whenever a valid completion B was found, a clause $\bigvee_{B \models p} \neg p$ was programmatically added to the SAT instance. The added clause blocked the completion from occurring again later in the search.

Finally, for each of the 58 completions of blocks 2–6 a SAT instance was generated that included the constraints from blocks 2–7 and a cube specifying the completion (i.e., $\bigwedge_{B \models p} p$). It was found that none of the completions of blocks 2–6 could be extended to block 7 and this final step required less than a second.

4.5 Certificate Verification

The runs from the solvers produced DRUP proofs totalling about 33 terabytes. These were verified using the proof verification tool DRAT-trim [39] which was also used to trim and compress the proofs. These optimized proofs were archived using 7z data compression and produced archives totalling about 3 terabytes. These archives are available from the authors by request.

In order for the proofs to be verified by DRAT-trim the clauses which were programmatically generated during the solver’s run also need to be provided to DRAT-trim. One way of doing this is to add the programmatic clauses directly into the CNF file provided to DRAT-trim. However, this method was found to

suffer from very poor performance because this significantly increased the size of the initial active clause database tracked by DRAT-trim.

To get around this we modified DRAT-trim to support the addition of “trusted” clauses midway through the proof. Normally, each step of a proof consists of either an *addition* or *deletion* to the active clause database. In the case of an addition, DRAT-trim verifies that the added clause is a logical consequence¹ of the current set of active clauses. In our proofs we have a third kind of step, a *trusted addition* that adds the clause into the current set of active clauses without checking its provability. The justification for these clauses relies on our symmetry breaking method and not on a property easily checkable in Boolean logic, so the symmetry breaking clauses were not verified by DRAT-trim. However, if you believe in the correctness of our SAT encoding, generation scripts, DRAT-trim, and the trusted additions (whose correctness relies on our symmetry breaking method and a call to nauty) then you must believe in our certificates.

These proofs were checked using a system configured to limit each core to at most 4 GB of memory. In order to meet this limit it was necessary to ensure that each proof did not grow too large. To do this, March_{cu} was used generate a second “toplevel” set of cubes that partitioned the i th SAT instance into $398 - i$ subinstances. (As the label increased fewer subinstances were used because the instances became easier due to symmetry breaking.) Each of the subinstances were solved and had their proofs verified separately (each using at most 4 GB of memory and 10 min of computing time).

5 Conclusions and Future Work

In this paper we have completed an independent search showing the nonexistence of ovals in a projective plane of order ten. This was accomplished using a reduction to the Boolean satisfiability problem along with a SAT solver to show the resulting SAT instances are unsatisfiable. However, in order to make the amount of computation feasible it was necessary to use a symmetry breaking method. We used a “programmatic” SAT solver coupled with the symbolic computation library nauty [34] in order to learn symmetry breaking clauses on-the-fly during the search.

Our implementation uses the SAT+CAS interface as developed by the MathCheck project [40]. We are currently working on using MathCheck to verify more of the searches that were necessary in order to show the nonexistence of projective planes of order ten [3]. To date, we have verified the searches of MacWilliams et al. [32], Carter [9], and Lam et al. [29] that show that the row space of a projective plane of order ten does not contain vectors of weight 15 or 16. A consequence of these searches is that the weight enumerator of the error-correcting code generated by a projective plane of order ten can be specified exactly [32].

¹ DRAT-trim also supports a more general kind of provability termed “resolution asymmetric tautology” that we did not use in our proofs.

In particular, the rowspace of a projective plane of order ten must contain exactly 24,675 vectors of weight 19. The search for such vectors is the only case that remains in order to provide a complete SAT-based independent verification of the nonexistence of projective planes of order ten. We are currently exploring the feasibility of this and believe MathCheck will be useful in this case as well. The same basic encoding can be used but it seems necessary to tailor the symmetry breaking method and the structure of the search. This will be the subject of future research.

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