Limit Analysis of Dry Masonry Block Structures with Non-associative Coulomb Friction: A Novel Computational Approach

Nicola A. Nodargi, Claudio Intrigila, and Paolo Bisegna

Abstract The limit analysis of dry-masonry block structures with non-associative Coulomb friction is formulated as a Mixed Complementarity Problem. After highlighting some of its peculiar features, such as the lack of uniqueness of the collapse multiplier, a fixed-point based algorithm is presented for constructing a solution, obtained by iteratively solving straightforward associative limit analysis problems. Supported by the comparison with benchmark problems, the resulting procedure is proven to be able to predict the collapse multiplier of masonry block structures with accuracy, robustness and effectiveness.

1 Introduction

The analysis of the mechanical behavior of historical masonry structures represents a significant research topic in computational mechanics, as related to the preservation and the restoration of architectural heritage and of historical buildings. Many computational strategies have been developed to date, aiming at modeling masonry response at different scales and levels of complexity. Among them, it is worth mentioning micromechanical approaches (e.g., see [13, 30]), multi-scale/homogenization approaches (e.g., see [1, 6, 21, 35]) and macromechanical/phenomenological approaches (e.g., see [14, 24, 27, 33, 34]), to be used in conjunction with finite element formulations suitable for the analysis of inelastic structures (e.g., see [7, 8, 23, 25, 26, 28]).

N. A. Nodargi (✉) · C. Intrigila · P. Bisegna
Department of Civil Engineering and Computer Science, University of Rome Tor Vergata, 00133 Rome, Italy
e-mail: nodargi@ing.uniroma2.it

C. Intrigila
e-mail: intrigila@ing.uniroma2.it

P. Bisegna
e-mail: bisegna@ing.uniroma2.it

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Renouncing to a constitutive description of masonry material, the collapse loading of masonry structures can be rigorously determined by limit analysis theory, as first shown by Kooharian and Heyman in their classical works (see [16, 18]). A fundamental contribution in translating limit analysis into a computational strategy was the seminal work by Livesley (see [19]), who proposed to consider a typical masonry structure as a system of rigid blocks, which interact through no-tension frictional interfaces. The attractiveness of such idealization, only requiring the friction angle of block interfaces as material parameter, was also motivated by the simple format of Linear Programming (LP) problem taken by limit analysis theorems. Unfortunately, limit analysis theorems intrinsically presuppose an associative flow law, and, correspondingly to Coulomb friction, the collapse mechanism exhibits interface dilatancy and the collapse multiplier is usually overestimated (as already shown in [10] and [37]).

Abandoning standard limit analysis theorems to assume a non-associative friction flow law, a non-associative limit analysis problem for the analysis of masonry block structures has been progressively formalized in [3, 4, 11, 12, 20]. That is obtained by explicitly considering equilibrium and compatibility equations pertaining to blocks, along with admissibility constraints (including Coulomb friction), flow laws (including non-associative Coulomb friction flow law) and Kuhn-Tucker complementarity conditions pertaining to block interfaces. However, due to the non-convex structure of the complementarity constraint, the resulting coupled static/kinematic Mixed Complementarity Problem (MCP) carries an ill-posedness issue related to lack of uniqueness of the collapse multiplier. Accordingly, by assuming the minimum collapse multiplier as the actual target, a constrained minimization problem is formulated, with constraints given by the MCP conditions. In particular, that can be interpreted as a Mathematical Program with Equilibrium Constraints (MPEC). As nowadays optimization tools for the solution of the MPEC are severely limited in the size of problems they can handle, ad-hoc solution strategies have been explored in [11, 15, 31, 38].

In the present work, a fixed-point based algorithm is discussed for solving the non-associative limit analysis MCP relevant to 2D masonry block structures (see [29]). Basic observation is that a solution can be derived by considering a fixed-point problem, with the fixed-point map involving the solution of a simple associative limit analysis problem. Accordingly, the proposed procedure achieves to construct a non-associative limit analysis solution by iteratively solving straightforward associative limit analysis problems. Numerical results are presented for assessing accuracy, robustness and effectiveness of the proposed computational approach. Possible extensions of the present approach deal with the limit analysis of non periodic block masonry structures (e.g., see [5]) and of 3D block masonry structures, also undergoing large displacements (e.g., see [17, 36]).

The present paper is organized as follows. In Sect. 2 the non-associative limit analysis MCP is formulated. In Sect. 3, a simple two-blocks model problem is presented to highlight some features of the relevant MCP. In Sect. 4 the present fixed-point based solution algorithm is discussed. Numerical simulations are reported in Sect. 5 and conclusions are outlined in Sect. 6.

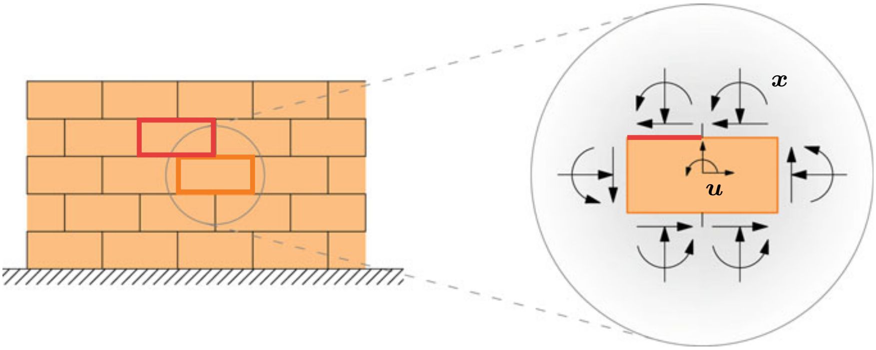

Fig. 1 Blocky model for dry-masonry structures. Block displacements $\boldsymbol{u}$ and block interface forces $\boldsymbol{x}$

2 Limit Analysis Problem for Block Structures

A system of 2D blocks is considered, as shown for example in Fig. 1, to model a typical dry-masonry block structure. It is assumed that the blocks are rigid and that they interact through Coulomb-frictional interfaces. Let b and c respectively denote the number of blocks and of interfaces, and let $\{O; x, y\}$ be a fixed Cartesian reference frame.

In case an associative flow law is considered for the frictional behavior of the interfaces, classical static and kinematic theorems of limit analysis hold. Specifically, assuming that external loads $\boldsymbol{f}_{\mathrm{d}} + \lambda \boldsymbol{f}_{\mathrm{l}}$ are applied at block centroids, with $\boldsymbol{f}_{\mathrm{d}}$ as dead loads, $\boldsymbol{f}_{\mathrm{l}}$ as basic live loads and λ as multiplier of live loads, the static theorem reads:

$$\begin{aligned} \max_{\lambda, \boldsymbol{x}} \quad & \lambda \\ \text{s.t.} \quad & \boldsymbol{E}\boldsymbol{x} + \boldsymbol{f}_{\mathrm{d}} + \lambda \boldsymbol{f}_{\mathrm{l}} = \boldsymbol{0} \\ & \boldsymbol{N}_{\mathrm{u}}^{T} \boldsymbol{x} \leq \boldsymbol{0}, \quad \boldsymbol{N}_{\mathrm{f}}^{T} \boldsymbol{x} \leq \boldsymbol{0}, \end{aligned} \tag{1}$$

where $\boldsymbol{x}$ is a $3c \times 1$ vector collecting interface shear forces, normal forces and bending moments, $\boldsymbol{E}$ is a $3b \times 3c$ block equilibrium operator, $\boldsymbol{N}_{\mathrm{u}}^{T}$ is a $3c \times 3c$ interface unilateral constraint operator and $\boldsymbol{N}_{\mathrm{f}}^{T}$ is a $2c \times 3c$ interface friction constraint operator (e.g., see [19]). Conversely, the kinematic theorem yields:

$$\begin{aligned} \min_{\boldsymbol{u}, \boldsymbol{z}_{\mathrm{u}} \geq \boldsymbol{0}, \boldsymbol{z}_{\mathrm{f}} \geq \boldsymbol{0}} \quad & -\boldsymbol{f}_{\mathrm{d}}^{T} \boldsymbol{u} \\ \text{s.t.} \quad & \boldsymbol{E}^{T} \boldsymbol{u} + \boldsymbol{N}_{\mathrm{u}} \boldsymbol{z}_{\mathrm{u}} + \boldsymbol{N}_{\mathrm{f}} \boldsymbol{z}_{\mathrm{f}} = \boldsymbol{0} \\ & 1 - \boldsymbol{f}_{\mathrm{l}}^{T} \boldsymbol{u} = 0, \end{aligned} \tag{2}$$

where $\boldsymbol{u}$ is a $3b \times 1$ vector collecting x-displacements, y-displacements and rotations (about block centroid) of blocks, $\boldsymbol{z}_{\mathrm{u}}$ is a $3c \times 1$ vector of interface unilateral flow multipliers, and z_f is a $2c \times 1$ vector of interface friction flow multipliers (e.g., see [19]). In a mechanical perspective, the static and kinematic theorems of limit analysis supply dual descriptions of collapse multipliers, respectively obtained by maximizing the load multipliers for which statically admissible equilibrium is feasible or by minimizing the (opposite of) resisting work related to kinematically admissible mechanisms. That duality also holds from an optimization standpoint, i.e. formulations (1) and (2) represent linear programming problems dual to each other, thus guaranteeing the existence of a unique collapse multiplier, which is the common optimal value of the two problems (e.g., see [11]).

Unfortunately, the duality of static and kinematic theorems is a consequence of the friction associative flow law. That is, of the fact that (up to a transposition) the same operator $\boldsymbol{N}_\mathrm{f}$, involving the interface friction angle φ, is used for expressing both the friction flow and the friction constraint. If a non-associative friction flow law is instead postulated to avoid spurious dilatancy, a distinct interface friction flow operator $\boldsymbol{V}_\mathrm{f}$ has to be considered in place of $\boldsymbol{N}_\mathrm{f}$, obtained by replacing the friction angle φ with the dilatancy angle ψ. As a consequence, static and kinematic problems are no longer uncoupled, and the limit analysis problem has to be formulated in the following form (e.g., see [3]):

$$
\begin{aligned}
&\boldsymbol{E}\boldsymbol{x} + \boldsymbol{f}_\mathrm{d} + \lambda \boldsymbol{f}_\mathrm{l} = \boldsymbol{0} \\
&\boldsymbol{E}^T\boldsymbol{u} + \boldsymbol{N}_\mathrm{u} z_\mathrm{u} + \boldsymbol{V}_\mathrm{f} z_\mathrm{f} = \boldsymbol{0} \\
&1 - \boldsymbol{f}_\mathrm{l}^T \boldsymbol{u} = 0 \\
&\boldsymbol{N}_\mathrm{u}^T \boldsymbol{x} \leq \boldsymbol{0}, \quad z_\mathrm{u} \geq \boldsymbol{0}, \quad z_\mathrm{u}^T \boldsymbol{N}_\mathrm{u}^T \boldsymbol{x} = \boldsymbol{0} \\
&\boldsymbol{N}_\mathrm{f}^T \boldsymbol{x} \leq \boldsymbol{0}, \quad z_\mathrm{f} \geq \boldsymbol{0}, \quad z_\mathrm{f}^T \boldsymbol{N}_\mathrm{f}^T \boldsymbol{x} = \boldsymbol{0}.
\end{aligned}
\tag{3}
$$

It is worth noticing that nonlinear (and nonconvex) complementarity constraints $(3)_{6,9}$ are here involved, thus turning the limit analysis into a Mixed Complementary Problem (MCP) (e.g., see [32]). In particular, excluding the simple case of associative friction flow law, it is affected by a ill-posedness issue related to the lack of uniqueness of the collapse multiplier. As several structural collapse states might exist, each attained for a distinct intensity of the live loads, a conservative possibility is to assume the minimum collapse multiplier as the actual target. Accordingly, the following optimization problem is introduced (e.g., see [11]):

$$
\begin{aligned}
&\min_{\lambda,\, \boldsymbol{x},\, \boldsymbol{u},\, z_\mathrm{u},\, z_\mathrm{f}} && \lambda, \\
&\text{s.t.} && \{\lambda,\ \boldsymbol{x},\ \boldsymbol{u},\ z_\mathrm{u},\ z_\mathrm{f}\} \text{ is a solution of (3),}
\end{aligned}
\tag{4}
$$

which is a special case of a Mathematical Program with Equilibrium Constraints (MPEC) (e.g., see [32]).

3 A Two-Blocks Model Problem

In this section a simple model problem is discussed to highlight some features of the non-associative limit analysis problem discussed in Sect. 2. As depicted in Fig. 2, a structure constituted by two blocks is considered, each block being characterized by width b and height h. The two blocks are supported on a base where unilateral and friction constraints, with friction coefficient $\mu = \tan \varphi$, hold. The same constraints are also enforced at the interface between the two blocks. It is assumed that each block is subjected to a dead load coinciding with its weight W, whereas a horizontal force of the same intensity is assumed as basic live load. Accordingly, a parametric analysis of the collapse multiplier λ of the basic live loads is conducted with respect to block slenderness $\eta = h/b$ and friction coefficient μ.

As a reference result, the associative limit analysis problem is initially considered. In such a case, a kinematic approach formulated as in Eq. (2) yields four possible collapse mechanisms, collected in Fig. 3:

- Panel (a) depicts a sliding mechanism with dilatancy, labelled as A1, corresponding to a collapse multiplier $\lambda = \mu$;
- Panel (b) depicts a single rocking mechanism with 'up' dilatancy, labelled as A2-1, corresponding to a collapse multiplier $\lambda = \left[\eta\left(1-\mu^2\right)+\mu\right]^{-1}$;
- Panel (c) depicts a single rocking mechanism with 'down' dilatancy, labelled as A2-2, corresponding to a collapse multiplier $\lambda = \left(1+2\mu^2\right)/\left[\eta\left(1+\mu^2\right)-\mu\right]$;

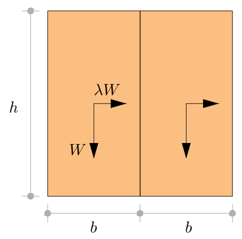

Fig. 2 A two-blocks model problem: geometry and loading conditions

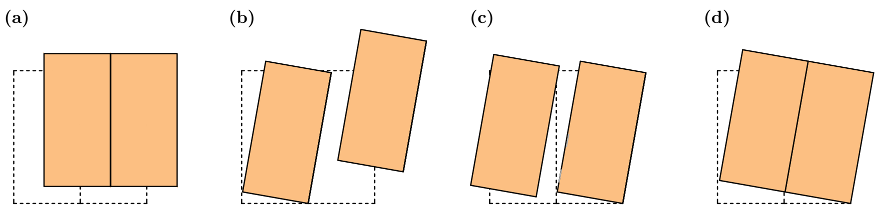

Fig. 3 A two-blocks model problem: collapse mechanisms for associative friction. **a** Sliding with dilatancy, labelled as A1, **b** single rocking with 'up' dilatancy, labelled as A2-1, **c** single rocking with 'down' dilatancy, labelled as A2-2, **d** coupled rocking, labelled as A3

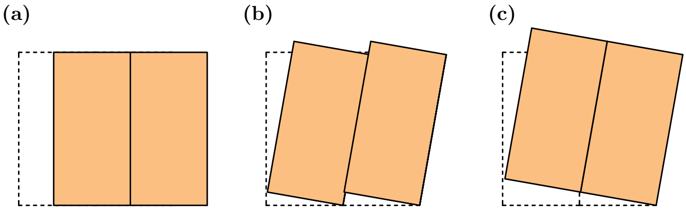

Fig. 4 A two-blocks model problem: collapse mechanisms for non-associative friction. **a** Sliding, labelled as NA1, **b** single rocking, labelled as NA2, **c** coupled rocking, labelled as NA3

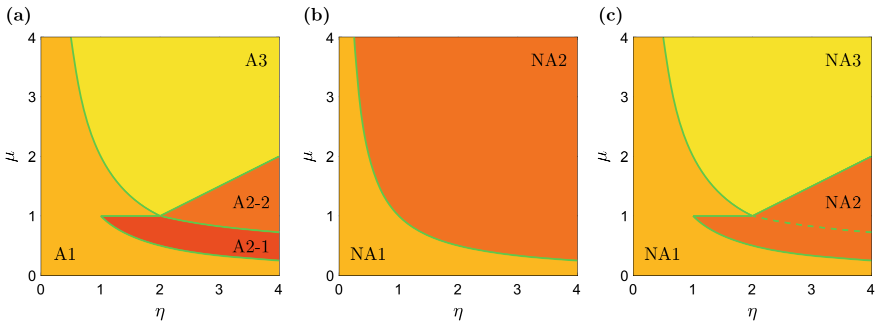

Fig. 5 A two-blocks model problem: partition of parameter space into regions corresponding to different collapse mechanisms, for **a** associative friction, **b** non-associative friction, minimum collapse multiplier, **c** non-associative friction, maximum collapse multiplier

– Panel (d) depicts a coupled rocking mechanism, labelled as A3, corresponding to a collapse multiplier $\lambda = 2/\eta$.

In Fig. 5a the partition of the parameter space into regions corresponding to the different collapse multipliers is shown, whereas in Fig. 6a the collapse multiplier is plotted versus the parameter space.

As for the non-associative limit analysis problem, here addressed under the assumption of vanishing dilatancy $\psi = 0$, the MCP (3) has to be solved. To such an aim, two distinct optimization problems are considered, consisting in the minimization formulation (4) and in the analogous maximization formulation obtained by replacing min with max in equation (4).

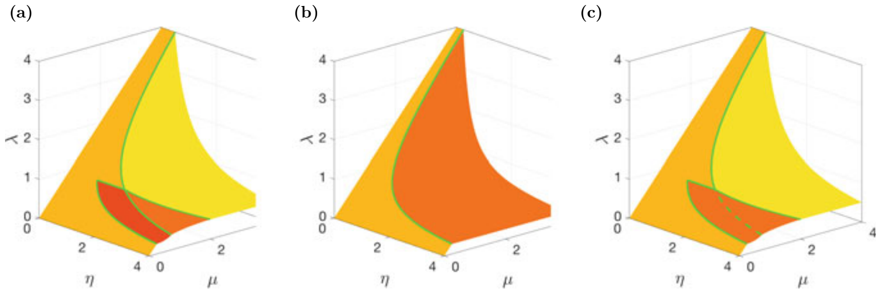

Fig. 6 A two-blocks model problem: collapse multiplier versus parameter space, for **a** associative friction, **b** non-associative friction, minimum collapse multiplier, **c** non-associative friction, maximum collapse multiplier

In detail, the minimum collapse multiplier is attained for:

- a sliding mechanism when $\mu \leq 1/\eta$, with collapse multiplier $\lambda = \mu$. That is labelled as NA1 and depicted in Fig. 4a;
- a single rocking mechanism when $\mu \geq 1/\eta$, with collapse multiplier $\lambda = 1/\eta$. That is labelled as NA2 and depicted in Fig. 4b.

Figures 5b and 6b respectively show the partition of the parameter space corresponding to the two mechanisms and the minimum collapse multiplier as a function of the parameters.

Regarding the maximum collapse multiplier, a more intricate situation emerges. In that case it is convenient to first discuss the plot of the collapse multiplier versus the parameter space, shown in Fig. 6c. Interestingly, the relevant results coincide with those pertaining to the associative limit analysis problem. However, the maximum collapse multiplier is attained by the collapse mechanisms illustrated in Fig. 5c. Specifically, the sliding mechanism with dilatancy A1 switches into the sliding mechanism NA1 and the two single rocking mechanism with 'up' and 'down' dilatancy, respectively A2-1 and A2-2, switch into the single rocking mechanism NA2. Of course, the coupled rocking mechanism A3 coincides with NA3, Fig. 4c, not implying any dilatancy.

For instance, let the point $\eta = 3$ and $\mu = 0.65$ be considered. In that case, the associative collapse multiplier corresponds to the mechanism A2-1 and results to be $\lambda = 0.41973$. That coincides with the maximum non-associative collapse multiplier, though the latter is attained by the mechanism NA2. Under different interface forces, the same mechanism NA2 also provides the minimum non-associative collapse multiplier, $\lambda = 1/3$, with a reduction of 25%. Accordingly, the present model problem exemplifies the following features of the non-associative limit analysis problem: (i) the MCP (3) can suffer from lack of uniqueness of the solution also in terms of collapse multipliers, and (ii) those collapse multipliers might be strictly (and significantly) lower than the associative one.

4 Numerical Solution Algorithm

State-of-the-art optimization tools for the solution of the MPEC (4) are severely limited in the size of problems they can handle, and their applicability to structures of practical interest is precluded (see [11]). On the other hand, also renouncing to global minimization and restricting to the solution of the MCP (3), a numerical solution strategy suitable to the limit analysis of real structures is still missing. As a matter of fact, it is nowadays possible to resort to numerical tools that make the solution of LP problems, also of large size, a straightforward task to accomplish. Accordingly, a novel solution strategy is here conceived to construct a solution of the MCP (3), whose main motive is to exploit the iterated solution of suitable associative limit analysis problems.

Basic idea of the algorithm is to assume the block interface normal forces $\boldsymbol{n}$ as iteration variables. Hence, let the current iterate $\boldsymbol{n}^*$ be given. Without loss of generality, it is assumed that all interfaces have the same friction and dilatancy angles, respectively φ and ψ. Then, an associative limit analysis problem is formulated, comprising the following cohesive-frictional criterion:

$$|\boldsymbol{t}| \leq -\boldsymbol{n} \tan \psi - \boldsymbol{n}^* (\tan \varphi - \tan \psi) \,, \tag{5}$$

where $\boldsymbol{t}$ collects the block interface shear forces. Accordingly, ψ is assumed as friction angle, whereas a vector of interface cohesions $-\boldsymbol{n}^* (\tan \varphi - \tan \psi)$ is prescribed. Interestingly, the solution of such LP problem fulfills the equilibrium condition $(3)_1$, the compatibility condition $(3)_2$ (in fact an associative flow law with friction angle ψ is assumed), the normalization condition $(3)_3$, and the unilateral constraint and complementarity conditions $(3)_{4\text{–}6}$. Contrarily, friction constraint and complementarity conditions $(3)_{7\text{–}9}$ are in general not satisfied, as being affected by the cohesive-frictional criterion under consideration. However, as shown in [29], the original Coulomb friction and the cohesive-frictional criterion result to be equivalent if the block interface normal forces $\boldsymbol{n}$ in solution of the LP problem coincide with $\boldsymbol{n}^*$.

The discussion above suggests to introduce the (continuous) function $\mathcal{F}$, mapping a given vector of block interface normal forces $\boldsymbol{n}^*$ into the block interface normal forces $\boldsymbol{n}$ in solution of the modified associative limit analysis problem. Consequently, a solution of the MCP (3) can be constructed by solving the following fixed-point problem:

$$\boldsymbol{n} := \mathcal{F}\big(\boldsymbol{n}^*\big) = \boldsymbol{n}^*. \tag{6}$$

Two concluding remarks are in order. First, problem (6) can be addressed by standard fixed-point iterations, or by a general-purpose derivative-free algorithm (e.g., see [22]). Second, in [15] a heuristic algorithm is proposed to construct a solution of the MPEC (4), by the iterated solution of associative limit analysis problems. As a main difference with respect to the present algorithm, in that case convergence is assumed when the collapse multiplier does not change in two successive iterations.

Contrarily, no check is therein required on the difference of block interface normal force, which might produce a not-negligible error in the friction admissibility condition (see [29]).

5 Numerical Simulations

In this section, numerical simulations are reported for assessing the performances of the present fixed-point algorithm. For comparison, other computational strategies available in the literature are also considered, specifically: the PATH solver proposed in [9], implementing a stabilized Newton method for the solution of general MCPs; the iterative relaxed nonlinear programming (NLP) algorithm proposed in [11], addressing the MPEC (4) by relaxation of the complementary constraints with a progressively reduced relaxation parameter; and the sequentially LP-based (SLP) algorithm proposed in [15], solving a succession of associative limit analysis sub-problems and controlling the difference of collapse multipliers in successive iterations.

Four benchmark problems are analyzed, modeling a round arch structure (Fig. 7a), an arch on buttresses structure (see [2]) (Fig. 8a), and two wall structures (see [3]) (Figs. 9a and 10a). Each structure is composed of equal-sized blocks (blocks having aspect ratio of 1 : 2) and is supposed to be supported on a base, where unilateral and friction constraints are enforced. Blocks are subjected to vertical dead loads $\boldsymbol{f}_{\mathrm{d}}$ and

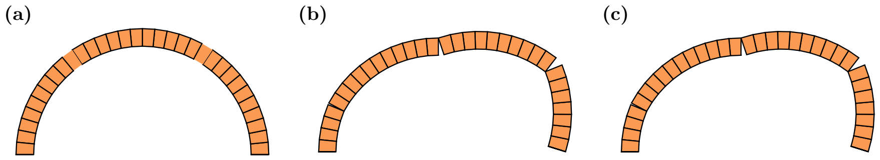

Fig. 7 Numerical simulations: round arch problem. **a** Geometry, **b** collapse mechanism with associative Coulomb friction and **c** collapse mechanism with non-associative Coulomb friction

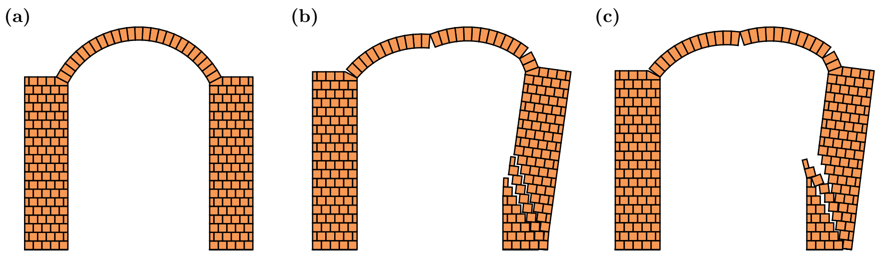

Fig. 8 Numerical simulations: arch on buttresses problem (see [2]). **a** Geometry, **b** collapse mechanism with associative Coulomb friction and **c** collapse mechanism with non-associative Coulomb friction

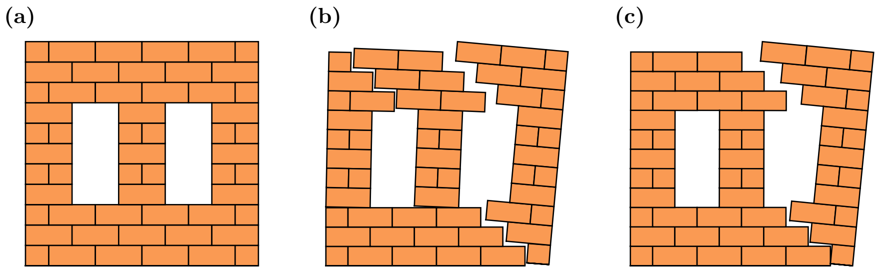

Fig. 9 Numerical simulations: 2 × 1 wall problem (see [3]). **a** Geometry, **b** collapse mechanism with associative Coulomb friction and **c** collapse mechanism with non-associative Coulomb friction

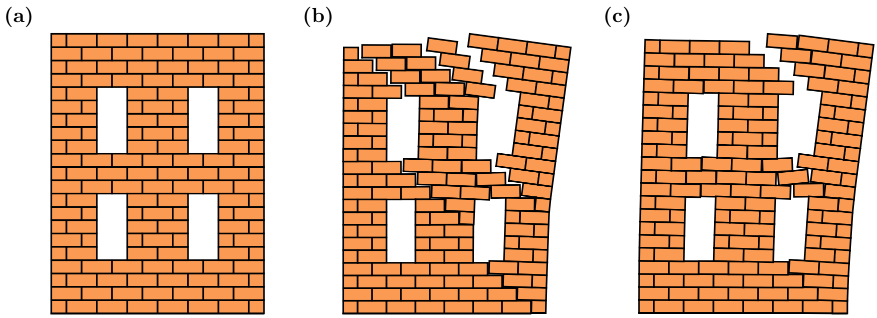

Fig. 10 Numerical simulations: 2 × 2 wall problem (see [3]). **a** Geometry, **b** collapse mechanism with associative Coulomb friction and **c** collapse mechanism with non-associative Coulomb friction

horizontal basic live loads $\boldsymbol{f}_1$, which are both proportional to the block volumes to mimic a pseudo-static earthquake loading. The material parameters are: the friction angle $\varphi = \arctan(0.65)$ and the dilatancy angle $\psi = 0$.

Panel (b) of Figs. 7–10 depicts the collapse mechanisms of the structures assuming associative friction flow law, whereas Panel (c) of the same figures depicts the collapse mechanisms predicted by the present fixed-point algorithm assuming non-associative friction flow law. The corresponding collapse multipliers are reported in Table 1, where also the results computed with the above competing algorithms are displayed.

Except for the round arch problem, whose collapse mechanism only involves unilateral failures, a reduced collapse multiplier corresponds to the non-associative friction flow law with respect to the associative one. Concerning the different estimation of non-associative collapse multipliers supplied by the algorithms under investigation, it can be noticed that the PATH solver converges to one of the (possibly many) solutions of the MCP (3), without any further specification. Conversely, the NLP algorithm explicitly seeks for a (local) minimum of the MPEC (4), thus justifying a demanding computational cost which precludes its use for large-size block structures (see [29]). On the other hand, the SLP and the present algorithms intend to construct a solution of the MCP (3) by iteratively addressing a static theorem formulation of modified associative limit analysis problems. Hence, those are methods

Table 1 Numerical simulations: collapse multiplier obtained by the present fixed-point algorithm. Corresponding values relevant to a in-house implementation of PATH algorithm (see [9]), NLP algorithm (see [11]) and SLP algorithm (see [15]) are reported (with × denoting lack of convergence)

Problem	Collapse multiplier				
	Associative	Non-associative			
		PATH	NLP	SLP	Present
Round arch	0.16034	0.16034	0.16034	0.16034	0.16034
Arch on buttresses	0.09085	0.08466	×	0.08195	0.08690
2 × 1 wall	0.33194	0.26374	0.26374	0.26374	0.26374
2 × 2 wall	0.34782	0.29725	0.29577	0.29649	0.29611

Table 2 Numerical simulations: relative errors in compatibility and friction admissibility conditions. For comparison, the corresponding values relevant to a in-house implementation of SLP algorithm (see [15]) are reported

Problem	Relative error in friction admissibility	
	SLP	Present
Round arch	0	0
Arch on buttresses	8.43×10^{-2}	1.84×10^{-4}
2 × 1 wall	4.56×10^{-2}	2.62×10^{-9}
2 × 2 wall	7.04×10^{-2}	1.97×10^{-4}

characterized by a reduced computational cost, which in turn cannot aim to a minimality property of the collapse multiplier. In order to highlight the main difference between the SLP and the present algorithms, the relevant solution quality has to be considered. Table 2 shows the relative error in the friction admissibility condition, defined as $\|(\boldsymbol{N}_{\mathrm{f}}^T \boldsymbol{x})_+\| / \|\boldsymbol{N}_{\mathrm{f}}^T \boldsymbol{x}\|$ (here $\|\cdot\|$ denotes the Euclidean norm and $(\cdot)_+$ the positive part operator), which results in the order of 10% for the SLP algorithm and much smaller for the present one. As that error is proportional to the difference of normal forces in successive iterations (see [29]), the improved solution quality of the proposed algorithm relies in explicitly assuming a convergence criterion on block interface normal forces, instead of collapse multipliers as in the SLP algorithm.

6 Conclusions

The limit analysis of dry-masonry block structures with non-associative Coulomb friction has been considered. Its formulation has been obtained as a Mixed Complementarity Problem, comprising equilibrium and compatibility equations pertaining to blocks, along with admissibility constraints (including Coulomb friction), flow laws (including non-associative Coulomb friction flow law) and Kuhn-Tucker complementarity conditions relevant to block interfaces. A simple two-blocks model problem has been presented to remark well-known peculiar features of non-associative limit analysis problem, such as the lack of uniqueness of the collapse multiplier and the fact that non-associative collapse multipliers are smaller than the associative one. A fixed-point based algorithm has been proposed for constructing a solution of the non-associative limit analysis problem, obtained by iteratively solving straightforward associative limit analysis formulations. Numerical simulations have been presented to investigate the performances of the resulting procedure. Compared to computational costly available methods, which explicitly seek for the minimum collapse multiplier, the proposed algorithm gives reasonable estimation of the collapse multiplier. Conversely, compared to similar procedures, which aim at deriving a solution iteratively attacking associative limit analysis problems, the present approach guarantees accuracy of the solution, also with respect to the friction admissibility condition.

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