Who Wins and Loses Under Approval Voting? An Analysis of Large Elections

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1 Introduction

Approval Voting (AV) is the method of election according to which a voter can vote for as many candidates as he wishes, the elected candidate(s) being the one(s) who receive(s) the most votes. This simple voting rule has attracted interest from scholars in political science and economics (see Laslier 2010) due to its flexibility: voters approve of each candidate independently of the rest of the candidates. As far as preference aggregation is concerned (on which we focus), one of the main features of equilibria is presented in Laslier (2009). It provides a strong argument for the use of AV in a model of large elections: AV selects the Condorcet Winner (CW) as long as the voters expect that no pair of candidates gets exactly the same number of votes.

Our main contribution is to fully characterize the set of equilibrium winners under Approval voting following Myerson (1993) model. This characterization is stated provided that the electorate is "large enough". By "large enough", we consider the benchmark for the study of large elections in which: (i) each voter does not affect the pivotal probabilities since his influence becomes negligible and (ii) yet his probability of affecting the outcome is strictly positive so that a rational voter selects the ballot that gives him the highest expected utility. Our results contrast with the previous ones

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M. Diss and V. Merlin (eds.), *Evaluating Voting Systems with Probability Models*,
Studies in Choice and Welfare, https://doi.org/10.1007/978-3-030-48598-6_14

in the strategic voting literature in its generality.[1] Most of the previous works focus on precise examples (for some given preference profile) and compute the set of equilibria under some voting rules. Few general results (that is that apply to any preference profile) are available. Among these results, one deals with Plurality and the other with Approval Voting. Under Plurality voting, one can construct equilibria in which any candidate who is not the Condorcet loser can win the election [see Myerson (2002) for instance]. Under Approval voting, the existence of an equilibrium in which the Condorcet winner wins, as previously described, is a salient feature. Yet, little is known about the rest of equilibria in elections with this rule. Informally, one might expect that Approval voting should reduce, when compared with other voting rules, the set of voting equilibria (and of equilibrium winners) and hence be more robust to the concept of focal manipulation as described by Myerson (1993). However, the previous results do not focus on the whole set of equilibrium winners under Approval as we do. Moreover, note that Brams (2016) perform a related analysis assuming that voters are sincere. Our work can hence be thought as an extension of their work to a strategic environment. Our approach is based on the candidates who are viable and unviable. A candidate a is unviable if, in the election, the number of voters who do not rank a last is smaller than the number of voters who rank some other candidate first. A viable candidate is one who is not unviable; as will be shown, only viable candidates can win when voters play best responses. Our notion of viability is somewhat related to the underlying idea of critical strategies as presented by Brams (2016). The equilibrium winners are as follows.

If there are at most two viable candidates, then the unique equilibrium winner is the Condorcet Winner (Theorem 1). Furthermore, we prove that if the unique equilibrium winner is the Condorcet winner for every utility representation of an ordinal preference profile, then there are at most two viable candidates. We hence derive necessary and sufficient conditions for implementing the Condorcet Winner as the unique equilibrium winner in terms of the number of viable candidates. To prove such a result, we need to impose two mild restrictions in the preference profile: the Simple Asymmetry (*SA*) and the Inverse Asymmetry (*IA*). According to *SA*, for any pair of candidates x, y the number of voters who prefer x to y are different from the number of voters who prefer y to x. The role of *SA* is simple: it removes equilibria with two winners. *IA* states that for any triple of candidates x, y, z, the number of voters who prefer x to y and y to z is different from the number of voters who prefer z to y and y to x. The role of *IA* is subtler as will be shown by Example 2, which proves that the Condorcet loser can be the only winner in equilibrium when *IA* fails to hold.

[1] In a recent computer science literature the use of probabilistic models in contexts with human participants has received quite a lot of criticism. Especially, they argue that the common approach to handle uncertainty is by maximizing expected utility, which requires a cardinal utility function as well as detailed probabilistic information. However, often such probabilities are not easy to estimate or apply. Therefore a number of alternative frameworks for modelling uncertainty (including for voting settings) have been proposed. For an up-to-date coverage of this literature, see Meir (2014) and Lev (2019).

On the contrary with at least three viable candidates, the situation is much more nuanced. We prove that for any such preference profile, we can build an equilibrium in which all viable candidates are tied for victory (Theorem 2). Note that this equilibrium exists for some set of cardinal utilities but need not exist for all utility profiles representing an ordinal preference profile.

We then go on to derive some implications of the previous results.

The first consequence is that when many candidates are viable, taking into account equilibria with more than one winner seems unavoidable. We believe that these equilibria are not degenerate but, on the contrary, are inherent to the voting system. One possible interpretation is that, unless the electorate is polarized over two candidates (there are at most two viable candidates included), the rule is unable to make a clear choice.

Secondly, we obtain a description of equilibrium winners in elections with three candidates. In these elections (on which most of the literature focus), we prove that either the CW wins or the three candidates are tied in any equilibrium. This reinforces our claim according to which equilibria with ties cannot be ignored. Indeed, doing so, leads us to conclude that some elections *do not admit an equilibrium at all*. For instance, any election without a CW only admits equilibria with ties (Theorem 1).

This work is structured as follows. After briefly reviewing the literature on models of large elections, Sect. 2 introduces the general framework and Sect. 3 describes the strategic behavior of the voters. Section 4 analyzes the relation between AV and the Condorcet Winner whereas Sect. 5 focus on elections with many viable candidates. Section 6 concludes the paper.

1.1 Related Literature

One common feature of the models dealing with the study of large elections with strategic voters are the pivotal probabilities. It is often assumed that a voter anticipates that with some small probability (even though strictly positive) his vote is relevant to modify the outcome of the election. Determining and comparing the magnitude of these probabilities is hence key to describe the voters' strategic behavior. Our model is no exception.

Yet, there are different approaches that have been taken to incorporate this assumption. One may either, as in the present model or as in Myerson (1993), make some simple assumptions about the pivotal probabilities, without explicitly incorporating a mechanism that actually leads to positive pivotal probabilities. Or one may add a certain uncertainty to the model that generates positive pivotal probabilities. Myerson (2000, 2002), among others, assumes that the actual number of voters is uncertain and follows a Poisson distribution (Poisson voting game). Laslier (2009) assumes that the actual number of voters is given, but that each voter's vote has a certain small probability of being wrongly recorded ("Florida-tremble"). Other than implicit versus explicit mechanism generating pivotal probabilities, Laslier (2009) and the present model are essentially the same.[2]

The previously discussed modelling approaches can be divided into two groups: those in which the voters' anticipations follow some intuitive behavioral assumption and those who do not. Our work belongs to the first group together with Myerson (1993) and Laslier (2009), whereas, in general, preference aggregation Poisson games belong to the second group.[3] Informally intuitive behavioral assumption entails that a voter believes that in case of being pivotal, it is way more probable to break a tie in which (at least) one of the winners is involved than a tie in which no winner is involved. In Laslier (2009), they prove that situation arises endogenously when the scores of the candidates are treated as independent random variables and the number of voters is large enough. In our model, we follow this behavioral assumption.

Two remarks can be made on this behavioral assumption. First, it is true that Myerson (2002) finds that a Poisson voting game is inconsistent with the model of "reduced form"": due to Myerson (1993). Moreover, Nunez (2010) has an example in which preferences satisfy the Simple Asymmetry and the Inverse Asymmetry, there exists a Condorcet winner, but it is not selected in an equilibrium with a unique winner. This is a mathematical objection to the current model. However, when using the ordering condition (which is quite intuitive), we interpret it as a behavioral one. We do not claim that it can be derived under very general conditions from a model. On the contrary, our claim is that, when the condition fails, the strategic reasoning might be highly unreasonable. For instance, in the example in Nunez (2010), the voters anticipate that the most probable pivot event takes place between the first and the third ranked candidate rather than between the two candidates with the most votes. This seems to be hardly sustainable with experimental data [see Forsythe (1993) among others]. Moreover, Lachat (2019) test one model of strategic voting in which the ordering condition holds on data of an AV election in the Zurich cantons (with several winners). They find substantial evidence that these models correctly predict strategic behavioral both at the individual and at the aggregate level.

2 Elections

The finite set of voters and candidates are respectively denoted by $\mathcal{N} = \{1, \ldots, n\}$ and $\mathcal{X} = \{a, b, \ldots, k\}$. Note that n is supposed to be large. The strict preferences of a voter are defined by a utility function $u : \mathcal{X} \to \mathbb{R}$, in which $u(x)$ denotes the utility a voter gets if candidate x wins the election. In other words, for each $i \in \mathcal{N}$ and for any pair of candidates $x, y \in \mathcal{X}$, x is strictly preferred to y, denoted $x \succ_i y$, if

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[2] In these models when the size of the electorate becomes large, the voter becomes almost certain of the distribution of the voters' preferences. In order to tackle this features, a recent strand of the literature is focusing on models of aggregate uncertainty such as the works of Fisher (2014) and Bouton (2016).

[3] So is the case if one focuses on classic equilibrium refinements such as perfection or Mertens' stability as proved by De Sinopoli (2006, 2014).

and only if $u_i(x) > u_i(y)$. Given that our focus is on whether the Condorcet Winner is selected, we consider exclusively strict preferences over alternatives under which this concept is unambigously defined.

Election $E := (\mathcal{X}, \mathcal{N}, u)$ is then characterized by its set of candidates $\mathcal{X}$, its set of voters $\mathcal{N}$ and the utility vector $u = (u_i)_{i \in \mathcal{N}}$ that depicts the utility function of each voter.

For any pair of candidates $x, y \in \mathcal{X}$, the majority relation M is defined as follows. We say that x is M-preferred to y, denoted xMy, if and only if $N(x, y) > N(y, x)$, with $N(x, y) = \#\{i \in \mathcal{N} \mid x \succ_i y\}$. The majority relation allows us to introduce the notion of Condorcet Winner (CW), that is a candidate who is majority preferred in any pairwise comparison. This concept will be of special importance in this work and its formal definition is as follows.

Definition 1 An election E admits a Condorcet winner if there exists some candidate $x \in \mathcal{X}$ such that:

$$xMy \quad \text{for any } y \in \mathcal{X} \setminus \{x\}.$$

Throughout the work, we make two (slight) assumptions that ensure that social preferences are asymmetric: Simple and Inverse Asymmetry. Note that both conditions are quite mild.

The first one concerns the preferences of the electorate over any pair of candidates, as follows.

Definition 2 An election E satisfies Simple Asymmetry (SA) if:

$$\text{for any } \ x, y \in \mathcal{X}, \ N(x, y) \neq N(y, x).$$

The assumption *SA* is rather weak. Its role is to remove knife-edge cases in which the electorate is divided in two exact halves: the ones who prefer x to y and the ones who prefer y to x. When the population is large, the probability of these knife-edge cases is typically very small.

The second one concerns preferences over triples of candidates and is defined as follows. For any triple of candidates $x, y, z \in \mathcal{X}$, we let $N(x, y, z)$ denote the number of voters who prefer x to y and y to z; formally, $N(x, y, z) = \#\{i \in \mathcal{N} \mid x \succ_i y \succ_i z\}$.

Definition 3 An election E satisfies Inverse Asymmetry (IA) if:

$$\text{for any } x, y, z \in \mathcal{X}, \ N(x, y, z) \neq N(z, y, x).$$

The role of *IA* and *SA* is to avoid non-generic situations in which the number of players of certain type exactly coincide with the number of players of a different type. This goes in line with many models of incomplete information where the number of voters of each of the different types are drawn from a common distribution. Indeed, in such models, when the number of voters goes to infinity, the probability that two types have exactly the same number of voters becomes negligible. Both our assumptions, *IA* and *SA* mimic these vanishingly small probabilities, in a setting where (*i*) there is a large number of voters and (*ii*) voters have complete information over the preferences of the rest of voters.

The next definition concerns some sort of candidates in an election. A candidate y in election E is unviable if and only if there exists some other candidate x such that the number of voters who rank x first (denoted $N(x, \ldots)$) is higher than the number of voters who do not rank y last (denoted $n - N(\ldots, y)$) so that

Definition 4 A candidate y in election E is unviable if:

$$\exists\, x \in \mathcal{X} \quad \text{with}\, N(x, \ldots) > n - N(\ldots, y).$$

with $N(\ldots, y)$ the number of voters who rank y last.

Any candidate who is not unviable is viable. The set of viable and unviable candidates are respectively denoted by $\mathcal{X}^v$ and $\mathcal{X}^{uv}$ so that

$$\mathcal{X} = \mathcal{X}^v \cup \mathcal{X}^{uv}.$$

Note that (Courtin, 2017) show that if an election E is such that $\mathcal{X}^v \leq 2$ and *SA* holds, then E admits a Condorcet Winner.

3 The Electoral Game

As previously discussed, we assume that the voters are strategic and vote simultaneously through the Approval voting method. In other words, each voter can approve of as many candidates as he wishes by choosing a ballot $v = (v_a, \ldots, v_k)$ where $v_x \in \{0, 1\}$ denotes the number of points given to candidate x by a voter. In the following the set of all possible ballots will be denoted V. We follow Myerson (1993) by assuming that each voter maximizes his expected utility to determine which ballot in the set V he will cast. In this model, his vote has an impact in his payoff if it changes the winner of the election. Therefore, a voter needs to estimate the probability of these situations: the *pivot* events. We say that two candidates are tied if their vote totals are equal. Furthermore, let H denote the set of all unordered pairs of candidates. We denote a pair $\{x, y\}$ in H as xy with $xy = yx$.

For each pair of candidates x and y, the xy-pivot probability p_{xy} is the probability of the outcome perceived by the voters that candidates x and y will be tied for first place in the election. A voter perceives that the probability that he will change the winner of the election from candidate x to candidate y by casting ballot v with $v_x \geq v_y$

to be linearly proportional to $\mid v_x - v_y \mid$. Moreover, the perceived chance of changing the winner from y to x is identical to the one of changing the winner from x to y.[4]

A pivot vector p is a vector listing the pivot probabilities for all pairs of candidates is denoted by $p = (p_{xy})_{xy \in H}$.

This vector p is assumed to be identical and common knowledge for all voters in the election. A voter with xy-pivot probability p_{xy} anticipates that submitting the ballot v can change the winner of the election from candidate x to candidate y with a probability of $p_{xy} \times \max\{v_x - v_y, 0\}$.

We let $U_i(v; p)$ denote the expected utility gain of voter i from casting ballot v given the pivot vector p with:

$$U_i(v; p) = \sum_{xy \in H} (v_x - v_y) \cdot p_{xy} \cdot [u_i(x) - u_i(y)]. \qquad (U)$$

A strategy profile $\sigma = (\sigma_i, \sigma_{-i})$ is any mapping from $\mathcal{N}$ into the set of probability distributions over V. That is, each σ_i describes the probability with which voter i chooses each ballot v in the set V. The expected utility gain of a voter when he plays the strategy σ_i equals $U_i(\sigma_i; p) = \sum_{v \in V} \sigma_i(v) U_i(v; p)$.
Given the strategy profile σ, the size of the electorate who casts ballot v is denoted by $\tau(v) = \sum_{i \in \mathcal{N}} \sigma_i(v)$. Therefore, the score of candidate x equals $S(x; \sigma) = \sum_{v \in V} v_x \tau(v)$ given the strategy profile σ.

Definition 5 For any strategy profile σ, the set of *winners* at σ, $W(\sigma) \subseteq \mathcal{X}$, contains the candidates whose score $S(x; \sigma)$ is maximal given σ.

Given a pivot vector p, the set of pure best responses of a voter equals $\text{BR}_i(p) = \{v \in V \mid v \in \arg\max_{v' \in V} U_i(v'; p)\}$. Given the strategy σ_i of a voter i, its support denotes the set of pure strategies played with positive probability according to σ_i: $\text{Supp}(\sigma_i) = \{v \in V \mid \sigma_i(v) > 0\}$;

For any candidate y, let $v^y = (v^y_x)_{x \in \mathcal{X}}$ represent the ballot that assigns 1 point to candidate y and zero to the rest of them ($v^y_y = 1$ and $v^y_x = 0$ if $x \neq y$). The following lemma will simplify the voter's expected utility and hence helps to understand his best responses.

Lemma 1 *For any ballot $v \in V$, any pivot vector p and any voter $i \in \mathcal{N}$,*

$$U_i(v\,;\,p) = \sum_{\{y:\, v_y = 1\}} U_i(v^y\,;\,p),$$

[4]This is roughly equivalent to assume that the probability of candidates x and y being tied for first place is the same as the probability of candidate x being in first place one point ahead of candidate y (and both candidates above the rest of the candidates), which is in turn the same one as the probability of candidate y being in first place one vote ahead of candidate x. (Myerson, 1993) justify this assumption by arguing that it seems reasonable when the electorate is large enough.

To see why Lemma 1 is correct, for any ballot $v = (v_x)_{x\in\mathcal{X}}$ which assigns no points to candidate x (i.e. $v_x = 0$), we let $v \cup \{x\}$ denote the ballot that assigns one point to x and v_y points to any candidate $y \neq x$. The linear expected utility of the voters given by (U) implies that for any ballot $v \in V$, $U_i(v \cup \{x\}\ ;\ p) - U_i(v\ ;\ p) = U_i(v^x; p)$. In other words, $U_i(v \cup \{x\}\ ;\ p) = U_i(v\ ;\ p) + U_i(v^x\ ;\ p)$ which implies the claim.

This lemma says that the expected utility of approving of a set of candidates equals the sum of the expected utilities of voting *independently for each of them*. Thus, any best response consists on approving of the candidates for which approving them marginally improves the voter's expected utility. Those candidates with null expected utility might be included but need not. The following lemma presents the structure of voters' best responses.

Lemma 2 *(Best Responses) For any pivot vector p and any voter $i \in \mathcal{N}$, the voter's set of best responses is as follows:*

(i) if $U_i(v^x\ ;\ p) > 0$ then $v_x = 1$ for any $v \in BR_i(p)$.
(ii) if $U_i(v^x\ ;\ p) < 0$ then $v_x = 0$ for any $v \in BR_i(p)$.
(iii) if $U_i(v^x\ ;\ p) = 0$, then there is some $v \in BR_i(p)$ with $v_x = 1$.

As depicted by the above lemma, the structure of voters' best responses is particularly simple since one only needs to compute the expected utility of each of the different candidates that are to be included in the ballot. Moreover, one can easily check that, in any best response, a voter always approves his most preferred candidate and never approves his least preferred one if the pivot vector p is such that $p_{xy} > 0$ for some $xy \in H$.

Given the pivot vector p, one can choose a best response σ such that the score of each candidate x can take any value in $[\min S(x;\sigma), \max S(x;\sigma)]$ with:

$$\min S(x;\sigma) = \#\{i \in N \mid U_i(v^x; p) > 0\}$$

and

$$\max S(x;\sigma) = \#\{i \in N \mid U_i(v^x; p) \geq 0\}.$$

Note that the minimal score of candidate x corresponds to the situation in which only the voters who get a *strictly* positive expected utility of voting x do vote for him. On the contrary, the maximal score is reached when every voter with a non-negative expected utility of voting x votes for x.

Proposition 1 *In any election E and any strategy profile σ in which voters use best responses, this defines an equilibrium state in which the set of winners only contains viable candidates.*

Proof Assume by contradiction that there is some unviable candidate y in some election E such that $y \in W(\sigma)$ for some strategy profile σ. By definition, since y is unviable in E then $\exists\, x \in \mathcal{X}$ with $N(x,\ldots) > n - N(\ldots, y)$. Moreover, if the voters play a best response, they always vote for their preferred candidate and never for their worst preferred one as shown by Lemma 2. Hence, $\min S(x;\sigma) > \max S(y;\sigma)$ so that $y \notin W(\sigma)$, as required. □

We now move to the equilibrium concept that we will follow. Following Myerson (1993), we assume that voters expect candidates with lower scores to be less likely serious contenders for first place than candidates with higher scores. In other words, if the score for some candidate x is strictly higher than the score for some candidate y, then the voters would perceive that candidate x's being tied for winning with any third candidate z is much more likely than candidate y's being tied for first place with candidate z.

Definition 6 Given any strategy profile σ and any candidate z, the pivot vector satisfies the *ordering condition* with respect to any $\varepsilon \in (0, 1)$ if

$$S(x;\sigma) > S(y;\sigma) \Longrightarrow \varepsilon p_{xz} \geq p_{yz},$$

for any two candidates x, y.

This implies that pivot probabilities involving candidates with low vote shares are zero in a similar fashion to the definition of proper equilibrium. We follow this assumption.[5]

Moreover, we also assume that the probability of three (or more) candidates being tied for first place is very small in comparison to the probability of a two-candidate tie.

As will be shown, as long as the election does not admit a Condorcet winner, the model prescribes that in any equilibrium, there are at least three winners. This might, at first glance, seem counterintuitive with the previous assumption according to which ties with more than two candidates are negligible, which is not the case of pivots with exactly two candidates. However, note that in Poisson games in which this assumption does not hold, Myerson (2002) writes that "[j]ust because all candidates have equal expected scores per voter in the limit does not imply that they have equal chance of winning in large equilibria" . Myerson (2002) then proves that the strategic reasoning in an equilibrium with three tied winners deals just with the two-candidate pivots. Our assumption is hence not unduly restrictive.

Given any strategy profile σ, a sequence of pivot vectors $\{p^\varepsilon\}_{\varepsilon\to 0}$ satisfies the ordering condition if, for each $\varepsilon > 0$, p^ε is a positive pivot vector that satisfies the ordering condition.

Definition 7 The strategy profile σ is an *equilibrium* of election E if and only if, there exists a sequence of pivot vectors p^ε with $p^\varepsilon_{xy} > 0$ for every $xy \in H$ that satisfies the ordering condition given σ and such that, for each ballot v and for each voter i,

$$v \in \text{Supp}_i(\sigma) \implies v \in \text{BR}_i(p^\varepsilon) \text{ for each } \varepsilon > 0.$$

[5]The reader also can refer to the recent contribution by Kavai (2013) for an empirical test of strategic voting in a model in which different weakening of the ordering condition is proposed. See also the recent experimental work by Bouton (2016).

As shown by Myerson (1993), an equilibrium exists for any possible distribution of the voters' utilities which makes the model very attractive for the study of voting rules. It should be stressed that, in this definition, the pivot probabilities p_{xy} are supposed to be constant when the voter contemplates casting one ballot or the other. More specifically, these pivot probabilities, from each voter's perspective, should be the probabilities of ties or 1-vote differences among all other voters' ballots, before his own ballot is counted. But then the independence of pivot probability on the perceiving voter can be justified if the true stochastic model treats all voters symmetrically. This is why the literature tends to use models where voters have independent identically-distributed types.

The focus of the paper is on the equilibrium winners under Approval voting. For any election E, the set of equilibrium winners is denoted by $W_E(\sigma)$.

In order to illustrate the model, we conclude this section by an example of an election E that gives an excellent description of the equilibria with ties that we describe throughout.

Example 1 Consider an election E with three alternatives $\mathcal{X} = \{a, b, c\}$ and three groups of voters. The first one is endowed with the utility profile[6] $u_A = (10, x, 0)$, the second one with $u_B = (0, 10, y)$ and the final one with $u_C = (z, 0, 10)$ with $0 < x, y, z < 10$. The shares of the different groups are, respectively, 40%, 35% and 25%. Hence, this election does not admit a Condorcet Winner. Therefore, at first sight, one can imagine that a will be the winner in some equilibrium. Indeed, the voters focus on the pair $\{a, b\}$ in the sense that these candidates are the ones with the two highest expected scores. This implies that the most likely pivot event occurs between a and b so that the voters in groups 1 and 3 approve of a and the ones in group 2 approve of b so that, a gets a higher score than b, namely $S(a; \sigma) = 65\% > S(b; \sigma) = 35\%$, and a is the winner. However, the logic of the model is more complex. Indeed, one still needs to consider the pivot events in which alternative c is involved since every possible pivot event occurs with positive probability. However, since a has a higher score than b, then it is infinitely more likely that c is involved in a pivot with a than with b (as described by the ordering condition). Hence, the voters in groups 2 and 3 approve of c since they all prefer c to a. However, this implies that $S(a; \sigma)=65\%>S(c; \sigma)=60\%>S(b; \sigma)=35\%$ so that it is not anymore rational that voters focus on the pair $\{a, b\}$ but rather on the pair $\{a, c\}$; in other words, there is no equilibrium in which the voters focus on the pair $\{a, b\}$. This reasoning applies to any pair of candidates so that there is no equilibrium σ in which the voters just focus on a pair of candidates in the sense that these candidates are the ones with the two highest expected scores. Moreover, one can prove that any equilibrium in this election leads to a tie among the three candidates so that $W_E(\sigma) = \{a, b, c\}$ for any equilibrium σ in E. As we will see in the next section, this inevitably generates ties in equilibrium.

[6]The utility values in each vector are the utilities of alternative a, b and c respectively.

4 Approval Voting and the Condorcet Winner

This section describes the conditions that ensure that the unique equilibrium winner is the Condorcet Winner under Approval voting. Moreover, it shows that when the election admits no Condorcet Winner, there are at least three equilibrium winners at any equilibrium. These results make more precise the relation between Approval voting and the Condorcet Winner.

The main characteristic of these results is that they do not depend *explicitly* on the voters' best responses. In other words, we do not need to completely define how the voters vote in order to predict who the equilibrium winners are. The main logic is driven by the voters' anticipations to the possible scores of the candidates, greatly simplifying the task at hand. As far as scenarios with a few number of viable candidates are considered, the main implication is summarized in the following theorem: the Condorcet Winner is the unique winner in equilibrium.

Within the proofs, we write $S(x)$ rather than $S(x;\sigma)$ to simplify notations. That is, we remove the explicit reference to the strategy profile.

Theorem 1 *If the election E satisfies both IA and SA, then:*

1. *If there are at most two viable candidates, then the unique equilibrium winner is the Condorcet Winner.*
2. *If there is no Condorcet winner, the set of equilibrium winners $W_E(\sigma)$ contains at least three candidates for any equilibrium σ.*

Theorem 1 is the main result of this section.

The two assumptions about the society, *IA* and *SA*, play a key role in the proof, although they do not have the same role.

Proposition 2 *If the election E satisfies SA, then there is no equilibrium with two winners.*

Proof Assume, by contradiction, that there is an equilibrium with two winners. W.l.o.g. we let x and y be this pair of candidates. Due to the ordering condition, the most probable pivot outcome in which x (resp. y) is involved is against y (resp. x). Therefore, the voters who strictly prefer x over y vote for x and the ones who strictly prefer y over x vote for y. Hence, the score of x equals $N(x, y)$ whereas the one of y equals $N(y, x)$. However, since *SA* holds, the scores of such candidates are different, contradicting the assumption that both x and y are both equilibrium winners. □

Consider now the role of *IA*. While the role of *SA* in selecting equilibria is intuitive, the role of *IA* is subtler. We first prove that it ensures that if there is an equilibrium with a unique winner, then this candidate is the Condorcet Winner. However, in order to see that this condition is necessary and important, Example 2 demonstrates that the Condorcet Loser (a candidate who is never M-preferred to any other candidate in the election) might be the unique winner when *IA* does not hold.

Example 2 Let $\mathcal{X} = \{a, b, c\}$, and consider a society with four possible utility functions: $u_A = (10, \mu, 0)$, $u_B = (10, 0, \mu)$, $u_C = (\mu, 10, 0)$ and $u_D = (0, \mu, 10)$ with $10 > \mu > 5$. The proportion of voters with each utility profile equals respectively 0.2, 0.35, 0.25 and 0.2. Therefore, b is the Condorcet Loser since aMb and cMb. Moreover *IA* does not hold since there is the same number of voters with utility vectors u_A and u_D so that $N(a, b, c) = N(c, b, a)$.

It is easy to see that the strategy profile σ with

$$\sigma_A \to \{a, b\},\ \sigma_B \to \{a, c\},\ \sigma_C \to \{b\} \quad \text{and} \quad \sigma_D \to \{b, c\},$$

leads to the victory of b. Moreover, the strategy profile σ is justified by the pivot vector $p^\varepsilon = (p^\varepsilon_{ab}, p^\varepsilon_{ac}, p^\varepsilon_{bc}) = (1/2 - \varepsilon, 2\varepsilon, 1/2 - \varepsilon)$ and hence is an equilibrium. Therefore, the Condorcet loser b is the unique equilibrium winner at σ. As the next result shows, this bad outcome does not occur when the election satisfies *IA*.

Proposition 3 *If the election E satisfies IA and there is an equilibrium with a unique winner, then this candidate is the Condorcet Winner.*

Proof Assume that there is a unique winner in equilibrium, denoted a. Due to the ordering condition, every voter knows that, when $\varepsilon \to 0$, the pivot outcome in which any candidate $x \neq a$ is involved against a becomes infinitely more likely than the rest of pivot events.

We have two cases: either there is a tie in the scores of two candidates (who are not the winners) or there is no tie.
Case 1: Assume first that, given σ, there is a tie in the score of two candidates who are not the winners. We denote them b and c w.l.o.g. As the most likely pivot outcome in which both are involved is against a, we know that the unique voters who vote for b (resp. c) are the ones who prefer b (resp. c) to a.

Therefore, the scores of both candidates are the following ones:

$$S(b) = N(b, a, c) + N(b, c, a) + N(c, b, a),$$

and

$$S(c) = N(c, a, b) + N(c, b, a) + N(b, c, a).$$

Since the condition *IA* holds, it follows that the scores of b and c cannot be equal, a contradiction.

In other words, when *IA* holds, there is not an equilibrium with a unique winner in which two candidates have the same score. So that, if there is a unique winner in equilibrium, the only possible case is that there is no tie in the scores, to be analyzed in the *Case* 2.

Case 2: Assume now that there are no ties in the scores. Note first that $N(x, a) \neq N(y, a)$ for any pair $x, y \in \mathcal{X}$. To prove this, it suffices to see that $N(x, a) = N(x, a, y) + N(x, y, a) + N(y, x, a)$ and $N(y, a) = N(y, a, x) + N(y, x, a) + N(x,$

$y, a)$. The condition *IA* implies that $N(x, a, y) \neq N(y, a, x)$. Therefore, $N(x, a) \neq N(y, a)$ for any pair $x, y \in \mathcal{X}$.

W.l.o.g. we assume that $N(b, a) > N(c, a) > \cdots > N(k, a)\ \forall\, b, c, \ldots, k \in \mathcal{X}$.

Since every voter anticipates that the most likely pivot outcome involving any candidate $x \neq a$ is against a, it follows that the score of each candidate $x \neq a$ equals $N(x, a)$ the share of voters who strictly prefer x to a whereas the one of a equals $N(a, b)$. Hence, the scores of the candidates satisfy $S(a) > S(b) > \cdots > S(k)$.

Assume that a is not the CW so that there is some candidate y with yMa. If $y = b$, then $N(b, a) > N(a, b)$ so that the score of b is higher than the score of a, a contradiction with a being the winner. If $y \neq b$, then $N(y, a) > 1/2$ so that $S(y) = N(y, a) > 1/2 > N(b, a) = S(b)$. Therefore, y is ranked second. In this case, the score of a equals $N(a, y) < 1/2$, a contradiction with a being the winner. Hence, it can only be the case that a is M-preferred to the rest of the candidates: for any $x \in \mathcal{X} \setminus \{a\}$, aMx. In other words, a is the Condorcet winner. □

Finally, *IA* entails that if there is a CW in the profile, there exists an equilibrium in which this candidate is the unique winner.

Proposition 4 *If the election E satisfies IA and admits a Condorcet winner, then there exists an equilibrium that uniquely selects this candidate.*

Proof Take a society in which there is a CW (denoted a) and in which *IA* holds. Since *IA* holds, we can assume w.l.o.g. that $N(b, a) > N(c, a) > \cdots > N(k, a)$. Indeed, as shown in the proof of Proposition 3 (case 2), if *IA* holds, then $N(x, a) \neq N(y, a)$ $\forall x, y \in \mathcal{X}$. Assume that the scores satisfy $S(a) > S(b) > \cdots > S(k)$. Due to the ordering condition, it follows that the most likely pivot in which a is involved is against b whereas the most likely pivot outcome in which any other candidate x is against a. Thus, the score of a equals $N(a, b)$ whereas the score of x ($x \neq a$) equals $N(x, a)$. As a is the CW, it follows that $N(a, b) > 1/2$ and that $N(x, a) < 1/2$ for any $x \neq a$. Finally, since $N(b, a) > N(c, a) > \cdots > N(k, a)$, the scores satisfy $S(a) > S(b) > \cdots > S(k)$ as wanted. Thus we have proved that there exists an equilibrium in which the CW is the unique winner, concluding the proof. □

Proof of Theorem 1 As previously mentioned, if $\#\mathcal{X}^v \leq 2$, then the election admits a CW. By Proposition 4 we have shown that if there is a CW, there exists an equilibrium in which he is the unique winner. Moreover, there is no other equilibrium with a unique winner as ensured by Proposition 3. As shown by Proposition 2, there is no equilibrium with two winners since *SA* is satisfied. Therefore, the only type of equilibrium that might exist is the one in which at least three candidates are tied. However, the candidates who are unviable cannot be in the set of winners. Hence, when $\#\mathcal{X}^v \leq 2$, there is no equilibrium in which at least three candidates wins, which concludes the proof of part 1 of the Theorem 1. The part 2 of the Theorem 1 is a direct implication of the different results of this section. □

One main implication of Theorem 1 is that in elections with three candidates, the equilibrium winners are as follows.

Corollary 1 *If the election E satisfies both IA and SA and there are three candidates in the election, there are at most two sets of equilibrium winners:*

1. *the Condorcet Winner,*
2. *the three candidates belong to the winning set.*

5 On the Indeterminacy of Approval Voting

We now focus on the elections that admit at least three viable candidates.

We first focus on elections with three candidates and then explain how to extend the results to elections with at least four candidates.

5.1 Three Candidates

The next theorem gives a simple condition for the existence of a tie among viable candidates: if three candidates are viable, such an equilibrium exists. Let $\mathcal{U}$ a set of utilities, then we have

Theorem 2 *Assume that the election has three candidates. If all candidates are viable, then there is a closed set of utilities $\hat{\mathcal{U}} \subseteq \mathcal{U}$ such that for any election with utility vector $u \in \hat{\mathcal{U}}$, there is an equilibrium in which all viable candidates are tied for victory.*

This subsection presents the proof of Theorem 2. In the second part of this subsection, we prove that the equilibrium described by Theorem 2 need not exist for every utility representation. Indeed, Example 3 discusses an election with three candidates that admits no tie among equilibrium winners for some set of utilities.

We first present one proposition and one technical lemmata that will be useful for proving Theorem 2.

Proposition 5 *For each election E, there exists some $m \in \mathbb{N}^+$ such that*

$$n - N(\ldots, a) \geq m \Longleftrightarrow a \in \mathcal{X}^v.$$

Proof of Theorem 1 Take any election E with $\mathcal{X}^{uv} = \emptyset$. Thus the result trivially follows. Consider now any election that $\mathcal{X}^{uv} \neq \emptyset$ and let a, b be two candidates such that $a = \arg\min_{x \in \mathcal{X}^v} n - N(\ldots, x)$ and $b = \arg\max_{x \in \mathcal{X}^{uv}} n - N(\ldots, x)$. Note that if we prove that $n - N(\ldots, a) > n - N(\ldots, b)$, then the result follows.

Thus, let us assume by contradiction that $n - N(\ldots, a) \leq n - N(\ldots, b)$. It follows that $a \in \mathcal{X}^v$ and $b \in \mathcal{X}^{uv}$. Thus, there exists some $y \in \mathcal{X}$ with $N(y, \ldots) > n - N(\ldots, b)$. Since we have assumed that $n - N(\ldots, a) \leq n - N(\ldots, b)$, it follows that $N(y, \ldots) > n - N(\ldots, a)$ and hence $a \in \mathcal{X}^{uv}$, showing the desired contradiction. □

Lemma 3 *For any candidate a, there exists some sequence of pivot probabilities p^ε that induces when $\varepsilon \to 0$, the following boundaries for the score of candidate a:*

$$\min S(a) = N(a, \ldots) \text{ and } \max S(a) = n - N(\ldots, a).$$

Proof of Theorem 1 Consider any candidate d and assume that

$$\lim_{\varepsilon \to 0} \frac{p^\varepsilon_{ad}}{p^\varepsilon_{ab}} = \lim_{\varepsilon \to 0} \frac{p^\varepsilon_{ad}}{p^\varepsilon_{ac}} = 0 \text{ for any } ad \neq ab, ac \quad (g).$$

Note that, given (g), when $\varepsilon \to 0$, the voter's decision concerning whether to cast a vote for a only depends on the pivotal events in which candidates b and c are involved, the rest of them becoming infinitely less likely.

We divide the voters into the usual six groups according their ordinal preference over a, b and c (as described in the primer of the proof).

Note that the N_1 and N_2 voters always vote for a and the N_4 and N_6 voters never vote for a independently of the pivot vector.

Moreover, we let $R_3 R_5 = 1$ and we assume that $U_3(a) = 0$ and $U_5(a) = 0$.

It follows that given p^ε,

$$\min S(a) = N(a, \ldots), \quad \text{and} \quad \max S(a) = n - N(\ldots, a)$$

Note that the proof is done with homogenous cardinal utilities but a similar argument applies with heterogeneous cardinal utilities. □

We can now present the proof of Theorem 2.

Proof of Theorem 1 Voters' preferences are strict so that we divide the voters into six groups as follows:

N_1	N_2	N_3	N_4	N_5	N_6
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a
n_1	n_2	n_3	n_4	n_5	n_6

with for example N_1 being the set of voters i with preference ordering $a \succ_i b \succ_i c$, with $\#N_1 = n_1$. A set of voters sharing the same preference ordering is denoted N_l with $l = 1, \ldots, 6$.

We first assume that the voters in the same group (i.e. sharing the same ordinal preferences) have the same cardinal utilities. This assumption simplifies the proof and will be relaxed in the second part of the proof.

Part 1. Homogenous cardinal utilities within each group

The proof proceeds as follows. It builds for any type distribution in which $\mathcal{X}^{uv}$ is empty (i.e. $\#\mathcal{X}^v = 3$), a set of utilities and a strategy profile such that the three candidates are tied. Moreover, it builds a pivot vector that justifies the strategy profile proving that in equilibrium the three candidates are tied.

W.l.o.g. we let $n - N(\ldots, a) \geq n - N(\ldots, b) \geq n - N(\ldots, c)$.

For each voter i, we let t_i, m_i and b_i respectively denote his top, middle and bottom-ranked candidate over a, b, c. Moreover, for each $i \in \mathcal{N}$ we let R_i denote the following ratio $\frac{u_i(t_i)-u_i(m_i)}{u_i(m_i)-u_i(b_i)}$. Since all the voters sharing the same ordinal preferences have the same cardinal utilities, it follows that for any $i, j \in N_l$, $R_i = R_j$.

Therefore w.l.o.g. R_l stands for the ratio R_i for each $i \in N_l$.

We consider the set of utilities $\hat{\mathcal{U}}$ defined as follows:

$$\hat{\mathcal{U}} = \{u_{\mathcal{N}} \in \mathcal{U} \mid R_1R_6 = 1,\ R_3R_5 = 1,\ R_1 = R_2R_3 \text{ and } R_3 = R_1R_4\}.$$

The set $\hat{\mathcal{U}}$ is closed with an empty interior since it is the intersection of lower dimensional hyperplanes. Note that this set is not empty since we can independently choose each R_l. Moreover, we implicitly assume that $n_l > 0$ for each $l = 1, \ldots, 6$. A similar argument applies if $n_l \geq 0$ for $l = 1, \ldots, 6$.

We set $p^\varepsilon = (\varepsilon p_{ab}, \varepsilon p_{ac}, \varepsilon p_{bc})$ with

$$p_{ab} = \frac{1}{1+R_1+R_3}, \quad p_{ac} = \frac{R_3}{1+R_1+R_3} \quad \text{and} \quad p_{bc} = \frac{R_1}{1+R_1+R_3}.$$

One can check that the previous pivot probabilities imply that:

$$p_{m_ib_i} = R_i p_{t_im_i} \quad \text{for each } \ i \in \mathcal{N},$$

which is equivalent to

$$U_i(v^{m_i}; p^\varepsilon) = 0 \quad \text{for each } i \in \mathcal{N},$$

where v^{m_i} stands for the ballot that assigns one point to m_i (the middle-ranked candidate of voter i) and zero points to the rest of the candidates.

Given the description of the best responses given by Lemma 2, we know that the previous equality implies that every voter i is indifferent between voting for his top candidate (t_i) and for his top-two candidates (t_i, m_i). Hence, given p^ε one can choose a best response σ such that the score of each candidate x can take any value in $[N(x, \ldots), n - N(\ldots, x)]$.

Since $\mathcal{X}^{uv}$ is empty, it follows that $n - N(\ldots, c) > N(a, \ldots), N(b, \ldots)$. Moreover, by assumption, $n - N(\ldots, a) \geq n - N(\ldots, b) \geq n - N(\ldots, c)$. Thus, one can choose the three scores equal to $n - N(\ldots, c)$.

So far we have proved that for each vector $u \in \hat{U}$ and given p^ε, there exists a best response σ that leads to three tied winners. Moreover the pivot probability vector p^ε satisfies the ordering condition since the three candidates are tied given σ. Therefore, σ is an equilibrium, concluding the proof with homogeneous utilities.

Part 2. Heterogenous cardinal utilities within each group

We now allow the voters to have different cardinal utilities while having the same ordinal preferences. Therefore, there is not anymore a unique R_i for each voter i in each group N_l.

For each pivot vector p, we can divide the voters in each group N_l in three possible categories: those for which $U_i(v^{m_i}; p) > 0$, those for which $U_i(v^{m_i}; p) = 0$ and finally those for which $U_i(v^{m_i}; p) < 0$.

For each group N_l, we denote by N_l^* the group of voters such that for each $i \in N_l^*$, $U_i(m_i) = 0$. We let $R_l^* = R_i$ for each $i \in N_l^*$ and for each l.

Consider a voter i in N_1^* with middle-ranked candidate b. Therefore,

$$U_i(v^b; p) = 0 \iff p_{bc} = R_1^* p_{ab}.$$

Any voter j in N_1 with $R_j > R_1^*$ is such that $U_j(v^b; p) < 0$, whereas if $R_j < R_1^*$, it is the case that $U_j(v^b; p) > 0$. The same reasoning applies for each voter in any of the N_l groups. Therefore, R_l^* determines the best responses of the other voters in the group N_l. Moreover, the number of voters in N_l who vote for their middle-ranked candidate can vary from 0 to n_l, since one can set R_l^* to be equal to any R_i for each $i \in N_l$.

We consider the set of utilities $\hat{\mathcal{U}}^*$ defined as follows:

$$\hat{\mathcal{U}}^* = \{u \in \mathcal{U} \mid R_1^* R_6^* = 1,\ R_3^* R_5^* = 1,\ R_1^* = R_2^* R_3^* \text{ and } R_3^* = R_1^* R_4^*\},$$

and $p^\varepsilon = (\varepsilon p_{ab}, \varepsilon p_{ac}, \varepsilon p_{bc})$ with

$$p_{ab} = \frac{1}{1 + R_1^* + R_3^*}, \quad p_{ac} = \frac{R_3^*}{1 + R_1^* + R_3^*} \text{ and } p_{bc} = \frac{R_1^*}{1 + R_1^* + R_3^*}.$$

Given p^ε, a similar reasoning to the one in Part 1 proves that each N_l^* is non-empty. It follows that given p^ε one can choose a best response σ such that the score of each candidate x can take any value in $[N(x, \ldots), n - N(\ldots, x)]$. Therefore, since p^ε satisfies the ordering condition, this proves that σ is an equilibrium, concluding the proof for heterogenous preferences. □

Theorem 2 proves that for some set of utilities, there is an equilibrium in which all the candidates in the race are tied. However, it should be noted that this sort of equilibria need not exist for *every utility* representation of the election. The following example illustrates this point with just three candidates.

Example 3 Let $\mathcal{X} = \{a, b, c\}$ and consider a society with the following proportions with $0 < \mu < 10$: $\frac{1}{9}$ of the voters with $u_A = (10, \mu, 0)$; $\frac{2}{9}$ of the voters with $u_B = (10, 0, \mu)$; $\frac{4}{9}$ of the voters with $u_C = (10 - \mu, 10, 0)$ and $\frac{2}{9}$ of the voters with $u_D = (10 - \mu, 0, 10)$. The candidate a is the CW and $\mathcal{X}^{uv} = \emptyset$. Note that both SA and IA

hold. Since $\mathcal{X}^{uv} = \emptyset$, every candidate is viable. Hence, there is a strategy profile under which the candidate wins with positive probability. However, whether this occurs in equilibrium depends on the intensity of the voters' utilities as will be shown in the next lines.

Indeed, Proposition 2 implies that there is no equilibrium with two winners. Moreover, since there is a CW, Proposition 4 ensures that there exists an equilibrium in which a is the unique winner. Finally, there is no other equilibrium with a unique winner as ensured by Proposition 3. In other words, neither b or c can win alone.

One question remains to be answered: is there an equilibrium with the three candidates tied for victory? These equilibria might or not exist as a function of the voters' intensities of preferences.

There is no equilibrium in this election in which the three candidates have the same score with no voter being indifferent between single and double voting. Indeed, when no voter is indifferent between single and double voting, it follows that all the voters with the same utility vector vote in the same way. One can check that in any strategy profile in which the voters play best responses (each voter voting for his top candidate or for his two top candidates), there is no equality between the scores of the candidates. Hence, there is no such equilibrium with a three-way tie.

Thus, in order to have such an outcome, some type of voters are indifferent between single and double voting. In equilibrium, voters always approve of their most preferred candidate and never approve of their worst preferred one.

If just one type of voters play a mixed strategy, then it is not possible to obtain a three-way tie. If at least two types play in mixed strategies, then either C or D voters vote also for their middle ranked candidate so that a has the highest score.

Indeed, assume first that a C voter plays a mixed strategy over his two best responses so that $U_C(0, 1, 0) = U_C(1, 1, 0)$. Due to (U), the previous equality is equivalent to $U_C(1, 0, 0) = 0$ so that

$$p^{\varepsilon}_{13}(10 - \mu) - p^{\varepsilon}_{12}\mu = 0. \qquad (*)$$

However, when $(*)$ holds, we have that $U_D(1, 0, 1) > U_D(0, 0, 1)$. To see why, note first that $U_D(1, 0, 1) > U_D(0, 0, 1) \iff U_D(1, 0, 0) > 0$. Moreover, remark that $U_D(1, 0, 0) = (10 - \mu)p^{\varepsilon}_{12} - \mu p^{\varepsilon}_{13}$ so that, when $(*)$ holds,

$$U_D(1, 0, 0) = \frac{10(10 - 2\mu)}{\mu} p^{\varepsilon}_{13} > 0.$$

which holds since $\mu < 5$.

Therefore, if a C voter plays a mixed strategy, D voters vote for their second ranked candidate a, leading to its victory. A symmetric argument applies when a D voter plays a mixed strategy. Therefore, in any mixed strategy profile in which either C or D voters play a mixed strategy between their two best responses, a is the sole winner of the election.

Hence, the only possibility for the existence of an equilibrium in which the three candidates get the same score is to assume that A and B voters both play a mixed strategy. However, this implies that

$$U_A(0,1,0) = 0 \iff -p_{12}^{\varepsilon}(10-\mu) + p_{23}^{\varepsilon}\mu = 0,$$

and

$$U_B(0,0,1) = 0 \iff -p_{13}^{\varepsilon}(10-\mu) + p_{23}^{\varepsilon}\mu = 0.$$

The previous two equalities imply that the unique pivot probability vector justifying such best responses equals $p^{\varepsilon} = (\frac{\mu}{10+\mu}\varepsilon, \frac{\mu}{10+\mu}\varepsilon, \frac{10-\mu}{10+\mu}\varepsilon)$. However, as previously noted, $U_C(1,0,0) = p_{13}^{\varepsilon}(10-\mu) - p_{12}^{\varepsilon}\mu$ which is strictly positive given p^{ε} since $\mu < 5$. Hence, as in the previous case, if both A and B voters play a mixed strategy, C voters give one point to a, leading to its victory. Therefore, there is no equilibrium with three winners. Moreover, by Proposition 4, we know that there exists an equilibrium in which a is the unique winner. Furthermore, Proposition 2 implies that there is no equilibrium with two winners. Hence, in any equilibrium, a is the unique winner as long as $\mu < 5$. Hence, the CW is the unique equilibrium winner.

This example illustrates then that that for some set of utilities, when there is a CW and $\mathcal{X}^{uv} = \emptyset$, the unique equilibrium winner is the CW. However, for a different utility representation, we can find an equilibrium in which the three candidates get the same score. For example, if we set $\mu = 6$, there is an equilibrium in which the three candidates are tied for victory with a score of 4/9 as long as $p^{\varepsilon} = (p_{ab}^{\varepsilon}, p_{ac}^{\varepsilon}, p_{bc}^{\varepsilon}) = (3/7\varepsilon, 2/7\varepsilon, 2/7\varepsilon)$.

5.2 Viable* Candidates and Many Candidates

We now move on to describe elections with at least four candidates and the sufficient conditions for the presence of ties among viable candidates.

To see why Theorem 2 does not hold with more candidates, we present now the following example.

Example 4 Let $\mathcal{X} = \{a, b, c, d\}$ and consider an election with $\frac{6}{10}$ of the voters with $u_A = (10, 9, 8, 0)$ and $\frac{4}{10}$ of the voters with $u_B = (9, 10, 0, 8)$. This election has 3 viable candidates: a, b, and c. In particular, the latter is viable because the number of voters who rank a first is equal to the number of candidates who do not rank c last. Candidate c cannot win in any equilibrium. This is because c only receives approval votes if there is a high enough probability pivot event where it is facing d. But d always receives fewer approval votes than a as long as voters use best responses. So, by the ordering condition, no voter approves of c. In fact, the only equilibrium winner in this election is the CW.

The previous example shows that one needs to introduce a stronger condition than viability to ensure the existence of ties among viable candidates. In particular, this condition takes into account the iterative reasoning described in the previous example. The rest of this section described the notion of $k-$viable and $k-$unviable which are needed to derive a version of Theorem 2 with at least four candidates.

We first introduce some notation that will be useful throughout concerning degrees of viability and then prove how these definitions help to describe the strategic behavior under the Approval rule.

For notational purposes, we respectively relabel the set of viable and unviable candidates by $\mathcal{X}_0^v$ and $\mathcal{X}_0^{uv}$ (rather than $\mathcal{X}^v$ and $\mathcal{X}^{uv}$). Indeed, we say that a candidate in $\mathcal{X}_0^v$ (resp. in $\mathcal{X}_0^{uv}$) is $0-$viable (res. $0-$unviable). It follows that

$$\mathcal{X} = \mathcal{X}_0^v \cup \mathcal{X}_0^{uv}.$$

The main reason for this relabelling is the introduction of viability degrees for the different candidates as follows.

Definition 8 For any integer $k \geq 1$, a candidate y in election E is $k-$unviable if:

$$\exists\, x \in \mathcal{X}_{k-1}^v \quad \text{with } N(x, \ldots) > n - N^k(\ldots, y).$$

with

$$N^k(\ldots, y) := \#\{i \in \mathcal{N} \mid x \succ_i y \quad \text{for any } x \in \mathcal{X}_{k-1}^v\},$$

where $\mathcal{X}_k^{uv}$ stands for the set of $k-$ unviable candidates and $\mathcal{X}_k^v = \mathcal{X} \setminus \bigcup_{j=1}^k \mathcal{X}_j^{uv}$ the set of $k-$viable candidates.

Definition 8 is hence defining, recursively, the sets of $k-$viable and $k-$unviable candidates. It should be remarked that, for any integer $k \geq 0$,

$$\mathcal{X}_k^v \subset \mathcal{X}_{k-1}^v \quad \text{since } \mathcal{X} \setminus \bigcup_{j=1}^{k} \mathcal{X}_j^{uv} \subset \mathcal{X} \setminus \bigcup_{j=1}^{k-1} \mathcal{X}_j^{uv}$$

so that any $k-$viable candidate is also $(k-1)-$viable. The converse does not hold. Building on the previous set inclusion, it is easy to see that for any non-negative integer k such that $\mathcal{X}_k^v \neq \emptyset$, $N^k(\ldots, z) > N^{k-1}(\ldots, z)$.

The next definition deals with the candidates which are viable for every degree.

Definition 9 A candidate x in election E is viable* if x is $k-$viable for any positive integer k.

The set of viable* candidates is denoted $\mathcal{X}^*$. Such a set is non-empty by construction.

Theorem 3 *For any non-negative integer k, any election E and any equilibrium σ,*

1. *a voter never approves of his least preferred $k-$viable candidate.*
2. *no $k-$unviable candidate belongs to the set of equilibrium winners $W(\sigma)$.*

3. *the set of equilibrium winners $W(\sigma)$ contains only viable* candidates.*

Proof of Theorem 1 The proof proceeds by induction. Steps A and B prove that (1) and (2) hold. Step A proves the claim for $k = 0$ and Step B proves how to iterate the same reasoning. The result (3) is an immediate consequence of both (1) and (2).

Step A: $k = 0$.

Step A. is divided in two parts. We first prove that no voter approves of his least preferred $0-$viable candidate (in A.1) and then show that this implies no $1-$unviable candidate belongs to the set of equilibrium winners for any equilibrium (in A.2).

Step A. 1.

Assume that there is some equilibrium σ in which some voter approves of his least preferred $0-$viable candidate z. It follows that, there exists a sequence of pivot vectors p^{ε} with

$$U_i(v^z\ ;\ p^{\varepsilon}) \geq 0 \quad \text{for any } \varepsilon \in (0, 1),$$

where $U_i(v^z; p^{\varepsilon}) = \sum_{xy \in H}(v^z_x - v^z_y) \cdot p_{xy} \cdot [u_i(x) - u_i(y)]$, with $v^z_x = 0$ for any $x \neq z$ and $v^z_z = 1$. Note that that for any pair xy in which z is not involved, $v^z_x - v^z_y = 0$. Hence the expected utility for voter i can be rewritten as:

$$U_i(v^z; p^{\varepsilon}) = \sum_{x \neq z}(0 - 1) \cdot p_{xz} \cdot [u_i(x) - u_i(z)].$$

Take some equilibrium σ. The ordering condition implies that for any $\varepsilon \in (0, 1)$, any $x \in W(\sigma)$ and $y \notin W(\sigma)$, $\varepsilon p^{\varepsilon}_{xz} \geq p^{\varepsilon}_{yz}$. Therefore,

$$\lim_{\varepsilon \to 0} \frac{p^{\varepsilon}_{yz}}{p^{\varepsilon}_{xz}} = 0. \quad (a)$$

Moreover, Proposition 1 implies that only $0-$viable candidates are in $W(\sigma)$. Therefore, since z is the least preferred $0-$viable candidate for voter i, this implies that

$$u_i(x) - u_i(z) > 0 \ \text{ for any } x \in W(\sigma) \subset X^v_0. \quad (b)$$

Combining (a) with (b), it follows that

$$\lim_{\varepsilon \to 0} U_i(v^z\ ;\ p^{\varepsilon}) < 0,$$

which proves that voter i does not approve of z in equilibrium. This concludes the proof of Step A.1.

Step A. 2. Assume that there is some $1-$unviable candidate y so that $\exists\, x \in \mathcal{X}$ with $N(x, \ldots) > n - N^1(\ldots, y)$. Step A.1 proves that no voter votes for his least preferred $0-$viable candidate in equilibrium. Moreover, Lemma 2 proves that no vote for his least preferred candidate in equilibrium. It follows that for any equilibrium

σ, $S(y;\sigma) \leq n - N^1(\dots, y)$ since $N^1(\dots, y)$ denotes the number of voters who rank y last among all candidates and last among the 0−viable candidates. Since, in equilibrium, all voters vote for their most preferred candidate, it follows that:

$$S(x;\sigma) \geq N(x,\dots) > n - N^1(\dots, y) > S(y;\sigma),$$

so that y is not in the winning set. Hence, no 1−unviable candidate is in the set of equilibrium winners, as wanted.

Hence, Step A. has proved that no voter votes for his least preferred 0−viable candidate and that this implies that no 1−unviable candidate is among the winners in equilibrium. We now move to Step B. that proves the induction argument.

Step B: Induction Argument.

Assume now that no voter approves of his least preferred j−viable candidate with $j \in \{0, \dots, k-1\}$ and an h−unviable candidate does not belong to the set of equilibrium winners for any $h \in \{0, \dots, k\}$. This Step proves that this implies that no voter approves of his least preferred k−viable candidate and that that an $(k+1)$−unviable candidate does not belong to the set of equilibrium winners.

Step B.1. We first prove that a voter never approves of his least preferred k−viable candidate.

Since no j−unviable candidate belongs to the set of equilibrium winners for any $j \in \{0, \dots, k\}$, it follows that just k−viable candidates are in $W(\sigma)$ since $\mathcal{X}_k^v = \mathcal{X} \setminus \bigcup_{j=1}^k \mathcal{X}_j^{uv}$.

Denote by z the least preferred k−viable candidate of some voter i. This implies that just k−viable candidates are in $W(\sigma)$, it follows that $\lim_{\varepsilon\to 0} U_i(v^z\ ;\ p^\varepsilon) < 0$ for any sequence of pivot vectors p^ε satisfying the ordering condition. Hence, it is not a best response to approve of z, as wanted.

Step B.2. We now prove that no $(k+1)$−unviable candidate belongs to the set of equilibrium winners. Assume, by contradiction that there is some $(k+1)$−unviable candidate y in some set $W(\sigma)$ of equilibrium winners. Since y is $(k+1)$−unviable, it follows that:

$$\exists\, x \in \mathcal{X} \quad \text{with}\, N(x,\dots) > n - N^k(\dots, y).$$

However, in Step B.1., we have proved that no voter approves his least preferred k−viable candidate y. Moreover, we have assumed that no voter approves of his least preferred j−viable candidate with $j \in \{0, \dots, k-1\}$.

It follows that for any equilibrium σ, $S(y;\sigma) \leq n - N^k(\dots, y)$ since $N^k(\dots, y)$ denotes the number of voters who rank y last among the set of candidates and the set of j−viable candidates for any $j = 0, \dots, k$.

Furthermore, in equilibrium, all voters approve of their first ranked candidate which implies that:

$$S(x;\sigma) \geq N(x,\dots) > n - N^k(\dots, y) > S(y;\sigma),$$

so that y is not in the winning set. Hence, no $(k+1)-$unviable candidate belongs to the set of equilibrium winners, entailing a contradiction and finishing the proof. □

Theorem 4 *Assume that the election has at least four candidates. If there are at least three viable* candidates, then there is a closed set of utilities* $\hat{\mathcal{U}} \subseteq \mathcal{U}$ *such that for any election with utility vector* $u \in \hat{\mathcal{U}}$*, there is an equilibrium in which all viable* candidates are tied for victory.*

Proof of Theorem 1 We prove that, for any profile with at least three viable* candidates, there exists some strategy profile σ and some sequence of pivot vectors $p^{\varepsilon} = (p^{\varepsilon}_{xy})_{xy \in H}$ that constitutes an equilibrium in which all candidates in $\mathcal{X}^*$ are tied for victory so that $W(\sigma) = \mathcal{X}^*$.

The proof is divided in three sections: the preferences, the pivot probabilities and the conclusion.

Section I: The voters' preferences

Take a preference profile with $\#\mathcal{X}^* \geq 3$. Moreover, take some candidate $c \in \mathcal{X}^*$ and such that $n - N(\dots, c) = \min_{x \in \mathcal{X}^*} n - N(\dots, x)$.

Due to Lemma 5, any $d \in \mathcal{X}^*$ satisfies $n - N(\dots, d) \geq n - N(\dots, c)$ whereas if $d \in \mathcal{X} \setminus \mathcal{X}^*$ then $n - N(\dots, d) < n - N(\dots, c)$.

Moreover, let $a, b \in \mathcal{X}^*$ with

$$a = \arg\max_{x \in \mathcal{X}^*} n - N(\dots, x) \text{ and}$$

$$b = \arg\max_{x \in \mathcal{X}^* \setminus \{a\}} n - N(\dots, x).$$

Consider the voters' preferences restricted to the set of candidates $M = \{a, b, c\}$. Moreover, for each $i \in N$ we recall that $R_i = \frac{u_i(t_i) - u_i(m_i)}{u_i(m_i) - u_i(b_i)}$.

Section II: The Pivot Probabilities

We assume that the sequence of pivot probabilities p^{ε} satisfies for any $xy \neq ab, ac, bc$,

$$\lim_{\varepsilon \to 0} \frac{p^{\varepsilon}_{xy}}{p^{\varepsilon}_{xa}} = 0, \ \lim_{\varepsilon \to 0} \frac{p^{\varepsilon}_{xy}}{p^{\varepsilon}_{xb}} = 0 \ \text{ and } \lim_{\varepsilon \to 0} \frac{p^{\varepsilon}_{xy}}{p^{\varepsilon}_{xc}} = 0. \ (f)$$

The condition (g) implies that when $\varepsilon \to 0$, the voter's decision concerning whether to cast a vote for $x \neq a, b, c$, only depends on the pivotal events in which candidates b and c are involved, the rest of them becoming infinitely less likely.

Given these assumptions, we have two implications concerning the voters' decisions. These implications are different if one considers the decision over a, b and c or a different candidate.

Section $II.1$**: Votes for** a,b,c.

Consider the expected utility for a voter i of casting ballot v^a which consists of a vote just for candidate a (and no points for the rest of the candidates):

$$U_i(v^a; p^\varepsilon) = \sum_{ax \in H} p^\varepsilon_{ax}(u_i(a) - u_i(x)).$$

However, since (f) applies, it follows that

$$\lim_{\varepsilon \to 0} \frac{p^\varepsilon_{xy}}{p^\varepsilon_{ab} + p^\varepsilon_{ac} + p^\varepsilon_{bc}} = 0,$$

whenever $xy \neq ab, ac, bc$. Therefore, the following limit equality holds

$$\lim_{\varepsilon \to 0} \frac{U_i(v^a; p^\varepsilon)}{p^\varepsilon_{ab} + p^\varepsilon_{ac} + p^\varepsilon_{bc}} = \frac{p^\varepsilon_{ab}}{p^\varepsilon_{ab} + p^\varepsilon_{ac} + p^\varepsilon_{bc}}(u_i(a) - u_i(b)) + \frac{p^\varepsilon_{ac}}{p^\varepsilon_{ab} + p^\varepsilon_{ac} + p^\varepsilon_{bc}}(u_i(a) - u_i(c)).$$

Hence, writing $q^\varepsilon_{xy} = \frac{p^\varepsilon_{xy}}{p^\varepsilon_{ab} + p^\varepsilon_{ac} + p^\varepsilon_{bc}}$, it follows that

$$\lim_{\varepsilon \to 0} \text{Sign}(U_i(v^a; p^\varepsilon)) = \text{Sign}\left(q^\varepsilon_{ab}(u_i(a) - u_i(b)) + q^\varepsilon_{ac}(u_i(a) - u_i(c))\right).$$

Note that the sign of the utility is the only information needed to determine the voter's best response (since under AV, no constraints are given on the number of dis/approved candidates). Therefore, following a similar reasoning, it can be deduced that:

$$\lim_{\varepsilon \to 0} \text{Sign}(U_i(v^b; p^\varepsilon)) = \text{Sign}\left(-q^\varepsilon_{ab}(u_i(a) - u_i(b)) + q^\varepsilon_{bc}(u_i(b) - u_i(c))\right),$$

and

$$\lim_{\varepsilon \to 0} \text{Sign}(U_i(v^c; p^\varepsilon)) = \text{Sign}\left(-q^\varepsilon_{ac}(u_i(a) - u_i(c)) - q^\varepsilon_{bc}(u_i(b) - u_i(c))\right).$$

Therefore, the decision of the voters over these candidates is equivalent to the one with just three candidates (a, b and c). As discussed in the primer of the proof, we can choose a set of utilities and conditions on p^ε_{ab}, p^ε_{ac} and p^ε_{bc} such that the three candidates are tied for victory for some best response σ. The score of the three candidates equals $n - N(\ldots, c)$.

Section $II.2$**: Votes for the rest of the viable candidates.**

Consider any candidate $d \in \mathcal{X}^v$. By assumption, note that $n - N(\ldots, c) \leq n - N(\ldots, d)$.

Moreover, it is the case that $N(d, \ldots) < n - N(\ldots, c)$. Indeed, assume by contradiction that $N(d, \ldots) > n - N(\ldots, c)$. Then, c is unviable, a contradiction since c in $\mathcal{X}^{in}$.

As proved by Lemma 3, we can choose a best response σ such that the score of candidate d can take any value in $[N(d, \ldots), n - N(\ldots, d)]$.

Moreover, since $N(d, \ldots) < n - N(\ldots, c) \leq n - N(\ldots, d)$, we can set $S(d) = n - N(\ldots, c)$.

In other words, for each candidate $d \in \mathcal{X}^v$, we can find pivot probabilities, cardinal utilities and voters' best responses such that $S(d) = n - N(\ldots, c)$.

Section $II.3$: **Votes for the rest of the unviable candidates.**

The votes for these candidates do not affect the pivot probabilities. Indeed, since an unviable candidate cannot win (by definition), this candidate is not in the winning set of the equilibrium.

Section III: **Conclusion.**

It follows that given p^ε we can choose a strategy profile σ such that every voter chooses among his best responses and $W(\sigma) = \mathcal{X}^v$. Since p^ε follows the ordering condition, σ is an equilibrium in which all the viable candidates are tied as wanted. □

6 Concluding Comments

This work focuses on the implications of allowing strategic voters to vote for as many candidates as they want. We divide the elections into two categories: the ones in which at most two candidates are viable (a) and the ones in which at least there are three viable candidates (b). Our work fully characterizes the set of equilibrium winners for each election.

The results are surprisingly different in both scenarios. It can be argued that the first scenario is much more plausible than the second one at an empirical level. However, note that this intuition holds for plurality elections in which the Duverger's law tends to hold. Within the model, the election takes place under approval voting so that defining a priori what is more plausible seems elusive.

In scenario (a), our model uniquely predicts that the unique equilibrium winner is the Condorcet Winner. Moreover, note that the existence of two viable candidates is a necessary and sufficient condition for the uniqueness of this equilibrium. This is a strong argument for the use of this rule since it coincides with the recommendation made by different fairness theories (i.e. tournament solutions) that entitle that such a candidate should win if it exists.

In contrast, in scenario (b), we prove that there is some equilibrium in which the set of viable candidate coincides with the set of equilibrium winners. More specifically, we show that for any preference profile that admits at least three viable candidates, we can build an equilibrium in which all these candidates are tied for victory. More precisely, our result states that, given the ordinal preferences, there are some cardinal preferences that would admit a voting equilibrium in which every viable candidate is a likely winner. Note that the set of cardinal preferences that admit such large sets of winners may be very small. Finally, our result suggests that this rule may exhibit some indecisiveness when many candidates might win.

A potentially interesting venue for the current model would be to test our model of strategic voting on experimental data or real data. Clearly, the main testable prediction that can be derived from our contribution is the presence of ties among winners with Approval voting when (*i*) there is no Condorcet Winner or (*ii*) when there are at least three viable candidates. Moreover, pushing further the notion of viable candidate and understanding how this concept can be adapted to other theoretical and empirical models seems also very pertinent.

Acknowledgements The authors would like to thank Micael Castanheira, Arnaud Dellis, Amrita Dhillon, Ani Guerdjikova, Fabian Gouret, Yukio Koriyama, Jean-Francois Laslier, Vincent Merlin, Roger Myerson, Carlos Pimienta, Francisco Ruiz-Aliseda and Fabrice Valognes for their useful comments and encouragements. This work has also benefited from comments of several conference and seminar participants. This research has been developed within the center of excellence MME-DII (ANR-11-LBX-0023-01).

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