Studies in Choice and Welfare

Mostapha Diss
Vincent Merlin *Editors*

Evaluating Voting Systems with Probability Models

Essays by and in Honor of William Gehrlein and Dominique Lepelley

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Editors
Mostapha Diss
Department of Economics
CRESE EA3190, Université Bourgogne Franche-Comté
Besançon, France

Vincent Merlin
Faculty of Economics, Management, and Geography
CNRS and Université de Caen Normandie
Caen, France

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Foreword

The history of social choice theory is mainly a history of negative results. The rebirth of this theory in modern times is essentially due to Kenneth J. Arrow. In 1948, he demonstrated the inconsistency of several properties of procedures of aggregation of individual preferences into a 'social' preference. Some of these properties are generally considered as basic elements of democratic procedures, such as the absence of a dictator and the respect of unanimity (this property is being associated with Pareto and the so-called Pareto optimality, a major concept of microeconomic theory). Other properties have been called into question such as the so-called independence of irrelevant alternatives and the transitivity of the social preference. Arrow's independence condition amounts to limit the information that can be used in the aggregation procedure to pairwise preference considerations.

A second negative result, due to Amartya Sen, concerns an inconsistency between collective rationality (for instance, some kind of transitivity property of the social preference), unanimity, and a degree of freedom conferred to individuals.

A third major negative result due to Alan Gibbard, Prasanta Pattanaik, and Mark Satterthwaite concerns a property of strategyproofness: Aggregation procedures should be immune to a strategic behavior of individuals (in the case of voters, this, in consequence, concerns 'useful or tactical' voting—it can advantageous for a voter to misrepresent her preference so that the voting rule generates an outcome—for instance, a candidate—that this voter prefers to the outcome that would have prevailed if she had not misrepresented her preference).

Long before these results were obtained (in 1948 and in the 1970s), Condorcet in 1785 had shown that majority rule, a procedure that obviously satisfies Arrow's properties, could generate a cycle of the social preference. A very simple example (a lot simpler than Condorcet's example!) could be the following. Consider three persons who want to have dinner together and have a choice between three restaurants *a*, *b*, and *c*. Person 1 prefers *a* to *b* and *b* to *c*, and since she is a rational person she prefers *a* to *c*. Persons 2 and 3 are also rational persons, and person 2 prefers *b* to *c* and *c* to *a* while person 3 prefers *c* to *a* and *a* to *b*. They agree that the choice of the restaurant will be made by using majority rule: If the number of persons who prefer say *a* to *b* is greater than the number of persons who prefer *b* to *a*, then *a* will be 'socially' preferred to *b*. But given the individual preferences just mentioned, one can see that *a* is socially preferred to *b* which is socially preferred to *c* which is socially to *a*. So there is no restaurant which is socially preferred to the other two, and the choice is problematic.

However, this outcome is based on a specific configuration of individual preferences. A natural question arises regarding the probability that such situations occur. For instance, given three options (the three restaurants), there are six possible rational preference orderings (excluding ties). If each person has one of these six preferences with probability 1/6, what is the probability to obtain a cyclic outcome or a situation where there is no option defeating the other two. The French mathematician G. Th. Guilbaud indicated in 1952 that for our example a cycle will be obtained in less than 6% of the situations. In a rather enigmatic (enigmatic at the time of its publication) footnote, Guilbaud gave a limit for a 'large' number of individuals, limit being less than 9%. Although Guilbaud's work was largely ignored in the English-speaking world, this kind of analysis took off at the end of the 1960s as indicated by Sen in his book of 1970 and by Peter Fishburn in his 1973 treatise on social choice theory.

The works of Fishburn and William Gehrlein in the 1970s establish a new sub-domain of the theory of social choice where various paradoxical situations generated by various voting rules under various combinatorial/probabilistic assumptions were studied. William Gehrlein has been the most prolific and most influential author in this sub-domain over the last decades, and he published a wonderful book in 2006 on Condorcet's paradox.

At the University of Caen, under the leadership of Dominique Lepelley, this sub-domain was eagerly developed (in particular by several of the contributors to this volume). I must outline that Dominique Lepelley and Boniface Mbih were, to the best of my knowledge, the pioneers regarding exact calculations related to the manipulation of voting rules (previous works were based on simulations and Monte Carlo techniques).

What should happen did happen: A collaboration between Gehrlein and Lepelley began (a collaboration which was extended to a few others). The result of this collaboration is the publication of many joint papers and of two exceptional books. Another outcome of this collaboration is the present volume partially based on a conference which took place at the University of Caen-Normandy in 2018.

I am very proud to have been a very minor element in the success of this scientific accomplishment.

Maurice Salles
Normandy University, UNICAEN, CNRS,
UMR 6211, CREM
Université de Caen Normandie
Caen, France
e-mail: maurice.salles@unicaen.fr

CPNSS, London School of Economics
London, UK

Murat Sertel Center for Advanced
Economic Studies
Bilgi University
Istanbul, Turkey

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Contents

Contributors

Fuad Aleskerov National Research University Higher School of Economics, Moscow, Russia;
Institute of Control Sciences of Russian Academy of Sciences, Moscow, Russia

Nicolas Gabriel Andjiga Ecole Normale Supérieure - Département de Mathématiques, Université de Yaoundé1, Yaoundé, Cameroun

Fabrice Barthélémy Department of Social Sciences, Université de Versailles Saint-Quentin en Yvelines, CEMOTEV, Guyancourt Cedex, France

Raouia Belayadi Normandie Université, Unicaen, CREM, UMR CNRS 6211, Caen, France

Felix Brandt Institut für Informatik, Technische Universität München, München, Germany

Michel Le Breton Institut Universitaire de France and Toulouse School of Economics, University of Toulouse Capitole, Toulouse, France

Sébastien Courtin Normandie Université, Unicaen, CREM, UMR CNRS, Caen, France

Olivier de Mouzon Toulouse School of Economics, INRA, University of Toulouse Capitole, Toulouse, France

Mostapha Diss CRESE EA3190, Université Bourgogne Franche-Comté, Besançon, France

Abdelhalim El Ouafdi University of Lyon, UJM Saint-Etienne, GATE UMR 5824, Saint-Etienne, France

Marc Feix Nantes, France

William V. Gehrlein Department of Business Administration, University of Delaware, Newark, DE, USA

Christian Geist Institut für Informatik, Technische Universität München, München, Germany

Johannes Hofbauer Institut für Informatik, Technische Universität München, München, Germany

Alexander Ivanov National Research University Higher School of Economics, Moscow, Russia

Eric Kamwa LC2S UMR CNRS 8053, Faculté de Droit et d'Economie de la Martinique, Université des Antilles, Schœlcher Cedex, France

Daniel Karabekyan National Research University Higher School of Economics, Moscow, Russia

Alexander Karpov National Research University Higher School of Economics, Moscow, Russia;
Russian Academy of Science, Institute of Control Science, Moscow, Russia

Thibault Laurent Toulouse School of Economics, CNRS, University of Toulouse Capitole, Toulouse, France

Dominique Lepelley CEMOI, Department of Economics, University of La Réunion, Saint-Denis Cedex 9, La Réunion, France

Mathieu Martin Department of Economics, CY Cergy Paris Université THEMA, Cergy-Pontoise Cedex, France

Boniface Mbih CREM UMR CNRS 6211, Université de Caen Normandie, Caen, France

Vincent Merlin Faculty of Economics, Management, and Geography, CNRS and Université de Caen Normandie, Caen Cedex, France

Issofa Moyouwou Ecole Normale Supérieure, Department of Mathematics, University of Yaounde I, Yaoundé, Cameroon

Matías Núñez CREST, Ecole Polytechnique & CNRS, Palaiseau, France

Patrizia Pérez-Asurmendi PRESAD Research Group and IMUVA, Department of Economic Analysis, UCM, Madrid, Spain;
ICAE, UCM, Madrid, Spain

Jean-Louis Rouet Université d'Orléans, CNRS/INSU, BRGM, ISTO, UMR7327, Orleans, France

Hatem Smaoui Université de La Réunion, CEMOI EA13, Saint Denis, La Réunion, France

Martin Strobel School of Computing, National University of Singapore, Singapore, Singapore

Abdelmonaim Tlidi National School of Applied Science-Safi, University Cadi Ayyad of Marrakesh, LMPEQ, Safi, Morocco

Laurent Vidu Normandy University, UNICAEN, CNRS, UMR 6211, CREM, MRSH bureau 131, Université de Caen Normandie, Caen cedex, France

Vyacheslav Yakuba National Research University Higher School of Economics, Moscow, Russia;
Institute of Control Sciences of Russian Academy of Sciences, Moscow, Russia

Introduction

Mostapha Diss and Vincent Merlin

The use of probability arguments to assess the qualities and flaws of decision procedures has a long history. The first book on the mathematics of democracy, *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*, was published by the Marquis de Condorcet (1785). In his famous essay, he exposes for the first time what is now known as the Condorcet's jury theorem: When a jury has to take a decision, if each juror has a probability to discern the truth superior to one half, the likelihood that this jury takes the right decision using the majority rule tends to one as the number of jurors increases.

In contemporary times, the birth of game theory and social choice theory in the 1940s saw the development of new mathematical tools and techniques to analyze voting rules and decision-making. It is needless to say that some of these arguments are based on the notion of probabilities. The next paragraphs will present several contributions that are worth mentioning.

The statistician Penrose (1946, 1952) is now considered as the pioneer of the literature on power indices, whose aim is to evaluate a priori influence a delegate could enjoy in a committee, an assembly or a parliament. Here, power is defined as the a priori probability for a voter to influence the outcome. His work, largely ignored at his time, nevertheless contains many results and intuitions that will be rediscovered in the next decades. In particular, similar concepts will become popular after the publication of Banzhaf's paper (1965). He proposes to evaluate the power of different cities in the Nassau County council, by counting the number of times a city could swing the council election divided by the total number of possible voting

M. Diss (✉)
CRESE EA3190, Université Bourgogne Franche-Comté, 25000 Besançon, France
e-mail: mostapha.diss@univ-fcomte.fr

V. Merlin
Faculty of Economics, Management, and Geography, CNRS and Université de Caen Normandie, 14032 Caen Cedex, France
e-mail: vincent.merlin@unicaen.fr

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situations. Shapley and Shubik (1954) also define power as a probability, here as the number of times a voter is pivotal divided by the total number of possible permutations among the voters. Later, Rae (1969) will define and measure the notion of success in the game theoretical framework as the probability of being on the winning side in a vote. Coleman (1971) played also a significant role in popularizing these concepts.

In voting theory, one of the earliest papers is due to the mathematician May (1948). In a short note published in Mathematics Magazine, he derives the probability for a two-tier voting system (such as the US Electoral College or the House of Commons) to give a majority of delegates for one party, while the other party would get more votes overall. Just after the publication of *Social Choice and Individual Values*, by Arrow (1951), a French mathematician, Guilbaud (1952), gave the first estimation of the probability of a cycle among three candidates while using the majority rule.[1] In the 1960s, several authors also tackled this issue (Campbell and Tullock 1965; Garman and Kamien 1968; Niemi and Weisberg 1968).

Clearly, these pioneers were excited by the new possibilities that the use of formal mathematical models in social sciences could open. In addition, at that time, precise electoral data were rare and the treatment software did not exist yet, so assessing the probability of an event with a reasonable a priori probability distribution was clearly a first step forward. Even if nowadays it is clear to everyone that a priori probabilistic models do not describe the reality, they enable us to evaluate and compare the decision processes on a normative basis, “before the player(s) enter the room”. As such, the probabilistic approach is a distinguished member of the family of the formal tools one can mobilize to analyze decision procedures.

Even if probabilistic arguments were from time to time mobilized in the 1960s, we had to wait till the 1970s to get a well-defined set of assumptions, terminology, tools, and methods; this development was concomitant to the “boom” of social choice theory and game theory applications that occurred at that time. Let us mention a few seminal contributions. Departing from the *Impartial Culture* (IC) model that was dominant in early works, Gehrlein and Fishburn (1976) developed a new probability assumption, the *Impartial Anonymous Culture* (IAC), to estimate the likelihood of various paradoxical events in voting. Berg (1985) showed that both assumptions belong to a larger family of probability models, the Polya-Eggenberger urn models. Straffin (1977) put the emphasis on the probabilistic foundations of the major voting power index, that is, the Penrose-Banzhaf index and the Shapley-Shubik index. Lepelley and Mbih (1994) extended the range of applications of probabilistic approaches in social choice by computing the vulnerability of famous voting rules to coalitional manipulations. More recently, these probability approaches were even borrowed by computer scientists, to understand whether NP-complexity is really a barrier against manipulation of voting schemes (Walsh 2010). In addition, several scholars proposed new original probability models to describe the behavior of individuals, models who can be of particular interest in terms of tractability and

[1] Surprisingly, his result is just stated in a footnote without any proof, and William Gehrlein admitted that it took him some time in the 1970s to understand how he proceeded. Fortunately, Guilbaud’s computation was right.

predictability for the resolution of certain issues. In this vein, Theil (1970) and Merrill (1977) built models based on the realization of a latent Gaussian variable, to capture phenomena that are regularly observed in political science (in a wave year, a party progresses more in its weakest territories than in its strongholds). In game theory, Feddersen and Pesendorfer (1997) introduced game theoretic arguments to revisit the literature on juries; Myerson (2000) proposed the large Poisson games as a way to model large elections with an unknown number of participants and examined the equilibria of the associated voting games.

This list of examples is of course a partial one, but they all clearly demonstrate that the notion of probability is a central one as we need to model the behavior of the voters for a better understanding of collective decision-making.

The aim of this volume is to present up to date contributions on the domain of probabilistic analysis of voting rules and decision mechanisms, broadly defined. The origin this book is to be found in the 8th Murat Sertel Workshop on Economic Design, Decision, Institutions, that was held in Caen, France, on the 22nd and 23rd of May 2018. As this workshop was dedicated to our colleague Dominique Lepelley, who had retired earlier in autumn 2017, his co-author, Bill Gehrlein kindly, accepted to come and present the plenary lecture. It became obvious for many colleagues during this event that the social choice community should pay a tribute to both of them, for their influence on the development of the probability approaches in voting theory. We find no better way than to gather a selection of recent papers, which clearly display the different facets of this field and illustrate their influence.

The volume contains 16 contributions by 32 authors. The chapters are organized into 6 parts.

1 Part I: The Condorcet Efficiency of Voting Rules and Related Paradoxes

In his book, Condorcet (1785) suggested that, when a society faces a choice among several candidates, the best choice would be to select the candidate who is able to defeat any other candidate in majority comparisons. Such a candidate is nowadays called a *Condorcet Winner*. Symmetrically, a candidate who is defeated by all his opponents is called a *Condorcet Loser*. As Condorcet (1785) noticed, the Condorcet winner may not exist for some preference patterns due to the existence of cycles in the majority relationships, a voting rule may not select a Condorcet winner even if it exists, and even worse, some voting rules may pick a Condorcet loser as a winner. This last phenomenon is called a *Borda Paradox,* in the honor of the scientist who first mentioned it (de Borda 1781). Evaluating the likelihood of the existence of a Condorcet winner and computing the *Condorcet Efficiency,* that is, the probability for a voting rule to select the Condorcet winner whenever it exists, became central issues in social choice theory. These issues triggered the development of computational techniques for the evaluation of the probabilities of voting outcomes. Gehrlein (2006) and Gehrlein and Lepelley (2011, 2017) have presented in recent books the state of the art in the domain. Obviously, the first four chapters of this volume concern recent contributions on these iconic issues.

In the second chapter of this volume, Gehrlein and Lepelley revisit the known results on the probability of electing the Condorcet winner and on the probability of the Borda paradox, for three-weighted scoring rules, the plurality rule (PR), the negative plurality rule (NPR), the Borda rule (BR), and their two-stage versions, the plurality elimination rule (PER), the negative plurality elimination rule (NPER), and the Borda elimination rule (BER). Their studies introduce an element that has been seldom examined in the literature, the possibility of abstention, and they provide precise formulas in the three-candidate case, both under the IC and IAC assumptions. When the participation rate is low, the behavior of the voting rules is almost random, as very few voters show up. When it reaches 100%, we recover the values that have been already derived in the literature. For each voting rule, they are able to show at which rate does the likelihood of these voting paradoxes converge to their 100% participation value.

The next two chapters also concern the evaluation of the Condorcet efficiency of weighted scoring rules. Diss, Kamwa, Moyouwou, and Smaoui study the effects of various scenarios under the IAC model for three candidates: the effect of indifferent voters, the effect of global abstention (voters from all possible types may abstain), the effect of self-confident abstention (voters who support the potential winner tend not to go to the pools), and the effect of pessimistic abstention (voters who support a potential loser tend not to go the pools). By focusing on the limit case of an infinite population, they are able to derive formulas for all the weighted scoring rules and check which rule has the highest Condorcet efficiency in each case. The authors show that in general, the Condorcet efficiency of the considered voting rules may change significantly when indifference or abstention is possible. This change depends on the voting rule under consideration and on the probability distribution on the set of observable voting situations in each case.

Diss, Pérez-Asurmendi, and Tlidi have another objective: They want to understand the effect of a close election on the Condorcet efficiency of PR, NPR, BR, PER, and NPER. Election closeness is measured in this chapter by an index calculated as a proportion of points obtained by the last ranked candidate divided by the aggregated scores of all competing candidates under the considered voting rule. The authors show that the Condorcet efficiency of some voting rules may significantly decrease as the results of elections become very close. However, such a reduction varies depending on the considered voting rule. They show, for instance, that the Condorcet efficiency of PR and NPR is more sensitive to the closeness of the election than the Condorcet efficiency of BR. One can also notice that this chapter and the previous ones are based on the geometrical techniques that were developed by Cervone et al. (2005) more than fifteen years ago, techniques that are now fully operational due to the development of computer computations.

In Chap. 5, together with the Borda paradox, Brandt, Geist, and Strobel consider another paradox, the Agenda Contraction Paradox (ACP). The ACP occurs when removing losing alternatives changes the set of winners. Together with the BR, they also consider six other rules, which respect the Condorcet Principle: the maximin rule (MMR), Young's rule (YR), Dodgson's rule (DR), Tideman's rule (TR), Copeland's rule (CR), and the essential set (ES). None of them satisfies the ACP condition, and four of them suffer from the Borda paradox (BR, MMR, YR, and DR). The contribution is particularly notable by the number of assumptions and the variety of techniques that the authors mobilize. On the assumption side, they use not only classic models such as IC and IAC, but also the Mallows-φ model, the Polya-Eggenberger urn model, and the spatial model. From the technical viewpoint, they use precise evaluations relying on Ehrhart techniques (which they are for the first time able to use in the four-candidate case), massive simulations (from 3 up to 10 candidates, 3 to 1001 voters), and analysis of elections collected from different database (PrefLib, Netflix prize library). While the Borda paradox seems a very unlikely event for all the rules studied, this is not the case for the ACP. For example, the ACP probability for Copeland's rule is 44% with 6 candidates and 50 voters. All the results depend heavily on the tested scenarios, but for no rule, we can neglect the ACP.

2 Part II: Other Voting Paradoxes

Chapters 6 and 7 consider other voting paradoxes, namely the Ostrogorski paradox and the violation of reversal symmetry.

Gehrlein and Merlin consider another well-known voting paradox, described for the first time by the political scientist Ostrogorski (1902). Consider a set of citizens, who have to vote "yes" or "no" on a series of issues via referendums. In a direct democracy, the policies implemented will be the ones which have received a majority of votes on each issue. However, in representative democracy, voters have to choose between parties and the winner will implement its electoral platform. Hence, a voter cannot vote anymore issue by issue, but he has to select the party which is the closest to his views. In two-party competitions, a Strict Ostrogorski Paradox (SOP) occurs each time the platform of the winning party would gather a minority of votes on each issue. An Ostrogorski paradox occurs if the winning party has a minority of the votes on a majority of issues. Gehrlein and Merlin evaluate the likelihood of these paradoxes when there are 3 to 5 issues and a large number of voters under the IC model. They conclude that the strict form of the paradox is extremely unlikely to happen.

The violation of reversal symmetry occurs when all voters reverse their preferences. One would guess that in that case, the collective result is reversed too: The candidate who was previously ranked last is the new winner, and the old winner should now be ranked last. Unfortunately, except for the Borda rule, no other weighted scoring rule satisfies this property. Belayadi and Mbih evaluate the likelihood of this paradox for three alternatives under the IAC model. They consider the whole class of weighted scoring rules and their two-stage counterparts. The lack of reversal symmetry cannot be neglected for weighted scoring rules: It can be as high as 20.3%

for NPR and almost reach 10% for the PR. Its impact is much more limited for two-stage scoring rules: It is less than 3% for PER, peaks above 4% for the rule which gives 1 point for a first place, 0.45 point for a second place, and 0 for a third place, and then decreases to 1.2% for NPER.

3 Part III: Binary Voting in Federations

The use of probabilistic arguments has been extremely successful and widespread in one particular context, the analysis of the decision procedure for federal unions. Typically, several states (elsewhere countries, cities, regions, etc.) have decided to form a political union to govern their common affairs. Each state is represented by a certain number of delegates at the council of the federal union. To name a few examples, the election of the US president by the Electoral College, the governance of the EU by its council of ministers, the board of governors of the International Monetary Fund, etc. follow this logic. Hence, the questions are: How many delegates should we allocate to each state? Which quota should we use to pass a decision? These issues have triggered many debates in political science. A vast number of solutions to the problem have been proposed in game theory and social choice theory.

The criteria used by Feix, Lepelley, Merlin, Rouet, and Vidu are inspired by the Condorcet efficiency literature. In a two-party competition, they say that a two-tier voting is majority efficient if the party which obtains a majority of votes in the federation is also the party which has a majority of delegates. In other terms, we wish to avoid situations similar to the election of George W. Bush in 2000 or Donald Trump in 2016. If we assume that the party, which obtains a majority of votes in one state, controls all its delegates, how should we allocate the delegates among the states in order to maximize the majority efficiency? The authors explore this issue when the number of delegates is proportional to n_t^α, where n_t is the population of state t and α is a positive parameter. Pure federalism is obtained, with one delegate per state independently of its population for $\alpha = 0$. Pure proportionality is recovered with the value $\alpha = 1$. For any value in between 0 and 1, there is degressive proportionality. Through simulations performed from 3 to 50 states, the authors show that the square root rule ($\alpha = 0.5$) seems to be optimal under the IC assumption. However, when they use the model initially proposed by May (1948), that is, when they use the IAC assumption within a state, but assume independence across the states, the proportional rule $\alpha = 1$ emerges as the optimal rule.

In their chapter, de Mouzon, Laurent, Le Breton, and Moyouwou revisit a classic theme, the fairness of the apportionment of delegates in the US Electoral College. Each American state is represented by a number of delegates which is equal to its number of representatives in the House (which is apportioned according to the Huntington-Hill rule after each census), plus its number of senators. This procedure is supposed to favor small states, as the smallest state controls at least 3 delegates (out of 535). A completely different picture emerges if we use a measure of fairness borrowed from game theory: A voter is supposed to be decisive, if, by changing his vote, he can modify the outcome of the voting procedure. Hence, fairness among the voters is achieved only if all the votes have the same probability of being decisive. Using the 2010 apportionment of the delegates among the American states, the authors evaluate the inequality among US citizens in the Electoral College by using different power indices. Again, the role of the underlying probabilistic models is crucial. Using the IC model, that is, the Banzhaf index, or the IAC model, that is, the Shapley-Shubik index, favors the larger states. In contrast, using a probabilistic model similar to May's one, where the IAC assumption holds within each state but the behaviors of the voters of the different states are independent, suggests that the citizens living in small states are favored by the Electoral College voting scheme.

4 Part IV: Resistance to Manipulations

In the 1990s, the tools and techniques from the literature on the evaluation of paradoxes started to be applied to a new theme, the resistance to manipulation. The Gibbard-Satterthwaite (Gibbard 1973; Satterthwaite 1975) theorem is a milestone in social choice theory: It asserts that as soon as there are at least three alternatives, all voting rules are manipulable, unless they are the constant rule or dictatorship. Said differently, for all the voting rules we use in everyday life, there always exist voting situations where a voter is better off by reporting a preference, which is different from his sincere ballot. Given that all the common voting rules are manipulable, can we say that some of them are less manipulable than other ones? The first contributions on the propensity to be manipulated dealt only with individual manipulations (Nitzan 1985; Chamberlin 1985; Kelly 1993). The papers by Lepelley and Mbih (1987, 1994) were a milestone in this literature. Lepelley and Mbih were the first authors who proposed to study the vulnerability to coalitional manipulations, and they derive precise formulas under the IAC assumption in evaluating this issue, rather than relying on computer simulations. Since then, many contributions have tackled these issues, and the volume gathers three chapters on this topic.

Aleskerov, Karabekyan, Ivanov, and Yakuba examine the vulnerability to individual manipulation of 12 different voting rules, namely PR, BR, q-approval, Black, Copeland (with three different versions), Threshold, Nanson, BER, and PER. Their results complement a previous study by Favardin and Lepelley (2006), in the sense that they do not use an a priori tie-breaking rule in case of multiple winners. Instead, they assume that voters are able to compare different subsets of tied outcomes with extra criteria (Leximin, Leximax, Preference Worst, and Preference Best extensions). In addition, as in Favardin and Lepelley (2006), they take into consideration the possibility of counter manipulations: A voter successfully manipulates the outcome if no other voter can use a counterthreat to reverse the effects of the initial manipulation. The authors obtain their results by computer simulations for 3, 4, and 5 alternatives and up to 30 voters. The main result is quite surprising: When counterthreats are considered, the Borda count fares very well compared to other voting rules.

Chapters 11 and 12 consider other ways to manipulate the elections. The so-called No-Show Paradox (NSP) occurs when a voter is better off by abstaining from the election instead of casting his sincere ballot. It is well known that scoring elimination rules, such as PER, NPER, and BER, suffer from this paradox, as well as Condorcet consistent rules. However, the extent to which this paradox is of practical concern needs to be assessed. In Chap. 11, Brandt, Hofbauer, and Strobel evaluate the vulnerability of six voting rules (Black's rule, Baldwin's rule, Nanson's rule, maximin rule, Tideman's rule, and Copeland's rule) to the NSP under a large variety of assumptions: IC, IAC, Mallow's model, and the spatial model. To get their results, they either derived exact formulas using Ehrhart theory, or rely on massive simulations. The key message is that, while it is very unlikely that a single voter can alter the outcome by abstaining when the number of alternatives is small, there are few scenarios where it may matter. For example, with 30 alternatives, Copeland's rule, Baldwin's rule, and Nanson's rule are manipulable by abstention for more than 30% of the voting situations.

Chapter 12 deals with another type of manipulation, the manipulation by truncation of preferences. Here, voters are not obliged to report their full preference and may choose to report only their top choice in order to manipulate the outcome. Almost all the voting rules are sensitive to the truncation paradox, with the exception of PR, PER, and Approval Voting. For three alternatives, Kamwa and Moyouwou characterize all the voting situations vulnerable to the truncation paradox by groups of voters for all the weighted scoring rules and the weighted scoring elimination rules. Next, they compute the corresponding limit probabilities under the IAC assumption. In particular, they show that these values can be very high for NPR and NPER. However, the probability of the truncation paradox is much lower when we concentrate on the single-peaked domain.

5 Part V: Game Theory

Aspects of weighted quota games and power indices are already studied in Chap. 9 when analyzing voting in federations. Typically, each player is endowed with a weight and a coalition of players is able to pass a decision if the sum of their weights is over some predefined threshold, called the quota. A power index will then indicate how frequently a player is pivotal, given some underlying probability distribution on the formation of the coalitions. Sometimes, even if a player is endowed with a positive weight in the weighted quota game, his power is null. Indeed, he is never in situation of altering the outcome. Such a player is called a dummy player. A famous example is given by the first decision scheme in the European Union, where Luxembourg, with one vote, was never a necessary player to reach the quota. How frequent are these situations? Barthélemy and Martin explore this issue, by giving some analytical results for games with 3, 4, and 5 players and also running computer simulations on randomly generated games. In particular, they show that the value of the quota has a strong influence on the probability of observing dummy players in a game.

Probabilistic considerations are also extremely important when one wants to predict the outcomes of a voting game. The concept of Nash equilibrium is often useless when one wants to describe the strategic behavior of the voters: Too many profiles are an equilibrium, as it is unlikely for a unique voter to affect the outcome as the number of electors grows. This phenomenon has been observed quite clearly in Chap. 10. Hence, the literature, which analyzes the voting rules as a game form, has to make stronger assumptions on the behavior of the voters, and on the way, they perceive the information. In this line of research, Courtin and Nuñez propose an extended analysis of the Approval Voting game. In particular, in three-candidate elections, when there are only two viable candidates, they show that the unique equilibrium winner is the Condorcet winner.

6 Part VI: Techniques for Probability Computations

The previous chapters have clearly displayed the large range of applications of the probabilistic approach in voting theory. The reader may also notice that these results were obtained by a variety of tools and techniques. In the three last chapters of this volume, we will focus on more computational aspects of the literature.

Indeed, the choice of a voting model is based on some specific probabilistic assumptions on the set of preference profiles. The IC model assumes that each voter picks randomly and independently his preference from a uniform distribution on the set of all possible preferences. The IAC model considers anonymous profiles. Two profiles are identical if one of them is obtained from another one by a permutation on the names of the voters. Hence, these two profiles belong to the same (anonymous) equivalent class, and each class should just count once for the probabilistic evaluation of the voting rule. We can even go further. Egecioglu and Giritligil (2013) suggested that two profiles are equivalent if one of them is obtained from another one via a permutation on the names of the candidates. Hence, two profiles belong to the same class if they are equivalent up to a permutation of the names of voters (anonymity condition) and up to a permutation of the names of candidates (neutrality condition). These symmetries partition the set of all the profiles in Anonymous and Neutral Equivalent Classes (ANECs). By considering that each ANEC counts for one and that they are equally likely, it is possible to define the Impartial Anonymous and Neutral Condition, IANC. This condition is of particular interest when one has to design laboratory experiments in voting, where the names of the candidates and the names the voters should not count (Sertel and Giritligil 2003; Giritligil and Sertel 2005). In Chap. 15, Karpov goes a step further, by considering two profiles as equivalent if one is the reversed version of the other. He defines the Self-Symmetric Anonymous and Neutral Equivalent Class, SSANEC (i.e., a set of profiles, which is invariant by reverse symmetry). Similarly, a pair of ANECs is reverse symmetric, if every profile in one class is the reverse profile of another profile in the other class. Considering reverse symmetric ANECs as equivalent, a set of Reverse Invariant Anonymous and Neutral Equivalent Classes (RIANECs) is defined. The chapter first presents new representations for preference profiles, which enable to compute the numbers of RIANECs and SSANECs for three alternatives, as a function of n, the number of voters. Next, Karpov partitions again the number of ANECs so that each ANEC in one class selects the same winner for a given voting rule. By comparing the number of classes obtained for PR, BR, and Kemeny rule for three alternatives, he evaluates the quantity of information needed to compute the output for each rule.

The subsequent chapter comes back to the tools and techniques susceptible to be used to derive the probabilities of events under the IAC assumptions. Indeed, the evaluation of such probabilities is equivalent to the following question: What is the exact number of integer solutions to a finite set of linear inequalities involving integer coefficients of bounded integer free variables and integer parameters as a function of the parameters? A French mathematician, Ehrhart (1962), has conducted the first study of this problem in the case of a unique parameter. He found that the solution of the problem was pseudo-polynomials functions, that is, a piecewise-defined function with polynomial expressions (modulo a positive integer period). Moreover, Ehrhart suggested that pseudo-polynomials still model the case of several parameters. Andjiga, Mbih, and Moyouwou propose here a direct proof of Ehrhart's conjecture, together with a companion algorithm. Next, they apply the algorithm to give exact formulas for the likelihood of some voting paradoxes.

The last chapter, by El Ouafdi, Moyouwou, and Smaoui, is a progress report on the techniques that have been used in voting theory to compute the likelihood of events under the IAC assumption, and it tells the story of almost 50 years of research in the domain. Gehrlein and Fishburn (1976) obtained the first representations of the probabilities of voting events under IAC by polynomials functions. Their simple algebraic techniques have been used until the early 2000s, when two new methods emerged. Observing that the number of voting situations satisfying certain constraints was described by a series of periodic polynomial, Huang and Chua (2000) were the first authors to systematically search the coefficients in these formulas. They provided a simple algorithm based on the interpolation techniques. Later, Cervone et al. (2005) developed a geometric approach in the limit case where the number of voters tends to infinity. An important breakthrough occurred when Wilson and Pritchard (2007) and Lepelley et al. (2008) drew attention to the existence of a well-established mathematical approach for performing the calculations under the IAC assumption, based on Ehrhart (1962) work. Since then, several mathematicians and economists have proposed more and more sophisticated algorithms to perform efficiently such computations.

Mostapha Diss and Vincent Merlin, editors.

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The Condorcet Efficiency of Voting Rules and Related Paradoxes

Analyzing the Probability of Election Outcomes with Abstentions

William V. Gehrlein and Dominique Lepelley

1 Introduction

We presented some preliminary results at the *Eighth Murat Sertel Workshop on Economic Design, Decision, Institutions, and Organization: In the honor of Dominique Lepelley* at University of Caen in May 2018 that were very disconcerting. Those results considered the very negative impact that was observed when dependence among voters' preferences was added to scenarios in which voters could abstain. The results were so disconcerting that the closing comment of the presentation was: "So, we leave you with chaos". There were comments suggesting that something had to be wrong with the models being used, given that polls with relatively small sample sizes, reflecting the case of high abstention rates, can quite accurately predict outcomes. This was all quite bewildering, to make it difficult for us to walk away from that meeting to settle into states of happy and content retirement. The objective of this current paper is to fully develop the preliminary results that were presented at that meeting, and then to explain with further analysis why those results are indeed correct and why they are not as shocking as they were initially thought to be. In fact, these results quite possibly should have been expected.

To begin, we consider the impact that voter abstention can have on elections with three candidates $\{A, B, C\}$ when there are n possible voters in the electorate. Define the preferences of these voters by using $A \succ B$ to denote the fact that any given voter prefers Candidate A to Candidate B. There are six possible linear voter preference

W. V. Gehrlein
Department of Business Administration, University of Delaware, Newark, DE 19716, USA
e-mail: wvg@udel.edu

D. Lepelley (✉)
Department of Economics, University of La Réunion, 97744 Saint-Denis Cedex 9, La Réunion, France
e-mail: dominique.lepelley@univ-reunion.fr

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A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
n_1	n_2	n_3	n_4	n_5	n_6

Fig. 1 Six possible linear preference rankings on three candidates

rankings on these candidates that are transitive and have no voter indifference between candidates, as shown in Fig. 1.

Here, for example, n_1 represents the number of possible voters with the preference ranking $A \succ B \succ C$, with $A \succ C$ being required by transitivity. The total number of possible voters is n with $n = \sum_{i=1}^{6} n_i$, and an *actual voting situation* defines any particular combination of these n_i terms. It is inherently assumed in the following analysis that any actual voting situation reflects the true preference rankings from informed voters, and that no loosely formed preferences are included from initially disengaged voters who might later be quite surprised and become very engaged as a result of the consequences that ultimately follow from their uninformed choices. More will be said about difficulties that can arise from considering such disengaged voters later in this study.

Much of what follows is based on the notion of using the preference rankings of all possible voters in an actual voting situation to perform majority rule comparisons on pairs of candidates, such that Candidate A beats B by *Pairwise Majority Rule* $[AMB]$ if $A \succ B$ more frequently than $B \succ A$ in the preference rankings of the possible voters. We assume throughout that n is odd to avoid the possibility of Pairwise Majority Rule ties. Candidate A is defined as the *Actual Condorcet Winner* (*ACW*) if both AMB and AMC, with:

$$n_1 + n_2 + n_4 > n_3 + n_5 + n_6[AMB] \quad (1)$$

$$n_1 + n_2 + n_3 > n_4 + n_5 + n_6[AMC]. \quad (2)$$

It is well known that an ACW does not necessarily exist (Condorcet 1785), but such a candidate would clearly be a very good choice for selection as the winner of an election whenever there is one, since a majority of the possible voters would oppose the choice of either of the other two candidates. The *Actual Condorcet Loser* (*ACL*) is then defined in the obvious manner, and such a candidate would be a terrible choice for selection as the winner.

We proceed to consider election outcomes when abstentions are allowed, so that some of the possible voters can choose not to participate for any reason. Let n_i^* denote the number of voters with the associated preference ranking in Fig. 1 who actually choose to participate in the election, with $0 \leq n_i^* \leq n_i$, for $i = \{1, 2, 3, 4, 5, 6\}$. The total number of voters who participate is defined by $n^* = \sum_{i=1}^{6} n_i^*$, so that the voter participation rate is $\frac{n^*}{n}$. An *observed voting situation* from an actual voting situation is then defined by any feasible combination of n_i^* terms that sum to n^*.

Candidate A is the *Observed Condorcet Winner* (*OCW*) based on the preference rankings of the participating voters if AM^*B and AM^*C, with:

$$n_1^* + n_2^* + n_4^* > n_3^* + n_5^* + n_6^* \left[AM^*B\right] \tag{3}$$

$$n_1^* + n_2^* + n_3^* > n_4^* + n_5^* + n_6^* \left[AM^*C\right]. \tag{4}$$

The *Observed Condorcet Loser* (*OCL*) is similarly defined in the obvious manner.

Gehrlein and Lepelley (2017a) perform an analysis to conclude that the likelihood that some very bad election outcomes might be observed is significantly increased when voter participation rates are low. For example, depending upon which voters choose to abstain in any particular case, the ACW and the OCW do not necessarily have to coincide, and this current study begins by considering the probability that the ACW and OCW will be the same candidate as a function of the voter participation rate (Sect. 2). We then proceed to significantly extend the analysis of just how serious the impact of abstentions might be on a number of other very negative election outcomes. That phase of our study begins with an evaluation of some commonly considered voting rules on the basis of their Condorcet Efficiency, which measures the conditional probability that each of these rules will elect the ACW given that one exists, based on the available results from observed voting situations after abstention takes place. The voting rules that we initially consider in Sect. 3 are the single-stage voting rules: Plurality Rule (*PR*), Negative Plurality Rule (*NPR*) and Borda Rule (*BR*). The potentially extreme negative impact of voter abstention is obvious in this case, since all voting rules effectively become random choosers for the winner of an election, with Condorcet Efficiencies of 33.3%, when voter participation rates are near zero! The point of particular interest is how quickly these voting rules recover to achieve acceptable Condorcet Efficiency values as voter participation rates increase from near-zero.

After the initial Condorcet Efficiency component of this analysis is completed, the probability that these voting rules will perform in a very poor manner to exhibit a Borda Paradox by electing the ACL is considered as a function of the voter participation rate in Sect. 4. We then proceed to extend in Sect. 5 this same overall evaluation to consider common two-stage voting rules: Plurality Elimination Rule (*PER*), Negative Plurality Elimination Rule (*NPER*) and Borda Elimination Rule (*BER*). We consider in Sect. 6 the case where indifference is allowed and this context allows us to investigate the impact of abstention on the performances of two additional voting rules: Approval Voting (*AV*) and Approval Elimination Voting (*AEV*). We limit our attention to cases in which the number of possible voters is very large as $n \to \infty$, and the overall results of our analysis typically present a somewhat pessimistic outlook for the performance of voting rules for scenarios in which voter participation rates are as low as those that can be observed in actual elections. But the most disturbing conclusion is the following: for each of the voting rules that we consider, the addition of dependence among voters' preferences leads to a decrease in the Condorcet Efficiency and an increase in the probability of observing a Borda Paradox with low participation rates. The penultimate section of the current study (Sect. 7) offers a discussion and an explanation of these disconcerting results. Section 8 concludes our study.

2 Probability of ACW and OCW Coincidence

Candidate A will be both the ACW and OCW whenever the actual and observed voting situations have preference rankings for voters that are simultaneously consistent with (1), (2), (3) and (4). The coincidence probability for the ACW and OCW is obviously driven both by the probability that various actual voting situations will occur and by the mechanism that determines the subset of the possible voters who choose to participate in the election. Two standard assumptions from the literature are used as a basis for models to consider these two components, and each will be seen to have its own interpretation of how the voter participation rate is defined. The first of these models is $IC(\alpha)$, with voter participation rate α, which is based on the *Impartial Culture Condition* (*IC*) that assumes complete independence between voters' preferences. The second model is $IAC(\alpha_*)$, with voter participation rate α_*, which is based on the *Impartial Anonymous Culture Condition* (*IAC*) that inherently assumes some degree of dependence among voters' preferences. Probability representations for the coincidence of the ACW and OCW will be considered in turn with these two models.

2.1 ACW and OCW Coincidence Results with IC

The basic form of IC uses p_i to denote the probability that a randomly selected possible voter will have the associated preference ranking in Fig. 1, with $p_i = \frac{1}{6}$ for all $i = \{1, 2, \ldots, 6\}$, so that each possible voter is equally likely to have any of the six linear rankings. Then, Gehrlein and Fishburn (1978) developed an extension of IC, such that $IC(\alpha)$ further assumes that each possible voter will independently have a probability α of participating in the election. In the limit $n \to \infty$, the Law of Large Numbers requires that the proportion of participating voters will have $\frac{n^*}{n} \to \alpha$ with $IC(\alpha)$.

The basic IC assumption without abstention being allowed was used by Guilbaud (1952) to develop a representation for the limiting probability $P_{\text{ACW}}(IC, \infty)$ that an ACW exists to begin with as $n \to \infty$, and

$$P_{\text{ACW}}(IC, \infty) = \frac{3}{4} + \frac{3}{2\pi} Sin^{-1}\left(\frac{1}{3}\right) \approx .91226. \tag{5}$$

The development of a representation for the limiting probability that the ACW and OCW coincide begins with the definitions of four variables $\{X_1, X_2, X_3, X_4\}$ that

Table 1 Definitions of X_1, X_2, X_3 and X_4

Ranking	$X_1[AMB]$	$X_2[AMC]$	$X_3[AM^*B]$	$X_4[AM^*C]$
$A \succ B \succ C(p_1)$	+1	+1	+1	+1
$A \succ C \succ B(p_2)$	+1	+1	+1	+1
$B \succ A \succ C(p_3)$	−1	+1	−1.	+1
$C \succ A \succ B(p_4)$	+1	−1	+1	−1
$B \succ C \succ A(p_5)$	−1	−1	−1	−1
$C \succ B \succ A(p_6)$	−1	−1	−1	−1
Abstention	–	–	0	0

have different values that are based on the linear preference ranking that is associated with a randomly selected voter and on whether, or not, that voter participates in the election. These variables are defined in Table 1, where the entries show that the value of X_1 for the linear preference ranking of a randomly selected voter is +1 whenever $A \succ B$ and it is -1 if $B \succ A$, so that *AMB* for *n* voters if the average value of X_1 has $\overline{X}_1 > 0$. In the same fashion *AMC* if $\overline{X}_2 > 0$, so Candidate *A* is the ACW whenever both $\overline{X}_1 > 0$ and $\overline{X}_2 > 0$. It then follows from the same logic that *A* will be the OCW for participating voters when both $\overline{X}_3 > 0$ and $\overline{X}_4 > 0$.

The expected values of these four variables with $IC(\alpha)$ are given by:

$$E(X_1) = +1p_1 + 1p_2 - 1p_3 + 1p_4 - 1p_5 - 1p_6$$

$$E(X_2) = +1p_1 + 1p_2 + 1p_3 - 1p_4 - 1p_5 - 1p_6$$

$$E(X_3) = +1p_1\alpha + 1p_2\alpha - 1p_3\alpha + 1p_4\alpha - 1p_5\alpha - 1p_6\alpha + 0(1-\alpha)$$

$$E(X_4) = +1p_1\alpha + 1p_2\alpha + 1p_3\alpha - 1p_4\alpha - 1p_5\alpha - 1p_6\alpha + 0(1-\alpha).$$

With the restriction that $p_i = \frac{1}{6}$ for all $i = \{1, 2, \ldots, 6\}$, $E(X_j) = 0$ for $j = \{1, 2, 3, 4\}$, so that all $E(\overline{X}_j) = 0$ also. The probability that Candidate *A* is both the ACW and OCW is therefore given by the joint probability that $\overline{X}_j > E(\overline{X}_j)$, or $\overline{X}_j\sqrt{n} > E(\overline{X}_j\sqrt{n})$, for $j = \{1, 2, 3, 4\}$. The Central Limit Theorem requires that the distribution of the $\overline{X}_j\sqrt{n}$ variables is multivariate normal in the limit as $n \to \infty$, where the correlation matrix for the $\overline{X}_j\sqrt{n}$ terms is obtained directly from the correlations between the original X_j variables.

The variance and covariance terms of the X_j variables in Table 1 with $IC(\alpha)$ follow from:

$$E(X_1^2) = E(X_2^2) = 1 \text{ and } E(X_3^2) = E(X_4^2) = \alpha$$

$$E(X_1X_2) = (+1)(+1)p_1 + (+1)(+1)p_2 + (-1)(+1)p_3 + (+1)(-1)p_4 + (-1)(-1)p_5 + (-1)(-1)p_6 = \frac{1}{3}$$

$$E(X_1X_3) = (+1)(+1)p_1\alpha + (+1)(+1)p_2\alpha + (-1)(-1)p_3\alpha + (+1)(+1)p_4\alpha + (-1)(-1)p_5\alpha + (-1)(-1)p_6\alpha + 0(1-\alpha) = \alpha$$

$$E(X_1X_4) = (+1)(+1)p_1\alpha + (+1)(+1)p_2\alpha + (-1)(+1)p_3\alpha + (+1)(-1)p_4\alpha + (-1)(-1)p_5\alpha + (-1)(-1)p_6\alpha + 0(1-\alpha) = \frac{\alpha}{3}$$

$$E(X_2X_3) = (+1)(+1)p_1\alpha + (+1)(+1)p_2\alpha + (+1)(-1)p_3\alpha + (-1)(+1)p_4\alpha + (-1)(-1)p_5\alpha + (-1)(-1)p_6\alpha + 0(1-\alpha) = \frac{\alpha}{3}$$

$$E(X_2X_4) = (+1)(+1)p_1\alpha + (+1)(+1)p_2\alpha + (+1)(+1)p_3\alpha + (-1)(-1)p_4\alpha + (-1)(-1)p_5\alpha + (-1)(-1)p_6\alpha + 0(1-\alpha) = \alpha$$

$$E(X_3X_4) = (+1)(+1)p_1\alpha + (+1)(+1)p_2\alpha + (-1)(+1)p_3\alpha + (+1)(-1)p_4\alpha + (-1)(-1)p_5\alpha + (-1)(-1)p_6\alpha + 0(1-\alpha) = \frac{\alpha}{3}.$$

The resulting correlation matrix for these four variables is then given by R_1, with:

$$R_1 = \begin{bmatrix} 1 & \frac{1}{3} & \sqrt{\alpha} & \frac{\sqrt{\alpha}}{3} \\ & 1 & \frac{\sqrt{\alpha}}{3} & \sqrt{\alpha} \\ & & 1 & \frac{1}{3} \\ & & & 1 \end{bmatrix}.$$

The probability that any $\overline{X}_j\sqrt{n}$ takes on a specific value, including $E\left(\overline{X}_j\sqrt{n}\right)$, is zero in any continuous distribution. So, the limiting probability that Candidate A is both the ACW and OCW is given by the multivariate normal positive orthant probability $\Phi_4(R_1)$ that $\overline{X}_j\sqrt{n} \geq E\left(\overline{X}_j\sqrt{n}\right)$, for $j = \{1, 2, 3, 4\}$. The symmetry of $IC(\alpha)$ with respect to the three candidates leads to the conclusion that the conditional limiting probability $P_{\text{ACW}}^{\text{OCW}}(IC(\alpha), \infty)$ that an OCW exists that coincides with the ACW, given that an ACW exists, is then directly obtained from

$$P_{\text{ACW}}^{\text{OCW}}(IC(\alpha), \infty) = \frac{3\Phi_4(R_1)}{P_{\text{ACW}}(IC, \infty)}. \tag{6}$$

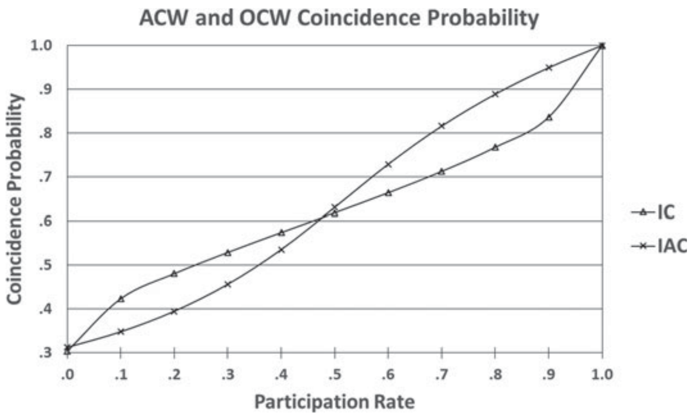

Fig. 2 Probability of ACW and OCW coincidence with $IC(\alpha)$ and IAC(α_*)

Closed form representations for these positive orthant probabilities only exist for special cases, and R_1 is one such case from Cheng (1969). Using that result with (5) and (6) leads to

$$P_{\mathrm{ACW}}^{\mathrm{OCW}}(IC(\alpha), \infty) = \frac{\begin{array}{c}\left[\frac{3}{16}+\frac{3}{4\pi}\left\{\mathrm{Sin}^{-1}\left(\frac{1}{3}\right)+\mathrm{Sin}^{-1}\left(\sqrt{\alpha}\right)+\mathrm{Sin}^{-1}\left(\frac{\sqrt{\alpha}}{3}\right)\right\}\right] \\ \hline +\frac{3}{4\pi^2}\left\{\left(\mathrm{Sin}^{-1}\left(\frac{1}{3}\right)\right)^2+\left(\mathrm{Sin}^{-1}\left(\sqrt{\alpha}\right)\right)^2-\left(\mathrm{Sin}^{-1}\left(\frac{\sqrt{\alpha}}{3}\right)\right)^2\right\}\end{array}}{\frac{3}{4}+\frac{3}{2\pi}\mathrm{Sin}^{-1}\left(\frac{1}{3}\right)}. \quad (7)$$

This representation corresponds to the result in Gehrlein and Fishburn (1978), and the derivation has been developed in detail here as an example, since the same basic procedure will be used later to develop additional new probability representations for other election outcomes.

There is no election output to evaluate without any voter participation when $\alpha = 0$, so Fig. 2 shows plotted values of $P_{\mathrm{ACW}}^{\mathrm{OCW}}(IC(\alpha), \infty)$ from (7) for $\alpha \to 0$ and for each $\alpha = .1(.1)1.0$.

Some results from the $P_{\mathrm{ACW}}^{\mathrm{OCW}}(IC(\alpha), \infty)$ values in Fig. 2 are predictable. In particular, the coincidence probability is one when all voters participate with $\alpha = 1$, and the coincidence probability decreases as this participation rate declines. When almost all voters independently abstain as $\alpha \to 0$, the candidate that becomes the OCW is effectively selected at random if there is an OCW, so the coincidence probability goes to $\frac{1}{3}P_{\mathrm{ACW}}(IC, \infty)$. What is surprising is the very steep rate of decline in $P_{\mathrm{ACW}}^{\mathrm{OCW}}(IC(\alpha), \infty)$ values as α decreases, with only about a 62% chance of coincidence when $\alpha = 0.5$. Low voter participation rates can clearly have a huge negative impact on election outcomes with the independent voter $IC(\alpha)$ scenario.

2.2 ACW and OCW Coincidence Results with IAC

It is well known that the introduction of a degree of dependence among voters' preferences generally reduces the probability that most paradoxical election outcomes will be observed, relative to the case of complete independence with IC [see for example Gehrlein and Lepelley (2011)]. Gehrlein and Lepelley (2017b) investigated this impact on the conditional probability for ACW and OCW coincidence with the assumption of IAC(α_*), such that all actual voting situations with all of their associated possible observed voting situations are equally likely to be observed, given that the voter participation rate is fixed at $\alpha_* = \frac{n^*}{n}$.

To begin developing this representation for the coincidence probability of ACW and OCW with IAC(α_*), we note that this requires that (1), (2), (3) and (4) must hold, along with

$$n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \tag{8}$$

$$n_1^* + n_2^* + n_3^* + n_4^* + n_5^* + n_6^* = n^* \tag{9}$$

$$0 \le n_i^* \le n_i, \text{ for } i = 1, 2, 3, 4, 5, 6. \tag{10}$$

We would start by obtaining a representation for the total number of voting situations for which the restrictions (1), (2), (3), (4), (8), (9) and (10) simultaneously apply as a function of n and n^*, and then divide that by the total number of voting situations for which (1), (2), (8), (9) and (10) simultaneously apply as a function of n and n^*. The final representation could then be expressed as a function of n and α_*.

In the limit as $n \to \infty$, this process of developing representations to count the number of voting situations that meet the specified restrictions reduces to computing volumes of subspaces. All observed voting situations with the same value of α_* are then assumed to be equally likely to be observed with IAC(α_*), but it is *not* assumed that all α_* are equally likely to be observed. This general procedure has been used many times to develop limiting IAC-based probability representations in the literature, and we rely throughout this study on a particular method for doing this that uses the multi-parameter version of Barvinok's Algorithm that is described in detail in Lepelley et al. (2008). The resulting representation for the limiting conditional probability that the ACW and OCW coincide with IAC(α_*) is denoted $P_{\text{OCW}}^{\text{ACW}}(\text{IAC}(\alpha_*), \infty)$, and it is given by:

$$\begin{aligned} P_{\text{OCW}}^{\text{ACW}}(\text{IAC}(\alpha_*), \infty) &= \frac{444\alpha_*^5 - 4376\alpha_*^4 + 11817\alpha_*^3 - 15576\alpha_*^2 + 10080\alpha_* - 2520}{128\left(42\alpha_*^5 - 274\alpha_*^4 + 603\alpha_*^3 - 624\alpha_*^2 + 315\alpha_* - 63\right)}, 0 \le \alpha_* \le \frac{1}{2}, \\ &= \frac{10812\alpha_*^5 - 364\alpha_*^4 - 3947\alpha_*^3 + 1761\alpha_*^2 - 185\alpha_* - 13}{128\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \frac{1}{2} \le \alpha_* \le 1. \end{aligned} \tag{11}$$

Table 2 Probability values of ACW and OCW coincidence with $IC(\alpha)$ and IAC(α_*)

Participation rate	$IC(\alpha)$	IAC(α_*)
0.0	0.3041	00.3125
0.1	0.4236	0.3482
0.2	0.4806	0.3949
0.3	0.5290	0.4564
0.4	0.5742	0.5357
0.5	0.6186	0.6310
0.6	0.6640	0.7291
0.7	0.7126	0.8160
0.8	0.7675	0.8887
0.9	0.8365	0.9492
1.0	1.0000	1.0000

Computed values of $P_{\text{OCW}}^{\text{ACW}}(\text{IAC}(\alpha_*), \infty)$ from (11) are plotted in Fig. 2 for $\alpha_* \rightarrow 0$ and for each $\alpha_* = 0.1(0.1)1.0$, to show some very surprising results. The introduction of a degree of dependence with IAC(α_*) can indeed increase the probability of ACW and OCW coincidence relative to the case of complete voter independence with $IC(\alpha)$. But, this only occurs for $\alpha_* \geq .5$, and the coincidence probability is actually reduced by introducing dependence for most cases with $\alpha_* \leq .5$! When a large proportion of voters choose not to participate in an election, it is now quite evident that very bad things really can happen with the resulting election outcomes. And, adding a degree of dependence, which was expected to improve the very negative impact of abstentions, can make things even worse. We now proceed to investigate this phenomenon further to see just how bad things can get in such cases, but we first list in Table 2 the numerical values for the probabilities of ACW and OCW coincidence that are shown graphically in Fig. 2. These probability values are included here, and in the following sections of this study, to give more precision to these probability values for others who might wish to attempt to reproduce them by other techniques.

3 Actual Condorcet Efficiency of Single-Stage Voting Rules

As mentioned above, the Condorcet Efficiency of a voting rule measures the conditional probability that the voting rule will elect the ACW, given that an ACW exists, based on the results of an observed voting situation. A single-stage voting rule determines the winner of an election in a single step from the information on the voters' ballots. A weighted scoring rule WSR(λ) requires voters to rank the three candidates and a weight of one is assigned to each voter's most preferred candidate, a weight of zero to the least preferred candidate and a weight of λ to the middle-ranked candidate.

The candidate who receives the greatest total score from the observed voting situation is then declared as the winner. Such a voting rule reduces to PR when $\lambda = 0$, so that voters must only report their most preferred candidate on a ballot. When $\lambda = 1$, voters must only report their two more-preferred candidates, which defines NPR since it is the same as having each voter cast a ballot against their least-preferred candidate. Voters obviously do not really have to report a complete ranking on the candidates when either PR or NPR is employed, but they must do so for BR which uses $\lambda = 1/2$.

3.1 Actual Single-Stage Rule Efficiency with IC

The development of a representation for the Condorcet Efficiency $CE_{\mathrm{WSR}(\lambda)}(IC(\alpha), \infty)$ of WSR(λ) as $n \to \infty$ with the assumption of $IC(\alpha)$ begins by defining four variables that are based on the likelihood that given voter preference rankings are observed, as shown in Table 3.

Following the logic of development of the representation for $P^{\mathrm{OCW}}_{\mathrm{ACW}}(IC(\alpha), \infty)$, Y_1 and Y_2 are identical to X_1 and X_2 respectively, so that Candidate A is the ACW if $\overline{Y_1} > 0$ and $\overline{Y_2} > 0$. Variables Y_3 and Y_4 define the difference in scores obtained by Candidate A over each of B and C respectively in voter preference rankings, so that A will be the winner by WSR(λ) in an observed voting situation whenever both $\overline{Y_3} > 0$ and $\overline{Y_4} > 0$.

By utilizing the methodology that led to (6), it follows that

$$CE_{\mathrm{WSR}(\lambda)}(IC(\alpha), \infty) = \frac{3\Phi_4(R_2)}{P_{\mathrm{ACW}}(IC, \infty)}. \tag{12}$$

Based on the representation in (12), computations with the Y_i definitions from Table 3 lead to:

Table 3 Definitions of Y_1, Y_2, Y_3 and Y_4

Ranking	$Y_1[AMB]$	$Y_2[AMC]$	$Y_3[AWB]$	$Y_4[AWC]$
$A \succ B \succ C(p_1)$	$+1$	$+1$	$1-\lambda$	$+1$
$A \succ C \succ B(p_2)$	$+1$	$+1$	$+1$	$1-\lambda$
$B \succ A \succ C(p_3)$	-1	$+1$	$\lambda-1$	$+\lambda$
$C \succ A \succ B(p_4)$	$+1$	-1	$+\lambda$	$\lambda-1$
$B \succ C \succ A(p_5)$	-1	-1	-1	$-\lambda$
$C \succ B \succ A(p_6)$	-1	-1	$-\lambda$	-1
Abstention	–	–	0	0

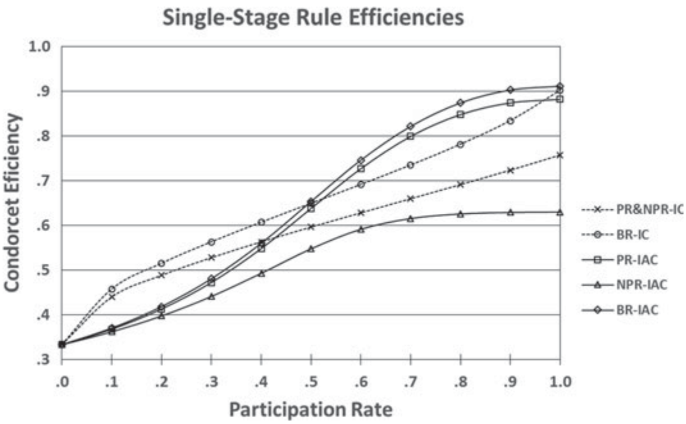

Fig. 3 Condorcet Efficiency of PR, NPR and BR with $IC(\alpha)$ and IAC(α_*)

$$R_2 = \begin{bmatrix} 1 & \frac{1}{3} & \sqrt{\frac{2}{3z}} & \sqrt{\frac{1}{6z}} \\ & 1 & \sqrt{\frac{1}{6z}} & \sqrt{\frac{2}{3z}} \\ & & 1 & \frac{1}{2} \\ & & & 1 \end{bmatrix}, \text{ with } z = \frac{1 - \lambda(1 - \lambda)}{\alpha}.$$

Since z is symmetric about $\lambda = 1/2$ for all α, it follows that PR and NPR must have the same limiting Condorcet Efficiency as $n \to \infty$ for any given α. The form of R_2 unfortunately does not meet the conditions for any special case with a closed form representation for $\Phi_4(R_2)$, as we observed above for $\Phi_4(R_1)$ in the development of (7). Gehrlein and Fishburn (1979) therefore used a representation from Gehrlein (1979) to obtain values of $\Phi_4(R_2)$ by using numerical integration over a single variable. Resulting values of $CE_{\mathrm{WSR}(\lambda)}(IC(\alpha), \infty)$ for PR, NPR and BR from (12) are plotted in Fig. 3 for the case that $\alpha \to 0$ and for each $\alpha = .1(.1)1.0$.

We see the expected result in Fig. 3 that Condorcet Efficiency values approach 1/3 to make all voting rules equivalent to random selection procedures when $\alpha \to 0$. We also note that BR dominates PR and NPR for all non-zero participation rates. The efficiencies decrease rapidly as voter participation rates decrease with the assumption of $IC(\alpha)$, and when this rate is 40% or less, all voting rules have Condorcet Efficiency values less than 61%. This level of efficiency is very disappointing, since voter participation rates as low as 40% are definitely observed in practice. For example, voter participation rates in the 2014 US elections [see McDonald (2018)] range from 28.7% in Indiana to 58.7% in Maine, with an overall national participation rate of only 36.7%. While these participation rates do increase for elections during years when a president is being chosen, they are even lower during the primary elections that select the final candidates for political parties. So, we certainly hope that the introduction of a degree of dependence among voters' preferences will improve this situation, just as it has typically done in previous studies.

3.2 Actual Single-Stage Rule Efficiency with IAC

The impact that the inclusion of some degree of dependence among voters' preferences might have on this dreary expected result is considered by using the assumption of IAC(α_*) to obtain representations for $CE_{\text{Rule}}(\text{IAC}(\alpha_*), \infty)$. It is not feasible to obtain such a representation for a generalized WSR(λ) with a reasonable effort, as we did with $CE_{\text{WSR}(\lambda)}(IC(\alpha), \infty)$, but we can obtain a representation for each specified voting rule Rule $\in \{PR, NPR, BR\}$.

Based on the definitions of PR, NPR and BR that were given above, the following restrictions on observed voting situations apply:

Candidate A is the PR winner if

$$n_1^* + n_2^* > n_3^* + n_5^* \; [APB] \tag{13}$$

$$n_1^* + n_2^* > n_4^* + n_6^* \; [APC]. \tag{14}$$

Candidate A is the NPR winner if

$$n_5^* + n_6^* < n_2^* + n_4^* \; [ANB] \tag{15}$$

$$n_5^* + n_6^* < n_1^* + n_3^* \; [ANC]. \tag{16}$$

Candidate A is the BR winner if

$$2\left(n_1^* + n_2^*\right) + n_3^* + n_4^* > 2\left(n_3^* + n_5^*\right) + n_1^* + n_6^* \; [ABB] \tag{17}$$

$$2\left(n_1^* + n_2^*\right) + n_3^* + n_4^* > 2\left(n_4^* + n_6^*\right) + n_2^* + n_5^* \; [ABC]. \tag{18}$$

The same procedure that was used to obtain the limiting representation for $P_{\text{OCW}}^{\text{ACW}}(\text{IAC}(\alpha_*), \infty)$ in (11) from the restrictions in (1), (2), (3), (4), (8), (9) and (10) is used to obtain representations for each $CE_{\text{Rule}}(\text{IAC}(\alpha_*), \infty)$. This is done by replacing (3) and (4) with (13) and (14) for PR to obtain:

$$CE_{PR}(\text{IAC}(\alpha_*), \infty) = \frac{\begin{matrix} 1517527\alpha_*^5 - 13088244\alpha_*^4 + 33868125\alpha_*^3 \\ -43056360\alpha_*^2 + 27051003\alpha_* - 6613488 \end{matrix}}{31492828\left(42\alpha_*^5 - 274\alpha_*^4 + 603\alpha_*^3 - 624\alpha_*^2 + 315\alpha_* - 63\right)}, \; 0 \le \alpha_* \le \frac{1}{2}$$

$$= \frac{\left[\begin{matrix} 18667651456\alpha_*^{10} - 94540776960\alpha_*^9 + 187428504960\alpha_*^8 \\ -169349253120\alpha_*^7 + 37682184960\alpha_*^6 + 57987062784\alpha_*^5 \\ -55553299200\alpha_*^4 + 21238744320\alpha_*^3 - 3741344640\alpha_*^2 \\ +158644980\alpha_* + 21907179 \end{matrix}\right]}{201553920(\alpha_* - 1)^5\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \; \frac{1}{2} \le \alpha_* \le \frac{3}{4}$$

$$= \frac{-\left(13661\alpha_*^5 - 72995\alpha^4 + 68740\alpha^3 - 26460\alpha^2 + 3735\alpha_* - 9\right)}{128\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \quad \frac{3}{4} \le \alpha_* \le 1. \tag{19}$$

A limiting representation is then obtained for the Condorcet Efficiency of NPR by replacing (3) and (4) with (15) and (16):

$$\begin{aligned} CE_{NPR}(IAC(\alpha_*), \infty) &= \frac{-\begin{pmatrix} 33037\alpha_*^5 + 3096510\alpha_*^4 - 10408905\alpha_*^3 \\ +14029200\alpha_*^2 - 8726130\alpha_* + 2066715 \end{pmatrix}}{98415\left(42\alpha_*^5 - 274\alpha_*^4 + 603\alpha_*^3 - 624\alpha_*^2 + 315\alpha_* - 63\right)}, \quad 0 \le \alpha_* \le \frac{1}{2} \\ &= \frac{\begin{bmatrix} 677320000\alpha_*^{10} - 5117808000\alpha_*^9 + 16793438400\alpha_*^8 \\ -31338161280\alpha_*^7 + 36609330240\alpha_*^6 - 27803103552\alpha_*^5 \\ +13822189920\alpha_*^4 - 4419227160\alpha_*^3 + 862705890\alpha_*^2 \\ -90246555\alpha_* + 3562623 \end{bmatrix}}{6298560(1-\alpha_*)^5\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \quad \frac{1}{2} \le \alpha_* \le \frac{3}{4} \\ &= \frac{459\alpha_*^5 + 365\alpha_*^4 - 400\alpha_*^3 + 270\alpha_*^2 - 120\alpha_* + 21}{15\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \quad \frac{3}{4} \le \alpha_* \le 1. \end{aligned} \tag{20}$$

A limiting representation for the Condorcet Efficiency of BR is obtained in the same fashion by replacing (3) and (4) with (17) and (18):

$$\begin{aligned} CE_{BR}(\text{IAC}(\alpha_*), \infty) &= \frac{\begin{matrix} 931991\alpha_*^5 - 11823930\alpha_*^4 + 33929685\alpha_*^3 \\ -45596520\alpha_*^2 + 29546370\alpha_* - 7348320 \end{matrix}}{349920\left(42\alpha_*^5 - 274\alpha_*^4 + 603\alpha_*^3 - 624\alpha_*^2 + 315\alpha_* - 63\right)}, \quad 0 \le \alpha_* \le \frac{1}{2}, \\ &= \frac{-\begin{bmatrix} 971901664\alpha_*^{10} - 4826652480\alpha_*^9 + 9189897120\alpha_*^8 \\ -7389999360\alpha_*^7 + 1632960\alpha_*^6 + 4616051328\alpha_*^5 \\ -3672527040\alpha_*^4 + 1323475200\alpha_*^3 - 223482240\alpha_*^2 \\ +8281440\alpha_* + 1422603 \end{bmatrix}}{11197440(1-\alpha_*)^5\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \quad \frac{1}{2} \le \alpha_* \le \frac{3}{4}, \\ &= \frac{-\left(187353\alpha_*^5 - 987625\alpha_*^4 + 527980\alpha_*^3 + 221880\alpha_*^2 - 253805\alpha_* + 56249\right)}{4320\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \quad \frac{3}{4} \le \alpha_* \le 1. \end{aligned} \tag{21}$$

The representations in (19), (20) and (21) are then used to obtain the $CE_{\text{Rule}}(\text{IAC}(\alpha_*), \infty)$ values with each *Rule* = PR, NPR and BR for values of $\alpha_* \to 0$ and for each $\alpha_* = 0.1(0.1)1$. The results are displayed graphically in Fig. 3, to show that the addition of dependence does improve efficiency results for PR and BR with voter participation of 50% or more. However, the addition of dependence has a negative impact on NPR efficiency at all non-zero levels of participation. The overall efficiency results for IAC(α_*) in Fig. 3 are even more discouraging for the case of lower voter participation rates for all voting rules than the results for $IC(\alpha)$. BR still outperforms PR and NPR for all non-zero participation rates with IAC(α_*), but the addition of a degree of dependence among voters' preferences has led to even lower efficiencies for the voting rules when voter participation rates are small. The efficiencies of all voting rules are now less than 56% when participation rates are

Table 4 Condorcet Efficiency values of PR, NPR and BR with $IC(\alpha)$ and IAC(α_*)

Participation	PR&NPR-IC	BR-IC	PR-IAC	NPR-IAC	BR-IAC
0.0	0.3333	0.3333	0.3333	0.3333	0.3333
0.1	0.4399	0.4576	0.3679	0.3621	0.3704
0.2	0.4882	0.5151	0.4131	0.3975	0.4184
0.3	0.5277	0.5630	0.4723	0.4411	0.4807
0.4	0.5630	0.6068	0.5481	0.4928	0.5600
0.5	0.5961	0.6490	0.6378	0.5475	0.6533
0.6	0.6280	0.6911	0.7267	0.5912	0.7457
0.7	0.6595	0.7345	0.7988	0.6153	0.8212
0.8	0.6911	0.7810	0.8476	0.6255	0.8737
0.9	0.7234	0.8337	0.8738	0.6289	0.9026
1.0	0.7572	0.9012	0.8815	0.6296	0.9111

40% or less with IAC(α_*). Adding dependence among voters' preferences has made things worse when participation rates are less than 50%!

Table 4 lists the numerical values for the Condorcet Efficiency values of PR, NPR and BR that are shown graphically in Fig. 3.

4 Borda Paradox Probabilities

The occurrence of a *Borda Paradox* is much more disconcerting than having a voting rule fail to elect the ACW. The worst possible outcome occurs when a Borda Paradox is observed, such that a voting rule elects the ACL. Given the potentially significant negative impact that voter abstentions have been seen to have on the Condorcet Efficiency of voting rules, it is natural to wonder how serious the impact of voter abstentions might be for observing examples of this even more dramatic Borda Paradox.

4.1 Borda Paradox Probabilities for Single-Stage Rules with IC

It is easy to develop a representation for the limiting probability $BP_{\text{WSR}(\lambda)}(IC(\alpha), \infty)$ that a Borda Paradox is observed when using WSR(λ) as $n \to \infty$ with the assumption of $IC(\alpha)$ by mirroring the development of the representation for $CE_{\text{WSR}(\lambda)}(IC(\alpha), \infty)$ in (12). All that has to be done is to reverse the signs of the Y_1 and Y_2 values in Table 3 to make Candidate A the ACL and the winner by WSR(λ). It obviously follows from the logic that led to (12) and the definition of R_2 that

$$BP_{\mathrm{WSR}(\lambda)}(IC(\alpha), \infty) = \frac{3\Phi_4(R_3)}{P_{\mathrm{ACW}}(IC, \infty)}, \tag{22}$$

where

$$R_3 = \begin{bmatrix} 1 & \frac{1}{3} & -\sqrt{\frac{2}{3z}} & -\sqrt{\frac{1}{6z}} \\ & 1 & -\sqrt{\frac{1}{6z}} & -\sqrt{\frac{2}{3z}} \\ & & 1 & \frac{1}{2} \\ & & & 1 \end{bmatrix}, \text{ with } z = \frac{1-\lambda(1-\lambda)}{\alpha}.$$

Since z remains symmetric about $\lambda = 1/2$ for all α, it follows as in the case of Condorcet Efficiencies, that PR and NPR have the same limiting probability of exhibiting a Borda Paradox for any specified value of α. The form of R_3 is not a special case for which $\Phi_4(R_3)$ has a closed form representation, so the representation from Gehrlein (1979) is used to obtain values of $\Phi_4(R_3)$. Computed $BP_{\mathrm{WSR}(\lambda)}(IC(\alpha), \infty)$ values from (22) are displayed graphically in Fig. 4 for PR, NPR and BR with $\alpha \to 0$ and for each $\alpha = 0.1(0.1)1.0$.

The expected result in Fig. 4 is that the Borda Paradox Probabilities approach 1/3 as $\alpha \to 0$, since all voting rules are equivalent to random selection procedures in this particular case, and that BR has a lower Borda Paradox probability than PR and NPR for all non-zero participation rates. A well-known result is that BR cannot elect the OCL Fishburn and Gehrlein (1976), so that the ACL cannot be elected by BR if all voters participate. However, this result is not true when some voters abstain so that the OCL and ACL do not necessarily coincide. The probability of observing a Borda Paradox consistently increases as voter participation rates decrease with $IC(\alpha)$, and when this rate is 40% or less, all voting rules have a Borda Paradox probability of

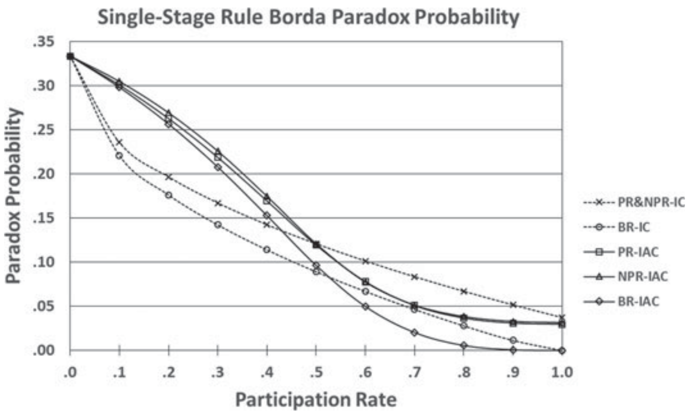

Fig. 4 Borda Paradox Probabilities for PR, NPR and BR with $IC(\alpha)$ and IAC(α_*)

greater than 11%! That is a shockingly large probability for such a very negative outcome, given that voter participation rates as low as 40% are actually observed in practice.

4.2 Borda Paradox Probabilities for Single-Stage Rules with IAC

We consider the impact that the introduction of some degree of dependence among voters' preferences will have on the probability that Borda's Paradox will be observed by extending this analysis with the assumption of IAC(α_*). This begins by noting that Candidate A is the ACL whenever:

$$n_3 + n_5 + n_6 > n_1 + n_2 + n_4[BMA] \tag{23}$$

$$n_4 + n_5 + n_6 > n_1 + n_2 + n_3[CMA]. \tag{24}$$

The representation for $CE_{PR}(\text{IAC}(\alpha_*), \infty)$ in (19) was obtained by considering actual and observed voting situations for which (1), (2), (13), (14), (8), (9) and (10) held simultaneously. The definitions for (1) and (2) respectively required that AMB and AMC to make Candidate A the ACW. This process is now repeated after replacing (1) and (2) with (23) and (24) to obtain a representation for $BP_{PR}(\text{IAC}(\alpha_*), \infty)$, with

$$BP_{PR}(IAC(\alpha_*), \infty) = \frac{\begin{matrix} 58215686\alpha_*^5 - 284362350\alpha_*^4 + 519458670\alpha_*^3 \\ -455939280\alpha_*^2 + 195419385\alpha_* - 33067440 \end{matrix}}{1574640\left(42\alpha_*^5 - 274\alpha_*^4 + 603\alpha_*^3 - 624\alpha_*^2 + 315\alpha_* - 63\right)}, \ 0 \le \alpha_* \le \frac{1}{2}$$
$$= \frac{\begin{bmatrix} 1687165696\alpha_*^{10} - 10095594240\alpha_*^9 \\ +23635307520\alpha_*^8 - 22382853120\alpha_*^7 \\ -8582837760\alpha_*^6 + 45924060672\alpha_*^5 \\ -53966062080\alpha_*^4 + 33709893120\alpha_*^3 \\ -12060167760\alpha_*^2 + 2317476420\alpha_* \\ -186417693 \end{bmatrix}}{201553920(\alpha_* - 1)^5\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \ \frac{1}{2} \le \alpha_* \le \frac{3}{4}$$
$$= \frac{-\left(1989\alpha_*^5 - 9340\alpha_*^4 + 14390\alpha_*^3 - 9900\alpha_*^2 + 2925\alpha_* - 288\right)}{120\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \ \frac{3}{4} \le \alpha_* \le 1. \tag{25}$$

This process is then repeated for both NPR and BR, and the resulting limiting representations for observing Borda's Paradox are given by:

$$BP_{\text{NPR}}(\text{IAC}(\alpha_*), \infty) = \frac{\begin{matrix} 2571059\alpha_*^5 - 14546670\alpha_*^4 + 28993545\alpha_*^3 \\ -26888760\alpha_*^2 + 11941020\alpha_* - 2066715 \end{matrix}}{98415\left(42\alpha_*^5 - 274\alpha_*^4 + 603\alpha_*^3 - 624\alpha_*^2 + 315\alpha_* - 63\right)}, 0 \le \alpha_* \le \frac{1}{2}$$

$$= \frac{\begin{bmatrix} 198413056\alpha_*^{10} - 1667619840\alpha_*^9 \\ +6862786560\alpha_*^8 - 17302947840\alpha_*^7 \\ +28555571520\alpha_*^6 - 31612472640\alpha_*^5 \\ +23544017280\alpha_*^4 - 11583051840\alpha_*^3 \\ +3587226750\alpha_*^2 - 629659170\alpha_* \\ +47731275 \end{bmatrix}}{25194240(1-\alpha_*)^5\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \frac{1}{2} \le \alpha_* \le \frac{3}{4}$$

$$= \frac{340\alpha_*^5 + 20\alpha_*^4 - 1060\alpha_*^3 + 1260\alpha_*^2 - 525\alpha_* + 84}{60\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \ \frac{3}{4} \le \alpha_* \le 1. \tag{26}$$

$$BP_{BR}(\text{IAC}(\alpha_*), \infty) = \frac{\begin{matrix} 2603797\alpha_*^5 - 12826098\alpha_*^4 + 23528961\alpha_*^3 \\ -20631672\alpha_*^2 + 8787366\alpha_* - 1469664 \end{matrix}}{69984\left(42\alpha_*^5 - 274\alpha_*^4 + 603\alpha_*^3 - 624\alpha_*^2 + 315\alpha_* - 63\right)}, 0 \le \alpha_* \le \frac{1}{2}$$

$$= \frac{\begin{bmatrix} 37038752\alpha_*^{10} - 252012096\alpha_*^9 \\ +721014048\alpha_*^8 - 1093927680\alpha_*^7 \\ +868408128\alpha_*^6 - 191382912\alpha_*^5 \\ -275970240\alpha_*^4 + 284850432\alpha_*^3 \\ -121223952\alpha_*^2 + 25345872\alpha_* \\ -2140425 \end{bmatrix}}{2239488(\alpha_*-1)^5\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \frac{1}{2} \le \alpha_* \le \frac{3}{4}$$

$$= \frac{(1-\alpha_*)^3\left(99387\alpha_*^2 - 94358\alpha_* + 24131\right)}{864\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \ \frac{3}{4} \le \alpha_* \le 1. \tag{27}$$

Values of $BP_{PR}(\text{IAC}(\alpha_*), \infty)$ from (25), $BP_{\text{NPR}}(\text{IAC}(\alpha_*), \infty)$ from (26) and $BP_{BR}(\text{IAC}(\alpha_*), \infty)$ from (27) are displayed graphically in Fig. 4 for $\alpha_* \rightarrow 0$ and for each $\alpha_* = .1(.1)1.0$. An analysis of the results in Fig. 4 shows the same general outcome that was observed when the Condorcet Efficiency of voting rules was analyzed earlier with $IC(\alpha)$ and IAC(α_*). That is, the addition of a degree of dependence among voters' preferences with IAC(α_*) actually makes things worse for lower levels of voter participation. We now see that the probability of observing a Borda Paradox is now greater than 15% for all levels of voter participation at 40% or less! The results for BR are uniformly the best of these options for all positive levels of voter participation, but BR still performs very poorly for lower levels of voter participation. If this very poor overall performance of single-stage voting rules at lower levels of voter participation is to be improved, the likely approach would be to consider the use of some other forms of voting rules.

Table 5 lists the numerical values for the Borda Paradox Probability values of PR, NPR and BR that are shown graphically in Fig. 4.

5 Two-Stage Voting Rules

Two-stage voting rules are sequential elimination procedures that utilize two steps. In the first stage, a voting rule determines the candidate that is ranked as the worst of the three candidates. That candidate is eliminated from consideration in the second stage,

Table 5 Borda Paradox Probability values for PR, NPR and BR

Participation	PR&NPR-IC	BR-IC	PR-IAC	NPR-IAC	BR-IAC
0.0	0.3333	0.3333	0.3333	0.3333	0.3333
0.1	0.2355	0.2208	0.3006	0.3046	0.2980
0.2	0.1965	0.1761	0.2626	0.2691	0.2564
0.3	0.1670	0.1423	0.2186	0.2257	0.2077
0.4	0.1423	0.1140	0.1693	0.1745	0.1526
0.5	0.1207	0.0892	0.1192	0.1207	0.0962
0.6	0.1012	0.0669	0.0778	0.0773	0.0497
0.7	0.0834	0.0466	0.0508	0.0512	0.0202
0.8	0.0669	0.0280	0.0366	0.0383	0.0055
0.9	0.0515	0.0115	0.0309	0.0328	0.0006
1.0	0.0371	0.0000	0.0296	0.0315	0.0000

where majority rule is used to determine the ultimate winner from the remaining two candidates. All that changes with the two-stage rules is the voting rule that is used to eliminate the loser in the first stage. We consider Plurality Elimination Rule (*PER*), Negative Plurality Elimination Rule (*NPER*) and Borda Elimination Rule (*BER*). It is definitely of interest to determine if these two-stage voting rules can exhibit better performance than the single-stage rules on the basis of Condorcet Efficiency and the probability that Borda's Paradox is observed with lower participation rates.

5.1 Condorcet Efficiency of Two-Stage Voting Rules with IC

The development of the representation for $CE_{\mathrm{WSR}(\lambda)}(IC(\alpha), \infty)$ in (12) was based on the definitions of the four Y_j variables in Table 3 that had $\overline{Y}_j > 0$ for $j = \{1, 2, 3, 4\}$. When we consider instead the more complex two-stage elimination rules WSER(λ), the resulting development of a representation for $CE_{\mathrm{WSER}(\lambda)}(IC(\alpha), \infty)$ requires the use of five variables that are denoted as Z_i for $i = \{1, 2, 3, 4, 5\}$ in Table 6.

By comparing Table 6 to Table 3, we see that Z_1 and Z_2 are identical to Y_1 and Y_2, so that Candidate A will be the ACW if $\overline{Z}_1 > 0$ and $\overline{Z}_2 > 0$. It is also true that Z_3 is identical to Y_3, so that A beats B by WSR(λ) if $\overline{Z}_3 > 0$. Variable Z_4 defines the relative margin by which C beats B in each ranking under WSR(λ). So, B will be the lowest ranked candidate by WSR(λ) when both $\overline{Z}_3 > 0$ and $\overline{Z}_4 > 0$, and it therefore will be eliminated under WSER(λ). Variable Z_5 accounts for the fact that the ACW Candidate A will then be the majority rule winner over C for participating voters in the second stage of WSER(λ) if $\overline{Z}_5 > 0$. Candidate A will therefore be the ACW and the winner by WSER(λ) when $\overline{Z}_i > 0$ for $i = \{1, 2, 3, 4, 5\}$.

The correlation matrix between these five variables is found to be given by R_5:

Table 6 Definitions of Z_1, Z_2, Z_3, Z_4 and Z_5

Ranking	$Z_1[AMB]$	$Z_2[AMC]$	$Z_3[AWB]$	$Z_4[CWB]$	$Z_5[AM^*C]$
$A \succ B \succ C(p_1)$	$+1$	$+1$	$1-\lambda$	$-\lambda$	$+1$
$A \succ C \succ B(p_2)$	$+1$	$+1$	$+1$	λ	$+1$
$B \succ A \succ C(p_3)$	-1	$+1$	$\lambda-1$	-1	$+1$
$C \succ A \succ B(p_4)$	$+1$	-1	λ	$+1$	-1
$B \succ C \succ A(p_5)$	-1	-1	-1	$\lambda-1$	-1
$C \succ B \succ A(p_6)$	-1	-1	$-\lambda$	$1-\lambda$	-1
Abstention	---	---	0	0	0

$$R_5 = \begin{bmatrix} 1 & \frac{1}{3} & \sqrt{\frac{2}{3z}} & \sqrt{\frac{1}{6z}} & \frac{\sqrt{\alpha}}{3} \\ & 1 & \sqrt{\frac{1}{6z}} & -\sqrt{\frac{1}{6z}} & \sqrt{\alpha} \\ & & 1 & \frac{1}{2} & \sqrt{\frac{1}{6z\alpha}} \\ & & & 1 & -\sqrt{\frac{1}{6z\alpha}} \\ & & & & 1 \end{bmatrix}, \text{ with } z = \frac{1-\lambda(1-\lambda)}{\alpha}.$$

There are three candidates that could be the ACW and there are two remaining candidates that could be eliminated when proceeding to the second stage of WSER(λ), so a representation for limiting Condorcet Efficiency is obtained from

$$CE_{\text{WSER}(\lambda)}(IC(\alpha), \infty) = \frac{6\Phi_5(R_5)}{P_{ACW}(IC, \infty)}. \tag{28}$$

As we have observed before, z is symmetric about $z = 1/2$, so the Condorcet Efficiency values of PER and NPER are identical for $IC(\alpha)$ for any given α. The process of obtaining values of multivariate-normal positive orthant probabilities becomes much more complicated for the case of five variables, but they can still be obtained by using numerical techniques over a series of integrals on a single variable, as described in Gehrlein (2017). That procedure is used here to obtain values of $\Phi_5(R_5)$ for use in (28), to obtain values of $CE_{\text{WSER}(\lambda)}(IC(\alpha), \infty)$ for PER, NPER and BER that are displayed graphically in Fig. 5 for $\alpha \to 0$ and for each $\alpha = .1(.1)1.0$.

The results in Fig. 5 produce some interesting outcomes. First of all, it is well known from Daunou (1803) that BR cannot rank the OCW in last place, so that it cannot be eliminated in the first stage with BER, and the OCW must then win in the second stage. The Actual Condorcet Efficiency of BER is therefore 100% when all voters participate so that the OCW and ACW must coincide. However, this is not true when some voters abstain so that the OCW and ACW are not necessarily the same.

When we compare the Condorcet Efficiency results for the two-stage rules with $IC(\alpha)$ in Fig. 5 to their counterpart single-stage rules in Fig. 3, BER dominates PER and NPER for all non-zero voter participation rates, just as BR dominated PR and

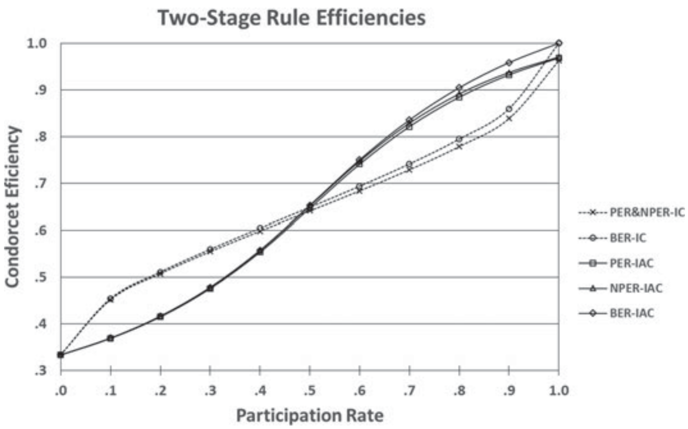

Fig. 5 Condorcet Efficiency of PER, NPER and BER with $IC(\alpha)$ and IAC(α_*)

NPR. However, the degree of this domination is significantly dampened with the two-stage rules, and BER has a less than 1% advantage over PER and NPER for voter participation rates of 60% or less. It seems logical to expect increased Condorcet Efficiency results with two-stage rules, and this is true for PER and NPER. But, it is a very surprising outcome to observe that computed BER efficiencies are actually marginally smaller than BR efficiencies for $0.1 \leq \alpha \leq 0.5$ and the Condorcet Efficiency values for two-stage rules remain less than 61% for voter participation rates that are 40% or less.

5.2 *Condorcet Efficiency of Two-Stage Voting Rules with IAC*

The same type of two-stage voting rule Condorcet Efficiency analysis is extended with the assumption of IAC(α_*) and the resulting representations for PER, NPER and BER are shown respectively in (29), (30) and (31).

$$CE_{PER}(IAC(\alpha_*), \infty) = \frac{\begin{array}{c} 3823792093\alpha_*^5 - 32968107660\alpha_*^4 \\ +85500757710\alpha_*^3 - 109197460800\alpha_*^2 \\ +68946837120\alpha_* - 16930529280 \end{array}}{806215680\left(42\alpha_*^5 - 274\alpha_*^4 + 603\alpha_*^3 - 624\alpha_*^2 + 315\alpha_* - 63\right)}, 0 \leq \alpha_* \leq \frac{1}{2}$$

$$= \frac{\left[\begin{array}{c} 69127262173\alpha_*^{10} - 348198438540\alpha_*^9 \\ +678673944270\alpha_*^8 - 579624310080\alpha_*^7 \\ +63656046720\alpha_*^6 + 284644523520\alpha_*^5 \\ -244169976960\alpha_*^4 + 91152760320\alpha_*^3 \\ -16099119360\alpha_*^2 + 755827200\alpha_* + 81881280 \end{array}\right]}{806215680(\alpha_* - 1)^5\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \frac{1}{2} \leq \alpha_* \leq \frac{2}{3}$$

$$= \frac{\begin{bmatrix} 759280879\alpha_*^{10} - 2333165430\alpha_*^9 \\ -1845638460\alpha_*^8 + 18995483340\alpha_*^7 \\ -39014909955\alpha_*^6 + 42548701734\alpha_*^5 \\ -28497933135\alpha_*^4 + 12208971240\alpha_*^3 \\ -3302020080\alpha_*^2 + 517269240\alpha_* - 36039573 \end{bmatrix}}{12597120(\alpha_* - 1)^5\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \frac{2}{3} \le \alpha_* \le \frac{3}{4}$$

$$= \frac{91279\alpha_*^5 + 132885\alpha_*^4 - 153065\alpha_*^3 + 47235\alpha_*^2 + 525\alpha_* - 1707}{1920\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \frac{3}{4} \le \alpha_* \le 1. \quad (29)$$

$$CE_{NPER}(IAC(\alpha_*), \infty) = \frac{\begin{matrix} 26697173\alpha_*^5 - 245571270\alpha_*^4 \\ +650959335\alpha_*^3 - 842387040\alpha_*^2 \\ +536121180\alpha_* - 132269760 \end{matrix}}{6298560\left(42\alpha_*^5 - 274\alpha_*^4 + 603\alpha_*^3 - 624\alpha_*^2 + 315\alpha_* - 63\right)}, 0 \le \alpha_* \le \frac{1}{2}$$

$$= \frac{\begin{bmatrix} 19679445664\alpha_*^{10} - 101681630400\alpha_*^9 \\ +209258419680\alpha_*^8 - 207447920640\alpha_*^7 \\ +79851744000\alpha_*^6 + 26877215232\alpha_*^5 \\ -39945467520\alpha_*^4 + 15947953920\alpha_*^3 \\ -2579260320\alpha_*^2 + 9054180\alpha_* + 30488967 \end{bmatrix}}{201553920(\alpha_* - 1)^5\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \frac{1}{2} \le \alpha_* \le \frac{3}{4}$$

$$= \frac{65013\alpha_*^5 - 16605\alpha_*^4 + 54380\alpha_*^3 - 72240\alpha_*^2 + 33285\alpha_* - 5145}{960\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \frac{3}{4} \le \alpha_* \le 1. \quad (30)$$

$$CE_{NPER}(IAC(\alpha_*), \infty) = \frac{\begin{matrix} 6379861\alpha_*^5 - 55971360\alpha_*^4 + 146241045\alpha_*^3 \\ -187945920\alpha_*^2 + 119257110\alpha_* - 29393280 \end{matrix}}{1399680\left(42\alpha_*^5 - 274\alpha_*^4 + 603\alpha_*^3 - 624\alpha_*^2 + 315\alpha_* - 63\right)}, 0 \le \alpha_* \le \frac{1}{2}$$

$$= \frac{\begin{bmatrix} 119753941\alpha_*^{10} - 603246240\alpha_*^9 \\ +1176055605\alpha_*^8 - 1004659200\alpha_*^7 \\ +110071710\alpha_*^6 + 494174520\alpha_*^5 \\ -423906210\alpha_*^4 + 158251320\alpha_*^3 \\ -27949860\alpha_*^2 + 1312200\alpha_* \\ +142155 \end{bmatrix}}{1399680(\alpha_* - 1)^5\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \frac{1}{2} \le \alpha_* \le \frac{3}{5}$$

$$= \frac{\begin{bmatrix} 187666713\alpha_*^{10} - 993390820\alpha_*^9 \\ +2139714765\alpha_*^8 - 2358795600\alpha_*^7 \\ +1333918530\alpha_*^6 - 285155640\alpha_*^5 \\ -45978030\alpha_*^4 + 15921360\alpha_*^3 \\ +10661625\alpha_*^2 - 5205060\alpha_* \\ +642249 \end{bmatrix}}{1866240(\alpha_* - 1)^5\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \frac{3}{5} \le \alpha_* \le \frac{3}{4}$$

$$= \frac{\begin{matrix} 170049\alpha_*^5 + 22465\alpha_*^4 + 42320\alpha_*^3 \\ -144960\alpha_*^2 + 88055\alpha_* - 16649 \end{matrix}}{2560\left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right)}, \frac{3}{4} \le \alpha_* \le 1. \quad (31)$$

Computed values of $CE_{\text{Rule}}(\text{IAC}(\alpha_*), \infty)$ for PER, NPER and BER from (29), (30) and (31) are displayed graphically in Fig. 5 for $\alpha_* \rightarrow 0$ and for each $\alpha_* = 0.1(0.1)1.0$. A comparison of the IAC(α_*) Condorcet Efficiency values in Fig. 5 to their counterpart single-stage rules in Fig. 3 produces very similar results to those observed immediately above with $IC(\alpha)$. BER typically dominates PER

Table 7 Condorcet Efficiency of PER, NPER and BER

Participation	PER&NPER-IC	BER-IC	PER-IAC	NPER-IAC	BER-IAC
0.0	0.3333	0.3333	0.3333	0.3333	0.3333
0.1	0.4516	0.4537	0.3687	0.3693	0.3692
0.2	0.5075	0.5109	0.4148	0.4163	0.4162
0.3	0.5546	0.5593	0.4755	0.4779	0.4779
0.4	0.5982	0.6043	0.5534	0.5571	0.5574
0.5	0.6407	0.6484	0.6465	0.6514	0.6527
0.6	0.6836	0.6934	0.7407	0.7470	0.7501
0.7	0.7288	0.7411	0.8213	0.8286	0.8354
0.8	0.7786	0.7944	0.8844	0.8914	0.9045
0.9	0.8387	0.8597	0.9323	0.9372	0.9584
1.0	0.9629	1.0000	0.9685	0.9704	1.0000

and NPER for all non-zero voter participation rates, but computed values give NPER a very small advantage over BER for α_* values of 0.1 and 0.2. All differences in efficiencies are significantly dampened with the two-stage rules, with a less than 1% difference between the efficiencies for voter participation rates of 50% or less. Increased Condorcet Efficiency results are observed for PER and NPER compared to PR and NPR over the range of non-zero voter participation rates, but BER efficiencies are marginally smaller than BR efficiencies for $0.1 \leq \alpha_* \leq 0.5$. Condorcet Efficiency values for the two-stage rules with dependence among voters' preferences are now less than 56% for voter participation rates that are 40% or less, to make things worse than we observed for the independent voter case with $IC(\alpha)$ with two-stage voting rules in Fig. 5.

Table 7 lists the numerical values for the Condorcet Efficiency of PER, NPER and BER that are shown graphically in Fig. 5.

5.3 *Borda Paradox Probability for Two-Stage Voting Rules with IC*

The development of a representation for the limiting probability that Borda's Paradox will be observed for two-stage voting rules with $IC(\alpha)$ directly follows the process that led to the representation for $CE_{\text{WSER}(\lambda)}(IC(\alpha), \infty)$ in (28). The only difference is that the signs for the variables Z_1 and Z_2 in Table 6 are reversed to make Candidate A the ACL, which is then elected as the ultimate winner by WSER(λ). It follows directly that

$$BP_{\text{WSER}(\lambda)}(IC(\alpha), \infty) = \frac{6\Phi_5(R_6)}{P_{ACW}(IC, \infty)}, \text{ where} \tag{32}$$

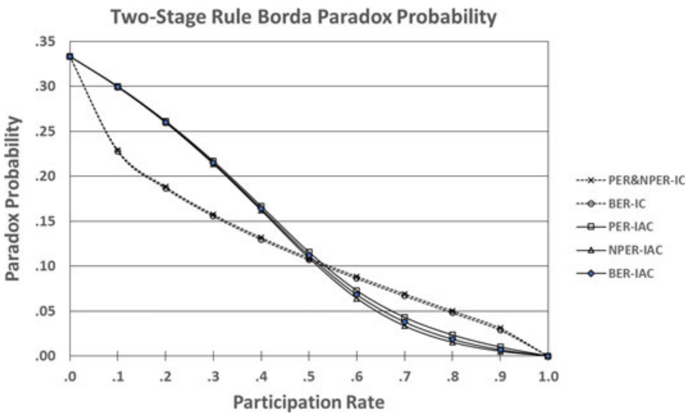

Fig. 6 Borda Paradox Probabilities for PER, NPER and BER with $IC(\alpha)$ and IAC(α_*)

$$R_6 = \begin{bmatrix} 1 & \frac{1}{3} & -\sqrt{\frac{2}{3z}} & -\sqrt{\frac{1}{6z}} & -\frac{\sqrt{\alpha}}{3} \\ & 1 & -\sqrt{\frac{1}{6z}} & \sqrt{\frac{1}{6z}} & -\sqrt{\alpha} \\ & & 1 & \frac{1}{2} & \sqrt{\frac{1}{6z\alpha}} \\ & & & 1 & -\sqrt{\frac{1}{6z\alpha_{VR}}} \\ & & & & 1 \end{bmatrix}, \text{ with } z = \frac{1-\lambda(1-\lambda)}{\alpha}.$$

Computed values of $BP_{\text{WSER}(\lambda)}(IC(\alpha), \infty)$ from (32) for each PER, NPER and BER as $\alpha \to 0$ and for each $\alpha = .1(.1)1.0$ are displayed graphically in Fig. 6.

It follows from the definition of two-stage elimination rules that the OCL cannot win with any of these rules, so that Borda's Paradox cannot be observed if all voters participate, so that the ACL and OCL coincide. However, this observation is not true when some voters abstain, so that the ACL and OCL might not be the same.

Very different behaviors are observed for the two-stage voting rules in Fig. 6 when they are compared to their single-stage rule counterparts in Fig. 4. The Borda Paradox probabilities with $IC(\alpha)$ are reduced at all levels of non-zero voter participation for PER and NPER compared to PR and NPR. However, the BER Borda Paradox probabilities all increase from their corresponding BR values for all $0 < \alpha < 1$. BER still has lower Borda Paradox probabilities than PER and NPER over the range of voter participation rates, but the differences are less than 0.3% in all cases. The particularly disturbing result from Fig. 6 is that when the voter participation rates is 40% or less, all two-stage voting rules now have a Borda Paradox probability of greater than about 13%, compared to the 11% that was observed in Fig. 4 for the single-stage rules!

5.4 Borda Paradox Probabilities for Two-Stage Rules with IAC

Representations are also obtained for the limiting probabilities $BP_{\mathrm{PER}}(\mathrm{IAC}(\alpha_*), \infty)$, $BP_{\mathrm{NPER}}(\mathrm{IAC}(\alpha_*), \infty)$ and $BP_{BER}(IAC(\alpha_*), \infty)$ by using the same process that lead respectively to the Condorcet Efficiency results in (29), (30) and (31). These resulting limiting probability representations are given by:

$$
\begin{aligned}
BP_{PER}(IAC(\alpha_*), \infty) &= \frac{\begin{array}{c} 30368797609\alpha_*^5 - 147521223660\alpha_*^4 \\ +268423789950\alpha_*^3 - 234843935040\alpha_*^2 \\ +100358455680\alpha_* - 16930529280 \end{array}}{806215680\left(42\alpha_*^5 - 274\alpha_*^4 + 603\alpha_*^3 - 624\alpha_*^2 + 315\alpha_* - 63\right)}, \; 0 \le \alpha_* \le \frac{1}{2} \\
&= \frac{\left[\begin{array}{l} 8600974249\alpha_*^{10} - 56418851820\alpha_*^9 \\ +150514746750\alpha_*^8 - 197354905920\alpha_*^7 \\ +95067665280\alpha_*^6 + 81478172160\alpha_*^5 \\ -161633646720\alpha_*^4 + 115641561600\alpha_*^3 \\ -43875768960\alpha_*^2 + 8692012800\alpha_* - 711737280 \end{array}\right]}{\begin{array}{c} 806215680(\alpha_* - 1)^5 \\ \left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right) \end{array}}, \; \frac{1}{2} \le \alpha_* \le \frac{2}{3} \\
&= \frac{\left[\begin{array}{l} 1484308489\alpha_*^{10} - 10365275280\alpha_*^9 \\ +32356702665\alpha_*^8 - 59403130380\alpha_*^7 \\ +70958056995\alpha_*^6 - 57586120074\alpha_*^5 \\ +32164285545\alpha_*^4 - 12237314760\alpha_*^3 \\ +3052242810\alpha_*^2 - 454677300\alpha_* \\ +30921993 \end{array}\right]}{\begin{array}{c} 12597120(\alpha_* - 1)^5 \\ \left(42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1\right) \end{array}}, \; \frac{2}{3} \le \alpha_* \le \frac{3}{4} \\
&= \frac{(1 - \alpha_*)\left(\begin{array}{l} 25239\alpha_*^4 - 36326\alpha_*^3 \\ +28314\alpha_*^2 - 6546\alpha_* - 6 \end{array}\right)}{1920\left(\begin{array}{l} 42\alpha_*^5 + 64\alpha_*^4 \\ -73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1 \end{array}\right)}, \; \frac{3}{4} \le \alpha_* \le 1. \qquad (33)
\end{aligned}
$$

$$
\begin{aligned}
BP_{NPER}(IAC(\alpha_*), \infty) &= \frac{\begin{array}{l} 238551113\alpha_*^5 - 1159474830\alpha_*^4 \\ +2109732075\alpha_*^3 - 1844208000\alpha_*^2 \\ +786576420\alpha_* - 132269760 \end{array}}{6298560\left(\begin{array}{l} 42\alpha_*^5 - 274\alpha_*^4 + 603\alpha_*^3 \\ -624\alpha_*^2 + 315\alpha_* - 63 \end{array}\right)}, \; 0 \le \alpha_* \le \frac{1}{2} \\
&= \frac{\left[\begin{array}{l} 388210400\alpha_*^{10} - 3308366400\alpha_*^9 \\ +14344659360\alpha_*^8 - 39847541760\alpha_*^7 \\ +74031874560\alpha_*^6 - 92271384576\alpha_*^5 \\ +76451921280\alpha_*^4 - 41116999680\alpha_*^3 \\ +13647404880\alpha_*^2 - 2517455700\alpha_* \\ +197676369 \end{array}\right]}{201553920(1 - \alpha_*)^5\left(\begin{array}{l} 42\alpha_*^5 + 64\alpha_*^4 \\ -73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1 \end{array}\right)}, \; \frac{1}{2} \le \alpha_* \le \frac{3}{4}
\end{aligned}
$$

$$= \frac{(1-\alpha_*)\begin{pmatrix} 24471\alpha_*^4 - 66184\alpha_*^3 \\ +66276\alpha_*^2 - 25284\,\alpha_* + 3171 \end{pmatrix}}{960\begin{pmatrix} 42\alpha_*^5 + 64\alpha_*^4 \\ -73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1 \end{pmatrix}}, \ \frac{3}{4} \le \alpha_* \le 1. \tag{34}$$

$$BP_{BER}(IAC(\alpha_*), \infty) = \frac{\begin{matrix} 10663043\alpha_*^5 - 51663168\alpha_*^4 \\ +93817521\alpha_*^3 - 81924048\alpha_*^2 \\ +34935138\alpha_* - 5878656 \end{matrix}}{279936\begin{pmatrix} 42\alpha_*^5 - 274\alpha_*^4 + 603\alpha_*^3 \\ -624\alpha_*^2 + 315\alpha_* - 63 \end{pmatrix}}, 0 \le \alpha_* \le \frac{1}{2}$$

$$= \frac{\begin{bmatrix} 3104771\alpha_*^{10} - 20030400\alpha_*^9 \\ +52876881\alpha_*^8 - 68907024\alpha_*^7 \\ +33098058\alpha_*^6 + 28291032\alpha_*^5 \\ -56122794\alpha_*^4 + 40153320\alpha_*^3 \\ -15234642\alpha_*^2 + 3018060\alpha_* \\ -247131 \end{bmatrix}}{279936(\alpha_* - 1)^5\begin{pmatrix} 42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 \\ +39\alpha_*^2 - 10\alpha_* + 1 \end{pmatrix}}, \ \frac{1}{2} \le \alpha_* \le \frac{3}{5}$$

$$= \frac{\begin{bmatrix} 794736\alpha_*^{10} - 7818275\alpha_*^9 \\ +33936246\alpha_*^8 - 86024484\alpha_*^7 \\ +140947128\alpha_{VR}^6 - 155365938\alpha_*^5 \\ +116001396\alpha_*^4 - 57518100\alpha_*^3 \\ +17990262\alpha_*^2 - 3186459\alpha_* \\ +243486 \end{bmatrix}}{186624(1 - \alpha_*)^5\begin{pmatrix} 42\alpha_*^5 + 64\alpha_*^4 - 73\alpha_*^3 \\ +39\alpha_*^2 - 10\alpha_* + 1 \end{pmatrix}}, \ \frac{3}{5} \le \alpha_* \le \frac{3}{4}$$

$$= \frac{(\alpha_* - 1)\begin{pmatrix} 528\alpha_*^4 - 581\alpha_*^3 \\ -1524\alpha_*^2 + 855\alpha_* - 118 \end{pmatrix}}{256\begin{pmatrix} 42\alpha_*^5 + 64\alpha_*^4 \\ -73\alpha_*^3 + 39\alpha_*^2 - 10\alpha_* + 1 \end{pmatrix}}, \ \frac{3}{4} \le \alpha_* \le 1. \tag{35}$$

Computed values for the Borda Paradox probabilities in (33), (34) and (35) for each PER, NPER and BER respectively are displayed graphically in Fig. 6 with $\alpha_* \to 0$ and for each $\alpha_* = 0.1(0.1)1.0$. When we compare the results of Fig. 6 to those in Fig. 4, the PER and NPER Borda Paradox probabilities with IAC(α_*) are smaller than the counterpart PR and NPR probabilities over the range of voter participation rates with $0 < \alpha_* < 1$, while the BER probabilities increase compared to BR. The end result is that the Borda Paradox probabilities in Fig. 6 display only a

Table 8 Borda Paradox Probability values for PER, NPER and BER

Participation	PER&NPER-IC	BER-IC	PER-IAC	NPER-IAC	BER-IAC
0.0	0.3333	0.3333	0.3333	0.3333	0.3333
0.1	0.2291	0.2278	0.3000	0.2993	0.2995
0.2	0.1883	0.1866	0.2613	0.2596	0.2601
0.3	0.1575	0.1556	0.2166	0.2137	0.2147
0.4	0.1318	0.1297	0.1665	0.1620	0.1638
0.5	0.1091	0.1070	0.1155	0.1089	0.1118
0.6	0.0885	0.0863	0.0727	0.0641	0.0681
0.7	0.0691	0.0668	0.0430	0.0334	0.0378
0.8	0.0503	0.0481	0.0234	0.0152	0.0186
0.9	0.0310	0.0288	0.0100	0.0053	0.0069
1.0	0.0000	0.0000	0.0000	0.0000	0.0000

small degree of variability across the voting rules within each of $IC(\alpha)$ and IAC(α_*), but the addition of voter dependence with IAC(α_*) again causes an increase in the probability of observing a Borda Paradox for lower levels of voter participation rates of 40% or less.

Table 8 lists the numerical values for the Borda Paradox Probability values of PR, NPR and BR that are shown graphically in Fig. 6.

6 Results for Other Voting Rules When Indifference Between Candidates is Permitted

A companion study of this phenomenon is performed by Gehrlein and Lepelley (2018) to investigate the impact that is observed on these results when we relax the requirement that voters must have complete linear preference rankings on candidates like those in Fig. 1. Voters are instead permitted the additional option of having some indifference between candidates, to have dichotomous preferences on candidates. There are six such possible dichotomous preference rankings on candidates, as shown in Fig. 7.

The notation (B, C) denotes that a voter is indifferent between Candidates B and C, and the case of complete indifference between all three candidates is ignored, since

A	B	C	(B,C)	(A,C)	(A,B)
(B,C)	(A,C)	(A,B)	A	B	C
n_7	n_8	n_9	n_{10}	n_{11}	n_{12}

Fig. 7 Six possible dichotomous preference rankings for eligible voters' preferences

any such voter would have absolutely no impact on the evaluation of any election outcomes.

The very important impact of this addition of possible dichotomous preferences is that it allows for a consideration of Approval Voting (*AV*), which is in some sense simpler for voters to respond to than the voting rules that have been considered above, since AV only requires voters to cast a ballot for as many candidates as they approve of. The winner is then determined as the candidate that receives the greatest total number of approvals from voters. This particular analysis is important since AV has been shown to have many very positive qualities when all voters have dichotomous preferences [see for example Brams and Fishburn (1978, 1983)]. As a result, AV has many strong advocates supporting its use, with the staunchest support coming from Steven Brams and Peter Fishburn. However, there are also individuals who are strongly opposed to the use of AV, with the strongest opposition coming from Donald Saari [see for example Saari and Van Newenhizen (1988)].

Laslier and Sanver (2010) present a thorough survey of work that is related to AV, and it contains many interesting remarks that are made in evaluating the more negative views of AV that are presented in the work of Saari relative to the more positive views of Brams and Fishburn. The primary argument that is used against AV is based on the general perception that it might tend to select mediocre candidates, only because most voters do not find them to be objectionable. Gehrlein and Lepelley (2017a) analyze some results that are available from elections to conclude that this overall objection does not appear to be valid, but that AV does appear to have a tendency to elect candidates that are political centrists.

It is not possible to ignore some of the very positive qualities of AV, but it is also not possible to ignore some of the strong opposition that exists to using it. So, Gehrlein and Lepelley (2018) suggest the use of a hybrid approach that uses AV to reduce the set of available candidates down to a smaller set of candidates that the overall electorate views as being the most generally acceptable candidates. After the elimination of the least acceptable candidates is performed by AV, some other voting rule would be used to select the ultimate winner from the reduced set. Such a voting rule might be considered to be an acceptable compromise between proponents and opponents of AV, by simultaneously retaining some of the known benefits of AV and eliminating the primary criticism of its use. This would be an effective mechanism to show the true level of support for all candidates in the first round of voting, while significantly reducing the possibility that fringe candidates with strong levels of support from small groups of voters might unduly influence the final outcome in the second stage of the election. However, the primary consideration is that the Approval Elimination Voting (*AEV*) procedure would eliminate the possibility of electing a candidates that is widely viewed as being unacceptable to the electorate.

Brams and Sanver (2009) consider two related election procedures that both require voters to report approval-disapproval results on all candidates, along with preference rankings on some subset of candidates. Their Preference Approval Voting requires voters to provide a ranking on all candidates, while Fallback Voting only requires each voter to rank those candidates that are in the approval set. Both of these voting rules employ a sequence of rather complex mechanisms to determine the winner from the reported results from voters, if a winner is not determined in the first step. The AEV procedure that is described above from Gehrlein and Lepelley (2018) does have the drawback of requiring two separate elections, but it has the great advantage of being much more understandable to the electorate, to lead to more confidence among voters in the end result of the election. The expense of using such a two-stage procedure might not be justifiable for low-level elections, but it would certainly be acceptable for situations like the determining the president of a country, or for other high-level positions.

For the case of three candidate elections, AEV is used in the first stage to remove one candidate from consideration, and the winner in the second stage is then determined by majority rule on the two remaining candidates. Gehrlein and Lepelley (2018) analyzed a number of different voting rules for the limiting case of voters with $n \to \infty$ when dichotomous preferences are allowable and voter abstention is an option, but the results for comparing AV to AEV are most relevant to our current discussion. Extensions of two different models based on $IC(\alpha)$ and IAC(α_*) served as the basis of this analysis. In both models, k defines the proportion of voters in an actual voting situation with complete linear preference rankings.

The IWOC(k) assumption from Fishburn and Gehrlein (1980) is IC-like, since it considers the scenario of independent voter preferences with $p_i = \frac{k}{6}$ for $i = \{1, 2, 3, 4, 5, 6\}$ and with $p_j = \frac{1-k}{6}$ for $j = \{7, 8, 9, 10, 11, 12\}$. These p_j probabilities follow from Fig. 7 in the same fashion as the p_i terms followed from Fig. 1. Then, IWOCA(k, α) is an extension of IWOC(k) that further assumes that that each voter independently chooses to participate in the election with probability α.

The IAWOC(k) model from Gehrlein and Lepelley (2015) is IAC-like, such that voters' preferences have some dependence when all voting situations with $n = \sum_{h=1}^{12} n_h$ potential voters and $k = \sum_{i=1}^{6} n_i$ voters with complete linear preferences are equally likely to be observed for a specified k. Then, the assumption of IAWOCA(k, α_*) is a natural extension of IAWOC(k), where all possible actual voting situations that account for n potential voters, where k of them have complete linear preference rankings, and n^* of them participate are assumed to be equally likely to be observed. As before, the proportion of participating voters is defined by $\alpha_* = n^*/n$ for each voting situation. The development of probability representations for which both k and α_* are allowed to vary becomes extraordinarily complicated, so only specific cases for which one of the two terms is fixed is considered. Since it was mentioned above that AV is known to have many very positive features when all voters have dichotomous preferences, we focus on the results in Gehrlein and Lepelley (2018) with $k = 0$.

It was found that CE_{AV}(IWOCA($0, \alpha$), ∞) $= CE_{AEV}$(IWOCA($0, \alpha$), ∞) for all α. The CE_{AEV}(IAWOCA($0, \alpha$), ∞) results meet expectations by consistently increasing as α increases, and the Condorcet Efficiency of AEV does show a marked improvement with the addition of dependence among voters' preferences for participation rates of 60% or more. However, the addition of dependence again leads to a marked reduction in AEV efficiency for participation rates of 50% or less. The addition of dependence among voters' preferences is also found to lead to a marked increase in the probability of observing a Borda Paradox for AEV with participation rates of 40% or less, so these results are very consistent with the disturbing trends that were observed for all other voting rules considered above.

7 Discussion and Explanation of These Disconcerting Results

7.1 A Further Evaluation

Earlier analysis above considered the probability that the ACW and OCW would coincide for the limiting case of voters with independent voters' preferences when voters have the option to abstain. This indicated that the probability of non-coincidence becomes relatively high as voter participation rates decline. Rather pessimistic results were also found for the Condorcet Efficiency and the probability that a Borda Paradox is observed with PR, NPR and BR under the same independent voter scenario. Two options were considered to improve this negative result. The first added a degree of dependence among voters' preferences. The second used more complex versions of these three voting rules by considering each as the basis for a two-stage elimination election procedure. The end result is that both of these options tended to make things better for both Condorcet Efficiency and the probability that a Borda Paradox is observed for larger values of voter participation rates. However, both options tended to make things worse for all voting rules with voter participation rates of 40% or less.

These results for lower participation rates are extraordinarily disconcerting, since earlier research strongly supports the notion that the addition of some dependence among voters' preferences will typically significantly reduce the expected high likelihood of observing bad election outcomes that occurs when independence among voters' preferences is assumed. The behavior of IAC-based probabilities is also critical since patterns of changes in such IAC-based probabilities as various parameters change typically mirror the patterns that are observed in empirically-based results [See for example Gehrlein et al. (2016a, b, 2018)]. It must also be pointed out that a possible resolution of this problem is not as simple as using mandatory voting. The disengaged voters who were discussed earlier most likely represent those who abstain from voting due to their having an initial lack of interest in the election outcome. Milner et al. (2007) perform an empirical analysis to find that mandatory voting will typically induce these disengaged voters to participate, but it is also found to be unlikely that such voters would be motivated to seriously evaluate the candidates before doing so. As a result, little would be accomplished in establishing the true ACW. It looks at this point like we really do have chaos based on these results.

7.2 An Explanation of the Results: Hindsight is 20–20

As things turn out, these results probably should have been anticipated from the onset. One explanation for these results is that the IAC-based models that are being used in this current study are introducing some statistical dependence among the preferences of the voters who choose to abstain. To explain this, the event outcomes have been defined above in terms of the six n_i terms for possible voters and the six n_i^* terms for participating voters. We know that the use of an IAC-like assumption will result in a scenario in which there is dependence among these twelve terms that identify possible voters and participating voters. It would be quite possible to instead redefine all of these same event outcomes in terms of the six n_i terms and six terms for non-participating voters. By using the same IAC-like assumptions, there would then be dependence between preferences of the possible voters and the preferences of non-participating voters. The resulting dependence among the preferences of non-participating voters could result in a situation in which their preference rankings are not necessarily being withheld from the election results in relative proportions that are consistent with the distribution of voters' preferences in the actual voting situation. This could then lead to having observed voting situations with very different properties than the actual voting situations, particularly with high rates of voter abstention.

From a practical perspective, an example of this could occur if some coalitions of voters with similar (dependent) preference rankings perceive that their preferred candidates have no chance to win, and they therefore choose not to participate in relatively large numbers, which therefore also removes their preference rankings on all other candidates from consideration. Any results from the observed voting situation could then be quite different than results from the actual voting situation.

It is possible to obtain an indication of how much of an influence such dependence among non-participating voters has on the disconcerting results that have been observed. We shall ultimately see that it is only feasible to do this analysis by considering computed probabilities with a finite n. Three different probability models are considered in this analysis, and the first two are IAC(α_*) and $IC(\alpha)$ that were developed above. The analysis starts by developing representations for $P_{\text{ACW}}^{\text{OCW}}(IAC(\alpha_*), n)$ and a minor variation of $P_{\text{ACW}}^{\text{OCW}}(IC(\alpha), n)$ as an example to indicate the significant impact that dependence among non-participating voters' preferences can have on election outcomes.

7.2.1 A Representation for $P_{ACW}^{OCW}(IAC(\alpha_*), n)$

We begin by noting the very simple limits that identify the bounds on the six n_i terms that are associated with the six possible voters' preference rankings on three candidates from Fig. 1.

$$
\begin{aligned}
0 &\le n_6 \le n \\
0 &\le n_5 \le n - n_6 \\
0 &\le n_4 \le n - n_6 - n_5 \\
0 &\le n_3 \le n - n_6 - n_5 - n_4 \\
0 &\le n_2 \le n - n_6 - n_5 - n_4 - n_3 \\
n_1 &= n - n_6 - n_5 - n_4 - n_3 - n_2.
\end{aligned} \tag{36}
$$

The total number $N(n)$ of possible voting situations for n voters can then be obtained directly from the relationship

$$
N(n) = \sum_1 1 = \sum_{n_6=0}^{n} \sum_{n_5=0}^{n-n_6} \sum_{n_4=0}^{n-n_6-n_5} \sum_{n_3=0}^{n-n_6-n_5-n_4} \sum_{n_2=0}^{n-n_6-n_5-n_4-n_3} 1 \tag{37}
$$

where Σ_1 denotes the five summation function with indices n_6, n_5, n_4, n_3, n_2 and index bounds from (36). We know from Feller (1957) that (37) reduces to

$$
N(n) = \frac{\prod_{i=1}^{5} (n+i)}{120}. \tag{38}
$$

The additional restrictions that must be placed on the bounds on n_i values in (36) to make Candidate A the ACW are identified in Gehrlein and Fishburn (1976) for odd n as:

$$
\begin{aligned}
0 &\le n_6 \le \tfrac{(n-1)}{2} \\
0 &\le n_5 \le \tfrac{(n-1)}{2} - n_6 \\
0 &\le n_4 \le \tfrac{(n-1)}{2} - n_6 - n_5 \\
0 &\le n_3 \le \tfrac{(n-1)}{2} - n_6 - n_5 \\
0 &\le n_2 \le n - n_6 - n_5 - n_4 - n_3 \\
n_1 &= n - n_6 - n_5 - n_4 - n_3 - n_2.
\end{aligned} \tag{39}
$$

The total number of voting situations such that A is the ACW with n possible voters is denoted by $N^A(n)$, and it can be obtained directly from the bounds in (39) by $N^A(n) = \Sigma_2 1$, where Σ_2 denotes the five summation function with indices n_6, n_5, n_4, n_3, n_2 and index bounds from (39). This representation is algebraically reduced to obtain

$$
N^A(n) = \frac{(n+1)(n+3)^3(n+5)}{384}. \tag{40}
$$

Since all voting situations are equally likely to be observed with IAC, the probability $P^A(\text{IAC}, n)$ that Candidate A is the ACW follows directly from

$$
P^A(\text{IAC}, n) = \frac{N^A(n)}{N(n)} = \frac{5(n+3)^2}{16(n+2)(n+4)}. \tag{41}
$$

We now expand on this by considering the additional bounds on (39) that will restrict voting situations with n possible voters when Candidate A is the ACW to the condition that only n^* voters participate in the election. The bounds on n_i^* values for participating voters are defined for each of the $N^A(n)$ voting situations, with:

$$\begin{aligned}
&\mathrm{Max}\{0, n^* - (n - n_6)\} \le n_6^* \le \mathrm{Min}\{n_6, n^*\}\\
&\mathrm{Max}\left\{0, n^* - n_6^* - (n - n_6 - n_5)\right\} \le n_5^* \le \mathrm{Min}\left\{n_5, n^* - n_6^*\right\}\\
&\mathrm{Max}\left\{0, n^* - n_6^* - n_5^* - (n - n_6 - n_5 - n_4)\right\} \le n_4^* \le \mathrm{Min}\left\{n_4, n^* - n_6^* - n_5^*\right\}\\
&\mathrm{Max}\left\{0, n^* - n_6^* - n_5^* - n_4^* - (n - n_6 - n_5 - n_4 - n_3)\right\} \le n_3^*\\
&\le \mathrm{Min}\left\{n_3, n^* - n_6^* - n_5^* - n_4^*\right\}\\
&\mathrm{Max}\left\{0, n^* - n_6^* - n_5^* - n_4^* - n_3^* - (n - n_6 - n_5 - n_4 - n_3 - n_2)\right\} \le n_2^*\\
&\le \mathrm{Min}\left\{n_2, n^* - n_6^* - n_5^* - n_4^* - n_3^*\right\}\\
&n_1^* = n^* - n_6^* - n_5^* - n_4^* - n_3^* - n_2^*.
\end{aligned} \tag{42}$$

The upper bounds for all participating voters' preference rankings simultaneously require that $n_i^* \le n_i$ while maintaining $\sum_{i=1}^{6} n_i^* = n^*$. The lower bounds for each n_i^* simultaneously require that $n_i^* \ge 0$, while it also makes certain that a sufficient number of participating voters are assigned to each associated ranking. In particular, we require that the number of unassigned participating voters that will remain after establishing n_i^*, which is $n^* - \sum_{j=i}^{6} n_j^*$, does not exceed the number of possible voters, $n - \sum_{j=i}^{6} n_i$, that have the preference rankings that these remaining participating voters must be assigned to.

The number $N^A(n, n^*)$ of possible combined voting situations that exist for n possible voters where Candidate A as the ACW from (37), and concurrently where there are n^* participating voters from (42) is obtained with a 10 summation function by following the same procedure that was used to obtain $N(n)$ in (37), with

$$N^A\left(n, n^*\right) = \sum_2 \sum_3 1. \tag{43}$$

Here, Σ_3 is a five summation function with indices $n_6^*, n_5^*, n_4^*, n_3^*, n_2^*$ and index bounds from (42). No simple closed form representation is available for $N^A(n, n^*)$ like the one for $N^A(n)$ in (40). However, the IAC(α_*) assumption that was used above for the limiting case as $n \to \infty$ can be extended to the case of finite n and n^*. This is done by assuming that all actual voting situations with n possible voters that have Candidate A as the ACW are combined with all of their associated feasible observed voting situations with n^* participating voters, to have an equally likely probability $\frac{1}{N^A(n,n^*)}$ of being observed. As before, the voter participation rate is $\alpha_* = \frac{n^*}{n}$, but it is obvious that not all values of α_* can be observed for any given finite n.

As a next step we introduce the restriction that Candidate A is also the OCW for the set of participating voters, so that there will be coincidence between the ACW and OCW. Following the format of the process leading to the bounds in (39), the bounds on observed voting situations in (42) are further restricted to make Candidate

A the OCW for odd n^*:

$$\begin{gathered}
\text{Max}\{0, n^* - (n - n_6)\} \le n_6^* \le \text{Min}\left\{n_6, \tfrac{n^*-1}{2}\right\} \\
\text{Max}\left\{0, n^* - n_6^* - (n - n_6 - n_5)\right\} \le n_5^* \le \text{Min}\left\{n_5, \tfrac{n^*-1}{2} - n_6^*\right\} \\
\text{Max}\left\{0, n^* - n_6^* - n_5^* - (n - n_6 - n_5 - n_4)\right\} \le n_4^* \le \text{Min}\left\{n_4, \tfrac{n^*-1}{2} - n_6^* - n_5^*\right\} \\
\text{Max}\left\{0, n^* - n_6^* - n_5^* - n_4^* - (n - n_6 - n_5 - n_4 - n_3)\right\} \le n_3^* \\
\le \text{Min}\left\{n_3, \tfrac{n^*-1}{2} - n_6^* - n_5^*\right\} \\
\text{Max}\left\{0, n^* - n_6^* - n_5^* - n_4^* - n_3^* - (n - n_6 - n_5 - n_4 - n_3 - n_2)\right\} \le n_2^* \\
\le \text{Min}\left\{n_2, n^* - n_6^* - n_5^* - n_4^* - n_3^*\right\} \\
n_1^* = n^* - n_6^* - n_5^* - n_4^* - n_3^* - n_2^*.
\end{gathered} \tag{44}$$

These modifications to the upper bounds in (44) unfortunately create a new set of problems for the bounds on some of the indices. We see that this modification for the upper bound on n_3^* now creates the necessary requirement that $\frac{n^*-1}{2} - n_6^* - n_5^* \ge n^* - n_6^* - n_5^* - n_4^* - (n - n_6 - n_5 - n_4 - n_3)$, so that we must have an additional restriction that $n_4^* \ge \frac{n^*+1}{2} - (n - n_6 - n_5 - n_4 - n_3)$. This, in turn, creates another problem, since this leads to the further restriction that $\frac{n^*-1}{2} - n_6^* - n_5^* \ge \frac{n^*+1}{2} - (n - n_6 - n_5 - n_4 - n_3)$, so that we must add $n_5^* \le n - n_6 - n_5 - n_4 - n_3 - n_6^* - 1$. It is easily shown that this addition does not create any conflicts with the lower bounds for n_5^*.

The resulting restrictions on (44) that are imposed for consistency to create the additional condition that Candidate A is also the OCW for the set of participating voters, so that there will be coincidence between the ACW and OCW, are given by

$$\text{Max}\left\{0, n^* - (n - n_6)\right\} \le n_6^* \le \text{Min}\left\{n_6, \frac{n^*-1}{2}\right\} \tag{45}$$

$$\begin{gathered}
\text{Max}\left\{0, n^* - n_6^* - (n - n_6 - n_5)\right\} \\
\le n_5^* \le \text{Min}\left\{n_5, \tfrac{n^*-1}{2} - n_6^*, n - n_6 - n_5 - n_4 - n_3 - n_6^* - 1\right\}
\end{gathered}$$

$$\begin{aligned}
&\text{Max}\left\{0, n^* - n_6^* - n_5^* - (n - n_6 - n_5 - n_4), \tfrac{n^*+1}{2} - (n - n_6 - n_5 - n_4 - n_3)\right\} \\
&\le n_4^* \le \text{Min}\left\{n_4, \tfrac{n^*-1}{2} - n_6^* - n_5^*\right\}
\end{aligned}$$

$$\begin{aligned}
&\text{Max}\left\{0, n^* - n_6^* - n_5^* - n_4^* - (n - n_6 - n_5 - n_4 - n_3)\right\} \\
&\le n_3^* \le \text{Min}\left\{n_3, \tfrac{n^*-1}{2} - n_6^* - n_5^*\right\}
\end{aligned}$$

$$\begin{gathered}
\text{Max}\left\{0, n^* - n_6^* - n_5^* - n_4^* - n_3^* - (n - n_6 - n_5 - n_4 - n_3 - n_2)\right\} \\
\le n_2^* \le \text{Min}\left\{n_2, n^* - n_6^* - n_5^* - n_4^* - n_3^*\right\}
\end{gathered}$$

$$n_1^* = n^* - n_6^* - n_5^* - n_4^* - n_3^* - n_2^*.$$

It directly follows that the number of voting situations $N_{MA}^A(n, n^*)$ with n possible voters with n^* participating voters for which there is mutual agreement with Candidate A being both the ACW and OCW is obtained from

$$N_{MA}^{A}(n, n^*) = \sum_2 \sum_4 1. \tag{46}$$

Here, Σ_4 is a five summation function with indices $n_6^*, n_5^*, n_4^*, n_3^*, n_2^*$ and index bounds from (45).

The symmetry of IAC-based assumptions with respect to candidates leads to the determination of the conditional probability $P_{\text{ACW}}^{\text{OCW}}(\text{IAC}(\alpha_*), n)$ that ACW and OCW coincide, given that an ACW exists, with the assumption of IAC(α_*) for finite n and n^* with $\alpha_* = \frac{n^*}{n}$ is given by

$$P_{\text{ACW}}^{\text{OCW}}(\text{IAC}(\alpha_*), n) = \frac{N_{MA}^{A}(n, n^*)}{N^{A}(n, n^*)}. \tag{47}$$

While it is theoretically possible to develop closed representations for each of these 10-summation functions, it is not practically possible, or even worthwhile, to do so. As a result, we consider numerical evaluations for $P_{\text{ACW}}^{\text{OCW}}(\text{IAC}(\alpha_*), n)$, which become extremely time consuming to obtain for large n. The largest practical limit that we find is to use $n = 31$ for each odd $n^* = 1(2)31$, and the computed results are displayed in Fig. 8, which also shows the corresponding limiting values for $P_{\text{ACW}}^{\text{OCW}}(\text{IAC}(\alpha_*), \infty)$ from (11) with $\alpha_* = \frac{n^*}{n}$.

The computed probability values in Fig. 8 indicate that the rate of convergence of $P_{\text{ACW}}^{\text{OCW}}(\text{IAC}(\alpha_*), n)$ to the limiting case is slow as n increases, most noticeably so for small n^*. But, we find that an additional computation shows that $P_{\text{ACW}}^{\text{OCW}}(IAC(\alpha_*), 51)$ is equal to 0.3525 for $n^* = 1$, so the actual convergence is evident.

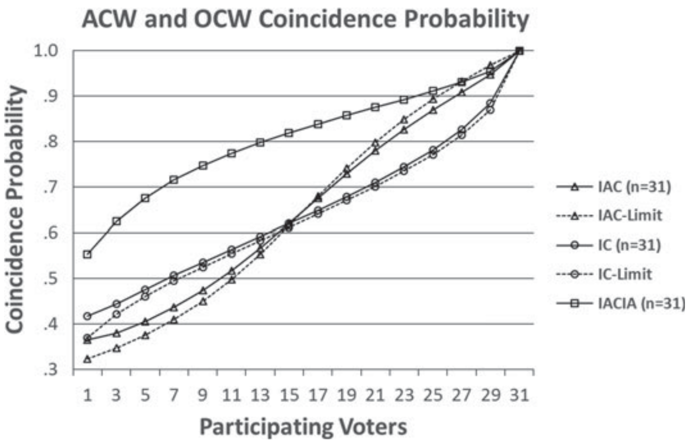

Fig. 8 ACW and OCW coincidence with $IC(\alpha)$, IAC(α_*) and IACIA(α_*)

7.2.2 A Representation for $P_{ACW}^{OCW}(IC(\alpha_*), n)$

Obtaining a probability model that is equivalent to $IC(\alpha)$ for finite n and n^* requires some additional development. The first step is to consider the probability that a given actual voting situation with n possible voters will be observed with IC with independent voters. Many studies have noted that this follows from a standard multinomial probability model as $\frac{n!}{\prod_{i=1}^{6} n_i!}\frac{1}{6^n}$ for the six n_i terms that are associated with the actual voting situation.

In the next step, each voter in this actual voting situation independently chooses to participate with probability α with the limiting $IC(\alpha)$ model. With finite n and n^*, there are $\binom{n}{n^*}$ combinations of ways that an assignment of n^* participating voters can be obtained from the n possible voters. Moreover, each of these combinations is equally likely to be observed with probability $\alpha^{n^*}(1-\alpha)^{n-n^*}$ with the assumption of independence for voter participation. The number of different ways that a given assignment of six n_i^* values for participating voters can be obtained from the specified set of n_i values of possible voters in the actual voting situation is $\prod_{i=1}^{6}\binom{n_i}{n_i^*}$, and each of them is equally likely to be observed with probability $\alpha^{n^*}(1-\alpha)^{n-n^*}$. As a result, the probability of observing a given assignment of six n_i^* values for the specified the n_i values reduces to $\frac{\prod_{i=1}^{6}\binom{n_i}{n_i^*}}{\binom{n}{n*}}$ for any α. However, these probabilities will change as the given participation rate $\frac{n^*}{n}$ changes, so we use the notation $IC(\alpha_*)$ when we are extending the limiting case $IC(\alpha)$ model to this scenario with finite n.

When all of this is accounted for, the probability $P_{\text{ACW}}^{A}(IC(\alpha_*), n, n^*)$ that Candidate A is the ACW with n possible voters when n^* voters choose to participate reduces to

$$P_{\text{ACW}}^{A}(IC(\alpha_*), n, n^*) = \sum\nolimits_2 \sum\nolimits_3 \frac{n!}{6^n\left[\prod_{i=2}^{6}\left(n_i - n_i^*\right)!\right]\left[n - \sum_{i=2}^{6} n_i - \left(n^* - \sum_{i=2}^{6} n_i^*\right)\right]!}. \tag{48}$$

We note that $P_{\text{ACW}}^{A}(IC(\alpha_*), n, n^*)$ does not actually change as n^* changes in (48), but our primary interest here is in the preliminary development of the probability representation for the likelihood for observing a particular assignment of six n_i^* values for specified n_i values.

When all of this is used with the basic discussion that led to (47), the conditional probability that the ACW and OCW coincide for n voters with the assumption of $IC(\alpha_*)$, given that an ACW exists, is given by:

$$P_{\text{ACW}}^{\text{OCW}}(IC(\alpha_*), n) = \frac{\sum_2 \sum_4 \frac{n!}{6^n\left[\prod_{i=2}^{6}(n_i - n_i^*)!\right]\left[n - \sum_{i=2}^{6} n_i - \left(n^* - \sum_{i=2}^{6} n_i^*\right)\right]!}}{P_{ACW}^{A}(IC(\alpha_*), n, n^*)}. \tag{49}$$

Computed values of $P_{\text{ACW}}^{\text{OCW}}(IC(\alpha_*), 31)$ from (49) for each odd $n^* = 1(2)31$ are displayed in Fig. 8, along with corresponding limiting values for $P_{\text{ACW}}^{\text{OCW}}(IAC(\alpha_*), \infty)$ from (11) with $\alpha_* = \frac{n^*}{n}$.

Just as we observed with IAC(α_*) probabilities, the rate of convergence of $P_{\text{ACW}}^{\text{OCW}}(IC(\alpha_*), n)$ to the limiting case is slow as n increases, most noticeably so for small n^*. An additional computation indicates that $P_{\text{ACW}}^{\text{OCW}}(IC(\alpha_*), 51)$ is equal to 0.3972 for $n^* = 1$, so that convergence is definitely occurring as n increases.

It is definitely not at all surprising to find that the same disconcerting phenomenon is now observed in Fig. 8 with $n = 31$ that we previously saw with limiting results in Fig. 2, where $IAC(\alpha_*)$ coincidence probabilities are lower than the associated $IC(\alpha_*)$ probabilities for α_* less than about 0.5. However, things will now become much more interesting when we proceed to consider a third model that considers dependence among possible voters' preferences, while there is complete independence for each voter's choice to participate.

7.2.3 A Representation for $P_{ACW}^{OCW}(IACIA(\alpha_*), n)$ with Independent Abstention

This third model starts with the standard IAC model to create actual voting situations, so that some dependence exists among the possible voters' preferences. Each of the $N^A(n)$ possible actual voting situations from (40) for which Candidate A is the ACW is assumed to be equally likely to be observed when we now consider the additional requirement that A is also the OCW. For each of these possible actual voting situations for n possible voters, we now assume that each individual voter will choose to participate independently with probability α, just as we did above in the development of the $IC(\alpha_*)$ model. We refer to this as the IAC-based Independent Abstention, or IACIA(α_*), model. Since this hybrid model simultaneously uses aspects of the IC-based and IAC-based models, neither of the two solution techniques that are used above for the limiting case as $n \to \infty$ can be used for this case. So, that is why it was stated above that the direct computation of probabilities for finite n is our only option for this stage of our analysis.

All of our earlier discussion leads directly to a representation for the conditional probability $P_{\text{ACW}}^{\text{OCW}}(\text{IACIA}(\alpha_*), n)$ that the ACW and OCW coincide with IACIA(α_*), given that an ACW exists, that is given by:

$$P_{\text{ACW}}^{\text{OCW}}(\text{IACIA}(\alpha_*), n) = \sum_2 \sum_4 \left[\frac{\left(\dfrac{n_2! n_3! n_4! n_5! n_6! (n - n_2 - n_3 - n_4 - n_5 - n_6)!}{\left[\prod_{i=2}^{6} \left(n_i - n_i^* \right)! \right] \left[n - \sum_{i=2}^{6} n_i - \left(n^* - \sum_{i=2}^{6} n_i^* \right) \right]!} \right)}{\left(\dfrac{n^*!(n - n^*)!}{n!} \right) \left(\dfrac{384}{(n+1)(n+3)^3(n+5)} \right)} \right]. \tag{50}$$

Computed values of $P_{\text{ACW}}^{\text{OCW}}(\text{IACIA}(\alpha_*), 31)$ from (50) for each odd $n^* = 1(2)31$ are displayed in Fig. 8, where it is very evident that these IACIA(α_*) probabilities

Table 9 Probability of ACW and OCW coincidence for 31 voters with independent abstentions

n^*	IAC ($n = 31$)	IAC-Limit	IC (n = 31)	IC-Limit	IACIA ($n = 31$)
1	0.3645	0.3230	0.4163	0.3694	0.5522
3	0.3793	0.3469	0.4432	0.4215	0.6246
5	0.4047	0.3753	0.4752	0.4601	0.6758
7	0.4364	0.4092	0.5058	0.4936	0.7152
9	0.4739	0.4497	0.5351	0.5245	0.7471
11	0.5173	0.4976	0.5636	0.5540	0.7741
13	0.5664	0.5531	0.5918	0.5827	0.7977
15	0.6199	0.6150	0.6200	0.6114	0.8190
17	0.6753	0.6793	0.6488	0.6403	0.8385
19	0.7292	0.7411	0.6786	0.6701	0.8569
21	0.7797	0.7976	0.7099	0.7012	0.8747
23	0.8262	0.8482	0.7437	0.7346	0.8924
25	0.8688	0.8930	0.7813	0.7714	0.9106
27	0.9086	0.9327	0.8255	0.8141	0.9305
29	0.9477	0.9682	0.8837	0.8687	0.9549
31	1.0000	1.0000	1.0000	1.0000	1.0000

are remarkably greater than their associated $IC(\alpha_*)$ probabilities, so the very disconcerting results that were observed before now vanish with this model. The value of $P_{\mathrm{ACW}}^{\mathrm{OCW}}(\mathrm{IACIA}(\alpha_*), 51)$ is found to be 0.5449 when $n^* = 1$.

Table 9 lists the numerical values for probability of ACW and OCW coincidence for 31 voters with independent abstentions that are shown graphically in Fig. 8.

8 Conclusion

Preliminary results were very disturbing regarding the extremely negative impact that adding dependence among voters' preferences with the IAC(α_*) model could have on election outcomes with low voter participation rates, relative to the case of complete independence among voters' preferences with $IC(\alpha)$. This concern followed from the fact that the introduction of dependence among voters' preferences almost always dampens the probabilities that negative election outcomes will be observed. However, it is now quite evident that these earlier unexpected results with IAC(α_*) must result from the dependence that it creates among preferences of voters who choose not to participate in the election. If a model requires that voters are restricted to abstain independently, then the addition of dependence among the preferences of possible voters actually vastly improves the very negative outcomes that were initially observed when the possible voters had independent preferences. The preliminary results were correct, and they were actually showing that extremely bad results can be expected to follow from the existence of dependence among abstaining voters' preferences for all voting rules, particularly for low voter participation rates.

As a result, we do not leave you with chaos, but we instead leave you with a very complex problem that calls for significant further investigation. This outcome now makes it much easier for us to settle into states of happy and content retirement.

It is also important to note that there were many insightful comments from the participants at the *Eighth Murat Sertel Workshop on Economic Design, Decision, Institutions, and Organization: In the honor of Dominique Lepelley* when the preliminary version of this work was presented. Those comments motivated us to continue working on this project and they were extremely beneficial to the presentation of the results in this final version of the paper.

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Condorcet Efficiency of General Weighted Scoring Rules Under IAC: Indifference and Abstention

Mostapha Diss, Eric Kamwa, Issofa Moyouwou, and Hatem Smaoui

1 Introduction

The simplest representation of a voting environment includes a set of voters, a set of candidates, a list of admissible individual preferences and a rule that aggregates each possible configuration of voters' preferences, into a social outcome. In this context, a *profile* is defined as a sequence of preferences of all the individuals taking part in the vote. The performance of a voting rule is then measured by its propensity to avoid counterintuitive results or to produce desirable electoral outcomes. For example, there may exist a candidate that is preferred to any other candidate by a majority of voters, *a Condorcet winner*; Condorcet (1875) advocated that when a Condorcet winner exists, he/she should be the outcome of any reasonable rule. But undertaking in practice all pairwise majority comparisons for a given profile of individual preferences is very demanding as the total number of voters or of candidates increases. An earlier alternative suggested by Borda (1781) consists of assigning an amount of points to each candidate each time he/she is ranked at a given position by a voter. For example with three candidates, 1 point is awarded to each candidate for each first place in an individual ranking, λ points for each second-place where $0 \leq \lambda \leq 1$ and no point for last position. The winner is then the candidate with the highest total score. However, all such weighted scoring rules may fail to select a Condorcet winner. Since then, the

M. Diss (✉)
CRESE EA3190, Université Bourgogne Franche-Comté, 25000 Besançon, France
e-mail: mostapha.diss@univ-fcomte.fr

E. Kamwa
LC2S UMR CNRS 8053, Faculté de Droit et d'Economie de la Martinique, Université des Antilles, 97275 Schœlcher Cedex, France
e-mail: eric.kamwa@univ-antilles.fr

I. Moyouwou
Ecole Normale Supérieure - Department of Mathematics, University of Yaounde I, BP 47, Yaounde, Cameroon
e-mail: imoyouwou2@yahoo.fr

H. Smaoui
CEMOI, Université de La Réunion, Saint-Denis, France
e-mail: hatem.smaoui@univ-reunion.fr

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ability of a weighted scoring rule to select a Condorcet winner has been the subject of abundant literature that aims at measuring the desirability of a voting rule with respect to its *Condorcet efficiency*; that is, the conditional probability that the rule will select a Condorcet winner assuming that one exists.

Cervone et al. (2005) investigated in three-candidate elections which (one-shot) weighted scoring rule exhibits the maximum Condorcet efficiency under the Impartial Anonymous Culture (IAC) assumption. First explored by Gehrlein and Fishburn (1976), the IAC assumption amounts to assuming that all anonymous profiles of individual preferences are equally probable. Cervone et al. (2005) showed that when individual preferences are linear orders, the weighted scoring rule that performs the best in selecting a Condorcet winner is a rule that lies between the Plurality rule ($\lambda = 0$) and the Borda rule ($\lambda = 0.5$). In this paper, we address a similar question when individual preferences are weak orders (some voters may be indifferent between two or more candidates); or when some voters may abstain (they freely decide to not participate in the election).

The possibility that voter indifference may be observable has already been considered by some other authors; see for example Diss et al. (2010), Gehrlein and Lepelley (2015), Kamwa (2019b), or Merlin and Valognes (2004) among others. More precisely, we propose an IAC counterpart of Gehrlein and Valognes (2001), who considered the same topic, when we assume that each voter uniformly picks his/her preference out of a predefined set of weak orders. This is known as an Impartial Culture (IC)-like probability distribution over the set of all configurations of voters' preferences. Here, two extreme cases are explored. We first consider only *concerned voters*, i.e., voters with strict preferences on at least a pair of candidates. We determine the exact Condorcet efficiency of each possible weighted scoring rule under the IAC assumption in three-candidate elections when the total number of voters tends to infinity. It appears that when we move from a weighted scoring rule with linear orders to its extended version on weak orders, the Condorcet efficiency increases for all weights λ ranged from 0 up to approximately 0.3765; but decreases for all weights ranged from 0.3765 to 1. Another salient point is that the maximum Condorcet efficiency in three-candidate elections under the IAC assumption is now observed for a new weight, approximately 0.4139, which is distinct from the optimal one provided by Cervone et al. (2005) which is approximately 0.3723. Finally, the maximum Condorcet efficiency is now equal to 0.9265, which is slightly greater than 0.9255, the one obtained with linear orders.

The possibility that some voters will abstain is also explored, and here we follow the recent framework of Gehrlein and Lepelley (2020, 2017). The authors measured the impact of indifference on voting rules with respect to the participation rate under both IC and IAC assumptions. We explore some new and extreme cases assuming that the participation rate is unknown and may be of any size. Three cases are considered: *global abstention* when voters from all possible types may abstain; *self-confident abstention* when only voters who prefer a Condorcet winner by self-confidence abstain—this may presumably be the case when a Condorcet winner exists and is acclaimed by almost all polls; and *pessimistic abstention* when only voters who prefer any other candidate to a Condorcet winner by discouragement abstain—this may be the case when some voters think their favorite candidate is lagging behind their less preferred candidate. In the global abstention setting, we would have expected a very low Condorcet efficiency for every weighted scoring rule; but in fact, we observe an honorable performance since some weighted scoring rules still record more than 60% of voting situations in which the Condorcet winner is selected after some voters abstain. The two other cases of abstention impact differently on the performances of weighted scoring rules. All those aspects are commented upon and discussed later in the chapter.

The rest of the paper is organized as follows: Sect. 2 underlines some key points of our investigations that differ from previous works. In Sect. 3, for a three-candidate election we provide the exact Condorcet efficiency of any weighted scoring rule as the total number of voters tends to infinity. Section 4 highlights some abstention patterns and the Condorcet efficiency of weighted scoring rules on those restricted domains. Section 5 concludes with a general comment on our investigations.

2 The Scope

Consider a three-candidate election with n voters ($n \geq 2$) and assume that each individual preference is a weak order (complete and transitive binary relations) over candidates A, B and C. In addition, each voter is assumed to act according to his/her true preferences, which means that strategic voting is not taken into consideration in our paper. There are thirteen possible types of preferences according to how candidates are ranked with or without indifference:

$A \succ B \succ C$ (x_1)	$A \succ (B \sim C)$ (x_7)	
$A \succ C \succ B$ (x_2)	$B \succ (A \sim C)$ (x_8)	
$B \succ A \succ C$ (x_3)	$C \succ (A \sim B)$ (x_9)	$(A \sim B \sim C)$ (x_{13})
$B \succ C \succ A$ (x_4)	$(A \sim B) \succ C$ (x_{10})	
$C \succ A \succ B$ (x_5)	$(A \sim C) \succ B$ (x_{11})	
$C \succ B \succ A$ (x_6)	$(B \sim C) \succ A$ (x_{12})	

In the notation $A \succ B \succ C$ (x_1), $A \succ B \succ C$ refers to the preference type of all voters who prefer A to B, A to C , and B to C; and x_1 is the proportion of such voters; that is the ratio $\frac{n_1}{n}$ where n_1 is the total number of voters who report $A \succ B \succ C$. Similarly, a voter endowed with the preference type $A \succ (B \sim C)$ prefers A to B, A to C and is indifferent between B and C. The proportion of all such voters is x_7. The collection $x = (x_1, x_2, \ldots, x_{13})$ will be called a voting situation when the thirteen terms x_j sum to 1. Voters having the preference type $(A \sim B \sim C)$ will be called *unconcerned voters* since each such voter is indifferent to the election of any of the three candidates. In case there is some evidence that allows each voter to have a strict ranking of the three competing candidates, only the first six preference types are observable. A voting situation will then be reduced to the 6-tuple $x = (x_1, x_2, \ldots, x_6)$ by setting $x_7 = x_8 = \cdots = x_{13} = 0$. This is the assumption taken into account by Cervone et al. (2005).

Indifference or abstention are possible factors that may justify alternative investigations. Given indifference, we carry out our investigations under two different but common settings. When all voters are concerned voters who may still be indifferent between at most two candidates, we identify a voting situation as the 12 -tuple $x = (x_1, x_2, \ldots, x_{12})$ assuming that $x_{13} = 0$. This is also the setting taken into consideration by Gehrlein (1983). We also consider the mixed case where unconcerned and concerned voters are involved. In this latter case, only voting situations $x = (x_1, x_2, \ldots, x_{12}, x_{13})$ such that $0 \leq x_{13} < 1$ are considered (the extreme case $x_{13} = 1$ removes the possibility of any objective differentiation among the three candidates). This is the setting developed by Gehrlein and Valognes (2001).

When indifference vanishes, some voters may abstain and a voting scenario is now a twofold vector (x, y) where $x = (x_1, x_2, \ldots, x_6)$ is the initial voting situation and $y = (y_1, y_2, \ldots, y_6)$ indicates the proportion y_j of voters who abstain among the voters having preference of type j. Note that $0 \leq y_j \leq x_j$ and $y_1 + y_2 + \cdots + y_6 < 1$ (the extreme case $y_1 + y_2 + \cdots + y_6 = 1$ is of no interest). The impact of abstention on voting procedures was also studied in Gehrlein and Fishburn (1978, 1979) and recently by Gehrlein and Lepelley (2020, 2017). We provide two extreme cases: (i) when all voters who abstain have the same top-ranked candidate (who is perceived as having no chance to win); and (ii) when all voters who abstain have the same bottom-ranked candidate.

In each of the preference settings we consider, our analysis is restricted to the set of voting situations that capture all the possible configurations of individual preferences. We denote this set by D and we assume for each case the uniform probability distribution over D: all voting situations in D are equally probable to be observed. This is known as an Impartial Anonymous Culture assumption over D and will be referred to as IAC_D. Given $\lambda \in [0, 1]$, the vector $w = (1, \lambda, 0)$ will be called the scoring vector. The weighted scoring rule on D is denoted by F_λ and assigns $x_j w(j, k)$ points to a candidate, say C, each time voters having type j rank C at the kth position given a voting situation x; where $w(j, k) = w_k$ in case preference of type j corresponds to a linear order ($j = 1, 2, \ldots, 6$ and $k = 1, 2, 3$); $w(j, 1) = \frac{(1+\lambda)}{2}$ and $w(j, 2) = 0$ if voters of type j are indifferent between their two first-ranked candidates ($j = 7, 8, 9$); $w(j, 1) = 1$ and $w(j, 2) = \frac{\lambda}{2}$ if voters of type j are indifferent between their two bottom-ranked candidates ($j = 9, 10, 11$); and $w(j, 1) = \frac{(1+\lambda)}{3}$ if $j = 13$. Obviously, the candidate who records the maximum sum of points wins.

A candidate X majority defeats another candidate Y in a pairwise comparison if there are more voters who strictly prefer X to Y than voters who strictly prefer Y to X. A Condorcet winner is a candidate who defeats any other candidate in pairwise majority voting. When a Condorcet winner exists, he/she is clearly a desirable election winner since he/she is immune to rejection by any majority of voters. It is well known that for a given weighted scoring rule F_λ, we may find some voting situation x in which a candidate, say C, is a Condorcet winner while $F_\lambda(x) \neq C$. Courtin et al. (2015a) show that this failure may be overcome in three-candidate elections by strengthening the size of the majority in favor of the Condorcet winner up to a threshold; see also Courtin et al. (2015b) for a more general framework.

In general, the Condorcet efficiency of the rule F_λ, given a domain D of observable voting situations with n voters and a probability distribution P_D over D, is the conditional probability $CE(\lambda, P_D, n)$ that the rule will select a Condorcet winner assuming that one exists. In particular, under the IAC assumption, the limit $CE(\lambda, IAC_D, \infty)$ of $CE(\lambda, IAC_D, n)$ as n tends to infinity is the ratio $\frac{vol(D_{CW,\lambda})}{vol(D_{CW})}$ where D_{CW} denotes the polytope of all voting situations in D in which a Condorcet winner exists[1] while $D_{CW,\lambda}$ is the polytope of all voting situations in D at which a Condorcet winner exists and is the winner for rule F_λ; for more details and a rich panel of related topics, interested readers are referred to Gehrlein (2006) or Gehrlein and Lepelley (2017, 2011). By symmetry, to evaluate $CE(\lambda, IAC_D, \infty)$, we can replace D_{CW} by $D_{CW,A}$, the subset of D_{CW} in which A is the Condorcet winner, and $D_{CW,\lambda}$ by $D_{CW,\lambda,A}$, the subset of $D_{CW,\lambda}$ in which A is the Condorcet winner and is selected by the voting rule F_λ.

3 Condorcet Efficiency of Weighted Scoring Rules When Indifference Is Observable

Giving a weight $\lambda \in [0, 1]$, we determine here the Condorcet efficiency of the weighted scoring rule associated with λ when some voters may be indifferent between two candidates.

3.1 With No Unconcerned Voters

When no voter is unconcerned and none of them abstains, the corresponding domain of observable voting situations is denoted by $\mathcal{D}$ and consists of all 12-tuples $x = (x_1, x_2, \ldots, x_{12})$ such that

$$\sum_{j=1}^{12} x_j = 1 \text{ and } x_j \geq 0 \text{ for all } j \in \{1, 2, \ldots, 12\}. \tag{1}$$

In this case, candidate A is a Condorcet winner in x if A beats B and A beats C in pairwise majority voting:

$$x_3 + x_4 + x_6 + x_8 + x_{12} - x_1 - x_2 - x_5 - x_7 - x_{11} < 0 \tag{2}$$

$$x_4 + x_5 + x_6 + x_9 + x_{12} - x_1 - x_2 - x_3 - x_7 - x_{10} < 0 \tag{3}$$

[1] i.e., the polytope defined by the linear system characterizing these voting situations.

Therefore, the set $\mathcal{D}_{CW,A}$ of all voting situations in $\mathcal{D}$ at which A is the Condorcet winner is the 11-dimensional polytope defined by (1), (2), and (3). The volume of $\mathcal{D}_{CW,A}$ as well as all other subsequent volumes in this chapter will be computed using the method presented in Moyouwou and Tchantcho (2017). Alternative methods are also available from Cervone et al. (2005) or Lepelley et al. (2008). We may also combine available packages such as Convex for convex geometry by Franz (2017) for a Maple implementation or well-established algorithms such as *Normaliz* by Bruns et al. (2017, 2019) and Bruns and Ichim (2010). These techniques have also recently been used under different forms by Bubboloni et al. (2019), Diss and Doghmi (2016), Diss et al. (2018), Diss and Gehrlein (2012, 2015), Kamwa (2019a), Kamwa and Moyouwou (2020), El Ouafdi et al. (2019); Lepelley et al. (2018b), and Lepelley and Smaoui (2019), among others. Up to a scaling constant that depends only on the dimension of $\mathcal{D}$,

$$vol\left(\mathcal{D}\right) = \frac{1}{11!} \text{ and } vol\left(\mathcal{D}_{CW}\right) = 3vol\left(\mathcal{D}_{CW,A}\right) = \frac{8821}{367873\,228\,800} \tag{4}$$

Then, the probability that a Condorcet winner exists under the assumption $IAC_{\mathcal{D}}$ is $\frac{vol(\mathcal{D}_{CW})}{vol(\mathcal{D})} \approx 0.95714$. As compared to the probability 0.9375 from Gehrlein and Fishburn (1976) of observing a Condorcet winner when individual preferences are linear orders, this confirms the observation by Gehrlein and Valognes (2001) that the possibility of indifference increases the probability that a Condorcet winner exists. Now candidate A is the winner for rule F_λ at x when the score of A is greater than both the score of B and the score of C; that is

$$\begin{aligned} &(\lambda - 1)(x_1 - x_3) - x_2 + x_4 - \lambda (x_5 - x_6) + \frac{\lambda - 2}{2}(x_7 - x_8) \\ &- \frac{1 + \lambda}{2}(x_{11} - x_{12}) < 0 \end{aligned} \tag{5}$$

$$\begin{aligned} &- x_1 + x_6 + (\lambda - 1)(x_2 - x_5) - \lambda (x_3 - x_4) + \frac{\lambda - 2}{2}(x_7 - x_9) \\ &- \frac{1 + \lambda}{2}(x_{10} - x_{12}) < 0 \end{aligned} \tag{6}$$

The subset $\mathcal{D}_{CW,A,\lambda}$ of $\mathcal{D}_{CW,A}$ that consists of all voting situations in which A is the Condorcet winner and at the same time is selected by F_λ at x is the polytope described by the constraints at (1), (2), (3), (5), and (6). Its volume is computed as a function of λ in order to derive the Condorcet efficiency $CE\left(\lambda, IAC_{\mathcal{D}}, \infty\right) = \frac{vol\left(\mathcal{D}_{CW,A,\lambda}\right)}{vol\left(\mathcal{D}_{CW,A}\right)}$ when the total number of voters tends to infinity. The corresponding formula is completely unreadable and is relegated to the Appendix. Numerical values of this function are reported in Table 1 and its graph appears in Fig. 1. The value of $CE\left(\lambda, IAC_{\mathcal{D}}, \infty\right)$ provided in Table 1 corresponds to some values of $\lambda = d_1 + d_2$, the first decimal (d_1) of which is indicated in the first column and the second decimal (d_2) in the first row. Moreover, the maximum of $CE\left(\lambda, IAC_{\mathcal{D}}, \infty\right)$ is for a unique value λ^* of λ between

Table 1 IAC-based Condorcet efficiency of weighted scoring rules when indifference is observable

λ	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.8575	0.8596	0.8618	0.8640	0.8662	0.8683	0.8705	0.8727	0.8749	0.8770
0.1	0.8792	0.8814	0.8835	0.8856	0.8878	0.8899	0.8920	0.8941	0.8961	0.8981
0.2	0.9001	0.9021	0.9040	0.9059	0.9078	0.9095	0.9113	0.9130	0.9146	0.9161
0.3	0.9176	0.9190	0.9202	0.9214	0.9225	0.9235	0.9243	0.9251	0.9256	0.9261
0.4	0.9264	0.9265	0.9265	0.9263	0.9260	0.9254	0.9247	0.9238	0.9227	0.9214
0.5	0.9199	0.9182	0.9163	0.9143	0.9121	0.9097	0.9071	0.9044	0.9016	0.8986
0.6	0.8955	0.8923	0.8889	0.8855	0.8819	0.8783	0.8746	0.8708	0.8669	0.8630
0.7	0.8590	0.8549	0.8508	0.8466	0.8424	0.8382	0.8338	0.8295	0.8251	0.8207
0.8	0.8163	0.8119	0.8074	0.8029	0.7984	0.7939	0.7894	0.7849	0.7803	0.7758
0.9	0.7713	0.7668	0.7622	0.7577	0.7532	0.7487	0.7442	0.7397	0.7353	0.7308
1	0.7264	–	–	–	–	–	–	–	–	–

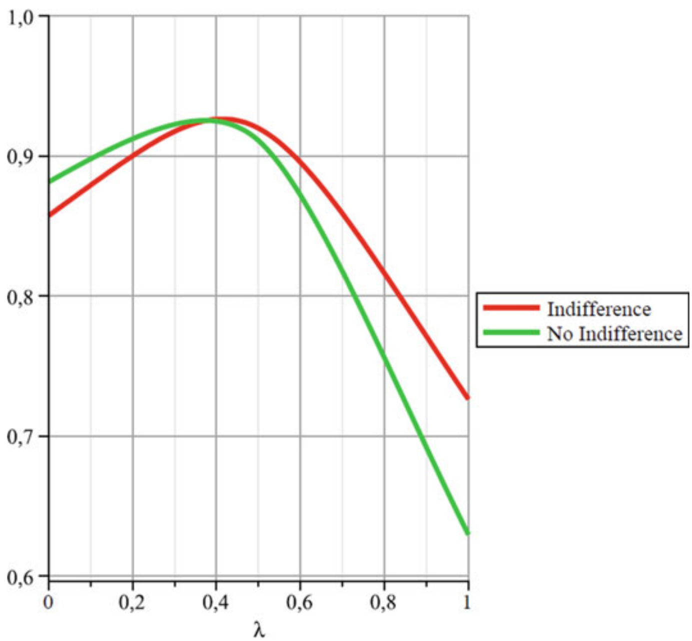

Fig. 1 IAC-based Condorcet efficiency of weighted scoring rules with and without indifference

$\frac{1}{3}$ and $\frac{1}{2}$. The exact value of λ^* is unreachable due to the intractable expressions of $CE(\lambda, IAC_{\mathcal{D}}, \infty)$ and of its first derivative. An approximation up to four decimal places gives $\lambda^* \approx 0.4139$ with $CE(\lambda^*, IAC_{\mathcal{D}}, \infty) \approx 0.9265$.

When no voter is indifferent between any pair of candidates, the function of the Condorcet efficiency of all weighted scoring rules with three candidates under the

IAC assumption comes from Cervone et al. (2005, Theorem 2). Its graph is also represented in Fig. 1. From this result, it appears that the rule that maximizes the Condorcet efficiency among weighted scoring rules corresponds to a value λ_0 of the weight λ such that $2\lambda_0 - 1 \approx -0.25544$; that is $\lambda_0 \approx 0.37228$. It follows that if we are looking for the optimal weighted scoring rule with respect to Condorcet efficiency under the IAC assumptions described above, the appropriate value of the weight λ differs when we admit indifference or only consider linear orders. With indifference, our results show that the optimal rule is nearer to the Borda rule ($\lambda = 0.5$) than it is with only linear orders. Finally, it is worth noting from our results that the maximal Condorcet efficiency among weighted scoring rules is approximately 0.9265. This value is greater than the maximal Condorcet efficiency among weighted scoring rules with only linear orders, which is approximately 0.9255 (Cervone et al. 2005, Theorem 2). Again, this is in accordance with earlier observations. However, the Condorcet efficiency of some weighted scoring rules decreases from linear orders to weak orders as shown in Fig. 1. More exactly, the Condorcet efficiency is greater for linear orders than for weak orders for all weighted scoring rules from $\lambda = 0$ (the Plurality rule) up to $\lambda \approx 0.3765$. Globally, the extensions of classical weighted scoring rules that permit us to handle the possible indifference of voters result in improvements of the Condorcet efficiency of weighted scoring rules for $1 \geq \lambda > 0.3765$ but not for $0 \leq \lambda < 0.3765$.

3.2 With Possibly Unconcerned Voters

When some voters are completely indifferent about the selection of one of the three candidates, the corresponding domain of observable voting situations is denoted by $\widetilde{\mathcal{D}}$ and consists of all 13-tuples $x = (x_1, x_2, \ldots, x_{13})$ such that

$$\sum_{j=1}^{13} x_j = 1 \text{ and } x_j \geq 0 \text{ for all } j \in \{1, 2, \ldots, 13\}. \tag{7}$$

Such a voting situation is completely determined by the 12-tuple $(x_1, x_2, \ldots, x_{12})$ which satisfies

$$\sum_{j=1}^{12} x_j = t \text{ with } t = 1 - x_{13} > 0. \tag{8}$$

More interestingly, the conditions that candidate A is the Condorcet winner or the winner for the weighted scoring rule F_λ do not change, since an unconcerned voter does not favor any of the three candidates. By setting $x_j = ty_j$ for $j = 1, 2, \ldots, 12$, it follows that A is the Condorcet winner at x if and only if $y = (y_1, y_2, \ldots, y_{12})$ lies in $\mathcal{D}_{CW,A}$ characterized by (1), (2) and (3). Similarly A is the winner at x for F_λ if and only if $y = (y_1, y_2, \ldots, y_{12})$ belongs to $\mathcal{D}_{CW,\lambda,A}$. Due to this homothetic transformation, we can recover the volumes of $\widetilde{\mathcal{D}}_{CW,A}$ and $\widetilde{\mathcal{D}}_{CW,\lambda,A}$ from the volumes of $\mathcal{D}_{CW,A}$ and $\mathcal{D}_{CW,\lambda,A}$ by noting that t varies from 0 to 1. That is

$$vol\left(\widetilde{\mathcal{D}}_{CW,A}\right) = \int_0^1 t^{11} vol\left(\mathcal{D}_{CW,A}\right) \mathrm{d}t = \frac{vol\left(\mathcal{D}_{CW,A}\right)}{12} \tag{9}$$

and

$$vol\left(\widetilde{\mathcal{D}}_{CW,\lambda,A}\right) = \int_0^1 t^{11} vol\left(\mathcal{D}_{CW,\lambda,A}\right) \mathrm{d}t = \frac{vol\left(\mathcal{D}_{CW,\lambda,A}\right)}{12}. \tag{10}$$

Since the Condorcet efficiency of the weighted scoring rule F_λ over $\widetilde{\mathcal{D}}$ is the ratio $\frac{vol\left(\widetilde{\mathcal{D}}_{CW,\lambda,A}\right)}{vol\left(\widetilde{\mathcal{D}}_{CW,A}\right)}$, the Eqs. (9) and (10) imply the following result.

Proposition 1 *For all* $\lambda \in [0, 1]$, *the IAC-based Condorcet efficiencies of the weighted scoring rule associated with* λ *with or without unconcerned voters coincide.*

In other words, Proposition 1 shows that the presence of unconcerned voters does not affect the Condorcet efficiency of weighted scoring rules under the IAC assumption as the total number of voters tends to infinity. It is clear that this is also the case for all other similar voting events that can be described by linear constraints with null constant terms. However, this is not necessarily the case with other probability distributions, such as the Impartial Culture assumption; see Gehrlein and Valognes (2001) where the authors include the possibility of having unconcerned voters.

4 Condorcet Efficiency with Abstention Allowed

In this section, we assume that all voters are concerned voters, individual preferences are linear orders and some voters may abstain. Out of the initial proportion x_j of voters of type j, we are now expecting that y_j voters will effectively take part in the election. Then, the participation rate can be calculated as the number of voters who will effectively take part in the election divided by the total number of voters. A voting scenario is a twofold vector (x, y) where $x = (x_1, x_2, \ldots, x_6)$ is a voting situation on linear orders and $y = (y_1, y_2, \ldots, y_6)$ satisfies

$$0 \leq y_j \leq x_j \text{ for } j = 1, 2, \ldots, 6. \tag{11}$$

The question is, assuming that a candidate X is a Condorcet winner at x, some voters abstain and y describes the proportion of voters from each type who finally participate in the election, what is the probability that X will be selected by a given weighted scoring rule? We evaluate this conditional probability over three distinct domains.

Without loss of generality, we assume that A is the Condorcet winner (or the *popular candidate*). We refer to the first domain as the *global abstention*: voters from any type may abstain. The second domain is called *self-confident abstention*: only voters who top-ranked the *popular candidate* may abstain. The third domain is called *pessimistic abstention*: only voters who prefer all other candidates to the popular candidate may abstain.

4.1 Global Abstention

In this setting, the set of voting scenarios is the set denoted by $\mathcal{S}$ that consists of all couples (x, y) such that

$$(\mathcal{S}) : \begin{cases} x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0 \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1 \\ y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0, y_5 \geq 0, y_6 \geq 0 \\ x_1 \geq y_1, x_2 \geq y_2, x_3 \geq y_3, x_4 \geq y_4, x_5 \geq y_5, x_6 \geq y_6 \end{cases} \tag{12}$$

This domain is an 11-dimensional polytope. We assume that all voting scenarios in $\mathcal{S}$ are equally probable and we refer to this probability distribution as $IAC_{\mathcal{S}}$. Now the subset $\mathcal{S}_{CW,A}$ of $\mathcal{S}$, that consists of all voting scenarios (x, y) in which A is the Condorcet winner at x, is the polytope characterized by the constraints at (12) and the following

$$\begin{cases} x_3 + x_4 + x_6 - x_1 - x_2 - x_5 < 0 \\ x_5 + x_6 + x_4 - x_1 - x_2 - x_3 < 0 \end{cases} \tag{13}$$

Its volume is $vol\left(\mathcal{S}_{CW,A}\right) = \frac{79}{10218700800}$. Since the volume of $\mathcal{S}$ is $\frac{1}{11!}$, it follows that the probability that a Condorcet winner exists in $\mathcal{S}$ under the $IAC_{\mathcal{S}}$ assumption is $\frac{vol(\mathcal{S}_{CW,A})}{vol(\mathcal{S})} = 0.30859$, which gives the proportion of voting situations (x, y) having candidate A as a Condorcet winner when the number of voters tends to infinity. Notice that using the symmetry of IAC-like assumptions with regards to candidates means that $0.30859 \times 3 = 0.92578$ is the proportion of voting situations (x, y) having a Condorcet winner when the number of voters tends to infinity. Finally, A is selected by the weighted scoring rule associated with λ at y if and only if

$$\begin{cases} (\lambda - 1)y_1 - y_2 + (1 - \lambda)y_3 - \lambda y_5 + y_4 + \lambda y_6 < 0 \\ -y_1 + (\lambda - 1)y_2 - \lambda y_3 + \lambda y_4 + (1 - \lambda)y_5 + y_6 < 0 \end{cases} \tag{14}$$

The subset $\mathcal{S}_{CW,A,\lambda}$ of $\mathcal{S}$ that consists of all voting scenarios (x, y) in which A is the Condorcet winner at x and is selected in y is the polytope described by the constraints at (12), (13), and (14). Its volume is computed in order to derive the Condorcet efficiency $CE\,(\lambda, IAC_{\mathcal{S}}, \infty) = \frac{vol(\mathcal{S}_{CW,A,\lambda})}{vol(\mathcal{S}_{CW,A})}$ when the total number of voters tends to infinity. The results of our calculations are given as follows:

For $0 \leq \lambda \leq \frac{1}{2}$, $CE(\lambda, IAC_{\mathcal{S}}, \infty) =$

$$\frac{\left(\begin{array}{l}218700\,\lambda^{21} - 1174320\,\lambda^{20} - 4535142\,\lambda^{19} + 18714908\,\lambda^{18} + 151671536\,\lambda^{17} \\ - 508196052\,\lambda^{16} - 1757330525\,\lambda^{15} + 9181808848\,\lambda^{14} - 2416926066\,\lambda^{13} \\ - 55062774610\,\lambda^{12} + 116439091808\,\lambda^{11} - 3497495094\,\lambda^{10} - 342404967208\,\lambda^{9} \\ + 608347988900\,\lambda^{8} - 430343075808\,\lambda^{7} - 70310632700\,\lambda^{6} + 424083710296\,\lambda^{5} \\ - 414925509984\,\lambda^{4} + 222950616032\,\lambda^{3} - 72115069504\,\lambda^{2} + 13241739264\,\lambda \\ - 1067873280\end{array}\right)}{204768(1-\lambda)^3(3\lambda-2)^2(\lambda-2)^4(-4+5\lambda)^2(\lambda^2+2\lambda-2)^3(1+\lambda)}$$

For $\frac{1}{2} \leq \lambda \leq 1$, $CE(\lambda, IAC_{\mathcal{S}}, \infty) =$

$$\frac{\left(\begin{array}{l}72900\,\lambda^{24} - 38460\,\lambda^{23} + 12014686\,\lambda^{22} - 284507414\,\lambda^{21} + 2137697548\,\lambda^{20} \\ - 7182774684\,\lambda^{19} + 9390687357\,\lambda^{18} + 4929800229\,\lambda^{17} - 18153355218\,\lambda^{16} \\ - 26293402260\,\lambda^{15} + 109425128388\,\lambda^{14} - 115472223994\,\lambda^{13} + 32683701680\,\lambda^{12} \\ + 26299843928\,\lambda^{11} - 23324433021\,\lambda^{10} + 2405339031\,\lambda^{9} + 6496354764\,\lambda^{8} \\ - 5222096538\,\lambda^{7} + 2209396698\,\lambda^{6} - 610438788\,\lambda^{5} + 115276342\,\lambda^{4} \\ - 14795282\,\lambda^{3} + 1234756\,\lambda^{2} - 60264\,\lambda + 1296\end{array}\right)}{204768\lambda^5(5\lambda-1)^2(2-\lambda)^3(1+\lambda)^4(-4\lambda+1+\lambda^2)^3(-1+3\lambda)}$$

Numerical results of $CE(\lambda, IAC_{\mathcal{S}}, \infty)$ are reported in Table 2 and sketched in Fig. 2.

4.2 *Self-confident Abstention*

Assume now that individual preferences are linear orders and that only voters of type *ABC* or *ACB* may abstain: due to some signals such as polls surveys, some of

Table 2 Condorcet efficiency of weighted scoring rules with distinct abstention scenarios

λ	Self-confident abstention	Global abstention	Pessimistic abstention
0	0.4979	0.6366	0.9722
0.1	0.5001	0.6427	0.9801
0.2	0.5018	0.6481	0.9865
0.3	0.5028	0.6522	0.9905
0.4	0.5029	0.6541	0.9906
0.5	0.5013	0.6521	0.9841
0.6	0.4970	0.6435	0.9662
0.7	0.4888	0.6268	0.9304
0.8	0.4765	0.6023	0.8744
0.9	0.4612	0.5718	0.8021
1	0.4443	0.5384	0.7209

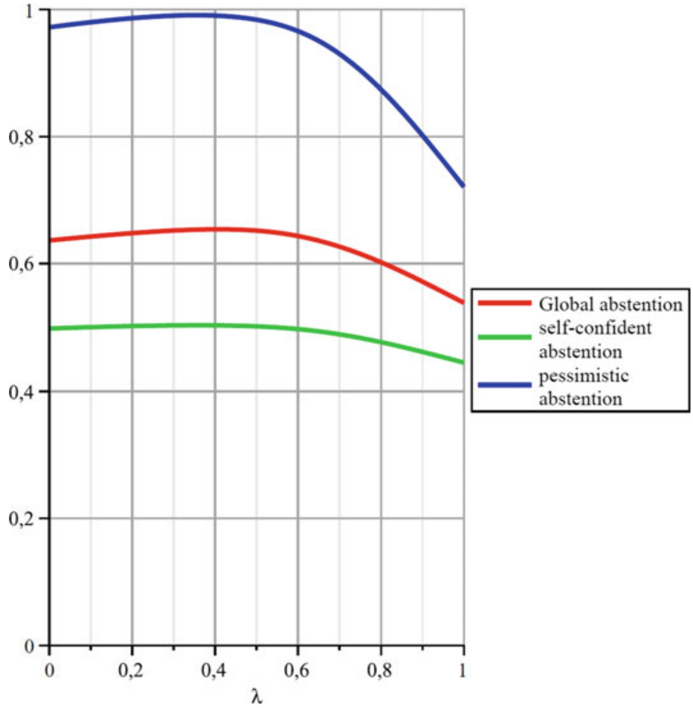

Fig. 2 Condorcet efficiency of weighted scoring rules with distinct abstention scenarios

these voters may be (erroneously or not) thinking that their favorite candidate A is sufficiently popular and does not especially need their votes to defeat B and C. With a similar notation as above, the corresponding set of voting scenarios (x, y) is denoted by $\mathcal{S}^*$ and is now 7-dimensional since we should have $y_j = 0$ for $j = 3, 4, 5, 6$. In the same way, the set $\mathcal{S}^*_{CW,A}$ and $\mathcal{S}^*_{CW,A,\lambda}$ are simply obtained, respectively, from $\mathcal{S}_{CW,A}$ and $\mathcal{S}_{CW,A,\lambda}$ by setting $y_j = 0$ for $j = 3, 4, 5, 6$. Finally $CE\,(\lambda, IAC_{\mathcal{S}^*}, \infty) = \frac{vol(\mathcal{S}^*_{CW,A,\lambda})}{vol(\mathcal{S}^*_{CW,A})}$ is obtained by performing a volume computation as before. Our results are described as follows:

$$\text{For } 0 \leq \lambda \leq \tfrac{1}{2},\ CE\,(\lambda, IAC_{\mathcal{S}^*}, \infty) = \frac{\begin{pmatrix} 34\,\lambda^{13} - 913\,\lambda^{12} - 3554\,\lambda^{11} + 36150\,\lambda^{10} + 15318\,\lambda^{9} \\ -384783\,\lambda^{8} + 458022\,\lambda^{7} + 1030506\,\lambda^{6} - 2830398\,\lambda^{5} \\ +1799251\,\lambda^{4} + 1181642\,\lambda^{3} - 2266676\,\lambda^{2} + 1182216\,\lambda - 216816 \end{pmatrix}}{13608(-1+\lambda)^3\left(\lambda^2+2\,\lambda-2\right)^2(2-\lambda)^3(1+\lambda)}$$

For $\frac{1}{2} \leq \lambda \leq 1$, $CE\,(\lambda, IAC_{\mathcal{S}^*}, \infty) =$

$$\frac{\begin{pmatrix} 31350\,\lambda^{16} + 424085\,\lambda^{15} - 5998520\,\lambda^{14} + 19077007\,\lambda^{13} \\ -5835650\,\lambda^{12} - 52134452\,\lambda^{11} + 46525702\,\lambda^{10} + 55349687\,\lambda^{9} \\ -78736170\,\lambda^{8} + 26324586\,\lambda^{7} + 8272256\,\lambda^{6} - 10772558\,\lambda^{5} \\ +4654654\,\lambda^{4} - 1146839\,\lambda^{3} + 169090\,\lambda^{2} - 13868\,\lambda + 488 \end{pmatrix}}{13608(-1+3\,\lambda)\left(-4\,\lambda+1+\lambda^2\right)^2(1+\lambda)^3(5\,\lambda-1)^2\lambda^3(2-\lambda)}$$

Numerical results of $CE\,(\lambda, IAC_{\mathcal{S}^*}, \infty)$ are also reported in Table 2 and sketched in Fig. 2.

4.3 Pessimistic Abstention

Finally, assume that individual preferences are linear orders and that only voters of type *BCA* or *CBA* may abstain: they may be (erroneously or not) feeling that *B* and *C* are lagging behind *A*, and that their votes for both *B* and *C* would be of no use. The corresponding set of voting scenarios (x, y) is denoted by $\mathcal{S}'$ and is now 7-dimensional since we should have $y_j = 0$ for $j = 1, 2, 3, 5$. The sets $\mathcal{S}'_{CW,A}$ and $\mathcal{S}'_{CW,A,\lambda}$ are simply obtained, respectively, from $\mathcal{S}_{CW,A}$ and $\mathcal{S}_{CW,A,\lambda}$ by setting $y_j = 0$ for $j = 1, 2, 3, 5$. We then get $CE\,(\lambda, IAC_{\mathcal{S}'}, \infty) = \frac{vol\left(\mathcal{S}'_{CW,A,\lambda}\right)}{vol\left(\mathcal{S}'_{CW,A}\right)}$ which is obtained by performing the same volume computations as before. Our results are given as follows:

For $0 \leq \lambda \leq \frac{1}{2}$, $CE\,(\lambda, IAC_{\mathcal{S}'}, \infty) =$

$$\frac{\begin{pmatrix} 4950\,\lambda^{13} - 52095\,\lambda^{12} + 196780\,\lambda^{11} + 220080\,\lambda^{10} - 3901650\,\lambda^{9} + 12926745\,\lambda^{8} \\ -20918022\,\lambda^{7} + 15291690\,\lambda^{6} + 4004808\,\lambda^{5} - 19395021\,\lambda^{4} \\ +18835758\,\lambda^{3} - 9447272\,\lambda^{2} + 2515488\,\lambda - 282240 \end{pmatrix}}{567(-1+\lambda)^3(2-\lambda)^3(-4+5\,\lambda)^2(3\,\lambda-2)^2(1+\lambda)}$$

For $\frac{1}{2} \leq \lambda \leq 1$, $CE\,(\lambda, IAC_{\mathcal{S}'}, \infty) =$

$$\frac{\begin{pmatrix} 1416\,\lambda^{11} + 27740\,\lambda^{10} - 30728\,\lambda^{9} - 73976\,\lambda^{8} \\ +29112\,\lambda^{7} + 162704\,\lambda^{6} - 113362\,\lambda^{5} + 12585\,\lambda^{4} \\ +19471\,\lambda^{3} - 10835\,\lambda^{2} + 2195\,\lambda - 162 \end{pmatrix}}{2268\lambda^4(1+\lambda)^3(2-\lambda)(-1+3\,\lambda)}$$

Numerical results of $CE\,(\lambda, IAC_{\mathcal{S}'}, \infty)$ are also displayed in Table 2 and Fig. 2.

Several lessons may be drawn from the probabilities corresponding to the three scenarios taken into account. First, it can be seen clearly that the Condorcet efficiency of the three considered scenarios exhibits the same behavior since the three curves first increase and then decrease. Every Condorcet efficiency stops rising and starts falling for a unique value λ^* of λ that maximizes the associated probability. An approximation up to four decimal places of the value of λ^* maximizing the Condorcet efficiency gives $\lambda^* \approx 0.4074$ with $CE\,(\lambda^*, IAC_{\mathcal{S}}, \infty) \approx 0.6542$ for the *global abstention* domain, $\lambda^* \approx 0.3567$ with $CE\,(\lambda^*, IAC_{\mathcal{S}'}, \infty) \approx 0.5030$ for the *self-confident* *abstention* scenario, and $\lambda^* \approx 0.3541$ with $CE\ (\lambda^*, IAC_{\mathcal{S}^*}, \infty) \approx 0.9912$ for the *pessimistic abstention* case. Third, it can be noted that on the one hand, the Condorcet efficiency remains approximately stable with regards to the value of λ when the setting of *self-confident abstention* is assumed. On the other hand, the change in the Condorcet efficiency is more pronounced when the *pessimistic abstention* domain is considered; its value steady declines, particularly when the value of λ exceeds 0.6. Finally, it is worth noting that all weighted scoring rules in three-candidate elections have highest performance with respect to the Condorcet criterion when the *pessimistic abstention* domain is assumed, the *self-confident abstention* domain is the worst scenario.

5 Conclusion

Given an arbitrary weighted scoring rule for three-candidate elections, the aim of this paper has been to provide the exact limit of its Condorcet efficiency as the total number of voters tends to infinity under some IAC-like assumptions over some domains of voting situations. More exactly, we have explored the impact of observing ties and abstention on the Condorcet efficiency of the whole class of weighted scoring rules in three-candidate elections under IAC-like assumptions. Some instructive observations have emerged. First, it appears that the weighted scoring rule that maximizes the Condorcet efficiency under IAC-type assumptions depends not only on the set of observable individual preferences, but also on the behavior of voters in the election such as abstention. Second, and more importantly, the scoring rule which tends to maximize the probability of selecting the Condorcet winner, when there is one, is not the well-known Borda rule. This result has also been shown in previous studies that have been conducted in other frameworks (see, for instance, Cervone et al. 2005; Lepelley et al. 2000, among others).

Many questions still remain unanswered. First, since ties and abstention have been treated separately in our framework, we believe that studying the weighted scoring rules that maximize the Condorcet efficiency, when both ties and abstention can be expressed at the same time by voters, remains a fruitful open line of research. Second, the extension of our results to multistage elimination scoring rules is also an important research direction. Under those voting rules, candidates are assigned scores according to their rank in the preferences of voters and then the candidate(s) with the lowest number of points are eliminated in each round. In this connection, other well-known voting rules widely studied in the literature can also be considered. Third, it is important to stress that the assumptions of IC and IAC have some subtle differences. For instance, results under many frameworks in the literature suggest that the Borda rule will maximize the limiting Condorcet efficiency with IC, but we have seen that it did not under IAC. As a consequence, it seems that the ways under which the voters' preferences are generated and their impact on the Condorcet efficiency of weighted scoring rules in all the scenarios considered in our paper is an important research direction to follow. Notice finally that analogous calculations would need to be done with more than three candidates, and it seems that certain new research techniques will make this possible. Results for more than three candidates will allow us to draw more accurate conclusions.

Throughout our analysis, we have assumed that voters abstain because they think they have no chance of changing the outcome. One might as well consider the case of a strategic behavior where they would try to change the result in their favor: such a work would join the recent analysis of Felsenthal and Nurmi (2019) and Kamwa et al. (2018) among others, which deal with the No-show paradox under various restrictions of domains like that of the existence of the Condorcet winner.

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Appendix: Condorcet Efficiency of Standard Weighted Scoring Rules for the Case of Indifference

For $0 \leq \lambda \leq 2-\sqrt{3}$, $CE\left(\lambda, IAC_{\mathcal{D}}, \infty\right) =$

$$\frac{\begin{pmatrix}
1820786\,688\,000\lambda^{48} - 69\,241916\,620\,800\lambda^{47} + 1026\,234629\,521\,920\lambda^{46} - 7284\,934967\,241\,672\lambda^{45} \\
+ 18\,277\,844939\,772\,940\lambda^{44} + 103\,592\,393347\,371\,022\lambda^{43} - 1238\,479\,837216\,933\,811\lambda^{42} \\
+ 7114\,690\,023954\,613\,055\lambda^{41} - 32211\,613\,928453\,148\,729\lambda^{40} + 115639\,088\,574725\,271\,854\lambda^{39} \\
- 236512\,647\,125549\,661\,383\lambda^{38} - 286151\,748\,427342\,093\,501\lambda^{37} + 4058202\,005\,007340\,429\,561\lambda^{36} \\
- 14\,318810\,047\,697903\,565\,096\lambda^{35} + 22\,349727\,410\,670132\,937\,793\lambda^{34} \\
+ 16\,543371\,732\,429846\,167\,336\lambda^{33} - 175\,555781\,573\,475047\,887\,373\lambda^{32} \\
+ 437\,422977\,910\,724619\,504\,115\lambda^{31} - 483\,555379\,875\,364616\,250\,580\lambda^{30} \\
- 317\,603525\,246\,580133\,257\,115\lambda^{29} + 2328\,610974\,249\,100702\,568\,584\lambda^{28} \\
- 4632\,327155\,285\,954449\,646\,038\lambda^{27} + 4727\,751430\,637\,011253\,403\,199\lambda^{26} \\
- 109\,245140\,931\,968759\,805\,205\lambda^{25} - 8885\,273274\,696\,887352\,596\,075\lambda^{24} \\
+ 17\,706\,774485\,498\,423664\,071\,476\lambda^{23} - 20\,017\,586527\,878\,097411\,380\,703\lambda^{22} \\
+ 12\,669\,590811\,026\,864041\,789\,277\lambda^{21} + 1136\,537775\,705\,418491\,635\,969\lambda^{20} \\
- 13\,778\,933630\,365\,017707\,246\,908\lambda^{19} + 18\,799\,513648\,879\,716877\,033\,827\lambda^{18} \\
- 15\,254\,787069\,146\,101371\,272\,038\lambda^{17} + 7332\,765302\,678\,185241\,972\,369\lambda^{16} \\
- 332\,383158\,733\,291688\,576\,093\lambda^{15} - 2996\,814688\,633\,054810\,513\,310\lambda^{14} \\
+ 3087\,233061\,146\,579368\,634\,651\lambda^{13} - 1810\,863957\,041\,637494\,575\,430\lambda^{12} \\
+ 641\,597719\,780\,426722\,446\,276\lambda^{11} - 56\,116471\,132\,277234\,004\,088\lambda^{10} \\
- 95\,214964\,467\,975593\,883\,012\lambda^{9} + 75\,729086\,200\,343544\,343\,640\lambda^{8} \\
- 34\,276146\,412\,782754\,811\,536\lambda^{7} + 11\,054599\,551\,772653\,125\,472\lambda^{6} \\
- 2669971\,755\,905471\,542\,464\lambda^{5} + 484686\,737\,258602\,283\,136\lambda^{4} \\
- 64625\,671\,172236\,822\,272\lambda^{3} + 6001\,143\,093138\,737\,664\lambda^{2} - 347\,714\,765843\,853\,312\lambda \\
+ 9483\,672291\,729\,408
\end{pmatrix}}{\begin{pmatrix}
38\,583\,054\left(-4\lambda+\lambda^{2}+2\right)^{2}(7\lambda-3)^{2}\left(-5\lambda+\lambda^{2}+2\right)(7\lambda-4)(\lambda-1)^{8}(\lambda+1)(\lambda+2) \\
\times(2\lambda-1)^{2}(2\lambda+1)(2-\lambda)\left(\lambda^{2}+1\right)(\lambda-3)(\lambda-4)(2\lambda-3)(3\lambda-2)^{2}(3\lambda+2) \\
\times\left(\lambda+\lambda^{2}-1\right)\left(-\lambda+\lambda^{2}+2\right)\left(2\lambda+\lambda^{2}-1\right)^{2}(\lambda-6)(3\lambda-4)(5\lambda-2)(5\lambda-3)^{2}
\end{pmatrix}}$$

For $2-\sqrt{3} \leq \lambda \leq \frac{1}{3}$, $CE\left(\lambda, IAC_{\mathcal{D}}, \infty\right) =$

$$\frac{\left(\begin{array}{l}
1820786\,688\,000\lambda^{48} - 74\,704276\,684\,800\lambda^{47} + 1244\,885099\,512\,320\lambda^{46} \\
- 10\,804\,581341\,381\,232\lambda^{45} + 47\,323\,169817\,532\,876\lambda^{44} - 11\,896\,189228\,265\,240\lambda^{43} \\
- 1295\,427\,558400\,669\,181\lambda^{42} + 10838\,602\,659769\,641\,280\lambda^{41} - 60932\,746\,397452\,834\,926\lambda^{40} \\
+ 272298\,567\,692948\,074\,279\lambda^{39} - 919748\,069\,915040\,272\,996\lambda^{38} \\
+ 1922729\,976\,378950\,744\,954\lambda^{37} - 107520\,324\,936731\,831\,476\lambda^{36} \\
- 16\,046349\,280\,257649\,212\,608\lambda^{35} + 63\,982392\,803\,835491\,246\,957\lambda^{34} \\
- 131\,594022\,183\,132278\,579\,333\lambda^{33} + 102\,985885\,243\,856862\,063\,807\lambda^{32} \\
+ 263\,130441\,491\,002792\,978\,434\lambda^{31} - 1120\,118363\,443\,620345\,250\,056\lambda^{30} \\
+ 2068\,147541\,140\,171435\,437\,810\lambda^{29} - 1911\,492390\,661\,346765\,109\,656\lambda^{28} \\
- 638\,819581\,233\,742237\,398\,078\lambda^{27} + 5387\,926806\,763\,347416\,361\,913\lambda^{26} \\
- 9728\,147692\,713\,242728\,280\,400\lambda^{25} + 10\,046\,559510\,469\,985452\,429\,584\lambda^{24} \\
- 4901\,586668\,964\,434084\,114\,939\lambda^{23} - 3110\,900254\,952\,426440\,166\,384\lambda^{22} \\
+ 9139\,448775\,320\,162952\,050\,832\lambda^{21} - 10\,021\,137586\,182\,738207\,585\,316\lambda^{20} \\
+ 6498\,420889\,415\,942163\,770\,942\lambda^{19} - 1840\,968960\,234\,262545\,229\,881\lambda^{18} \\
- 1178\,787637\,750\,470723\,251\,707\lambda^{17} + 1930\,389777\,296\,413710\,058\,071\lambda^{16} \\
- 1355\,027846\,210\,208450\,503\,438\lambda^{15} + 571\,078810\,313\,953333\,722\,844\lambda^{14} \\
- 103\,122899\,029\,894401\,764\,380\lambda^{13} - 49\,101913\,858\,193610\,558\,036\lambda^{12} \\
+ 52\,958716\,465\,121060\,088\,760\lambda^{11} - 26\,428806\,100\,020453\,419\,248\lambda^{10} \\
+ 9041586\,474\,056435\,631\,264\lambda^{9} - 2282625\,313\,621030\,973\,760\lambda^{8} \\
+ 429966\,791\,749254\,817\,664\lambda^{7} - 59239\,187\,188309\,597\,440\lambda^{6} + 5669\,413\,681460\,636\,160\lambda^{5} \\
- 337\,940\,174689\,744\,896\lambda^{4} + 9462\,296226\,484\,224\lambda^{3} \\
+ 1113258\,442\,752\lambda^{2} - 36643\,995\,648\lambda + 573\,308\,928
\end{array}\right)}{\left(\begin{array}{c}
38\,583\,054\,(7\lambda-4)\,\lambda^{3}\,(\lambda-1)^{9}\,(\lambda+1)\,(\lambda-2)\,(\lambda+2)\,(2\lambda-1)^{2}\,(2\lambda+1)\,(\lambda-3)\,(4-\lambda) \\
(2\lambda-3)\,(3\lambda-2)^{2}\,(3\lambda+2)\left(\lambda+\lambda^{2}-1\right)\left(-\lambda+\lambda^{2}+2\right)\left(2\lambda+\lambda^{2}-1\right)(\lambda-6) \\
(3\lambda-4)\,(5\lambda-2)\,(5\lambda-3)^{2}\left(-4\lambda+\lambda^{2}+2\right)^{2}(7\lambda-3)^{2}\left(-5\lambda+\lambda^{2}+2\right)
\end{array}\right)}$$

For $\frac{1}{3} \leq \lambda \leq \frac{1}{2}$, $CE\left(\lambda, IAC_{\mathcal{D}}, \infty\right) =$

$$\frac{\left(\begin{array}{l}
9289728000\,\lambda^{36} - 297619596000\,\lambda^{35} + 3377989143360\,\lambda^{34} - 13963337093844\,\lambda^{33} \\
+ 3370375264444\,\lambda^{32} - 402215454286943\,\lambda^{31} + 8963866330410706\,\lambda^{30} - 69941639381396539\,\lambda^{29} \\
+ 288643976788058119\,\lambda^{28} - 627477867299131010\,\lambda^{27} + 248517337184893361\,\lambda^{26} \\
+ 2609323768109145402\,\lambda^{25} - 7932461061154644246\,\lambda^{24} + 10019745050192949855\,\lambda^{23} \\
- 1158379395278615758\,\lambda^{22} - 15662441259201903711\,\lambda^{21} + 25118070398698305635\,\lambda^{20} \\
- 21072798121128351708\,\lambda^{19} + 19280898166689692789\,\lambda^{18} - 31894466931214607084\,\lambda^{17} \\
+ 42410957812210078536\,\lambda^{16} - 28745451043046875662\,\lambda^{15} - 1002323686662473050\,\lambda^{14} \\
+ 19442887622216608928\,\lambda^{13} - 16497433408345540744\,\lambda^{12} + 5138910439439921036\,\lambda^{11} \\
+ 1381560072929906496\,\lambda^{10} - 1869267824450225872\,\lambda^{9} + 648301094886938816\,\lambda^{8} \\
- 33718969352126832\,\lambda^{7} - 39306911200860128\,\lambda^{6} + 8546068707872128\,\lambda^{5} + 1061504521622784\,\lambda^{4} \\
- 556852979448576\,\lambda^{3} + 45154967560704\,\lambda^{2} + 2464451039232\,\lambda - 107254554624
\end{array}\right)}{\left(\begin{array}{c}
38583054\,(2-\lambda)\,(\lambda+2)\,(2\,\lambda-3)\,(-1+\lambda)^{8}\,\lambda^{2}\,(1+\lambda)\,(-3+\lambda)\,(5\,\lambda+1)\,(7\,\lambda-4)\,(5\,\lambda-3)^{2} \\
(3\,\lambda-4)\,(\lambda-6)\,(1+3\,\lambda)\left(\lambda^{2}-4\,\lambda+2\right)^{2}\left(\lambda^{2}+\lambda-1\right)(\lambda-4)\,(2\,\lambda+1)\,(3\,\lambda-2)\,(2+3\,\lambda)
\end{array}\right)}$$

For $\frac{1}{2} \leq \lambda \leq \frac{2}{3}$, $CE\left(\lambda, IAC_{\mathcal{D}}, \infty\right) =$

$$
\frac{\left(\begin{array}{l}
599298932736000\,\lambda^{43} - 24894797263257600\,\lambda^{42} + 321747726609561600\,\lambda^{41} \\
- 2460940238278107648\,\lambda^{40} + 15644469789077275008\,\lambda^{39} - 71043427323828455136\,\lambda^{38} \\
+ 115060892883434995088\,\lambda^{37} + 582465500014126288756\,\lambda^{36} - 3460843041196876374588\,\lambda^{35} \\
+ 5537814448809200666515\,\lambda^{34} + 8081840578573892819103\,\lambda^{33} - 46604545871578657603575\,\lambda^{32} \\
+ 64323974809259448671634\,\lambda^{31} + 25765871448120618830351\,\lambda^{30} - 207163457256228192262959\,\lambda^{29} \\
+ 284655633057587011364481\,\lambda^{28} - 103519599030222189252188\,\lambda^{27} \\
- 199154409908822765983565\,\lambda^{26} + 335884880646892989888899\,\lambda^{25} - 221784089193151064338067\,\lambda^{24} \\
+ 26791602419781197740670\,\lambda^{23} + 76500190435038303873815\,\lambda^{22} - 71634510774411743501567\,\lambda^{21} \\
+ 30495317238798878121765\,\lambda^{20} - 3503051930164397658360\,\lambda^{19} - 3862253035621582133922\,\lambda^{18} \\
+ 2852085814734176360086\,\lambda^{17} - 1049312702815311771754\,\lambda^{16} + 237959024404663565076\,\lambda^{15} \\
- 36579576228179144954\,\lambda^{14} + 7873967213479742182\,\lambda^{13} - 2593795952554456134\,\lambda^{12} \\
- 280924365248561364\,\lambda^{11} + 878391638981584720\,\lambda^{10} - 478889908254439808\,\lambda^{9} \\
+ 141550488288565984\,\lambda^{8} - 22652083592277440\,\lambda^{7} + 328316466849536\,\lambda^{6} + 790713386228736\,\lambda^{5} \\
- 212706949874688\,\lambda^{4} + 30558160300032\,\lambda^{3} - 2681179729920\,\lambda^{2} + 136280309760\,\lambda - 3105423360
\end{array}\right)}{\left(\begin{array}{c}
38583054\,(-1+\lambda)^2\,\lambda^8\,(2\lambda-3)\,(1+\lambda)\,(-3+\lambda)\,(5\lambda+1)\,(3\lambda-4)\,(\lambda-6)\,(1+3\lambda)\,(\lambda-4) \\
(2\lambda+1)\,(2+3\lambda)\left(\lambda^2+2\lambda-1\right)^2 \\
(5\lambda-2)^2\,(4\lambda-1)^3\,(\lambda-2)\,(\lambda+2)\,(-1+3\lambda)^3\left(\lambda^2-3\lambda+1\right)(-3+8\lambda)\,(7\lambda-3)^2
\end{array}\right)}
$$

For $\frac{2}{3} \le \lambda \le 1$, $CE\,(\lambda, IAC_{\mathcal{D}}, \infty) =$

$$
\frac{\left(\begin{array}{l}
50341110349824000\,\lambda^{53} - 1143202667456102400\,\lambda^{52} + 7217916749335756800\,\lambda^{51} \\
- 3467394048046373683\,2\,\lambda^{50} + 68513905408513569638\,4\,\lambda^{49} \\
- 762865387837079635238\,4\,\lambda^{48} + 357605274525381046809\,28\,\lambda^{47} \\
- 2144432915768524199205\,6\,\lambda^{46} - 5646109373261245355269\,68\,\lambda^{45} \\
+ 28230555839169956255829\,70\,\lambda^{44} - 48965274900682644949831\,10\,\lambda^{43} \\
- 7786116812937700030681990\,\lambda^{42} + 62037058533532306175024075\,\lambda^{41} \\
- 142318716263463772744482147\,\lambda^{40} + 82378226897358021316912587\,\lambda^{39} \\
+ 423869774526473167985740546\,\lambda^{38} - 1431119085114352349851417586\,\lambda^{37} \\
+ 2114065260450619432591021068\,\lambda^{36} - 782761018910366605213795998\,\lambda^{35} \\
- 3665557112402327685220357888\,\lambda^{34} + 9738380253026331395580532754\,\lambda^{33} \\
- 13266369722048681013025587978\,\lambda^{32} + 10457018999081962154072364758\,\lambda^{31} \\
- 1601104779572709668337318232\,\lambda^{30} - 8583229481587012991716442476\,\lambda^{29} \\
+ 14662789000331251552467633806\,\lambda^{28} - 14553236525078891871569897018\,\lambda^{27} \\
+ 10197513563186020546902075854\,\lambda^{26} - 5063994764702656999570698641\,\lambda^{25} \\
+ 1475784958290852263225596477\,\lambda^{24} + 150395038438085270702579407\,\lambda^{23} \\
- 500692297156210203731473490\,\lambda^{22} + 360356751917040727163107406\,\lambda^{21} \\
- 169259472314848100587046748\,\lambda^{20} + 56500451258403497688241462\,\lambda^{19} \\
- 11762016210680671213151624\,\lambda^{18} - 1294548001329312321905 80\,\lambda^{17} \\
+ 1395198996472301470596248\,\lambda^{16} - 719392290744137621324904\,\lambda^{15} \\
+ 229407208897037257843552\,\lambda^{14} - 48721375615928156525712\,\lambda^{13} \\
+ 5019213997220757059168\,\lambda^{12} + 901850049462101175680\,\lambda^{11} \\
- 570630537012716250880\,\lambda^{10} + 145064253910526063872\,\lambda^{9} \\
- 21483823004236570112\,\lambda^{8} + 1132069178749464576\,\lambda^{7} + 317788368055607296\,\lambda^{6} \\
- 101765096940740608\,\lambda^{5} + 15941551110832128\,\lambda^{4} \\
- 1610771917012992\,\lambda^{3} + 106787829841920\,\lambda^{2} - 4275313311744\,\lambda + 78989230080
\end{array}\right)}{\left(\begin{array}{c}
38583054\left(\lambda^2+2\lambda-1\right)^2(\lambda-2)\,(\lambda+2)\,\lambda^8\,(2\lambda-3)\,(1+\lambda)\,(-3+\lambda)\,(5\lambda+1)\,(7\lambda-4) \\
(1+3\lambda)\,(\lambda-4)\,(2\lambda+1)\left(\lambda^2-4\lambda+2\right)^2(3\lambda-1)^3\,(-1+2\lambda)^2\left(\lambda^2+3\lambda-2\right)(5\lambda-2)^2 \\
(8\lambda-3)\,(7\lambda-3)^2\left(-\lambda+2+\lambda^2\right)\left(\lambda^2-3\lambda+1\right)(9\lambda-5)\left(\lambda^2-2\lambda+2\right)(4\lambda-1)^3\,(3\lambda-4)
\end{array}\right)}
$$

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The Effect of Closeness on the Election of a Pairwise Majority Rule Winner

Mostapha Diss, Patrizia Pérez-Asurmendi, and Abdelmonaim Tlidi

1 Introduction

In the literature of social choice theory, a wide variety of voting rules are proposed in order to determine the winner of an election with more than two competing candidates. Plurality rule (voting only for top choice candidate) is the most widely used voting procedure in practice when the aim is the selection of a single winner. However, it is well known that Plurality rule can produce election results that do not reflect the whole preferences of the electorate. Condorcet (1785) and Borda (1781) are the first authors who exhibited these difficulties through many configurations of individual preferences that could create noncoherent social outcomes when we use Plurality rule. Much of the debate between Borda and Condorcet surrounds the discussion of which voting rule should be implemented as a replacement for Plurality rule in order to select the members of the French Academy of Sciences, of which Borda and Condorcet were members. On the one hand, the voting rule suggested by Borda (1781) consists in assigning an amount of points to each candidate each time she is ranked at a given position by a voter and then choose as a winner the candidate with the highest total number of points. On the other hand, according to Condorcet (1785), it seems essential to use pairwise majority comparisons between all of the

M. Diss (✉)
CRESE EA3190, Université Bourgogne Franche-Comt, 25000 Besançon, France
e-mail: mostapha.diss@univ-fcomte.fr

P. Pérez-Asurmendi
PRESAD Research Group and IMUVA, Department of Economic Analysis, UCM, Madrid, Spain
e-mail: patrizip@ucm.es

ICAE, UCM, Madrid, Spain

A. Tlidi
National School of Applied Science-Safi, University Cadi Ayyad of Marrakesh, LMPEQ, Route Sidi Bouzid, 63, 46000 Safi, Morocco
e-mail: mtlidi2010@gmail.com

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candidates. More precisely, if a candidate wins a simple majority election over every other candidate, then this candidate should be the winner of an election involving all candidates. A winner of these pairwise majority comparisons is called the *Condorcet winner* and any voting method conforming to the Condorcet criterion is known as a *Condorcet method*. After more than two centuries, the Borda-Condorcet debate is still well documented in the social choice literature (see, e.g., Gehrlein and Lepelley 2011, 2017).

The results presented in this chapter belong to the long tradition, issue from the Borda-Condorcet debate, which consists of evaluating voting rules on their propensity to select the Condorcet winner.[1] Indeed, it is well known that the existence of the Condorcet winner is not always guaranteed. Therefore, it is now common to consider the *Condorcet efficiency* (CE) as a measure of partial fulfillment of the Condorcet criterion. Note that the CE of a voting rule is defined as the conditional probability that the voting rule selects the Condorcet winner, given that such a candidate exists. As the CE involves the computation of probabilities of electoral events, it is usually required to make some assumptions on the likelihood of the different possible individual votes that could be observed. One of the most widespread used assumptions in the literature is the *Impartial Anonymous Culture* (IAC) introduced by Kuga and Nagatani (1974) and later developed by Gehrlein and Fishburn (1976).

The CE was extensively studied under the IAC assumption. In particular, a whole body of literature can be found regarding the CE of the *Weighted Scoring Rules* (WSRs). Under those rules, each candidate is awarded with a number of points according to her relative position within each individual voter's preference ranking and the winner is the candidate with the highest total score. The *Plurality Rule* (PR), the *Negative Plurality Rule* (NPR), and the *Borda Rule* (BR) are well-known examples of WSRs. Those three WSRs will be formally defined later in the chapter. It is also usual to find the IAC assumption in many papers dealing with the CE of the *Weighted Scoring Elimination Rules* (WSERs) which also constitute an important class of voting rules. Those rules also give points to the candidates according to their rank in voters' preference orders and eliminate the candidate(s) with the lowest number of points. The number of rounds is determined by the number of candidates and the implemented method. The elected candidate is the majority winner between the two remaining candidates in the last round.[2] The scoring rules that follow this process are called *Weighted Scoring Elimination Rules* (WSERs). The *Plurality Elimination*

[1] It is worth to note that there is a debate about the election (or not) of the Condorcet winner. For some authors, this can matter while for others it does not. However, there is a clear consensus, at all, that the candidate who is defeated by every other candidate in pairwise majority comparisons (Condorcet loser) is not a good outcome for an election.

[2] Notice that we only deal in this chapter with the classical form of eliminations before the choice of the overall winner of the election. Other ways of eliminations can be found in the literature such as the one developed by Kim and Roush (1996) when we eliminate all candidates who obtain strictly less than the average score. We also refer the reader to Favardin and Lepelley (2006) and Lepelley and Valognes (2003).

Rule (PER), the *Negative Plurality Elimination Rule* (NPER) and the *Borda Elimination Rule* (BER) are widely known examples of WSERs. Those voting rules will also be formally defined later in the chapter.

1.1 Related Literature

Many research papers already analyzed the CE on various voting rules taking into account different assumptions on individuals' preferences. Taking into account the aim of this chapter, we will only recall some relating results. The interested reader can find an exhaustive review of this topic in the recent books by Gehrlein and Lepelley (2011, 2017). First of all, Gehrlein (1982, 1992) calculated the CE values of BR, PR, PER, NPR, and NPER in three-candidate elections under the IAC assumption for large electorates (Table 1).

Also in the context of three-candidate elections under the IAC hypothesis, Gehrlein and Lepelley (2001) obtained a closed - form representation of the CE of BR as a function of the number of voters, and Cervone et al. (2005) developed a representation for the CE of every WSR. As a quite natural extension, many studies were carried out to deal with the effect of some additional assumptions on the CE of several voting rules. For instance, Lepelley (1995) provided an exact representation for the CE of WSRs when voters are endowed with *single-peaked preferences*. Intuitively, voters are said to have single-peaked preferences if there is an ideal outcome that they prefer the most, and alternatives that are further away from this ideal outcome (according to some linear ordering) are less preferred (see, for instance, Black 1948). Gehrlein et al. (2012) and Gehrlein and Lepelley (2015) dealt with the CE in the presence of degrees of *group mutual coherence* which measures a voting situation's propensity to specific underlying rational behavior models that may govern voter preferences. Finally, notice that many researchers reconsidered the CE of voting rules by using IAC-like models. For instance, Diss et al. (2020) and Gehrlein and Lepelley (2020) consider IAC assumptions that capture all the possible configurations of individual preferences in the presence of abstention among voters. Diss and Gehrlein (2015), for their part, consider a different IAC-like assumption by removing from consideration the subset of individual preferences for which all WSRs elect the same winner.

Table 1 Condorcet efficiency values under the IAC assumption for large electorates

Voting rules	Condorcet efficiency
PR	0.8815[a]
BR	0.9111[b]
NPR	0.6296[a]
PER	0.9685[a]
NPER	0.9704[a]

[a]Gehrlein (1982); [b]Gehrlein (1992)

It is important to mention that many studies focused on other assumptions on the individuals' preferences when calculating the CE of several voting rules. For more information on the research papers related to those assumptions and their refinements, we refer the reader to Diss and Merlin (2010), Diss et al. (2010), Gehrlein (1999), Gehrlein and Fishburn (1978a, b), Gehrlein and Lepelley (2014, 1999), Gehrlein and Roy (2014), and Gehrlein and Valognes (2001), among others.

1.2 Our Contribution

The central concern of this chapter is to deal with the problem of the CE of some common voting rules when the results of an election are closely contested. Specifically, we focus on the following WSRs and WSERs voting rules: PR, PER, NPR, NPER, and BR. Up to our knowledge, only a little attention was paid to this issue in the literature. Recently, Miller (2017) studied, in the context of three-candidate elections, the effect of closeness on the occurrence of the monotonicity paradox that is, getting more points from voters can make a candidate a loser or getting fewer points can make a candidate a winner. The main results obtained by Miller (2017) reveal that the probability of this paradox can be very high under PER when the results of the elections become very close. Lepelley et al. (2018) extended the previous study to other WSERs and showed that the probability of occurrence of monotonicity paradox remains also very high for BER and NPER. To measure the election closeness, Miller (2017) and Lepelley et al. (2018) considered the ratio between the lowest score and the sum of the scores of all competing candidates. It seems from these research papers that closeness deserves more consideration in social choice theory.

In what follows, we derive an exact representation for the CE as a function of the same closeness index that was used by Miller (2017) and Lepelley et al. (2018). Our results show that closeness also matters in our context since it affects negatively the CE of the five considered voting rules. However, the effect of the closeness varies through the considered voting rules. Specifically, the CE of BR remains relatively stable with the range of the closeness index when compared to the one of PR, PER, NPR, and NPER.

The chapter is organized as follows. In Sect. 2, we introduce the basic notation and definitions. In Sect. 3, we derive the analytical representations for the CE of the five considered WSRs and WSERs and discuss our results. Section 4 summarizes our findings, and to end, the proofs are presented in the Appendix.

2 Preliminaries

Throughout this chapter, we consider $n \geq 2$ voters in three-candidate elections and denote by $\mathbb{C} = \{a, b, c\}$ the set of candidates. Each voter is endowed with a linear preference ordering on the candidates, i.e., she is able to rank the set of candidates from the most desirable one to the least desirable one. We also assume that voters vote according to their true preferences, which means that the strategic behaviors are not allowed in this chapter. In this setting, there are six possible linear preference rankings that voters might have on $\mathbb{C}$:

Ranking	#	Ranking	#	Ranking	#
$a \succ b \succ c$	n_1	$a \succ c \succ b$	n_2	$b \succ a \succ c$	n_3
$b \succ c \succ a$	n_4	$c \succ a \succ b$	n_5	$c \succ b \succ a$	n_6

(1)

The notation $a \succ b \succ c$ in (1) means that the most preferred candidate of a voter is a, the middle-ranked candidate is b, and the least preferred one is c. A voting situation can be defined by the 6-tuple $\tilde{n} = (n_1, n_2, \ldots, n_6)$, where n_i denotes the number of voters endowed with the associated i^{th} preference ranking, such that $\sum_{i=1}^{6} n_i = n$. Let us denote by $a\mathbf{M}b$ the event that candidate a defeats b in a pairwise majority comparison, i.e., when more voters are endowed with $a \succ b$ in their preference rankings than with $b \succ a$. A Condorcet winner exists in a voting situation if there is a candidate who would be able to defeat any other opponent in pairwise majority comparisons. For instance, candidate a is a Condorcet winner if both $a\mathbf{M}b$ and $a\mathbf{M}c$ hold, which is equivalent to respectively $n_1 + n_2 + n_5 > n_3 + n_4 + n_6$ and $n_1 + n_2 + n_3 > n_4 + n_5 + n_6$ following our notation in (1). It is well known that such a candidate does not necessarily exist which means that cycles of types $a\mathbf{M}b$, $b\mathbf{M}c$ and $c\mathbf{M}a$ or $b\mathbf{M}a$, $a\mathbf{M}c$ and $c\mathbf{M}b$ can be observed in our framework. The Condorcet efficiency of any given voting rule is the conditional probability that such a voting rule selects the Condorcet winner, given that such a candidate exists.

In our setting of three-candidate elections, Weighted Scoring Rules (WSRs) can be represented by the vector of weights $(1, \lambda, 0)$ such that $0 \leq \lambda \leq 1$. In other words, each of the n voters assigns 1 point to her most preferred candidate, λ points to her middle-ranked candidate, and 0 points to her least preferred candidate. A candidate's score is the total number of points summed over all voters, and the winner is the candidate with the highest total score from the voters. Let $S(a)$, $S(b)$ and $S(c)$ be the scores of candidates a, b, and c, respectively, under the WSR with weights $(1, \lambda, 0)$. Taking into account our notation in (1), the scores of the candidates a, b, and c are the following:

$$S(a) = n_1 + n_2 + \lambda\,(n_3 + n_5) \tag{2}$$

$$S(b) = n_3 + n_4 + \lambda\,(n_1 + n_6) \tag{3}$$

$$S(c) = n_5 + n_6 + \lambda\,(n_2 + n_4) \tag{4}$$

To illustrate how a WSR works, let us assume that candidate a is the winner under the WSR with weights $(1, \lambda, 0)$. In such a case, $S(a)$ has to be greater than both $S(b)$ and $S(c)$, i.e., $n_1 + n_2 + \lambda\,(n_3 + n_5) > n_3 + n_4 + \lambda(n_1 + n_6)$ and $n_1 + n_2 + \lambda\,(n_3 + n_5) > n_5 + n_6 + \lambda\,(n_2 + n_4)$, respectively. Moreover, Weighted Scoring Elimination Rules (WSERs) can also be represented by the vector of weights $(1, \lambda, 0)$ in three-candidate elections. The two-stage election process works as follows: at the first step, the lowest scored candidate under the corresponding WSR is eliminated; in the second step, the candidate, with the highest number of votes between the two remaining candidates, wins. For instance, assuming that candidate c is the last ranked one under the WSR with weights $(1, \lambda, 0)$, the candidate a will be the winner under the corresponding WSER if the three following inequalities hold: $n_1 + n_2 + \lambda\,(n_3 + n_5) > n_5 + n_6 + \lambda\,(n_2 + n_4)$, $n_3 + n_4 + \lambda\,(n_1 + n_6) > n_5 + n_6 + \lambda\,(n_2 + n_4)$ and $n_1 + n_2 + n_5 > n_3 + n_4 + n_6$.

In this chapter, we focus on the following well-known WSRs and WSERs:

1. Plurality Rule (PR) which is the WSR with $\lambda = 0$. PR counts the number of times that each candidate is first ranked, and the winner is the candidate that gets the highest number of first ranks.
2. Plurality Elimination Rule (PER) which is the WSER with $\lambda = 0$. In the first step, the candidate with the fewest number of votes under PR is eliminated; in the second step, the winner is the candidate with the highest number of votes between the two remaining candidates.
3. Negative Plurality Rule (NPR) which is the WSR with $\lambda = 1$. NPR counts the number of times that each candidate is ranked last, and the winner is the candidate that gets the fewest number of last ranks.
4. Negative Plurality Elimination Rule (NPER) which is the WSER with $\lambda = 1$. This two - stage voting rule operates in the same fashion as PER, with NPR being used in the first step to determine the candidate to be excluded from further consideration.
5. Borda Rule (BR) which is the WSR with $\lambda = \frac{1}{2}$. Under this voting rule, voters assign one point to their most preferred candidate, one-half point to their middle-ranked candidate, and zero points to their least preferred candidate. The winner is the candidate who receives the greatest total number of points from the voters.

As mentioned before, BER is another well-known WSER. More precisely, in a three-candidate election, BER is defined as a two-step voting rule with BR being used in the first stage following the same reasoning as PER and NPER. However, as it is shown in Gehrlein and Lepelley (2015), this rule is the only WSER that guarantees the selection of the Condorcet winner when such a candidate exists. Consequently, the study of this rule is out of the scope of this chapter.

In order to measure the closeness of an election, recall that the closeness index that we consider is the ratio between the score of the last ranked candidate and the sum of the scores of all competing candidates. Notice that the considered index increases when elections become closer, reaching the value of $\frac{1}{3}$ when the three candidates obtain approximately the same score. Clearly, PR and PER share the same closeness index since the two voting rules use the same weight, $\lambda = 0$. This is also true when considering NPR and NPER where $\lambda = 1$.

Without loss of generality, assume that candidate c is last ranked under the considered voting rule and let α_1, α_2, and α_3 denote the closeness indices of PR/PER, NPR/NPER, and BR, respectively. Taking into account our notation in (1), the closeness indices α_1, α_2, and α_3 are computed as follows:

$$\alpha_1 = \frac{n_5 + n_6}{n} \tag{5}$$

$$\alpha_2 = \frac{n_2 + n_4 + n_5 + n_6}{2\,n} \tag{6}$$

$$\alpha_3 = \frac{2\,(n_5 + n_6) + n_2 + n_4}{3\,n} \tag{7}$$

It is easy to check that $0 \leq \alpha_i \leq \frac{1}{3}$, with $i = 1, 2, 3$, and recall that the objective of this chapter is to derive the effect of closeness on the CE of PR, PER, NPR, NPER, and BR. To find our probabilities, we need to assume a probability distribution that underlies how individual preferences are considered. The probabilities that we investigate are driven by the well-known Impartial Anonymous Culture (IAC) condition (Gehrlein and Fishburn 1976). In our three-candidate setting, it states that all voting situations $\tilde{n} = (n_1, n_2, \ldots, n_6)$, such that $\sum_{i=1}^{6} n_i = n$, for a specified number of voters n are equally likely to be observed. As long as we take into account only voting events where elections are supposed to be closely contested, we define the α_i-IAC assumption, where α is the closeness index and $i = 1, 2, 3$, based on the IAC condition as follows: all possible voting situations $\tilde{n} = (n_1, n_2, \ldots, n_6)$ having a concrete value of α_i are equally likely to be observed. To derive our probabilities, we use the *parameterized Barvinok's algorithm* (see, for instance, Verdoolaege et al. 2004; Bruynooghe et al. 2005; Lepelley et al. 2008). This algorithm allows us to compute the number of integer solutions for systems of inequalities with parameters. The representation of this number is given by *quasi-polynomials* with periodic coefficients (see, for instance, Ehrhart 1962, 1967). Further results based on this algorithm are provided by Bubboloni et al. (2020), Diss (2015), Diss et al. (2018), Diss et al. (2012), Diss and Pérez-Asurmendi (2016), Kamwa (2013, 2017), Kamwa and Valognes (2017), among others. For our concern of large electorates, it is possible to obtain the representation of the CE of the considered WSRs and WSERs as a function of the corresponding closeness index α_i, with $i = 1, 2, 3$.

3 Results and Discussions

Proposition 1 provides the CE of PR under the α_1-IAC assumption as a function of its closeness index α_1.

Proposition 1 *Consider a three-candidate election with large electorates and α_1 the proportion of points obtained by the last ranked candidate over the total number of points under PR. Then, the CE of PR under the α_1-IAC assumption is given as follows:*

$$CE_{PR}^{\infty}(\alpha_1) = \begin{cases} \dfrac{72\,{\alpha_1}^3 - 22\,{\alpha_1}^2 - 27\,\alpha_1 + 8}{8(11\,{\alpha_1}^3 - 4\,{\alpha_1}^2 - 3\,\alpha_1 + 1)} & \textit{for } 0 \leq \alpha_1 < \dfrac{1}{4} \\[2ex] \dfrac{(2\,\alpha_1 - 1)\left(12\,{\alpha_1}^2 - 15\,\alpha_1 + 5\right)}{4(18\,{\alpha_1}^3 - 18\,{\alpha_1}^2 + 6\,\alpha_1 - 1)} & \textit{for } \dfrac{1}{4} \leq \alpha_1 \leq \dfrac{1}{3} \end{cases}$$

The proof of Proposition 1 is presented in the Appendix. Proposition 2 provides the CE of PER under the α_1-IAC assumption as a function of its closeness index α_1.

Proposition 2 *Consider a three-candidate election with large electorates and α_1 the proportion of points obtained by the last ranked candidate over the total number of points under PR. Then, the CE of PER under the α_1-IAC assumption is given as follows:*

$$CE_{PER}^{\infty}(\alpha_1) = \begin{cases} \dfrac{10\,\alpha_1{}^3 - 4\,\alpha_1{}^2 - 3\,\alpha_1 + 1}{11\,\alpha_1{}^3 - 4\,\alpha_1{}^2 - 3\,\alpha_1 + 1} & \text{for } 0 \le \alpha_1 < \dfrac{1}{4} \\[2ex] \dfrac{(2\,\alpha_1 - 1)\left(6\,\alpha_1{}^2 - 3\,\alpha_1 + 1\right)}{18\,\alpha_1{}^3 - 18\,\alpha_1{}^2 + 6\,\alpha_1 - 1} & \text{for } \dfrac{1}{4} \le \alpha_1 \le \dfrac{1}{3} \end{cases}$$

Notice that $CE_{PR}^{n}(0) = 1$ for all n. To show this, let us assume, without loss of generality, that a is the Condorcet winner and c is the last ranked candidate under PR. If $\alpha_1 = 0$, then $n_5 = n_6 = 0$. Therefore, the scores of the candidates a and b under PR are $S(a) = n_1 + n_2$ and $S(b) = n_3 + n_4$, respectively. Since a beats b in pairwise majority comparisons ($n_1 + n_2 > n_3 + n_4$), we have $S(a) > S(b)$. In such a case, candidate a will be elected under PR with absolute certainty. Similarly, it is also possible to show that $CE_{PER}^{n}(0) = 1$ for any given number of voters n. In Fig. 1, we represent graphically the results from Propositions 1 and 2. Specifically, we illustrate the CE of PR and PER according to their closeness index α_1. Clearly, closeness significantly affects the CE of both voting rules. The CEs of PR and PER tend to dramatically decline as the election becomes closely contested. Notice that, in both cases, the decrease is stronger when α_1 belongs to the interval $[\frac{1}{4}, \frac{1}{3}]$. Nevertheless, the decrease is larger for PR than for PER. In the case of PR, the CE tends to a value of $\frac{1}{3}$ whereas in the case of PER, the CE tends to a value of $\frac{2}{3}$. In other words, PER remains more Condorcet consistent than PR over all the range of the closeness index α_1.

Proposition 3 provides the CE of NPR under the α_2-IAC as a function of the closeness index α_2.

Proposition 3 *Consider a three-candidate election with large electorates and α_2 the proportion of points obtained by the last ranked candidate over the total number of points under NPR. Then, the CE of NPR under the α_2-IAC assumption is given as follows:*

$$CE_{NPR}^{\infty}(\alpha_2) = \begin{cases} \dfrac{5\,\alpha_2 - 4}{8\,(2\,\alpha_2 - 1)} & \text{for } 0 \le \alpha_2 < \dfrac{1}{4} \\[2ex] \dfrac{510\,\alpha_2{}^3 - 510\,\alpha_2{}^2 + 165\,\alpha_2 - 17}{2\,(144\,\alpha_2{}^3 - 144\,\alpha_2{}^2 + 48\,\alpha_2 - 5)} & \text{for } \dfrac{1}{4} \le \alpha_2 \le \dfrac{1}{3} \end{cases}$$

Proposition 4 gives the CE of NPER under the α_2-IAC as a function of the closeness index α_2.

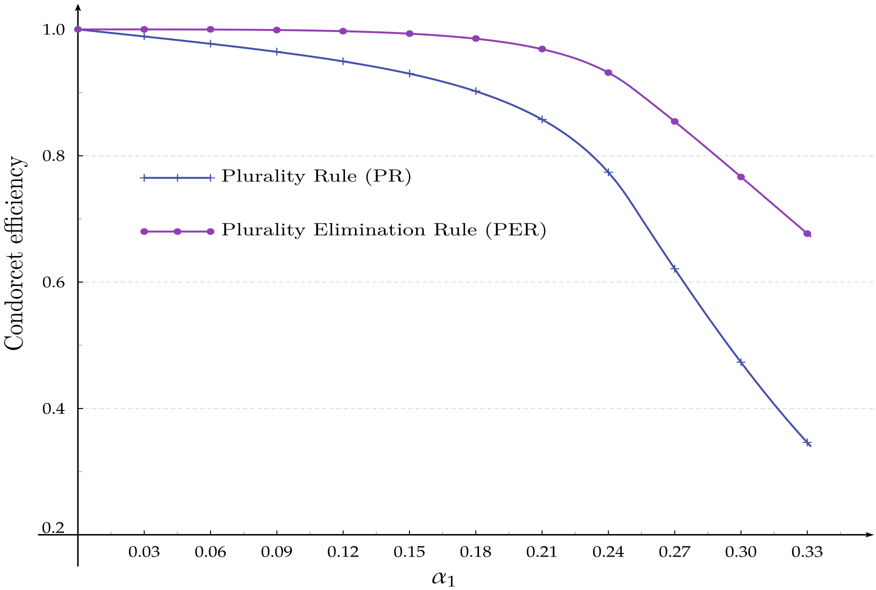

Fig. 1 Condorcet efficiency of PR and PER as a function of their closeness index for large electorates

Proposition 4 *Consider a three-candidate election with large electorates and* α_2 *the proportion of points obtained by the last ranked candidate over the total number of points under NPR. Then, the CE of NPER under the* α_2*-IAC assumption is given as follows:*

$$CE^{\infty}_{NPER}(\alpha_2) = \begin{cases} 1 & \textit{for } 0 \leq \alpha_2 < \dfrac{1}{4} \\ \dfrac{2\,(120\,\alpha_2{}^3 - 120\,\alpha_2{}^2 + 39\,\alpha_2 - 4)}{144\,\alpha_2{}^3 - 144\,\alpha_2{}^2 + 48\,\alpha_2 - 5} & \textit{for } \dfrac{1}{4} \leq \alpha_2 \leq \dfrac{1}{3} \end{cases}$$

Notice that for any given number of voters n, $CE^{n}_{NPR}(0) = \frac{1}{2}$. Indeed, when $\alpha_2 = 0$ (i.e., $n_2 = n_4 = n_5 = n_6 = 0$) and $\lambda = 1$, the scores of candidates a, b, and c are given by $S(a) = S(b) = n_1 + n_3$ and $S(c) = 0$. Thus, NPR will elect a and b with the same probability due to the symmetry of IAC-like assumptions with respect to candidates. This means that $CE^{n}_{NPR}(0) = \frac{1}{2}$. Notice also that for any given number of voters n, $CE^{n}_{NPER}(\alpha_2) = 1$ for $0 \leq \alpha_2 < \frac{1}{4}$. To prove this statement, suppose that the Condorcet winner, say a, is not elected under NPER. This implies that a is eliminated in the first round under NPR; otherwise, she would win the election since she is supposed to be the Condorcet winner. In addition, α_2 is supposed to be less than $\frac{1}{4}$, which implies that $n_1 + n_2 + n_3 + n_5 < n_4 + n_6$ (i). Since $a\mathbf{M}b$, then we can show that $n_3 + n_4 + n_6 + n_3 < n_1 + n_2 + n_3 + n_5$ (ii). From (i) and (ii), we

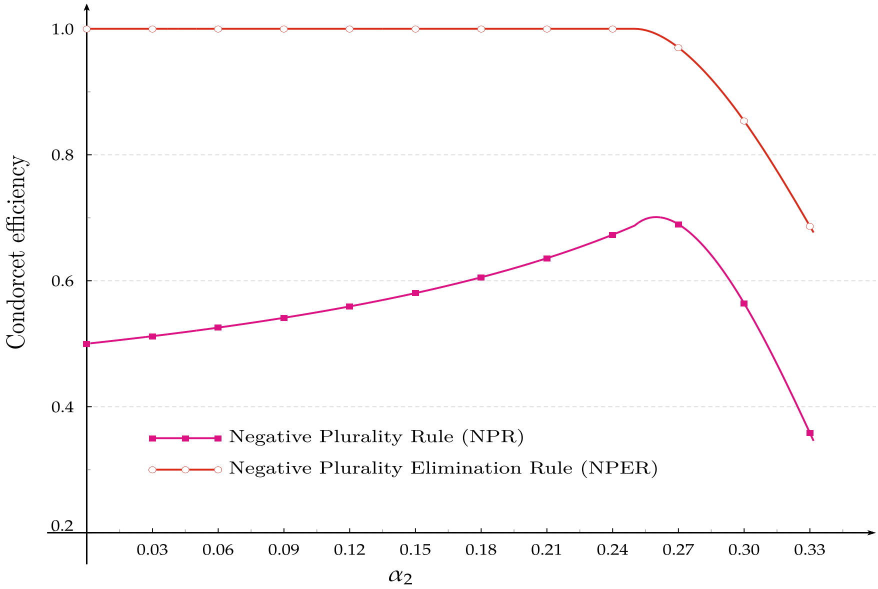

Fig. 2 Condorcet efficiency of NPR and NPER as a function of their closeness index for large electorates

deduce that $n_3 + n_4 + n_6 + n_3 < n_4 + n_6$, which implies that $n_3 < 0$. Because of this contradiction, candidate a will be elected with absolute certainty under NPER.

We plot the results from Propositions 3 and 4 in Fig. 2 supplying a graphical representation of the CE of NPR and NPER as a function of the closeness index α_2. From Fig. 2, it is clear that the performance of NPER in terms of the CE is significantly better than the one of NPR independently on the value of α_2. More specifically, the CE of NPER takes values in the interval $]\frac{2}{3}, 1]$ whereas in the case of NPR the CE values are located in the range $]\frac{1}{3}, 0.6875]$. Indeed, we note that the maximum of the CE for NPR over all range of the closeness index is $0.6875 = \frac{11}{16}$. This value is reached when the closeness index takes the value $\alpha_2 = 0.2500$. Recall that in the case of NPER the CE reaches the value of 1 over the range $0 \leq \alpha_2 \leq \frac{1}{4}$; for values of α_2 greater than $\frac{1}{4}$, that is, as elections are very close, we found that the CE decreases until the value of $\frac{2}{3}$. In the case of NPR, the behavior of the CE is slightly different for lower values of α_2. To be more concrete, it increases from 0 to 0.2500 and decreases from 0.2500 to $\frac{1}{3}$.

Finally, Proposition 5 provides the CE of BR under the α_3-IAC assumption as a function of the closeness index α_3.

Proposition 5 *Consider a three-candidate election with large electorates and α_3 the proportion of points obtained by the last ranked candidate over the total number of points under BR. Then, the CE of BR under the α_3-IAC assumption is given as follows:*

$$CE^{\infty}_{BR}(\alpha_3) = \begin{cases} \dfrac{21\,\alpha_3 - 8}{2\,(9\,\alpha_3 - 4)} & \text{for } 0 \le \alpha_3 < \dfrac{1}{9} \\[2ex] \dfrac{243\,{\alpha_3}^4 + 324\,{\alpha_3}^3 - 486\,{\alpha_3}^2 + 36\,\alpha_3 - 1}{648\,{\alpha_3}^3\,(9\,\alpha_3 - 4)} & \text{for } \dfrac{1}{9} \le \alpha_3 < \dfrac{1}{6} \\[2ex] \dfrac{70227\,{\alpha_3}^4 - 46332\,{\alpha_3}^3 + 11178\,{\alpha_3}^2 - 1260\,\alpha_3 + 53}{75816\,{\alpha_3}^4 - 49248\,{\alpha_3}^3 + 11664\,{\alpha_3}^2 - 1296\,\alpha_3 + 54} & \text{for } \dfrac{2}{9} \le \alpha_3 < \dfrac{5}{18} \\[2ex] \dfrac{20331\,{\alpha_3}^4 - 23436\,{\alpha_3}^3 + 9234\,{\alpha_3}^2 - 1500\,\alpha_3 + 89}{2\,(13608\,{\alpha_3}^4 - 15120\,{\alpha_3}^3 + 5832\,{\alpha_3}^2 - 936\,\alpha_3 + 55)} & \text{for } \dfrac{5}{18} \le \alpha_3 < \dfrac{2}{9} \\[2ex] \dfrac{25587\,{\alpha_3}^2 - 11466\,\alpha_3 + 1339}{16\,(1647\,{\alpha_3}^2 - 732\,\alpha_3 + 85)} & \text{for } \dfrac{2}{9} \le \alpha_3 \le \dfrac{1}{3} \end{cases}$$

It can be noticed that $CE^{n}_{BR}(0) = 1$ for any given number of voters n. To show this statement, let us assume, without loss of generality, that candidate a is the Condorcet winner while c is the last ranked candidate under BR. If $\alpha_3 = 0$, it follows that $n_2 = n_4 = n_5 = n_6 = 0$. In such a case, $S(a) - S(b) = \frac{n_1 - n_3}{3} > 0$ because a beats c in pairwise majority comparisons ($n_1 > n_3$). Since candidate c is supposed to receive zero points, candidate a is elected under BR with absolute certainty. It is also worth to note that the minimum value of the CE under BR over all range of the closeness index is 0.8983. This value is reached when the closeness index takes the value $\alpha_3 = 0.2103$. We represent graphically the results from Proposition 5 in Fig. 3. As it can be seen, the CE of BR ranges within the interval [0.8983, 1]. The CE decreases when the closeness index takes values from 0 to 0.2103 whereas it increases when the closeness index ranges from 0.2103 to $\frac{1}{3}$.

Finally, Table 2 shows computed values of $CE^{\infty}_{PR}(\alpha_1)$, $CE^{\infty}_{PER}(\alpha_1)$, $CE^{\infty}_{NPR}(\alpha_2)$, $CE^{\infty}_{NPER}(\alpha_2)$, and $CE^{\infty}_{BR}(\alpha_3)$ for various values of the closeness index α_i, with $i = 1, 2, 3$. As we can see from the previous graphical representations, the results in Table 2 show very different behaviors of the Condorcet efficiency of the five considered WSRs and WSERs with respect to their closeness index. For instance, the Condorcet efficiency of BR seems to be stable over all range of the closeness index α_3: Their values decrease from 1 to 0.9375. However, the Condorcet efficiency of the other studied rules tends to decrease substantially especially when the election results became very close: Their values decrease from 1 to 0.3333 for PR and from 0.5 to 0.3333 for NPR while the Condorcet efficiency of PER and NPER decreases from 1 to 0.6667 as the correspondent closeness index tends to $\frac{1}{3}$. Recall that BER always selects the CW when such a candidate exists for all range of the closeness index α_3 which means that the Condorcet efficiency of BER is equal to 1. Taking into consideration this remark, it is clear that the Condorcet efficiencies of the two-stage rules studied in this chapter are substantially greater than those of single-stage voting rules. This result remains true when the closeness assumption is not taken into consideration as it was previously shown by Gehrlein (1982).

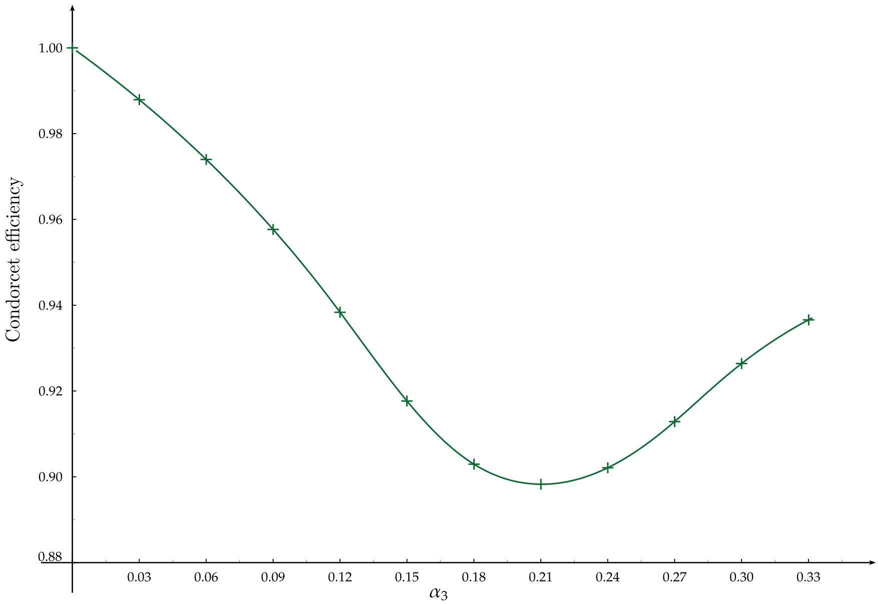

Fig. 3 Condorcet efficiency of BR as a function of its closeness index for large electorates

Table 2 Computed values of the CE of PR, BR, NPR, PER, and NPER for large electorates

α_i	$CE^{\infty}_{PR}(\alpha_1)$	$CE^{\infty}_{PER}(\alpha_1)$	$CE^{\infty}_{NPR}(\alpha_2)$	$CE^{\infty}_{NPER}(\alpha_2)$	$CE^{\infty}_{BR}(\alpha_3)$
0	1	1	0.5	1	1
0.02	0.9925	1.0000	0.5078	1	0.9921
0.04	0.9850	0.9999	0.5163	1	0.9835
0.06	0.9772	0.9997	0.5256	1	0.9740
0.08	0.9689	0.9993	0.5357	1	0.9634
0.10	0.9598	0.9985	0.5469	1	0.9516
0.12	0.9494	0.9971	0.5592	1	0.9384
0.14	0.9370	0.9948	0.5729	1	0.9243
0.16	0.9218	0.9911	0.5882	1	0.9118
0.18	0.9020	0.9852	0.6055	1	0.9030
0.20	0.8750	0.9756	0.6250	1	0.8988
0.22	0.8357	0.9596	0.6473	1	0.8987
0.24	0.7736	0.9315	0.6731	1	0.9021
0.25	0.7273	0.9091	0.6875	1	0.9050
0.26	0.6737	0.8821	0.7011	0.9917	0.9087
0.28	0.5698	0.8254	0.6606	0.9390	0.9175
0.3	0.4731	0.7665	0.5640	0.8537	0.9264
0.32	0.3857	0.7067	0.4320	0.7457	0.9336
$(\frac{1}{3})^{-}$	0.3333	0.6667	0.3333	0.6667	0.9375

4 Conclusion

The main purpose of this study was to provide new evidence of the effect of election closeness on the theoretical probability of electoral events. We considered the impact of election closeness on the probability of selecting the Condorcet winner, when such a candidate exists, under several well-known voting rules. In other words, the main objective of our study is to measure at which extend the CE of a given voting rule changes when the elections become more closely contested. To that end, we focussed on five popular WSRs and WSERs in the context of three-candidate elections. The first three rules, PR, NPR, and BR, choose the winner in one step while the last two rules, PER and NPER, do in a two-step iterative process. Election closeness is measured in our chapter by an index calculated as a proportion of points obtained by the last ranked candidate divided by the aggregated scores of all competing candidates under the given WSR/WSER. We followed an IAC-like assumption, by considering that every voting situation, with a given value of election closeness index, is equally likely to occur. As a result, we calculate the CE of the considered WSRs and WSERs for large electorates as a function of the corresponding closeness index. We show that the CE of some WSRs and WSERs may significantly decrease as the results of elections become very close. However, such a reduction varies depending on the considered voting rule; the CE does not substantially decrease under BR as it does in the case of the other analyzed WSRs and WSERs.

Finally, many open questions still remain unanswered given that we only studied the CE of some common WSRs and WSERs. For instance, the extension of our results to other voting rules remains open. In addition, we analyzed in this chapter the performance of several voting rules according to their CE but it is worthy to analyze the impact of election closeness on other interesting voting paradoxes. The reader can find an overview of different voting paradoxes that can be considered for this topic in Gehrlein and Lepelley (2011, 2017) and Nurmi (1999).

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5 Appendix

We only provide the proof of Proposition 1 which will allow the reader to understand the steps followed to derive the analytical representations for the CE under the α_i-IAC assumption. Complete proofs for the other considered voting rules are available upon request.

Let us assume, without loss of generality, that candidate c is the last ranked one under PR. In such a case, the closeness index is given by $\alpha_1 = \frac{n_5+n_6}{n} = \frac{k}{n}$ where

$k = n_5 + n_6$ is the score of candidate c under PR. Recall that the CE of a giving voting rule is a conditional probability. In order to compute the CE of PR under the α_1-IAC assumption, we first need to count the number of voting situations for which the Condorcet winner exists under the α_1-IAC assumption when candidate c is the last ranked one under PR. In order to accomplish this goal, we need to consider the three following independent events:

$X_1 =$ "a is the Condorcet winner, and c is last ranked under PR".
$X_2 =$ "b is the Condorcet winner, and c is last ranked under PR".
$X_3 =$ "c is the Condorcet winner, and c is last ranked under PR".

Let us denote by $|D_{X_j}(k,n)|$ the number of voting situations for which event X_j is observed under the α_1-IAC assumption, i.e., when $\alpha_1 = \frac{k}{n}$ takes a given value. The number $|D_{X_j}(k,n)|$ depends on the number of voters n and the score k of the candidate c under PR. Using those notations, the number of voting situations for which the Condorcet winner exists under the α_1-IAC assumption when candidate c is the last ranked one under PR can be written as follows:

$$|D_{X_1}(k,n)| + |D_{X_2}(k,n)| + |D_{X_3}(k,n)| \tag{8}$$

Due to the symmetry of IAC-like assumptions with respect to candidates, we can easily show that $|D_{X_1}(k,n)| = |D_{X_2}(k,n)|$. This means that the number of voting situations in (8) can also be written as follows:

$$2\,|D_{X_1}(k,n)| + |D_{X_3}(k,n)| \tag{9}$$

Thus, all that we have to do is to calculate $|D_{X_1}(k,n)|$ and $|D_{X_3}(k,n)|$. Notice first that $|D_{X_1}(k,n)|$ corresponds to the number of voting situations satisfying the following system of (in)equalities:

$$\begin{cases} n_1 + n_2 - n_5 - n_6 > 0 \\ n_3 + n_4 - n_5 - n_6 > 0 \\ n_5 + n_6 = k \\ n_1 + n_2 - n_3 - n_4 + n_5 - n_6 > 0 \\ n_1 + n_2 + n_3 - n_4 - n_5 - n_6 > 0 \\ n > 3k \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \\ n_i \geq 0 \text{ for } i \in \{1, \ldots, 6\} \\ k \geq 0 \end{cases} \tag{10}$$

As noticed before, we compute the number of voting situations that fulfill these conditions using the Parametrized Barvinok's algorithm. This algorithm allows us to quantify the number of integer solutions for systems of (in)equalities with parameters. In our study, given the two parameters n and k, the number of voting situations for the system (10) is provided by bivariate quasi-polynomials in n and k with 2-periodic coefficients meaning that such coefficients depend on the parity of the parameters n and k. We represent these coefficients by a list of two-rational numbers enclosed in square brackets. To illustrate, the coefficient $[a, b]_n$ will be either a when n is even or b when n is odd. The program indicates that the corresponding quasi-polynomial for the system (10) is given as follows:

1. If $\frac{n}{4} \leq k \leq \frac{n-2}{3}$:

$$|D_{X_1}(k,n)| = \frac{3}{2}k^4 + f_1 k^3 + f_2 k^2 + f_3 k + f_4$$

where,

$$f_1 = -2n + \left[\frac{7}{2}, 2\right]_n$$
$$f_2 = \frac{9}{8}n^2 + \left[-\frac{7}{2}, -\frac{3}{2}\right]_n + \left[1, \frac{7}{8}\right]_n$$
$$f_3 = -\frac{1}{3}n^3 + \left[\frac{9}{8}, \frac{1}{8}\right]_n n^2 + \left[-\frac{2}{3}, -\frac{7}{6}\right]_n n + \left[0, -\frac{5}{8}\right]_n$$
$$f_4 = \frac{1}{24}n^4 + \left[-\frac{1}{8}, \frac{1}{24}\right]_n n^3 + \left[\frac{1}{12}, \frac{5}{24}\right]_n n^2 + \left[0, -\frac{1}{24}\right]_n n + \left[0, -\frac{1}{4}\right]_n$$

2. If $0 \leq k \leq \frac{n-4}{4}$:

$$|D_{X_1}(k,n)| = \frac{5}{6}k^4 + g_1 k^3 + g_2 k^2 + g_3 k + g_4$$

where,

$$g_1 = -\frac{1}{3}n + \left[\frac{19}{6}, \frac{10}{3}\right]_n$$
$$g_2 = -\frac{1}{4}n^2 + \left[-\frac{11}{4}, -3\right]_n n + \left[\frac{5}{3}, \frac{17}{12}\right]_n$$
$$g_3 = \frac{1}{12}n^3 + \left[\frac{1}{8}, \frac{1}{4}\right]_n n^2 + \left[-3, -\frac{11}{4}\right]_n n + \left[-\frac{5}{3}, -\frac{19}{12}\right]_n$$
$$g_4 = \frac{1}{12}n^3 + \left[\frac{3}{8}, \frac{1}{2}\right]_n n^2 + \left[-\frac{7}{12}, -\frac{1}{12}\right]_n n + \left[-1, -\frac{1}{2}\right]_n$$

$|D_{X_3}(k,n)|$ corresponds to the number of voting situations satisfying the following system of (in)equalities:

$$\begin{cases} n_1 + n_2 - n_5 - n_6 > 0 \\ n_3 + n_4 - n_5 - n_6 > 0 \\ n_5 + n_6 = k \\ -n_1 + n_2 - n_3 - n_4 + n_5 + n_6 > 0 \\ -n_1 - n_2 - n_3 + n_4 + n_5 + n_6 > 0 \\ n > 3k \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \\ n_i \geq 0 \text{ for } i \in \{1, \ldots, 6\} \\ k \geq 0 \end{cases} \tag{11}$$

Using again the Parametrized Barvinok's algorithm, the program indicates that the corresponding quasi-polynomial for the system (11) is given as follows:

1. If $\frac{n-1}{4} \leq k \leq \frac{n-2}{3}$:

$$|D_{X_3}(k, n)| = \frac{3}{2}k^4 + h_1\, k^3 + h_2\, k^2 + h_3\, k + h_4$$

where,

$$\begin{aligned} h_1 &= -2\,n + \left[\frac{7}{2}, 2\right]_n \\ h_2 &= \frac{3}{4}\,n^2 + \left[-\frac{7}{2}, -3\right]_n n + \left[\frac{5}{2}, -\frac{1}{4}\right]_n \\ h_3 &= -\frac{1}{12}\,n^3 + n^2 + \left[-\frac{5}{3}, -\frac{11}{12}\right]_n n + \left[\frac{1}{2}, -1\right]_n \\ h_4 &= -\frac{1}{12}\,n^3 + \frac{1}{4}\,n^2 + \left[-\frac{1}{6}, \frac{1}{12}\right]_n n + \left[0, -\frac{1}{4}\right]_n \end{aligned}$$

2. If $0 \leq k \leq \frac{n-2}{4}$:

$$|D_{X_3}(k, n)| = \frac{1}{6}\,k^4 + \left[\frac{1}{6}, \frac{2}{3}\right]_n k^3 + \left[-\frac{1}{6}, \frac{5}{6}\right]_n k^2 + \left[-\frac{1}{6}, \frac{1}{3}\right]_n k$$

In order to calculate the CE of PR under the α_1-IAC assumption, the three following independent events have to be taken into consideration:

Y_1 = "a is the Condorcet winner, a is chosen under PR, and c is last ranked under PR".
Y_2 = "b is the Condorcet winner, b is chosen under PR, and c is last ranked under PR".
Y_3 = "c is the Condorcet winner, c is chosen under PR, and c is last ranked under PR".

It follows that the CE of PR under the α_1-IAC assumption is given in general by the following function in n and k:

$$\frac{|D_{Y_1}(k, n)| + |D_{Y_2}(k, n)| + |D_{Y_3}(k, n)|}{2\,|D_{X_1}(k, n)| + |D_{X_3}(k, n)|} \tag{12}$$

We can show that $|D_{Y_3}(k,n)| = 0$ because when candidate c is chosen under PR it cannot be last ranked by this voting rule. Again, due to the symmetry of IAC-like assumptions with respect to candidates, we can also show that $|D_{Y_1}(k,n)| = |D_{Y_2}(k,n)|$. It follows that the CE of PR under the α_1-IAC assumption in (12) can also be calculated as follows:

$$\frac{2\,|D_{Y_1}(k,n)|}{2\,|D_{X_1}(k,n)| + |D_{X_3}(k,n)|} \tag{13}$$

$|D_{Y_1}(k,n)|$ corresponds to the number of voting situations satisfying the following system of (in)equalities:

$$\begin{cases} n_1 + n_2 - n_5 - n_6 > 0 \\ n_3 + n_4 - n_5 - n_6 > 0 \\ n_1 + n_2 - n_3 - n_4 > 0 \\ n_5 + n_6 = k \\ n_1 + n_2 - n_3 - n_4 + n_5 - n_6 > 0 \\ n_1 + n_2 + n_3 - n_4 - n_5 - n_6 > 0 \\ n > 3k \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \\ n_i \geq 0 \text{ for } i \in \{1,\ldots,6\} \\ k \geq 0 \end{cases} \tag{14}$$

The program indicates that the corresponding quasi-polynomial of the system (14) is given as follows:

1. If $k \leq (n-4)/4$:

$$|D_{Y_1}(k,n)| = \frac{3}{4}k^4 + F_1\,k^3 + F_2\,k^2 + F_3\,k + F_4$$

where,

$$F_1 = -\frac{11}{48}n + \left[\left[\frac{19}{6}, \frac{53}{16}\right]_n, \left[\frac{79}{24}, \frac{51}{16}\right]_n\right]_k$$

$$F_2 = -\frac{9}{32}n^2 + \left[\left[-\frac{21}{8}, -\frac{23}{8}\right]_n, \left[-\frac{45}{16}, -\frac{43}{16}\right]_n\right]_k n + \left[\left[2, \frac{53}{32}\right]_n, \left[\frac{7}{4}, \frac{59}{32}\right]_n\right]_k$$

$$F_3 = \frac{1}{12}n^3 + \left[\left[\frac{1}{16}, \frac{3}{16}\right]_n, \left[\frac{1}{8}, \frac{1}{8}\right]_n\right]_k n^2 + \left[\left[-\frac{19}{6}, -\frac{35}{12}\right]_n, \left[-\frac{149}{48}, -\frac{137}{48}\right]_n\right]_k n + \left[\left[-5/3, -\frac{27}{16}\right]_n, \left[-\frac{43}{24}, -\frac{21}{16}\right]_n\right]_k$$

$$F_4 = \frac{1}{12}n^3 + \left[\left[\frac{3}{8}, \frac{1}{2}\right]_n, \left[\frac{13}{32}, \frac{13}{32}\right]_n\right]_k n^2 + \left[\left[-\frac{7}{12}, -\frac{1}{12}\right]_n, \left[-\frac{25}{48}, -\frac{19}{48}\right]_n\right]_k n + \left[\left[-1, -\frac{1}{2}\right]_n, \left[-1, -\frac{23}{32}\right]_n\right]_k$$

2. If $\frac{n-3}{4} \leq k \leq \frac{n-3}{3}$:

$$|D_{Y_1}(k,n)| = \frac{3}{4}k^4 + G_1 k^3 + G_2 k^2 + G_3 k + G_4$$

where,

$$G_1 = -\frac{25}{16}n + \left[\left[\frac{1}{2}, -\frac{11}{16}\right]_n, \left[\frac{5}{8}, -\frac{13}{16}\right]_n\right]_k$$
$$G_2 = \frac{39}{32}n^2 + \left[\left[-\frac{5}{8}, \frac{9}{8}\right]_n, \left[-\frac{13}{16}, \frac{21}{16}\right]_n\right]_{kn} + \left[\left[0, \frac{5}{32}\right]_n, \left[-\frac{1}{4}, \frac{11}{32}\right]_n\right]_k$$
$$G_3 = -\frac{5}{12}n^3 + \left[\left[\frac{5}{16}, -\frac{9}{16}\right]_n, \left[\frac{3}{8}, -\frac{5}{8}\right]_n\right]_k n^2 + \left[\left[\frac{1}{6}, -\frac{1}{12}\right]_n, \left[\frac{11}{48}, -\frac{1}{48}\right]_n\right]_k n$$
$$+ \left[\left[0, \frac{1}{16}\right]_n, \left[-\frac{1}{8}, \frac{7}{16}\right]_n\right]_k$$
$$G_4 = \frac{5}{96}n^4 + \left[-\frac{1}{16}, \frac{1}{12}\right]_n n^3 + \left[\left[-\frac{1}{12}, -\frac{1}{48}\right]_n, \left[-\frac{5}{96}, -\frac{11}{96}\right]_n\right]_k n^2 + \left[\left[0, -\frac{1}{12}\right]_n, \left[\frac{1}{16}, -\frac{19}{48}\right]_n\right]_k n$$
$$+ \left[\left[0, -\frac{1}{32}\right]_n, \left[0, -\frac{1}{4}\right]_n\right]_k$$

3. Otherwise, $|D_{Y_1}(k,n)| = 0$.

Notice that it is possible to represent the above results as functions of the closeness index α_1. If we assume large electorates and replace k by $\alpha_1 n$ in the above results, we obtain functions in α_1 by only considering the terms of higher degree in each function. Let us then denote by $|D^{\infty}_{X_j}(\alpha_1)|$ the number of voting situations for which event X_j is observed under the α_1-IAC assumption with large electorates. It follows that:

$$|D^{\infty}_{X_1}(\alpha_1)| = \begin{cases} \dfrac{\alpha_1\left(10\alpha_1{}^3 - 4\alpha_1{}^2 - 3\alpha_1 + 1\right)n^4}{12} & \text{for } 0 \leq \alpha_1 \leq \dfrac{1}{4} \\ \dfrac{(3\alpha_1 - 1)(2\alpha_1 - 1)\left(6\alpha_1{}^2 - 3\alpha_1 + 1\right)n^4}{24} & \text{for } \dfrac{1}{4} \leq \alpha_1 \leq \dfrac{1}{3} \end{cases} \quad (15)$$

$$|D^{\infty}_{X_3}(\alpha_1)| = \begin{cases} \dfrac{\alpha_1^4 n^4}{6} & \text{for } 0 \leq \alpha_1 \leq \dfrac{1}{4} \\ \dfrac{\alpha_1(3\alpha_1 - 1)\left(6\alpha_1{}^2 - 6\alpha_1 + 1\right)n^4}{12} & \text{for } \dfrac{1}{4} \leq \alpha_1 \leq \dfrac{1}{3} \end{cases} \quad (16)$$

$$|D_{Y_1}^{\infty}(\alpha_1)| = \begin{cases} \dfrac{\alpha_1\left(72\,\alpha_1{}^3 - 22\,\alpha_1{}^2 - 27\,\alpha_1 + 8\right)\, n^4}{96} & \text{for } 0 \le \alpha_1 \le \dfrac{1}{4} \\[2ex] \dfrac{(3\,\alpha_1 - 1)\,(2\,\alpha_1 - 1)\; n^4}{96} & \text{for } \dfrac{1}{4} \le \alpha_1 \le \dfrac{1}{3} \end{cases} \tag{17}$$

By replacing (15), (16), and (17) in (13) we derive the CE of PR as a function of the closeness index α_1 for large electorates as follows:

$$CE_{PR}^{\infty}(\alpha_1) = \begin{cases} \dfrac{72\,\alpha_1{}^3 - 22\,\alpha_1{}^2 - 27\,\alpha_1 + 8}{8(11\,\alpha_1{}^3 - 4\,\alpha_1{}^2 - 3\,\alpha_1 + 1)} & \text{for } 0 \le \alpha_1 < \dfrac{1}{4} \\[2ex] \dfrac{(2\,\alpha_1 - 1)\left(12\,\alpha_1{}^2 - 15\,\alpha_1 + 5\right)}{4(18\,\alpha_1{}^3 - 18\,\alpha_1{}^2 + 6\,\alpha_1 - 1)} & \text{for } \dfrac{1}{4} \le \alpha_1 \le \dfrac{1}{3} \end{cases}$$

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Analyzing the Practical Relevance of the Condorcet Loser Paradox and the Agenda Contraction Paradox

Felix Brandt, Christian Geist, and Martin Strobel

Abstract A large part of the social choice literature studies voting paradoxes in which seemingly mild properties are violated by common voting rules. In this chapter, we investigate the likelihood of the Condorcet Loser Paradox (CLP) and the Agenda Contraction Paradox (ACP) using Ehrhart theory, computer simulations, and empirical data. We present the first analytical results for the CLP on four alternatives and show that our experimental results, which go well beyond four alternatives, are in almost perfect congruence with the analytical results. It turns out that the CLP—which is often cited as a major flaw of some Condorcet extensions such as Dodgson's rule, Young's rule, and MaxiMin—is of no practical relevance. The ACP, on the other hand, frequently occurs under various distributional assumptions about the voters' preferences. The extent to which it is real threat, however, strongly depends on the voting rule, the underlying distribution of preferences, and, somewhat surprisingly, the parity of the number of voters.

1 Introduction

A large part of the social choice literature studies voting paradoxes in which seemingly mild properties are violated by common voting rules. Moreover, there are a number of sweeping impossibilities, which entail that there exists no "optimal" voting rule that avoids all paradoxes. As a consequence, much of the research in social choice theory is concerned with whether a paradox can appear for a given voting rule or not. However, it turns out that some paradoxes—while possible in principle—will almost never appear in practice.

F. Brandt (✉) · C. Geist
Institut für Informatik, Technische Universität München, 85748 München, Germany
e-mail: brandtf@in.tum.de

C. Geist
e-mail: geist@in.tum.de

M. Strobel
School of Computing, National University of Singapore, Singapore 117417, Singapore
e-mail: mstrobel@comp.nus.edu.sg

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An extreme example of this phenomenon occurred for the voting rule *TEQ* (Schwartz 1990). Due to its unwieldy recursive definition, it was unknown for more than 20 years whether *TEQ* satisfies any of a number of very basic desirable properties. In 2013, Brandt et al. (2013) have shown that *TEQ* violates all of these properties. However, their proof is non-constructive and only shows the existence of astronomically large counterexamples requiring about 10^{136} alternatives. While smaller computer-generated counterexamples exist, extensive simulations have shown that these counterexamples are *extremely* rare and that *TEQ* satisfies the desirable properties for all practical purposes (Brandt et al. 2010). These findings motivated us to provide analytical, experimental, and empirical justifications for such statements.

In this chapter, we study two voting paradoxes. The first is the well-known Condorcet loser paradox (CLP), which occurs when a voting rule selects the Condorcet loser, an alternative that loses against every other alternative in pairwise majority contests. Perhaps surprisingly, this paradox affects some Condorcet extensions, i.e., voting rules that are guaranteed to select an alternative that *wins* against every other alternative in pairwise majority contests. Common affected Condorcet extensions are Dodgson's rule, Young's rule, and MaxiMin (Fishburn 1977). The second paradox, called agenda contraction paradox (ACP), occurs when removing losing alternatives changes the set of winners. There are only few voting rules that do not suffer from this paradox, one of them being the essential set (Dutta and Laslier 1999). In fact, all common voting rules that violate the CLP also violate the ACP.

In principle, quantitative results on voting paradoxes can be obtained via three different approaches. The analytical approach uses theoretical models to quantify paradoxes based on certain assumptions about the voters' preferences. Analytical results usually tend to be quite hard to obtain and are limited to simple—and often unrealistic—assumptions. The experimental approach uses computer simulations based on underlying stochastic models of how the preference profiles are distributed. Experimental results have less general validity than analytical results, but can be obtained for arbitrary distributions of preferences. Finally, the empirical approach is based on evaluating real-world data to analyze how frequently paradoxes actually occur or how frequently they would have occurred if certain voting rules had been used for the given preferences. Unfortunately, only very limited real-world data for elections is available.

Our main results are as follows.

Using Ehrhart theory, we compute upper bounds for the CLP as well as the exact probabilities under which the CLP occurs for MaxiMin when there are four alternatives and preferences are distributed according to the Impartial Anonymous Culture (IAC) distribution. This approach also yields the exact limit probabilities (for the CLP and the ACP) when the number of voters goes to infinity. To the best of our knowledge, these are the first analytical results for the CLP on four alternatives (which is the minimal number of alternatives for which the voting rules we consider exhibit the CLP).

For both the CLP and the ACP, we throughly analyze a variety of other settings with more alternatives and other stochastic preference models using computer simulations. For those settings in which the analytical approach is also feasible, our results are in almost perfect congruence with the analytical results. This is strong evidence for the accuracy of our simulation results.

It turns out that the CLP—which is often cited as a major flaw of some Condorcet extensions —is of no practical relevance. The maximum probability under all preference models we studied is 2.2% (for MaxiMin, three voters, four alternatives, and IAC). In more realistic settings, it is much lower. For Dodgson's rule, it never exceeds 0.01%. We did not find any occurrence of the paradox in real-world data, neither in the PrefLib library (Mattei and Walsh 2013) nor in millions of elections based on data from the Netflix Prize (Bennett and Lanning 2007).

The ACP, on the other hand, frequently occurs under various distributional assumptions about the voters' preferences. The extent to which it is real threat, however, strongly depends on the voting rule, the underlying distribution of preferences, and the parity of the number of voters. If the number of voters is much larger than the number of alternatives, less discriminating voting rules seem to fare better than more discriminating ones. For example, when there are 1,000 voters and four alternatives, the probability for the ACP under Copeland's rule and IAC is 9% while it occurs with a probability of 33% for Borda's rule. When there are fewer voters, the parity of the number of voters plays a surprisingly strong role. For example, if there are 6 alternatives, the ACP probability for Copeland's rule is 44% for 50 voters, but only 26% for 51 voters. These results are in line with the empirical data we analyzed.

2 Related Work

There is a huge body of research on the quantitive study of voting paradoxes. Gehrlein (2006) focusses on the non-existence of Condorcet winners, arguably the most studied voting paradox. An overview of many paradoxes with an analysis of group coherence is provided by Gehrlein and Lepelley (2011). On top of that, Gehrlein and Lepelley (2011, 2017) survey different tools and techniques that have been applied over the years for the quantitive study of voting paradoxes.

The analytical study of voting paradoxes under the assumption of IAC is most effectively done via Ehrhart theory, which goes back to the year 1962 and the French mathematician Eugène Ehrhart (Ehrhart 1962). Interestingly, parts of these results have been reinvented (in the context of social choice) by Huang and Chua (2000), before Ehrhart's original work was independently rediscovered for social choice by Wilson and Pritchard (2007) and Lepelley et al. (2008) more than forty years later.

Current research on the probability of voting paradoxes under IAC is based on algorithms that build upon Ehrhart's results, such as the algorithm developed by Barvinok (1994). For many years, these approaches were limited to cases with three or fewer alternatives. Recent advances in software tools and mathematical modeling enabled the study of elections with four alternatives. Bruns and Söger (2015) and Schürmann (2013) provide such results for Condorcet's paradox, the Condorcet efficiency of plurality and the similarity between plurality and plurality with runoff. Schürmann (2013) further shows how symmetries in the formulation of the paradoxes can be exploited to facilitate the corresponding computations. Finally, Bruns et al. (2019b) study Condorcet and Borda paradoxes, as well as the Condorcet efficiency of plurality voting with runoff.

For the CLP (sometimes also referred to as "Borda's paradox") many quantitive results are known (Gehrlein and Lepelley 2011; Diss and Gehrlein 2012), which are, however, limited to simple voting rules and scoring rules in particular. These results also include some empirical evidence for the paradox under plurality (Gehrlein and Lepelley 2011, p. 15) and suggest that it is an unlikely yet possible problem in practice. Interestingly, the CLP for *Condorcet extensions* has—to the best of our knowledge—only been considered by Plassmann and Tideman (2014). However, they restrict their analysis to the 3-alternative case and find that the CLP never occurs, which is unsurprising since provably four alternatives are required for the Condorcet extensions they considered. In a more recent work, Bubboloni et al. (2019) consider the probability of the CLP for extensions of MaxiMin to the committee selection setting.

The ACP appears to have received less attention in the quantitative literature on voting paradoxes. Some limit probabilities for scoring rules were obtained by Gehrlein and Fishburn (see Gehrlein and Lepelley 2011, pp. 282–284). Fishburn (1974) experimentally studied a variant of this paradox called "winner turns loser paradox" for Borda's rule under Impartial Culture. For Condorcet extensions, Plassmann and Tideman (2014) considered another variant of the ACP under a spatial model, but again limit their experiments to three alternatives. These few results already seem to indicate that the ACP might occur even under realistic assumptions. However, there are no results for more than three alternatives, Condorcet extensions, and the ACP in its full generality.

3 Models and Definitions

Let A be a set of m alternatives and $N = \{1, \ldots, n\}$ a set of voters. Each voter is equipped with a *(strict) preference relation* $\succ_i$, i.e., a complete, transitive, and asymmetric binary relation on A. We read $x \succ_i y$ as voter i (strictly) preferring alternative x to alternative y.

A *(preference) profile* (or an *election*) is an n-tuple of preference relations and will be denoted by $R := (\succ_1, \ldots, \succ_n)$. We will sometimes consider the restriction of $\succ_i$ to a subset of alternatives $B \subseteq A$, called an *agenda*. Such a restriction will be denoted by $R|_B := (\succ_1|_B, \ldots, \succ_n|_B)$.

3.1 Stochastic Preference Models

In this chapter we consider five of the most common stochastic preference models. These models vary in their degree of realism. Impartial culture (IC) and impartial anonymous culture (IAC), for example, are usually considered as rather unrealistic. However, the simplicity of these models enables the use of analytical tools that cannot be applied to the other models. IC and IAC typically yield higher probabilities for paradoxes than other preference models and can therefore be seen as worst-case estimates (see, e.g., Regenwetter et al. 2006). We only give informal definitions here; for more extensive treatments see, e.g., Critchlow et al. (1991) and Marden (1995).

Impartial culture. The most widely-studied distribution is the so-called *impartial culture (IC)*, under which every possible preference relation has the same probability of $\frac{1}{m!}$. Thus, every preference profile is equally likely to occur.

Impartial anonymous culture. In contrast to IC the *impartial anonymous culture (IAC)* is not based on the probabilities of individual preferences but on the probabilities of whole profiles. Under IAC one assumes that each possible *anonymous* preference profile on n voters is equally likely to occur. A more formal definition is given in Sect. 4.1.

Mallows-ϕ model. In *Mallows-ϕ model*, the distance to a reference ranking (or ground truth) is measured by means of the Kendall-tau distance[1] and a parameter ϕ is used to indicate the dispersion. The case of $\phi = 1$ means absolute dispersion and coincides with IC, the case $\phi = 0$ corresponds to no dispersion and every voter always picks the "true" ranking. We chose $\phi = 0.8$ to simulate voters with relatively bad estimates, which leads to situations in which paradoxes are more likely to occur.

Pólya-Eggenberger urn model. In the Pólya-Eggenberger *urn model*, each possible preference relation is represented by a ball in an urn from which individual preferences are drawn. After each draw, the chosen ball is put back and $\alpha \in \mathbb{N}_0$ new balls of the same kind are added to the urn. While the urn model subsumes both impartial culture ($\alpha = 0$) and impartial anonymous culture ($\alpha = 1$), we set $\alpha = 10$ to obtain a reasonably realistic interdependence of individual preferences.

Spatial model. In the *spatial model* alternatives and agents are placed in a multi-dimensional space uniformly at random and the agents' preferences are then determined by the Euclidean distances to the alternatives (closer alternatives are preferred to more distant ones). The spatial model is considered particularly realistic in political science where the dimensions are interpreted as different aspects of the alternatives (Tideman and Plassmann 2012). We chose the simple case of two dimensions for our analysis.[2]

[1]The Kendall-tau distance counts the number of pairwise disagreements.

[2]In a related study, Brandt and Seedig (2016) have found that the number of dimensions does not seem to have a large impact on the results as long as it is at least two.

3.2 Voting Rules

A *voting rule* is a function f that maps a preference profile to a non-empty *set of winners*. For a preference profile R, let $g_{xy} := |\{i \in N : x \succ_i y\}| - |\{i \in N : y \succ_i x\}|$ denote the *majority margin* of x against y. A very influential concept in social choice is the notion of a Condorcet winner, an alternative that wins against any other alternative in a pairwise majority contest. Alternative x is a *Condorcet winner (CW)* of a profile R if $g_{xy} > 0$ for all $y \in A \setminus \{x\}$. Conversely, alternative x is a *Condorcet loser (CL)* if $g_{yx} > 0$ for all $y \in A \setminus \{x\}$. Neither CWs nor CLs necessarily exist, but whenever they do they are unique. A voting rule f is called a *Condorcet extension* if $f(R) = \{x\}$ whenever x is the CW in R.

In the following paragraphs we briefly introduce the voting rules considered in this chapter.

Borda's Rule. Under *Borda's rule* each alternative receives from 0 to $|A| - 1$ points from each voter (depending on the position the alternative is ranked in). The alternatives with highest accumulated score win.

MaxiMin. The *MaxiMin* rule is only concerned with the highest defeat of each alternative in a pairwise majority contest. It yields all alternatives x which have the maximal value of $\min_{y \in A} g_{xy}$.

Young's Rule. *Young's rule* yields all alternatives that can be made a CW by removing a minimal number of voters.

Dodgson's Rule. *Dodgson's rule* selects all alternatives that can be made a CW by a minimal number of pairwise swaps of adjacent alternatives in the individual preference relations.

Tideman's Rule. *Tideman's rule* was introduced as an approximation of Dodgson's rule by Tideman (1987). It yields all alternatives x for which the sum of pairwise majority defeats $\sum_{y \in A} \max(0, g_{yx})$ is minimal.

Copeland's Rule. *Copelands's rule* selects all alternatives where the number of majority wins plus half the number of majority draws is maximal.

Essential Set. Consider the symmetric two-player zero-sum game G given by the skew-symmetric matrix with entries g_{xy} for all pairs of alternatives x, y. The *essential set* is the set of all alternatives that are played with positive probability in some mixed Nash equilibrium of G.[3]

Except for Borda's rule, all presented voting rules are in fact Condorcet extensions. While Borda's rule, MaxiMin, and the essential set can be computed efficiently, Young's rule and Dodgson's rule have been shown to be computationally intractable. The essential set is one of the few voting rules that do suffer from neither the CLP nor the ACP, and is merely included as a reference. For more formal definitions and computational properties of these rules, we refer to Brandt et al. (2016).

[3]These mixed equilibria are also known as *maximal lotteries* in probabilistic social choice.

3.3 Voting Paradoxes

In this chapter we focus on two voting paradoxes whose occurrence can be determined given a voting rule f and a preference profile R.

Let f be a voting rule. Formally, a *(voting) paradox* is a characteristic function that maps a preference profile to 0 or 1. In the latter case, we say the paradox *occurs* for voting rule f at profile R.

The *Condorcet Loser Paradox (CLP)* occurs when a voting rule selects the CL as a winner.

Definition 1 Given a voting rule f the *Condorcet loser paradox* CLP_f is defined as

$$\text{CLP}_f(R) = \begin{cases} 1 & \text{if } f(R) \text{ contains a CL} \\ 0 & \text{otherwise.} \end{cases}$$

The *agenda contraction paradox (ACP)* occurs when reducing the set of alternatives, by eliminating unchosen alternatives, influences the outcome of an election.

Definition 2 Given a voting rule f the *agenda contraction paradox* ACP_f is a paradox defined as

$$\text{ACP}_f(R) = \begin{cases} 1 & \text{if } f(R|_B) \neq f(R) \text{ for some } B \supseteq f(R) \\ 0 & \text{otherwise.} \end{cases}$$

4 Quantifying Voting Paradoxes

In this section we present the three general approaches for quantifying voting paradoxes: the analytical approach via Ehrhart theory, the experimental approach via computer simulations, and the empirical approach via real-world data.

4.1 Exact Analysis via Ehrhart Theory

Anonymous preference profiles only count the number of voters for each of the $m!$ possible rankings on m alternatives. An anonymous preference profile can hence be viewed as an integer point in a space of $d := m!$ dimensions. Formally, the set $S_{m,n}$ of anonymous preference profiles on m alternatives with n voters can be identified with the set of all integer points $z = (z_1, \ldots, z_{m!}) \in \mathbb{Z}^{m!}$ which satisfy

$$z_i \geq 0 \text{ for all } i \in \{1, \ldots, m!\}, \text{ and } \sum_{i=1}^{m!} z_i = n.$$

Under IAC each anonymous preference profile is assumed to be equally likely to occur. Hence, in order to determine the probability of a paradox *under IAC* it is enough to compute the number of points belonging to preference profiles in which the paradox occurs and compare them to the total number of points in $S_{m,n}$, which is known to be $|S_{m,n}| = \binom{m!+n-1}{m!-1}$.[4]

In this framework, many paradoxes X can be described with the help of linear constraints, i.e., the set of points belonging to the event can be described with the help of (in)equalities, a polytope. For variable n, this approach then describes a *dilated polytope* $P_n = nP := \{n\mathbf{x} : \mathbf{x} \in P\}$. Hence, we know that the probability of a paradox X_n under IAC is given by:

$$\mathbb{P}(X_n) = \frac{|nP \cap \mathbb{Z}^d|}{|S_{m,n}|}.$$

and we can determine the probability of (many) voting paradoxes under IAC by evaluating the function $L(P, n) := |nP \cap \mathbb{Z}^d|$, which describes the number of integer points inside the dilation nP. This can be done with the help of Ehrhart theory. Ehrhart (1962) was the first to show that $L(P, n)$ can be described by special functions, called quasi- or Ehrhart-polynomials. A function $f : \mathbb{Z} \to \mathbb{Q}$ is a *quasi-polynomial* of degree d and period q if there exists a list of q polynomials $f_i : \mathbb{Z} \to \mathbb{Q}$ $(0 \leq i < q)$ of degree d such that $f(n) = f_i(n)$ if $n \equiv i \mod q$.

Quasi-polynomials can be determined with the help of computer programs such as LATTE (De Loera et al. (2004)) or NORMALIZ (Bruns et al. (2019a)). Unfortunately, the computation of our quasi-polynomials is computationally very demanding, especially because the dimension of the polytopes grows super-exponentially in the number of alternatives. This limits analytical results under IAC to rather small numbers of alternatives. To the best of our knowledge, NORMALIZ is the only program which is able to compute polytopes corresponding to elections with up to *four* alternatives. And even NORMALIZ is not always able to compute the whole quasi-polynomial, but sometimes we had to resort to computing the leading coefficients only of the polynomial, which fortunately suffices for determining the limit probability of a paradox when the number of voters goes to infinity. The problem of calculating the limit probability is equivalent to computing the volume of polytopes, for which there are also other software solutions (e.g., Convex by Franz (2016))

An overview of our analytical findings obtained in this way is provided in Table 1.

[4] For most preference models other than IAC this approach does not work. While for specific combinations of (simple) distributions and voting rules there are some highly tailor-made computations in the literature (cf. Sect. 2), these have to be redesigned for each individual setting.

Table 1 Theoretical results obtained via Ehrhart theory (for four alternatives and under IAC)

Paradox	Voting rule(s)	Result
CLP	Condorcet extensions	upper bound ($\forall n \in \mathbb{N}$)
	MaxiMin	probability ($\forall n \in \mathbb{N}$)
	Tideman's rule	limit prob. ($n \to \infty$)
ACP	MaxiMin	limit prob. ($n \to \infty$)

4.2 Finding a Quasi-polynomial for MaxiMin

As an example for the method just described, we consider the $\text{CLP}_{\text{MaxiMin}}$ in 4-alternative elections under IAC, the probabilities of which can be computed from a quasi-polynomial with degree 23 and a period of 5,040.[5]

In order to determine the polynomial, we first need to describe the corresponding polytope with equalities and inequalities. Recall the definition of MaxiMin from Sect. 3.2:

$$f_{\text{MaxiMin}}(R) := \arg\max_{x \in A} \min_{y \in A} g_{xy}.$$

For $\text{CLP}_{\text{MaxiMin}}(R) = 1$ the CL of R has to have the lowest highest defeat. Formally, there is $x \in A$ such that for all $y \in A \setminus \{x\}$,

$$g_{yx} > 0, \text{ and} \tag{1}$$

$$\max_{z \in A \setminus \{x\}} g_{zx} \leq \max_{z \in A \setminus \{y\}} g_{zy}. \tag{2}$$

Now let $A = \{a, b, c, d\}$ and assume $x = d$. We then have that $g_{ad}, g_{bd}, g_{cd} > 0$, which implies $\max_{z \in A \setminus \{d\}} g_{zd} > 0$. Furthermore,

$$\max_{z \in A \setminus \{y\}} g_{zy} > 0 \text{ for all } y \in \{a, b, c\},$$

from which it follows that either $g_{ab}, g_{bc}, g_{ca} > 0$ or $g_{ba}, g_{cb}, g_{ac} > 0$. In both cases there is a majority cycle between a, b, and c. Due to symmetry we can choose one direction of the cycle arbitrarily and assume $g_{ab}, g_{bc}, g_{ca} > 0$. Then,

$$\max_{z \in A \setminus \{a\}} g_{za} = g_{ca}, \max_{z \in A \setminus \{b\}} g_{zb} = g_{ab}, \text{ and } \max_{x \in A \setminus \{c\}} g_{zc} = g_{ac}.$$

Condition (1) is already represented in the form of linear inequalities. In order to model condition (2) we determine $\max_{z \in A \setminus \{d\}} g_{zd}$ and distinguish cases for the seven possible outcomes. The inequalities for the case $\max_{z \in A \setminus \{d\}} g_{zd} = \{g_{ad}\}$ are

[5] In theory, the analysis can be adapted to also cover more complex rules (e.g., Dodgson's and Young's rule, which involve solving an integer linear program). It is unclear, however, how one would translate their definitions to linear inequalities.

$$g_{ad} - g_{bd} > 0 \quad \text{and} \quad g_{ad} - g_{cd} > 0.$$

Condition (2) furthermore yields

$$g_{ca} - g_{ad} \geq 0, \quad g_{ab} - g_{ad} \geq 0, \quad \text{and} \quad g_{bc} - g_{ad} \geq 0.$$

Each case belongs to a different polytope and the polytopes are pairwise distinct, so we can compute each quasi-polynomial separately and later combine them to one. To get the final polynomial we have to multiply by eight for the four different possible choices of a CL and the two possible directions of the majority cycle. This then enables us to efficiently evaluate the *exact* probabilities for any number of voters. The results are depicted in Fig. 2. The leading coefficient of the quasi-polynomial can also be used to determine the limit probability which is given by

$$\mathbb{P}(\text{CLP}_{\text{MaxiMin}} = 1 \mid m = 4, n \to \infty) = 8 \cdot \frac{485052253637930099}{6443662124777472000000} \approx 0.06\%.$$

4.3 *Experimental Analysis*

As we will see, simulating elections with the help of computers is a viable way of achieving very good approximations for the probabilities we are looking for. It even turns out that the results of our simulations are almost indistinguishable from the theoretical result obtained via Ehrhart theory (with the exception of the limit case, which cannot be realized via simulations).

More specifically, the experimental approach works as follows: a profile source creates random preference profiles according to a specific preference model. The profiles are then used to compute the winner(s) according to a given voting rule and to determine if the paradox occurs. Any such experiment is carried out for each pair of n and m and repeated frequently. In many cases in which we covered a wide range of voters, we did not consider every possible value of n but, more economically, only simulated the values: 1–30, 49–51, 99–101, 199–201, 499–501, 999–1,001.

Since we are particularly concerned about the statistical significance of our experimental results, we also computed 99%-confidence intervals for each data point we generated. To this end, we used the `binofit` function in MATLAB which is based on the standard approach by Clopper and Pearson (1934). It shows that, based on our sampling rate of 10^5 and 10^6, respectively, the 99%-confidence intervals are pleasantly small. Hence, even though they are depicted in all of the figures throughout this chapter, sometimes it can be difficult to recognize them.

4.4 Empirical Analysis

The most valuable quantification of voting paradoxes would be their actual frequency in real-world elections. As mentioned before, real-world election data is generally relatively sparse, incomplete, and inaccurate. This makes empirical research on this topic rather difficult. Otherwise, the empirical approach strongly resembles the experimental approach.

For this chapter we used two sources of empirical data. First, we used the 314 profiles with strict order preferences from the PrefLib library (Mattei and Walsh 2013). Second, we had access to the 54,650 preference profiles over four alternatives without a CW which belong to the roughly 11 million 4-alternative elections which Mattei et al. (2012) derived from the Netflix Prize data (Bennett and Lanning 2007). Non-existence of Condorcet winners is a prerequisite for the paradoxes we study.

5 Condorcet Loser Paradox

In this section we present our findings on the CLP. We conclude that—even though the CLP is possible in principle—it is so unlikely that it cannot be used as a serious argument against any of the Condorcet extensions we considered.

5.1 An Upper Bound

Before analyzing the CLP for concrete voting rules, we discuss an upper bound valid for all Condorcet extensions. For a Condorcet extension to choose the CL a profile obviously has to satisfy two conditions. First, there has to exist a CL in the profile. Second, no CW may exist in the profile. In the case of 4-alternative elections—which is the first interesting case—we can compute the quasi-polynomial via Ehrhart theory and hence know the exact probabilities for any number of voters. Similar to the example in Sect. 4.1, we can assume that alternative d is the CL and obtain the inequalities $g_{ad}, g_{bd}, g_{cd} > 0$. The event that none of the remaining alternatives is the CW can be formalized as

$$(g_{ba} \geq 0 \vee g_{ca} \geq 0) \wedge (g_{ab} \geq 0 \vee g_{cb} \geq 0) \wedge (g_{ac} \geq 0 \vee g_{bc} \geq 0).$$

This leads to 27 satisfiable cases all belonging to disjoint polytopes, since $g_{xy} \geq 0$ and $\neg(g_{xy} \geq 0)$ are exclusive. Each quasi-polynomial can be computed separately and (attributing for the four different possible CLs) they can be combined to a single quasi-polynomial, which has degree 23 and contains 24 polynomials. The coefficients take up several pages and we omit them here. The resulting probabilities for up to

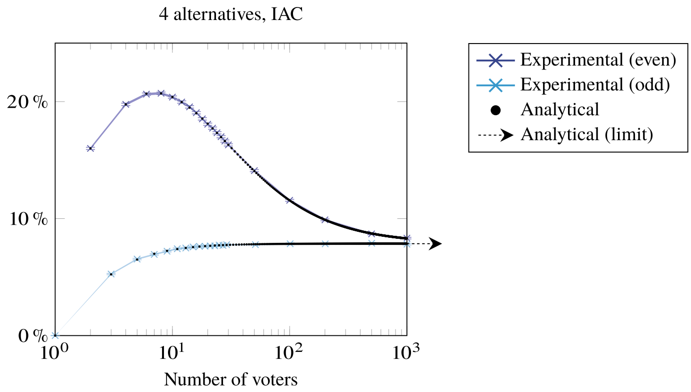

Fig. 1 Probability of the event that a Condorcet extension could choose a CL in 4-alternative elections under IAC

1, 000 voters—and a comparison with the results of an experimental analysis—can be obtained from Fig. 1. The value of the limit probability is approximately 8%.

Especially for small even numbers of voters, where the probability is around 20%, the upper bound is too high to discard the CLP for Condorcet extensions altogether, and even the limit probability of 8% is relatively large. Also, for an increasing number of alternatives this problem does not vanish (for elections with 50 and 51 one voters and up to 100 alternatives the probabilities range between 5 and 25%).[6]

Note that differences between odd and even number of voters were to be expected since even numbers allow for majority ties, which have significant consequences for the paradoxes; this effect decreases for larger electorates. In the specific case under consideration, the upper bound is generally higher for an even number of voters because the much higher likelihood of not having a CW more than counterbalances the lower likelihood of having a CL.

5.2 Results Under IAC

Despite the high upper bounds from the previous section, the picture is quite clear for concrete Condorcet extensions: even under IAC, the risk of the considered Condorcet extensions selecting the CL is very low, as shown in Fig. 2 for 4-alternative elections. The highest probability was found for CLP_{MaxiMin} with 2.2% for three voters (CLP_{Young} with about 0.9%). The limit probability of CLP_{MaxiMin}, with 0.06% is so low that for sufficiently large electorates it would occur in only one out of 10,000 elections. The same seems to hold for the limit probability for CLP_{Young}. The probability of CLP_{Dodgson} is even significantly lower, with a maximum of about 0.01% in

[6]These upper bounds turn out to be relatively independent from the underlying preference distribution (among the models we considered, cf. Sect. 5.3).

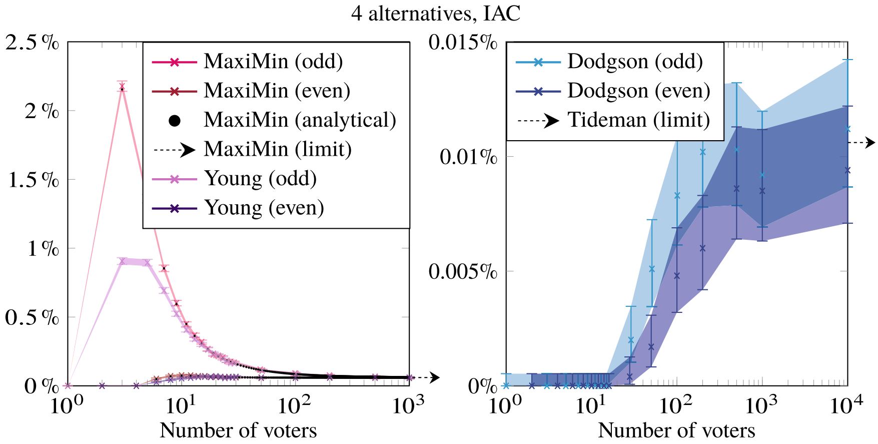

Fig. 2 Comparison between CLP probabilities for MaxiMin, Young's rule (left) and Dodgson's rule (right) under IAC in 4-alternative elections

elections with 9,999 voters. We could determine the limit probability of 0.01% only for an approximation of Dodgson's rule by Tideman (1987), which seems to be close to that for Dodgson's rule, based on our experimental data.

When increasing the number of alternatives the probabilities drop even further. For elections with more than ten alternatives they reach a negligibly small level of less than 0.005% for all considered rules and in no simulations with twelve or more alternatives we could find any occurrence of the paradox.

5.3 Results Under Other Preference Models

Figure 3, as one would expect, shows that under more realistic assumptions the probability of the CLP decreases further in 4-alternative elections with 50/51 voters, with the highest probability occurring under the unrealistic assumption of IC and the lowest probability under what may be the most realistic model in many settings, the spatial model. In our experiments, Dodgson's rule *never* selected a CL in the spatial model.

Similarly, we could not find any occurrence of the CLP in real-world data, which may be considered the strongest evidence that the CLP virtually never materializes in practice.[7]

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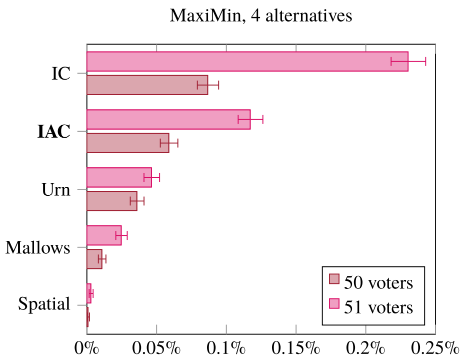
[7] We tested 314 preference profiles with strict orders from the PrefLib library as well as the roughly 11 million 4-alternative elections which Mattei et al. (2012) derived from the Netflix Prize data. While about 54,000 of those elections were susceptible to the CLP, it never occurred under the rules we considered in this chapter. In contrast, under plurality it already occurred in twelve out of the 314 PrefLib-instances.

Fig. 3 CLP probabilities in 4-alternative elections for varying preference models and MaxiMin



6 Agenda Contraction Paradox

Recall that the *agenda contraction paradox (ACP)* occurs when a reduced set of alternatives (created by the unavailability of losing alternatives) influences the outcome of an election. For many cases, it may be considered a generalization of the CLP as the following argument shows. Suppose the CL x is uniquely selected by a voting rule which implements majority rule on 2-alternative choice sets. Then restricting A to $\{x, y\}$ for some alternative $y \neq x$ yields the new winner y (since $g_{xy} > 0$).

As we will see, the ACP is much more of a practical problem than the CLP. The picture, however, is not black and white. Whether or not it is a serious threat depends on the voting rule, the underlying preference distribution, and on the parity of the number of voters.

6.1 Varying Voting Rules

The ACP probability strongly varies for different voting rules (see Fig. 4). Borda's rule generally exhibits the worst behavior of the rules studied, with probabilities of up to 56%, and with 34% for large electorates with 1, 000 voters. In contrast, Copeland's rule is quite robust to the ACP for large electorates (with only about 8% occurrence probability for 1, 000 voters).[8]

The reason for this gap between Borda's and Copeland's rule appears to be twofold: First, Condorcet extensions are safe from this paradox as long as a CW exists; Borda's rule, by contrast, is not. Second, the discriminatory power of voting rules (i.e., their ability to select small winning sets) strongly supports the paradox. As soon as a single majority-dominated alternative is selected, the ACP has to occur. For large numbers of voters, this is in line with Copeland's rule being least discriminating among those evaluated. The essential set is among the most discriminating

[8]For small *even* numbers of voters, Copeland's rule also frequently fails agenda contraction, which is also visible in Fig. 5 and explains the seemingly high values in Table 2.

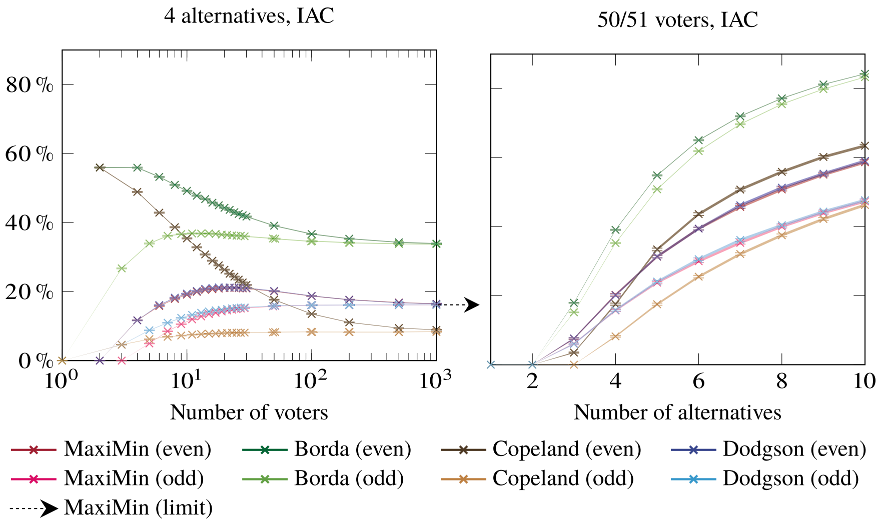


Fig. 4 Comparison between ACP probabilities for different voting rules under IAC

known voting rules immune to the ACP, but presumably less discriminating than Copeland's rule.

The behavior of MaxiMin is almost identical to that of Young's and Dodgson's rule. Confirming our approximate "limit" results of 1,000 voters, we were able to analytically compute the limit probability for MaxiMin as $\frac{331}{2048} \approx 16\%$. This is in perfect congruence with the (rounded) values for MaxiMin, Young's rule, and Dodgson's rule.

It should also be noted that with fewer than 100 voters, the parity of the number of voters plays a major role. For even numbers, significantly higher probabilities arise (which is particularly true for Copeland's rule, see above). At least part of this can be explained by a reduced probability for CWs in these cases.

For more alternatives (see the right-hand side of Fig. 4), the relative behavior remains vastly unchanged with probabilities further increasing to values larger than 40–80% (mostly since the likelihood of a CW decreases roughly at the same rate).

6.2 *Varying Preference Models*

Figure 5 extends the analysis of the previous section by additionally considering preference models beyond IAC. The overall picture regarding the different rules remains the same. For large electorates Copeland's rule outperforms the other rules, whereas Borda's rule performs worst. Regarding the different preference models, three classes emerge from Fig. 5.

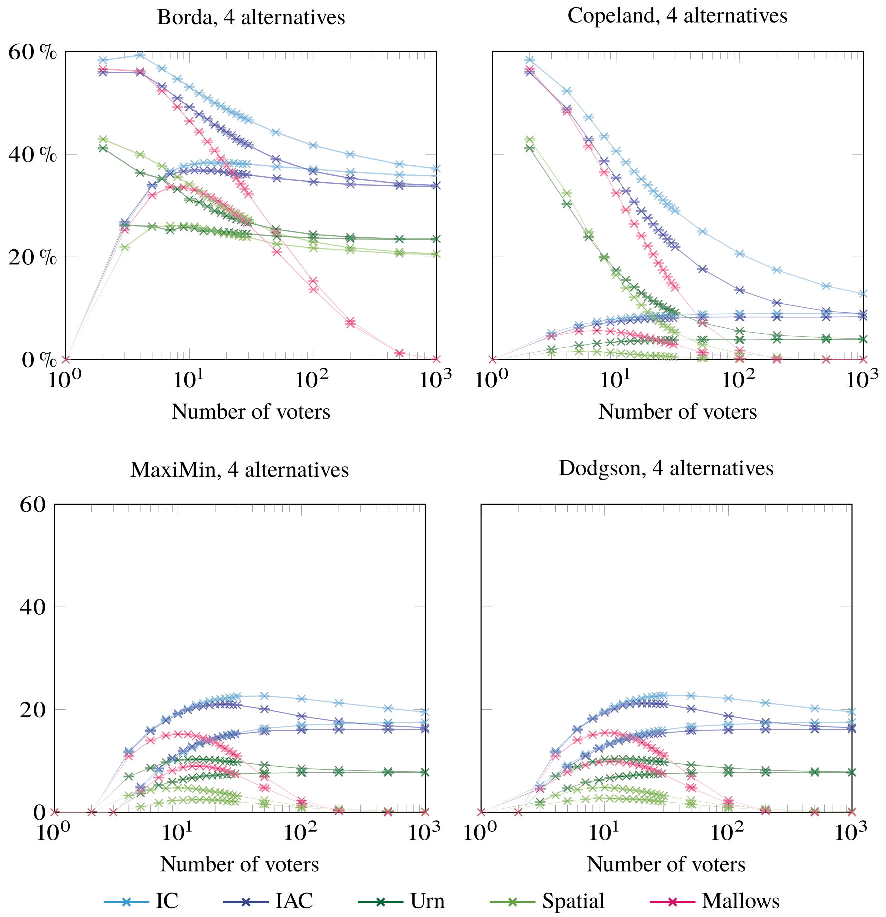


Fig. 5 Comparison between ACP_{Borda}, $ACP_{Copeland}$, $ACP_{MaxiMin}$, and $ACP_{Dodgson}$ for varying preference models in 4-alternative elections; the values of ACP_{Young} are omitted since they strongly resemble the ones of $ACP_{MaxiMin}$ and $ACP_{Dodgson}$

First, for Mallows-ϕ we observe probabilities that are vanishing with increased numbers of voters. Under the spatial model this is true as well, with the surprising exception of Borda's rule, for which the picture looks completely different and the probability does not go below 20% in the spatial model. Presumably, this can be explained by Borda's inability to select the CW in this setting, a hypothesis that deserves further study, however. On the contrary, the other rules appear to be benefitting from the fact that the existence of a CW becomes very likely under models with high voter interdependence.

Second, as expected, the assumption of IC serves as an upper bound for all other preference models. The results for IAC are not much lower, fostering the impression that IAC could also be an unrealistic upper bound.

Table 2 Rounded maximal CLP and ACP probabilities which occurred during our simulations

Paradox	Condorcet loser paradox (CLP)			Agenda contraction paradox (ACP)		
Model	IAC		IC	IAC		IC
n	$\{1, \ldots, 1000\}$	$\{50, 51\}$	$\{50, 51\}$	$\{1, \ldots, 1000\}$	$\{50, 51\}$	$\{1, \ldots, 1000\}$
m	4	$\{1, \ldots, 10\}$	4	4	$\{1, \ldots, 10\}$	4
Essential set	0%	0%	0%	0%	0%	0%
Borda	0%	0%	0%	56%	84%	59%
Copeland	0%	0%	0%	56%	63%	58%
Dodgson	0.01%	0.005%	0.005%	21%	59%	23%
Young	1%	0.15%	0.25%	21%	59%	23%
MaxiMin	2.2%	0.15%	0.25%	21%	59%	23%

Third, the urn model yields much lower values compared to IAC and IC. The absolute numbers, however, are still beyond acceptable levels (between 4 and 23% for 1,000 voters).

The findings in the empirical data corroborate our experimental findings. In PrefLib the ACP occurs 17 times for Borda, three times for Copeland and exactly once for MaxiMin as well as Young's and Dodgson's rule. In the Netflix data set, where the number of voters is at least 350, Copeland performs much better than the other Condorcet extensions (4, 400 compared to 18, 470 occurrences for the other Condorcet extensions). Borda's rule virtually always suffers from the ACP on this data set: there are 54, 620 instances of ACPs already when considering profiles that do not have a CW (there are 54, 650 of such).

7 Conclusion

We investigated the likelihood of the CLP and the ACP using Ehrhart theory, computer simulations, and empirical data. The CLP is often cited as a major flaw of some Condorcet extensions such as Dodgson's rule, Young's rule, and MaxiMin. For example, Fishburn regards Condorcet extensions that suffer from the CLP (specifically referring to the three rules mentioned above) as "'dubious' extensions of the basic Condorcet criterion" (Fishburn 1977, p. 480).[9] While this is intelligible from a theoretical point of view, our results have shown that the CLP is of virtually no practical concern. The ACP, on the other hand, frequently occurs under various distributional assumptions about the voters' preferences. The extent to which it is real threat, however, strongly depends on the voting rule, the underlying distribution of preferences, and, surprisingly, the parity of the number of voters. Our main quantitative results for the worst case are summarized in Table 2.

[9] Fishburn (1977) actually analyzes violations of "Smith's Condorcet principle", which are weaker than the CLP.

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Other Voting Paradoxes

On the Probability of the Ostrogorski Paradox

William V. Gehrlein and Vincent Merlin

1 Introduction

Suppose that there are two parties {R, L} that have opposing positions on each of m different issues that are being considered. Each of n voters has preferences on the individual issues that are in agreement with the position of either Party R or of Party L, but a given voter does not necessarily agree with the position of the same party on every issue. However, a given voter will be considered to have an overall alignment with a particular party whenever that voter is in agreement with that party's positions over a majority of the issues that are being considered. If a voter has preferences that agree with both Party R and Party L on $m/2$ issues each when m is even, then there is a tie and that voter is not considered to be aligned with either party. The majority party is the one with which the greater number of voters has an overall alignment, excluding voters that are not aligned with either party. Each issue is voted on individually, and the outcome of the vote will be in agreement with the position of either Party R or Party L, based on the outcome of majority rule voting on the issue. Voters are assumed to vote sincerely on each of the individual issues.

We are interested in developing representations for the probability, PA_m^k, that the voting results on the m different issues produce exactly k outcomes on issues that are in agreement with the positions of the majority party. PA_m^m is the probability that there is complete agreement between voting outcomes on issues and the majority party positions. PA_m^0 is the probability that there is complete disagreement between voting outcomes on issues and the positions of the majority party. The situation of complete disagreement is referred to as an occurrence of a *Strict Ostrogorski*

W. V. Gehrlein
Department of Business Administration, University of Delaware, Newark, DE 19716, USA

V. Merlin (✉)
Faculty of Economics, Management, and Geography, CNRS and Université de Caen Normandie, Caen, France
e-mail: vincent.merlin@unicaen.fr

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Paradox, loosely following some terminology that is introduced in Deb and Kelsey (1987). The general problem of such disagreement was originally presented in Ostrogorski (1902), and it was discussed in Daudt and Rae (1976). This phenomenon is often referred to in research that is related to paradoxical outcomes that can produce problems in group decision making. For example, see Brams et al. (1998), Nurmi (1999), Saari and Sieberg (2001), List (2005) and Gehrlein (2006). Laffond and Laine (2006) consider restrictions on voters' preferences that preclude the existence of an occurrence of Ostrogorski's Paradox. Mbih and Valeu (2016) derive the likelihood of related paradoxes, Ostrogorski's Paradox and Anscombe's Paradox, under the impartial anonymous culture assumption for $m = 3$ issues, as a function of n. They also provide estimations for these paradoxes under the impartial culture assumption via Monte-Carlo simulations for small values of n.

Values of PA_m^k for $0 < k < m$ represent probabilities of increasingly less paradoxical outcomes as k increases, with partial agreement between voting outcomes and the positions of the majority party. Attention is focused on developing these representations for the limiting case in voters, with $n \to \infty$, when voters form their preferences on each of the issues independently of the preferences of other voters. The study begins by considering specific cases with a small number of issues and then considers general results.

2 The Case of Three Issues

There are $2^3 = 8$ different possible combinations of voters' preference agreement with party positions on issues with $m = 3$. Let q_i denote the probability that a randomly selected voter will have the associated ith such combination for $1 \leq i \leq 8$, as shown in Table 1.

The probability of complete voter agreement with the positions of Party R (L) for a randomly selected voter is q_1 (q_8) in Table 1. Let $a = q_1 + q_8$ measure the probability that a randomly selected voter shows complete agreement with the issue positions of one of the parties. Such voters can be viewed as staunch supporters of one party's positions. There is greater degree of complete voter agreement with party positions as a increases, but the preferences of the voters will also reflect an increasing degree of polarization as a increases, since we further assume that $q_1 = q_8 = a/2$ so that neither party has any advantage over the other party. The motivation behind this

Table 1 Feasible voter preferences for sequential elections on three issues

	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8
Issue 1	R	R	R	R	L	L	L	L
Issue 2	R	R	L	L	R	R	L	L
Issue 3	R	L	R	L	R	L	R	L
Overall	R	R	R	L	R	L	L	L

party parity assumption will be explained in detail later. The remaining six rankings represent more moderate voters who show less than complete agreement with any party, with position agreement with some party on two issues and disagreement on one issue. The parity concept is also used here, since it is assumed that $q_i = (1 - a)/6$ for $2 \le i \le 7$.

We start the development of a representation for the limiting probability of complete agreement, $PA_3^3(a)$, as $n \to \infty$ for a specified value of a by defining a binary variable Y_i^j for the jth voter, to denote the party association of that voter's preference on the ith issue. For Issue 1:

$$\begin{aligned} Y_1^j &= +1 : q_1 + q_2 + q_3 + q_4 \\ &\quad - 1 : q_5 + q_6 + q_7 + q_8. \end{aligned} \tag{1}$$

This definition of Y_1^j requires that Issue 1 must have an outcome that is in agreement with Party R by majority rule for n voters whenever $\sum_{j=1}^{n} Y_1^j > 0$ or when the average value $\overline{Y_1^j} > 0$. The definitions of the q_i probabilities also lead to the expected values $E\left(Y_1^j\right) = 0$ and $E\left({Y_1^j}^2\right) = 1$ with party parity.

Issues 2 and 3 have corresponding binary variables Y_2^j and Y_3^j with

$$\begin{aligned} Y_2^j &= +1 : q_1 + q_2 + q_5 + q_6 \\ &\quad - 1 : q_3 + q_4 + q_7 + q_8 \\ Y_3^j &= +1 : q_1 + q_3 + q_5 + q_7 \\ &\quad - 1 : q_2 + q_4 + q_6 + q_8. \end{aligned} \tag{2}$$

As above, $E\left(Y_2^j\right) = E\left(Y_3^j\right) = 0$ and $E\left({Y_2^j}^2\right) = E\left({Y_3^j}^2\right) = 1$.

Variable Y_4^j is then defined to account for the party alignment of voter j, with

$$\begin{aligned} Y_4^j &= +1 : q_1 + q_2 + q_3 + q_5 \\ &\quad - 1 : q_4 + q_6 + q_7 + q_8. \end{aligned} \tag{3}$$

Party R will be the majority party if $\sum_{j=1}^{n} Y_4^j > 0$, or $\overline{Y_4^j} > 0$. We also note that $E\left(Y_4^j\right) = 0$ and that $E\left({Y_4^j}^2\right) = 1$ with the adopted definition of the q_i probabilities.

Issues 1, 2 and 3 will then each have outcomes that are in majority rule agreement with Party R, while Party R is the majority party with the joint probability that $\overline{Y_i^j} > 0$ for each $1 \le i \le 4$. The q_i definitions make this equivalent to the joint probability that $\overline{Y_i^j}\sqrt{n} > E\left(\overline{Y_i^j}\sqrt{n}\right)$ for $1 \le i \le 4$ since $E\left(Y_i^j\right) = E\left(\overline{Y_i^j}\right) = 0$ for $1 \le i \le 4$. The central limit theorem requires that the distribution of the $\overline{Y_i^j}\sqrt{n}$ variables is multivariate normal as $n \to \infty$, and the corresponding correlation terms for these variables are obtained directly from the $E\left(Y_i^j Y_k^j\right)$ of the original variables, since $E\left(Y_i^j\right) = 0$ and $E\left({Y_i^j}^2\right) = 1$ for all $1 \leq i \leq 4$, with:

$$\begin{aligned}
E\left(Y_1^j Y_2^j\right) &= q_1 + q_2 - q_3 - q_4 - q_5 - q_6 + q_7 + q_8 = (4a-1)/3 \\
E\left(Y_1^j Y_3^j\right) &= q_1 - q_2 + q_3 - q_4 - q_5 + q_6 - q_7 + q_8 = (4a-1)/3 \\
E\left(Y_2^j Y_3^j\right) &= q_1 - q_2 - q_3 + q_4 + q_5 - q_6 - q_7 + q_8 = (4a-1)/3 \\
E\left(Y_1^j Y_4^j\right) &= q_1 + q_2 + q_3 - q_4 - q_5 + q_6 + q_7 + q_8 = (2a+1)/3 \\
E\left(Y_2^j Y_4^j\right) &= q_1 + q_2 - q_3 + q_4 + q_5 - q_6 + q_7 + q_8 = (2a+1)/3 \\
E\left(Y_3^j Y_4^j\right) &= q_1 - q_2 + q_3 + q_4 + q_5 + q_6 - q_7 + q_8 = (2a+1)/3.
\end{aligned} \tag{4}$$

Since the probability of observing any specific value, including zero, in a continuous distribution is equal to zero, the limiting probability as $n \to \infty$ that Issues 1, 2 and 3 have outcomes that are in majority rule agreement with Party R, while Party R is the majority party is equivalent to the four-variate normal positive orthant probability, $\Phi_4(\boldsymbol{R}_1)$, that $\overline{Y_i^j}\sqrt{n} \geq E\left(\overline{Y_i^j}\sqrt{n}\right)$ for $1 \leq i \leq 4$ with correlation matrix $\mathbf{R}_1$:

$$\boldsymbol{R}_1 = \begin{bmatrix} 1 & \frac{4a-1}{3} & \frac{4a-1}{3} & \frac{2a+1}{3} \\ - & 1 & \frac{4a-1}{3} & \frac{2a+1}{3} \\ - & - & 1 & \frac{2a+1}{3} \\ - & - & - & 1 \end{bmatrix}. \tag{5}$$

The probability of having complete agreement for the case of three issues with two parties and a given a is therefore obtained from $PA_3^3(a) = 2\Phi_4(\boldsymbol{R}_1)$, and some results follow immediately. Since all correlation terms in $\boldsymbol{R}_1$ increase as a increases, a result from Slepian (1962) can be used to show that $PA_3^3(a)$ does not decrease as a increases. The correlation terms in $\boldsymbol{R}_1$ fit the special case for four-variate normal positive orthant probabilities in Gehrlein (1979), and it leads to a representation for $PA_3^3(a)$, with

$$\begin{aligned}
PA_3^3(a) = \frac{1}{8} + \frac{3}{4\pi}\left\{\sin^{-1}\left(\frac{2a+1}{3}\right) + \sin^{-1}\left(\frac{4a-1}{3}\right)\right\} \\
+ \frac{3}{2\pi^2}\int_0^{\frac{2a+1}{3}} \sqrt{\frac{1}{1-z^2}}\sin^{-1}\left(\frac{4a-1-3z^2}{4a+2-6z^2}\right)\mathrm{d}z.
\end{aligned} \tag{6}$$

For the specific case with $a = 1$, an exact result is obtained from direct integration of (6), with $PA_3^3(1) = 1$. Table 2 lists values of $PA_3^3(a)$ for each value of $a = 0.00(0.10)1.00$ that were obtained by numerical integration from (6). The computed

Table 2 Computed values of $PA_3^3(a)$, $PA_3^2(a)$, $PA_3^1(a)$ and $PA_3^0(a)$

A	$PA_3^3(a)$	$PA_3^2(a)$	$PA_3^1(a)$	$PA_3^0(a)$
0.00	0.0877	0.6491	0.2632	0.0000
0.10	0.1499	0.5971	0.2490	0.0040
0.20	0.2097	0.5527	0.2291	0.0085
0.25	0.2396	0.5312	0.2187	0.0104
0.30	0.2697	0.5098	0.2083	0.0121
0.40	0.3314	0.4665	0.1873	0.0148
0.50	0.3959	0.4215	0.1662	0.0164
0.60	0.4648	0.3735	0.1446	0.0170
0.70	0.5407	0.3207	0.1220	0.0166
0.80	0.6282	0.2598	0.0971	0.0149
0.90	0.7392	0.1823	0.0671	0.0114
1.00	1.0000	0.0000	0.0000	0.0000

value for $a = 0.25$ is also included since this corresponds to the situation in which all voter preference combinations are equally likely to be observed, which is referred to as the *impartial culture condition* (*IC*) in the general literature.

The voting outcomes on all three issues agree with the position of Party R under simple majority rule if $\overline{Y_i^j}\sqrt{n} \geq E\left(\overline{Y_i^j}\sqrt{n}\right)$ for all $i = 1, 2, 3$, and Party L is the majority party if $\overline{Y_4^j}\sqrt{n} \leq E\left(\overline{Y_4^j}\sqrt{n}\right)$. A representation for this probability of the Strict Ostrogorski Paradox with three issues, $PA_3^0(a)$, can be obtained quite easily by going through the analysis that was presented above while replacing variable Y_4^j with $-Y_4^j$, which leads to

$$PA_3^0(a) = \frac{1}{8} - \frac{3}{4\pi}\left\{\sin^{-1}\left(\frac{2a+1}{3}\right) - \sin^{-1}\left(\frac{4a-1}{3}\right)\right\} - \frac{3}{2\pi^2}\int_0^{\frac{2a+1}{3}}\sqrt{\frac{1}{1-z^2}}\sin^{-1}\left(\frac{4a-1-3z^2}{4a+2-6z^2}\right)\mathrm{d}z. \qquad (7)$$

An exact solution of (7) can be obtained for the special case of $a = 1$, with $P_3^0(1) = 0$. Computed values of $P_3^0(a)$ are listed in Table 2 for each $a = 0.00(0.10)1.00$ that were obtained from numerical integration from (7), along with the value for $a = 0.25$. These results indicate that the probability of observing a Strict Ostrogorski Paradox is quite small over the range of all possible values of a.

An alternative representation for $P_3^0(a)$ is found by using a different approach to the problem that follows Merlin and Tataru (1997), Saari and Tataru (1999) and Merlin et al. (2000, 2002), with

$$PA_3^0(a) = \frac{3}{2\pi^2}\int_0^a \left\{ \frac{2\cos^{-1}\left(\sqrt{\frac{2t+1}{8t+1}}\right)}{\sqrt{2+2t-4t^2}} - \frac{\cos^{-1}\left(\frac{t+1}{2t+1}\right)}{\sqrt{2-t-t^2}} \right\} dt. \tag{8}$$

This particular representation for $P_3^0(a)$ is useful, since it can be used to find that the maximum value of $P_3^0(a)$ exists at $P_3^0(0.60976) \approx 0.01702.$, which is not equivalent to the IC scenario.

For any a, the sum $PA_3^3(a) + PA_3^0(a)$ is simply obtained as two times the joint probability that $\overline{Y_i^j}\sqrt{n} \geq E\left(\overline{Y_i^j}\sqrt{n}\right)$ for all $i = 1, 2, 3$. This joint probability is the orthant probability of the multivariate normal distribution $\Phi_3(\boldsymbol{R}_2)$ where all correlation terms in $\boldsymbol{R}_2$ are obtainable from $\boldsymbol{R}_1$ and are equal to $(4a-1)/3$. Using the three-variate extension of Sheppard's Theorem of Median Dichotomy (see Johnson and Kotz 1972, p. 92), it is simple to show that

$$PA_3^3(a) + PA_3^0(a) = 2\Phi_3(\boldsymbol{R}_2) = \frac{1}{4} + \frac{3}{2\pi}\sin^{-1}\left(\frac{4a-1}{3}\right). \tag{9}$$

3 The Case of Partial Agreement on Three Issues

Party R will be the majority party, and Issue 1 will have the only election outcome that is in agreement with the position of Party L if $\overline{Y_1^j}\sqrt{n} \leq E\left(\overline{Y_1^j}\sqrt{n}\right)$ and if $\overline{Y_i^j}\sqrt{n} \geq E\left(\overline{Y_i^j}\sqrt{n}\right)$ for $2 \leq i \leq 4$. This is the same as the joint probability that $\overline{Y_i^j}\sqrt{n} \geq E\left(\overline{Y_i^j}\sqrt{n}\right)$ for $2 \leq i \leq 4$ minus the probability that $\overline{Y_i^j}\sqrt{n} \geq E\left(\overline{Y_i^j}\sqrt{n}\right)$ for $2 \leq i \leq 4$. The first probability can be obtained directly from the three-variate extension of Sheppard's Theorem, and the second probability is $\Phi_4(\boldsymbol{R}_1)$. We must also account for the fact that there are three issues that could be the single issue that is not in agreement with the majority party and for the fact that there are two parties that could be the majority party. After algebraic reduction, we get a representation for $PA_3^2(a)$ as

$$PA_3^2(a) = \frac{3}{8} + \frac{3}{4\pi}\left\{\sin^{-1}\left(\frac{2a+1}{3}\right) - \sin^{-1}\left(\frac{4a-1}{3}\right)\right\}$$
$$- \frac{9}{2\pi^2}\int_0^{\frac{2a+1}{3}} \sqrt{\frac{1}{1-z^2}}\sin^{-1}\left(\frac{4a-1-3z^2}{4a+2-6z^2}\right)dz. \tag{10}$$

An exact integral solution of (10) can be found for the special case of $a = 1$ with $PA_3^2(1) = 0$. Computed values of $PA_3^2(a)$ are listed in Table 2 for each $a =$

0.00(0.10)1.00 that were obtained from numerical integration of (10), along with the value for $a = 0.25$. A representation for $PA_3^1(a)$ is obtained from the identity relation $PA_3^1(a) = 1 - PA_3^3(a) - PA_3^2(a) - PA_3^0(a)$. After substituting the representations from (9) and (10) into this identity relation, algebraic reduction leads to

$$PA_3^1(a) = \frac{3}{8} - \frac{3}{4\pi}\left\{\sin^{-1}\left(\frac{2a+1}{3}\right) + \sin^{-1}\left(\frac{4a-1}{3}\right)\right\} + \frac{9}{2\pi^2}\int_0^{\frac{2a+1}{3}} \sqrt{\frac{1}{1-z^2}}\sin^{-1}\left(\frac{4a-1-3z^2}{4a+2-6z^2}\right)\mathrm{d}z. \quad (11)$$

An exact integral solution of (11) can be found for the special case of $a = 1$, with $PA_3^1(1) = 0$. Computed values of $PA_3^1(a)$ are listed in Table 2 for each $a =$ 0.00(0.10)1.00 that were obtained from numerical integration from (11), along with the value for $a = 0.25$.

4 The Impact of the Party Parity Assumption

Some discussion is in order regarding the impact of the party parity assumption that $q_1 - q_8 = a/2$ and $q_i = (1-a)/6$ for $2 \leq i \leq 7$. The possible voters' preferences on party positions on issues in Table 1 indicate that this assumption is equivalent to saying that the probability that any voter has a given set of preferences on issues is identical to the probability that the voter has preferences on issues with all of the R and L entries reversed. This leads to parity in voters' preferences for issues positions of Parties R and L such that voters are equally likely to have an overall party alignment with either party and the majority party is equally likely to be either party. Situations of this nature with a complete balance of outcome possibilities will obviously tend to exaggerate the probability that paradoxical events are observed for large electorates, since the introduction of any consistent bias that favors the position of either party on the issues will typically lead to a very high probability of complete agreement with the majority party position on issues as $n \to \infty$. However, such a parity situation is not a completely implausible scenario, despite the fact that it does represent an extreme case.

Other more extreme theoretical models can be developed to obtain significantly greater probabilities that a Strict Ostrogorski's Paradox is observed. For example, consider a scenario in which the q_i probabilities are obtained with the following process. Randomly generate two variables, δ and ε, from some probability distribution on the interval [0, 1/8], and let $\delta(\varepsilon)$ denote the propensity of voters to lean toward Party R (L) partisanship. Thus, voters are generally more disposed to favor the issue positions of Party R than Party L whenever $\delta > \varepsilon$. The q_i probabilities for a *partisanship model* can then defined on the basis of δ and ε, as shown in Table 3.

Table 3 Feasible voter preferences with a partisanship model

	$\frac{1}{8}+3\delta$	$\frac{1}{8}-\delta$	$\frac{1}{8}-\delta$	$\frac{1}{8}-\varepsilon$	$\frac{1}{8}-\delta$	$\frac{1}{8}-\varepsilon$	$\frac{1}{8}-\varepsilon$	$\frac{1}{8}+3\varepsilon$
Issue 1	R	R	R	R	L	L	L	L
Issue 2	R	R	L	L	R	R	L	L
Issue 3	R	L	R	L	R	L	R	L
Overall	R	R	R	L	R	L	L	L

The definitions from Table 2 lead to $E\left(Y_i^j\right) = 2(\delta - \varepsilon)$ for $i = 1, 2, 3$ and $E\left(Y_4^j\right) = 0$. If we suppose without a loss of generality that $\delta > \varepsilon$ as $n \to \infty$, then Party R will be the majority rule winner on all issues with probability approaching one. However, Party R will only be the majority party with probability 0.5 with this model since $E\left(Y_4^j\right) = 0$, so there is a very significant chance that a Strict Ostrogorski Paradox will be observed. However, the impact of this striking observation must be weighed against the relative degree of rationality that this partisanship model associates with the electorate.

Suppose that δ is significantly greater than ε so that we have a population that has a strong bias toward adopting the issue positions that are taken by Party R. A randomly selected voter is very predictably most likely to have preferences that are in complete agreement with Party R on all issues, which is quite a rational outcome for this model. Unfortunately, Table 3 then tells us that a randomly selected voter is least likely to have agreement with Party R on two out of three issues, suggesting an electorate that displays very odd behavior for a group that is supposedly predisposed to be highly favorable toward the issue positions of Party R. So, while it is possible to define such a theoretical model, it falls out of the realm of plausibility.

In the same vein, it is possible to develop other models that give a significantly large probability of observing other forms of an Ostrogorski Paradox by making assumptions about different intensities of importance that parties might place on the passage of the various issues that are being considered. While these models can indeed fall into the realm of plausibility, they typically rely on the assumption that the majority party has some subset of issues for which it takes a position, but where it has a low intensity of concern about the ultimate vote outcome. However, it would not be particularly paradoxical or disconcerting for the majority party if the minority party position won on such issues that are considered to be of little importance.

The party parity model that we use in the current study attempts to give an upper bound on the estimate of the paradox probabilities with a *not* implausible model that assumes that the parties take issue positions with a real concern about the voting outcome on the issues, without making any a priori assumptions that are intentionally creating a specific situation that is tailored to produce the paradoxical outcome that is being studied.

5 The Case of Two Issues

When there are only two issues that are being considered, there are four possible sets of preferences that each voter might have on issues, as listed in Table 4. The possibility of ties in determining the party alignment of voters exists with even m, and no designation for party alignment is made for the voter in such cases.

Let b denote the probability that there is complete agreement between a randomly selected voter's preference and the position of the same party on each issue. The party parity assumption is then applied with $t_1 = t_4 = b/2$ and $t_2 = t_3 = (1-b)/2$, and the special case with $b = 1/2$ is equivalent to IC.

Following the same general logic that led to the development of the representation for $PA_3^3(a)$, the probability $PA_2^2(b)$ is given by $2\Phi_3(\boldsymbol{R}_3)$, where

$$\boldsymbol{R}_3 = \begin{bmatrix} 1 & 2b-1 & \sqrt{b} \\ - & 1 & \sqrt{b} \\ - & - & 1 \end{bmatrix}. \tag{12}$$

As a result of Slepian (1962), $PA_2^2(b)$ does not decrease as b increases, and the three-variate extension of Sheppard's Theorem leads to

$$PA_2^2(b) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}(2b-1) + \frac{1}{\pi}\sin^{-1}\left(\sqrt{b}\right). \tag{13}$$

We note from identities for special cases that $PA_2^2(0) = 0$, $PA_2^2(1) = 1$ and $PA_2^2(1/4) = 1/3$. Computed values of $PA_2^2(b)$ from (13) are listed in Table 5 for each value of $b = 0.00(0.10)1.00$.

The results in Table 5 clearly show that the probability of complete agreement increases dramatically as b increases for the case of two issues.

Using the logic from earlier arguments, we obtain a representation for the probability, $PA_2^0(b)$, that the Strict Ostrogorski Paradox occurs with two items

$$PA_2^0(b) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}(2b-1) - \frac{1}{\pi}\sin^{-1}\left(\sqrt{b}\right). \tag{14}$$

Direct integration of (14) shows that $PA_2^0(0) = 0$, $PA_2^0(1) = 0$, and $PA_2^2(1/4) = 0$, which suggests that a Strict Ostrogorski Paradox cannot be observed when $m = 2$. This general result is proved rigorously in Corollary 1 of Deb and Kelsey (1980).

Table 4 Feasible voter preferences for sequential elections on two issues

	t_1	t_2	t_3	t_4
Issue 1	R	R	L	L
Issue 2	R	L	R	L
Overall	R	–	–	L

Table 5 Computed values of $PA_2^2(b)$ and $PA_2^1(b)$

b	$PA_2^2(b)$	$PA_2^1(b)$
0.00	0.0000	1.0000
0.10	0.2048	0.7952
0.20	0.2952	0.7048
0.30	0.3690	0.6310
0.40	0.4359	0.5641
0.50	0.5000	0.5000
0.60	0.5641	0.4359
0.70	0.6310	0.3690
0.80	0.7048	0.2952
0.90	0.7952	0.2048
1.00	1.0000	0.0000

We can use this observation with (13) and (14) to obtain a simpler representation for $PA_2^2(b)$ as

$$PA_2^2(b) = \frac{1}{2} + \frac{1}{\pi}\sin^{-1}(2b-1). \tag{15}$$

It then follows directly from earlier discussion that

$$PA_2^1(b) = \frac{1}{2} - \frac{1}{\pi}\sin^{-1}(2b-1). \tag{16}$$

Then, (15) and (16) require that $PA_2^2(b) = PA_2^1(1-b)$, as shown in Table 5.

6 The General Case of *M* Issues

The development of these types of representations becomes significantly more complex as the number of issues increases, since the number of possible combinations of voter agreements with party positions increases as 2^m. Attention is therefore restricted to the IC assumption when we consider $m \geq 4$. The initial analysis that was presented above for the two- and three-issue cases is generalized by defining m binary variables to determine if the position of Party R is the winner by majority rule on each issue. Variable Z_i^j takes a value of $+1$ (-1) when the jth voter is in agreement with the position of Party R (L) on the ith issue. The m variables are formally defined as

$$\begin{aligned} Z_i^j &= +1 : \text{For } j\text{th voter agreement with the position of Party R on Issue } i. \\ &\quad -1 : \text{For } j\text{th voter agreement with the position of Party L on Issue } i. \end{aligned} \tag{17}$$

There are an equal number of possible voters' party position agreement combinations in both the $+1$ and -1 categories in the variable definitions in (17). Since each possible combination is equally likely with IC, it then follows directly that $E\left(Z_i^j\right) = 0$ and $E\left({Z_i^j}^2\right) = 1$ for all $1 \leq i \leq m$.

In order to determine $E\left(Z_i^j Z_k^j\right)$ for $1 \leq I < k \leq m$, we partition the set of all possible combinations of voter agreements with party positions on issues into 2^{m-2} subsets of cardinality four such that each of Z_i^j and Z_k^j can have values of $+1$ or -1 in each subset, while the party agreements on issues on the remaining $m - 2$ issues are identical within each of the subsets. Obviously, $E\left(Z_i^j Z_k^j\right) = 0$ within each of these subsets with IC, so it follows that $E\left(Z_i^j Z_k^j\right) = 0$ over the entire set of all possible combinations of voter agreements with party positions. Let $\omega_{i,k}$ denote the correlation between variables Z_i^j and Z_k^j, and the fact that $E\left(Z_i^j\right) = 0$ for all $1 \leq i \leq m$ coupled with earlier discussion leads to the observation that $\omega_{i,k} = 0$ for all $1 \leq i \leq k \leq m$.

Variable Z_{m+1}^j is then defined to denote the contribution that the party alignment of the jth voter makes toward Party R being the majority party, with

$$\begin{aligned} Z_{m+1}^j &= +1 : \text{If } j\text{th voter is aligned with Party R} \\ &\quad 0 : \text{If } j\text{th has no party alignment} \\ &\quad -1 : \text{If } j\text{th voter is aligned with Party L.} \end{aligned} \tag{18}$$

Each of the possible combination of a voter's party agreement on issues can be paired with the equally likely combination in which the Party R and L positions are interchanged. Then, either both members of this pair do not have a party alignment or one is aligned with Party R while the other is aligned with Party L. It is therefore obvious that $E\left(Z_{m+1}^j\right) = 0$ with IC.

If m is odd, ties for party alignment of a voter cannot exist, so it must be true that

$$E\left({Z_{m+1}^j}^2\right) = 1, \text{ for odd } m. \tag{19}$$

When m is even, there are exactly $C_{m/2}^m$ different combinations of the 2^m possible voter agreements on the m issues for which a voter has no party alignment, with $Z_{m+1}^j = 0$. Since IC assigns an equally likely probability of $1/2^m$ to each possible combination,

$$E\left({Z_{m+1}^j}^2\right) = \frac{2^m - C_{m/2}^m}{2^m}, \text{ for even } m. \tag{20}$$

Consider the voters' party agreement on Issue h in the 2^m different possible combinations of voter agreements to obtain $E\left(Z_h^j Z_{m+1}^j\right)$. Half of these combinations have an alignment with Party R on Issue h, and this subset is denoted as S(R). There are C_i^{m-1} combinations of voter agreements on the remaining $m - 1$ party positions for issues in S(R) that will have exactly i issues in agreement with the position of Party L, and then Party R will then be the majority party if $0 \le i \le m/2$. The total number of combinations in S(R) for which Party R is the majority party is therefore given by #S(R), with

$$\#S(R) = \sum_{i=0}^{(m-1)/2} C_i^{m-1}, \text{for odd } m \tag{21}$$

$$\#S(R) = \sum_{i=0}^{(m-2)/2} C_i^{m-1}, \text{ for even } m. \tag{22}$$

The remaining half of the possible combinations have agreement with Party L on Issue h, and we denote this subset as S(L). There are C_i^{m-1} combinations that have exactly i issues in agreement with the position of Party L in the remaining $m - 1$ issues, and each such combination will have Party R as the majority party if $0 \le i \le (m - 2)/2$. The total number of combinations in S(L) for which Party R is the majority party is therefore given by #S(L), with

$$\#S(L) = \sum_{i=0}^{(m-3)/2} C_i^{m-1}, \text{ for odd } m \tag{23}$$

$$\#S(L) = \sum_{i=0}^{(m-4)/2} C_i^{m-1}, \text{ for even } m. \tag{24}$$

The value of variable Z_h^j will then be +1 [−1] for each combination of possible voter agreements in S(R)[S(L)]. The expected value $E\left(Z_h^j Z_{m+1}^j\right)$ with IC is then obtained from

$$E\left(Z_h^j Z_{m+1}^j\right) = [(+1)\#S(R) + (-1)\#S(L)]/2^m. \tag{25}$$

The correlation between Z_h^j and Z_{m+1}^j for all $1 \le h \le m$ follows from all of the above as

$$\omega_{h,m+1} = \frac{C_{(m-1)/2}^{m-1}}{2^{m-1}}, \text{ for all } 1 \le h \le m (\text{odd}) \tag{26}$$

$$\omega_{h,m+1} = \frac{C_{(m-2)/2}^{m-1}}{\sqrt{2^{m-2}\left(2^m - C_{m/2}^m\right)}}, \text{ for all } 1 \le h \le m (\text{even}). \tag{27}$$

Let $\boldsymbol{W}_{m+1}$ denote the correlation matrix for the $m+1$ variables that are defined in (17) and (18), with components $\omega_{i,j}$. The neutrality of IC toward the two parties gives the limiting probability of complete agreement with the majority party when $n \to \infty$ as $PA_m^m(IC) = 2\Phi_{m+1}(\boldsymbol{W}_{m+1})$.

7 The General Case of *M* Issues—Partial Agreement

Suppose that we are interested in the probability that there is nearly complete agreement, in the sense that only Issue 1 has a majority rule outcome for a party position that is in disagreement with the majority party. This would be determined by finding the resulting correlation matrix $\boldsymbol{W}_{m+1}^1$ where the signs of the variable values for Z_1^j are reversed, which would reverse the sign on all correlation terms in $\boldsymbol{W}_{m+1}$ that involve Z_1^j. As a result, it is still true that $\omega_{h,k}^1 = \omega_{h,k} = 0$ for all $1 \le h < k \le m$ and $\omega_{g,m+1}^1 = \omega_{g,m+1}$ for all $1 < g \le m$. The only difference between $\boldsymbol{W}_{m+1}^1$ and $\boldsymbol{W}_{m+1}$ is that $\omega_{1,m+1}^1 = -\omega_{1,m+1}$. With the neutrality of IC toward the two parties and the symmetry of IC with respect to the m possible issues that could be the single issue in disagreement with the majority party position, it follows that $PA_m^{m-1}(IC) = 2m\Phi_{m+1}\left(\boldsymbol{V}_{m+1}^1\right)$.

This logic can easily be extended to the general case in which exactly k issues have majority rule agreement with party positions that are in disagreement with the majority party positions. The correlation matrix that is applicable to the associated probability is $\boldsymbol{W}_{m+1}^k$, which comes from $\boldsymbol{W}_{m+1}$ simply by negating the $\omega_{i,m+1}$ correlation values for $1 \le i \le k$. The same probability value will be obtained, regardless of which specific set of k issues are selected to have their $\omega_{i,m+1}$ terms negated in order to obtain the $\omega_{i,m+1}^1$ values. There are C_k^m different sets of k issues, and there are two parties that could be the majority party, so $PA_m^{m-k}(IC) = 2C_k^m\Phi_{m+1}\left(\boldsymbol{W}_{m+1}^k\right)$.

This observation can be extended to produce some interesting results.

Theorem 1 $PA_m^{m-k}(IC) \ge PA_m^k(IC)$, for $0 \le k \le m/2$.

Proof Given the definition of $\boldsymbol{W}_{m+1}^k, \omega_{i,j}^k \ge \omega_{i,j}^{k*}$ for all $1 \le i < j \le m+1$ when $k < k*$. This observation is contingent upon the requirement that $\omega_{i,m+1} > 0$, which is true from (26) and (27). It then follows from Slepian (1962) that $\Phi\left(\boldsymbol{W}_{m+1}^k\right) \ge \Phi\left(\boldsymbol{W}_{m+1}^{k*}\right)$. Given that $C_k^m = C_{m-k}^m, 2C_k^m\Phi\left(\boldsymbol{W}_{m+1}^k\right) \ge 2C_{m-k}^m\Phi\left(\boldsymbol{W}_{m+1}^{m-k}\right)$ if $k \le m/2$. **QED**.

Theorem 2 $PA_m^{m-k}(IC) + PA_m^k(IC) = C_k^m\left(\frac{1}{2}\right)^{m-1}$ for $m \ge 2$ with $0 \le k \le m$.

Proof The limit probability $PA_m^{m-k}(IC)$ is obtained from the positive orthant probability $\Phi_{m+1}\left(\boldsymbol{W}_{m+1}^k\right)$, which is the probability that Party R is the majority party and that there are exactly k majority rule outcomes on issues that are in disagreement with the position of Party R. This orthant probability can alternatively be obtained as the difference in two probabilities. The first of these probabilities represents the situation in which there are exactly k majority rule outcomes on issues that are in disagreement with the position of Party R. This situation makes no determination of the majority party, and the associated correlation matrix $\boldsymbol{Z}_m$ on this joint distribution is obtained from $\boldsymbol{W}_{m+1}^k$ by removing all correlation terms that are related to Z_{m+1}^j. Given the definition of $\boldsymbol{W}_{m+1}^k$, all correlations in $\boldsymbol{Z}_m$ are equal to zero, which gives $\Phi_m(\boldsymbol{Z}_m) = \left(\frac{1}{2}\right)^m$.

We then subtract the second probability that there are exactly k majority rule outcomes on issues that are in disagreement with the position of Party R, when Party L is the majority party. This second probability is obtained by using the assumptions that led to the development of $\Phi_{m+1}\left(\boldsymbol{W}_{m+1}^k\right)$, except that the signs on variable Z_{m+1}^j are reversed. This reverses the signs on all correlation terms that involve Z_{m+1}^j and leads to an associated positive orthant probability that is equivalent to $\Phi_{m+1}\left(\boldsymbol{W}_{m+1}^{m-k}\right)$.

As a result, we find that

$$\Phi_{m+1}\left(\boldsymbol{W}_{m+1}^k\right) = \left(\frac{1}{2}\right)^m - \Phi_{m+1}\left(\boldsymbol{W}_{m+1}^{m-k}\right)$$

$$2C_k^m \Phi_{m+1}\left(\boldsymbol{W}_{m+1}^k\right) = C_k^m \left(\frac{1}{2}\right)^{m-1} - 2C_k^m \Phi_{m+1}\left(\boldsymbol{W}_{m+1}^{m-k}\right)$$

$$2C_k^m \Phi_{m+1}\left(\boldsymbol{W}_{m+1}^k\right) + 2C_{m-k}^m \Phi_{m+1}\left(\boldsymbol{W}_{m+1}^{m-k}\right) = C_k^m \left(\frac{1}{2}\right)^{m-1}. \qquad \textbf{QED}$$

A related observation then follows directly from the proof of Theorem 2.

Corollary 1 $PA_m^{m/2}(IC) = C_{m/2}^m \left(\frac{1}{2}\right)^m$ for all even $m \geq 2$.

8 The Case of Four Issues with IC

Some results can be obtained for $P_m^k(IC)$ in the special case of four issues. Corollary 1 directly leads to $P_4^2(IC) = 3/8$. A representation can be obtained the for limit probability $PA_4^4(IC)$ from the identity $PA_4^4(IC) = 2\Phi_5(\boldsymbol{W}_5)$. Representations for multivariate normal positive orthant probabilities generally become extremely complex in cases with more than four variables, except for special cases in which very restrictive conditions can be placed on the associated correlation matrix for the distribution.

A reasonable representation is obtained for $\Phi_5(\boldsymbol{W}_5)$ by appealing to Boole's equation (see Johnson and Kotz 1972, p. 52), which describes a procedure that can be used to express positive orthant probabilities with an odd number of dimensions in terms of a linear combination of positive orthant probabilities with fewer dimensions. With the correlation matrix $\boldsymbol{W}_5$, Boole's equation results in

$$\Phi_5(W_5) = \frac{1}{2}\Bigg[1 - 5\left(\frac{1}{2}\right) + \{6\Phi_2(Z_2) + 4\Phi_2(U_2)\} \\ -\{4\Phi_3(Z_3) + 6\Phi_3(U_3)\} + \{\Phi_4(Z_4) + 4\Phi_4(U_4)\}\Bigg] \quad (28)$$

Here, $\boldsymbol{Z}_j$ denotes a correlation matrix for a distribution on j variables with all correlation terms are equal to zero, as above. The correlation matrix $\boldsymbol{U}_j$ is defined on j variables with terms $u_{i,h} = 0$ for all $1 \le i < h < j$ and $u_{i,j} = \sqrt{\frac{9}{40}}$ for all $1 \le i \le j - 1$. The term $\sqrt{\frac{9}{40}}$ is obtained from (27).

Sheppard's Theorem can be used to obtain simple representations for $\Phi_2(\boldsymbol{U}_2)$ and $\Phi_3(\boldsymbol{U}_3)$, while $\Phi_4(\boldsymbol{U}_4)$ is a special case of a representation that is considered in Gehrlein (1979). After substitution and algebraic reduction, (28) reduces to

$$\Phi_5(\boldsymbol{W}_5) = \frac{1}{32} + \frac{1}{4\pi}\sin^{-1}\left(\sqrt{\frac{9}{40}}\right) + \frac{3}{2\pi^2}\int_0^{\sqrt{\frac{9}{40}}} \sqrt{\frac{1}{1-z^2}}\sin^{-1}\left(\frac{-z^2}{1-2z^2}\right)\mathrm{d}z. \quad (29)$$

Using the fact that $PA_4^4(IC) = 2\Phi_5(\boldsymbol{W}_5)$ with (29) yields

$$PA_4^4(IC) = \frac{1}{16} + \frac{1}{2\pi}\sin^{-1}\left(\sqrt{\frac{9}{40}}\right) - \frac{3}{\pi^2}\int_0^{\sqrt{\frac{9}{40}}} \sqrt{\frac{1}{1-z^2}}\sin^{-1}\left(\frac{z^2}{1-2z^2}\right)\mathrm{d}z \approx 0.1245. \quad (30)$$

Numerical integration is used to obtain the value of 0.1245 for $PA_4^4(IC)$ from (30).

Theorem 2 can then be used in conjunction with (30) to obtain a representation for $PA_4^0(IC)$, with

$$PA_4^0(IC) = \frac{1}{16} - \frac{1}{2\pi}\sin^{-1}\left(\sqrt{\frac{9}{40}}\right) + \frac{3}{\pi^2}\int_0^{\sqrt{\frac{9}{40}}} \sqrt{\frac{1}{1-z^2}}\sin^{-1}\left(\frac{z^2}{1-2z^2}\right)\mathrm{d}z \approx 0.0005. \quad (31)$$

Attention is turned to the situation in which there is only partial agreement with four issues, with the development of a representation for $PA_4^3(IC)$. Issue 1 will have the only majority rule outcome that is in disagreement with the issue position of the majority party, Party R, when both $\overline{Z_1^j}\sqrt{n} \le E\left(\overline{Z_1^j}\sqrt{n}\right)$ and $\overline{Z_i^j}\sqrt{n} \ge E\left(\overline{Z_i^j}\sqrt{n}\right)$ for each $2 \le i \le 5$. This probability is equivalent to the probability that $\overline{Z_i^j}\sqrt{n} \ge$

$E\left(\overline{Z_i^j}\sqrt{n}\right)$ for each $2 \le i \le 5$ minus the probability that $\overline{Z_i^j}\sqrt{n} \ge E\left(\overline{Z_i^j}\sqrt{n}\right)$ for each $1 \le i \le 5$, which is $\Phi_4(\boldsymbol{U}_4) - \Phi_5(\boldsymbol{W}_5)$. There are four issues that could be the single issue that is in disagreement with the issue position of the majority party, and there are two parties that could be the majority party. The symmetry of IC with respect to issues and parties leads to the conclusion that $PA_4^3(IC) = 8\{\Phi_4(\boldsymbol{U}_4) - \Phi_5(\boldsymbol{W}_5)\}$. After performing all necessary substitution and algebraic reduction,

$$PA_4^3(IC) = \frac{1}{4} + \frac{1}{\pi}\sin^{-1}\left(\sqrt{\frac{9}{40}}\right) + \frac{6}{\pi^2}\int_0^{\sqrt{\frac{9}{40}}} \sqrt{\frac{1}{1-z^2}}\sin^{-1}\left(\frac{z^2}{1-2z^2}\right)\mathrm{d}z \approx 0.4406. \tag{32}$$

A representation for the remaining probability $PA_4^1(IC)$ can be obtained from the identity relationship $\sum_{i=0}^4 PA_4^i(IC) = 1$, which leads to

$$PA_4^1(IC) = \frac{1}{4} - \frac{1}{\pi}\sin^{-1}\left(\sqrt{\frac{9}{40}}\right) - \frac{6}{\pi^2}\int_0^{\sqrt{\frac{9}{40}}} \sqrt{\frac{1}{1-z^2}}\sin^{-1}\left(\frac{z^2}{1-2z^2}\right)\mathrm{d}z \approx 0.0594. \tag{33}$$

9 Conclusion

The possibility of the existence of a Strict Ostrogorski Paradox presents a very interesting phenomenon that could lead to very unsettling outcomes in group decision-making situations. However, this phenomenon cannot exist in two-issue voting situations for any n. When three-issue situations are considered, the results of Table 2 indicate that the probability of such an outcome never reaches as much as a two percent for large electorates, regardless of the propensity of voters to align their views with the standards of political parties. The results of (31) indicate that the probability of observing a Strict Ostrogorski Paradox in four-issue situations is nearly zero with IC for large electorates.

Given our discussion regarding the propensity of models with the assumption of part parity to exaggerate the likelihood that such paradoxical outcomes will be observed, we can conclude that it is very unlikely that a Strict Ostrogorski Paradox, or any other extreme form of Ostrogorski's Paradox, would ever be observed in any real situation with large electorates over the range of the number of issues that we have been considering.

An alternative approach to the problem of deriving these probability representations in the limit as $n \to \infty$ that was mentioned previously has been used to obtain alternative forms to verify numerical results from all of the representations that are given above for $m = 2, 3, 4$. Details of these derivations are available from the authors upon request.

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Violations of Reversal Symmetry Under Simple and Runoff Scoring Rules

Raouia Belayadi and Boniface Mbih

1 Introduction

A story reported by Donald Saari (1995) is about a situation that once happened in his department: when asked to give a preference ordering on a set of alternatives, every member of the department misunderstood the instructions given by the chair, and ranked their top-ranked alternative first, the second-ranked second and so forth, while the chair expected the opposite. Saari concludes the story with the following words : “Instead of holding still another election (which we did), it was suggested that we reverse the original election outcome. After all, if everyone reverses their ranking, then surely the outcome also should be reversed”. As another example, consider the following situation: Suppose the inhabitants of a municipality are enthusiastic after they have been informed of an entrepreneur′s desire to set up an activity in their region (for example because this will provide jobs to unemployed persons). They are then asked by the mayor to vote on one location for the installation of the activity, among three possibilities, *a*, *b*, and *c*. Initially believing that the activity is a commercial mall, each inhabitant ranks the proposed sites in favor of those closest to her place of residence. One site, say *a*, is selected. Then, suppose the inhabitants are later informed that the activity is in fact a plant susceptible to emit pollution. They still accept the activity (for the same reason as above), but they now rank the sites in favor of those farthest from their place of residence. By so doing each individual

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R. Belayadi · B. Mbih (✉)
Normandie Université, Unicaen, CREM, UMR CNRS 6211, Esplanade de la Paix, 14032 Caen, France
e-mail: boniface.mbih@unicaen.fr

R. Belayadi
e-mail: raouia.belayadi@unicaen.fr

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reverses her ranking of the sites and if a is again selected, then reversal symmetry is violated. To put it another way, reversal symmetry requires that if some candidate is elected and then each voter reverses her preference order, then that same candidate must not be elected.
This notion was first introduced by Saari (1995). He presents it as a natural extension of neutrality; neutrality means that when all individual preferences concerning two alternatives are reversed, then the social ranking of those two alternatives also should be reversed. Saari and Barney (2003) focus on scoring rules; these are voting rules in which voters give each alternative a score from a given list, the scores are then added for each alternative and the alternative with the highest total is elected. In particular, they provide an example showing that in three-candidate elections, the only scoring rule that does not violate reversal symmetry is the Borda rule. A formal proof of this statement in the general case of m alternatives can be found in Llamazares and Peña (2015). Now, since all scoring rules, except Borda rule, violate reversal symmetry, does this violation often occur ? This is the problem tackled by Guèye (2014), for two specific rules, namely plurality and plurality with runoff ; he computes, under a specific probabilistic assumption (see IAC below), the frequency of violation of reversal symmetry for these two rules. In the context of committee selection Bubboloni et al. (2019) compare two extensions of the Simpson voting rule according to their ability to satisfy a set of social choice axioms, among which reversal symmetry, which they call "reversal bias"; they provide frequencies of violations of reversal symmetry by those extensions rules, both in the single winner and multi winner frameworks, under the IAC hypothesis. In this paper, we consider scoring rules, especially plurality and plurality with runoff, anti-plurality and anti-plurality with runoff, and the runoff version of the Borda rule; moreover, we provide frequencies of violation of reversal symmetry for all possible scoring rules for large electorates, and more precisely when the number of voters tends to infinity,

This work is organized as follows. The next section introduces some notations and presents the main definitions. Section 3 is devoted to the study of violations of reversal symmetry in universal domain, that is when individuals are allowed to report any possible preference order; Sect. 4 studies reversal symmetry in bipolar domain, a restricted domain where alternatives can be divided into two groups and each individual prefers all alternatives of one group to each alternative of the other group; finally, Sect. 5 concludes the paper.

2 Preliminaries

We consider a finite set N of n, $n \geq 2$, individuals, or voters. Each voter i is assumed to report a linear order, that is a complete, anti-symmetric and transitive binary relation R^i on a finite set A of alternatives, or candidates. A *profile* is an $n-$tuple $(R^1, \ldots, R^n)$ of individual preferences. A *voting rule* is a mapping the domain of which is the set of all possible profiles, and the range of which is the set A of alternatives. A *scoring voting rule* assigns a vector of weights or a *scoring vector*

$w = (w_1, w_2, \ldots, w_m)$ to each voter's first, second, …and mth ranked alternative in a profile, with $w_h \geq w_{h+1}$ for all $h = 1, \ldots, m-1$, $w_1 > w_m$, and then the winning alternative is the one with the greatest sum of weights over the set of voters. For scoring rules with runoff, the two alternatives with the greatest scores are selected for a second round, where a simple majority rule determines the winner. In the sequel we will use the standard notation where $w_1 = 1$ and $w_m = 0$. Then, *plurality rule* (also known as the widely used *first-past-the-post* voting system) is the scoring rule where $w_1 = 1$ and $w_h = 0$ for all $h = 2, \ldots, m$, the *anti-plurality rule* is defined by $w_h = 1$ for all $h = 1, \ldots, m-1$, and $w_m = 0$ and the *Borda rule* by $w_h = \frac{m-h}{m-1}$ for all $h = 1, \ldots, m$; clearly, for $m = 3$, all possible scoring rules are based on scoring vectors $(1, \lambda, 0)$, with $\lambda \in [0, 1]$, and for instance plurality, anti-plurality and Borda rules are based on scoring vectors $(1, 0, 0)$, $(1, 1, 0)$, and $(1, \frac{1}{2}, 0)$, respectively. The *reverse* of a linear order R is the linear order R' such that for all $x, y \in A$, $xR'y$ if and only if yRx. Given some original profile π, we shall denote its reverse by the profile π^{-1} where all individual preferences in π have been reversed. A voting rule F satisfies *reversal symmetry* if for all possible profiles π and π^{-1} such that $\pi \neq \pi^{-1}$, $F(\pi) \neq F(\pi^{-1})$. Example 1 below illustrates violations of reversal symmetry by scoring rules.

Example 1 Consider 100 voters, 3 alternatives a, b, and c, a profile π and its reverse π^{-1}. For these two profiles, we can determine the winner under all possible scoring voting rules.

π:

score	*number of voters*					
	9	22	21	21	20	10
1	*a*	*a*	*b*	*b*	*c*	*c*
λ	*b*	*c*	*a*	*c*	*a*	*b*
0	*c*	*b*	*c*	*a*	*b*	*a*

We then compute the total score of each alternative:

number of voters→	9	22	21	21	20	10	*total score*
score of a	9	22	21λ		20λ		$31 + 41\lambda$
score of b	9λ		21	21		10λ	$42 + 19\lambda$
score of c		22λ		21λ	20	10	$30 + 43\lambda$

π^{-1}:

score	*number of voters*					
	10	21	20	22	21	9
1	*a*	*a*	*b*	*b*	*c*	*c*
λ	*b*	*c*	*a*	*c*	*a*	*b*
0	*c*	*b*	*c*	*a*	*b*	*a*

Again, the total scores are given below:

							total score
nb of voters→	10	21	20	22	21	9	
score of a	10	21	20λ		21λ		$31 + 41\lambda$
score of b	10λ		20	22		9λ	$42 + 19\lambda$
score of c		21λ		22λ	21	9	$30 + 43\lambda$

First note that in our example, for each alternative the total score is the same in π and in π^{-1}, which means that the winning candidate is the same in both profiles. The reader can easily check that for all $\lambda < \frac{1}{2}$, alternative b is the winner in both profiles. Similarly, with $\lambda > \frac{1}{2}$, c is the winner in both profiles. And this shows that reversal symmetry is violated for all $\lambda \neq \frac{1}{2}$. In other terms, every scoring rule other than the Borda rule violates reversal symmetry. Also note that the social ranking in both profiles is *bac* if $\lambda < \frac{1}{2}$ and *cab* if $\lambda > \frac{1}{2}$, which illustrates the violation of the axiom not only in single-seat elections, but also if the goal is to determine the social ranking from individual preferences.

Now, admitting that reversal symmetry is a good property for a voting rule (since as explained by Schulze (2011) its violation means that voting to determine the socially best as well as the socially worst alternatives produces the same outcome), should we be so much worried by the possibility of violation of this property by a voting rule ? Indeed, if cases of violation are not so frequent, there is no real reason to care about that; but on the contrary maybe should we be careful about the outcome of voting if this often occurs. That is the main concern of this paper. We evaluate the frequency of violation of reversal symmetry by scoring rules. More precisely, for each rule under consideration we compute the proportion of preference profiles at which reversal symmetry is violated. Notice that no strategic behavior is taken into account in our analysis since we are only interested in reported preferences, and for intuition, the reader can consider those preferences as sincere ones. We consider two alternative usual assumptions: impartial culture (IC) and impartial anonymous culture (IAC). Under the former assumption, profiles are defined as above, that is, two profiles are different as soon as two individuals interchange their preferences, which means that the names of the voters matter. Under the latter assumption, profiles are *anonymous*, in the sense that we do not care about which voter reports some given preference relation, the sole criterion we use to distinguish two anonymous profiles is the number of voters who report every preference relation. To distinguish anonymous profiles from usual profiles, we shall call the former *voting situations*. It is also worth noting that, as a difference with IC, IAC entails some dependence between voters' opinions. These two models are commonly used to evaluate the frequency of paradoxes, and their properties have been extensively studied in the literature, see for example (Berg 1985; Lepelley et al. 2008; Regenwetter et al. 2006; Gehrlein and Lepelley 2011, 2017, among others). In both cases, we assume equiprobability of profiles (under IC) and of voting situations (under IAC). Then, the frequency of violation of reversal symmetry is given by the following ratio:

$$\text{Frequency} = \frac{\substack{\text{number of profiles (resp. of voting situations)} \\ \text{at which reversal symmetry is violated}}}{\text{total number of profiles (resp. of voting situations)}}$$

We shall come back to computation techniques in the next section.

3 Reversal Symmetry in Universal Domain

In universal domain we consider all logically possible profiles and voting situations. With m alternatives, the total number of preference relations (linear orders) is $m!$, and the total number of profiles is $(m!)^n$. Then, under the IC assumption, in order to compute the value of the frequency, we have to determine the value of the numerator, that is the number of profiles at which the voting rule under study violates reversal symmetry. That number is obtained by complete computer enumeration, and frequencies are computed as the ratio of that number and the total number of profiles. Under the IAC assumption, every preference relation is numbered, from 1 to $m!$, and every voting situation describes the number of voters reporting each preference relation; it can be written as an $m!-$tuple $s = (n_1, n_2, \ldots, n_{m!})$, $\sum_{k=1}^{k=m!} n_k = n$, where for all $k = 1, \ldots, m!$, n_k is the number of voters who report preference relation number k. Considering voting situations instead of profiles amounts to ranking profiles into equivalence classes in which every type of preference relation is reported by exactly the same number of voters for all distinct profiles in the class. Let S be the set of all possible voting situations for a given number n of voters and a given number m of candidates. The cardinality of S, denoted $|S|$, is the total number of voting situations, given by the formula:

$$|S| = \binom{n + m! - 1}{m! - 1}$$

The first subsection of this section is devoted to the computation of the frequency of violation of reversal symmetry for the five voting rules mentioned in the introduction of this paper. Then in Sect. 3.2 we consider the infinite electorate in the three-candidate case.

3.1 *Violations of Reversal Symmetry According to the Number of Voters*

With $A = \{a, b, c\}$ we have $m = 3$, and the following six linear orders:

$$R_1 = abc; \quad R_2 = acb; \quad R_3 = bac;$$
$$R_4 = bca; \quad R_5 = cab; \quad R_6 = cba.$$

Every possible voting situation can then be written as $(n_1, n_2, n_3, n_4, n_5, n_6)$, and its reverse as $(n_6, n_4, n_5, n_2, n_3, n_1)$. Then, our first task is to determine, for each of the scoring rules under consideration, the set of all voting situations (and all profiles) that lead to the same outcome as their reverses. And from this set—and the sets of all possible profiles and situations as well—we remove the set of situations that are identical to their reverses, since in such cases having the same outcome is clearly not surprising, and this is an additional difference with Guèye (2014), who does not take this difficulty into consideration. For a scoring rule based on the scoring vector $(1, \lambda, 0)$, let S_λ^a be the subset of S at which alternative a is elected at some situation and its reverse, and S_λ the subset of S at which the same candidate is elected at some situation and its reverse; clearly, S_λ can be written as the union of the three following disjoint sets:

$$1.\ a \text{ is the winner } (S_\lambda^a) : \begin{cases} n_1 + n_2 + \lambda n_3 + \lambda n_5 > \lambda n_1 + n_3 + n_4 + \lambda n_6 & (a.1) \\ n_1 + n_2 + \lambda n_3 + \lambda n_5 > \lambda n_2 + \lambda n_4 + n_5 + n_6 & (a.2) \\ \lambda n_3 + n_4 + \lambda n_5 + n_6 > \lambda n_1 + n_2 + n_5 + \lambda n_6 & (a.3) \\ \lambda n_3 + n_4 + \lambda n_5 + n_6 > n_1 + \lambda n_2 + n_3 + \lambda n_4 & (a.4) \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \end{cases}$$

$$2.b \text{ is the winner } (S_\lambda^b) : \begin{cases} \lambda n_1 + n_3 + n_4 + \lambda n_6 > n_1 + n_2 + \lambda n_3 + \lambda n_5 & (b.1) \\ \lambda n_1 + n_3 + n_4 + \lambda n_6 > \lambda n_2 + \lambda n_4 + n_5 + n_6 & (b.2) \\ \lambda n_1 + n_2 + n_5 + \lambda n_6 > \lambda n_3 + n_4 + \lambda n_5 + n_6 & (b.3) \\ \lambda n_1 + n_2 + n_5 + \lambda n_6 > n_1 + \lambda n_2 + n_3 + \lambda n_4 & (b.4) \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \end{cases}$$

$$3.\ c \text{is the winner } (S_\lambda^c) : \begin{cases} \lambda n_2 + \lambda n_4 + n_5 + n_6 > n_1 + n_2 + \lambda n_3 + \lambda n_5 & (c.1) \\ \lambda n_2 + \lambda n_4 + n_5 + n_6 > \lambda n_1 + n_3 + n_4 + \lambda n_6 & (c.2) \\ n_1 + \lambda n_2 + n_3 + \lambda n_4 > \lambda n_3 + n_4 + \lambda n_5 + n_6 & (c.3) \\ n_1 + \lambda n_2 + n_3 + \lambda n_4 > \lambda n_1 + n_2 + n_5 + \lambda n_6 & (c.4) \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \end{cases}$$

Inequality (a.1) says that a beats b, that is, the score of alternative a is strictly greater than the score of alternative b; this means that, for the sake of simplicity, we ignore all voting situations (and profiles) at which there are ties between candidates: we only consider voting situations where there are unambiguous winners; and every other inequality can be interpreted similarly. From the set of all those voting situations (resp. profiles), we remove the situations (resp. profiles) that are identical to their reverses. These latter voting situations are described by the following set of equalities:

$$S' = \begin{cases} n_1 = n_6 \\ n_2 = n_4 \\ n_3 = n_5 \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \end{cases}$$

It remains to compute the cardinalities of all these sets. Note that, clearly, $|S_\lambda^a| = |S_\lambda^b| = |S_\lambda^c|$. It then suffices to compute $|S_\lambda^a|$ and to multiply the result by 3. $|S_\lambda^a|$ is equal to the number of solutions of the corresponding system of linear inequalities. We use the same techniques as in many previous works (Gehrlein and Fishburn 1976; Lepelley and Mbih 1994; Regenwetter et al. 2006, among others). These techniques allow to compute the number of solutions of a system of linear inequalities. A system of linear inequalities defines a combination of convex polytopes in a lattice of rational points. Then, computing the cardinality of S_λ^h, $h \in \{a, b, c\}$ amounts to counting the number of integer solutions of the corresponding system of inequalities; such a computation yields pseudo-polynomials, that is polynomials in which constant coefficients vary according to the modulos of the size parameters. Further, our results are checked by complete computer enumeration, for small values of n.
We begin with the set of all situations that are identical to their reverses. The number of all such situations is given in Proposition 1. Notice that such situations occur only when n is even. Further, the results given in all propositions in this text are obtained under the IAC assumption. The results under IC given in the tables are all obtained by complete computer enumeration.

Proposition 1 *For all strictly positive even values of n, the number of voting situations identical to their reverses is*

$$|S'| = \frac{(n+4)(n+2)}{8}.$$

It follows that, subtracting this number from the total number of situations gives the denominator we will use in universal domain for all even values of n and for all voting rules, that is:

$$|S| - |S'| = \frac{n(n+2)(n+4)(n^2+9n+23)}{120}$$

while the total number $|S|$ of situations will be used for all odd values of n.

With $\lambda = 0$, that is under plurality rule, the number of situations at which reversal symmetry is violated when alternative a is the winner is obtained from system S_λ^a above; that number is then multiplied by 3, and the result is given in Proposition 2. A very similar approach is used for all other rules for the subsequent propositions.

Proposition 2 *Suppose n is the number of voters in the society. Then, under the plurality rule the number of voting situations leading to the same outcome as their reverses is given by*

$$|S_0| = \begin{cases} \frac{(n-1)(n+5)(n+3)(2n^2-n+17)}{2592} & \textit{if} \quad n \equiv 1 \mod 6 \\ \frac{(n-2)(n+4)(2n^3+9n^2-42n-292)}{2592} & \textit{if} \quad n \equiv 2 \mod 6 \\ \frac{(n-3)(n+5)(n+3)(2n^2+3n+27)}{2592} & \textit{if} \quad n \equiv 3 \mod 6 \\ \frac{(n+2)(2n^4+9n^3+6n^2-92n+480)}{2592} & \textit{if} \quad n \equiv 4 \mod 6 \\ \frac{(n-5)(2n+5)(n+7)(n+1)^2}{2592} & \textit{if} \quad n \equiv 5 \mod 6 \\ \frac{n(2n^4+13n^3+24n^2-144n-216)}{2592} & \textit{if} \quad n \equiv 6 \mod 6 \end{cases}$$

Note that in Proposition 2, for all $n \in \{0, 1, 2, 3, 5\}$, $|S_0| = 0$. Thus, for all those values of n, under plurality, there is no voting situation leading to the same outcome as its reverse; but this does not mean that for all those values and more precisely for the only even value $n = 2$ there is no voting situation identical to its reverse; for example, the reader can easily check that for $n = 2$ there are three voting situations $s_1 = (0, 0, 1, 0, 1, 0)$, $s_2 = (0, 1, 0, 1, 0, 0)$, and $s_3 = (1, 0, 0, 0, 0, 1)$ such that $s_1 = s_1^{-1}$, $s_2 = s_2^{-1}$ and $s_3 = s_3^{-1}$. Thus, removing these three voting situations from the set of voting situations at which reversal symmetry is violated would lead to a negative number. This is so because all those voting situations—where there are ties—are *ab initio* out of the range of our analysis because we only consider voting situations at which there are strict winners. But for none of these voting situations is reversal symmetry violated since according to the three systems of inequalities S_λ^a, S_λ^b and S_λ^c we rule out voting situations like s_1, s_2 and s_3 where there is no unambiguous winner. As a consequence and more generally, removing them from the set S_λ, $0 \le \lambda \le 1$, would amount to a double-counting. It follows that the set (and the number) of voting situations identical to their reverses when some candidate, say a, is elected, will depend on the voting rule under use; to illustrate, when $n = 2$ and candidate a is elected, the reader can check that the set of all such situations is empty under plurality rule as noted above, but is the singleton $\{(0, 0, 1, 0, 1, 0)\}$ under anti-plurality rule.

More generally, for any possible simple scoring voting rule, the set $S_\lambda^{'a}$ of all voting situations identical to their reverses when alternative a is the winner is given by

$$S_\lambda^{'a} : \begin{cases} n_1 + n_2 + \lambda n_3 + \lambda n_5 > \lambda n_1 + n_3 + n_4 + \lambda n_6 \\ n_1 + n_2 + \lambda n_3 + \lambda n_5 > \lambda n_2 + \lambda n_4 + n_5 + n_6 \\ \lambda n_3 + n_4 + \lambda n_5 + n_6 > \lambda n_1 + n_2 + n_5 + \lambda n_6 \\ \lambda n_3 + n_4 + \lambda n_5 + n_6 > n_1 + \lambda n_2 + n_3 + \lambda n_4 \\ n_1 = n_6, n_2 = n_4, n_3 = n_5 \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \end{cases}$$

For plurality rule, when n is even and a, b, or c are elected, the number of all voting situations similar to those described by the preceding set of linear inequalities is

$$\left|S_0'\right| = 3|S_0^{'a}| = \begin{cases} \frac{(n-2)(n+4)}{8} & \textit{if} \quad n \equiv 2 \mod 6 \\ \frac{n(n+2)}{8} & \text{otherwise} \end{cases}$$

and for anti-plurality

$$\left|S_1'\right| = 3|S_1^{'a}| = \begin{cases} \frac{n^2+4n+12}{8} & \textit{if} \quad n \equiv 2 \mod 12 \\ \frac{n^2+4n-8}{8} & \textit{if} \quad n \equiv 4 \mod 12 \\ \frac{n^2+4n+12}{8} & \textit{if} \quad n \equiv 6 \mod 12 \\ \frac{n(n+4)}{8} & \textit{if} \quad n \equiv 8 \mod 12 \\ \frac{(n+2)^2}{8} & \textit{if} \quad n \equiv 10 \mod 12 \\ \frac{n(n+4)}{8} & \textit{if} \quad n \equiv 12 \mod 12 \end{cases}$$

And after removing all situations identical to their reverses when a, b or c are elected, we obtain the number of voting situations at which reversal symmetry is violated:

$$|S_0| - \left|S_0'\right| = \begin{cases} \frac{(n-1)(n+5)(n+3)(2n^2-n+17)}{2592} & \textit{if} \quad n \equiv 1 \mod 6 \\ \frac{(n-2)(n+4)(2n^3+9n^2-42n-616)}{2592} & \textit{if} \quad n \equiv 2 \mod 6 \\ \frac{(n-3)(n+5)(n+3)(2n^2+3n+27)}{2592} & \textit{if} \quad n \equiv 3 \mod 6 \\ \frac{(n+2)(n-4)(2n^3+17n^2+74n-120)}{2592} & \textit{if} \quad n \equiv 4 \mod 6 \\ \frac{(2n+5)(n+7)(n-5)(n+1)^2}{2592} & \textit{if} \quad n \equiv 5 \mod 6 \\ \frac{n(2n^4+13n^3+24n^2-468n-864)}{2592} & \textit{if} \quad n \equiv 6 \mod 6 \end{cases}$$

for plurality, and

$$|S_1| - \left|S_1'\right| = \begin{cases} \frac{(n-1)(n+2)(n+5)(11n^2+44n-7)}{6480} & \textit{if} \quad n \equiv 1 \mod 12 \\ \frac{(n-2)(n+4)(2n^3+9n^2-42n-616)}{6480} & \textit{if} \quad n \equiv 2 \mod 12 \\ \frac{(n+3)(11n^4+77n^3+219n^2+243n+810)}{6480} & \textit{if} \quad n \equiv 3 \mod 12 \\ \frac{(11n^5+110n^4+290n^3-830n^2-3280n-7360)}{6480} & \textit{if} \quad n \equiv 4 \mod 12 \\ \frac{(n+1)(11n^4+99n^3+351n^2+629n+1230)}{6480} & \textit{if} \quad n \equiv 5 \mod 12 \\ \frac{11n^5+110n^4+450n^3+90n^2-1296n-3240}{6480} & \textit{if} \quad n \equiv 6 \mod 12 \\ \frac{(n+5)(n-1)(n+2)(11n^2+44n-7)}{6480} & \textit{if} \quad n \equiv 7 \mod 12 \\ \frac{(n+4)(11n^4+66n^3+186n^2-574n+1320)}{6480} & \textit{if} \quad n \equiv 8 \mod 12 \\ \frac{(n+3)(11n^4+77n^3+219n^2+243n+810)}{6480} & \textit{if} \quad n \equiv 9 \mod 12 \\ \frac{(n+2)(11n^4+88n^3+114n^2-1058n-1180)}{6480} & \textit{if} \quad n \equiv 10 \mod 12 \\ \frac{(n+1)(11n^4+99n^3+351n^2+629n+1230)}{6480} & \textit{if} \quad n \equiv 11 \mod 12 \\ \frac{n(11n^4+110n^3+450n^2+90n-1296))}{6480} & \textit{if} \quad n \equiv 12 \mod 12 \end{cases}$$

for anti-plurality.

It clearly appears that for all $n < 6$, $|S_0| - |S_0'| = 0$, while $|S_1| - |S_1'| = 0$ for all $n < 3$. Also note that for anti-plurality, the periodicity of polynomials is equal to 6 for odd values of n, but is equal to 12 for even values. The frequencies of violation of reversal symmetry for these two rules are given in Propositions 3 and 4. For every scoring vector $(1, \lambda, 0)$, $f_\lambda(n)$ will denote the frequency of violation of the corresponding simple scoring rule.
We start with the plurality rule.

Proposition 3 *Suppose n is the number of voters in the society. Then, the frequency of violation of reversal symmetry by plurality rule under IAC is given by*

$$f_0(n) = \begin{cases} \frac{5(n-1)(2n^2-n+17)}{108(n+4)(n+2)(n+1)} & \text{if} \quad n \equiv 1 \mod 6 \\ \frac{5(n-2)(2n^3+9n^2-42n-616)}{108n(n+2)(n^2+9n+23)} & \text{if} \quad n \equiv 2 \mod 6 \\ \frac{5(n-3)(2n^2+3n+27)}{108(n+4)(n+2)(n+1)} & \text{if} \quad n \equiv 3 \mod 6 \\ \frac{5(n-4)(2n^3+17n^2+74n-120)}{108n(n+4)(n^2+9n+23)} & \text{if} \quad n \equiv 4 \mod 6 \\ \frac{5(n-5)(n+1)(2n+5)(n+7)}{108(n+5)(n+4)(n+3)(n+2)} & \text{if} \quad n \equiv 5 \mod 6 \\ \frac{5(2n^4+13n^3+24n^2-468n-864)}{108(n+2)(n+4)(n^2+9n+23)} & \text{if} \quad n \equiv 6 \mod 6 \end{cases}$$

The formulas in Proposition 3 clearly indicate that for all n such that $1 \leq n \leq 5$, the plurality rule does not violate reversal symmetry; this fact appears in Table 1 with values of frequencies equal to 0. By contrast, as soon as the number of voters is equal to at least six, then violation of that property is susceptible to occur, up to $\frac{10}{108} \approx 9.26\%$, as the number of voters tends to infinity. The next step is about anti-plurality rule, and then about scoring rules with runoff. Proposition 4 summarizes the frequencies for anti-plurality rule.

Proposition 4 *Suppose n is the number of voters in the society. Then, the frequency of violation of reversal symmetry by anti-plurality rule under IAC is given by*

$$f_1(n) = \begin{cases} \frac{(n-1)(11n^2+44n-7)}{54(n+4)(n+3)(n+1)} & \text{if} \quad n \equiv 1 \mod 12 \\ \frac{(n-2)(11n^4+132n^3+714n^2+1598n+2220)}{54n(n+2)(n+4)(n^2+9n+23)} & \text{if} \quad n \equiv 2 \mod 12 \\ \frac{11n^4+77n^3+219n^2+243n+810}{54(n+5)(n+4)(n+2)(n+1)} & \text{if} \quad n \equiv 3 \mod 12 \\ \frac{11n^5+110n^4+290n^3-830n^2-3296n+7360}{54n(n+2)(n+4)(n^2+9n+23)} & \text{if} \quad n \equiv 4 \mod 12 \\ \frac{11n^4+99n^3+351n^2+629n+1230}{54(n+5)(n+4)(n+3)(n+2)} & \text{if} \quad n \equiv 5 \mod 12 \\ \frac{11n^5+110n^4+450n^3+90n^2-1296n-9720}{54n(n+2)(n+4)(n^2+9n+23)} & \text{if} \quad n \equiv 6 \mod 12 \\ \frac{(n-1)(11n^2+44n-7)}{54(n+4)(n+3)(n+1)} & \text{if} \quad n \equiv 7 \mod 12 \\ \frac{11n^4+66n^3+186n^2-574n+1320}{54n(n+2)(n^2+9n+23)} & \text{if} \quad n \equiv 8 \mod 12 \\ \frac{11n^4+77n^3+219n^2+243n+810}{54(n+5)(n+4)(n+2)(n+1)} & \text{if} \quad n \equiv 9 \mod 12 \\ \frac{11n^4+88n^3+114n^2-1058n-1180}{54n(n^2+9n+23)(n+4)} & \text{if} \quad n \equiv 10 \mod 12 \\ \frac{11n^4+99n^3+351n^2+629n+1230}{54(n+5)(n+4)(n+3)(n+2)} & \text{if} \quad n \equiv 11 \mod 12 \\ \frac{11n^4+110n^3+450n^2+90n-1296}{54(n+2)(n+4)(n^2+9n+23)} & \text{if} \quad n \equiv 12 \mod 12 \end{cases}$$

For $n < 3, f_1(n) = 0$, but as soon as there are at least 3 candidates, the probability of violation of reversal symmetry is greater than 10%, and tends to $\frac{11}{54} \approx 20.37\%$ as n tends to infinity: for every value of n, the probability of violation of reversal symmetry is significantly greater for anti-plurality than for plurality.

We now turn to scoring rules with runoff. Voting situations leading to the same alternative, say a, as their reverses are described by the following system of inequalities:

$$\begin{cases} n_1 + n_2 + \lambda n_3 + \lambda n_5 > \lambda n_2 + \lambda n_4 + n_5 + n_6 \\ \lambda n_1 + n_3 + n_4 + \lambda n_6 > \lambda n_2 + \lambda n_4 + n_5 + n_6 \\ n_1 + n_2 + n_5 > n_3 + n_4 + n_6 \\ \lambda n_3 + n_4 + \lambda n_5 + n_6 > n_1 + \lambda n_2 + n_3 + \lambda n_4 \\ \lambda n_1 + n_2 + n_5 + \lambda n_6 > n_1 + \lambda n_2 + n_3 + \lambda n_4 \\ n_3 + n_4 + n_5 > n_1 + n_2 + n_6 \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \end{cases}$$

or

$$\begin{cases} n_1 + n_2 + \lambda n_3 + \lambda n_5 > \lambda n_1 + n_3 + n_4 + \lambda n_6 \\ \lambda n_2 + \lambda n_4 + n_5 + n_6 > \lambda n_1 + n_3 + n_4 + \lambda n_6 \\ n_1 + n_2 + n_3 > n_4 + n_5 + n_6 \\ \lambda n_3 + n_4 + \lambda n_5 + n_6 > \lambda n_1 + n_2 + n_5 + \lambda n_6 \\ n_1 + \lambda n_2 + n_3 + \lambda n_4 > \lambda n_1 + n_2 + n_5 + \lambda n_6 \\ n_3 + n_4 + n_6 > n_1 + n_2 + n_5 \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \end{cases}$$

Let $F_\lambda(n)$ be the frequency of violation of reversal symmetry by the scoring rule with runoff based on the scoring vector $(1, \lambda, 0)$. As for simple scoring rules, replacing λ by any real value between 0 and 1, we get a specific scoring rule with runoff. With the same tools as above, we obtain the following propositions, for plurality, Borda, and anti-plurality rules with runoff.

We begin with plurality with runoff.

Proposition 5 *Suppose n is the number of voters in the society. Then, the frequency of violation of reversal symmetry by plurality rule with runoff under IAC is given by*

$$F_0(n) = \begin{cases} \frac{(n-1)(49n^4+539n^3+849n^2+2969n+33610)}{1728(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 1 \mod 12 \\ \frac{(n-2)(49n^4-147n^3-174n^2-8468n+21480)}{1728(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 2 \mod 12 \\ \frac{(n-3)(49n^4+637n^3+3501n^2+4743n+5670)}{1728(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 3 \mod 12 \\ \frac{(n-4)(49n^4-49n^3-1356n^2+7136n+21760)}{1728(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 4 \mod 12 \\ \frac{(n+7)(49n^4+147n^3+561n^2+5233n+14490)}{1728(n+5(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 5 \mod 12 \\ \frac{(n-6)(49n^4+49n^3+414n^2-4356n+9720)}{1728(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 6 \mod 12 \\ \frac{49n^4+245n^3-915n^2-9505n+13366}{1728(n+4)(n+3(n+2)(n+1)} & \text{if} \quad n \equiv 7 \mod 12 \\ \frac{49n^4-441n^3+1884n^2+544n+3840}{1728(n+5)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 8 \mod 12 \\ \frac{49n^4+343n^3+561n^2+8757n+29970}{1728(n+5)(n+4)(n+2)(n+1)} & \text{if} \quad n \equiv 9 \mod 12 \\ \frac{49n^4-343n^3-474n^2-2692n+31000}{1728(n+5)(n+4)(n+3)(n+1)} & \text{if} \quad n \equiv 10 \mod 12 \\ \frac{49n^4+441n^3+1149n^2-8189n-5490}{1728(n+2)(n+3)(n+4)(n+5)} & \text{if} \quad n \equiv 11 \mod 12 \\ \frac{n(49n^4-245n^3+120n^2+9360n+3456)}{1728(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 12 \mod 12 \end{cases}$$

Then for anti-plurality with runoff we have:

Proposition 6 *Suppose n is the number of voters in the society. Then, the frequency of violation of reversal symmetry by anti-plurality rule with runoff under IAC is given by*

$$F_1(n) = \begin{cases} \frac{(n-1)(11n^4+66n^3+216n^2+766n+12765)}{864(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 1 \mod 12 \\ \frac{(n-2)(11n^4-88n^3-336n^2+2048n+11600)}{864(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 2 \mod 12 \\ \frac{(n-3)(11n^4+88n^3+414n^2+1152n+20655)}{864(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 3 \mod 12 \\ \frac{(n-4)(11n^4-66n^3+216n^2+64n+11520)}{864(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 4 \mod 12 \\ \frac{(n-5)(n+7)(n+1)(11n^2+22n-193)}{864(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 5 \mod 12 \\ \frac{(n-6)(11n^4-44n^3+216n^2-144n+19440)}{864(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 6 \mod 12 \\ \frac{(n+5)(11n^4+150n^2-200n+19479)}{864(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 7 \mod 12 \\ \frac{(n+4)(n-2)(n-8)(11n^2-44n-160)}{864(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 8 \mod 12 \\ \frac{(n+3)(11n^4+22n^3+84n^2-342n+11745)}{864(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 9 \mod 12 \\ \frac{(n+2)(11n^4-132n^3+744n^2-2288n+22320)}{864(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 10 \mod 12 \\ \frac{(n+1)(11n^4+44n^3-534n^2-1156n+13235)}{864(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 11 \mod 12 \\ \frac{n(11n^4-110n^3+480n^2-1440n+13824)}{864(n+5)(n+4)(n+3)(n+2)(n+1)} & \text{if} \quad n \equiv 12 \mod 12 \end{cases}$$

And for Borda rule with runoff:

Proposition 7 *Suppose n is the number of voters in the society. Then, the frequency of violation of reversal symmetry by Borda rule with runoff under IAC is given by*

$$F_{\frac{1}{2}}(n) = \begin{cases} \frac{(n-1)(3n^2+12n+89)}{72(n+4)(n+3)(n+1)} & \textit{if} \quad n \equiv 1 \; \textit{mod}\; 6 \\ \frac{(n-2)(3n^3-21n^2+146n-640)}{72(n+5)(n+3)(n+2)(n+1)} & \textit{if} \quad n \equiv 2 \mod 6 \\ \frac{(n-3)(3n^3+30n^2+197n+810)}{72(n+5)(n+4)(n+2)(n+1)} & \textit{if} \quad n \equiv 3 \mod 6 \\ \frac{(n-1)(n-4)(3n^2-6n+80)}{72(n+5)(n+4)(n+3)(n+1)} & \textit{if} \quad n \equiv 4 \mod 6 \\ \frac{(n+7)(3n^3+6n^2+101n-310)}{72(n+2)(n+3)(n+4)(n+5)} & \textit{if} \quad n \equiv 5 \mod 6 \\ \frac{n(n-6)(3n^3+3n^2+98n+408)}{72(n+5)(n+4)(n+3)(n+2)(n+1)} & \textit{if} \quad n \equiv 6 \mod 6 \end{cases}$$

As a difference with simple scoring rules, for scoring rules with runoff situations in which $\pi = \pi^{-1}$ correspond to cases where the two alternatives selected for the second round are necessarily ties. As a consequence, removing them from the set of voting situations would lead to a double-counting, exactly like for some cases indicated in the comments of Proposition 2. Comparing the limit values of frequencies when n tends to infinity, we observe that $F_0(\infty) = \frac{49}{1728} \approx 2.83\%$, $F_1(\infty) = \frac{11}{864} \approx 1.27\%$, and $F_{\frac{1}{2}}(\infty) = \frac{3}{72} \approx 4.17\%$, which means that, at least for large electorates, the hierarchy between the three rules is completely reversed.

Table 1 summarizes the results in the propositions of this subsection. It clearly appears that for all rules in the table frequencies rise as the number of voters rises; it is also noticeable that the dependence among voters' opinions by the move from IC to IAC increases the difference in the violation of reversal symmetry, between plurality and anti-plurality for their simple versions, while it reduces this difference for their runoff versions. Moreover, except for the Borda rule, runoff versions of the scoring rules are strongly less susceptible to violate reversal symmetry than the corresponding simple versions. It can also be noted that anti-plurality rule is susceptible to violate reversal symmetry as soon as there are three voters in the society.

3.2 Infinite Electorate and Three Candidates

In this subsection, we provide a more general view of the vulnerability of scoring rules to reversal symmetry. The infinite electorate case is the usual framework for the evaluation of frequencies of a phenomenon for all possible rules within a class of voting rules. Here, we first consider simple, and then runoff scoring rules.

Let $f_\lambda(\infty) = f(\lambda)$ (resp. $F_\lambda(\infty) = F(\lambda)$) denote the frequencies of violation of reversal symmetry by a simple scoring rule (resp. a scoring rule with runoff) based on a scoring vector $(1, \lambda, 0)$ when the number n of voters tends to infinity. The propositions and figures below summarize the result of our computations.

Proposition 8 *Under IAC, the frequency of violation of reversal symmetry by every simple scoring rule based on the scoring vector* $(1, \lambda, 0)$ *when the number of voters tends to infinity is given by*

$$f(\lambda) = \begin{cases} -\frac{1}{27}\frac{(5\lambda^3-9\lambda+5)(2\lambda-1)^2}{(\lambda-2)(\lambda-1)^2} & \textit{if} \quad \lambda \in [0, \frac{1}{2}] \\ -\frac{1}{54}\frac{(\lambda^2-16\lambda+4)(2\lambda-1)^2}{\lambda^2} & \textit{if} \quad \lambda \in [\frac{1}{2}, 1] \end{cases}.$$

Table 1 Frequencies of violation of reversal symmetry for 3 candidates under scoring rules and universal domain)

IAC Model						IC Model					
Simple scoring rules			Scoring rules with runoff			Simple scoring rules			scoring rules with runoff		
n	plurality	Anti-plurality	Plurality	Borda	Anti-plurality	n	Plurality	Anti-plurality	Plurality	Borda	Anti-plurality
3	0	0.107142	0	0	0	3	0	0.083333	0	0	0
4	0	0.05	0	0	0	4	0	0.044776	0	0	0
5	0	0.119047	0.023809	0.023809	0	5	0	0.104166	0.023148	0.023148	0
6	0.013274	0.106195	0	0	0	6	0.024109	0.075141	0	0	0
7	0.037879	0.10606	0.007575	0.030303	0.007575	7	0.063014	0.056862	0.003000	0.034507	0.013503
8	0.014151	0.127358	0.004662	0.004662	0	8	0.016441	0.088300	0.002000	0.002000	0
9	0.041958	0.137862	0.020979	0.029970	0.005994	9	0.061514	0.103375	0.007501	0.039384	0.007501
10	0.040241	0.122736	0.001998	0.007992	0.001998	10	0.058438	0.065368	0.000312	0.004480	0.001667
11	0.037087	0.145604	0.017857	0.031593	0.004120	11	0.040877	0.110990	0.017542	0.041534	0.003247
12	0.030303	0.145130	0.006787	0.009696	0.001939	12	0.062183	0.098902	0.002185	0.006769	0.001146
13	0.053221	0.141456	0.018907	0.032913	0.007002	13	0.078414	0.082582	0.008470	0.042586	0.012280
14	0.043478	0.151656	0.000708	0.011867	0.001547	14	0.047415	0.105842	0.003296	0.008726	0.000576
15	0.056115	0.155959	0.020510	0.033281	0.007352	15	0.078823	0.112239	0.015857	0.043188	0.010670
21	0.064031	0.166372	0.023168	0.034934	0.008391	21	0.088942	0.117844	0.016692	0.044171	0.012243
27	0.069124	0.173079	0.023716	0.036022	0.009117	27	0.095917	0.121835	0.017286	0.044821	0.013300
75	0.082931	0.191161	0.026564	0.039204	0.011072	75	0.119376	0.135816	0.019548	0.047715	0.016993
∞	0.092592	0.203704	0.028356	0.041667	0.012731	–	–	–	–	–	–

The reader can check that for $\lambda = 0$ and $\lambda = 1$, we obtain exactly the same values as for $f_0(n)$ and $f_1(n)$, respectively, when n tends to infinity. From these formulas we obtain the curve in Fig. 1.

Proposition 9 *Under IAC, the frequency of violation of reversal symmetry by every scoring rule with runoff based on the scoring vector* $(1, \lambda, 0)$ *when the number of voters tends to infinity is given by*

$$F(\lambda) = \begin{cases} -\frac{6\lambda^5+16\lambda^4-146\lambda^3+273\lambda^2-197\lambda+49}{432(\lambda-2)(3\lambda-2)(\lambda-1)^3} & \text{if } \lambda \in [0, \frac{1}{2}] \\ \frac{324\lambda^9-2187\lambda^8+5835\lambda^7-8013\lambda^6+6007\lambda^5}{216(\lambda-2)(3\lambda-1)^2(3\lambda^2-5\lambda+1)\lambda^3} + \frac{-2332\lambda^4+367\lambda^3+22\lambda^2-13\lambda+1}{216(\lambda-2)(3\lambda-1)^2(3\lambda^2-5\lambda+1)\lambda^3} & \text{if } \lambda \in [\frac{1}{2}, 1] \end{cases}.$$

Comparing the previous proposition with Propositions 5, 6 and 7, it is easy to check that $F(0) = \lim_{n\to\infty} F_0(n)$, $F(\frac{1}{2}) = \lim_{n\to\infty} F_{\frac{1}{2}}(n)$ and $F(1) = \lim_{n\to\infty} F_1(n)$, respectively. From these formulas we obtain the curve depicted in Fig. 2.

Figures 1 and 2 illustrate in the three-alternative case how frequent reversal symmetry is violated by all possible simple and runoff scoring rules, respectively, as the number of voters tends to infinity. In the simple scoring voting rules case, the curve is U-shaped; frequencies decrease from plurality to the Borda rule, where with no surprise $F(\lambda) = 0$, the minimum value. They then rise from the Borda rule up to anti-plurality. The curve also shows that the limit frequency for plurality is smaller than the limit value for anti-plurality. On the contrary, for scoring rules with runoff we have a bell-shaped curve, and the maximum value is reached at $\lambda \approx 0.45$. And the limit value of plurality is greater than that of anti-plurality.

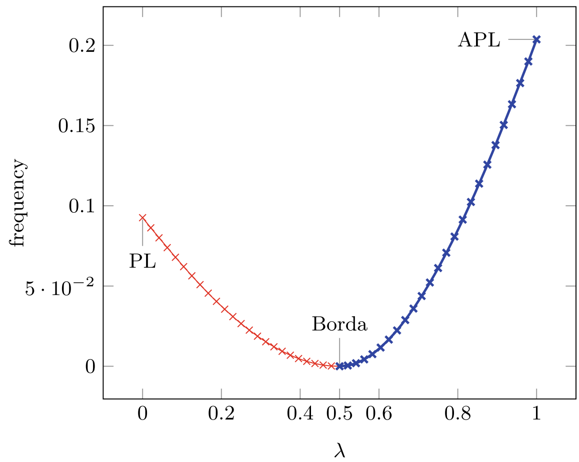

Fig. 1 Frequencies, under the IAC assumption, of violation of reversal symmetry in the three-candidate case by all simple scoring rules in universal domain when $n \to \infty$

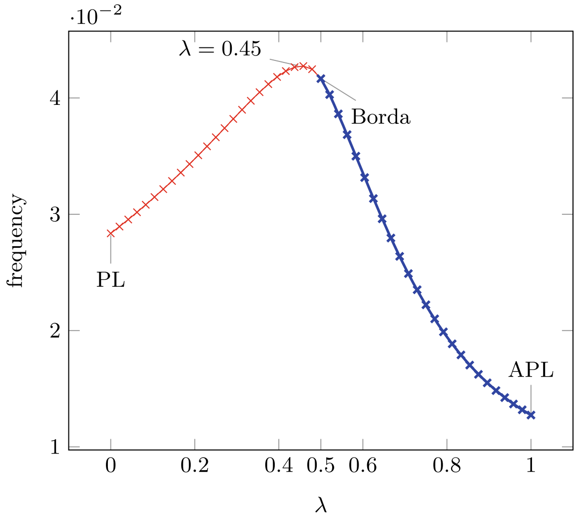

Fig. 2 Frequencies, under the IAC assumption, of violation of reversal symmetry in the three-candidate case by scoring rules with runoff in universal domain when $n \to \infty$

4 Restricted Domain: Bipolar Preferences

Remember that according to our definition of reversal symmetry, for this axiom to be violated in a domain by some voting rule, there must exist some profile π in the domain whose inverted profile π^{-1} is (i) different from the original one and (ii) leads to the same outcome as the original one. Now, consider for example the restriction consisting in only taking into account single-peaked[1] preferences with the left-right order *abc*; it is clear that the only profiles π such that π^{-1} is in the domain are those in which only extreme preferences *abc* and *cba* can be reported both in the original and the reversed profiles. But, then, in such a domain, there is no possibility of violation of reversal symmetry. In this section, we consider a specific domain restriction whose intuition is as follows: voters are not always able to have a precise idea (or the same perception) of a complete pairwise ranking of candidates according to the left-right ideological axis. Here, we only suppose that voters have the same perception of which candidates are on the left-wing and which ones are on the right-wing. Then, with p left-wing and q right-wing candidates we have only $(p! \times q!) \times 2$ (instead of $m! = (p+q)!$) preference orders. We shall study violations of reversal symmetry in such a domain, successively for three and for four candidates.

[1]Preferences over a set $A = \{a, b, c\}$ of candidates are said to be single-peaked according to the left-right ideological axis *abc* if no voter ranks the centrist candidate b last, so that the only possible preference orders are *abc*, *bac*, *bca* and *cba*.

4.1 Three Candidates

With a single left-wing candidate, say a, and two candidates, b and c, as right-wing candidates, voters will have the choice between four preference relations:

$$P_1 : abc;\ P_2 : acb;\ P_3 : bca;\ P_4 : cba$$

Let us denote by p_k the number of individuals reporting preference relation P_k; all voting situations at which reversal symmetry is violated under simple scoring rules, when candidate a is the winner in bipolar domain are described by the following system of linear inequalities:

$$\begin{cases} p_1 + p_2 > \lambda p_1 + p_3 + \lambda p_4 \\ p_1 + p_2 > \lambda p_2 + \lambda p_3 + p_4 \\ p_3 + p_4 > \lambda p_1 + p_2 + \lambda p_4 \\ p_3 + p_4 > p_1 + \lambda p_2 + \lambda p_3 \\ p_1 + p_2 + p_3 + p_4 = n \end{cases}$$

The reader can easily check that under plurality rule ($\lambda = 0$) reversal symmetry is violated only when a is the winner. Similarly, it is easy to check that under anti-plurality rule reversal symmetry can be violated when the winner is b or c, but not when a is the winner.

As for universal domain, when the number n of individuals is even, we exclude situations whose reverses are identical to the original ones, that is situations such that $p_1 = p_4$ and $p_2 = p_3$. The total number of all such situations is

$$\frac{(n+2)}{2}$$

. Consequently, subtracting this number from the total number of situations gives the denominator we will use in restricted domain for all even values of n and for all voting rules, that is:

$$\binom{n+3}{3} - \frac{(n+2)}{2} = \frac{n(n+2)(n+4)}{6}$$

Let $g_\lambda(n)$ be the frequency of violation of reversal symmetry under simple scoring rule based on scoring vector $(1, \lambda, 0)$. We obtain the propositions below.

Proposition 10 *Suppose n is the number of voters in the society. Then, under IAC the frequency of violation of reversal symmetry by plurality rule in bipolar domain is given by*

$$g_0(n) = \begin{cases} \frac{(n-1)(2n^2-n-19)}{9(n+3)(n+2)(n+1)} & \textit{if} \quad n \equiv 1 \mod 6 \\ \frac{2n^3-3n^2-84n+56}{9n(n+4)(n+2)} & \textit{if} \quad n \equiv 2 \mod 6 \\ \frac{(n-3)(2n-3)}{9(n^2+3n+2)} & \textit{if} \quad n \equiv 3 \mod 6 \\ \frac{2n^3-3n^2-72n+100}{9n(n+4)(n+2)} & \textit{if} \quad n \equiv 4 \mod 6 \\ \frac{(2n+5)(-5+n)}{9(n^2+5n+6)} & \textit{if} \quad n \equiv 5 \mod 6 \\ \frac{(n+6)(n-6)(2n-3)}{9n(n+4)(n+2)} & \textit{if} \quad n \equiv 6 \mod 6 \end{cases}$$

Proposition 11 *Suppose n is the number of voters in the society. Then, under IAC the frequency of violation of reversal symmetry by anti-plurality rule in bipolar domain is given by*

$$g_1(n) = \begin{cases} \frac{(n-1)(7n^2+22n+7)}{18(n^3+6n^2+11n+6)} & \textit{if} \quad n \equiv 1 \mod 6 \\ \frac{7n^3+15n^2-69n-115}{18n(n+4)(n+2)} & \textit{if} \quad n \equiv 2 \mod 6 \\ \frac{7n^2-6n+27}{18(n^2+3n+2)} & \textit{if} \quad n \equiv 3 \mod 6 \\ \frac{7n^2+n-62}{18n(n+4)} & \textit{if} \quad n \equiv 4 \mod 6 \\ \frac{(n+1)(7n+1)}{18(n^2+5n+6)} & \textit{if} \quad n \equiv 5 \mod 6 \\ \frac{7n^3+15n^2-36n-108}{18n(n+4)(n+2)} & \textit{if} \quad n \equiv 6 \mod 6 \end{cases}$$

Next, we consider scoring rules with runoff. A scoring rule with runoff based on the scoring vector $(1, \lambda, 0)$ violates reversal symmetry at a voting situation (p_1, p_2, p_3, p_4) when alternative a is elected in bipolar domain if and only if:

$$\begin{cases} p_1 + p_2 > p_4 + \lambda p_2 + p_3 \\ p_3 + \lambda(p_1 + p_4) > p_4 + \lambda(p_2 + p_3) \\ p_1 + p_2 > p_3 + p_4 \\ p_3 + p_4 > p_2 + \lambda(p_1 + p_4) \\ p_1 > p_2 + \lambda(p_1 + p_4) \\ p_3 + p_4 > p_1 + p_2 \\ p_1 + p_2 + p_3 + p_4 = n \end{cases} .$$

It however appears that the previous system contains two incompatible inequalities, the third and the sixth, which means that candidate a cannot be simultaneously selected in the corresponding profiles π and π^{-1}. In other words, violation of reversal symmetry is not possible in that case.

A scoring rule with runoff defined by the scoring vector $(1, \lambda, 0)$ violates reversal symmetry at a voting situation (p_1, p_2, p_3, p_4) when alternative b is elected in bipolar domain if and only if:

$$\begin{cases} p_3 + \lambda(p_1 + p_4) > p_1 + p_2 \\ p_4 + \lambda(p_2 + p_3) > p_1 + p_2 \\ p_3 + p_1 > p_2 + p_4 \\ p_2 + \lambda(p_1 + p_4) > p_1 + \lambda(p_2 + p_3) \\ p_3 + p_4 > p_1 + \lambda(p_2 + p_3) \\ p_1 + p_2 > p_3 + p_4 \\ p_1 + p_2 + p_3 + p_4 = n \end{cases}$$

The system of inequalities above characterizes situations at which reversal symmetry is violated under scoring rules with runoff, when candidate b is elected. We obtain a contradiction for all values of λ. This conclusion is also true for candidate c. The system therefore satisfies reversal symmetry for all three candidates in bipolar domain.

Table 2 below provides frequencies for plurality and anti-plurality rules in bipolar domain; the table shows the effect of homogeneity on the violation of reversal symmetry. Not only the hierarchy in the violation of the property is reversed when we move from dependence (IAC) to independence (IC), but furthermore, under plurality rule almost all profiles tend to violate reversal symmetry as soon as the number of voters is greater than 75, while less than half of profiles violate the property under anti-plurality.

Table 2 Frequencies of violation of reversal symmetry under simple scoring rules in bipolar domain (3 candidates)

IAC model			IC model		
n	Plurality	Anti-plurality	n	Plurality	Anti-plurality
3	0	0.200000	3	0	0.0.187500
5	0	0.214285	5	0	0.175781
7	0.066666	0.233333	7	0.205078	0.203369
9	0.090909	0.272727	9	0.307617	0.271133
11	0.098901	0.285714	11	0.352478	0.279622
13	0.121428	0.121428	13	0.452493	0.289735
15	0.132352	0.308823	15	0.515499	0.323222
21	0.154150	0.328063	21	0.653124	0.355805
27	0.167487	0.339901	27	0.748367	0.377460
75	0.200956	0.369788	75	0.976952	0.434376
∞	0.222222	0.388889	–	–	–

4.2 Large Electorates and Three Candidates

In this subsection we come back to the three-candidate case but we consider all possible simple scoring rules in bipolar domain. We provide formulas and graphs showing how the frequencies of violation of reversal symmetry by scoring rules change in bipolar domain as n tends to ∞.

Let $g(\lambda) = \lim_{n\to\infty} g_\lambda(n)$ be the frequency of violation of reversal symmetry under simple scoring rule based on scoring vector $(1, \lambda, 0)$. We obtain the propositions below.

Proposition 12 *Given a simple scoring rule defined by a scoring vector* $(1, \lambda, 0)$*, under IAC the frequency of violation of reversal symmetry* $g(\lambda, \infty)$ *in bipolar domain when* $n \to \infty$ *is given by the following polynomials:*

$$g(\lambda) = \begin{cases} -\frac{(\lambda+1)(5\lambda-4)(2\lambda-1)^2}{9(\lambda-2)(\lambda-1)} & \text{if } \lambda \in [0, \frac{1}{2}] \\ -\frac{(\lambda-8)(2\lambda-1)^2}{18\lambda} & \text{if } \lambda \in [\frac{1}{2}, 1] \end{cases} .$$

Again, the reader can check that for $\lambda = 0$, $\lambda = \frac{1}{2}$ and $\lambda = 1$, we obtain exactly the same values as for $g_0(n)$, $g_{\frac{1}{2}}(n)$ and $g_1(n)$, respectively, when n tends to infinity. From these formulas we obtain the curve in Fig. 3.

The curve in Fig. 3 is U-shaped, as for Fig. 1, but limit values are greater than for universal domain; in other terms, restricting the domain to bipolar preferences in the three-alternative case increases the limit frequencies of violation of reversal symmetry.

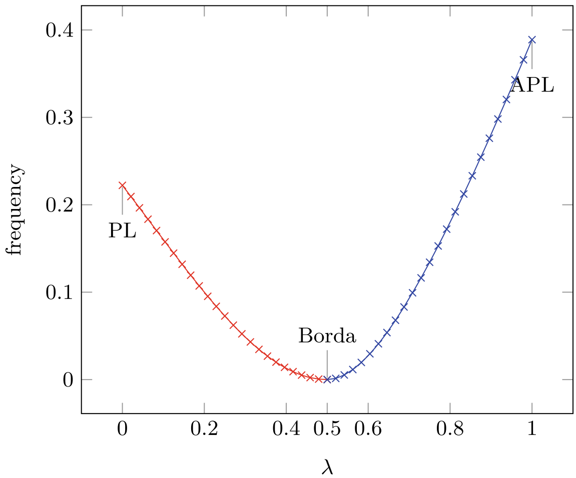

Fig. 3 Frequencies, under the IAC assumption, of violation of reversal symmetry in the three-candidate case by simple scoring rules in bipolar domain when $n \to \infty$

4.3 Four Candidates

With four candidates scoring rules are now defined according to scoring vectors $(1, \lambda_1, \lambda_2, 0)$ with $\lambda_1, \lambda_2 \in [0, 1]$ and $\lambda_2 \leq \lambda_1$. The number of possible individual preferences depends on how we distribute the candidates along the bipolar domain; we can distinguish two cases: (i) we first assume that we have two candidates on the left-wing and two candidates on the right-wing, resulting in $(2! \times 2!) \times 2 = 8$ eight possible preference orders, and (ii) we then consider the case where there is a single candidate in one ideological camp and all others are in the other camp; this ultimately results in $(1! \times 3!) \times 2 = 12$ possible preference orders.
We first consider case (i). With $A = \{a, b, c, d\}$ we have $m = 4$, and the following eight linear orders:

$$T_1 = abcd;\ R_2 = abdc;\ T_3 = bacd;\ T_4 = badc;$$
$$T_5 = cdab;\ T_6 = cdba;\ T_7 = dcab;\ T_8 = dcba;$$

Let t_k be the number of individuals reporting preference order T_k. A simple scoring rule defined by a scoring vector $(1, \lambda_1, \lambda_2, 0)$, violates reversal symmetry at voting situation $(t_1, \ldots, t_8)$ when alternative a is elected in bipolar domain if and only if

$$\begin{cases} t_1 + t_2 + \lambda_1(t_3 + t_4) + \lambda_2(t_5 + t_7) > t_3 + t_4 + \lambda_1(t_1 + t_2) + \lambda_2(t_6 + t_8) \\ t_1 + t_2 + \lambda_1(t_3 + t_4) + \lambda_2(t_5 + t_7) > t_7 + t_8 + \lambda_1(t_5 + t_6) + \lambda_2(t_2 + t_4) \\ t_1 + t_2 + \lambda_1(t_3 + t_4) + \lambda_2(t_5 + t_7) > t_5 + t_6 + \lambda_1(t_7 + t_8) + \lambda_2(t_1 + t_3) \\ t_6 + t_8 + \lambda_1(t_5 + t_7) + \lambda_2(t_3 + t_4) > t_5 + t_7 + \lambda_1(t_6 + t_8) + \lambda_2(t_1 + t_2) \\ t_6 + t_8 + \lambda_1(t_5 + t_7) + \lambda_2(t_3 + t_4) > t_1 + t_3 + \lambda_1(t_2 + t_4) + \lambda_2(t_5 + t_6) \\ t_6 + t_8 + \lambda_1(t_5 + t_7) + \lambda_2(t_3 + t_4) > t_2 + t_4 + \lambda_1(t_1 + t_3) + \lambda_2(t_7 + t_8) \\ t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8 = n \end{cases}$$

All violation voting situations for specific scoring rules can be deduced by assigning precise values to λ_1 and λ_2. Since the approach used to evaluate the frequency of violation of reversal symmetry is very similar to the one used for three alternatives in Sect. 3 and in Subsection 4.2, we will then simply give numerical values of frequencies in Table 3, obtained by complete computer enumeration.

The next step is the examination of violations of reversal symmetry by scoring rules with runoff for four candidates, in bipolar domain. Violation of reversal symmetry will depend on scenarios described by a system of inequalities. For each scenario, it is possible to know whether the corresponding system contains or not a contradiction, implying that the property is satisfied.
These different scenarios are obtained by successively assuming that a candidate from an ideological group, say a, is selected at the first round, and is at the second round in a contest against the other candidate of the same ideological group, b, or against one of the ideological group $\{c, d\}$. In order to avoid repetition, and since all

four scenarios are based on the same reasoning, we give some details for only one of these scenarios.

Scenario: *Consider a profile* π, *with four candidates* $\{a, b, c, d\}$ *in a bipolar domain. A candidate, say a, from the first pole* $\{a, b\}$ *is faced with another candidate, say c, from the second pole* $\{c, d\}$ *in the second round in profile* π; *these same two candidates will face each other in the second round in the reversed profile. The question is then as follows: will it be possible to have the same winner in both profiles* π *and* π^{-1}?

$$\begin{cases} t_1 + t_2 + \lambda_1(t_3 + t_4) + \lambda_2(t_5 + t_7) > t_3 + t_4 + \lambda_1(t_1 + t_2) + \lambda_2(t_6 + t_8) \\ t_1 + t_2 + \lambda_1(t_3 + t_4) + \lambda_2(t_5 + t_7) > t_7 + t_8 + \lambda_1(t_5 + t_6) + \lambda_2(t_2 + t_4) \\ t_5 + t_6 + \lambda_1(t_7 + t_8) + \lambda_2(t_1 + t_3) > t_3 + t_4 + \lambda_1(t_1 + t_2) + \lambda_2(t_5 + t_7) \\ t_5 + t_6 + \lambda_1(t_7 + t_8) + \lambda_2(t_1 + t_3) > t_7 + t_8 + \lambda_1(t_5 + t_6) + \lambda_2(t_2 + t_4) \\ t_1 + t_2 + t_3 + t_4 > t_5 + t_6 + t_7 + t_8 \\ t_6 + t_8 + \lambda_1(t_5 + t_7) + \lambda_2(t_3 + t_4) > t_5 + t_7 + \lambda_1(t_6 + t_8) + \lambda_2(t_1 + t_2) \\ t_6 + t_8 + \lambda_1(t_5 + t_7) + \lambda_2(t_3 + t_4) > t_1 + t_3 + \lambda_1(t_2 + t_4) + \lambda_2(t_5 + t_6) \\ t_2 + t_4 + \lambda_1(t_1 + t_3) + \lambda_2(t_7 + t_8) > t_5 + t_7 + \lambda_1(t_6 + t_8) + \lambda_2(t_1 + t_2) \\ t_2 + t_4 + \lambda_1(t_1 + t_3) + \lambda_2(t_7 + t_8) > t_1 + t_3 + \lambda_1(t_2 + t_4) + \lambda_2(t_5 + t_6) \\ t_5 + t_6 + t_7 + t_8 > t_1 + t_2 + t_3 + t_4 \\ t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8 = n \end{cases}$$

It is worth noting that independently of the values of λ_1 and λ_2 the fifth and the tenth inequalities are incompatible, which means that the system has no solution, and establishes the fact that reversal symmetry is satisfied in this case by all scoring rules with runoff.

The same remark is true for all other scenarios: a contradiction appears between two inequalities, which excludes any possible solution, for all values of λ_1 and λ_2. It follows that for four alternatives in bipolar, all scoring rules with runoff, satisfy the axiom of reversal symmetry.

Table 3 sheds additional light on the effect of opinions dependence in bipolar domain: when the number of candidates increases from 3 to 4 the frequencies decrease, and the difference is much less pronounced between the IC and IAC assumptions. This is particularly true under equal distribution of candidates between the two poles. Unequal distribution between poles leads to higher violation frequencies. For the $a(bcd)$ distribution, our techniques and computer program have been inefficient to deliver results under the IC assumption.

Table 3 Frequencies of violation of reversal symmetry under simple scoring rules in bipolar domain (4 candidates)

(ab)(cd) distribution					(a)(bcd) distribution		
IAC model			IC model		IAC model		
n	Plurality	Anti-plurality	Plurality	Anti-plurality	*n*	Plurality	Anti-plurality
4	0	0.012121	0	0.023437	4	0.123076	0.023443
5	0	0.020020	0	0.029296	5	0.134615	0.065934
7	0.018648	0.032634	0.025634	0.030441	7	0.249874	0.103318
9	0.019580	0.047552	0.021929	0.043258	9	0.310073	0.133960
11	0.024635	0.055806	0.033044	0.043868	11	0.362943	0.153948
15	0.032554	0.070925	0.039887	0.052071	15	0.433227	0.184548
21	0.039444	0.086037	0.046227	0.062367	21	0.496694	0.215491
27	0.044565	0.095156	0.053791	0.065718	27	0.535646	0.235180
75	0.057948	0.121524	–	–	75	0.633391	0.290180

5 Conclusion

The main contribution of this chapter is the evaluation of scoring rules vis-à-vis the axiom of reversal symmetry. We know from Saari's work that Borda rule is the only scoring rule satisfying reversal symmetry. Our results show that: (i) for three alternatives, in universal domain the anti-plurality rule is more sensitive to violation of reversal symmetry than the plurality rule; (ii) the hierarchy between these two rules in terms of their vulnerability to this property is however reversed under runoff scoring rules, that is, plurality with runoff becomes more sensitive to violation of reversal symmetry than anti-plurality with runoff; (iii) the introduction of a second round in universal domain reduces frequencies of violation for plurality and anti-plurality, but does not completely solve the problem; (iv) furthermore, the Borda rule, the only scoring rule that in its simple version is immune to violation of reversal symmetry now becomes one of the most vulnerable rules when there is a second round; (v) in bipolar domain, the frequencies of violation of reversal symmetry rise, especially when there is an asymmetry between the two poles (one candidate on one pole and two or three on the other pole); it then appears that restricting individual preference in a bipolar domain is clearly not the good solution if we look for a way to escape from the violation of reversal symmetry for simple scoring rules; (vi) nevertheless, the introduction of a second round on a bipolar domain eliminates all the cases of violation of reversal symmetry; (vii) with three alternatives and large electorates - more precisely in situations where the number of voters is so high than it can be considered to tend to infinity - it is possible to compare all possible scoring rules and to observe that for simple scoring rules we obtain a U-shaped curve and for scoring rules with runoff a bell-shaped curve as the value of λ changes from 0 to 1. Many Condorcet voting rules also violate reversal symmetry, and it would be interesting to contribute to the comparison of the two families of voting rules by evaluating also Condorcet voting rules relative performances vis-à-vis this property.

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Binary Voting in Federations

Majority Efficient Representation of the Citizens in a Federal Union

Marc Feix, Dominique Lepelley, Vincent Merlin, Jean-Louis Rouet, and Laurent Vidu

1 Introduction

In federal unions, a decision (or an election) often involves two steps, either because it is impossible to call the electors (e.g. the decisions in the European Union, where a minister represents his country and holds a certain number of mandates in the decision process) or for historical reasons (e.g. the US presidential election case, as the states did not want to lose their sovereignty in the early years of the union). In both cases, a crucial question is the choice of the "best" two tier voting system. More specifically, how many weights (or mandates, delegates, representatives, etc.) should be given to countries or states in a two tier voting system? And what quota should be required for a decision to be passed?

Marc Feix passed away on July 4th 2005 at the age of 78. Though he never saw the last version of this paper, he greatly contributed to an early version of this piece of work.

M. Feix
Nantes, France

D. Lepelley
CEMOI, Department of Economics, Université de la Réunion,
97715 Saint-Denis cedex 9, France
e-mail: lepelley@univ-reunion.fr

V. Merlin (✉) · L. Vidu
Faculty of Economics, Management, and Geography, CNRS and Université de Caen Normandie, Caen, France
e-mail: vincent.merlin@unicaen.fr

L. Vidu
e-mail: laurent.vidu@unicaen.fr

J.-L. Rouet
Université d'Orléans, CNRS/INSU, BRGM, ISTO, UMR7327, 45071 Orleans, France
e-mail: jean-louis.rouet@univ-orleans.fr

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Very different answers to these questions have been adopted by the different federal structures. When we look at the American and European cases, we find systems that try to navigate in between the pure federal system of “one state-one vote” (the threshold of 65% of the states in the European constitution and the +2 premium per state component in the US Electoral College) and the more democratic representation of the states proportionally to their population. Clearly, for all these schemes, the outcome was the result of a political bargain between the small states and the big states.

The purpose of this paper is to study a precise criterion to evaluate different two-tier voting methods and to study its impact on the analysis of two tier voting rules. Our criterion, the *majority efficiency*, requires the method to *minimize the probability that a decision is taken with a majority of weights at the federal level though it is supported by a minority of voters over the whole union*. This majority efficiency is inspired by the concept of *Condorcet efficiency* that has been developed in Social Choice Theory in order to discriminate between voting rules on their capacity to pick out the Condorcet winner whenever it exists. The main difference being that the majority winner is always well defined in a two candidate election.[1] More specifically, we will look for the best two-tier voting systems, using this normative criterion, in the particular case where all of the decisions are made with majority voting between two candidates. In doing so, we connect our study to the study of the so called *referendum paradox* (Nurmi 1999) and try to minimize its likelihood of occurrence. A referendum paradox occurs whenever a decision taken by representatives elected in local jurisdictions conflicts with the decision that would have been adopted if the voters had directly given their opinion through a referendum. In political science, Miller (2012) calls this phenomena an *election inversion* but also notices that public commentary uses terms such as “reversal winner”, “wrong winner”, “divided verdict”, etc.

Majority efficiency is not merely a theoretical object but a fairly natural criterion and a very significant issue. Indeed, such paradoxical situations occurred in past elections. For instance, with the election of George W. Bush against Al Gore in 2000 and the more recent victory of Donald Trump against Hilary Clinton in 2016.[2] Moreover, the existence of a federal union may be put in danger if it is plagued too often by these situations: A majority of the citizens would lose confidence in the institutions, leading to a political crisis. We also believe that, as the referendum paradox has been popularized by the media since year 2000, the criterion of majority efficiency could be more easily accepted by the public opinion than other normative criteria that have been presented in the social choice literature. We will review this literature and present some related definitions and concepts in Sect. 2.

Though the criterion of majority efficiency is simple to identify and popularize, the search of the apportionment rule which minimizes the occurrence of the referendum paradox is not that simple. An analytical solution of the problem remains out of reach. The situation is similar to the one we encounter in Social Choice Theory for the evaluation of the Condorcet efficiency of different voting rules: Unless the

[1] For more on this literature, see Gehrlein (2006) and Gehrlein and Lepelley (2011).

[2] For other examples in US, United Kingdom and France, see Feix et al. (2004).

number of parameters (voters, alternatives, and here states) is very small, we have to rely on computer simulations. Since we cannot test all of the possible apportionment methods, we will focus our analysis on the particular class of δ-rules. Let n_i be the population in the state i. Through simulations, we will try to identify which parameter δ minimizes the probability of the paradox if we allocate a_i seats for state i according to the law $a_i = n_i^{\delta}$. Although this formula does not take into account all of the possibilities, it covers the pure federal case ($\delta = 0$), the square-root rule ($\delta = 1/2$,), the proportional case ($\delta = 1$), and even the dictatorship of the biggest state ($\delta \to \infty$). Notice that we shall not study the question of the best threshold required to pass a decision. Throughout the paper, we will only consider the 50% quota.

In order to compute the probability of the referendum paradox, we need to set some *a priori* assumptions regarding the behavior of the voters. We will focus our analysis on two models previously introduced in the voting theory, the Penrose-Banzhaf model (Penrose 1946, 1952; Banzhaf 1965) and May's model (May 1948). These models, the probability assumptions regarding the actions of the voters and the methodology concerning simulations will be presented in detail in Sect. 3. Section 4 is devoted to the study of the δ-rules. In particular, we test some conjectures on what should be the best voting rule according to the probabilistic assumptions, based upon the results of the numerous simulations we have carried on. We first prove that the square root rule is not exactly the optimal rule under the Penrose-Banzhaf assumptions, the optimal values of δ always being slightly less than 0.5. On the other hand, the proportional rule always emerges as the optimal rule for the model proposed by May. Section 5 concludes.

2 Normative Criteria for Two-Tier Voting Rules

In this section, we present the different normative criteria that have been proposed to evaluate two-tier voting systems. An extensive survey has been recently published in French (Le Breton et al. 2017). We will just outline some major contributions in this section. They can be gathered in two categories. Historically, the first contributions were linked to the literature on power indices with the purpose to give to each voter the same influence on the decision process. At the turn of the millennium, the election of George W. Bush and the debates about the European constitution have seen a renewed interest in the economic literature for this issue. New criteria were proposed, all based on various utility principles.

2.1 *Equalizing Power and Influence: The Penrose-Banzhaf Model*

The most widely publicized normative claim about two-tier systems comes from the voting power literature, where many scholars endorse the so called *Penrose square root law* (Penrose 1946, 1952; Banzhaf 1965). In this literature, the key concept is the notion of a *decisive player*. A voter is decisive each time he can modify the outcome by changing his vote. Thus, the voting power or influence of a voter is defined as his a priori probability of being decisive, given some probability assumptions on the actions of the voters.

In the Penrose-Banzhaf model,[3] we assume that n voters have to choose between two exclusive proposals A and B. Abstention is not allowed, and there is no bias in favor of either alternative (such as a *statu quo* alternative). We assume that each vote is determined by independently flipping a fair coin randomly; In game theory, this hypothesis has been called the Independence assumption (Straffin 1977), and it is equivalent to the Impartial Culture (IC) model used in social choice literature for the computation of voting paradox probabilities (Gehrlein 2006; Gehrlein and Lepelley 2011). A vote configuration is the list of the ballots chosen by the voters. Under the Penrose-Banzhaf random voting model, all of the 2^n vote configurations are equally likely, and the power of voter j is simply the proportion of the configurations of the other $n-1$ votes for which voter j is decisive.[4] The power of voter j is then given by the number of situations in which he is decisive divided by the total number of vote configurations, 2^n. This is exactly the well-known (non normalized) *Banzhaf Power index*:

$$\text{Banzhaf power of voter}\, j = \frac{\text{Number of configurations for which voter}\, j \text{ is decisive}}{\text{Total number of voting configurations}} \tag{1}$$

How can we extend this power index to the framework of two tier decision models? In a federal union, a voter casts his vote in his home state for party A or B. The winner in state i is the party which obtains a majority of votes (abstention is not allowed) among the n_i citizens. Each state i is represented at the federal level by a_i weights,[5] and the winner in state i catches all these weights. Then, the position that is officially adopted by the union is the one which obtains a majority of weights at the federal level. Since the voters cast their ballots independently, the probability that a voter will be decisive is the product of the probability that he is decisive in his home state times the probability his state is decisive at the federal level.

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[3]The reader can find very nice introductions to these concepts in a series of papers (Gelman et al., 2002, 2004).

[4]If the number of weights is even, ties may occur with majority rule. A way to avoid such situation is to assume that the number of weights is odd, or to flip a fair coin to take a decision in case of a draw, or to ask for a new election until a clear decision is obtained, etc.

[5]Or, equivalently, a_i representatives are elected to seat at the federal level (European assembly).

$$\text{Probability that } j \text{ is decisive in the union} = \text{Probability that state } i \text{ is decisive} \times \text{Probability that voter } j \text{ is decisive in state } i \tag{2}$$

Penrose's limit theorem (Penrose 1952) says that, in a weighted majority game,[6] if the number of voters increases indefinitely and the relative quota is pegged, then, under certain conditions, the ratio between the voting power of any two voters converges to the ratio between their weights. However, this result is only implicit in Penrose's work, and the result is only valid "most of the time", as counter examples exist.[7] However, when we draw randomly the populations of the different states of the union, it has been proven that in 99% of the cases (Chuang et al. 2006) that

$$\frac{\text{Banzhaf power of state } i \text{ in the union}}{\text{Banzhaf power of state } i' \text{ in the union}} \approx \frac{a_i}{a_{i'}} \tag{3}$$

But in two-tier voting, we also have to estimate the power of voter j in state i. If n_i is odd, voter j is decisive for

$$\binom{n_i - 1}{(n_i - 1)/2} \tag{4}$$

configurations among the $2^{n_i - 1}$ possible vote configurations of the other citizens of state i. For n_i large, this can be approximated as $\sqrt{\frac{2}{\pi n_i}}$. When the behavior of the voters is governed by the Penrose-Banzhaf assumptions, we immediately deduce from Eqs. (2), (3) and (4) that

$$\frac{\text{Probability that voter } j \text{ from state } i \text{ is decisive in the union}}{\text{Probability that voter } j' \text{ from state } i' \text{ is decisive in the union}} \approx \frac{a_i}{a_{i'}} \frac{\sqrt{n_{i'}}}{\sqrt{n_i}} \tag{5}$$

and that equal treatment in term of power is, approximately achieved if a_i is proportional to $\sqrt{n_i}$.

Admittedly, in real life, voters seldom flip coins independently before casting their vote. The analysis of electoral data for a fifty years period, has showed that Straffin's Independence assumption had to be rejected for the elections of senators, representatives and president in the United States (Gelman et al. 2004). Similar

[6] In a weighted majority game, each player is endowed with a positive weights a_i. A coalition is winning if the sum of the weights of its members exceeds some predefined quota q.

[7] A formal version of Penrose's statement has been proposed recently (Lindner and Machover 2004). Using simulations, the validity of the approximation for numerous partition of the population among states has been tested; It has been shown that it is valid with a probability close to one for the non normalized Banzhaf index and a quota of 50% (Chang et al. 2006). The proportionality between the weight and the Banzhaf index is even better for some super majority rules (Feix et al. 2007; Słomiczyǹski and Życzkowski 2007). To give an example, the proportionality between the mandantes and the Banzhaf power is almost perfectly met for the enlarged European Union if we attribute to each state a number of weight in proportion to the square root of its population and use a quota of 61.5% (Feix et al. 2007). Recent works with applications to the European Union are perfect examples of this tradition (Felsenthal and Machover 1998, 2001, 2004).

conclusions are drawn from the electoral data collected over Europe. A way out of this problem is to recognize that the probability of being decisive depends on the probability distribution of the configurations for which a voter is decisive. That is to say that different probability assumptions, modeling different versions of the veil of ignorance, can be used.

2.2 Alternative Models

An immediate alternative to the Penrose-Banzhaf model (and the Independence assumption) is proposed by the Shapley-Shubik index (Shapley and Shubik 1954). Its probabilistic interpretation is slightly different: before casting their vote, all the voters choose a common probability p to vote for A from the uniform distribution on [0, 1]. This probabilistic interpretation is known as the Homogeneity assumption (Straffin 1977). As a consequence, each partition of the votes between A and B is equally likely. That is, the probability that A gets 0% of the vote is equal to the probability that he gets 15%, 51% or 89% of the vote. Hence, the probability that is attached to a configuration with t votes for A and $(n - t)$ votes for B is no longer $1/2^n$, but it is now given by:

$$\frac{1}{(n+1)\binom{n}{t}} \tag{6}$$

One should notice that the Impartial Anonymous Culture (IAC) used in Social Choice theory to compute the likelihood of the Condorcet paradox gives the same probability when used in binary elections (Berg 1999). However, switching from the Penrose-Banzhaf model to the Shapley-Shubik model does not modify the normative recommendation when we wish to equalize the influence of the different voters. Though Eq. (2) is no longer valid as the votes of the different states are now correlated, it can be proved that the square root principle still applies as a first approximation in order to equalize power (Owen 1975). However, using the Shapley-Shubick index still leaves us unsatisfied, as the same common prior p will be used over the whole Union. That is, different opinions cannot emerge from state to state.

Thus, we chose to use another model as a second benchmark model in this paper. We will assume that the probability of voting for A in state i, p_i, is drawn independently from state to state from a uniform distribution on [0, 1]. In other words, we assume homogeneity within each state and independence across the states. This model was used as early as 1948, in a contribution that had been ignored for a long time (May 1948).[8] May considers N states with n voters each. He presents his model in very simple terms, as a statistical problem. In each state, there are $n + 1$ balls, each one marked with a number from 0 to n. They correspond to all the possible results for party A. Then, an election consists of drawing one of these balls independently

[8]We are indebted to John Roemer and Hannu Nurmi, who mentioned this reference to us ten years ago.

in each state. Candidate A may win a majority of votes in a majority of states, but the sum of the numbers may be less than $(nN)/2$, meaning that candidate B gets more votes on average. This is a referendum paradox in modern terms, and May, using several central limit theorems, proves that its probability tends to 1/6 as n and N tend to infinity.These results were rediscovered much later in the literature (Feix et al. 2004). Notice that, as soon as n_i is large enough, the probabilistic interpretation of an independent p_i drawn from [0, 1] is equivalent to independent draws in m urns, with each urn having $n_i + 1$ balls marked from 0 to n_i.

Hence, under May's assumption (revisited to cope with unequal populations), Eq. (2) still holds and the power of a voter inside his state is given by the Shapley-Shubik index. Given a voter j, there are $\binom{n_i-1}{(n_i-1)/2}$ configurations of the other voters which split equally between A and B in state i, so that voter j is pivotal whether he votes for A or B. Thus:

$$\text{Shapley-Shubick power for player } j \text{ in state } i = \frac{2\binom{n_i-1}{(n_i-1)/2}}{(n_i+1)\binom{n_i}{(n_i-1)/2}} = \frac{1}{n_i}. \quad (7)$$

As each state will cast its vote for A or B with probability $1/2$ due to the independence of the draws across the states, we can still use the non normalized Banzhaf index to evaluate the power of state i. Penrose's approximation still applies and:

$$\frac{\text{Probability that } j \text{ from state } i \text{ is decisive in the union}}{\text{Probability that } j' \text{ from state } i' \text{ is decisive in the union}} \approx \frac{a_i}{n_i}\frac{n_{i'}}{a_{i'}} \quad (8)$$

Contrary to Eq. (7), equal treatment is now obtained when the number of weights is proportional to the population.

It should be clear by now that the objective of equalizing the probability of being decisive in two-tier systems has no clear answer. For example the square root rule law has been rediscovered in a model with a continuum of options (Napel and Maaser 2007), but a generalized version of the model proved that this result is fragile (Kurz et al. 2017). Similarly a completely different picture emerges by using a variation of the Shapley Shubik index that copes with the US electoral data (Owen et al. 2006). In this volume, de Mouzon et al. (2020) find the same diverse conclusions when they evaluate the power of the citizens in the US Electoral College with different probability assumptions. More than 70 years after the first contributions, it is clear that the choice of the right apportionment rule is completely driven by the characteristics of the underlying probability model governing the behavior of the voters, which in turn defines a particular measure of the influence.

2.3 Utility Based Arguments

The argument that the citizens of the different states should be given equal power, that is equal probability to be decisive, was for a long time the only mathematical argument to evaluate the merits of a federal constitution. Nevertheless, the literature based upon the notion of decisive voters has been often criticized: a classical argument against the power indices approach is that the influence of a single voter is in any case extremely low in federal bodies like the USA or the EU, of magnitude $a_i/\sqrt{n_i}$ or a_i/n_i ! Thus, it is unlikely that citizens would fight for some fairer representation based upon the notion of power. As a consequence, alternative concepts such as equal satisfaction or equal opportunity of success were introduced. Using the independence assumption, Rae defines his index of success as the probability of being in the winning side (Rae 1969). It is well known that the Rae index can be linked to the Penrose-Banzhaf index and should then give equivalent normative recommendation (Felsenthal and Machover 1998; Laruelle and Valenciano 2005). Again, the same criticism applies: for many voting rules, the differences in term of success are so thin among the citizens that this concept does not bite either (Laruelle and Valenciano 2008).

More recently, new criteria have been proposed. These developments consist of a shift in focus to criteria based upon the sum of the utilities of the members of the society.

Felsenthal and Machover (1999) suggest that, for a federal union, the average difference between the size of the majority camp among all of the citizens and the number of citizens who agree with the decision made by the majority of the representatives in the states should be minimized. This criterion can be considered as an utilitarian one, in the sense that a satisfied (dissatisfied) voter gets a $+1(-1)$ utility level. It gives an estimation for the loss of utility of the society when the decision is not supported by a majority of voters. They prove that, under the independence assumption, the Penrose square root rule still applies as a solution to the problem of the choice of the best two tier voting rule.

Barberà and Jackson (2006) generalize the same idea. They assume that in a two party election, candidate A's partisans obtain a utility $u_j = 1$ if he is elected (and 0 in the other case), whereas the partisans of B obtain a utility $u_j = v,\ v \in [0, +\infty[$ if their preferred candidate is elected (and 0 in the other case). They also assume, that, at the federal level, a motion passes if it is supported by $q\%$ of the sum of the weights, with q possibly different from 1/2. Then, the optimal voting rule for two-tier election systems is the one that maximizes the expected total utility of voters. Barberà and Jackson call their criterion the *efficient utility* principle. Their first results are very general in the sense that they do not depend on a particular model of probability. They retrieve the square root rule (proportional rule) when they use a voting model similar to Penrose's one (May's one). More recently, two extensions of this model support the degressive proportionality principle in apportionment problems. In the first one, the voters face a series of binary choice, and their utility function is concave in the number of votes they won (Koriyama et al. 2013). In the second one, it is shown that accounting for participation constraints before entering the federal union entails overweighting the smallest states (Macé and Treibich 2019).

Beisbart et al. (2005) compared seven various possible decision rules for the European Union with respect to their capacity to choose motions which will have a positive total utility for its citizens, while rejecting the bad policies. But the probabilistic foundations of their model are different, in the sense that each country is modeled by a unique representative agent.

As noted in the introduction, for two candidate elections, maximizing the majority efficiency is tantamount to reducing the probability of the referendum paradox. The main difference between the utilitarian criterion discussed above and the majority criterion is that the latter ignores the importance of the paradox. It only attempts to estimate the number of situations where a majority of voters are frustrated, but does not evaluate the magnitude of the paradox, either by counting unhappy voters, as in Felsenthal and Machover (1999) or by summing-up the utilities, as in Barberà and Jackson (2006) and Beisbart et al. (2005). Moreover, since the referendum paradox has been popularized by the media after the U.S. elections of 2000 and 2016, this criterion could be accepted by public opinion more easily than any other criteria.

Another important conclusion can be drawn from this review of the literature. Whatever the criteria (equalizing power or success, maximizing the utility), the choice of the optimal apportionment rule seems to be driven by the underlying probability assumptions. The independence assumption seems to always point toward the square root rule, while models which assume that the variance of the results grows as n_i favor weights proportional to the size of the population. Thus, it is clear that one of the questions at stake in this paper is to know whether the square root rule (resp. the proportional rule) will be the optimal apportionment rule in terms of majority efficiency when the Penrose-Banzhaf's (resp. May's) assumption is used.

3 Methodology

3.1 The Model

Consider a finite set $I = \{1, \ldots, i, \ldots, N\}$ of states (or regions, districts, etc.) which have to make decisions altogether in a political union. We assume that n_i voters live in state i, and $\sum_{i=1}^{N} n_i = n$. The vector $\tilde{n} = (n_1, \ldots, n_i, \ldots, n_N)$ describes the partition of the population among the N states. Without loss of generality, we will assume throughout the paper that $n_1 \geq n_2 \ldots \geq n_N > 0$. Two parties, A and B, compete in all the states; the winner in state i is the party that obtains a majority of voters on its side (abstention is not allowed). Each state is represented by a_i weights in the union, and the winner in state i gets all the weights. For the sake of simplicity, we set that $a_1 \geq a_2 \geq \ldots \geq a_N \geq 0$, with at least a_1 strictly positive. Thus, the position that is officially adopted by the union is the one which obtains a majority of weights at the federal level. Notice that we always use throughout the paper the quota of 50% for all the decisions (votes in the states, vote of the delegates and popular vote nationwide).

In our search of the apportionment rules that minimize the probability of the referendum paradox, we have decided to focus our study on the family of δ-rules. That is, we assume that the vector of weights, $\tilde{a} = (a_1, a_2, \ldots, a_N)$, is entirely characterized by the parameter $\delta \in [0, \infty[$ as $a_i = n_i^{\delta}$ $\forall i = 1, \ldots, N$. As already stated, we will recover the most famous apportionment rules (pure federalism, square root, proportionality and dictatorship) by changing the value of δ. But clearly, our main objective is to check whether the recommendations we should adopt when we wish to minimize the likelihood of the referendum paradox are compatible with solutions that have been put forward when one wishes to equalize the power of the citizens, by setting $\delta \approx 1$ for May's model and $\delta \approx 0.5$ for Penrose-Banzhaf.

3.2 On Probability Assumptions

As seen in Sect. 2, there are several ways to theoretically model the behavior of the citizens; we present here these assumptions with more details. We model the peoples' vote inside each state and we assume that their behavior is described by the same probability distribution in every state. Furthermore, we assume that the votes from state to state are always drawn independently. Thus, the probabilistic behavior of a given state at the federal level is totally driven by the behavior of its own voters.

In the Penrose-Banzhaf model, each citizen votes independently of the others and selects among the two issues with equal probability. Due to the law of large numbers, the scale of the mean difference of ballots between the two issues will vary in $n_i^{1/2}$. The other classical model is the one proposed by May. It assumes that every partition of the votes between A and B is equally likely. Many interpretations of this model have been given (going from Polya urns to quantum Bose-Einstein statistics), a good one being a probabilistic interpretation (Straffin 1977; Berg 1999). The idea is the following: In state i, for a given election, a "public opinion" emerges, i.e. an individual probability p_i for selecting one of the issue is drawn from the uniform distribution on [0, 1]. Thus, the probability of picking A may be 0.1, 0.5, 0.7 or whatever you want in [0,1], with equal probability. Of course, p_i varies from one election to the other, but on average, there is no bias in favor of any alternative.

It is possible to generalize Berg's and Straffin's reasoning by assuming that the choice of a probability p is itself of probabilistic nature through the introduction of a probability distribution function $f(p)$. The choice of $f(p)$ is a first step for a better description of the electorate behavior. In particular, it could be determined from the study of real data. The distribution $f(p)$ is defined on $0 \leq p \leq 1$, with $f(p) \geq 0$ and $\int_0^1 f(p)dp = 1$. The probability of a given configuration of n identifiable voters with t votes for A and $(n-t)$ votes for B is $p^t(1-p)^{n-t}$, and for a large number of elections it reads

$$\int_0^1 f(p)p^t(1-p)^{n-t}dp. \tag{9}$$

Following the social choice terminology, we will call this model the Generalized Impartial Anonymous Culture (GIAC) assumption.

If $f(p) = \delta(p - 1/2)$, where δ is the Dirac distribution function, that is if p is equal to $1/2$ for all elections, then the Penrose-Banzhaf model (IC in social choice terminology) is recovered with a probability of $1/2^n$ for all configurations.

For $f(p) = 1$, we get May's model (IAC in social choice terminology) and we obtain

$$\frac{1}{(n+1)\binom{n}{t}} \tag{10}$$

for the probability of a configuration with t votes for A and $n - t$ votes for B. As there are $\frac{n!}{t!(n-t)!}$ voting configurations with t A's and $(n - t)$ B's, we recover the fact that any result is equally likely, as in May's paper.

The Rescaled IAC assumption (RIAC) for which the p_i are independent and are drawn from the distribution

$$f_i(p_i) = \begin{cases} \frac{1}{2\Delta_i} & \text{if } \frac{1}{2} - \Delta_i < p_i < \frac{1}{2} + \Delta_i \\ 0 & \text{elsewhere} \end{cases} \tag{11}$$

for $i = 1, \ldots, N$ is also of special interest. Here, Δ_i is a strictly positive value less than $1/2$. It has been introduced under the name of Biased and Rescaled IAC (BRIAC) (Feix et al. 2004), with the possibility of a bias in favor of one candidate that we omit in this paper. If $\Delta_i = 1/2$, for all i, then May's model is recovered, while in the limit $\Delta_i \to 0$, for all i, the RIAC model tends toward the Penrose-Banzhaf model if the population is finite according to Berg's interpretation.

RIAC can be interpreted as follows: It means that a percentage $1/2 - \Delta_i$ of the population of state i always votes for A, while the same percentage always votes for B; only a fraction $2\Delta_i$ of the population hesitates between both alternatives. Thus, the RIAC model can be used to overcome the limitations of both the IC model (elections are too close) and the IAC model (the range of the results is too spread out). For example, $\Delta_i = 0.2$ means that the results vary from 30% of the votes for A to 70% of the votes for A from election to election, quite a realistic pattern!

3.3 Simulation Techniques

The probability of observing the referendum paradox has been studied analytically, and through Monte Carlo simulations as soon as N is greater than 5, when all the states were assumed to have the same population and consequently the same number of weights (Feix et al. 2004). Only small differences were found between IC and IAC models: with 101 states, the probabilities of the referendum paradox seem to stabilize around 16.5% for IAC and around 21.5% for IC. The results of these simulations are confirmed with approximations based upon the law of large numbers (Lepelley et al. 2011). The value under IAC is consistent with the $\frac{1}{6}$ asymptotic result (May 1948). Here we study the conflict frequency between the direct (popular) vote and the two-tier decision when the size of the population can differ from state to state, for a given apportionment of the weights, and under different probability assumptions; we denote this probability by $P(N, \tilde{n}, \tilde{a}, I(A)C)$. Here, N is the number of states,

$\tilde{n}$ the distribution of the population among the N states, $\tilde{a}$ the apportionment rule we study and I(A)C is either IC or (R)IAC. When $a_i = n_i^\delta$, we will simply write the probability $P(N, \tilde{n}, \delta, GIAC)$. Notice that all of the probabilities will be estimated for such large n_i's so that the discrete nature of the population size does not play any role. Thus, $\Delta_i \to 0$ does not mean that the RIAC model converges to the IC case, as it would be if the populations were finite.

For the IC model, each voter selects a party with equal probability. When n_i is sufficiently large, the distribution of the votes follows a normal distribution. In each state, the excess of ballots for A or B is then given by

$$\varepsilon_i \sqrt{n_i} \tag{12}$$

where ε_i is drawn randomly according to the Gauss distribution:

$$(2\pi)^{-1/2} \exp(-\varepsilon^2/2). \tag{13}$$

The popular vote over the whole union is given by

$$\operatorname{sgn}\left(\sum_i \varepsilon_i \sqrt{n_i}\right), \tag{14}$$

while the decision taken by the representatives is given by

$$\operatorname{sgn}\left(\sum_i a_i \operatorname{sgn}(\varepsilon_i)\right) \tag{15}$$

The sgn function is defined by

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{si } x \geq 0 \\ -1 & \text{si } x < 0 \end{cases} \tag{16}$$

By convention, a value of one (minus one) results in the selection of candidate A (B). A difference in sign between (14) and (15) means that we observe a paradox.

Next, we will study the RIAC models, defined by Eq. (11). Thus, the excess of ballots for A (or B) is given by $\varepsilon_i 2\Delta_i n_i$ where ε_i is drawn from the uniform distribution on $[-1, 1]$. Notice that we can assume different $\Delta_i \neq 0$ for the different states. Then, we have to compare:

$$\operatorname{sgn}\left(\sum_i \varepsilon_i 2\Delta_i n_i\right) \tag{17}$$

for the popular vote with the vote of the representatives

$$\operatorname{sgn}\left(\sum_i a_i \operatorname{sgn}(\varepsilon_i)\right) \tag{18}$$

We can reinterpret the RIAC model with the n_i's as an IAC model with new populations given by $n'_i = 2\Delta_i n_i$. Let us draw ε_i from an uniform distribution on $[-1, 1]$. Thus, we have to compare the total excess ballots, given by:

$$\operatorname{sgn}\left(\sum_i \varepsilon'_i n'_i\right) \tag{19}$$

with the vote of the representatives, given by

$$\operatorname{sgn}\left(\sum_i a'_i \operatorname{sgn}(\varepsilon'_i)\right) \tag{20}$$

Equations (17)–(18) describe the same problem as Eqs. (19)–(20) and the two interpretations are equivalent. In words, the probability of the paradox for a RIAC model with respective populations $n_1 = 4, n_2 = 3$ and $n_3 = 3$ while $\Delta_1 = 0.1, \Delta_2 = 0.2$ and $\Delta_3 = 0.3$, is given by a simulation with the IAC model with new populations $n'_1 = 0.8$, $n'_2 = 1.2$ and $n'_3 = 1.8$. Moreover, when $\Delta_i = \Delta$ for all the states, the results are directly given by an IAC simulation.

Thus, we can focus on the two models IC and IAC to test the optimality of the different apportionment rules in various scenarios using Monte Carlo techniques to simulate a large number of votes for a large number of states. Also, exact formulas for the three state case with unequal populations have been obtained for both the IC case (Lepelley et al. 2014) and the IAC case (Kaniovski and Zaigraev 2018). The simulations results we will obtain are in perfect agreement with their theoretical results.

4 The General Study of the δ-Rules Via Simulations

4.1 Staircase Curves

We assume that $a_i = n_i^\alpha$ for the IC case and $a_i = (2\Delta_i n_i)^\beta$ for the RIAC case. Our objective is to check whether the recommendations we should adopt when we wish to minimize the likelihood of the referendum paradox are compatible with approximate solutions that have been suggested in the power index literature ($\beta = 1$ for (R)IAC, $\alpha = 0.5$ for IC). But before getting to the heart of the matter, we first study two cases of unequal populations in detail, to highlight several facts about the shapes of the curves we obtain.

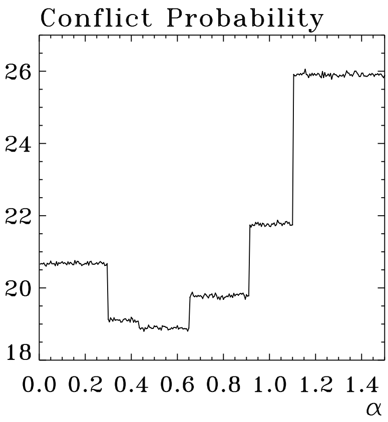

Fig. 1 IC model : Percentage of conflicts between states and population decision for $N = 5$ states of population $\tilde{n}^{\star} = (10.24, 5.29, 3.24, 1.96, 1)$ respectively

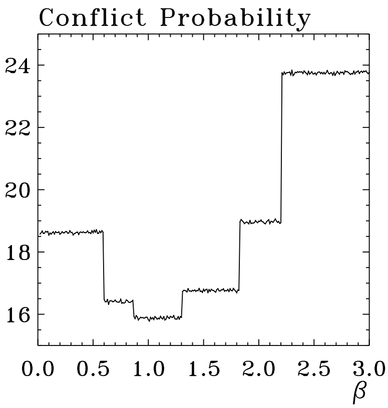

Fig. 2 RIAC model : same as figure 1. $2\Delta_i n_i$ =3.2, 2.3, 1.8, 1.4,1, for $i = 1, \ldots, 5$ (that is the square root of the values taken for Fig. 1)

Let us consider first the populations $(2\Delta_i n_i) = \tilde{n}' = (3.2, 2.3, 1.8, 1.4, 1)$ for a RIAC simulation, and the populations equal to the square of the previous ones for an IC simulation (i.e. $(n_i) = \tilde{n}^{\star} = (10.24, 5.29, 3.24, 1.96, 1)$).

For each value of α and β, represented by a point, 1,000,000 elections were generated randomly. Then, α (resp β) is incremented by step of 0.005 (resp 0.01) on the range [0, 1.5] (resp [0, 3]). Figures 1 and 2 display the results of a Monte Carlo simulation, with all of the points being connected for clarity.

First, the δ-rules can only be associated to a limited number of underlying weighted majority games.[9] We immediately recognize on the Figs. 1 and 2 six plateaus, with each one corresponding to an underlying weighted majority game. These plateaus give to the curve a "staircase" shape.

More surprising is the fact that we obtain very similar shapes for both cases, though the magnitudes along the horizontal and vertical axis are different. As we

[9]Recall that different weights can lead to the same set of winning coalitions (Taylor and Zwicker 1999).

have chosen $\tilde{n}^\star$ such that $n_i^\star = (n_i')^2$, if we assume furthermore that $\alpha = \beta/2$, Eqs. (14) and (15) become equivalent to Eqs. (21) and (22).

$$\text{sgn}\left(\sum_i \varepsilon_i \sqrt{(n_i^\star)}\right) = \text{sgn}\left(\sum_i \varepsilon_i \sqrt{(n_i')^2}\right) = \text{sgn}\left(\sum_i \varepsilon_i (n_i')\right) \tag{21}$$

$$\text{sgn}\left(\sum_i a_i \,\text{sgn}(\varepsilon_i)\right) = \text{sgn}\left(\sum_i (n_i^\star)^\alpha \,\text{sgn}(\varepsilon_i)\right) = \text{sgn}\left(\sum_i (n_i')^\beta \,\text{sgn}(\varepsilon_i)\right) \tag{22}$$

In other words, the games that we will encounter for an IAC simulation with a population (n_i) as β varies are the same ones that will encounter for an IC simulation with $\alpha = \beta/2$ and populations $(n_i)^2$. This explain why we recover a very similar staircase structure in both cases. However, one difference remains: on the one hand the ε_i are drawn from a normal distribution, and on the other hand, they are drawn from a uniform distribution. As a consequence, the minimal values of the paradox may not appear for the same plateau.

Let us now comment more precisely regarding Fig. 2. For the RIAC case, we encounter 6 plateaus. Easy computation shows they correspond to 6 of the 7 possible 5-person weighted majority games with an odd number of weights.[10] The vectors of weights are $\tilde{a}^1 = (1, 1, 1, 1, 1)$, $\tilde{a}^2 = (2, 2, 1, 1, 1)$, $\tilde{a}^3 = (3, 2, 2, 1, 1)$, $\tilde{a}^4 = (4, 2, 2, 2, 1)$, $\tilde{a}^5 = (5, 2, 2, 2, 2)$ and $\tilde{a}^6 = (1, 0, 0, 0, 0)$. We first meet a majority game with equal weights, and then, progressively move toward the dictatorship of state 1. The optimal value, which leads to a probability of about 16%, is obtained for values of β in between 0.9 and 1.3. It is superior to the exact value of 14.32 % that Feix et al. (2004) derived for five state with equal population. The pure federal case leads to a paradox in about 19% of the simulations, and the dictatorial case in about 24%.

It is also possible to encounter games with an even number of weights. For example, in between $\tilde{a}^1$ and $\tilde{a}^2$, the δ-rule is equivalent to the extra game $\tilde{a}^7 = (3, 3, 2, 2, 2)$ for a unique value of $\delta \approx 0.595420$. The reader will immediately realize that such games exist in between games with an odd number of weights, but that they are only realized at specific values of $\delta \approx 0.862767,\ 1.307213,\ 1.822085,\ 2.209592$. Then, the computer simulations, with finite steps for α or β, will never be able to catch them.

The picture for the IC case is very similar as we assumed $n_i^\star = n_i'$ and $2\alpha = \beta$. We recover the six plateaus, corresponding to the same six different voting games. However, the magnitudes are different. We start with a value of 21.5% in the federal case, and next obtain a minimum slightly lower than 20% for α in between 0.43 and 0.65 approximately, and then progressively go up to 25.5% for the dictatorial case.

Figures 3 and 4 present a much more complicated situation with 20 states. Due to the large number of possible majority weighted games with 20 players, the curves become almost continuous. The pattern of the two curves are similar: first a plateau

[10] The only "missing" game is defined by $\tilde{a} = (3, 3, 3, 0, 0)$.

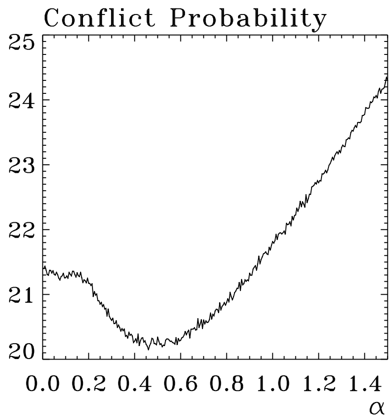

Fig. 3 IC model : same as Fig. 1 but for $N = 20$ states of population $n_i^\star$ = 35.88, 32.72, 30.69, 23.52, 18.84, 17.72, 17.31, 15.44, 14.06, 11.83, 11.49, 9.92, 8.82, 6.45, 5.81, 4.49, 3.13, 2.34, 1.72, 1, for $i = 1, \dots, 20$

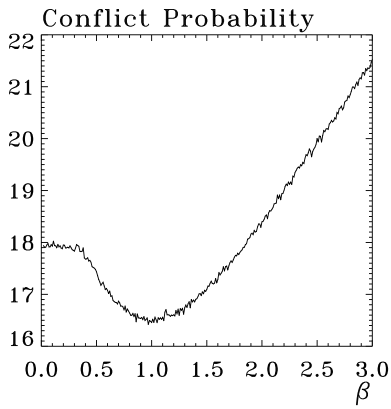

Fig. 4 RIAC model: same as Fig. 2 but for $N = 20$ states with n_i' =5.99, 5.72, 5.54, 4.85, 4.34, 4.21, 4.16, 3.93, 3.75, 3.44, 3.39, 3.15, 2.97, 2.54, 2.41, 2.12, 1.77, 1.53, 1.31, 1.,for $i = 1, \dots, 20$

around the federal case, then a decline till the optimal value (around $\beta = 1$ in the IAC case, and around $\alpha = 0.5$ in the IC model) and then a regular increase, at least in the range of values of α and β shown Figs. 3 and 4. Notice that we do not reach the dictatorial case for both simulations.[11]

4.2 Toward a General Result Under IAC

Until now, we have observed with the formulas for $N = 3$ states and two specific cases of unequal population (one for $N = 5$ states and one for $N = 20$) that $\beta = 1$ seems to be the optimal choice for a δ-rule under IAC in order to minimize the occurrence

[11] Some simulations, not presented in this paper, have shown that a second local minima, above the first one, may exist for higher values of α and β, before reaching the dictatorial case.

of the referendum paradox. In this section, we try to obtain general conclusions by systematically drawing several distributions $\tilde{n}$ for a given number of states.

Conjecture For the RIAC model, the minimum probability of conflicts between the popular and states among the class of δ-rules is obtained by taking $a_i = \Delta_i n_i$, that is, $\beta = 1$. □

We will systematically test the conjecture by simulations for $N = 3$ to $N = 33$, with an extra simulation for $N = 50$. For each value of N, we will first draw randomly 1,000 different federations from the uniform distribution on the unit simplex. Next, we will consider 100,000 elections for each federation[12] in order to estimate the likelihood of the paradox for different values of β, as explained in Sect. 3.3. β will vary from 0 to 2, with a step of 0.05, and an increased precision of 0.01 in between 0.9 and 1.1.

In Table 1, we summarize the results of these simulations. First, for each federation, we will check whether $\beta = 1$ leads to the minimal value of the paradox. Let $F(N, \beta, IAC)$ be the percentage of federations of size N for which β gives the minimal probability of the paradox. Column 2 gives the value of this frequency for $\beta = 1$. Column 3 and 4 gives the optimal $\beta^\star$ and the corresponding value $F(N, \beta^\star, IAC)$. We observe that $\beta = 1$ is almost all the time the value that maximizes the probability of getting the minimal occurrence of the referendum paradox, and when it is not the case, the optimal value is either $\beta = 0.99$ or $\beta = 1.01$! For $N = 3$, we observe 4 cases out of 1,000 where $\beta = 1$ is not optimal. Indeed, theoretical results have proven that $\beta = 1$ is the optimal value (Kaniovski and Zaigraev 2018). These value less than one are just a consequence of the fluctuations of the simulations. However, as N grows, $F(N, 1, IAC)$ plunges. But in fact, the number of possible underlying weighted majority games also increases with N, and now each value of β around 1 leads to a specific game; due to fluctuations of the random trials, values of β close to 1 can also frequently be designated as the optimal game. Thus, to check the optimality of $\beta = 1$, we consider column 5, which reports the maximal deviation for the optimal value of the paradox for the 1,000 federations of size N. With the exception of $N = 3$, this maximal deviation is always less than 0.15%.

At last, at the aggregated level, we will report in column 6 $P(N, \beta, IAC)$ for $\beta = 1$, the mean value of the $P(N, \tilde{n}, \beta, IAC)$ among the 1,000 federations. $\beta = 1$ has always produced the minimal value of the paradox, except for $N = 3$ where we observe a value of $P(N, \tilde{n}, 1.01, IAC) = 11.8738\%$, which can hardly be considered as significatively different from $P(N, \tilde{n}, 1, IAC)$.

To give a broader perspective on the behavior of $P(N, \beta, IAC)$, we display it for several values of N as β varies on Fig. 5. All the curves display the same pattern. We encounter the maximal value of the paradox for the pure federalism case ($\beta = 0$), and, after a plateau, the probabilities decline to their optimal values obtained for

[12] Ideally, we should have drawn a new set of elections for each value of β, as we did in the previous section. This would have allowed us to define a clear statistical test about the optimality of $\beta = 1$ compared to other values of β. However, one immediately realizes that drawing a new batch of data for each β would have enormously increased the computation time.

Table 1 Testing the optimality of $\beta = 1$ under the IAC assumption

N	$F(N, 1, IAC)$ (in %)	$\beta^\star$	$F(N, \beta^\star, IAC)$ (in %)	Max Dev (in %)	$P(N, 1, IAC)$ (in %)
3	99.6	1.00	99.6	0.252	11.8740
4	99.3	1.00	99.3	0.103	13.4914
5	97.4	1.00	97.4	0.113	14.2053
6	91.0	0.99	91.0	0.148	14.4684
7	78.8	1.00	78.8	0.104	14.9255
8	60.7	1.01	61.5	0.122	15.0010
9	37.0	1.00	37.0	0.106	15.3418
10	22.9	1.01	23.1	0.094	15.3632
11	18.8	1.00	18.8	0.111	15.5899
12	15.3	1.00	15.3	0.150	15.6656
13	12.6	1.00	12.6	0.088	15.8195
14	14.6	1.00	14.6	0.078	15.8253
15	14.1	1.00	14.1	0.094	15.8624
16	14.2	1.00	14.2	0.080	15.9419
17	13.0	0.99	13.5	0.096	15.9763
18	12.2	1.01	12.2	0.102	15.9965
19	14.8	1.00	14.8	0.080	16.0845
20	14.7	1.00	14.7	0.110	16.0911
21	13.0	1.00	13.0	0.079	16.1345
22	14.0	1.00	14.0	0.096	16.1118
23	13.4	1.00	13.4	0.106	16.1944
24	13.3	1.00	13.3	0.086	16.1965
25	12.5	1.00	12.5	0.091	16.2290
26	13.0	0.99	14.3	0.092	16.2487
27	14.9	1.00	14.9	0.108	16.2221
28	13.3	1.00	13.3	0.087	16.2842
29	15.5	1.00	15.5	0.093	16.2781
30	15.6	1.00	15.6	0.092	16.2790
31	14.0	1.00	14.0	0.093	16.3033
32	15.0	1.00	15.0	0.086	16.2947
33	12.7	0.99	13.6	0.092	16.3364
50	15.1	1.00	15.1	0.092	16.4370

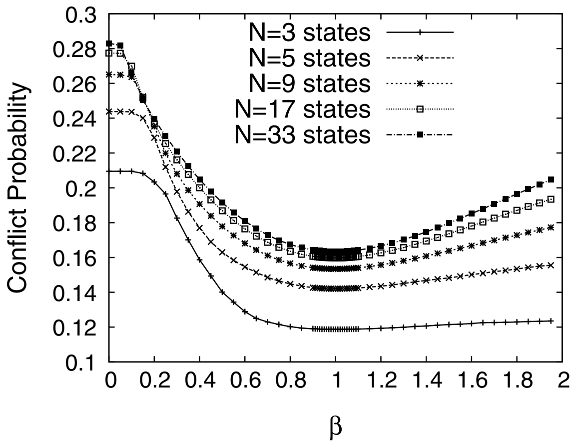

Fig. 5 The Probability of the referendum paradox as a function of β under the IAC assumption for an odd number of states

$\beta = 1$. After this point, the occurrence of the referendum paradox increases again. As N grows, the curves move up progressively and seem to reach a limit. In fact, in Table 1, the values of $P(N, 1, IAC)$ appear to converge to 16.5%, the limit value of the paradox for equally populated states that has been already observed (May 1948; Feix et al. 2004, Lepelley et al. 2011).

4.3 How Far from Optimality Is the Square Root Rule?

The method we use to simulate results under IC is similar to the one we use for the IAC assumption. For $N = 3$ to $N = 33$, we still draw 1,000 population vectors from the simplex, and then generate randomly 100,000 voting situations. We will add a simulation at $N = 50$ to have a glance at the large federation case. The only difference is that now ε_i is drawn from the Gaussian:

$$(2\pi)^{-1/2} \exp(-\varepsilon^2/2).$$

Different values of α are tested in between 0 and 1, with an increment 0.05, reduced to 0.005 in between 0.4 and 0.6.

Table 2 summarizes our findings. Let $F(N, \alpha, IC)$ be the percentage of federations of size N for which α gives the minimal probability of the paradox. Columns 2 and 3 display the optimal $\alpha^\star$ and the corresponding value, while Columns 4 gives the value of $F(N, 0.5, IC)$. The optimal α tends to be slightly smaller than 0.5, ranging from 0.435 to 0.505 according to N. This is consistent with the theoretical finding, which also suggested that the optimal value should be slightly below 0.5 (Lepelley et al. 2014). Again, the values for $F(N, \alpha, IC)$ decline quickly as N grows, as each specific value of α becomes associated with a specific game. Contrary to the IAC

Table 2 Testing the optimality of $\alpha = 0.5$ under the IC assumption

N	$\alpha^\star$	$F(N, \alpha^\star, IC)$ (in %)	$F(N, 0.5, IC)$ (in %)	Max Dev (in %)	$\alpha^\star$	$P(N, \alpha^\star, IC)$ (in %)	$P(N, 0.5, IC)$ (in %)
3	0.435; 0.445	98.9	91.8	1.750	0.440	17.1220	17.1925
4	0.440; 0.445	97.9	92.8	1.130	0.440	18.2866	18.3132
5	0.465	94.0	85.4	0.736	0.465	18.7005	18.7189
6	0.465; 0.470	86.0	76.4	0.509	0.470	19.0435	19.0569
7	0.47	67.7	59.3	0.338	0.475	19.2775	19.2858
8	0.48	50.5	44.4	0.220	0.480	19.4480	19.4544
9	0.475; 0.490	27.8	22.4	0.183	0.480	19.5767	19.5830
10	0.485	18.6	15.1	0.158	0.480	19.6675	19.6733
11	0.475	11.8	6.7	0.145	0.480	19.7516	19.7565
12	0.495	9.6	7.5	0.124	0.480	19.8325	19.8363
13	0.48	8.6	6.7	0.119	0.485	19.8877	19.8912
14	0.475	8.6	5.8	0.130	0.490	19.9342	19.9357
15	0.485	8.5	5.9	0.144	0.485	19.9679	19.9707
16	0.475	7.7	6.4	0.100	0.485	20.0278	20.0303
17	0.48	8.5	6.4	0.088	0.490	20.0554	20.0572
18	0.49	8.9	7.3	0.092	0.490	20.0839	20.0859
19	0.485	7.9	6.2	0.106	0.490	20.1099	20.1130
20	0.5	8.5	8.5	0.111	0.490	20.1395	20.1409
21	0.485	8.3	7.4	0.104	0.495	20.1523	20.1537
22	0.485; 0.495	8.1	6.2	0.101	0.495	20.1795	20.1803
23	0.5	8.1	8.1	0.097	0.490	20.2019	20.2026
24	0.5	8.4	8.4	0.107	0.490	20.2195	20.2209
25	0.495	8.5	6.2	0.084	0.495	20.2241	20.2245
26	0.49	9.0	7.3	0.109	0.490	20.2360	20.2368
27	0.505	8.5	7.3	0.091	0.500	20.2616	20.2616
28	0.49	8.3	7.8	0.096	0.490	20.2590	20.2598
29	0.49	7.9	7.0	0.098	0.490	20.2758	20.2766
30	0.475	8.9	8.5	0.111	0.490	20.2909	20.2919
31	0.485	8.3	7.3	0.084	0.490	20.2958	20.2964
32	0.485	8.9	8.1	0.098	0.495	20.3062	20.3065
33	0.48	8.4	7.7	0.108	0.495	20.3163	20.3165
50	0.49	8.7	7.8	0.094	0.495	20.4046	20.4053

case, we really observe in column 5 deviations from the optimum which are quite significant for small values of N.[13] However, the maximal deviations stay below 0.15 from $N = 10$.

At the aggregated level, we study the mean value of the paradox over the 1,000 federations, $P(N, \alpha, IC)$. Values for $P(N, 0.5, IC)$ are displayed on column 8. They are almost never the optimal values $P(N, \alpha^\star, IC)$ for small values of N, though the

[13] Again, we cannot propose a proper test as we use the same batch of elections to compute the $P(N, \tilde{n}, \alpha, IC)$ for different values of α.

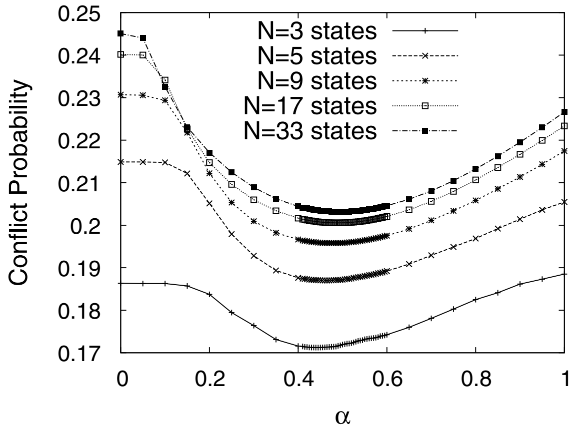

Fig. 6 The Probability of the referendum paradox as a function of α under the IC assumption for an odd number of states

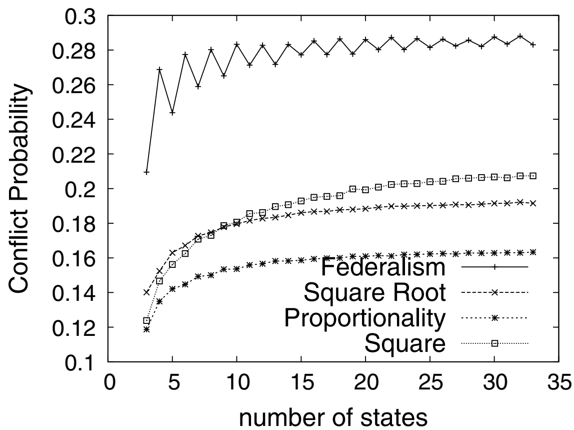

Fig. 7 Comparing different voting rules on their ability to avoid the referendum paradox under the IAC assumption

differences are tiny. The optimal value is $\alpha^{\star} = 0.44$ for $N = 3$, and it seems to converge toward 0.5. *Thus, the statement that the square root rule is "on average" the optimal two-tier voting rule under the IC assumption may be true for N sufficiently large* (Fig. 6).

As we did for the IAC case, we display in Fig. 6 the evolution of the paradox as α varies for different values of N. The curves exhibit a similar pattern: after a plateau around the value $\alpha = 0$, they decline continuously to their minimal value reached just before $\alpha = 0.5$. Then, the values of the paradox increase again as α increases.

5 Conclusion

In this paper, we have first discussed the literature about the optimal weight that a state should receive in a two-tier majority voting rule. We emphasized the fact that most of the criteria which are used to compare voting rules are all based upon abstract concepts like "pivotal players" or "utility". A simpler concept, like "equal opportunity of success" is more appealing but does not really discriminate among the voting rules as long as they treat the candidates equally. The concept of *majority efficiency* that we proposed in this paper is, in our opinion, easier to defend. First, it is just the application of the well known Condorcet criterion to a two candidate election. Secondly, a referendum paradox can be observed by all the citizens when it occurs, as show the 2000 and 2016 US presidential elections. As a consequence, it can be studied not only from a theoretical point of view, as we did in this paper, but could also be the subject of empirical studies, as soon as one possesses a sufficiently large and consistent electoral database. To some extent, studying this paradox would be a good way to build a bridge between formal *a priori* models used in game theory, economics and social choice theory, and empirical facts described in political sciences.

One of the conclusion of the paper is that the square root rule, which stood for a long time as the only normative recommendation for voting in federations, can be seriously contested. We have seen in Sect. 2 that other normative criteria that the idea of equalizing power can be used. In Sect. 3, we put forward May's model as an alternative to the classical Penrose-Banzhaf model. Section 4 clearly demonstrated that even under the Independence assumption, the square root rule could only hold for sufficiently large values of the number of states, N. In contrast, the proportional rule has always emerged as the uncontested apportionment method under May's model.

At last, one may wonder what happens when we consider rules which are different from the δ-rules. At this stage, we cannot guarantee that other two-tier voting schemes

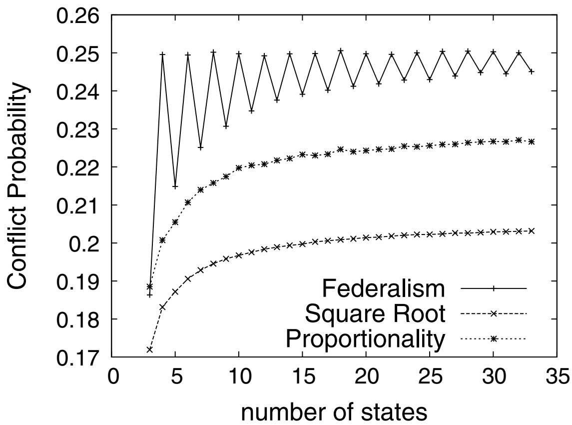

Fig. 8 Comparing different voting rules on their ability to avoid the referendum paradox under the IC assumption

may beat the proportional rule or the square root rule. But, we can already directly compare three famous voting rules, the federal rule ($\delta = 0$), the square root ($\delta = 0.5$) and the proportional rule ($\delta = 1$). The results, for the IC assumption, are displayed on Fig. 8. We recover that the square root rule does better than the proportional rule or the federal rule, but the margins are not impressive: As N grows, its advantage over the proportional rule (resp. the federal rule) stabilizes around 2.5% (resp. 4%). Such differences may not be perceived empirically unless someone gathers a huge set of data. Moreover, as it has been noticed that the IC model does not fit with electoral data (Gelman *et al.* 2004), it seems imprudent to set normative recommendations on this sole basis. On the other hand, by describing more homogeneous societies, the IAC could be more reliable. The results, displayed in Fig. 7 are quite shocking for the federalism.[14] With 33 states, the probability of paradoxes reaches 28% with the federal rule, while it stays below 16.5% for the proportional rule. Even the cube rule ($\beta = 2$) performs better! Over a long series of elections, such a difference could be detected, and unless the "one state – one weight" principle enjoys strong support in the society, our study suggests it should be abandoned in federations.

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[14] To compute the values for an even number of states, we flip a fair coin in case of a tie in terms of weights. This explains the discontinuities we observe for $\alpha = \beta = 0$ between odd and even values of N.

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"One Man, One Vote" Part 1: Electoral Justice in the U.S. Electoral College: Banzhaf and Shapley/Shubik Versus May

Olivier de Mouzon, Thibault Laurent, Michel Le Breton, and Issofa Moyouwou

1 Introduction

While there are controversies about the appropriate precise definition of a perfect democratic electoral system, it is fair to say that a consensus exists among scholars and commentators on two properties that any such system should possess. These two properties, typically referred to as anonymity and neutrality in the jargon of social choice theory, are the two main pillars of any democratic electoral system.

Anonymity calls for an equal treatment of voters[1] and neutrality calls for an equal treatment of candidates. If an electoral system is described in full mathematical generality as a mapping from the profiles of ballots into the set of candidates describing both the set of ballots available to each person and the selection of the winner for each profile of individual choices, anonymity implies first that all the persons have access to the same set of ballots and second that two profiles of votes which can deduced from each other through a permutation of names lead to the same electoral outcome.

[1] It requires that the "votes" of any two persons should have the same influence.

O. de Mouzon
Toulouse School of Economics, INRA, University of Toulouse Capitole, Toulouse, France
e-mail: olivier.demouzon@tse-fr.eu

T. Laurent
Toulouse School of Economics, CNRS, University of Toulouse Capitole, Toulouse, France
e-mail: thibault.laurent@tse-fr.eu

M. L. Breton (✉)
Institut Universitaire de France and Toulouse School of Economics,
University of Toulouse Capitole, Toulouse, France
e-mail: michel.lebreton@tse-fr.eu

I. Moyouwou
Ecole Normale Supérieure, Department of Mathematics,
University of Yaounde I, Yaounde, Cameroon
e-mail: imoyouwou@gmail.com

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The second part of the definition amounts to an axiom of invariance to permutations of the names of the persons. Neutrality is an axiom of invariance to permutations of the names of the candidates. In this paper, we will focus exclusively on the anonymity property to which we will refer to alternatively as the "one person, one vote's" principle[2] or electoral justice.[3]

Let us discuss briefly the first component of the anonymity axiom. For a meaningful discussion of that dimension, we need first the definition of a reference population listing all the persons who are considered to be part of the process on objective grounds. In practice, this set is a proper subset of the all population since for instance persons who are considered too young may not be listed in the reference population. Anonymity imposes as a necessary condition that all the persons in the reference population have access to the same set of ballots.

The most extreme violation of that principle appears when some persons are totally excluded from the process (i.e., their set of ballots is empty) on the basis of one or several observable criteria like for instance gender, age, race, wealth, or education. Most existing democracies went through a long period of time during which this basic version of anonymity has been violated. Even often, democracies have added temporarily an extra layer of departure from anonymity. Instead of having two classes of citizens: those (the non- citizens) who do not vote at all and those (the voters) who do vote on equal grounds, the electoral system subdivides the second class into several classes defined by different ballot structures.

Three examples from the European electoral history illustrate the second departure from anonymity: the law of double vote which was used from 1820 to 1830 in France,[4] the three-class franchise system[5] used from 1848 to 1918 in the Kingdom of Prussia and the "university constituency"[6] electoral system (university constituencies represent the members of one or more universities rather than residents of a

[2]One man, one vote (or one person, one vote) is a slogan used by advocates of political equality through various electoral reforms such as universal suffrage, proportional representation, or the elimination of plural voting, malapportionment, or gerrymandering.

[3]For deep discussions of the notions of equity and justice, see Balinski (2005) and Young (1994).

[4]For an analysis, see Le Breton and Lepelley (2014), Newman (1974) and Spitzer (1983).

[5]This indirect election system (In German: Dreiklassenwahlrecht) has also been used for shorter intervals in other German states. Voters were grouped into three classes such that those who paid most tax formed the first class, those who paid least formed the third, and the aggregate tax revenue of each class was equal. Voters in each class separately elected one-third of the electors (Wahlmänner) who in turn voted for the representatives in Prussia from 1849 to 1909 and the law of sieges(called the law of double vote) created among the voters (only old enough males paying a critical amount of taxes were voters). For more on this electoral law, see Droz (1963) and Schilfert (1963).

[6]University constituencies originated in Scotland, where the representatives of the ancient universities of Scotland sat in the unicameral Estates of Parliament. When James VI inherited the English throne in 1603, the system was adopted by the Parliament of England. It was also used in the Parliament of Ireland, in the Kingdom of Ireland, from 1613 to 1800, and in the Irish Free State from 1922 to 1936. It is still used in elections to Irish "senate". For more on this electoral law, see Beloff (1952).

geographical area) which has been used in the Parliament of Great Britain (from 1707 to 1800) and the United Kingdom Parliament, until 1950.[7]

Having access to the same ballots is necessary but not sufficient to obtain anonymity. A third and more subtle departure from anonymity arises when the ballots are the same but do not have the same influence on the final electoral outcome. The mathematical definition of anonymity is that for any given profile of ballots, the electoral outcome remains the same for all possible permutations of these ballots across voters. This condition is violated in two-tier electoral systems. A (single-seat) two-tier electoral system is a system where the population is partitioned into areas. In each area, the citizens elect a number of representatives who then meet in the upper tier to elect the winner. In such system, the final outcome is sensitive to the geography of the votes, i.e., to the distribution of the votes across the units composing the first tier.

Two identical ballots will not have the same influence and the crucial question becomes: How to measure the differences across voters and the departure of the electoral system from perfect anonymity? In this paper, we will follow a popular approach pioneered by Banzhaf (1964) and Shapley and Shubik (1954) which consists in evaluating the power of a voter as the probability of this voter being pivotal where probability refers to a probability model where the profiles of preferences or utilities of the voters are the elementary events of the state space.[8]

Precisely, our methodology to evaluate the degree of electoral justice will consist in the computation of the values of these power indices (there is one value per class of voters) and then the ratios of the numbers with respect to the smallest one.

An alternative and equivalent way to present the same information would be to compute the relative shares of power. In this paper, we do not attempt to end up with a one-dimensional uncontroversial measure of electoral inequity as we will consider several probability models. This concern together with some related statistical developments is the main topic addressed in our companion paper (De Mouzon et al. 2020a).

This paper will focus on an extremely popular and important two-tier electoral mechanism, namely the US Electoral College which is the electoral mechanism used by the United States of America to elect their president. In his pioneering and must

[7]These may or may not involve plural voting, in which voters are eligible to vote in or as part of this entity and their home area's geographical constituency.

[8]We refer the readers to Felsenthal and Machover (1998) and Laruelle and Valenciano (2011) for overviews of the theory and its main applications. An alternative measurement approach could be based upon utilities. From the Penrose's formula (see for instance, Penrose 1946, 1952; Felsenthal and Machover 1998), under IC, utility is an affine function of power. This simple relationship ceases to hold true for other probabilistic models (see, e.g., Laruelle and Valenciano 2011; Le Breton and Van Der Straeten 2015). We have not explored the conclusions in terms of electoral justice drawn from utilities.

read paper on the Electoral College, Miller (2012) offers a very clear presentation of the issue of unequal representation in the context of this specific electoral institution. He writes:

> Does the transformed Electoral College system give voters in different states unequal voting power? If so, are voters in large or small states favored and by how much? With respect to this question, directly contradictory claims are commonly expressed as a result of the failure by commentators to make two related distinctions: the theoretical distinction between 'voting weight' and 'voting power', and the practical distinction between how electoral votes are apportioned among the states (which determines their voting weights), and how electoral votes are cast by states (which influences their voting power).
>
> Those claiming that the Electoral College system favors voters in small states point to the advantage small states have with respect to the apportionment of electoral votes. States have electoral votes equal to their total representation in Congress. Since every state is guaranteed at least one seat in House and has two Senators, every state is entitled to at least three electors regardless of population. Approximate proportionality to population takes effect only beyond this three-electoral-vote floor, and this creates a substantial small-state advantage in the apportionment of electoral votes.
>
> However, other commentators (starting with like Luther Martin) emphasize that voting power is not proportional to voting weight (e.g., electoral voters), for two reasons. First, the voting power of a state depends not only on its share of electoral votes but on how the remaining electoral votes are distributed among the other states. Second, the voting power of a state depends on whether it casts its electoral votes as a bloc for a single candidate or splits them among two or more candidates, as well as how other states cast their votes. Intuition seems to tell us that the fact that elector slates are elected on a general ticket and therefore cast as bloc produces a large-state advantage—but intuition doesn't tell us how big this advantage may be. Moreover, we saw earlier that this intuition is only weakly supported in the state voting power calculations. The large-state advantage in the 51-state weighted voting game resulting from winner-take-all is not great enough to counterbalance the small-state advantage with respect to apportionment except in the case of the megastate of California, so those claiming a (modest) small-state advantage may appear to be correct. However, the top-tier 51-state weighted voting game entailed by the transformed Electoral College is a chimera, and the picture changes dramatically when we consider the more realistic 130-million-voter two-tier popular election.

The literature on the qualities and weaknesses of the Electoral College is vast. We will here focus our attention on two questions:

1. How to compare the voters from the different states in the Electoral College?
2. Is there an advantage to small states or large states?

To address, these questions we will follow the vast area of research based on the use several distinct a priori probability models on top of which the two most popular ones: the Banzhaf/IC probability model (Banzhaf 1964) and the Shapley-Shubik/IAC probability model (Shapley and Shubik 1954).

These a priori models have been criticized on several grounds among which the lack of empirical support in favor of these models.[9] We think that theoretical and empirical probability models serve different purposes. It depends whether[10] the

[9]See Gelman et al. (2004), Gelman et al. (2002a), Gelman et al. (2002b), Gelman et al. (1998). See also the empirical analysis of pivotality conducted by Mulligan and Hunter (2003).

[10]We refer to Miller (2009) and De Mouzon et al. (2019) for a defense of the a priori approach.

emphasis of the analysis is either positive, i.e., on predicting the power of the citizens on the basis of the current electoral data or normative, i.e., on evaluating on a priori neutral grounds the current electoral system and its potential contenders.

In this paper, we revisit and complement the pioneering work of Owen (1975) who writes:

> Discussion has frequently centered on the excessive power which this system seems to give to one group or another (the large states, the small states, organized minorities within one or another of the kinds of states, etc.), though there is also frequent disagreement about the identity of these favored groups.

Owen computes for both the 1960 and 1970 apportionments and census, the 51−dimensional vector of power indices of US citizens (as a function of the US state where they vote) for the two most popular probability models to which we have already alluded: the Banzhaf/*IC* model and the Shapley-Shubik/*IAC* probability model. From these calculations, he derives both for 1960 and 1970, the 50−dimensional ratios of the power of the citizen of any given state by the smallest power (which corresponds to the District of Columbia). In 1960, the highest ratio is obtained for New York with a value equal to 3.312 for IC and 3.287 for IAC followed by California with a value of 3.162 for *IC* and 3.143 for *IAC*. In 1970, the situation is reversed with California on top with a value of 3.177 for *IC* and 3.166 for *IAC* followed by New York with a value of 3.004 for *IC* and 2.976 for *IAC*. Between 1960 and 1970, California gained 5 electoral votes (from 40 to 45) while New York lost 2 electoral votes (from 43 to 41). More precisely, Owen obtains for the two probability models a complete numerical ranking of the states according to these ratios. Two conclusions emerge from his work:

1. For the two models and the two periods, citizens from large states have more influence on the electoral process than citizens from small states (around three times more for citizens in California and New York State).
2. The numerical rankings (and *de facto* their ordinal implications) attached to *IC* and *IAC* are almost the same.

The first conclusion has been widely commented and criticized by many authors. This conclusion seems at odds with the conventional wisdom asserting that the conclusion should be opposite since the small states are endowed with at least three electoral votes irrespective of their populations. This conclusion is shared by political scientists.

For instance, by calculating an *advantage ratio* for each state by simply dividing its share of the total electoral vote by its share of the national population, Shugart (2004) obtains that "these ratios range from 0.85 for California and Texas to 3.18 for Wyoming. In other words, California's weight in electing a president is only 85% of its contribution to the national population, while Wyoming's is more than three times as great as its population". On a figure, he plots each state's advantage ratio against its population and claims that "it shows very clearly how the smallest states are significantly overrepresented... There are 13 states with an advantage ratio greater than 1.5...".[11]

Our paper revisits the two conclusions and elucidates the difference between Owen's conclusions that large states are overrepresented and traditional views about overrepresentation of small states.

First, to the best of our knowledge, the second result in Owen's paper has not received any attention. This result, that we suggest to call *Owen's coincidence result*, is first mathematically intriguing. How could it be the case that two models which are very different[12] lead both to overrepresentation of large states. The *IC* model postulates complete independence among voters while the *IAC* model displays correlation among voters within and across states.

This coincidence is not obvious at all and is not empathized as such in Owen's work who derives his results through ingenious and sophisticated numerical approximation arguments. Our paper will revisit this coincidence as without any mathematical general result, we could indeed speculate that it may just well be the case that this coincidence is specific to the 1960 and 1970 data. Our first main result is that this coincidence extends to more recent data as well.

Besides the mathematical curiosity, this coincidence also obliges to have a different view about the so-called Banzhaf's fallacy (Margolis 1983). Conclusions derived from Banzhaf are often disregarded as the *IC* model is very special. It is very special indeed, and it is fair to say that the square root law often attached to it should be considered with caution. With the *IAC* model, the order of magnitude of the probability of influence for any voter in any state is $\frac{1}{n}$ instead of $\frac{1}{\sqrt{n}}$.

But the surprise is that when we compute the ratios for any pair of different states, they are the same for the two models. *This means that the conclusions in terms of electoral justice do not depend exclusively upon the choice of IC*.

The second contribution of the paper is to point out that the statement that small states are overrepresented can be obtained as the result of the methodology adopted here for a third probability model which has been invented first by May (1948) and used by several authors and that we will call the May's probability model.[13] This probability model is identical to IAC within states and IC across states, i.e., correlations exist between voters from the same state and are absent between voters from different states.

[11] In the same vein, see also Durran (2017).

[12] Among the differences, note in particular that, as demonstrated by De Mouzon et al. (2020b), the probability of an election inversion (that is an Electoral College winner different from the popular winner) in the Electoral College tends to 0 with the population size for the IAC model while the limit is positive for the IC model.

[13] It is sometimes called the IAC* probability model (Le Breton et al. 2016).

The ranking of the states according to influence is now almost a complete reversal of the ranking attached to the IC and IAC models. Further the largest ratio is now about three times the smallest one. The mathematical side of this result is easy but it is quite surprising to observe that the ordering of the states depends very much on whether the preferences of the voters are correlated or not across states. *With no correlation across states, electoral justice is against small states while with enough correlation electoral justice is against large states.* There is likely a critical level of interstate correlation separating the two conclusions!

The paper is organized as follows. In Sect. 2, we present the main notations and definitions. Then in Sect. 3, we present the main results of the paper. They are all based on simulated elections based upon the 2010 apportionment and census for the three a priori probability models which are considered. Our two main results on electoral justice (identical overrepresentation of large states for both Banzhaf and Shapley-Shubik and over representation of small states for May) are presented in that section.[14]

2 Notations and Definitions

The purpose of this section is twofold. First, we introduce the main notations and definitions with a special emphasis on the notion of two-tier weighted majority mechanism. Second, we present the measure of influence of a voter which is used in this paper and the three main probability models that are considered to conduct the computations.

2.1 Two-Tier Weighted Majority Mechanisms

We consider a society N of n voters which must chose among two alternatives[15]: D versus R. Each member i of N is described by his/her preference P_i. There are two possible preferences: D or R. We assume that N is partitioned into K states: $N = \cup_{1 \leq k \leq K} N^k$. The n^k voters of state k are endowed with w^k electoral votes. The electoral outcome $F(P) \in \{D, R\}$ attached to the profile of preferences

[14]In addition, the working paper version contains an appendix that does two things. First, we discuss the issue of their asymptotic coincidence in the case of discrete versions of the May and Shapley-Shubik models, and we sketch an explanation of their coincidence/difference in the case of the continuous version. Second, we examine the validity of Penrose's approximation in the second tier of the Electoral College by comparing the exact ratios of power indices of the states with the ratio of weights.

[15]In our simplified setting, like Owen (1975), we neglect the spoiler effects due to the existence of candidates in addition to the two main ones.

$P = (P_1, \ldots, P_n) \in \{D, R\}^n$ is determined by the following[16] two-tier weighted majority mechanism.[17] Let $F^k(P^k)$ be the majority winner in state k, i.e.,[18]

$$F^k(P^k) = \begin{cases} D \text{ if } \left|\left\{i \in N^k : DP_iR\right\}\right| \geq |\{i \in N : RP_iD\}| \\ R \text{ if } \left|\left\{i \in N^k : DP_iR\right\}\right| < |\{i \in N : RP_iD\}| \end{cases}$$

Then:

$$F(P) = \begin{cases} D \text{ if } \sum_{1 \leq k \leq K : F^k(P^k) = D} w^k \geq \sum_{1 \leq k \leq K : F^k(P^k) = R} w^k \\ R \text{ if } \sum_{1 \leq k \leq K : F^k(P^k) = D} w^k < \sum_{1 \leq k \leq K : F^k(P^k) = R} w^k \end{cases}$$

Candidate D or R wins the election if the total number of electoral votes attached to the states where he/she wins a majority of the votes is larger than the total number of electoral votes attached to the states where his/her opponent wins a majority of the votes.

The simple game attached to a two-tier weighted majority mechanism is called a *compound simple game*[19] (Owen 2001; Shapley 1962). From that perspective, the ingredients of the two-tier electoral mechanism consist of $K + 1$ simple games:

- K ordinary majority games (N^k, W^k_{maj}) $k = 1, \ldots, K$: In each state, the allocation of the totality of the w^k electoral votes of the state is decided by ordinary majority voting within the state.
- The weighted majority game $(\{1, \ldots, K\}, \mathcal{W}(q, w))$ where $w = \left(w^1, \ldots, w^K\right)$ and $q = \frac{\sum_{1 \leq k \leq K} w^k}{2}$: In the second tier, the representatives of each state (voting as a block) elect the president through majority voting.

This mechanism can receive two interpretations: either, it describes the election of a president through an Electoral College or it describes the election of a parliamentary house through a plurality formula.

In the presidential interpretation, the states represent the (geographical) states in the Federal Union. The majority winner in state k wins all[20] the electoral votes in

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16 A general electoral mechanism F is defined as a monotonic mapping from $\{0, 1\}^n$ into $\{0, 1\}$ where $D \equiv 1$ and $R \equiv 0$.

17 Alternatively and equivalently, any electoral mechanism F can be described in terms of winning coalitions. A coalition of voters $S \subseteq N$ is winning, denoted $S \in \mathcal{W}$, iff $F(P) = D$ whenever $P_i = D$ for all $i \in S$. It is straightforward to check that the family $\mathcal{W}$ of winning coalitions is monotonic with respect to inclusion. The pair $(N, \mathcal{W})$ is called a *simple game* (Owen, 2001). Among those, weighted majority games are central. A weighted majority game on N is described by a vector of weights $w = \left(w^1, \ldots, w^n\right)$ and a quota q: $S \subseteq N$ is winning, denoted $S \in \mathcal{W}(q, w)$, iff $\sum_{i \in S} w^i \geq q$. When $w = (1, \ldots, 1)$ and $q = \frac{n}{2}$,, we obtain the ordinary majority game.

18 In this definition, in both tiers, ties are broken in favor of D. The details of the tie-breaking rule do not impact our results. In fact, our simulations are conducted under the assumption that in both tiers, and ties are broken through a fair random choice between D and R. We will offer further comments on that, later in the paper.

19 The notion of composition is quite general and can be applied to very abstract simple games.

20 This is the "winner takes all" feature of the mechanism. In our paper, we ignore the fact that for Maine and Nebraska"winner takes all" does not fully apply. Strictly speaking, congressional

state k. The upper tier, called the Electoral College, is composed of $\sum_{1 \leq k \leq K} w^k$ electors who are either on the D side or on the R side. It is assumed that they elect the president through an ordinary majority vote. In the parliamentary interpretation, the states represent the electoral districts of the country and w^k is the district magnitude of district k. If all the seats of district k go to the majority winner[21] in district k, then $F(P)$ denotes the majority "color" of the parliament while $\sum_{1 \leq k \leq K : F^k(P^k)=D} w^k$ and $\sum_{1 \leq k \leq K : F^k(P^k)=R} w^k$ denote the number of seats won, respectively, by D and R. In the main real-world applications of this second interpretation (U.S.; U.K.,...), the district magnitude of all the districts is equal to 1.

In this paper, we will focus on the first interpretation. The above formal definition calls for a comment as, strictly speaking, it deviates at the margin from the real one. Indeed, since we cannot exclude a priori the cases where there is a tie either within a state (this may happen if n^k is even) or more seriously within the Electoral College (this may happen if $\sum_{1 \leq k \leq K : F^k(P^k)=D} w^k = \sum_{1 \leq k \leq K : F^k(P^k)=R} w^k$), we need to define how these ties are broken. In this event, to make our presentation simple, we have broken deterministically the tie in favor of the candidate D. Instead, we could have decided to break the tie in favor of the candidate R or to use a probabilistic device like flipping a fair coin. In our simulations, to make the rule as neutral as possible, we opted for the random draw but in this general presentation, we decided not to do so as this calls for some cumbersome notational adjustments in the definition of pivotality that we wanted to avoid.

For the unbiased probabilistic models on preferences that will be introduced in the next section, all three tie-breaking rules lead to the same computations and conclusions. The US Electoral College uses a different rule in the case of such a contingent presidential election. The Twelfth Amendment requires the House of Representatives to go into session immediately to vote for a president if no candidate for president receives a majority of the electoral votes. In this event, the House of Representatives is limited to choosing from among the three candidates who received the most electoral votes for president. Each state delegation votes "en bloc": each delegation having a single vote; the District of Columbia does not receive a vote. A candidate must receive an absolute majority of state delegation votes (i.e., at present, a minimum of 26 votes) in order for that candidate to become the president-elected. The House continues balloting until it elects a president.[22] Like Owen (1975), we depart from the "real" tie-breaking rule. In Sect. 4, we speculate that the results that we will obtain for the three equivalent and simple theoretical breaking rules defined above are identical to those that would be obtained for the Twelfth Amendment rule.

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districts should be treated as additional states for the purpose of the modeling. We conjecture that our results are not significantly impacted by this simplification.

[21] In the real-world electoral systems which are used to elect the representatives, when the district magnitude is larger than 1, it is often the case that the "winner takes all" principle is replaced by a proportional principle. In such a case, the formal description of the electoral mechanism differs from the one considered here. For a general approach, when the district magnitude is equal to 2, the reader is referred to Le Breton et al. (2017).

[22] The House of Representatives has chosen the president only once in 1825 under the Twelfth Amendment. Senate is involved along similar principles in the election of the vice president.

2.2 Probability Models

To evaluate the power and utilities of voters and the properties of a voting mechanism F, we introduce a probability model π on the set of profiles $\{D, R\}^n$: $\pi(P)$ denotes the probability (frequency, ...) of profile P. Let us examine the situation from the perspective of voter i.

To evaluate how often i is influential, we consider the frequency of profiles P such that $F(D, P_{-i}) \neq F(R, P_{-i})$ or equivalently the frequency of coalitions S, such that $S \in \mathcal{W}$ and $S \setminus \{i\} \notin \mathcal{W}$ or $S \notin \mathcal{W}$ and $S \cup \{i\} \in \mathcal{W}$. In such situation , we say that voter i is *pivotal*. The probability $Piv_i(i, F, \pi, n)$[23] of such an event is

$$\sum_{T \notin \mathcal{W} \text{ and } T \cup \{i\} \in \mathcal{W}} \pi(T) + \sum_{T \in \mathcal{W} \text{ and } T \setminus \{i\} \notin \mathcal{W}} \pi(T) = \sum_{T \subseteq N \setminus \{i\} : T \notin \mathcal{W} \text{ and } T \cup \{i\} \in \mathcal{W}} \pi_{-i}(T)$$

where π_{-i} denotes the (marginal) probability induced by π on the product subspace $\{D, R\}^{N \setminus \{i\}}$. This formula[24] makes clear that the evaluation depends upon the probability π which is considered. Two popular specifications have attracted most of the attention and dominate the literature.

The first (known under the heading *Impartial Culture (IC))* leads to the Banzhaf's index.[25] It corresponds to the setting where all the preferences P_i proceed from independent Bernoulli draws with parameter $\frac{1}{2}$. In this case, for all $T \subseteq N \setminus \{i\}$: $\pi_{-i}(T) = \frac{1}{2^{n-1}}$. The Banzhaf power $B(i, F, n)$ of voter i is equal to

$$\frac{\eta_i(\mathcal{W})}{2^{n-1}},$$

where $\eta_i(\mathcal{W})$ denotes the number of coalitions $T \subseteq N \setminus \{i\}$ such that $T \notin \mathcal{W}$ and $T \cup \{i\} \in \mathcal{W}$ (in the literature, any such coalition T is referred to as a "swing" for voter i).

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[23] $Piv(i, F, \pi, n)$ contains a little abuse in notation since π and n cannot be separated as π is defined on $\{D, R\}^n$. $Piv(i, F, \pi, n)$ could also be denoted $Piv(i, \mathcal{W}, \pi, n)$, and it is often called the power of voter i for the voting rule F/W according to the probability model π. When the reference to F/W will be clear, we will drop it from the notation.

[24] This definition needs to be adjusted when the voting mechanism and when ties are not broken deterministically. Let us denote by T (T for ties) the set of profiles $P \in \{D, R\}^n$ such that D is elected with probability $0 < \chi(P) < 1$. Assuming that a tie is broken as soon as a single voter changes her mind, then the probability of pivotality is the probability over subprofiles P_{-i} of having $F(P_{-i}, D) \neq F(P_{-i}, R)$. When both outcomes are deterministic, then this happens only when $F(P_{-i}, D) = D$ and $F(P_{-i}, R) = R$. But with ties, this may also happen when : $F(P_{-i}, D) = T$ and $F(P_{-i}, R) = R$ or when $F(P_{-i}, R) = T$ and $F(P_{-i}, D) = D$. In the last two cases, the probability of having different outcomes is not equal to 1 anymore but to $\chi(P)$.

[25] Here, we have only two candidates. The wording *IC* is used more generally to define the situation of independent and identically draws of preferences over an arbitrary number of candidates. Here, we use the terms Banzhaf and *IC* equivalently.

The second model (known under the heading *Impartial Anonymous Culture (IAC) Assumption)* leads to the Shapley-Shubik's index.[26] It is defined as follows. Conditionally to a draw of the parameter p in the interval [0, 1], according to the uniform distribution,[27] the preferences P_i proceed from independent Bernoulli draws with parameter p. In such a case, for all $T \subseteq N \setminus \{i\} : \pi_{-i}(T) = \int_0^1 p^t (1-p)^{n-1-t} dp$ where $t \equiv \#T$. The Shapley-Shubik power $Sh(i, F, n)$ of voter i is equal to

$$\int_0^1 \left(\sum_{T \notin \mathcal{W} \text{ and } T \cup \{i\} \in \mathcal{W}} p^t (1-p)^{n-1-t} \right) dp.$$

In addition to these two models, we consider a third one, called here IAC^*, which is intermediate between *IC* and *IAC*. It was first introduced by May (1948) in his analysis of election inversions and pivotality was studied recently by Le Breton et al. (2016) when F is popular majority. This model is defined as follows.

Assume from now that N is partitioned into K states: $N = \cup_{1 \le k \le K} N^k$. Conditionally on K independent and identically distributed, draws $p_1, \ldots, p_K$ in the interval [0, 1], according to the uniform distribution, the preferences in group N_k proceed from independent Bernoulli draws with parameter p_k. In such a case, for all $T \subseteq N \setminus \{i\}$ such that i belongs to state $k(i)$:

$$\pi_{-i}(T) = \left(\prod_{1 \le k \le K : k \ne k(i)} \int_0^1 p_k^{t^k} (1-p_k)^{n^k - t^k} dp_k \right) \times \int_0^1 p_{k(i)}^{t^{k(i)}} (1-p_{k(i)})^{n^{k(i)}-1-t^{k(i)}} dp_{k(i)},$$

where $t^k \equiv \left| N^k \cap T \right|$ for all $k = 1, \ldots, K$. The May power $M(i, F, n)$ of voter i is then equal to

$$\sum_{T \notin \mathcal{W} \text{ and } T \cup \{i\} \in \mathcal{W}} \left(\prod_{1 \le k \le K : k \ne k(i)} \int_0^1 p_k^{t^k} (1-p_k)^{n^k - t^k} dp_k \right) \times \int_0^1 p_{k(i)}^{t^{k(i)}} (1-p_{k(i)})^{n^{k(i)}-1-t^{k(i)}} dp_{k(i)}.$$

In the case where F is the direct/popular (i.e., one tier) majority mechanism, all the voters have the same influence. In such case, we can drop the reference to i. When n, the number of voters, is large, it is well known that the Banzhaf power of a

[26]It can be proved that the Shapley-Shubik model amounts drawing uniformly the number of voters who prefer D to R. It can also be showed that for the *IAC* model the preferences display some correlation. Here we have only two feasible preferences. For an arbitrary number of candidates, the wording *IAC* is used more generally to define the situation where the draws of the vectors describing the numbers of preferences of each type are uniform. Here, we use the terms Shapley-Shubik and *IAC* equivalently.

[27]If we take an arbitrary absolutely continuous distribution, we obtain a generalized version of the Shapley-Shubik's probability model which has been analyzed by Chamberlain and Rothschild (1981) and Good and Mayer (1975).

voter is[28] approximatively equal to $\sqrt{\frac{2}{\pi n}}$ while the Shapley-Shubik index of a voter is equal to $\frac{1}{n}$. The combinatorics of the May's index are more involved and explored in Le Breton et al. (2016).

In the case where F is the two-tier weighted majority mechanism considered in this paper, note first that we cannot drop the reference to i anymore but all voters from the same state will have the same power as long as the probability model displays symmetry across players. As this is the case for the Banzhaf, Shapley-Shubik, and May probability models, we will have to compute K different values for these three models.

From now on, we will focus on the two-tier electoral mechanism F defined in Sect. 2.1, and we drop the reference to F in the coming computations of pivotality.[29] For any $i \in N$, we will denote by $Piv\ (i, \pi, n)$ the probability that voter i is pivotal according to the probability model π. We denote by $\underline{Piv}(i, \pi, n)$ the probability that i is pivotal according to π in his/her state $k(i)$.[30] When k is a representative in the upper tier, we will denote by $\overline{Piv}(k, \pi)$ the probability that k is pivotal in the upper tier. Note that when we examine *per se* the upper tier, we do not need the full knowledge of π but the probability induced by π on the set of representative preference profiles $\{D, R\}^K$. Let us illustrate that point for IC, IAC^*, and IAC when $K = 2$. When $\pi = IC$, the probability that both representatives vote D, i.e., the probability that there is a majority of representatives voting D in both states is equal to $\frac{1}{4}$. In such a case, the probability induced by π on the upper tier is simply IC on the set of representatives $\{1, 2\}$. By the same token, we obtain that when $\pi = IAC^*$, the probability induced by π on the upper tier is simply IC on the set of representatives $\{1, 2\}$. In contrast, when π is IAC, things get far more subtle. Under the assumption that the two states are equipopulated with $n^1 = n^2 \equiv m$ (m odd), the probability that both states vote democrat is equal to

$$\sum_{k=\frac{m+1}{2}}^{m} \sum_{r=\frac{m+1}{2}}^{m} \frac{m!}{k!(m-k)!} \times \frac{m!}{r!(m-r)!} \times \frac{(k+r)!(2m-k-r)!}{(2m+1)!}.$$

We do not know any closed form. When $m = 11$ and therefore $n = 22$, this probability is equal to 0.42107 which is much larger than the value 0.25 obtained for

[28] If n is odd, then for all i, $B_i = \binom{n-1}{\frac{n-1}{2}}/2^{n-1}$. If n is even, $B_i = \left[\binom{n-1}{\frac{n}{2}}/2^{n-1} + \binom{n-1}{\frac{n-2}{2}}/2^{n-1}\right] \times \frac{1}{2}$. The assertion follows from Stirling's formula.

[29] This means also that we will not explicitly refer to the division $n^1, \ldots, n^K$ of the n voters into the K states and to the electoral votes $w^1, \ldots, w^K$ of the states.

[30] Truly only the restrictions of F and π on the subset $N^{k(i)}$ matter. Since the restriction of F onto $N^{k(i)}$ is the ordinary majority mechanism with $n^{k(i)}$ voters, the computation of $\underline{Piv}(i, \pi, n)$ amounts to the computation of the pivotality according to π for the ordinary majority mechanism.

IC and IAC^*. Note that the probability induced by IAC on $\{1, 2\}$ is not IAC on $\{1, 2\}$. Indeed, if we consider IAC on $\{1, 2\}$, the probability of the profile (D, D) is $\int_0^1 p^2 dp = \frac{1}{3} = 0.3333...$

Let us consider the computation of $Piv\ (i, \pi, n)$ for $\pi = IC, IAC$ and IAC^*. From the description of F as a compound simple, it is straightforward that for a voter to be influential/pivotal, we need the combination of two events: The voter must be pivotal in his state, and the representatives of his state must be themselves pivotal in the electoral college. In general, unfortunately the two events are not independent. If the two events are independent for some probability model π, then the computation of the pivotality P_i of voter i proceeds from a simple multiplicative formula:

$$Piv\ (i, \pi, n) = \underline{Piv}\left(i, \pi, n^{k(i)}\right) \times \overline{Piv}\ (k(i), \pi)\,,$$

where $\underline{Piv}\left(i, \pi, n^{k(i)}\right)$ denotes the pivotality power of voter i in his state $k(i)$ and $\overline{Piv}\ (k(i), \pi)$ denotes the pivotality power of the representative(s) of state $k(i)$ in the second tier. For the Banzhaf and May probability models, the two events are independent. Further given the neutrality of these two probability models between the candidates and the neutrality of the ordinary majority mechanism between the two candidates, the pivotality power of representative k in the second tier is simply his Banzhaf power in the second tier. Therefore, when the number of voters in each state is large

$$B(i, n) = \underline{B}(i, n^{k(i)}) \times \overline{B}(k(i)) \simeq \sqrt{\frac{2}{\pi n^{k(i)}}}\overline{B}(k(i))$$

and

$$M\,(i, n) = \underline{Sh}(i, n^{k(i)}) \times \overline{B}(k(i)) = \frac{1}{n^{k(i)}}\overline{B}(k(i)).$$

In such case, we are left with the computation of $\overline{B}(k)$ for $k = 1, \ldots, K$. This can be done in several ways. Either by using existing software which works well as long as K is not too large. Another road is to use (if possible) Penrose's theorem which asserts that under some conditions, the Banzhaf power of player k in the weighted majority game $(\{1, \ldots, K\}, \mathcal{W}(q, w))$ is proportional to w^k.[31] Under the presumption that the Penrose's approximation is valid, we obtain for all $i, j \in N$:

$$\frac{B(i, n)}{B(j, n)} = \frac{\sqrt{n^{k(j)}}}{\sqrt{n^{k(i)}}} \times \frac{w^{k(i)}}{w^{k(j)}} \tag{1}$$

[31]The exact computation of these values as well as the validity of the Penrose's approximation is presented and discussed in appendix 3 of the working paper version.

and

$$\frac{M(i,n)}{M(j,n)} = \frac{n^{k(j)}}{n^{k(i)}} \times \frac{w^{k(i)}}{w^{k(j)}} \tag{2}$$

Unfortunately, we cannot proceed similarly for the Shapley-Shubik model. Remember that the preferences of the voters within and across states are correlated. This means that for that probability model, we cannot in principle separate the two computations on pivotality.[32] Owen (1975, 2001) presents developments on how to calculate the Shapley value of a compound simple game.[33] There is no easy way to proceed, and Owen states some results on the relationships of the multilinear extension of a compound simple game to the multilinear extensions of the simple games used in the composition. He uses these results to conduct his numerical computations and to obtain estimates of

$$\frac{Sh(i,n)}{Sh(j,n)}$$

for all $i, j \in N$.

3 Electoral Justice in the Electoral College

The main purpose of this section is to present our computational results on electoral justice in the Electoral College for the three probability models that have been defined. For reproducing figures and tables presented in that chapter, the codes of our computer program are available at http://www.thibault.laurent.free.fr/code/DL_issue/. In the first section, we present the 2010 apportionment and census data which is used in our analysis. Then, in three distinct subsections, we present and comment separately our results for Banzhaf, Shapley-Shubik, and May. All these computations are derived through a simulator that works as follows:

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[32]When $K = 2$, and $n^1 = n^2 \equiv m$, the probability that any player is pivotal for IAC is equal to $\frac{(m-1)!}{(\frac{m-1}{2})!(\frac{m-1}{2})!} \times \sum_{r=\frac{m+1}{2}}^{m} \frac{m!}{r!(m-r)!} \times \frac{(\frac{m-1}{2}+r)!(2m-\frac{m-1}{2}-r)!}{(2m+1)!}$, while it equals to $\frac{1}{m} \times \frac{1}{2}$ for IAC^* and to $\frac{\binom{m-1}{\frac{m-1}{2}}}{2^{m-1}} \times \frac{1}{2}$ for IC. When $m = 11$, we obtain the values 0.019, 0.05 and 0.123.

[33]There is the place to remind to the reader that Sh_i is also the Shapley value of the TU simple game $(N, V_{\mathcal{W}})$ where $V_{\mathcal{W}}(S) = 1$ iff $S \in \mathcal{W}$ and 0 otherwise.

Algorithm 1: Main steps of our algorithm

```
Initialization of program constants
K = number of states (51)
B = number of simulations (10^12)
∀k ∈ {1, ..., K}, Seats_k denotes the number of votes of State k for the
presidential election and n_k denotes the number of voters in State k.
Computation
for Model in {IC, IAC, IAC*} do
    Initialize the number of pivotal voters (at the presidential level):
    ∀k ∈ {1, ..., K}, Piv_k := 0.
    for b in 1 to B do
        Initialize the total number of seats for D or R: Seats_D := 0 and Seat_R := 0.
        for k in 1 to K do
            Simulate the choice of each voter in State k between D and R, following
            the chosen Model distribution (IC, IAC or IAC*)
            Compute Piv_k^State, the number of pivotal voters in State k, considering
            only State k choice.
            Compute State k choice for the presidential election: C_k ∈ {D, R}.
            Update the total number of seats for D or R: ∀P ∈ {D, R}, if C_k = P,
            then Seats_P := Seats_P + Seats_k.
        for k in 1 to K do
            If State k has pivotal voters (Piv_k^State > 0) and Seats of State k are
            pivotal, then update the number of Pivotal voters:
            Piv_k := Piv_k + Piv_k^State.
    Compute the estimated probability of a voter to be pivotal at the
    presidential election:
    ∀k ∈ {1, ..., K}, the probability for a voter in State k to be pivotal at the
    presidential election is: Piv_k / (B*n_k).
```

Table 1 US Electoral College and population data per state

State	Population in 2010	Electoral College in 2012
Alabama	4,802,982	9
Alaska	721,523	3
Arizona	6,412,700	11
Arkansas	2,926,229	6
California	37,341,989	55
Colorado	5,044,930	9
Connecticut	3,581,628	7
Delaware	900,877	3
District of Columbia	601,766	3
Florida	18,900,773	29
Georgia	9,727,566	16
Hawaii	1,366,862	4
Idaho	1,573,499	4
Illinois	12,864,380	20
Indiana	6,501,582	11
Iowa	3,053,787	6
Kansas	2,863,813	6
Kentucky	4,350,606	8
Louisiana	4,553,962	8
Maine	1,333,074	4
Maryland	5,789,929	10
Massachusetts	6,559,644	11
Michigan	9,911,626	16
Minnesota	5,314,879	10
Mississippi	2,978,240	6
Missouri	6,011,478	10
Montana	994,416	3
Nebraska	1,831,825	5
Nevada	2,709,432	6
New Hampshire	1,321,445	4
New Jersey	8,807,501	14
New Mexico	2,067,273	5
New York	19,421,055	29
North Carolina	9,565,781	15
North Dakota	675,905	3
Ohio	11,568,495	18
Oklahoma	3,764,882	7
Oregon	3,848,606	7
Pennsylvania	12,734,905	20

(continued)

Table 1 (continued)

State	Population in 2010	Electoral College in 2012
Rhode Island	1,055,247	4
South Carolina	4,645,975	9
South Dakota	819,761	3
Tennessee	6,375,431	11
Texas	2,5268,418	38
Utah	2,770,765	6
Vermont	630,337	3
Virginia	8,037,736	13
Washington	6,753,369	12
West Virginia	1,859,815	5
Wisconsin	5,698,230	10
Wyoming	568,300	3

Source http://www.thegreenpapers.com/Census10/HouseAndElectors.phtml

3.1 The 2010 US Electoral College and Population Data

Table 1 presents[34] the[35] number of voters and seats[36] which have been used in our simulator. It corresponds to the 2010 population census and 2012 Electoral College (which holds also for 2016 and 2020).

Figure 1 shows that a number of representatives are allocated proportionally to the population of the state (census 2010).[37] The exact distribution is derived from the Huntington–Hill method. The average is around 1.4 representative per million of inhabitants (see red-dashed line). Due to integer rounding effects, the actual number of representatives per million of inhabitants varies from one state to another between 1.0 (Montana) and 1.9 (Rhode Island). Due to the distribution rule, the variability is higher among states with small number of representatives.

Although the distribution seems as fair as possible among states, it is clear that some voters have more representatives than others. Hence, a voter in Rhode Island has almost twice as many representatives than a voter in Montana.

[34]This number is often 0. Considering, on the one hand, a State k with an odd number of voters, n_k, there are either no pivotal voters or $\frac{n_k+1}{2}$ (when there is almost a tie). Considering, on the other hand, a State k with an even number of voters, either there is a tie and half of the voters are pivotal or there is almost a tie and $\frac{n_k+2}{2}$ voters are pivotal in half of the cases. In all other cases, there are no pivotal voters.

[35]Seats of State k are pivotal if $Seats_{C_k} - Seats_k <= Seats_{-C_k} + Seats_k$, $-C_k$ denoting the non-chosen party by State k. In presence of a tie ($Seats_{C_k} = Seats_{-C_k}$ or $Seats_{C_k} - Seats_k = Seats_{-C_k} + Seats_k$), only half of the cases are pivotal.

[36]The number of electoral votes (called hereafter "seats") of a state is the sum of its number of representatives and number of senators (which is 2 for all states). The District of Columbia is allocated three seats.

[37]To be consistent, we have assumed that District of Columbia has 1 representative.

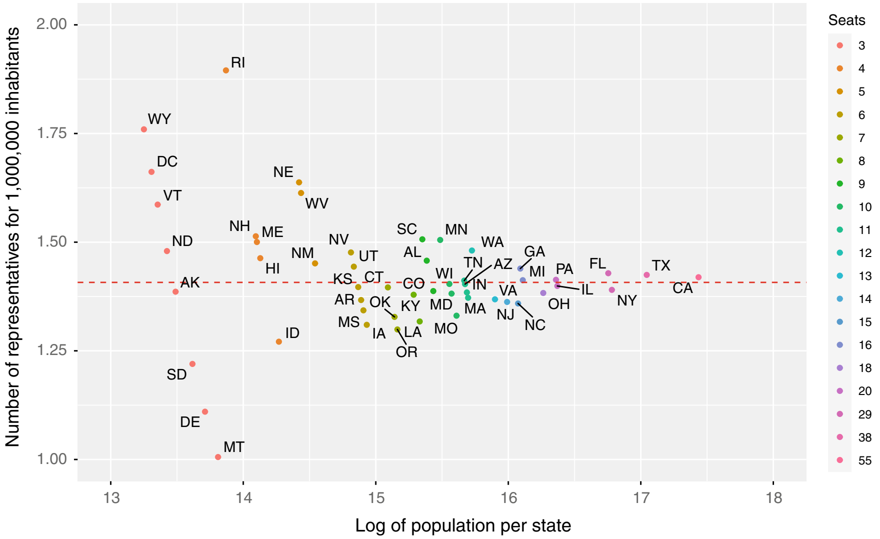

Fig. 1 Electoral representatives per inhabitant ratio in each state in years 2012, 2016, and 2020

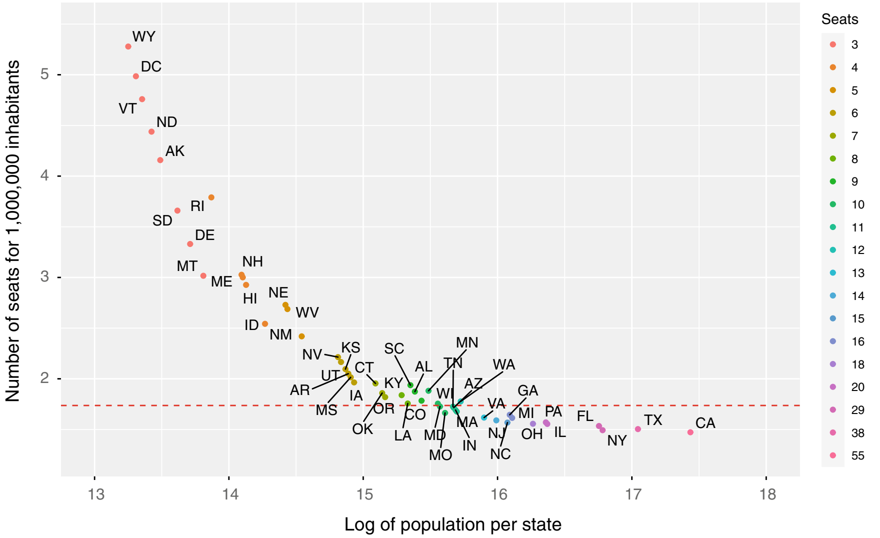

Fig. 2 Electoral seats per inhabitant ratio in each state in years 2012, 2016, and 2020

Moreover, the number of seats in the presidential election is the number of representatives added to the two senator votes. Hence, the distribution of seats per inhabitants is even more distorted, as shown on Fig. 2. The red-dashed line represents the average of 1.7 seats per million of inhabitants in the USA. Depending on the state, this number goes from 1.5 (California) to 5.3 (Wyoming). Hence, a voter from Wyoming seems to have around 3.6 more representation than a voter from California. Again, the variability in Fig. 2 is higher for small number of seats. Yet, it seems that the average number of seats per inhabitants is almost always decreasing in population!

In the end, the question is whether this distortion biases the outcome of the presidential election or if it corrects another distortion as a big state might be more often pivotal than a small one.

The aim of our simulator is exactly to study this question in the case of different standard probability models.

3.2 Electoral Justice with Respect to Banzhaf

In this section, the simulations have been made by keeping the exact population per state. We have done 10^{12} simulations, and the computation time was around 5 days, using 40 cores on a server of 58 logical cores at 3.07 GHz.

As shown in Table 2, the obtained probabilities to be pivotal are between 1.810×10^{-5} (Montana) and 6.153×10^{-5} (California). According to Bienaymé–Tchebychev, those results are significant and accurate ($\pm 3.5 \times 10^{-8}$) at a confidence level better than 95%.

Unsurprisingly, the results derived from our simulations are consistent with the theoretical ones (available for Banzhaf). Note that the maximal difference between the two values is less than $+/-10^{-8}$. So the significance and accuracy of our simulations are even better than what could be guaranteed according to Bienaymé–Tchebychev.

From Bienaymé–Tchebychev, we know that the ranking of states according to the probability for a voter to be pivotal is significant and accurate at a confidence level better than 95%, except for four groups of states:

- pivotality around 2.09×10^{-5} for New Mexico < Mississippi < New Hampshire
- pivotality around 2.17×10^{-5} for Utah < Oklahoma (the confidence level in this ranking is of 83.2%)
- pivotality around 2.19×10^{-5} for Nevada < North Dakota
- pivotality around 2.22×10^{-5} for Nebraska < Connecticut (the confidence level in this ranking is of 93.5%)

Yet, we know that the ranking obtained through the simulations matches perfectly the theoretical one, even in those four groups. Hence, the ranking presented in Fig. 3 is not debatable.

Figure 3 presents for each state the ratio of pivotality ordered from the maximum to the minimum. Colors correspond to the number of electoral seats in the states. It seems that the ratio of pivotality is higher for states with larger number of seats.

Hence, in the case of Banzhaf's vote distribution model, the distortion of seats in favor of small populated states does not compensate the electoral advantage of a

Table 2 Probability for a voter to be pivotal at the presidential election in each state, with respect to Banzhaf model

State	Theoretical	Simulation results (IC)	
	value	Prob to be pivot	Advantage ratio
Alabama	2.476e−05	2.475e−05	1.368
Alaska	2.125e−05	2.124e−05	1.174
Arizona	2.622e−05	2.622e−05	1.449
Arkansas	2.112e−05	2.113e−05	1.167
California	6.152e−05	6.153e−05	3.399
Colorado	2.416e−05	2.415e−05	1.334
Connecticut	2.228e−05	2.228e−05	1.231
Delaware	1.902e−05	1.901e−05	1.050
District of Columbia	2.327e−05	2.327e−05	1.286
Florida	4.111e−05	4.111e−05	2.272
Georgia	3.108e−05	3.107e−05	1.717
Hawaii	2.059e−05	2.059e−05	1.138
Idaho	1.919e−05	1.919e−05	1.060
Illinois	3.392e−05	3.392e−05	1.874
Indiana	2.604e−05	2.603e−05	1.438
Iowa	2.067e−05	2.067e−05	1.142
Kansas	2.135e−05	2.135e−05	1.180
Kentucky	2.311e−05	2.311e−05	1.277
Louisiana	2.259e−05	2.259e−05	1.248
Maine	2.085e−05	2.085e−05	1.152
Maryland	2.507e−05	2.506e−05	1.385
Massachusetts	2.592e−05	2.592e−05	1.432
Michigan	3.079e−05	3.079e−05	1.701
Minnesota	2.616e−05	2.616e−05	1.445
Mississippi	2.093e−05	2.094e−05	1.157
Missouri	2.460e−05	2.460e−05	1.359
Montana	1.810e−05	1.810e−05	1.000
Nebraska	2.224e−05	2.224e−05	1.229
Nevada	2.195e−05	2.195e−05	1.212
New Hampshire	2.094e−05	2.093e−05	1.157
New Jersey	2.853e−05	2.853e−05	1.576
New Mexico	2.093e−05	2.093e−05	1.156
New York	4.055e−05	4.055e−05	2.240
North Carolina	2.935e−05	2.935e−05	1.622
North Dakota	2.195e−05	2.196e−05	1.213
Ohio	3.212e−05	3.212e−05	1.775
Oklahoma	2.173e−05	2.173e−05	1.201

(continued)

Table 2 (continued)

State	Theoretical	Simulation results (IC)	
	value	Prob to be pivot	Advantage ratio
Oregon	2.149e−05	2.149e−05	1.187
Pennsylvania	3.409e−05	3.408e−05	1.883
Rhode Island	2.343e−05	2.343e−05	1.295
South Carolina	2.517e−05	2.517e−05	1.391
South Dakota	1.994e−05	1.994e−05	1.102
Tennessee	2.629e−05	2.629e−05	1.453
Texas	4.744e−05	4.744e−05	2.621
Utah	2.170e−05	2.171e−05	1.199
Vermont	2.273e−05	2.273e−05	1.256
Virginia	2.771e−05	2.771e−05	1.531
Washington	2.789e−05	2.789e−05	1.541
West Virginia	2.207e−05	2.206e−05	1.219
Wisconsin	2.527e−05	2.527e−05	1.396
Wyoming	2.394e−05	2.394e−05	1.323

voter living in a high populated state. This is very clear on Fig. 4.[38] For instance, a voter from California has more than two and a half more chances to be pivotal than a voter from Wyoming, although the one from Wyoming accounts for more than three and a half more seats than the one from California.

Of course, for a given number of seats, the order between states seen on Fig. 2 still holds on Fig. 3. For instance, for three seats, Wyoming is better off than District of Columbia, then Vermont, North Dakota, Alaska, South Dakota, Delaware, and finally Montana (where a voter has minimal power, in the case of Banzhaf, in all the USA, for the presidential election). But the comparison does not hold between states with different number of seats. For instance, Rhode Island is in between Alaska and South Dakota on Fig. 2, but much higher on Fig. 3, where it is in between Wyoming and District of Columbia.

In order to better understand the mechanisms at stake, Fig. 5 decomposes the pivotality part due to being a pivotal voter in his state (middle figure) and the part due to pivotality in the second tier (bottom figure). In Banzhaf's case, the total pivotality (top figure) is computed as the product of the two parts. For instance, for California (the sole pink dot corresponding to the state with 55 electoral votes): $(1.3 \times 10^{-4}) \times 0.47 \approx 6.15 \times 10^{-5}$. It is obvious that the second part plays the biggest role in the probability of being pivotal. Indeed, the second part is increasing with respect to the size of the population as well as the probability of being pivotal; whereas, the first

[38]We have also drawn figures with the same y-axis but with the number of electoral votes on the x-axis. These three figures should be compared to the one derived by Gelman et al. (2012) for an econometric model of elections.

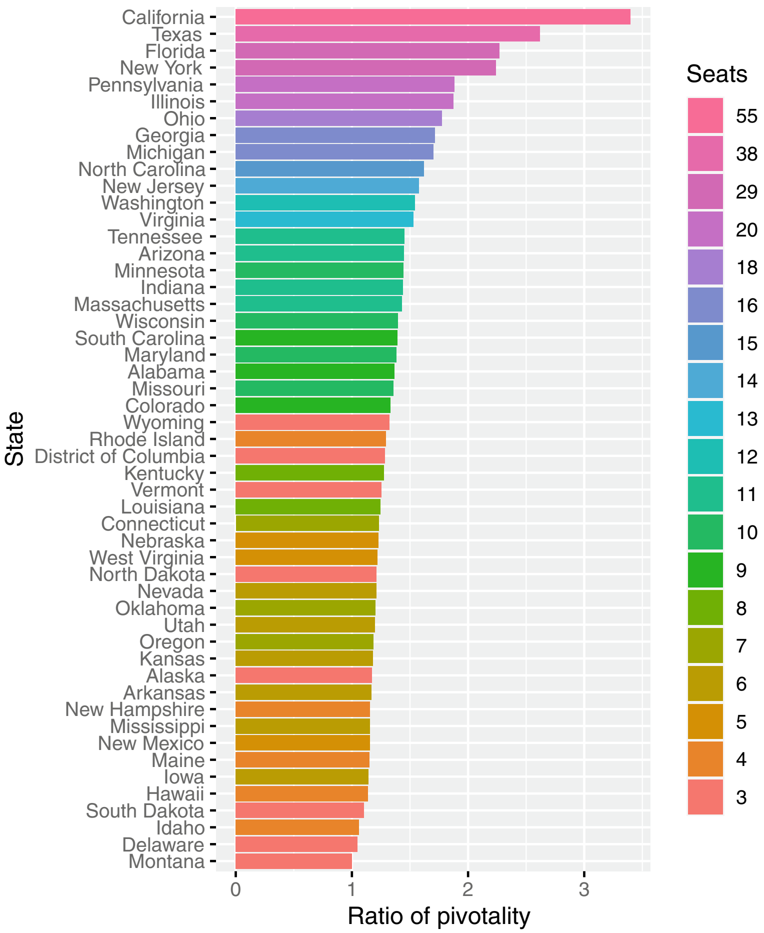

Fig. 3 Pivotality ratio by state and number of seats in the case of Banzhaf, ordered by decreasing pivotality ratio

part is decreasing proportionally to $1/\sqrt{n^{k(i)}}$. Besides, for the states which have an equal number of seats, the second part is constant; in that configuration, this is the first part which differentiates the probability of being pivotal, and the states with a lower size of population are advantaged. This can be seen in the top figure by the linear shapes which appear by group of states with the same number of seats. Finally, we have plotted in pink (resp. blue) -dashed line the average mean for a voter to be pivotal in the Electoral College (resp. the popular vote) case. Most states are below the two lines which confirms in the case of Banzhaf model an inequality between citizens belonging to small states and citizens belonging to large states. Only the two biggest states (California and Texas) are above the blue line, and two more additional states (New York and Florida) are above the pink line.

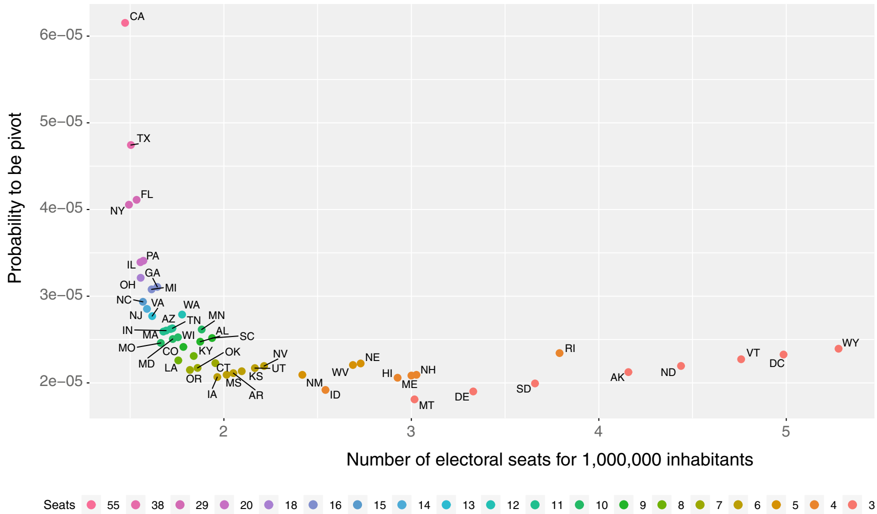

Fig. 4 Pivotality ratio and number of seats per 1,000,000 inhabitants in the case of Banzhaf, depending on the state

3.3 Electoral Justice with Respect to Shapley-Shubik

In this section, the simulations have been made by keeping the exact population per state. Again, we have done 10^{12} simulations, and the computation time was around 5 days, using 40 cores on a server of 58 logical cores at 3.07 GHz.

As shown in Table 3, the obtained probabilities to be pivotal are between 1.73×10^{-9} (Montana) and 5.72×10^{-9} (California). According to Bienaymé–Tchebychev, those results are significant and accurate ($\pm 2 \times 10^{-10}$) at a confidence level better than 85%. As the probabilities to be pivotal are much smaller in the Shapley-Shubik case than in the Banzhaf case, with the same setting the results are not as precise as in the Banzhaf case.

If we had wanted a precision of the same order ($\pm 3.5 \times 10^{-12}$ at a confidence level better than 95%), we should have done 10^{16} simulations. But this would take more than a century to obtain the results!

Yet, as the actual precision in the Banzhaf case was proven (from theoretical values) much better than what was guaranteed from Bienaymé–Tchebychev, it might also be the case in the Shapley-Shubik case. So we decided to use the results, even though they certainly are less precise.

The same problem occurs for the ranking of states according to the probability for a voter to be pivotal. In fact, the ranking of the top four states is known with confidence level better than 95% (according to Bienaymé–Tchebychev), except for third and fourth which could be reversed. But for the other states, the ranking is known only up to $+/-15$ positions in average (more precise for top states and less

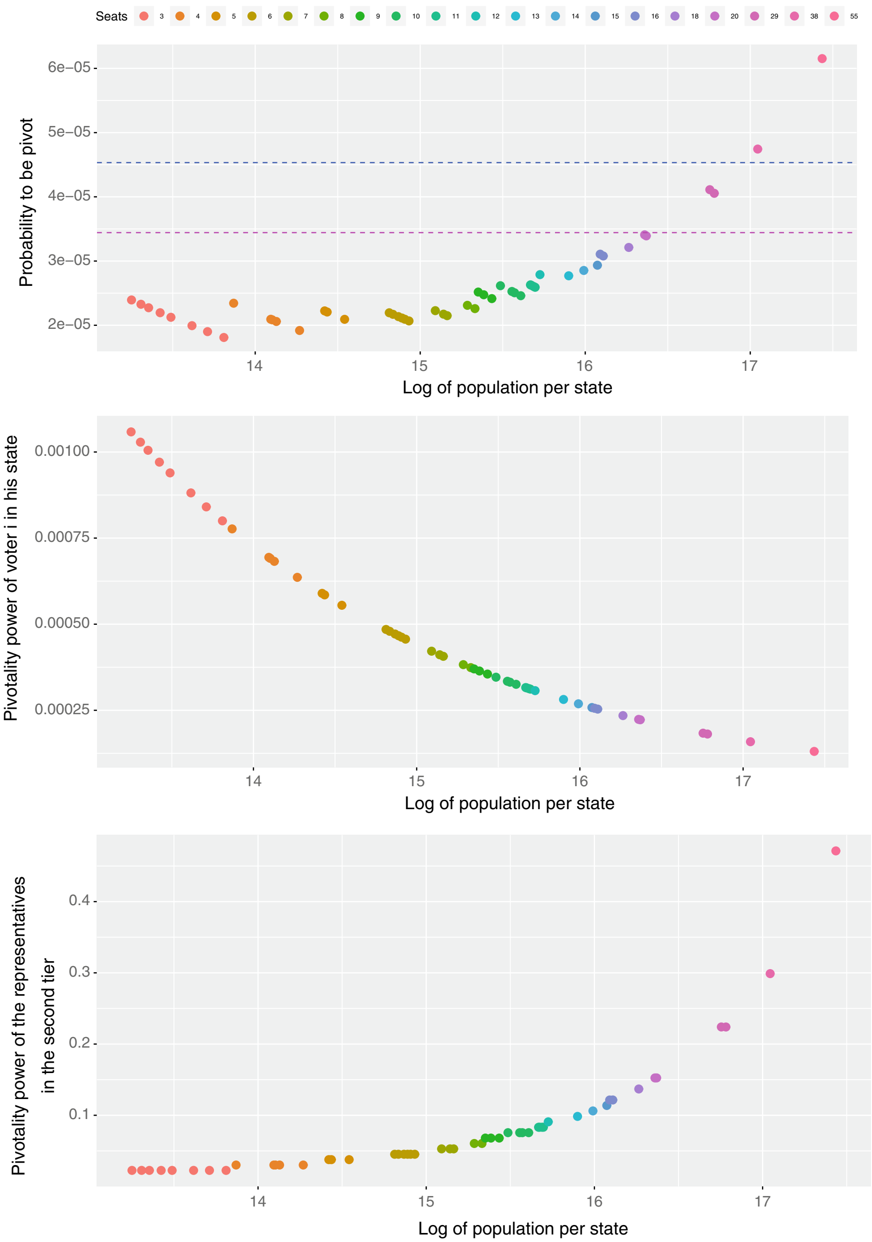

Fig. 5 Pivotality probabilities overall or inside the state or among states by state and number of seats in the case of Banzhaf

Table 3 Probability for a voter to be pivot at the presidential election in each state, with respect to Shapley-Shubik model

State	Simulation results (IAC)	
	Probability to be pivot	Advantage ratio
Alabama	2.311e−09	1.336
Alaska	1.905e−09	1.101
Arizona	2.465e−09	1.425
Arkansas	1.941e−09	1.122
California	5.721e−09	3.307
Colorado	2.263e−09	1.308
Connecticut	2.098e−09	1.213
Delaware	1.823e−09	1.054
District of Columbia	2.169e−09	1.253
Florida	3.873e−09	2.239
Georgia	2.990e−09	1.728
Hawaii	1.929e−09	1.115
Idaho	1.842e−09	1.065
Illinois	3.182e−09	1.839
Indiana	2.469e−09	1.427
Iowa	1.942e−09	1.123
Kansas	2.055e−09	1.188
Kentucky	2.178e−09	1.259
Louisiana	2.066e−09	1.194
Maine	1.955e−09	1.130
Maryland	2.377e−09	1.374
Massachusetts	2.448e−09	1.415
Michigan	2.851e−09	1.648
Minnesota	2.490e−09	1.439
Mississippi	1.951e−09	1.128
Missouri	2.306e−09	1.333
Montana	1.730e−09	1.000
Nebraska	2.061e−09	1.191
Nevada	2.091e−09	1.209
New Hampshire	1.955e−09	1.130
New Jersey	2.601e−09	1.503
New Mexico	2.036e−09	1.177
New York	3.882e−09	2.244
North Carolina	2.694e−09	1.557
North Dakota	2.061e−09	1.191
Ohio	3.029e−09	1.751
Oklahoma	2.050e−09	1.185

(continued)

Table 3 (continued)

State	Simulation results (IAC)	
	Probability to be pivot	Advantage ratio
Oregon	2.072e−09	1.197
Pennsylvania	3.227e−09	1.865
Rhode Island	2.250e−09	1.301
South Carolina	2.391e−09	1.382
South Dakota	1.896e−09	1.096
Tennessee	2.530e−09	1.463
Texas	4.491e−09	2.596
Utah	2.071e−09	1.197
Vermont	2.161e−09	1.249
Virginia	2.565e−09	1.482
Washington	2.568e−09	1.484
West Virginia	2.081e−09	1.203
Wisconsin	2.389e−09	1.381
Wyoming	2.226e−09	1.287

precise for bottom states) if we want to maintain a 95% confidence level (but it is still +/ − 10 positions with a lower confidence level of 85%).

Hence, as the results for Banzhaf case were better than the minimum guaranteed, it is likely that the situation of Shapley-Shubik case is also better than the minimum guaranteed, but the ranking presented in Fig. 6 can be debatable.

Figure 6 presents for each state the ratio of pivotality ordered from the maximum to the minimum. Note that, as in the Banzhaf setting, California and Montana are found in the top and bottom positions, with the same ratio (around 3.4). More generally, it seems that the Shapley-Shubik and Banzhaf ratios of pivotality behave similarly: Figures 3 and 6 are almost indistinguishable. The biggest states are advantaged over others.

In order to highlight the small differences between the two figures, we have represented in Fig. 7 the ranking of the states according to each probability model. It appears that the ranks are indeed very similar (particularly for the biggest states). The small differences concern essentially the states which have less than ten seats. These states have very close values of probability of being pivotal, and the differences of ranking could be attributed to the precision of our simulations, as explained in introduction of this section.

Indeed, we have seen, in the case of Shapley-Shubik, that the top four states were known (confidence level of 95%) except that it was not sure between the third and fourth which actually came firstk; whereas, in the case of Banzhaf, the order was known for sure. Figure 7 shows that both models rank identically the top four states, except that third and fourth are in reverse order, which could absolutely be due to the lack of precision between which state is third or fourth in the case of Shapley-Shubik.

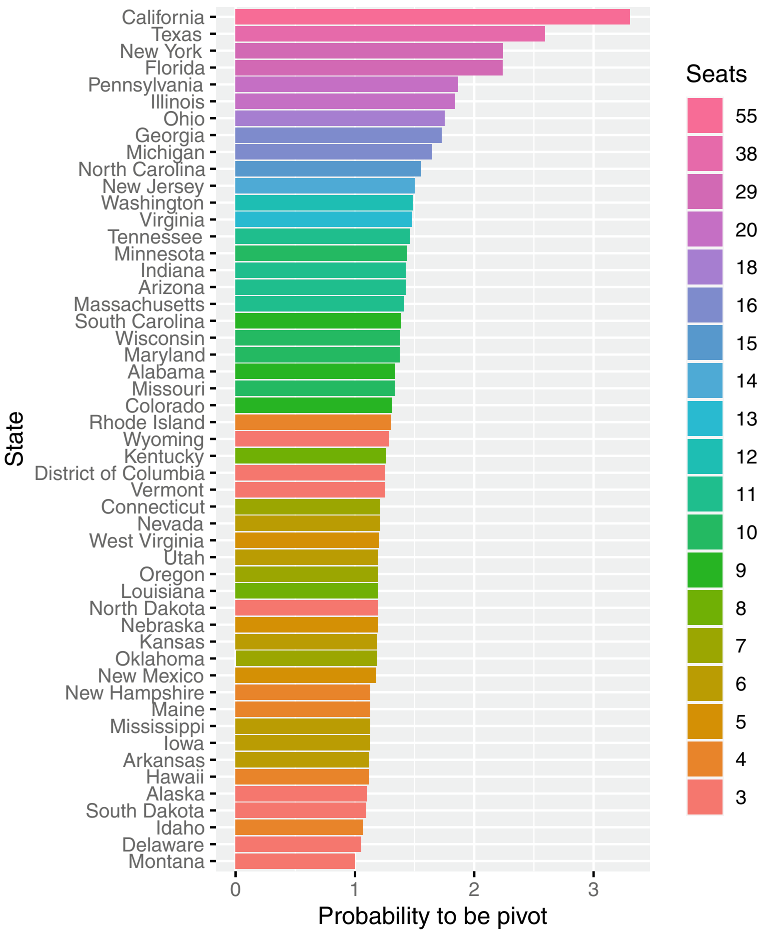

Fig. 6 Pivotality ratio by state and number of seats in the case of Shapley-Shubik, ordered by decreasing pivotality ratio

So this "inversion"-like shape does not necessarily mean that both models disagree. With more simulations, Shapley-Shubik's ranking could be the same as Banzhaf's.

The same applies for all other "inversion"-like shapes. For instance, the difference in ranking between the two models is at most of *seven* positions (Alaska), which is far below the 17 that would still be nonsignificant (confidence level of 85% or even 25 positions with confidence level of 95%).

The fact that the two models give similar results is also confirmed in Fig. 8 which represents the probability for being pivotal with respect to the number of seats per 100,000 inhabitants: This figure is almost indistinguishable from equivalent Fig. 4 for the Banzhaf setting. It clearly appears that the biggest states have the highest probabilities for being pivotal. It is also interesting to notice that for the states with

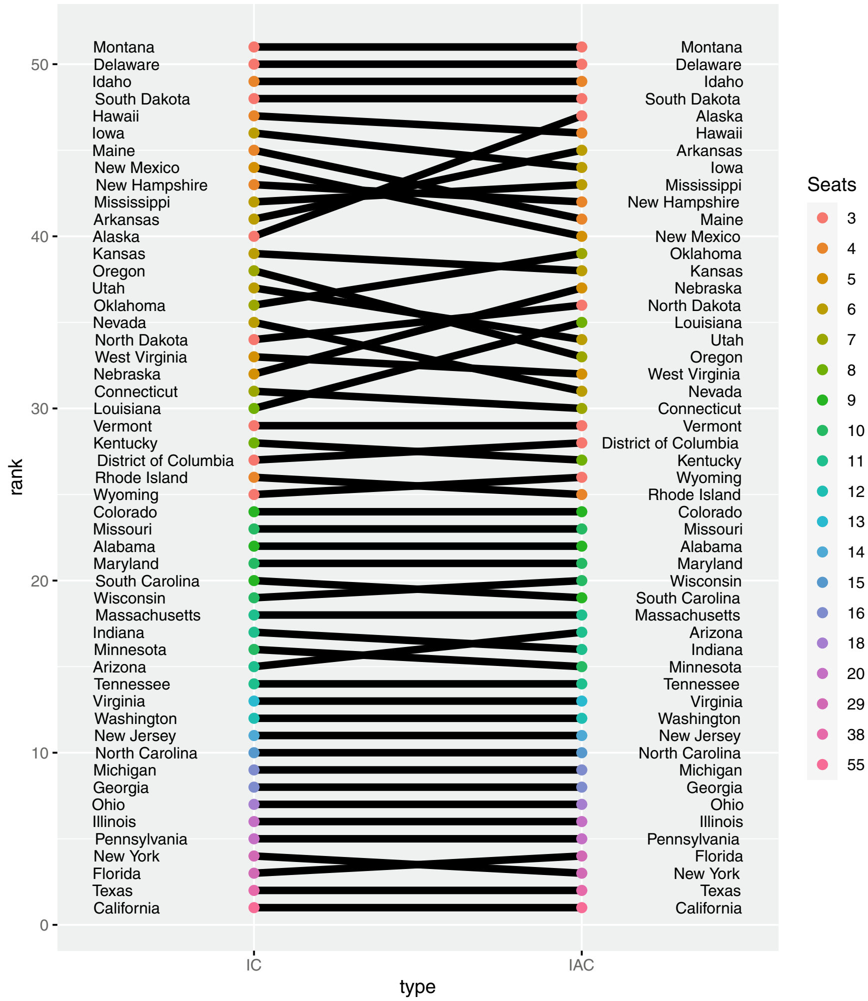

Fig. 7 Comparison of the ranking between Banzhaf and Shapley-Shubik

less than nine seats, the differences of pivotality are very weak. Still, for the states which have the same number of seats, the size of population has a negative effect on the probability of being pivotal, as it can be seen for the lowest states with three seats. Thus, the smaller state of the USA (Wyoming) in terms of population has a probability of being pivotal nearly equivalent to the states which have around ten seats.

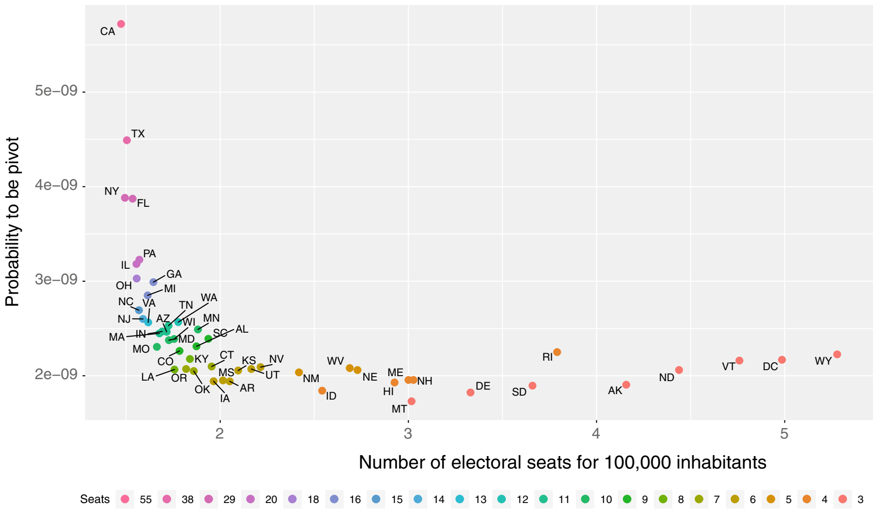

Fig. 8 Pivotality ratio and number of seats per 100,000 inhabitants in the case of Shapley-Shubik, depending on the state

To summarize the findings of this section, we believe that Banzhaf and Shapley-Shubik cases lead to identical ratio differences in the probability for a voter of being pivotal in a given state. The mathematical proof of this is beyond the scope of this paper, and the number of simulations (leading to $5 + 5 = 10$ days of computations) does not give empirical certainty of this,[39] but it also does prove that our belief is not ruled out.[40] Hence, although it might seem puzzling, as the two models behave very differently in many other ways, we still believe that they lead to identical ratio differences in the probability for a voter of being pivotal in a given state.

[39]To achieve this certainty, we have seen that more than a century of computations would be necessary. Another idea, which was not fully implemented here, would be to test the hypothesis with a lower number of voters: for example, dividing state population by 1000. The expected probabilities should be around 10^{-6} instead of 10^{-9}, so we could obtain accurate and more precise results with the same number of simulations (10^{12}, and so again in 5 days). Of course, for the sake of comparison, we should do the same with the Banzhaf case. In a former, less efficient version of our simulator, we tested for Shapley-Schubik a population divide factor of 5683, leading to states with population in the interval (100; 6571). But we were able to perform only 10^8 simulations, at that time, and so ended with the same type of conclusion as here. The obtained probabilities to be pivotal were between 3.284×10^{-5} (California) and 9.680×10^{-6} (Montana). According to Bienaymé–Tchebychev, those results were significant and accurate ($\pm 3 \times 10^{-6}$) at a confidence level better than 95%..

[40]We call the attention of the reader on the fact that appendices 1 and 2 in the working paper version shed some light on these questions.

3.4 Electoral Justice with Respect to May

In this section, the simulations have been made by keeping the exact population per state. Once again, we have done 10^{12} simulations, and the computation time was around 5 days, using 40 cores on a server of 58 logical cores at 3.07 GHz.

As shown in Table 4, the obtained probabilities to be pivotal are between 1.158×10^{-8} (New York) and 3.977×10^{-8} (Wyoming). According to Bienaymé–Tchebychev, those results are significant and accurate ($5 \times \pm 10^{-10}$) at a confidence level better than 85%. As the probabilities to be pivotal are much smaller in the May case than in the Banzhaf case, but higher than in the Shapley-Schubik case, with the same setting the results precision will also be in between those of the two previous cases.

If we had wanted a precision of the same order of Banzhaf case ($\pm 3.5 \times 10^{-11}$ at a confidence level better than 95%), we should have done close to 10^{15} simulations. But this would take more than a decade to obtain the results!

Yet, as the actual precision in the Banzhaf case was proven (from theoretical values) much better than what was guaranteed from Bienaymé–Tchebychev, it might also be the case in the May case.

Indeed, and unsurprisingly, the results derived from our simulations are consistent with the theoretical ones (available for May). Note that the maximal difference between the two values is less than $\pm 2.4 \times 10^{-10}$. Hence, the observed accuracy is twice better as the guaranteed one and for all of the 51 states.

The same question arises for the ranking of states according to the probability for a voter to be pivotal. In fact, the ranking of the top 16 states is known with confidence level better than 95% (according to Bienaymé–Tchebychev), except for a few group of states for which there are still some uncertainties. For instance, between states 15 and 16, the order could be reversed (but not if we allow for a 90.9% confidence level). Same situation with states 2 and 3 (but not if we allow for a 93.3% confidence level). But then there are still 3 groups of states (in the top 16) for which the inside group ranking is debatable (even at confidence level of 80%):

- pivotality around 2.0×10^{-8} for states 13 and 14
- pivotality around 2.2×10^{-8} for states 9 to 12
- pivotality around 2.8×10^{-8} for states 6 and 7.

Yet, we know that the ranking obtained through the simulations matches almost perfectly the theoretical one, even in those three groups, except for states 9 and 10: New Hampshire should be ranked just before Montana (and not the other way around, as found by the simulations, but the difference found between the two States is very small: around 3×10^{-11} only).

As for the other states (ranked 17 or below), the ranking, according to Bienaymé–Tchebychev, is known only up to $+/-7$ positions in average (more precise for top states and less precise for bottom states) if we want to maintain a 95% confidence level (but it is still $+/-4$ positions with a lower confidence level of 85%). Yet, the

Table 4 Probability for a voter to be pivotal at the presidential election in each state, with respect to May model

State	Theoretical	Simulation results (May)	
	value	Prob to be pivot	Advantage ratio
Alabama	1.416e−08	1.415e−08	1.222
Alaska	3.135e−08	3.116e−08	2.691
Arizona	1.297e−08	1.304e−08	1.126
Arkansas	1.547e−08	1.541e−08	1.331
California	1.262e−08	1.271e−08	1.098
Colorado	1.348e−08	1.338e−08	1.155
Connecticut	1.475e−08	1.475e−08	1.273
Delaware	2.511e−08	2.530e−08	2.185
District of Columbia	3.759e−08	3.748e−08	3.236
Florida	1.185e−08	1.202e−08	1.038
Georgia	1.249e−08	1.245e−08	1.075
Hawaii	2.207e−08	2.210e−08	1.908
Idaho	1.917e−08	1.893e−08	1.635
Illinois	1.185e−08	1.187e−08	1.025
Indiana	1.280e−08	1.272e−08	1.099
Iowa	1.483e−08	1.477e−08	1.275
Kansas	1.581e−08	1.588e−08	1.372
Kentucky	1.389e−08	1.384e−08	1.195
Louisiana	1.327e−08	1.319e−08	1.139
Maine	2.263e−08	2.259e−08	1.951
Maryland	1.306e−08	1.307e−08	1.128
Massachusetts	1.268e−08	1.256e−08	1.085
Michigan	1.226e−08	1.227e−08	1.059
Minnesota	1.422e−08	1.421e−08	1.227
Mississippi	1.520e−08	1.531e−08	1.322
Missouri	1.257e−08	1.259e−08	1.087
Montana	2.275e−08	2.283e−08	1.971
Nebraska	2.059e−08	2.062e−08	1.781
Nevada	1.671e−08	1.671e−08	1.443
New Hampshire	2.283e−08	2.279e−08	1.968
New Jersey	1.205e−08	1.224e−08	1.057
New Mexico	1.825e−08	1.803e−08	1.557
New York	1.153e−08	1.158e−08	1.000
North Carolina	1.189e−08	1.201e−08	1.037
North Dakota	3.347e−08	3.342e−08	2.886
Ohio	1.184e−08	1.198e−08	1.034
Oklahoma	1.404e−08	1.406e−08	1.214

(continued)

Table 4 (continued)

State	Theoretical	Simulation results (May)	
	value	Prob to be pivot	Advantage ratio
Oregon	1.373e−08	1.370e−08	1.183
Pennsylvania	1.197e−08	1.195e−08	1.032
Rhode Island	2.859e−08	2.837e−08	2.450
South Carolina	1.464e−08	1.463e−08	1.263
South Dakota	2.760e−08	2.764e−08	2.387
Tennessee	1.305e−08	1.296e−08	1.119
Texas	1.183e−08	1.185e−08	1.023
Utah	1.634e−08	1.633e−08	1.410
Vermont	3.589e−08	3.599e−08	3.108
Virginia	1.225e−08	1.226e−08	1.059
Washington	1.345e−08	1.339e−08	1.156
West Virginia	2.028e−08	2.039e−08	1.761
Wisconsin	1.327e−08	1.328e−08	1.146
Wyoming	3.981e−08	3.977e−08	3.434

simulated ranking is much closer to the theoretical one. In fact, there are only 5 groups of states where the inside group ranking would be different with more simulations

- pivotality around 1.19×10^{-8} for states 45 to 49: The real order should be Pennsylvania (45), North Carolina, Illinois, Florida, and Ohio (49)
- pivotality around 1.26×10^{-8} for states 38 to 40: The real order should be Massachusetts (38), Missouri, and California (40)
- pivotality around 1.29×10^{-8} for states 35 and 36: The real order should be Tennessee (35), and Arizona (36)
- pivotality around 1.32×10^{-8} for states 32 and 33: The real order should be Louisiana (32), and Wisconsin (33)
- pivotality around 1.33×10^{-8} for states 30 to 31: The real order should be Colorado (30), and Washington (31).

Each time, the difference to obtain the correct order is quite small (between 5×10^{-12} only and 1.5×10^{-10}). Hence, the ranking presented in Fig. 3 is not perfect, but not so bad either.

Figure 9 presents for each state the ratio of pivotality ordered from the maximum to the minimum. Interestingly, the results seem somewhat opposite of those derived in the Banzhaf and Shapley-Shubik settings. For instance, California flipped from upper position to one of the smallest ratios (1.1), while Wyoming (previously with a ratio of 1.3) is now on the top (with a ratio of 3.4). Notice also that the highest ratio in the three models is 3.4.

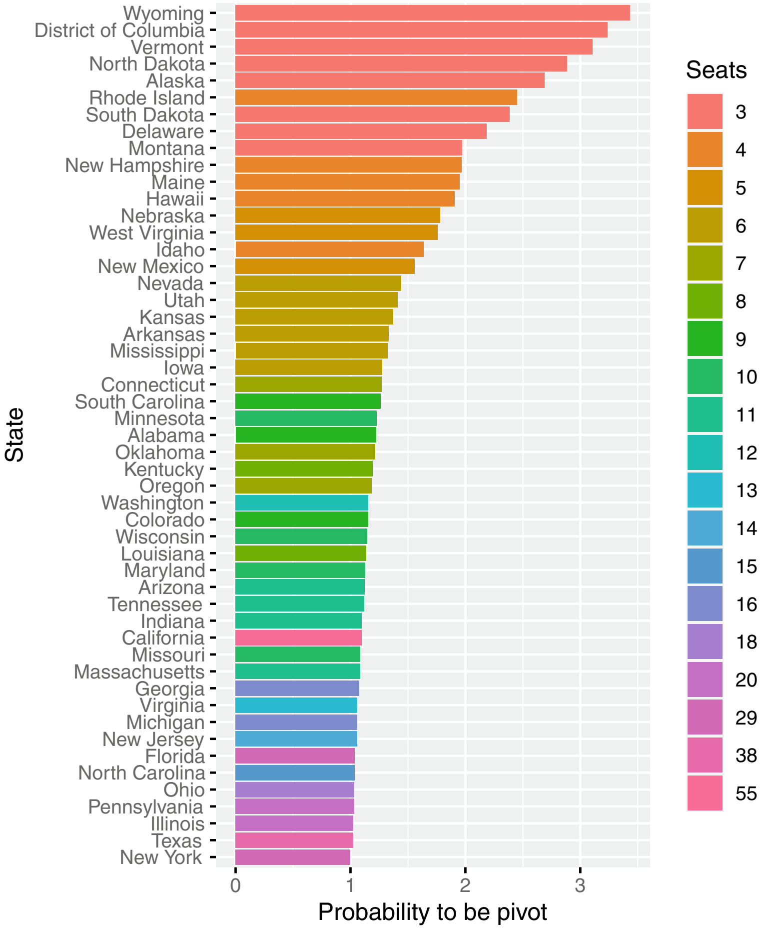

Fig. 9 Pivotality ratio by state and number of seats in the case of May, ordered by decreasing pivotality ratio

Hence, with May's model, the states with the highest probability of being pivotal are those with the smallest number of seats. This is in line with the common popular wisdom concerning the current representation of states in the Electoral College.

Figure 10 shows the ranking difference between the states according to Shapley-Shubik and May. It appears that the ranks have been drastically changed. The states with high (resp. low) probability of being pivotal in the Shapley-Shubik case are often those which have low (resp. high) probabilities in the May case.

Figure 11 decomposes the pivotality part due to being a pivotal voter in his state (middle figure) and the part due to pivotality in the second tier (bottom figure). In May's case, the total pivotality (top figure) is computed as the product of the two parts. The first part of the equation is decreasing proportionally to $1/n^{k(i)}$. It appears

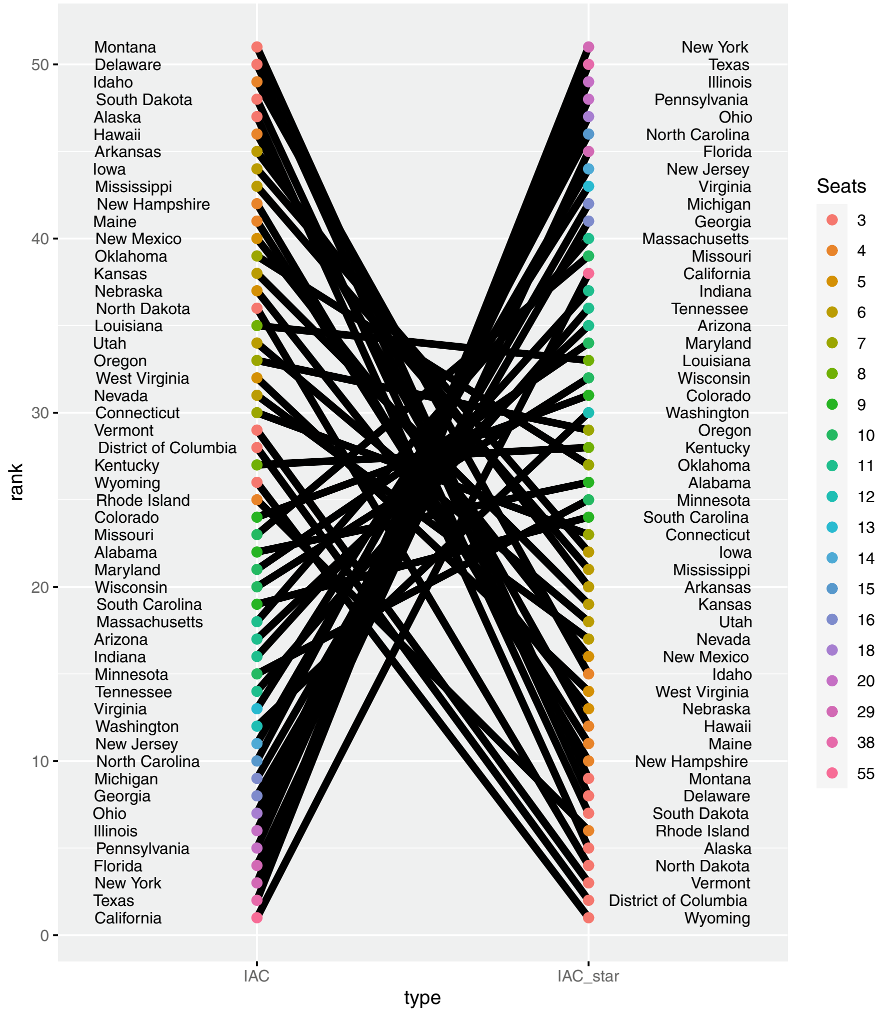

Fig. 10 Comparison of the ranking between Shapley-Shubik and May

that this part has a higher effect on the pivotality compared to the Banzhaf case. This explains why the small states have such a high probability of being pivotal.

Another way to understand the phenomenon is to represent the probability of being pivotal with respect to the number of seats per 100,000 inhabitants, as shown in Fig. 12. It appears that the trend is linear and increasing.

To summarize the findings of this section, we believe that May case can lead to almost opposite rankings to the two other cases, concerning the ratio differences in the probability for a voter of being pivotal in a given state. When states have the same number of seats, the order is maintained, but when they have a different number of

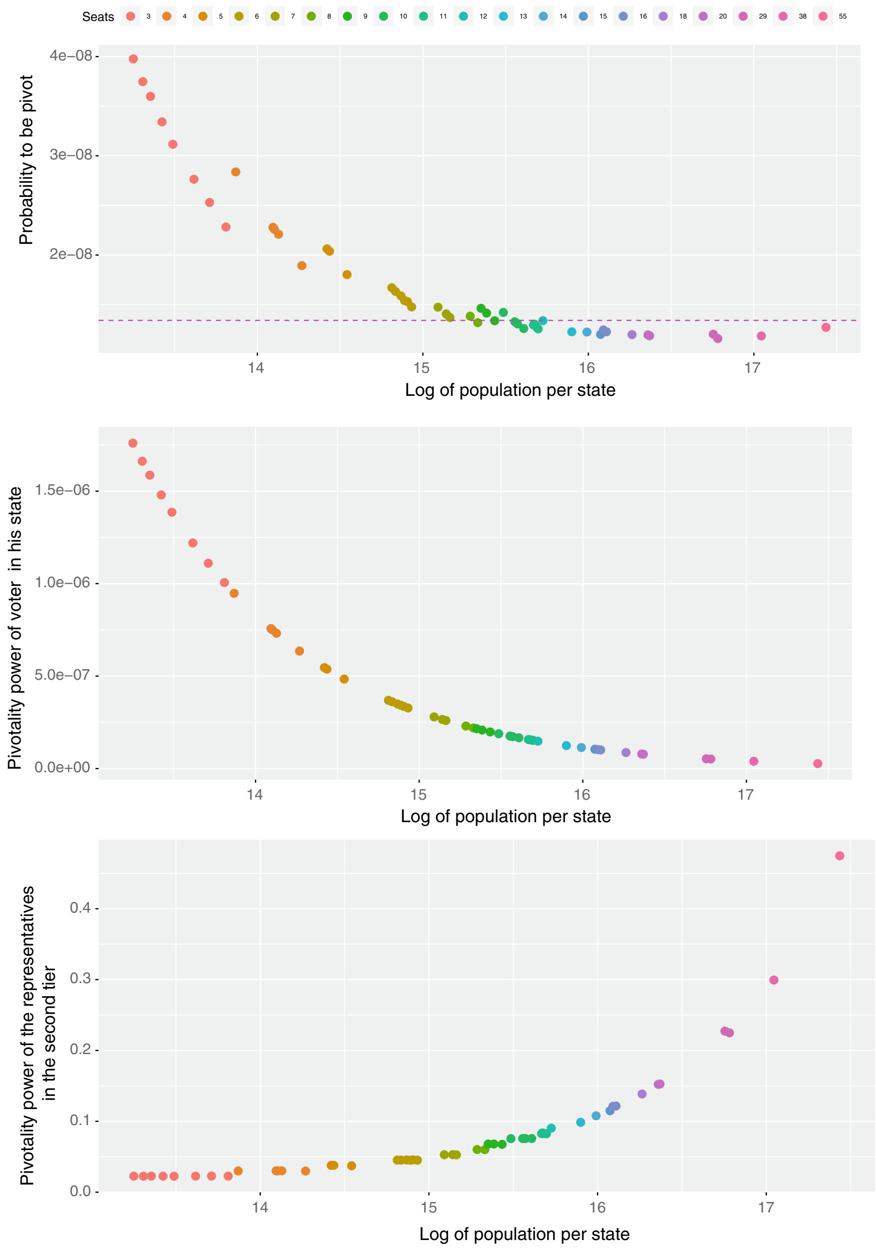

Fig. 11 Pivotality probabilities overall or inside the state or among states by state and number of seats in the case of May

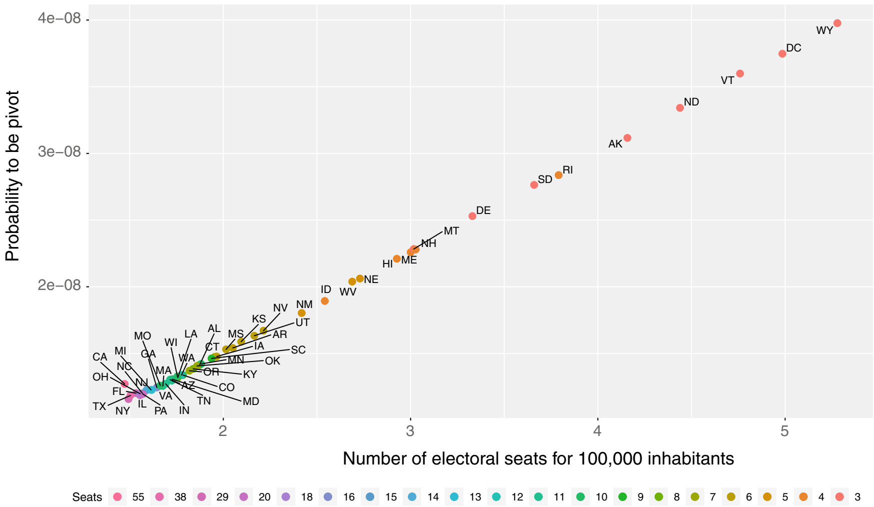

Fig. 12 Pivotality ratio and number of seats per 100,000 inhabitants in the case of May, depending on the state

seats, the order is reversed. The mathematical proof of this is beyond the scope of this paper, and the number of simulations (leading to $5 + 5 = 10$ days of computations) does not give empirical certainty of this,[41] but already give some insights into this almost reverse ranking. It is important to stress that not all models agree on the ranking. So conclusions on whether big or small states take most advantage of the current electoral system are not straightforward.

4 The Twelve Amendment

As already pointed out several times in Sect. 2, our description of the Electoral College departs slightly from the real one. First, two states (Maine and Nebraska) do not allocate their electoral votes according to the "winner take all" rule but use instead the following rule : The two "senators" electoral votes go to the state winner while the congressional electoral votes go the congressional district winners. We do not think that this difference has a great impact on our analysis but we have not done any estimations of the differences. The second and seemingly more serious difference has to do with our treatment of ties. In our simulations, we have assumed that in case

[41]To achieve this certainty, we have seen that more than a decade of computations would be necessary. Another idea, which was not fully implemented here, would be to test the hypothesis with a lower number of voters, as suggested in footnote 39.

of a tie in either a state election and/or in the Electoral College,[42] and the winner was determined through the random draw of fair coin. In reality, a new simple game with a different set of players (the members of the house of representatives[43] at the time of the presidential election with each state represented by a single voter with only one vote) with preferences possibly different from those of the voters (the members of the house have been elected few years before the presidential election).[44]

To speculate on the effects of this tie-breaking rule, consider the *IC* probability model and assume first that the preferences of the representatives are independent from those of the voters today and second that there is a clear majority in each state and a clear majority in the second tier. In such case, note that if a tie occurs in the electoral college, then the two candidates are elected both with a probability $\frac{1}{2}$. So under the above assumptions, there are no differences between the version of the Electoral College considered in this paper and the true one.

5 Concluding Remarks

In this paper, dedicated to the measurement of electoral justice (or lack of) in the 2010 Electoral College, we have obtained several results. First, we have seen that the results obtained by Owen in the case of the 1960 and 1970 Electoral College, on top of which the coincidence between the conclusions drawn, respectively, from Banzhaf and Shapley-Shubik's probability models, remain valid in 2010. Both probability models conclude to a violation of electoral justice at the expense of small states. Second, we have also shown that this conclusion has completely flipped upside down when we use instead May's probability model: This model concludes to a violation of electoral justice at the expense of large states. Besides unifying through a common measurement methodology disparate approaches, one main lesson is that the conclusion on electoral justice is sensitive to the probability models which are used and to the type and magnitude of correlation between voters that they carry.

Acknowledgements We are very pleased to offer this paper as a contribution to this volume honoring Bill Gehrlein and Dominique Lepelley with whom we had the pleasure to cooperate in the recent years. Both have made important contributions to the evaluation of voting rules and electoral systems through probability models. Power measurement and two-tier electoral systems, the two topics of our paper, are parts of their general research agenda. We hope that they will find our paper respectful of the approach that they have been promoting in their work. Last, but not least, we would like to thank two anonymous referees who have attracted our attention on the imperfections and

[42] According to Hayes (2012), there are $0.16976480564070 \times 10^{14}$ ways of arriving at a tie roughly 0.75 percent of 2^{51}, the total number of profiles in the upper tier.

[43] The district of Columbia is not part of the game and in fact in case of a tie, and the rule is to have further votes (as much as needed) as long as a fixed deadline has not expired.

[44] So truly, this tie-breaking simple game is itself a compound simple game where the second tier is the ordinary majority game with 50 players $(\{1, \ldots, 50\}, \mathcal{W}(q, w))$ where $w = (1, \ldots, 1)$ and $q = 25$, and the first tiers are the majority games with the representatives of the states as the basic players.

shortcomings contained in an earlier version and whose comments and suggestions have improved a lot the exposition of the ideas developed in our paper. Of course, they should not be held responsible for any of the remaining insufficiencies. The working paper version, available on the Web sites of the authors, contains three appendices offering supplementary mathematical developments on the intricacies of the Shapley-Shubik's probability model and its discrete counterpart. The three first authors acknowledge funding from the French National Research Agency (ANR) under the Investments for the Future program (Investissements d' Avenir, grant ANR-17 EURE-0010).

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Resistance to Manipulations

Further Results on the Manipulability of Social Choice Rules—A Comparison of Standard and Favardin–Lepelley Types of Individual Manipulation

Fuad Aleskerov, Daniel Karabekyan, Alexander Ivanov, and Vyacheslav Yakuba

1 Introduction

The problem of manipulation in voting is inevitable for all non-dictatorial voting rules as was shown by Gibbard (1973) and Satterthwaite (1975). In other words, a voter or a group of voters can get better results for themselves by purposely stating insincere preferences. This very problem was widely studied in the literature (see, e.g., Barbera et al. (2004), Benoît (2002), Chamberlin (1985), Ching and Zhou (2002), Duggan and Schwartz (2000), Durand et al. (2016), Favardin and Lepelley (2006), Favardin et al. (2002), Kim and Roush (1996), Lepelley and Mbih (1994), Lepelley and Valognes (2003), Pritchard and Wilson (2007), Slinko (2004)). All those papers estimate the degree of manipulability for different setups: different rules, number of alternatives, probabilistic structures, tie-breaking rules, and types of manipulability.

One of the most important papers in this field is "Some Further Results on the Manipulability of Social Choice Rules" by Favardin and Lepelley (2006). They considered not only many voting rules but also new type of manipulation: possibility of reactions and counterthreats. In other words, when voter decides to manipulate, she does not take preferences of others as given but considers the possible reaction,

F. Aleskerov (✉) · D. Karabekyan · A. Ivanov · V. Yakuba
National Research University Higher School of Economics, 101000 Moscow, Russia
e-mail: alesk@hse.ru

D. Karabekyan
e-mail: dkarabekyan@hse.ru

A. Ivanov
e-mail: aaivanov@hse.ru

V. Yakuba
e-mail: yakuba@ipu.ru

F. Aleskerov · V. Yakuba
Institute of Control Sciences of Russian Academy of Sciences, 117997 Moscow, Russia

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when other voters can either return sincere voting outcome or even make it worse to the manipulator.

In this paper, we consider the case of individual manipulation with and without reactions (Type I and Type IV in terms of Favardin and Lepelley (2006)) and try to extend their results in several ways. First, we study the influence of tie-breaking rule on the results by introducing extended preferences over the sets of alternatives. Second, we also look at impartial culture model in addition to impartial anonymous culture. Third, we extend results for 4 alternatives and, for some cases, for 5 alternatives.

Next section gives a detailed overview of the manipulation model and rules under study.

2 Social Choice Rules

The manipulability is evaluated for the following procedures (Aleskerov et al. (2011, 2012), Aleskerov and Kurbanov (1999)): Plurality (Pl), q-Approval Rule, $q = 2$ (A2), Borda (Bo), Black (Bl), Copeland I (C1), Copeland II (C2), Copeland III (C3), Threshold (T), Nanson (N), Inverse Borda (IvBo), and Hare (H).

Let $P = (P_1 \dots P_n)$ be a profile of linear preferences of n agents for the set of alternatives A, $|A| = m$. The majority relation μ is defined as a binary relation in which an alternative a dominates an alternative b if the number of agents for whom a is more preferable than b is greater than the number of agents for whom b is more preferable than a. $C(P)$ denotes the choice made by the rules.

Plurality Rule (Pl) chooses alternatives that are on the first place by the maximum number of agents, i.e., $a \in C(P) \Leftrightarrow [\forall x \in A \quad n^+(a, P) \geq n^+(x, P)]$,

where $n^+(a, P) = card\{i \in N | \forall y \in A \; aP_i y\}$

***q*-Approval Rule, *q*= 2, (A2)** chooses alternatives which are on the first or second places of the preferences of the maximum number of agents.

Borda's Rule (Bo). First, the Borda's score $r(a, P) = \sum_{i=1}^{n} r_i(a, P_i)$ is constructed, where $r_i(x, P)$ is the cardinality of the lower contour set of alternative x for agent i, i.e., $r_i(x, P) = |L_i(x)| = |\{b \in A : xP_i b\}|$.

The alternatives with maximum Borda's score are chosen

$$a \in C(P) \Leftrightarrow [\forall b \in A, \quad r(a, P) \geq r(b, P)].$$

Black's Procedure (Bl). An alternative is said to be a Condorcet winner CW, if it is undominated by any other alternative in the majority relation μ,

$$CW = [a | \neg \exists x \in A, \; x \mu a]$$

If the Condorcet winner exists, then it is chosen; otherwise, Borda's rule is applied.

Copeland's Rule I (C1). Alternatives with the maximum value of the function $u(x)$ are chosen. The function $u(x)$ is defined to be the cardinality of lower contour set minus cardinality of upper contour set of the alternative x in the majority relation μ, i.e., $u(x)$ is defined as the number of alternatives, which are dominated by the alternative x in the majority relation μ, minus the number of alternatives, which dominate the alternative x in the majority relation μ.

Copeland's Rule II (C2). Alternatives with the maximum value of the function $u(x)$ are chosen, and the function $u(x)$ is defined by the cardinality of lower contour set of alternative x in the majority relation μ.

Copeland's Rule III (C3). Alternatives with the minimum value of the function $u(x)$ are chosen. The function $u(x)$ is constructed as a cardinality of upper contour set of alternative x in the majority relation μ.

Threshold Rule (T). First, the number of worst places of the alternatives among agents is compared. If these numbers for 2 alternatives are equal, then the number of second worst places is compared and so on. If on any step of comparison the number of places is not equal, then the alternative with less number of such places dominates the other. The alternatives which are not dominated by others are chosen.

Nanson's Procedure (N). For each alternative, Borda's count is calculated, and alternatives, for which Borda's count is less than the mean value, are omitted. The procedure repeats on the rest of the alternatives until choice will not be empty.

Inverse Borda's Procedure (InvBo). For each alternative, Borda's count is calculated, the alternative with the minimum Borda's count is omitted, and the procedure is repeated until the choice is not empty.

Hare's Procedure (H). First, a simple majority rule is applied. If none of the alternatives gets simple majority of votes, then the alternative with the minimum number of votes is omitted, and the procedure is applied to the rest of alternatives until the simple majority choice is found or the choice is empty.

In Favardin and Lepelley (2006), scoring rules and scoring elimination rules were considered for 3 alternatives. In order to connect our results with those of that paper, we point out the correspondence between our voting rules and ones used in it. In Favardin and Lepelley (2006), the Plurality rule is called scoring rule $SR(0)$, where "0" refers to the number of points, assigned to the second place in the agent's ordering. The two-approval rule, which is identical to the inverse Plurality for 3 alternatives, is denoted as the $SR(1)$ procedure and Borda's rule as $SR(0.5)$. From the iterative scoring procedures, or procedures in which the alternatives are excluded successively, having the minimum number of first votes, Hare rule is called $ISR(0)$ in their notation, and the Nanson rule, which for any alternative evaluates the sum of points, assigned for each place, and excludes those, for which this sum is less than average, is called $\overline{ISR_1}(0.5)$. One of the three Copeland procedures, which are based on the majority relation, Copeland I is called in Favardin and Lepelley (2006) as the COP procedure. The rest of the considered procedures, Black, Copeland II, III, Threshold, and Inverse Borda's procedures do not have counterparts in their notation.

3 Extended Preferences and Manipulation

For the case of multiple choice, the extended preferences are constructed. For 3 alternatives, the manipulability is evaluated for Leximin, Leximax, PWorst, and PBest extensions defined below. For 4 and 5 alternatives, only Leximin and Leximax are considered. For these and other extensions, one can see the definitions and discussion in Aleskerov et al. (2011).

For 3 alternatives, the Leximin extension for the agent with preferences $a \succ b \succ c$ looks as follows $\{a\} \succ \{a, b\} \succ \underline{\{b\} \succ \{a, c\} \succ \{a, b, c\}} \succ \{b, c\} \succ \{c\}$. For the Leximax extension, the ordering of the sets is different: $\{a\} \succ \{a, b\} \succ \underline{\{a, b, c\} \succ \{a, c\} \succ \{b\}} \succ \{b, c\} \succ \{c\}$. PWorst extension for 3 alternatives is different in underlined part and looks like $\{a\} \succ \{a, b\} \succ \underline{\{b\} \succ \{a, b, c\} \succ \{a, c\}} \succ \{b, c\} \succ \{c\}$. PBest extension is defined as $\{a\} \succ \{a, b\} \succ \underline{\{a, c\} \succ \{a, b, c\} \succ \{b\}} \succ \{b, c\} \succ \{c\}$.

Leximin for 4 alternatives looks like:

$$\{a\} \succ \{a, b\} \succ \{b\} \succ \{a, c\} \succ \{a, b, c\} \succ \{b, c\} \succ \{c\} \succ \{a, d\} \succ$$
$$\succ \{a, b, d\} \succ \{b, d\} \succ \{a, c, d\} \succ \{a, b, c, d\} \succ \{b, c, d\} \succ \{c, d\} \succ \{d\}$$

Leximax for 4 alternatives looks as follows.

$$\{a\} \succ \{a, b\} \succ \{a, b, c\} \succ \{a, b, c, d\} \succ \{a, b, d\} \succ \{a, c\} \succ \{a, c, d\} \succ$$
$$\succ \{a, d\} \succ \{b\} \succ \{b, c\} \succ \{b, c, d\} \succ \{b, d\} \succ \{c\} \succ \{c, d\} \succ \{d\}$$

For 5 alternatives, we present the extension ordering for Leximin:

$$\{a\} \succ \{a, b\} \succ \{b\} \succ \{a, c\} \succ \{a, b, c\} \succ \{b, c\} \succ \{c\} \succ \{a, d\} \succ \{a, b, d\}$$
$$\succ \{b, d\} \succ \{a, c, d\} \succ \{a, b, c, d\} \succ \{b, c, d\} \succ \{c, d\} \succ \{d\} \succ \{a, e\}$$
$$\succ \{a, b, e\} \succ \{b, e\} \succ \{a, c, e\} \succ \{a, b, c, e\} \succ \{b, c, e\}$$
$$\succ \{c, e\} \succ \{a, d, e\} \succ \{a, b, d, e\} \succ \{b, d, e\} \succ \{a, c, d, e\}$$
$$\succ \{a, b, c, d, e\} \succ \{b, c, d, e\} \succ \{c, d, e\} \succ \{d, e\} \succ \{e\}$$

for Leximax:

$$\{a\} \succ \{a, b\} \succ \{b\} \succ \{a, c\} \succ \{a, b, c\} \succ \{b, c\} \succ \{c\} \succ \{a, d\} \succ \{a, b, d\} \succ$$
$$\succ \{b, d\} \succ \{a, c, d\} \succ \{a, b, c, d\} \succ \{b, c, d\} \succ \{c, d\} \succ \{d\} \succ \{a, e\} \succ$$
$$\succ \{a, b, e\} \succ \{b, e\} \succ \{a, c, e\} \succ \{a, b, c, e\} \succ \{b, c, e\} \succ \{c, e\} \succ$$
$$\succ \{a, b, d, e\} \succ \{b, d, e\} \succ \{a, c, d, e\} \succ \{a, b, c, d, e\} \succ$$
$$\succ \{b, c, d, e\} \succ \{c, d, e\} \succ \{d, e\} \succ \{e\}$$

For 3 alternatives, we also consider alphabetical tie-breaking rule used in Favardin and Lepelley (2006). For the alphabetical tie-breaking rule, the first alternative in alphabetical ordering of the choice set is taken, no matter what the orderings of the agents are. This means, for example, that if the multiple choice of some rule is $\{a, b\}$, then the alphabetical tie-breaking rule makes the choice to be $\{a\}$, no matter if some of the agents have the preferences $b \succ c \succ a$ or $c \succ a \succ b$, etc.

For the individual manipulability (Type I), only one agent in the considered profile manipulates. The manipulating agent thus does not collaborate with other agents, and she does not take into account possible response of other agents. The manipulating agent takes the others' preferences for granted and considers them fixed. Type IV manipulability model implies possible countermeasures from other agents. The social choice for a profile is calculated (call it SC1). For each agent, we try all possible ways of misrepresentation of her preferences. After each attempt, a new social choice is calculated (SC2). If given the preferences of the agent, the new social choice is better than the initial one, this situation is temporarily called manipulation, but we try all possible countermeasures. For each of the remaining agents, we generate all possible reactions. For each reaction, we calculate the updated social choice (SC3). If for the reacting agent the choice SC3 is better or the same as SC2, and for the manipulating agent the choice SC3 is worse than or equal to SC1, then the profile is considered to be a non-manipulable one (due to the counterthreat). Thus, in Type IV manipulation, a profile is considered non-manipulable if no agent can manipulate, or if for every manipulation attempt that leads to a better social choice for the manipulating agent, there exists at least one other agent with a counterthreat (i.e., if there exists SC2, which is better than SC1 for the manipulating agents, then there exists SC3 that is better than SC2 for some other reacting agent, and, at the same time, SC3 is worse or equal than SC1 for the manipulating agent). We calculate Nitzan–Kelly index, defined below, as the share of manipulable profiles (excluding non-manipulable profiles) in the set of all generated profiles.

Let us provide an example for 3 alternatives, 5 agents, and Approval $q = 2$ rule. Let us assume that the initial sincere preferences are

Agent 1: $c \succ b \succ a$
Agent 2: $b \succ a \succ c$
Agent 3: $b \succ c \succ a$
Agent 4: $b \succ c \succ a$
Agent 5: $b \succ c \succ a$

The social choice under Approval $q = 2$ rule will be $\{b\}$.

In the individual manipulability (Type I), we assume that it is possible that the first agent may manipulate. Currently, she has her second choice as a result of voting. Let us assume that she misrepresents her preferences by putting in the ballot not $c \succ b \succ a$, but $c \succ a \succ b$. The profile with misrepresented preferences will look as the following one

Agent 1: $c \succ a \succ b$

Agent 2: $b \succ a \succ c$

Agent 3: $b \succ c \succ a$

Agent 4: $b \succ c \succ a$

Agent 5: $b \succ c \succ a$

Now the result under Approval $q = 2$ rule will be different, and there is a tie between $\{b\}$ and $\{c\}$, so the result will be $\{b, c\}$ which is better for Agent 1. This profile will be marked as manipulable under Type I, because we found at least one case when an agent may misrepresent her preferences to obtain a better social choice.

In the counterthreat model (Type IV), we additionally check whether any countermeasures from other agents are possible. Let us assume that seeing Agent 1 manipulating, Agent 3 will also misrepresent her preferences and present not sincere preference $b \succ c \succ a$, but insincere one $b \succ a \succ c$. In this case, the profile will be

Agent 1: $c \succ a \succ b$

Agent 2: $b \succ a \succ c$

Agent 3: $b \succ a \succ c$

Agent 4: $b \succ c \succ a$

Agent 5: $b \succ c \succ a$

The social choice under Approval $q = 2$ will be $\{b\}$, and this choice is the same as initial choice in the profile. Because we found a way, how another agent may misrepresent her preferences in response to the manipulation attempt of the first agent, this profile will not be marked as manipulable in this case (but we need to check further to make sure that for every manipulation attempt, there is at least one way of response that will ruin the manipulation attempt).

To evaluate the degree of manipulability, the Nitzan–Kelly index (NK) is used.

The evaluation is performed for both impartial culture (IC) and impartial anonymous culture (IAC) models. While in impartial culture model all profiles are assumed to be equally likely, in impartial anonymous culture all voting situations are equally likely. In this model, profiles obtained by permutation of agents are indistinguishable and treated as one. For the considered models, the share of manipulable profiles (or voting situations) is calculated.

This index is defined as the share of the manipulable profiles in the total number of the profiles under consideration, i.e.,

$$NK = \frac{d_0}{d_{\text{total}}}$$

where d_0 is the number of manipulable profiles (or voting situations), d_{total} is the total number of profiles (voting situations).

4 Results

The results are estimates obtained via computer simulations. At least, ten thousands of random profiles are generated for each combination of alternatives and agents. Then for each profile, the degrees of manipulability and non-manipulability are calculated. A profile is considered as manipulable if there is at least one agent who can misrepresent her preferences in order to obtain a better choice. A profile is considered to be non-manipulable if all possible changes of preferences lead to either the same social choice or to a worse social choice, according to the notion of the upper bound of manipulability. The calculated numbers of manipulable and non-manipulable profiles are normed to the total number of considered profiles. An ensemble average is taken for each rule, for each number of alternatives and number of agents, and for IC and IAC cultures.

In Tables 1 and 2, we present results of the estimation of NK index for 3 alternatives and 25 agents, both IC and IAC models and both manipulation types. From this table, we can observe that the values of NK index for IC are higher than the ones for IAC for Type I manipulability. This happens because in IAC more unanimous profiles have higher probability. In other words, in the profile, the more agents have same preferences, the more probable the profile is in IAC, comparing to IC. At the same time, individual manipulation is less possible in those profiles. However, this is no longer true for Type IV manipulability. It seems that while the more diverse profiles are more vulnerable to individual manipulation, counterthreats are not always possible in those profiles. That is why for some rules, Type IV manipulability under

Table 1 NK values for 3 alternatives, 25 agents, Leximin, Leximax

	Leximin				Leximax			
	IC		IAC		IC		IAC	
	I	IV	I	IV	I	IV	I	IV
Plurality	0.36	0.01	0.13	0.041	0.36	0.01	0.132	0.041
Appr. 2	0.3	0.1482	0.2	0.046	0.3	0.1482	0.202	0.046
Borda	0.24	**0.0002**	0.11	**0.002**	0.23	0.0291	0.108	0.008
Black	0.13	0.0014	0.06	0.006	0.15	0.0573	0.057	0.014
Copl. I	0.08	0.0761	0.04	0.043	0.18	0.1845	0.057	0.057
Copl. II	0.08	0.0761	0.04	0.043	0.18	0.1845	0.057	0.057
Copl. III	0.08	0.0761	0.04	0.043	0.18	0.1845	0.057	0.057
Threshold	0.24	0.0498	0.14	0.021	0.24	0.0462	0.137	0.021
Nanson	0.07	0.0031	0.03	0.005	0.13	**0.0002**	**0.039**	0.005
Inv. Borda	**0.07**	0.0031	**0.03**	0.003	0.13	**0.0002**	0.04	**0.003**
Hare	0.11	0.049	0.05	0.013	**0.11**	0.049	0.046	0.013

Minimal values are highlighted with bold

Table 2 NK values for 3 alternatives, 25 agents, alphabet

	Alphabet			
	IC		IAC	
	I	IV	I	IV
Plurality	0.256	0.005	0.083	0.02
Appr. 2	0.221	0.08	0.135	0.02
Borda	0.162	**0.00004**	0.072	**0.001**
Black	0.131	0.0001	0.051	0.005
Copl. I	0.103	0.0004	0.038	0.004
Copl. II	0.103	0.0004	0.038	0.004
Copl. III	0.103	0.0004	0.038	0.004
Threshold	0.222	0.05	0.134	0.02
Nanson	**0.09**	0.0003	**0.033**	0.004
Inv. Borda	0.091	0.0002	0.034	0.002
Hare	0.097	0.05	0.046	0.01

IAC is higher than under IC. This can be seen from Fig. 1a, b. More detailed pictures of the least manipulable rules are given in Fig. 1c, d.

As one can see from Fig. 1a, c, for Type IV manipulability, for IC, the Borda's rule reaches the minimal manipulability. However, Black, Copeland II (for odd number of agents), Nanson, and Inverse Borda's rules are quite close to the minimum.

For the IAC, Borda and Inverse Borda are minimal ones, and Black and Nanson are close to minimal manipulability.

Consider also the alphabet extension for 3 alternatives, as in Favardin and Lepelley (2006). The counterparts of some of the rules are presented in Fig. 1e.

Note, that Borda's rule, or $SR(0.5)$, is minimally manipulable, along with some close to minimum rules, like Black, Copeland I,II,III, Threshold, Nanson, and Inverse Borda.

Results, in general, are consistent with the findings of Favardin and Lepelley (2006), but not for all preferences extensions. For the cases, considered in it, there is the same trend in the manipulability and the same dominating rules. For example, they consider the case of IAC, alphabetical tie-breaking rule, Type I and Type IV manipulability and found that for a higher number of agents, Nanson rule is least manipulable for Type I case and Borda's rule—for Type IV. In our case, results vary with the extension axiom. For Leximin extension, the results are the same, but for Leximax extension, the Hare rule and the Inverse Borda's rule are among the least manipulable ones. This can be seen in Fig. 2.

At Fig. 3a, b, there is a difference between Type IV and Type I indices for IC and IAC. It demonstrates the percentage of profiles where manipulation is no longer possible if we consider reactions. One can see that Borda's rule is among rules where this percentage is higher. This is based on the nature of the manipulable situations for those rules. Profiles which score of top alternatives are very close manipulable,

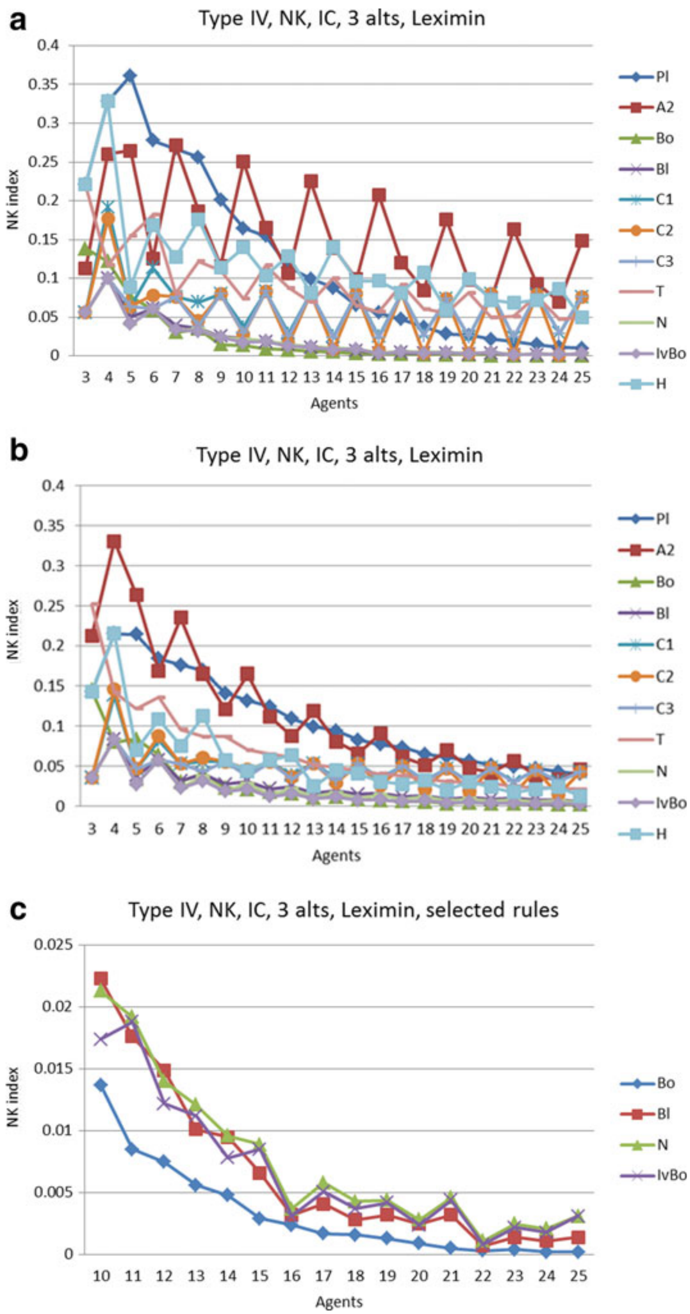
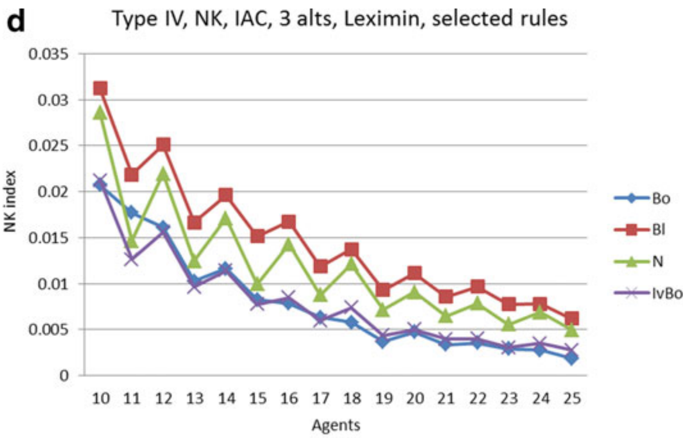


Fig. 1 **a** Manipulability of Type IV for IC, 3 alternatives, Leximin, **b** manipulability of Type IV for IAC, 3 alternatives, Leximin, **c** manipulability of Type IV for IC, 3 alternatives, from 10 agents, Leximin, selected procedures, close to minimum manipulability, **d** manipulability of Type IV for IAC, 3 alternatives, from 10 agents, Leximin, selected procedures, close to minimal manipulability, **e** manipulability of Type IV for IAC, 3 alternatives, alphabet



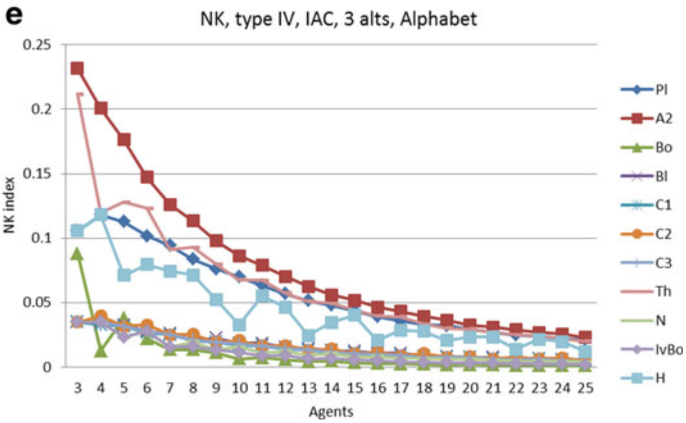


Fig. 1 (continued)

but in many cases, if someone can change her vote to change social choice, the others can react and at least get the situation back. Borda's rule gives a lot of options for all voters, for example, changing your second and third alternatives in declared preferences can lower score of your second best alternative without influencing on first best one.

For IC, the highest difference between Type IV and Type I manipulability show the Borda's and Inverse Borda's rules.

For IAC, the difference is high for Borda's, Inverse Borda's, and Nanson rules.

From Tables 3 and 4, we can see the least manipulable rules for each case. For Type I manipulability, Nanson and Hares rules are among the least manipulable in most cases. For Type IV, Inverse Borda's and Borda's rules are among the most stable. Those results are consistent with Favardin and Lepelley (2006), but we have more rules as the least manipulable ones for some cases.

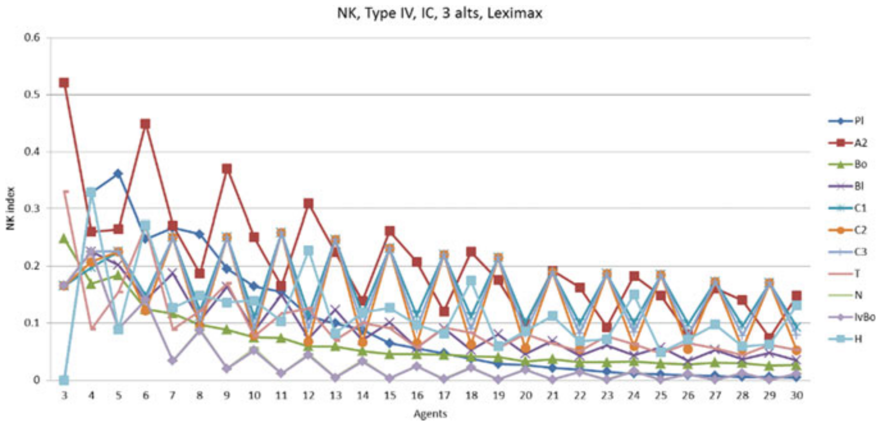


Fig. 2 Manipulability of Type IV for IC, 3 alternatives, Leximax

At Figs. 4a–c, 5a, b, results for 4 alternatives are presented. Least manipulable rules are pointed out in Tables 5 and 6. As one can see, while for Type I manipulability still the best rules are Nanson and Hare ones, for Type IV Borda's rule seems to be better than any other rule even for Leximax extension.

For 5 alternatives (Fig. 6a, b and Tables 7 and 8), results are even closer to the one of Favardin and Lepelley (2006).

We notice that Borda's rule for Leximin and Leximax extensions, being among minimally manipulable ones for both IC and IAC, also reaches the high difference in relative values in NK index between Type IV and Type I manipulability for 3, 4, and 5 alternatives.

For 4 alternatives for Type IV manipulability, Borda's rule is the least manipulable one starting from 4 agents. For 3 agents, for Leximin and alphabet extensions, despite the fact that the Black rule is the least manipulable one for IC, and Inverse Borda for IAC, both these two rules have low values of manipulability for both cultures.

For 5 alternatives, Leximin extension along with minimally manipulated Borda's rule, quite close to the minimum, is Inverse Borda's, Black, and all three Copeland rules.

For Leximax extension, the Borda's rule is minimally manipulated, while the manipulability of Inverse Borda's, Black, and Copeland I, II, III rules is quite scattered being close to minimum.

Contrary to Type I manipulability, for Type IV manipulability for 5 alternatives, Borda's rule is minimally manipulable for all extensions starting from 4 agents; only for 3 agents, the Plurality and Hare rules are minimally manipulable for Leximax extension.

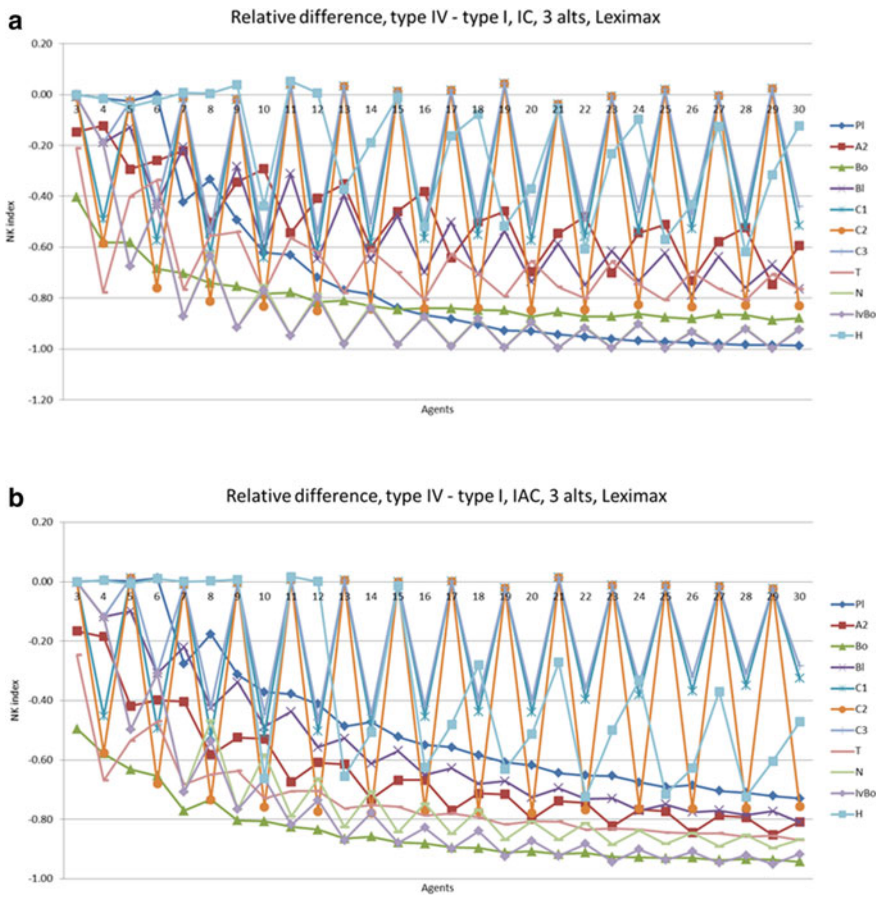


Fig. 3 **a** Difference between Type IV and Type I manipulability, IC, 3 alternatives, Leximax, **b** difference between Type IV and Type I manipulability, IAC, 3 alternatives, Leximax

5 Conclusion

We considered several extensions for manipulation model proposed in Favardin and Lepelley (2006). We considered multi-valued choice, both impartial and impartial anonymous culture, and the cases of 3, 4, and 5 alternatives. One can see that for Type I manipulability, the manipulation index for IC is higher, then for IAC. However, for Type IV manipulability the results are opposite, and the manipulation under IAC is higher. While for multi-valued choice and 3 alternatives results are even less certain than for alphabetical tie-breaking rule (more rules are among the least manipulable ones), for 4 and 5 alternatives Borda is the least manipulable one for Type IV manipulability even for multiple choice. The main reason behind it is that Borda's rule is the most "flexible" scoring rule allowing one to influence scores of

Table 3 Minimally manipulated procedures for Type I manipulability for 3 alternatives for several number of agents

Culture	Extensions	8	11	22	25
IC	Leximin	H	N, IvBo	C3, N	N
IC	Leximax	H	H	C3, N	H
IC	PWorst	H	N, IvBo	C3, N	N
IC	PBest	H	H	C3, N	H
IC	Alphabet	H	H	H	N
IAC	Leximin	H	N	N	N
IAC	Leximax	H	H	N	N
IAC	PWorst	H	N	N	N
IAC	PBest	H	H	N	N
IAC	Alphabet	C3, N	N	N	N

Table 4 Minimally manipulated procedures for Type IV manipulability for 3 alternatives for several number of agents

Culture	Extensions	8	11	22	25
IC	Leximin	IvBo	Bo	Bo	Bo
IC	Leximax	IvBo	N, IvBo	N, IvBo	N, IvBo
IC	PWorst	Bo	Bo	Bo	Bo
IC	PBest	IvBo	N, IvBo	IvBo	N, IvBo
IC	Alphabet	Bo	Bo	Bo	Bo
IAC	Leximin	IvBo	IvBo	Bo	Bo
IAC	Leximax	IvBo	IvBo	IvBo	IvBo
IAC	PWorst	Bo	IvBo	Bo	Bo
IAC	PBest	IvBo	IvBo	IvBo	IvBo
IAC	Alphabet	Bo	Bo	Bo	Bo

several alternatives in the most diverse way. For 4 and 5 alternatives, this flexibility becomes even more important. For the general case of Type I manipulability, more options to influence scores give more options to manipulate, so the Borda is never one of the least manipulable rules, but for Type IV manipulability, this flexibility gives opportunity to others to react and, as we can see from Fig. 5a, b, it quickly reduces manipulability options. These results for Borda's rule are in accordance with the conclusions made in Gehrlein and Lepelley (2017). For Type I manipulability, the Nanson rule is also between the least manipulable rules.

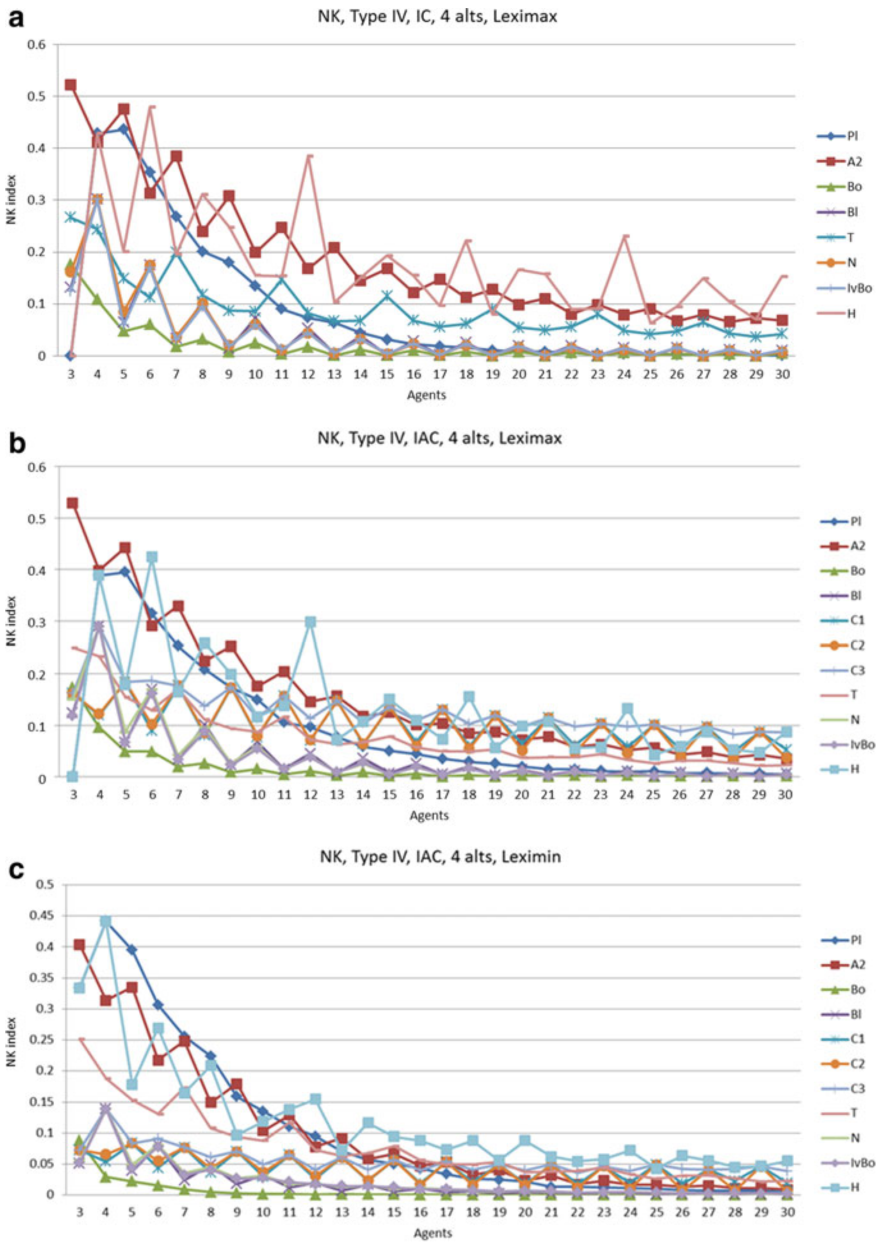


Fig. 4 **a** Manipulability of Type IV for IC, 4 alternatives, Leximax, **b** manipulability of Type IV for IAC, 4 alternatives, Leximax, **c** manipulability of Type IV for IAC, 4 alternatives, Leximin

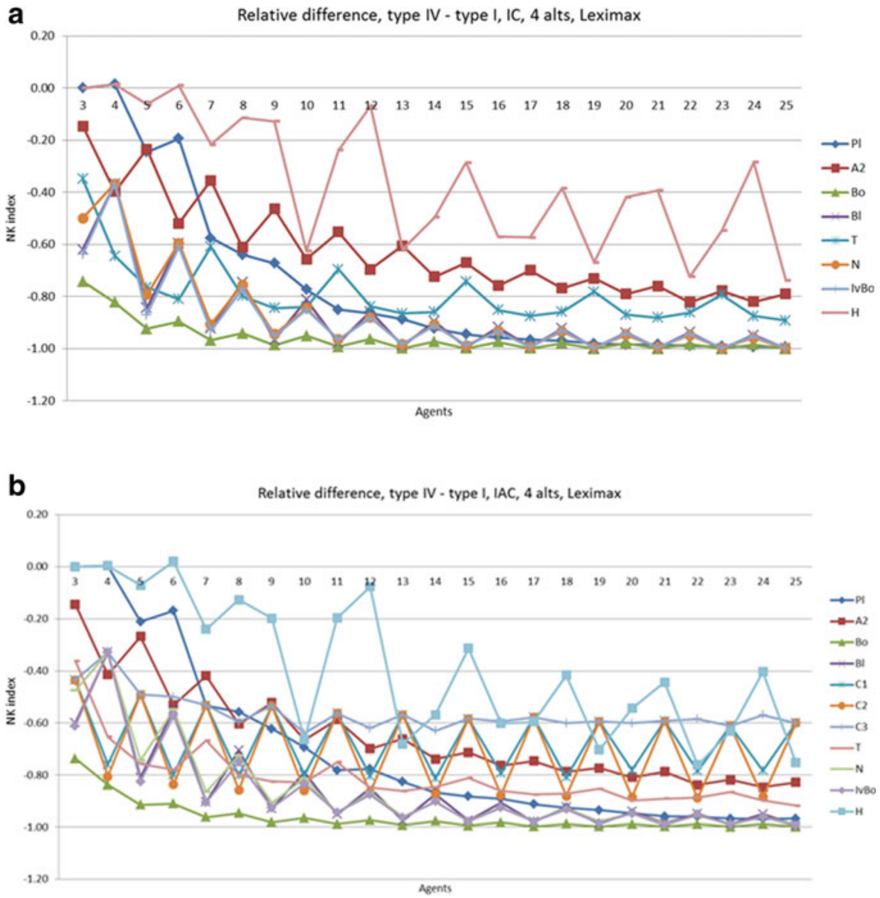


Fig. 5 **a** Difference between Type IV and Type I manipulability, IC, 4 alternatives, Leximax, **b** Difference between Type IV and Type I manipulability, IAC, 4 alternatives, Leximax

Table 5 Minimally manipulated procedures for Type I manipulability for 4 alternatives for several number of agents

Culture	Extensions	7	12	19	20
IC	Leximin	IvBo	H	IvBo	N
IC	Leximax	H	C3, S, N	H	H
IC	PWorst	IvBo	C3, N	IvBo	N
IC	PBest	H	H	H	H
IC	Alphabet	H	H	N, IvBo	IvBo
IAC	Leximin	IvBo	N, H	IvBo	N
IAC	Leximax	H	N	N	H
IAC	PWorst	IvBo	N	IvBo	N
IAC	PBest	H	H	N	H
IAC	Alphabet	N	N	N	N

Table 6 Minimally manipulated procedures for Type IV manipulability for 4 alternatives and several number of agents

Culture	Extensions	3	4–25
IC	Leximin	Bl	Bo
IC	Leximax	Pl, H	Bo
IC	Alphabet	Bl	Bo
IAC	Leximin	IvBo	Bo
IAC	Leximax	Pl, H	Bo
IAC	Alphabet	IvBo	Bo

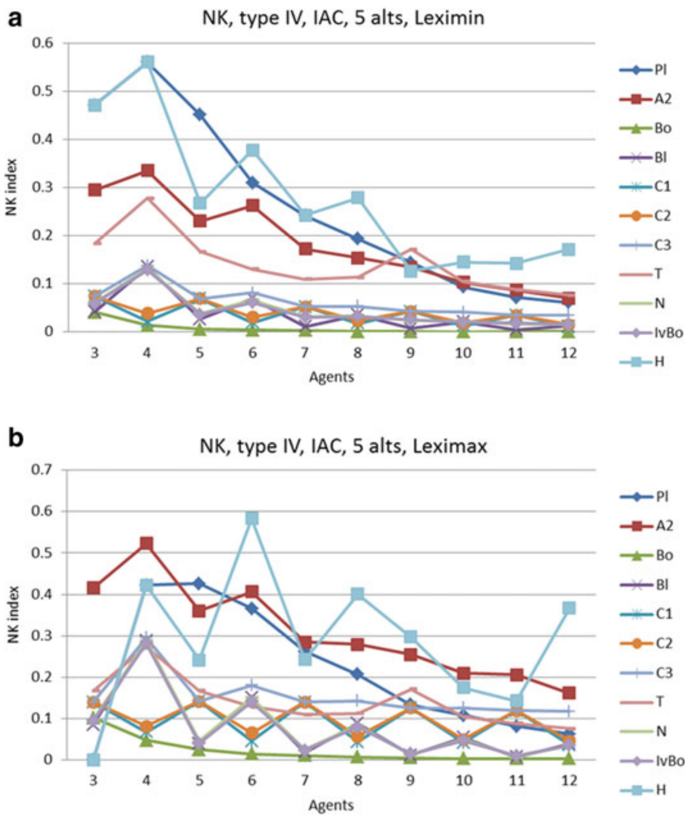


Fig. 6 **a** Manipulability of Type IV for IAC, 5 alternatives, Leximin, **b** manipulability of Type IV for IAC, 5 alternatives, Leximax

Table 7 Minimally manipulated procedures for Type I manipulability for 5 alternatives for several number of agents

Culture	Extensions	7	12	19	20
IC	Leximin	N	C3, N	N, IvBo	N
IC	Leximax	H	Bl, N	H	N
IC	PWorst	N	C3, N	N, IvBo	N
IC	PBest	H	H	H	H
IC	Alphabet	H	H	IvBo	IvBo
IAC	Leximin	N	N	N, IvBo	N
IAC	Leximax	H	Bl, N	H	N
IAC	PWorst	N	N	N, IvBo	N
IAC	PBest	H	H	H	H
IAC	Alphabet	H	N	N	N

Table 8 Minimally manipulated procedures for Type IV manipulability for alternatives for several number of agents

Culture	Extensions	3	4–6
IC	Leximin	Bo	Bo
IC	Leximax	Pl, H	Bo
IC	Alphabet	Bo	Bo
IAC	Leximin	Bo	Bo
IAC	Leximax	Pl, H	Bo
IAC	Alphabet	Bo	Bo

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Exploring the No-Show Paradox for Condorcet Extensions

Felix Brandt, Johannes Hofbauer, and Martin Strobel

Abstract An important and surprising phenomenon in voting theory is the *No-Show Paradox (NSP)*, which occurs if a voter is better off by abstaining from an election. While it is known that certain voting rules suffer from this paradox in principle, the extent to which it is of practical concern is not well understood. We aim at filling this gap by analyzing the likelihood of the NSP for six Condorcet extensions (Black's rule, Baldwin's rule, Nanson's rule, Max-Min, Tideman's rule, and Copeland's rule) under various preference models using Ehrhart theory as well as extensive computer simulations. We find that, for few alternatives, the probability of the NSP is rather small (less than 4% for four alternatives and all considered preference models, except for Copeland's rule). As the number of alternatives increases, the NSP becomes much more likely and which rule is most susceptible to abstention strongly depends on the underlying distribution of preferences.

1 Introduction

Voting theory has shown that every voting rule can result in outcomes that seem undesirable. An important research question is how often these phenomena—known as voting paradoxes—occur and how relevant they are for real-world elections. In this chapter, we employ sophisticated analytical and experimental methods to assess the frequency of the No-Show Paradox (NSP), which occurs if a voter is better off

F. Brandt (✉) · J. Hofbauer
Institut für Informatik, Technische Universität München, München 85748, Germany
e-mail: brandtf@in.tum.de

J. Hofbauer
e-mail: hofbauej@in.tum.de

M. Strobel
School of Computing, National University of Singapore, Singapore 117417, Singapore
e-mail: mstrobel@comp.nus.edu.sg

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by abstaining from an election (Fishburn and Brams 1983). The question we address goes back to Fishburn and Brams (1983), who write that “although probabilities of paradoxes have been estimated in other settings, we know of no attempts to assess the likelihoods of the paradoxes of preferential voting discussed above, and would propose this as an interesting possibility for investigation. Is it indeed true that serious flaws in preferential voting such as the No-Show Paradox [...] are sufficiently rare as to cause no practical concern?” It is well-known that all Condorcet extensions, a large class of attractive voting rules, suffer from the NSP and this is often used as an argument against Condorcet extensions. Our analysis covers six Condorcet extensions: Black’s rule, Baldwin’s rule, Nanson’s rule, MaxiMin, Tideman’s rule, and Copeland’s rule.

In principle, quantitative results on voting paradoxes can be obtained via three different approaches. The analytical approach uses theoretical models to quantify paradoxes based on certain assumptions about the voters’ preferences such as the *impartial anonymous culture (IAC)* model, in which every anonymous preference profile is equally likely. Analytical results usually tend to be quite hard to obtain and are limited to simple—and often unrealistic—assumptions. The experimental approach uses computer simulations based on underlying stochastic models of how the preference profiles are distributed. Experimental results have less general validity than analytical results, but can be obtained for arbitrary distributions of preferences. Finally, the empirical approach is based on evaluating real-world data to analyze how frequently paradoxes actually occur or how frequently they would have occurred if certain voting rules had been used for the given preferences. Unfortunately, only very limited real-world data for elections is available.

We analytically study the NSP under the assumption of IAC via *Ehrhart theory*, which goes back to the French mathematician Eugène Ehrhart (Ehrhart 1962). The idea of Ehrhart theory is to model the space of all preference profiles as a discrete simplex and then count the number of integer points inside of the polytope defined by the paradox in question. The number of these integer points can be described by so-called quasi- or Ehrhart-polynomials, which can be computed with the help of computers. The computation of the quasi-polynomials that arise in our context is computationally very demanding, because the dimension of the polytopes grows super-exponentially in the number of alternatives and was only made possible by recent advances of the computer algebra system NORMALIZ (Bruns et al. 2019a). We complement these results by very elaborate simulations using four common preference models in addition to IAC (IC, urn, spatial, and Mallows). In contrast to existing results, our analysis goes well beyond three alternatives.

2 Related Work

The NSP was first observed by Fishburn and Brams (1983) for a voting rule called *single-transferrable vote (STV)*. Moulin (1988) later proved that all Condorcet extensions are prone to the NSP; the corresponding bound on the number of voters was recently tightened by Brandl et al. (2017). Similar results were obtained for weak preferences and stronger versions of the paradox (Pérez 2001; Duddy 2014). The NSP was also transferred to other settings including set-valued voting rules (see, e.g., Jimeno et al. (2009); Pérez et al. (2010, 2015); Brandl et al. (2019a)), probabilistic voting rules (see, e.g., Brandl et al. (2015, 2019b)) and random assignment rules (Brandl et al. 2017).[1]

The frequency of the NSP was first studied by Ray (1986), who, in line with Fishburn and Brams's classic paper, analyzed situations where STV can be manipulated in elections with three alternatives. A similar goal was pursued by Lepelley and Merlin (2000) who quantified occurrences of the NSP assuming preferences are distributed according to IC or IAC. However, in contrast to the present approach, Lepelley and Merlin employed different statistical techniques to estimate the likelihood of multiple variants of the paradox and focused on score-based runoff rules in elections with three alternatives. In a recent paper, this setting was revisited by Kamwa et al. (2018) who focused on *single-peaked* preferences, where alternatives can be ordered on a one-dimensional axis and voters' preferences are determined by proximity to their optimal point on this axis. Under this assumption, they found that multiple scoring runoff rules do not suffer from any variant of the NSP while for others, e.g., plurality runoff, the probabilities of a paradox to occur are significantly lower compared to the unrestricted domain.

The general idea to quantify voting paradoxes via IAC has been around since the formal introduction of this preference model by Gehrlein and Fishburn (1976) (see, e.g., Lepelley et al. (1996); Le Breton et al. (2016); Lepelley et al. (2018)). Still, it took a good 30 years until the connection to Ehrhart theory (Ehrhart 1962) was established by Lepelley et al. (2008). We refer to Gehrlein and Lepelley (2011, 2017) for a more profound explanation of all details and an overview of results subsequently achieved (see also, e.g., Wilson and Pritchard (2007); Schürmann (2013); Le Breton et al. (2016)). The step from three to four alternatives, i.e., from six to 24 dimensions, was only made possible through recent advances in computer algebra systems by De Loera et al. (2012) and Bruns and Söger (2015). Brandl et al. (2016b) used a framework similar to ours to study the frequency of two single-profile paradoxes (the Condorcet Loser Paradox and the Agenda Contraction Paradox). In a recent paper, Bruns et al. (2019b) also made use of the possibility to analyze situations with four alternatives and looked at the Condorcet efficiency of plurality and plurality with runoff as well as the structure of majority graphs and varying Borda paradoxes.

Plassmann and Tideman (2014) conducted computer simulations for various voting rules and paradoxes under a modified spatial model in the three-alternative case. To the best of our knowledge, this is—apart from Brandl et al. (2016b) and Bruns et al. (2019b)—the only study of Condorcet extensions from a quantitative angle.

[1] Interestingly, when considering set-valued or probabilistic voting rules, there are Condorcet extensions immune to the NSP under suitable assumptions (Brandl et al. 2019a, b).

3 Preliminaries

Let A be a set of m alternatives and $N = \{1, \ldots, n\}$ a set of voters. We assume that every agent $i \in N$ is endowed with a preference relation $\succ_i$ over the alternatives A. More formally, $\succ_i$ is a complete, asymmetric and transitive binary relation, $\succ_i \in A \times A$, which gives a strict ranking over the alternatives. If $x \succ_i y$, we say that i prefers x to y.

A *preference profile* $\succ$ is a tuple consisting of one preference relation per voter, i.e., $\succ = (\succ_1, \ldots, \succ_n)$. By $\succ_{-i}$ we denote the preference profile resulting of voter i abstaining the election, $\succ_{-i} = (\succ_1, \ldots, \succ_{i-1}, \succ_{i+1}, \ldots, \succ_n)$. For most purposes, however, the ordering within the tuple of preference relations is irrelevant and one can alternatively consider multisets of preference relations, so-called *anonymous preference profiles*.

For two alternatives $x, y \in A$ and a preference profile $\succ$ we define the *majority margin* $g_{xy}(\succ)$ as

$$g_{xy}(\succ) = |\{i \in N : x \succ_i y\}| - |\{i \in N : y \succ_i x\}|.$$

Whenever $\succ$ is clear from the context we only write g_{xy}. A *voting rule* is a function f mapping a preference profile $\succ$ to a single alternative, $f(\succ) \in A$.

3.1 Condorcet Extensions

Alternative $x \in A$ is a *Condorcet winner* if it beats all other alternatives in pairwise majority comparisons, i.e., $g_{xy} > 0$ for all $y \in A \setminus \{x\}$. Similarly, x is a *weak Condorcet winner* if it beats or ties all other alternatives, i.e., $g_{xy} \geq 0$ for all $y \in A$. If a voting rule always selects the Condorcet winner whenever one exists, it is called a *Condorcet extension*. A *weak Condorcet extension* returns a weak Condorcet winner whenever (at least) one exists. Clearly, every weak Condorcet extension is a Condorcet extension. A wide variety of Condorcet extensions has been studied in the literature (see, e.g., Fishburn (1977); Brandl et al. (2016a)). In this chapter, we consider six Condorcet extensions: Black's rule, Baldwin's rule, Nanson's rule, MaxiMin, Tideman's rule, and Copeland's rule. The main criteria for selecting these rules were *discriminability* (in order to minimize the influence of lexicographic tie-breaking), *simplicity* (to allow for Ehrhart analysis and because voters generally prefer 'simpler' rules), and *efficient computability* (to enable rigorous and comprehensive simulations).[2] In the following, we briefly define the rules.

Black's rule (Black 1958) selects the Condorcet winner whenever one exists and otherwise returns a winner according to Borda's rule, where each voter assigns $m - 1$ points to his most preferred alternatives, $m - 2$ points to his second most preferred

[2]Note that other discriminating Condorcet extensions such as Kemeny's rule, Dodgson's rule, and Young's rule are NP-hard to compute see, e.g., Brandl et al. (2016a).

alternative, etc., and an alternative with highest accumulated score wins (Borda's rule itself is no Condorcet extension). For the formal definition below, we use affinely equivalent Borda scores based on majority margins.

$$f_{\text{Black}}(\succ) \in \begin{cases} x & \text{if } x \text{ is a Condorcet winner in } \succ \\ \arg\max_{x \in A} \sum_{y \in A} g_{xy} & \text{otherwise.} \end{cases}$$

Baldwin's rule (Baldwin 1926) proceeds in multiple rounds. In each round, we drop all alternatives with the lowest Borda score and then continue with the reduced preference profile, which is used to calculate updated scores. If multiple—but not all—alternatives are tied last, we delete all of them. Baldwin's rule chooses one of the alternatives that remains when no more alternative can be removed.

Nanson's rule (Nanson 1883; Niou 1987) is similar to Baldwin's rule in so far as it also focuses on the Borda scores and gradually eliminates alternatives. However, in contrast to before, we now remove all alternatives with average or below-average Borda score in every round. Nanson's rule returns an alternative out of those that remain when all alternatives have identical score.

The *MaxiMin* rule (Black 1958), which is also known as the Simpson-Kramer method (Simpson 1969; Kramer 1977), looks at the worst pairwise majority comparison for each alternative. It then returns an alternative with maximal such score, formally

$$f_{\text{MaxiMin}}(\succ) \in \arg\max_{x \in A} \min_{y \in A \setminus \{x\}} g_{xy}.$$

Tideman's rule (Tideman 1987) focuses on the sum of all pairwise majority defeats. It yields an alternative where this sum is closest to zero in terms of absolute value,[3] i.e.,

$$f_{\text{Tideman}}(\succ) \in \arg\max_{x \in A} \sum_{y \in A} \min(0, g_{xy}).$$

Copeland's rule (Copeland 1951) only relies on the signs of the majority margins. It chooses an alternative where the number of majority wins plus half the number of majority draws is maximal:

$$f_{\text{Copeland}}(\succ) \in \arg\max_{x \in A} |\{y \in A : g_{xy} > 0\}| + 1/2\,|\{y \in A : g_{xy} = 0\}|$$

In order to obtain well-defined voting rules we employ alphabetic tie-breaking for all rules defined above. Note that the actual tie-breaking ordering does not influence our results as long as this ordering is fixed. This is not the case if we would allow for tie-breaking based on the preference profile or the choice set. All presented voting rules can be computed in polynomial time and do not rely on the exact preference

[3]Tideman's rule is arguably the least well-known voting rule presented here. It was proposed to efficiently approximate Dodgson's rule and is not to be confused with *ranked pairs* which is sometimes also called Tideman's rule. Also note that the 'dual' rule returning alternatives for which the sum of weighted pairwise majority *wins* is maximal is not a Condorcet extension.

profile $\succ$ but only on the majority margins that can conveniently be represented by a skew-symmetric matrix or a weighted directed graph.

In order to illustrate these definitions, consider an example with seven voters and four alternatives given by the preference profile and the matrix of pairwise majority margins below. The preference profile is given as a table where a column with header k represents a group of k voters with preferences given in decreasing order.

3	3	1
a	d	b
c	c	d
b	b	a
d	a	c

$$(g_{xy})_{x,y\in A} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & -5 & 1 \\ -1 & 5 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix} \end{array}$$

In the absence of a Condorcet winner, Black's rule relies on the Borda scores which can be computed to be $s(\succ) = (10, 9, 12, 11)$ or, affinely equivalent, $(-1, -3, 3, 1)$ when determining them based on the majority margins only. Hence, $f_{\text{Black}}(\succ) = c$.

Having the lowest Borda score, b consequently is the first alternative to be eliminated when applying Baldwin's rule. After dropping c next, we have a strict majority in favor of d against a and thus $f_{\text{Baldwin}}(\succ) = d$.

In the first round of Nanson's rule, we eliminate a and b since both alternatives have a Borda score which is below average. Thereafter, we obtain a strict majority for d against c, meaning d has higher Borda score and it follows $f_{\text{Nanson}}(\succ) = d$.

For MaxiMin, we analyze all alternatives' worst pairwise majority comparison and see that a, c, and d are tied with -1. Due to alphabetic tie-breaking we have $f_{\text{MaxiMin}}(\succ) = a$.

Tideman's rule counts the sum of all pairwise majority defeats, which we find to be 2, 5, 2, and 1 for a, b, c, and d, respectively. The alternative with minimal sum is chosen, hence, $f_{\text{Tideman}}(\succ) = d$.

Lastly, Copeland's rule selects an alternative based on the number of pairwise majority wins and here breaks the tie between b and d alphabetically leading to $f_{\text{Copeland}}(\succ) = b$.

3.2 Strategic Abstention

A voting rule f is *manipulable by strategic abstention* if there exist some N, A, and $\succ$ such that for some $i \in N, f(\succ_{-i}) \succ_i f(\succ)$. Given an occurrence of manipulability by strategic abstention, f is said to suffer from the *No-Show Paradox* (NSP) (for N, A, $\succ$). Slightly abusing notation, we also say that $\succ$ is prone to the NSP whenever f, N, and A are clear from the context. All rules defined here are Condorcet extensions and therefore manipulable by strategic abstention. Occurrences of the NSP for Black's, Baldwin's, and Copeland's rule require three alternatives while four alternatives are needed for MaxiMin as well as Nanson's and Tideman's rule.

It is interesting to note that whenever a Condorcet winner exists, no *weak Condorcet extension* allows for manipulation by strategic abstention by a single voter. To see this, assume alternative x is the Condorcet winner, i.e., x wins in a pairwise majority comparison against all other alternatives. While some of these strict majority preferences might turn to indifferences if voter i abstains from the election procedure, this can only happen for comparisons to alternatives less preferred than x according to $\succ_i$. Hence, every alternative strictly more preferred than x still loses at least the pairwise majority comparison against x, which remains a weak Condorcet winner. We deduce that irrespective of other possible weak Condorcet winners and the underlying tie-breaking, no alternative preferred to x can be chosen. Of the rules defined above, MaxiMin and Tideman's rule are weak Condorcet extensions.[4]

3.3 Stochastic Preference Models

When analyzing properties of voting rules, it is a common approach to sample preferences according to some underlying model. Various concepts to model preferences have been introduced over the years; we refer to Critchlow et al. (1991) and Marden (1995) for a detailed discussion. We focus on three parameter-free models, *impartial culture (IC)* where each voter's preferences are drawn uniformly at random, *impartial anonymous culture (IAC)* where anonymous preference profiles are drawn uniformly at random, and the two-dimensional *spatial model* where we uniformly sample points in the unit square and their proximity determines the voters' preferences. Furthermore, we consider two model families that allow to simulate different degrees of voter correlation. These are the *urn model* (Berg 1985) with parameter 10 (i.e., whenever a preference relation is drawn, 10 copies of the same relation are added to the urn) and *Mallows' model* (Mallows 1957) with $\phi = 0.8$. These parameters induce stronger voter correlation than IC and IAC, which is widely considered to be more realistic.

4 Quantifying the No-Show Paradox

The goal in this chapter is to quantify the frequency of the NSP, i.e., to investigate for how many preference profiles a voter is incentivized to abstain from an election. In order to achieve this goal, we employ an exact analysis via Ehrhart Theory and experimental analysis via sampled preference profiles.

---

[4]For both MaxiMin and Tideman's rule, this holds by the observation that a weak Condorcet winner does not lose any pairwise majority comparison. Black's rule fails to be a weak Condorcet extension by definition; a counterexample for Baldwin's, Nanson's, and Copeland's rule is given by Fishburn (1977).

4.1 Exact Analysis via Ehrhart Theory

The imminent strength of exact analysis is that it gives reliable theoretical results. On the downside, precise computation is only feasible for very simple preference models and for small values of m. We focus on IAC and make use of Ehrhart theory.

First, note that an anonymous preference profile is completely specified by the number of voters sharing each of the $m!$ possible rankings on m alternatives. Hence, we can uniquely represent an anonymous profile by an integer point x in a space of $m!$ dimensions. We interpret x_i as the number of voters who share ranking i, where rankings are ordered lexicographically. For example, when there are three voters, an anonymous profile is of the following type.

x_1	x_2	x_3	x_4	x_5	x_6
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

For fixed m, our goal is to describe all profiles that are prone to the NSP by using linear (in)equalities that describe a polytope P_n.[5] Given that this is possible, the fraction of profiles prone to the NSP can be computed by dividing the number of integer points contained in P_n by the total number of profiles for n voters, i.e., the number of integer points x satisfying $x_i \geq 0$ for all $1 \leq i \leq m!$ and $\sum_{1 \leq i \leq m!} x_i = n$.

While the latter number is known to be $\binom{m!+n-1}{m!-1}$, the former can be determined using Ehrhart theory. Ehrhart (1962) shows that it can be found by so-called *Ehrhart-* or *quasi-polynomials* f—a collection of q polynomials f_i of degree d such that $f(n) = f_i(n)$ if $n \equiv i \mod q$. Obtaining f is possible via computer programs like LATTE (De Loera et al. 2004) or NORMALIZ (Bruns et al. 2019a).

4.1.1 Copeland's Rule

In order to illustrate this method, first consider Copeland's rule in elections with three alternatives under IAC. For the modeling we need to give linear constraints in terms of voter types—or equivalently majority margins—that describe polytopes containing all profiles prone to the NSP.

We first distinguish between the six possible manipulations from x to y, $x \neq y \in A = \{a, b, c\}$. A case-by-case analysis shows that, due to alphabetic tie-breaking, only manipulations from a to b or c and from either b or c to a are possible. In particular, for each of these cases, there is exactly one voting situation admitting an occurrence of the NSP. We find that we can specify the respective profiles using one polytope each:

[5] More precisely, P_n is a *dilated polytope* depending on n, $P_n = nP = \{n\mathbf{x} \colon \mathbf{x} \in P\}$.

$$
\begin{array}{llll}
g_{ba} \geq 2, & g_{ac} \geq 1, & g_{cb} = 1, & x_6 \geq 1 \quad (\mathrm{P}_1) \\
g_{ca} \geq 2, & g_{ab} \geq 1, & g_{bc} = 1, & x_4 \geq 1 \quad (\mathrm{P}_2) \\
g_{ac} \geq 2, & g_{ba} \geq 1, & g_{bc} = 0, & x_1 \geq 1 \quad (\mathrm{P}_3) \\
g_{ab} \geq 1, & g_{ca} \geq 1, & g_{bc} = 0, & x_2 \geq 1 \quad (\mathrm{P}_4)
\end{array}
$$

For the sake of readability we here omit but implicitly assume that the total number of voters present is n and there is a nonnegative number of voters per voter type. Polytope P_1, for example, describes a manipulation from a to b. The first three (in)equalities ensure that there is no Condorcet winner in the current profile and that a wins by tie-breaking. The first inequality ensures that b beats a even after the manipulator leaves, while the equality in the third column ensures that b will become a Condorcet winner. The last column demands the presence of a voter of type x_6, the only type able to manipulate. Note that P_1 and P_2 require n to be odd while every profile contained in P_3 or P_4 contains an even number of voters. The total number of anonymous preference profiles admitting a manipulation by abstention is given by the number of integer points contained in polytopes P_1 to P_4.

4.1.2 Black's Rule

When considering different rules or a larger number of alternatives, we find that the number of polytopes as well as the number of linear constraints defining them grows rapidly. Black's rule, for instance, can only be manipulated from a Condorcet winner to a Borda winner or *vice versa*. This distinction is also one of voter parity: a manipulation away from a Condorcet winner is possible for odd n, while n is required to be even in the converse case. In contrast to before, Black's rule allows for a manipulation between any pair of alternatives regardless of n. Hence, we obtain a total of 12 polytopes, one for every possible manipulation and parity of n. The polytopes for even n look as follows.

$$
\begin{array}{lll}
g_{ab} + g_{ac} \geq g_{ba} + g_{bc}, & g_{ba} \geq 1, & x_6 \geq 1, \\
g_{ab} + g_{ac} \geq g_{ca} + g_{cb}, & g_{bc} = 0 \quad (\mathrm{P}_1) &
\end{array}
$$

$$
\begin{array}{lll}
g_{ab} + g_{ac} \geq g_{ba} + g_{bc}, & g_{ca} \geq 1, & x_4 \geq 1, \\
g_{ab} + g_{ac} \geq g_{ca} + g_{cb}, & g_{cb} = 0 \quad (\mathrm{P}_2) &
\end{array}
$$

$$
\begin{array}{lll}
g_{ba} + g_{bc} \geq g_{ab} + g_{ac} + 1, & g_{ab} \geq 1, & x_5 \geq 1, \\
g_{ba} + g_{bc} \geq g_{ca} + g_{cb}, & g_{ac} = 0 \quad (\mathrm{P}_3) &
\end{array}
$$

$$g_{ba} + g_{bc} \geq g_{ab} + g_{ac} + 1, \qquad g_{cb} \geq 1, \qquad x_2 \geq 1,$$
$$g_{ba} + g_{bc} \geq g_{ca} + g_{cb}, \qquad g_{ca} = 0 \quad (\mathrm{P}_4)$$

$$g_{ca} + g_{cb} \geq g_{ab} + g_{ac} + 1, \qquad g_{ac} \geq 1, \qquad x_3 \geq 1,$$
$$g_{ca} + g_{cb} \geq g_{ba} + g_{bc} + 1, \qquad g_{ab} = 0 \quad (\mathrm{P}_5)$$

$$g_{ca} + g_{cb} \geq g_{ab} + g_{ac} + 1, \qquad g_{bc} \geq 1, \qquad x_1 \geq 1,$$
$$g_{ca} + g_{cb} \geq g_{ba} + g_{bc} + 1, \qquad g_{ba} = 0 \quad (\mathrm{P}_6)$$

Polytope P_1, for example, describes a manipulation from a to b (when n is even). The inequalities in the left column model that a currently is the Borda winner. The (in)equalities in the second column guarantee that a manipulator can make b Condorcet winner by abstaining as well as that with him being present, there is no Condorcet winner. The last column demands the presence of a voter of type x_6, the only type able to manipulate.

4.1.3 MaxiMin

When moving to MaxiMin and four alternatives, determining the necessary polytopes becomes tedious. Since alphabetic tie-breaking rules out most symmetries, we need 168 disjoint polytopes of varying sizes to encompass all profiles prone to the NSP. Each of these is defined by 8–10 constraints, not counting the total number of voters and nonnegative number per type.

Recall the definition of MaxiMin from Sect. 3 and assume $f_{\text{MaxiMin}} = x$. For the NSP to occur, two intrinsic conditions have to be satisfied: (i) There is a voter i such that $f_{\text{MaxiMin}}(\succ_{-i}) = y \neq x$ and (ii) for voter i, we have $y \succ_i x$. We find that for $A = \{a, b, c, d\}$, conditions (i) and (ii) entail that manipulation from a to b is only possible for $\succ_i$: c, b, a, d and $\succ_j$: d, b, a, c. It can be shown that no instance exists in which both voter types can influence the outcome in their favor. For the sake of this example, let us focus on $\succ_i$.

A first analysis shows that a's highest defeat has to be against d while b's highest defeat necessarily is against c with $g_{ad} = g_{bc}$,[6] and any other defeat of b lower by at least two. This gives rise to a first set of essential constraints.[7]

$$g_{ad} = g_{bc}, \qquad g_{ad} \leq 0,$$
$$g_{ab} \geq g_{ad}, \qquad g_{ba} \geq g_{ad} + 2 \quad (\text{basis})$$
$$x_i \geq 1$$

[6]Theoretically, we only require $g_{ad} - 1 \leq g_{bc} \leq g_{ad}$. As either all g_{xy} are even or all g_{xy} are odd, this collapses to $g_{ad} = g_{bc}$.

[7]Some inequalities are omitted to remove redundancies when taken together with later constraints.

At this point, we distinguish between $g_{cd} = 0$, $g_{cd} \leq -1$, and $g_{cd} \geq 1$. In case $g_{cd} = 0$, we trivially only need bounds on the defeats of c against a and d against b:

$$g_{cd} = 0, \qquad g_{ca} \leq g_{ad}, \qquad g_{db} \leq g_{ad} \quad \text{(A)}$$

If $g_{cd} \leq -1$, c's highest defeat could be against a, d, or both. We consequently need a case distinction to accommodate for these possibilities.

$$\begin{aligned}
&g_{cd} \leq -1, && g_{db} \leq g_{ab} \quad \text{(B)} && \\
&g_{cd} \leq g_{ad}, && g_{ca} \leq g_{ad} \quad \text{(B.1)} && \\
&g_{cd} \leq g_{ad}, && g_{ca} \geq g_{ad} + 1, && g_{ac} \geq g_{ad} \quad \text{(B.2)} \\
&g_{cd} \geq g_{ad} + 1, && g_{ca} \leq g_{ad} \quad \text{(B.3)} &&
\end{aligned}$$

For $g_{cd} \geq 1$ and an almost symmetric reasoning with reversed arguments for c and d we obtain (C), (C.1), (C.2), and (C.3).

Finally, the total set of profiles admitting a manipulation from a to b by i can be described by seven polytopes making use of the constraints developed above. We obtain

- $P_1 =$ (basis) + (A),
- $P_2 =$ (basis) + (B) + (B.1), $P_3 =$ (basis) + (B) + (B.2), $P_4 =$ (basis) + (B) + (B.3),
- $P_5 =$ (basis) + (C) + (C.1), $P_6 =$ (basis) + (C) + (C.2), and $P_7 =$ (basis) + (C) + (C.3).[8]

As we are interested in not only voter type $\succ_i$ but also $\succ_j$ and equivalently not only manipulations from a to b but also all different combinations, we need to undergo a similar reasoning 24 times. This amounts to a total of 168 disjoint polytopes to encompass all profiles prone to the NSP. We remark that even though manipulation instances are roughly in line for all 24 types of voters, there are no exact symmetries that allow for reducing the number of polytopes. This is due mostly to lexicographic—i.e., non-symmetric—tie-breaking and the required presence of a certain voter type in the electorate. Both effects diminish as n grows but discrepancies between different types of manipulators are significant up to lower three-digit n.

This approach of modeling profiles prone to the NSP is substantially more involved than using Ehrhart theory for single-profile paradoxes such as the Condorcet Loser Paradox because of three reasons.

(i) An occurrence of the NSP requires the presence of a certain type of voter.
(ii) Preference profiles for which different types of voters are able to manipulate must be counted only once.[9]

---

[8]We choose this informal notation for the sake of readability. It is to be understood in a way that P_1 is the polytope described by (in)equalities labelled (basis) as well as (A). We additionally assume for all polytopes that the sum of voters per type adds up to n and each type consists of a nonnegative number of voters.

[9]This effect is only relevant when there are at least four alternatives.

(iii) Possible manipulations not only rely on the winning alternative itself but on all majority margins that have to adhere to different constraints.

4.2 Experimental Analysis

In contrast to exact analysis, the experimental approach relies on simulations to grasp the development of different phenomena under varying conditions. On the upside, this usually allows for results for more complex problems or a larger scale of parameters, both of which might be prohibitive for exact calculations. At the same time, however, we find that we need a huge number of simulations per setting to get sound estimates which in turn often requires a high-performance computer and a lot of time. Also, there remains the risk that even a vast amount of simulations fails to capture one specific, possibly crucial, effect.

Regarding the pivotal question of our chapter, the frequency of the NSP for various voting rules, we sample preference profiles for different combinations of n and m using the modeling assumptions explained in Sect. 3. Our simulations were conducted on XeonE5-2697 v3 multi-core processors with 2 GB memory per job. The total runtime easily accumulates to thirty years on a single-core processor.

5 Results and Discussion

In this section we present our results obtained by both exact analysis and computer simulations.

5.1 Analytical Results Under IAC

We first focus on Copeland's rule with three alternatives, as our modeling in Sect. 4.1 allows for an exact analysis of the NSP. In particular, we compute the following Ehrhart-polynomial $f(n)$ with period $q = 2$:

$$f_0(n) = {}^1\!/_{192}\, n^4 - {}^1\!/_{48}\, n^3 - {}^1\!/_{48}\, n^2 + {}^1\!/_{12}\, n$$
$$f_1(n) = {}^1\!/_{192}\, n^4 - \frac{5}{96} n^2 + \frac{3}{64}$$

Recall that $f(n) = f_i(n)$ if $n \equiv i \mod q$. Consequently, the fraction of profiles that admit a manipulation by strategic abstention is given by

$$\frac{f_0(n)}{\binom{n+5}{5}} \quad \text{if } n \text{ is even and} \qquad \frac{f_1(n)}{\binom{n+5}{5}} \quad \text{if } n \text{ is odd.}$$

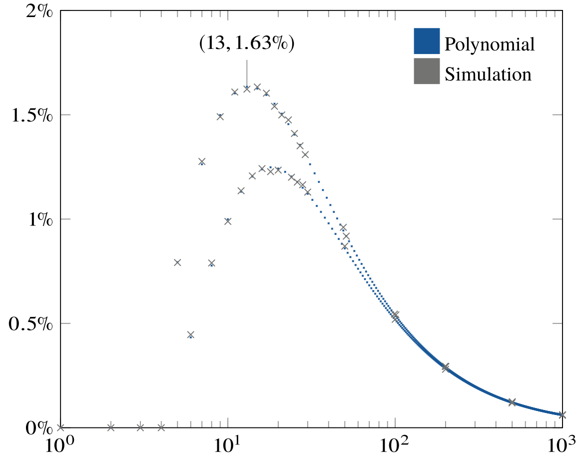

Fig. 1 Fraction of profiles prone to the NSP for Copeland's rule and $m = 3$. The alternating parity of the number of voters has a significant effect on the occurrence of the paradox and gives the appearance of two separate curves

This frequency of the NSP for Copeland's rule and $m = 3$ is plotted in Fig. 1, together with results obtained by computer simulations.

With respect to Black's rule and $m = 3$, we obtain an Ehrhart-polynomial with slightly larger period $q = 6$. Once more, we can explicitly give $f(n)$ which looks as follows:

$$
\begin{aligned}
f_0(n) &= {}^1\!/_{192}\, n^4 - {}^5\!/_{48}\, n^2\\
f_1(n) &= {}^1\!/_{192}\, n^4 - {}^1\!/_{48}\, n^3 - {}^7\!/_{96}\, n^2 + {}^3\!/_{16}\, n - {}^{19}\!/_{192}\\
f_2(n) &= {}^1\!/_{192}\, n^4 - {}^5\!/_{48}\, n^2 + {}^1\!/_{3}\\
f_3(n) &= {}^1\!/_{192}\, n^4 - {}^1\!/_{48}\, n^3 - {}^7\!/_{96}\, n^2 + {}^3\!/_{16}\, n + {}^{15}\!/_{64}\\
f_4(n) &= {}^1\!/_{192}\, n^4 - {}^5\!/_{48}\, n^2 + {}^1\!/_{3}\\
f_5(n) &= {}^1\!/_{192}\, n^4 - {}^1\!/_{48}\, n^3 - {}^7\!/_{96}\, n^2 + {}^3\!/_{16}\, n + {}^{15}\!/_{64}
\end{aligned}
$$

The fraction of profiles prone to the NSP for Black's rule and $m = 3$ is visualized in Fig. 2.

Similar connections between analytical and experimental results for MaxiMin can be observed in Fig. 3. Note that, while we are able to explicitly give the Ehrhart-polynomials for Copeland's and Black's rule and $m = 3$ here, this is not possible for MaxiMin and $m = 4$ due to space constraints. The corresponding polynomial $f(n)$ has a period of $q = 55\,440$, i.e., it consists of 55 440 different polynomials. We deduce that no two points in the MaxiMin chart of Fig. 3 are computed via the same polynomial, which makes the regularity of the curve even more remarkable.

A couple of points come to mind when closely studying these graphs. First, we note that the results obtained by simulation almost perfectly match the exact calculations, which can be seen as strong evidence for the correctness of both. On the one hand,

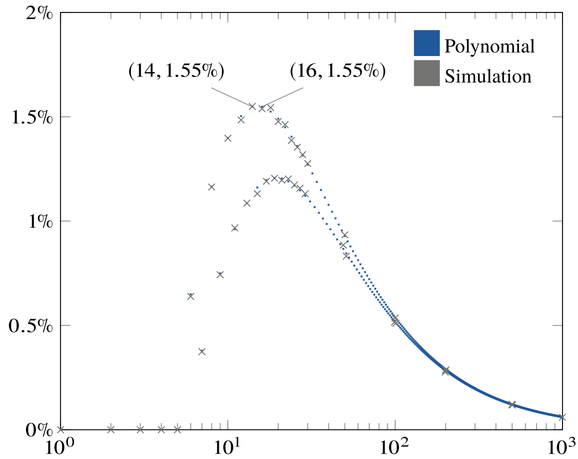

Fig. 2 Fraction of profiles prone to the NSP for Black's rule and $m = 3$

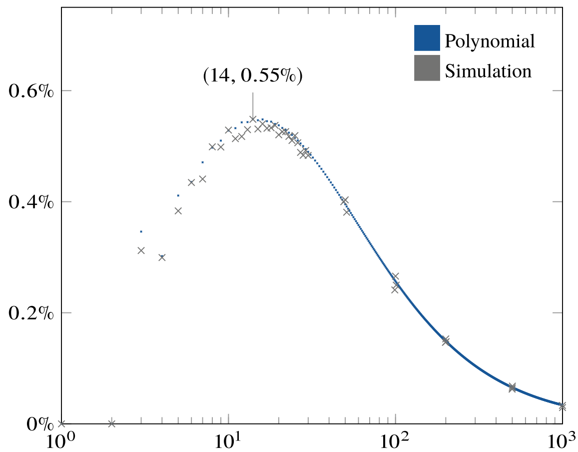

Fig. 3 Fraction of profiles prone to the NSP for MaxiMin and $m = 4$

it confirms our modeling via polytopes, and at the same time highlights that we are running a sufficiently large number of simulations. While this does not bear definite testimony to the correctness for larger m, we highlight that our implementation is both generic (with respect to m and n) and not particularly complex, which minimizes the risk of errors. We additionally believe that the perfect smoothness of Fig. 4 together with the fact that the NSP is independent of n, m, and the underlying voting rule strongly suggests that our experimental results are sound and reliable.

We see that for Black's rule the maximum is attained at 14 and 16 voters with 1.55% of all profiles, for Copeland's rule the maximum is at 13 voters with 1.63% of all profiles, while for MaxiMin and $m = 4$ it is at 14 voters with 0.55% of all profiles. Hence, we can argue that for elections with very few alternatives, the NSP seems to hardly cause a problem, independent of the number of voters or the voting rule considered. Strikingly, the maxima occur at roughly the same number of voters,

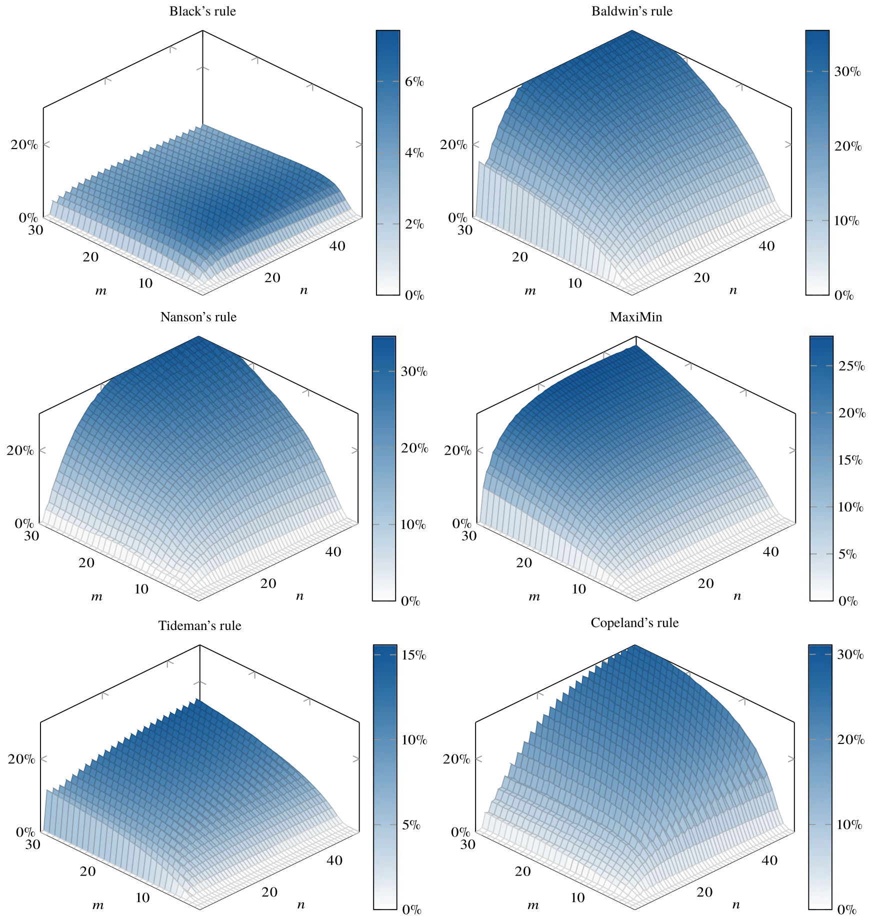

Fig. 4 Fraction of profiles prone to the NSP for different rules and increasing *n* and *m*

with this number varying between being even or odd. Also observe that Black's and Copeland's rule are more sensitive to the parity of *n* than MaxiMin.

Furthermore, we note that the probability for the NSP to occur converges to zero as *n* goes to ∞; this holds true for all voting rules considered and all fixed *m*. Intuitively, this is to be expected as for larger electorates, a single voter's power to sway the result diminishes. This first idea can be confirmed by considering the respective modeling via polytopes. Each modeling will contain at least one equality constraint, e.g., in the third column of our modeling of Copeland's rule in Sect. 4.1. Consequently, the polytopes describing profiles for which a manipulation is possible are of dimension at most $m! - 1$. By Ehrhart (1962), this means that the number of those profiles can be described by a polynomial of *n* of degree at most $m! - 1$. The total number of profiles, on the other hand, can equivalently be determined via a polynomial of degree $m!$. Hence, the fraction of profiles prone to the NSP is upper-bounded by $O(1/n)$. Following the intuitive argument, similar behavior is to be expected for all reasonable preference models and voting rules.

For $m = 4$, determining the Ehrhart polynomials for both Black's as well as Tideman's rule proved to be infeasible, even when using a custom-tailored version of NORMALIZ and employing a high-performance cluster.[10] Copeland's rule unfortunately causes problems even earlier: for four alternatives the modeling via linear (in)equalities quickly becomes very challenging due to the rule only caring about unweighted majority comparisons. For all rules, $m \geq 5$ appears to be out of scope for years to come.

5.2 Experimental Results Under IAC

In this section, we rely on simulations to grasp how often the NSP can occur for different combinations of n and m up to 50 voters and 30 alternatives. Our results can be found in Fig. 4 and allow for the following observations to be made.

To begin with, the relatively low fraction of profiles prone to the NSP for Copeland's rule, Black's rule, and MaxiMin with a small number of alternatives increases as m grows. This increase is quite dramatic for Copeland's rule and MaxiMin. In particular, for only 20 alternatives and both rules, a rough quarter of all profiles admit manipulation by abstention for a medium count of voters. This number is too large to discard the NSP as merely a theoretical problem. Black's rule, on the other hand, remains stable on a comparatively moderate level. Felsenthal and Nurmi (2018) argue in favor of Nanson's rule as it is—in contrast to the related Baldwin's rule—not prone to the NSP for three alternatives. We show that this difference between the two rules becomes moot for larger numbers of alternatives: the fractions of profiles allowing for a manipulation are on a roughly identical, severely high level.[11] This shows that voting rules based on Borda scores do not necessarily fare better with respect to the NSP.

When examining Baldwin's rule in Fig. 4, the ridge at $n = 3$ immediately catches the observer's eye.

We conjecture this unique behavior of Baldwin's rule is due to preference profiles similar in structure to the one depicted below. In case voter 3 places sufficiently many alternatives over x, x is going to be eliminated on the way causing y to eventually be chosen. Then again, if voter 3 abstains, x is always going to be selected as long as it beats y in the tie-breaking order. Note that x and y can be chosen almost freely, all other alternatives placed virtually arbitrarily,

[10] For Black's rule, we find that the polynomial would be of period $q \approx 2.7 \times 10^7$ corresponding to a mid two-digit GB file size.

[11] Felsenthal and Nurmi (2018) also show that none of the two rules fares strictly better than the other. Indeed, there are profiles where a manipulation is possible according to Baldwin's rule but not using Nanson's rule and *vice versa*.

1	1	1
x	y	⋮
y		x
⋮	⋮	⋮
		y
	x	⋮

Especially when considering Black's, Tideman's, and Copeland's rule, we see that the parity of n crucially influences the results. However, the parity of n does not affect the fractions in a consistent way: higher fractions occur for Black's and Copeland's rule when n is even, in contrast to Tideman's rule where this happens when n is odd. For Black's rule, this is most probably due to the fact that there are more suitable profiles close to having a Condorcet winner ($g_{xy} = 0$) than profiles close to not having one ($g_{xy} = 1$).[12] Copeland score's are integers when the number of voters is odd and half-integers when the number of voters is even. Hence, differences between alternatives are potentially more distinct for an odd number of voters which we assume makes manipulations harder to achieve. For Tideman's rule, we currently lack a convincing explanation for the observed behavior, mostly because it is hard to intuitively grasp when exactly a preference profile is manipulable.

Regarding Baldwin's and Nanson's rule as well as MaxiMin, the parity of n seems to have little effect on the numbers. More detailed analysis shows that at least for MaxiMin this appearance is deceptive: when manipulating towards an alphabetically preferred alternative, fractions are higher for even n, while the contrary holds for manipulations towards an alphabetically less preferred alternative. In sum, these two effects approximately cancel each other out.

The flawless smoothness and regularity of all plots in Fig. 4 are due to 10^6 runs per data point. This large number allows for all 95% confidence intervals to be smaller than 0.2%. Our simulations took 35–48 single-core hours for *each* data point and there are 1 500 data points per plot.

5.3 Comparing Different Preference Models

In order to get an impression of the frequency of the NSP under different preference models, we fix the number of alternatives to be $m = 4$ or $m = 30$ and sample 10^6

[12] For Black's rule, manipulation is only possible either towards or away from a Condorcet winner since Borda's rule is immune to strategic abstention and manipulation is impossible from Condorcet winner to Condorcet winner.

profiles for increasing n up to 1 000 or 200, respectively.[13] Fig. 5 gives the fraction of profiles prone to the NSP.

A close inspection of these graphs allows for multiple conclusions. First, we see that in particular Black's rule shows a severe dependency on the parity of n. For better illustration, we depict two lines per preference model to highlight this effect; which line stands for odd and which for even n is easiest checked using their corresponding point of intersection with the x-axis, which is either 1, 2, or 3 throughout. Apart from explanations given earlier, it is not completely clear why differences are more prominent for some voting rules, why we sometimes see higher percentages for odd n and other times for even n, or why for some instances there is a large discrepancy for one preference model but hardly any for another.

IC and IAC are often criticized for being unrealistic and only giving worst-case estimates (see, e.g., Tsetlin et al. (2003); Regenwetter et al. (2006)). This criticism is generally confirmed by our experiments, which show that the highest fractions of profiles is prone to the NSP when the sampling is done according to IC or IAC. A notable exception is Black's rule for 30 alternatives, where a different effect prevails: for many alternatives and comparably few voters, situations in which a Condorcet winner (almost) exists appear less frequently under IC or IAC than under the other preference models. In absence thereof, Black's rule collapses to Borda's rule, which is immune to the NSP. Note that were we to conduct a dual experiment with fixed n and increasing m, the fraction of profiles prone to the NSP using Black's rule and IC or IAC would converge to zero for similar reasons.

We moreover see that IC, IAC, and the urn model exhibit identical behavior for $m = 30$. The second and fourth column of Fig. 5 therefore seem to only feature three preference models, even though all five are depicted. This may be surprising at first but is to be expected since IC and IAC can equivalently be seen as urn models with parameters 0 and 1, respectively. For $30! \approx 2.7 \times 10^{32}$ voter types and a comparatively small n the difference between parameters 0, 1, and 10 is simply too small for a visible difference.

The large conceptual similarities between Baldwin's and Nanson's rule are also reflected in the corresponding charts. Apart from the peak at $n = 3$ for Baldwin's rule, both look almost identical for all preference models with the small difference being that Nanson's rule appears to feature a slightly lower manipulability. Fewer rounds for winner determination thus do not seem to come at a cost with respect to the NSP.

Finally, Copeland's, Baldwin's, and Nanson's rule as well as MaxiMin to a lesser extent appear to fare exceptionally bad with respect to the NSP and IC, IAC, and the urn model. At the same time, none of these rules exhibits overly conspicuous behavior for the spatial and Mallows' model. This suggests that the risk of a possible manipulation is reduced by structural similarities in the individual preferences compared to a greater likelihood for very diverse rankings. Though generally in line

[13] For increasing m the computations quickly become very demanding. The values for $m = 30$ and $n \geq 99$ are determined with 50 000 runs each only. The size of all 95% confidence intervals is, however, still within 0.5%.

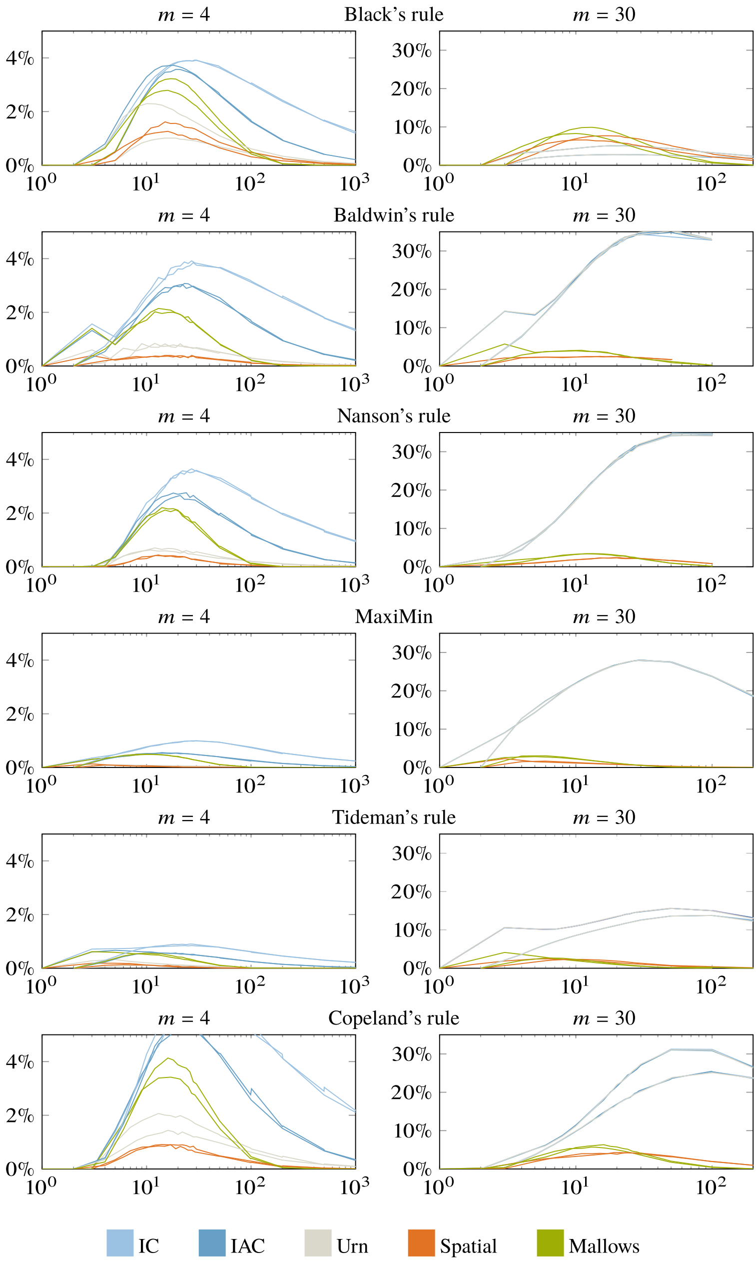

Fig. 5 Profiles prone to the NSP for different rules, fixed m, and increasing n on the x-axis; two lines per preference model depending on the parity of n; IC, IAC and the urn model collapse for $m = 30$, resulting in a bluish grey line

Table 1 Maximal percentage of total profiles prone to the NSP for different combinations of voting rules and preference models with $m = 4$ or $m = 30$; the number of voters n for which the maximum occurs attached in parentheses

	m	IC	IAC	Spatial	Urn	Mallows
Black	4	3.92 (29)	3.73 (18)	1.62 (15)	2.30 (10)	3.23 (17)
	30	5.12 (22)	5.12 (22)	7.70 (17)	5.14 (20)	9.90 (13)
Baldwin	4	3.92 (27)	3.07 (23)	0.40 (15)	0.84 (12)	2.14 (13)
	30	35.4 (49)	35.4 (51)	2.54 (21)	35.7 (49)	5.73 (3)
Nanson	4	3.64 (27)	2.76 (24)	0.44 (13)	0.68 (16)	2.20 (14)
	30	34.9 (51)	34.8 (51)	2.38 (21)	34.7 (99)	3.40 (12)
MaxiMin	4	1.00 (30)	0.56 (14)	0.14 (3)	0.13 (3)	0.50 (10)
	30	28.0 (30)	28.0 (30)	2.31 (3)	28.0 (30)	3.01 (6)
Tideman	4	0.80 (26)	0.67 (5)	0.19 (5)	0.32 (5)	0.62 (3)
	30	15.6 (51)	15.6 (49)	2.42 (7)	15.6 (49)	4.12 (3)
Copeland	4	6.96 (29)	5.54 (20)	0.91 (14)	2.07 (13)	4.13 (16)
	30	31.2 (50)	31.0 (50)	4.28 (21)	31.1 (50)	6.33 (16)

with expectations, we currently do not have a profound explanation for the magnitude of this effect. For Copeland's rule, it is plausible to assume that its particularly bad performance results from using less information, i.e., among all considered rules, Copeland's rule is the only one whose outcome only depends on unweighted majority comparisons.

The maximal fraction of total profiles prone to the NSP for $m = 4$, $m = 30$, different voting rules, preference models, and varying values of n is given in Table 1. Among other things, we for instance note that the maxima constantly occur for a higher number of voters for IC (26–51 voters) than for Mallows' model (3–17 voters), a fact probably due to an increasing (expected) structure under Mallows' model and larger n.

5.4 Empirical Analysis

We have also analyzed the NSP for empirical data obtained from real-world elections. Unfortunately, such data is generally relatively sparse and imprecise and often only fragmentarily available. A check of all 315 strict profiles contained in the PrefLib library (Mattei and Walsh 2013) for occurrences of the NSP shows that two profiles admit a manipulation by abstention when Black's rule is used, one profile for each Copeland's, Baldwin's, and Nanson's rule, and that no manipulation is possible for

MaxiMin as well as Tideman's rule.[14] While this suggests a low susceptibility to the NSP in real-world elections, much more data would be required to allow for meaningful conclusions.

6 Conclusion

We analyzed the likelihood of the NSP for six Condorcet extensions (Black's, Baldwin's, and Nanson's rule, MaxiMin, and Tideman's as well as Copeland's rule) under various preference models using Ehrhart theory as well as extensive computer simulations and some empirical data. Our main results are as follows.

- When there are few alternatives, the probability of the NSP is almost negligible (when $m = 4$, less than 1% for MaxiMin and Tideman's rule, less than 4% for Black's, Baldwin's, and Nanson's rule, and less than 7% for Copeland's rule under all considered preference models).
- When there are 30 alternatives and preferences are modeled using IC, IAC, and the urn model, Black's rule is least susceptible to the NSP ($<6\%$), followed by Tideman's rule ($<16\%$), MaxiMin ($<29\%$), Copeland's rule ($<32\%$) Nanson's rule ($<35\%$) , and Baldwin's rule ($<36\%$).
- For 30 alternatives and the spatial and Mallows' model, this ordering is roughly reversed. MaxiMin and Nanson's rule are least susceptible ($<4\%$), followed by Tideman's rule ($<5\%$), Baldwin's rule ($<6\%$), Copeland's rule ($<7\%$), and Black's rule ($<10\%$).
- The parity of the number of voters significantly influences the manipulability of Black's, Tideman's, and Copeland's rule. Black's and Copeland's rule are more manipulable for an even number of voters whereas MaxiMin is more manipulable for an odd number of voters (under the IAC assumption).
- Whenever analysis via Ehrhart theory is feasible, the results are perfectly aligned with our simulation results, highlighting the accuracy of the experimental setup.
- Only four (out of 315) strict preference profiles in the PrefLib database are manipulable by strategic abstention (manipulations only occur for Black's, Baldwin's, Nanson's, and Copeland's rule, but not for MaxiMin and Tideman's rule).

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[14]For instance the profile allowing for a manipulation under Copeland's rule is immune to the NSP for all other rules. It features 10 alternatives and 30 voters. Baldwin's and Nanson's rule exhibit the NSP for the same profile.

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Susceptibility to Manipulation by Sincere Truncation: The Case of Scoring Rules and Scoring Runoff Systems

Eric Kamwa and Issofa Moyouwou

1 Introduction

During an election or a referendum, some people may choose not to vote. In case of a high level of abstention, the legitimacy of the results of an election may be challenged. The motivations of an abstainer may be dictated by various considerations among which strategic behavior plays a central role. It has been known since Doron and Kronick (1977) and Fishburn and Brams (1983) that a voter may do better to abstain than to vote since abstaining may result in the victory of a more preferable or desirable candidate. This counterintuitive voting event is known in the literature as the *No-Show paradox*. Following Nurmi (1999) and Felsenthal (2012), the few voting rules that are not vulnerable to the No-Show paradox include the *Plurality rule*, the *Borda rule* and *Approval voting*.[1] According to Smith (1973), all the scoring runoff systems are sensitive to the No-show paradox. With at least four candidates, Moulin (1988) showed that all the Condorcet consistent rules are also vulnerable to the No-Show paradox (see also Brandt et al. (2018); Duddy (2013); Jimeno et al. (2009)). A *Condorcet consistent* voting rule always elects the Condorcet winner when

[1]Under the *Plurality rule*, every voter casts one vote for only one candidate, and the one with the greatest number of votes wins. With $m \geq 3$ candidates, the Borda rule gives $m - j$ points to a candidate each time he is ranked j-th in a voter's ranking; the winner is the candidate with the largest total number of points. Under *Approval voting*, each voter can approve as many candidates as he wants. The winner is the candidate with the greatest number of approvals.

E. Kamwa (✉)
LC2S UMR CNRS 8053, Université des Antilles, Faculté de Droit et d'Economie de la Martinique, Schœlcher Cedex 97275, Martinique
e-mail: eric.kamwa@univ-antilles.fr

I. Moyouwou
Ecole Normale Supérieure-Department of Mathematics, University of Yaounde I, Yaoundé BP47, Cameroon
e-mail: issofamoyouwou@ens.cm

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she exists. A *Condorcet winner* is a candidate who beats all the others in pairwise majority contests.

For voting situations with three candidates, Lepelley and Merlin (2001) computed the likelihood of the No-show paradox for three well-known scoring runoff rules; they concluded that when the electorate tends to infinity, the likelihood[2] of the No-show paradox is equal to 2.14% for the Borda runoff, 5.40% for the Plurality runoff and 4.25% for the Antiplurality runoff.[3] For their part, Kamwa et al. (2018) analyzed the No-show paradox for three-candidate elections with single-peaked preferences and found: (i) in three-candidate elections with single-peaked preferences, all the scoring runoff rules located between the Borda runoff and the Antiplurality runoff are not sensitive to the No-show paradox; (ii) single-peakedness of preferences greatly reduces the likelihood of the No-show paradox which nevertheless remains considerable.

It is perhaps to counter abstention behavior that several states (for instance, Bolivia, Belgium, Luxembourg, and Romania) have decided to render voting compulsory. Notwithstanding compulsory voting, some voters may still manipulate the vote by using a weak version of abstention behavior called "*sincere truncation*". The *sincere truncation* of preferences, also called the *truncation paradox*, was first introduced in the social choice literature by Brams (1982). Let us assume a group of voters who are asked to rank (sincerely) a list of candidates from the most preferred to the least preferred and that voters are allowed to submit incomplete rankings; all the candidates not ranked or listed on a ballot are assumed to be less preferred than all those who are ranked (Fishburn and Brams 1983, 1984). A voting rule is said to be vulnerable to sincere truncation if there are some configurations of ballots such that there is at least one voter who prefers the outcome obtained when he submits a sincere but incomplete ranking (truncated ranking) to the outcome obtained when he casts a complete sincere ranking.

A voting rule that is vulnerable to the No-Show paradox is also vulnerable to the truncation paradox, but the reverse is not necessarily true (see Nurmi 1999). Almost all the well-known voting rules are vulnerable to the truncation paradox. Fishburn and Brams (1984, p. 402) showed that, as a consequence of Moulin's theorem, that all the Condorcet consistent rules are sensitive to the truncation paradox. For a non-exhaustive list of the voting rules vulnerable to the truncation paradox, the reader may refer to Felsenthal (2012); Nurmi (1999) and Fishburn and Brams (1984). The few exceptions are the *Plurality rule*, *Plurality runoff* and *Approval voting*.

Is the truncation paradox a rare oddity or a generalized behavior? To our knowledge, the only work that has tried to evaluate the likelihood of the truncation paradox is that of Plassmann and Tideman (1999); for three-candidate elections, they focused on certain voting rules that include, amongst others, some Condorcet consistent rules, some scoring rules (Borda, Antiplurality) and some iterative scoring rules (iterative Plurality, iterative Antiplurality). They based their calculations on the spatial model for drawing voting situations. In this paper, we do the same job for the whole family of scoring rules and scoring runoff rules in three-candidate elections both under

[2]Under the assumption of Impartial and Anonymous Culture (defined later).

[3]The Plurality runoff, the Borda runoff, and the Antiplurality runoff will be defined later.

the universal and the single-peaked domains. Thus, we characterize all the paradoxical voting situations and then compute the exact likelihood of the paradox. We perform our analysis under the assumption of Impartial Anonymous Culture (IAC), which is one of the well-known assumptions often used for such a study. Under IAC, first introduced by Kuga and Hiroaki (1974) and later developed by Gehrlein and Fishburn (1976), each voting situation is assumed to be equally likely to occur. The likelihood of a given event is calculated with respect to the ratio between the number of voting situations in which the event is likely and the total number of possible voting situations. The number of voting situations associated with a given event can be reduced to the solutions of a finite system of linear constraints with (or without) rational coefficients. As recently pointed out in the social choice literature, when one is dealing with rational coefficients, the appropriate mathematical tools to find these solutions are Ehrhart polynomials. The background to this notion and its connection with the polytope theory can be found in (Gehrlein and Lepelley 2011, 2017; Lepelley et al. 2008). This technique has been widely used in numerous studies analyzing the probability of electoral events in the case of three-candidate elections under the IAC assumption. In this chapter, we will follow the technique initiated by Cervone et al. (2005) for our computations. We say few words on this technique in the appendices.

The rest of the chapter is organized as follows: Sect. 2 is devoted to basic definitions. In Sect. 3, given a three-candidate election where voters have strict rankings, for all the one-shot and runoff scoring rules, we characterize all the voting situations vulnerable to the truncation paradox and then we compute the limiting probabilities. We do the same job in Sect. 4 by assuming that voters' preferences are single-peaked. Section 5 concludes.

2 Notation and Definitions

2.1 *Preferences*

Let N be a set of n voters ($n \geq 2$) and A a set of m candidates ($m \geq 3$). Individual preferences are linear orders, these are complete, asymmetric and transitive binary relations on A. With m candidates, there are exactly $m!$ linear orders $P_1, P_2, \ldots, P_{m!}$ on A. A *voting situation* is an $m!$-tuple $\pi = (n_1, n_2, \ldots, n_t, \ldots, n_{m!})$ that indicates the total number n_t of voters casting each complete linear order P_t, $t = 1, 2, \ldots, m!$ in such a way that $\sum_{t=1}^{m!} n_t = n$. In the sequel, we consider three candidates a, b and c. In this case, we will simply write abc to denote the linear order on A according to which a is strictly preferred to b, b is strictly preferred to c; and by transitivity a is strictly preferred to c. Table 1 describes a voting situation with three candidates: there are six preference types and for $t = 1, 2, \ldots, 6$, n_t is the total number of voters of type t.

Table 1 Voting situation and possible preference types with three candidates

Type 1: *abc* (n_1)	Type 3: *bac* (n_3)	Type 5: *cab* (n_5)
Type 2: *acb* (n_2)	Type 4: *bca* (n_4)	Type 6: *cba* (n_6)

Given $a, b \in A$ and a voting situation π, we denote by $n_{ab}(\pi)$ (simply n_{ab}) the total number of voters who strictly prefer a to b. If $n_{ab} > n_{ba}$, we say that a majority dominates candidate b; or equivalently, a beats b in a pairwise majority voting. In such a case, we will simply write $aM(\pi)b$.

Possible actions of a voter include (i) *ranking all candidates* from the top-ranked candidate to the least preferred one; (ii) *abstaining*: no ranking is provided; or (iii) *truncating*: an incomplete ranking is provided. With the last action, it is assumed that all the candidates not ranked on a ballot are less preferred to all those who are ranked. With three candidates, when a voter truncates, he just states his most preferred candidate. For example with Table 1, if some voters of type 1 truncate, this leads to a new voting situation π' in which these voters only state $a - -$ as their ranking. Note that when some voters truncate, this does not alter the size of the electorate as is the case when some voters abstain.

2.2 Voting Rules

Scoring rules are voting systems that give points to candidates according to the position they have in voters' rankings. For a given scoring rule, the total number of points received by a candidate defines her score for this rule. The winner is the candidate with the highest score. In general, with $m \geq 3$ and complete strict rankings, a scoring vector is an m-tuple $w = (w_1, w_2, \ldots, w_k, \ldots, w_m)$ of real numbers such that $w_1 \geq w_2 \geq \cdots \geq w_k \geq \cdots \geq w_m$ and $w_1 > w_m$. Given a voting situation π, each candidate receives w_k each time she is ranked k^{th} by a voter. The score of a candidate $x \in A$ is the sum $S(\pi, w, x) = \sum_{t=1}^{m!} n_t w_{r(t,x)}$ where $r(t, x)$ is the rank of candidate x according to voters of type t.

For uniqueness, we use the normalized form $(1, \frac{w_2-w_m}{w_1-w_m}, \ldots, \frac{w_k-w_m}{w_1-w_m}, \ldots, 0)$ of each scoring vector w. With three candidates, a normalized scoring vector has the shape $w_\lambda = (1, \lambda, 0)$ with $0 \leq \lambda \leq 1$. For $\lambda = 0$, we obtain the *Plurality rule*; for $\lambda = 1$, we have the *Antiplurality rule* and for $\lambda = \frac{1}{2}$, we get the *Borda rule*. From now on, we will denote by $S(\pi, \lambda, x)$, the score of candidate x when the scoring vector is $w_\lambda = (1, \lambda, 0)$ and the voting situation is π. Table 2 gives the score of each candidate in $A = \{a, b, c\}$ given the voting situation of Table 1.

Table 2 Scores with three candidates

$S(\pi, \lambda, a) = n_1 + n_2 + \lambda(n_3 + n_5)$
$S(\pi, \lambda, b) = n_3 + n_4 + \lambda(n_1 + n_6)$
$S(\pi, \lambda, c) = n_5 + n_6 + \lambda(n_2 + n_4)$

In one-shot voting, the winner is just the candidate with the largest score. Runoff systems involve two rounds of voting: at the *first round*, the candidate with the smallest score is eliminated; at the *second round*, a majority contest determines who is the winner.

With three candidates, when a voter of type 1 with the ranking *abc* truncates and submits $a--$, candidate a receives 1 point in the new voting situation while both b and c receive zero points. Similar considerations hold for other types. Note that when some voters truncate, only the scores of candidates ranked second by some of these voters are affected and diminish. Moreover, truncation is only possible at the first round under runoff systems.

In our setting, we assume that ties among candidates will be broken alphabetically, e.g. a wins all ties against other candidates; while b wins all ties against c. Note that this special tie-breaking rule does not affect our results as we only deal with voting situations where the total number of voters tends to infinity. Let us now use an example in order to illustrate the truncation paradox for the voting rules we focus on.

2.3 Illustrating the Truncation Paradox

As stated above, among the scoring and the scoring runoff rules we focus on, only the Plurality rule and the Plurality runoff are not vulnerable to the truncation paradox. So, in our analysis of three-candidate elections, we will focus on $\lambda \in]0\ 1]$.

Now consider the following sincere voting situation π with three candidates and 45 voters:

$$11 : abc \quad 4 : acb \quad 7 : bac \quad 8 : bca \quad 10 : cab \quad 5 : cba$$

According to Table 2, the scores are as follows:

$$S(\pi, \lambda, a) = 15 + 17\lambda; \qquad S(\pi, \lambda, b) = 15 + 16\lambda; \qquad S(\pi, \lambda, c) = 15 + 12\lambda$$

It comes out that for all $\lambda \in]0\ 1]$, we get $S(\pi, \lambda, a) > S(\pi, \lambda, b) > S(\pi, \lambda, c)$.

- *The case of one-shot scoring rules.*

As $S(\pi, \lambda, a) > S(\pi, \lambda, b) > S(\pi, \lambda, c)$ for all $\lambda \in]0\ 1]$, candidate a is the winner. Assume that two voters with *bac* (type 3) truncate. Then, the new scores are:

$$S(\pi', \lambda, a) = 15 + 15\lambda; \quad S(\pi', \lambda, b) = 15 + 16\lambda; \quad S(\pi', \lambda, c) = S(\pi, \lambda, c) = 15 + 12\lambda$$

Candidate a is no longer the winner since $S(\pi', \lambda, b) > S(\pi', \lambda, a) > S(\pi', \lambda, c)$; the new winner is candidate b. Since the two voters of type 3 benefit from the truncation, the truncation paradox can occur for all $\lambda \in]0\ 1]$.

- *The case of runoff scoring rules.*

Given that $S(\pi, \lambda, a) > S(\pi, \lambda, b) > S(\pi, \lambda, c)$, candidate c is eliminated in the first round. At the second round, candidate a wins with $n_{ab} = 25$ favorable votes against $n_{ba} = 20$ votes in favor of b.

Assume that all the 5 voters with cba (type 6) truncate: they just state $c - -$. In this case, the new scores are:

$$S(\pi', \lambda, a) = 15 + 17\lambda; \quad S(\pi', \lambda, b) = 15 + 11\lambda; \quad S(\pi', \lambda, c) = 15 + 12\lambda$$

For all $\lambda \in]0\ 1]$, we get $S(\pi', \lambda, a) > S(\pi', \lambda, c) > S(\pi', \lambda, b)$: candidate b is eliminated in the first round. Since $n_{ac} = 22$ and $n_{ca} = 23$, c wins the second round. So, by truncating their true preferences, the five voters of type 6 obtain a better outcome: the truncation paradox occurs.

3 The Vulnerability of Scoring Runoff Rules to the Truncation Paradox in Three-Candidate Elections

Prior to the determination of the likelihood of the truncation paradox, we need to characterize all the voting situations under which this paradox is liable to occur.

3.1 The Case of One-Shot Scoring Rules

Consider a voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ on $A = \{a, b, c\}$ and the one-shot rule with $0 < \lambda \leq 1$. Let $\pi\left(\left[R_{j_1}, R_{j_2}, \ldots\right]\right)$ stands for the voting situation obtained from π when all type $R_{j_1}, R_{j_2}, \ldots$ voters truncate their preferences. For example, $\pi\ [abc]$ differs from π only in the fact that at $\pi\ [abc]$, candidate a receives 1 point from each type 1 voter while the two others receive 0 points. Similarly, from π to $\pi\ [abc, acb]$ the only change that occurs is that all type 1 voters and all type 2 voters now truncate their preferences to report $a \ldots$. For one-shot scoring rules, the following result identifies all voting situations in which the truncation paradox is possible.

Proposition 1 *Consider a voting situation* $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ *on* $A = \{a, b, c\}$*, the one-shot rule associated with* $0 < \lambda \leq 1$ *and a pair* $\{x, y\}$ *of candidates with* $A \setminus \{x, y\} = \{z\}$.

If x is the election winner at π, then the truncation paradox is liable to occur at π in favor of y if and only if y is the election winner at $\pi\left(\left[yxz, yzx\right]\right)$.

Proof See Appendix A. □

Remark 1 Given a voting situation with three candidates, only voters having the same top-ranked candidate can effectively benefit from truncating preferences by truly reporting their best candidate. For example, if a is the winning outcome, it appears from Proposition 1 that truncating preferences may benefit either voters of type 3 and type 4 with bac and bca respectively; or else voters of type 5 and type 6 with the orderings cab and cba respectively. This is in contrast with other strategic misrepresentations of preferences which allow successful coordination among voters who may report a fake ranking with possibly a false best candidate—see Lepelley and Mbih (1996); Pritchard and Wilson (2007) or Mbih et al. (1996).

Proposition 2 *Consider a one-shot scoring rule F_λ, $0 < \lambda \leq 1$. As the total number n of voters tends to infinity, the limit probability of observing a voting situation in which the truncation paradox may occur is given by:*

If $0 < \lambda \leq \frac{1}{2}$,

$$P_{TP}(F_\lambda) = \frac{\begin{pmatrix} 10\lambda^{14} - 37\lambda^{13} - 179\lambda^{12} + 1310\lambda^{11} - 1778\lambda^{10} - 6319\lambda^9 \\ +26773\lambda^8 - 25735\lambda^7 - 67880\lambda^6 + 259941\lambda^5 - 408078\lambda^4 \\ +356643\lambda^3 - 166536\lambda^2 + 31833\lambda \end{pmatrix}}{6(3+\lambda)^2\left(3-2\lambda+\lambda^2\right)^2(\lambda-2)^2(2\lambda-3)^2(\lambda-1)(-3+5\lambda)}$$

If $\frac{1}{2} \leq \lambda \leq 1$,

$$P_{TP}(F_\lambda) = \frac{\begin{pmatrix} 2\lambda^{13} + 50\lambda^{12} - 194\lambda^{11} - 190\lambda^{10} + 2548\lambda^9 - 5560\lambda^8 \\ -662\lambda^7 + 26915\lambda^6 - 62174\lambda^5 + 73636\lambda^4 - 48132\lambda^3 \\ +16425\lambda^2 - 3564\lambda + 324 \end{pmatrix}}{12(3+\lambda)^2\left(3-2\lambda+\lambda^2\right)^2(\lambda-2)^2\lambda^2(2\lambda-3)}$$

Proof See Appendix B for details of computations. □

As the total number n of voters tends to infinity, it appears from Proposition 2 that the limit probability, under the IAC assumption, of observing a voting situation in which the truncation paradox may occur given a one-shot scoring rule F_λ increases from 0 to $\frac{3}{4}$ as the weight λ increases from 0 (the Plurality rule) to 1 (the Antiplurality rule); for an overview of the behavior of $P_{\text{TP}}(F_\lambda)$, see Fig. 1 or Table 3 where we report some numerical evaluations.

3.2 The Case of Scoring Runoff Rules

Consider the voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ and a runoff rule with $0 < \lambda \leq 1$. Assume that at π, z is eliminated at the first round and that x wins against y at the second round. For simplicity, we say that x is the winner, y is the challenger and z is the (first-round) loser. To see how the truncation paradox arises under a runoff

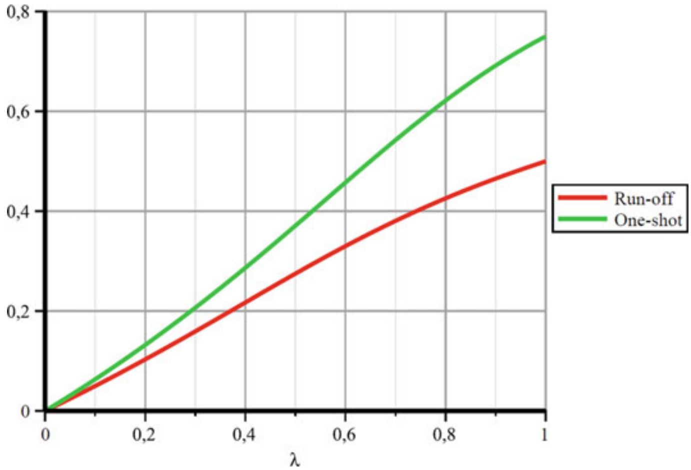

Fig. 1 Vulnerability of scoring rules to the truncation paradox

Table 3 Values of $P_{TP}(F_\lambda)$ and $P_{TP}(F'_\lambda)$

λ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$P_{TP}(F_\lambda)$	–	0.06334	0.1322	0.2067	0.2866	0.3710	0.4575	0.5423	0.6215	0.6916	0.7500
$P_{TP}(F'_\lambda)$	–	0.0499	0.1032	0.1593	0.2172	0.2750	0.3300	0.3806	0.4257	0.4652	0.5000

rule, recall that this paradox can be seen as a strategic behavior by some voters. Taking into account the specificity of runoff rules that combine both counting points at the first round and majority voting at the second round, successful truncations of preferences are either (i) in favor of the challenger when, by truncating their rankings, some voters make x lose at the first round and cause the loser to be beaten by the challenger at the second round; or (ii) in favor of the loser who defeats the winner or the challenger in the second round.

Proposition 3 *Consider a voting situation* $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ *on* $A = \{a, b, c\}$ *and a runoff rule with* $0 < \lambda \leq 1$. *Assume that* x *is the winner,* y *is the challenger and* z *is the first-round loser.*

1. *The truncation paradox is liable to occur at* π *in favor of* y *if and only if* y *wins the majority duel against* z *and* x *is the first-round loser at* $\pi\left(\left[yxz\right]\right)$.
2. *The truncation paradox is liable to occur at* π *in favor of* z *if and only if* z *wins the majority duel against* y *and* x *is the first-round loser at* $\pi\left(\left[zxy\right]\right)$*; or if* z *wins the majority duel against* x *and* y *is the first-round loser at* $\pi\left(\left[zyx\right]\right)$.

Proof See Appendix C. □

In contrast with one-shot scoring rules, when the truncation paradox occurs under a runoff rule with three candidates, it is always reachable by a coalition of voters of the same type. Proposition 3 completely describes all the possible scenarios that support possible occurrence of the truncation paradox given a voting situation. These conditions lead us to some sets of linear constraints that characterize all possible occurrences of the truncation paradox under a runoff rule. Details are available in Appendix D. Computing the volume of all the corresponding polytopes leads to Proposition 4.

Proposition 4 *Consider the scoring runoff rule F'_λ associated with the scoring vector $w_\lambda = (1, \lambda, 0)$ with $0 < \lambda \leq 1$. As the total number n of voters tends to infinity, the limit probability $P_{TP}\left(F'_\lambda\right)$ of observing a voting situation in which the truncation paradox may occur is given by : If $0 \leq \lambda \leq \frac{1}{2}$,*

$$P_{TP}\left(F'_\lambda\right) = -\frac{\begin{pmatrix} 996\,096\lambda^{20} - 25\,010\,368\lambda^{19} + 286\,101\,152\lambda^{18} - 2000\,804\,220\lambda^{17} \\ +9664\,972\,152\lambda^{16} - 34\,453\,144\,125\lambda^{15} + 94\,322\,255\,778\lambda^{14} \\ -203\,353\,434\,975\lambda^{13} + 350\,716\,379\,871\lambda^{12} - 488\,312\,722\,095\lambda^{11} \\ +551\,142\,449\,552\lambda^{10} - 504\,159\,008\,281\lambda^{9} + 372\,136\,194\,567\lambda^{8} \\ -219\,653\,377\,992\lambda^{7} + 102\,140\,474\,607\lambda^{6} - 36\,558\,733\,185\lambda^{5} \\ +9711\,109\,602\lambda^{4} - 1801\,641\,852\lambda^{3} + 208\,222\,083\lambda^{2} - 11\,278\,359\lambda \end{pmatrix}}{96\,(\lambda-1)^2\,(\lambda-2)^2\,(2\lambda-3)^2\,(4\lambda-3)^2\,(5\lambda-3)^2\left(-2\lambda+\lambda^2+3\right)\left(-5\lambda+\lambda^2+3\right)^2\left(-4\lambda+2\lambda^2+3\right)\left(-7\lambda+3\lambda^2+3\right)}$$

If $\frac{1}{2} \leq \lambda \leq 1$,

$$P_{TP}\left(F'_\lambda\right) = \frac{\begin{pmatrix} 132\lambda + 9346\lambda^2 - 55\,961\lambda^3 + 161\,587\lambda^4 - 283\,660\lambda^5 \\ +330\,502\lambda^6 - 265\,921\lambda^7 + 149\,437\lambda^8 - 57\,766\lambda^9 \\ +14\,560\lambda^{10} - 2112\lambda^{11} + 128\lambda^{12} - 180 \end{pmatrix}}{288\lambda^3\,(\lambda-2)^2\,(3-2\lambda)\left(-2\lambda+\lambda^2+3\right)\left(-4\lambda+2\lambda^2+3\right)}$$

Proof See Appendix D for further details on the computation. □

The limit, as the total number n of voters tends to infinity, and under the IAC assumption, of the probability of observing a voting situation in which the truncation paradox may occur given the runoff scoring rule F'_λ increases from 0 to 0.5 as the weight λ increases from 0 (the Plurality runoff rule) to 1 (the Antiplurality runoff rule). Moreover, each one-shot scoring rule is more vulnerable to the truncation paradox than its corresponding runoff version. For an overview of the behavior of $P_{\text{TP}}\left(F'_\lambda\right)$, see Fig. 1 or Table 3 where some numerical evaluations are reported. Finally, while the Plurality rule is not vulnerable to the truncation paradox as a one-shot rule or a runoff rule, the Antiplurality rule appears to be the most vulnerable rule among both the one-shot and runoff scoring rules with three candidates.

An analysis of the vulnerability of runoff scoring rules to profitable abstention in three-candidate elections is available from Lepelley and Merlin (2001) for the main scoring runoff rules, and from Kamwa et al. (2018) for the whole family of scoring runoff rules. In Table 4, we report the limiting probabilities of the no-show paradox obtained by Lepelley and Merlin (2001) for the universal and by Kamwa et al. (2018) for the single-peaked domain.

Table 4 Limiting probabilities of the no-show paradox under the universal and the single-peaked domains for scoring runoff rules

	λ										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Universal	0.0408	0.0382	0.0351	0.0313	0.0272	0.0243	0.0263	0.0299	0.0341	0.0383	0.0425
Single-peaked	0.0278	0.0208	0.0136	0.0067	0.0016	0	0	0	0	0	0

Comparing the probabilities by Lepelley and Merlin (2001) and Kamwa et al. (2018) on the No-show paradox with those we obtain on the truncation paradox, it emerges that the truncation paradox is always more likely to occur than the No-show paradox for all $0 < \lambda \leq 1$. This is consistent with the fact that the truncation paradox is a weak version of the No-show paradox.

4 The Impact of Single-Peaked Preferences

Kamwa et al. (2018) showed that when preferences are single-peaked in three-candidate elections, the No-show paradox never occurs with all the scoring runoff rules located between the Borda runoff and the Antiplurality runoff, i.e., for all $\lambda \in [\frac{1}{2} \quad 1]$. It emerges from their probability computations that the likelihood of the No-show paradox is drastically reduced with single-peaked preferences. In this section, we also want to check what happens with the truncation paradox when preferences are single-peaked.

With three candidates, when preferences are single-peaked, there is one candidate that is not bottom-ranked. On $A = \{a, b, c\}$, we assume without loss of generality that candidate c is never bottom-ranked. Table 5 describes a voting situation with three candidates and single-peaked preferences.

In the sequel, we assume that only the four preference types in Table 5 are observable. Note that when candidate a wins in a voting situation, only voters of type 4 and 6 who strictly prefer b to a may truncate their rankings in order to favor the election

Table 5 Single-peaked preferences and scores on $A = \{a, b, c\}$

Preference types	
$n_2 : acb$	$n_4 : bca$
$n_5 : cab$	$n_6 : cba$
Scores at the first round	
$S(\pi, \lambda, a) = n_2 + \lambda n_5$	
$S(\pi, \lambda, b) = n_4 + \lambda n_6$	
$S(\pi, \lambda, c) = n_5 + n_6 + \lambda(n_2 + n_4)$	

of b. Since candidate a is bottom-ranked by all those voters, there is no way left to favor candidate b by preference truncation. Similarly, when candidate b wins under a one-shot scoring rule, there is no opportunity to favor candidate a by preference truncation. The conditions of Proposition 1 for one-shot scoring rules still apply for viable sincere truncation of preference when preferences are single-peaked, except for the restriction just outlined and reported in Proposition 5.

Proposition 5 *Consider a voting situation on $A = \{a, b, c\}$ with single-peaked preferences such that candidate c is never bottom-ranked in the individual preferences. Assume that the voting rule is a one-shot scoring rule.*

- *When candidate a (or candidate b) is the election winner in π for $\lambda \in]0\ 1[$, then the truncation paradox is liable to occur only in favor of c.*
- *When candidate c is the election winner in π for $\lambda \in]0\ 1]$, it is possible to favor a candidate in $A \setminus \{c\}$ by sincere truncation of preferences.*

Proof See Appendix E. □

In the same way, the conditions of Proposition 3 also identify all the scenarios in which a runoff rule is vulnerable to the truncation paradox when individual preferences are single-peaked except for the restriction provided in Proposition 6.

Proposition 6 *Consider a voting situation on $A = \{a, b, c\}$ with single-peaked preferences such that candidate c is never bottom-ranked in individual preferences.*

(i) Assume that candidate c is eliminated after the first run. In this case, the truncation paradox can occur only in favor of candidate c; and only for all the scoring runoff rules associated with $\lambda \in]0, \frac{1}{2}[$.
(ii) Assume that candidate a or b wins the second run versus candidate c. The truncation paradox never occurs for all the scoring runoff rules with $\lambda \in]0, 1]$.
(iii) Assume that candidate c wins the second run versus candidate a or b. The truncation paradox can occur for all the scoring runoff rules such that $\lambda \in]0, 1[$.

Proof See Appendix F. □

What emerges from Proposition 6 is that single-peaked preferences do not vitiate the truncation paradox in the same manner as they do with the No-show paradox; as with the No-show paradox, they totally obviate the truncation paradox for voting situations under which the never-bottom-ranked candidate loses at the second stage. For three-candidate elections with single-peaked preferences, Table 6 gives all the scoring runoff rules vulnerable to the truncation paradox and to the No-show paradox for all the possible configurations: (i) a wins the second stage versus b; (ii) a or b wins the second stage versus c; and (iii) c wins the second stage versus a or b. The reader can then see from Table 6 that the impact of single-peaked preferences on the truncation paradox is not the same as on the Abstention paradox although the first paradox is the weaker version of the second.

Table 6 Vulnerable scoring runoff rules with three candidates and single-peaked preferences

Second-round opponents	a versus b	a or b wins versus c	c wins versus a or b
Abstention*	$\lambda \in [0\ \frac{1}{2}[$	–	$\lambda \in]0\ \frac{1}{2}[$
Truncation	$\lambda \in]0\ \frac{1}{2}[$	–	$\lambda \in]0\ 1[$

*From Kamwa et al. (2018)

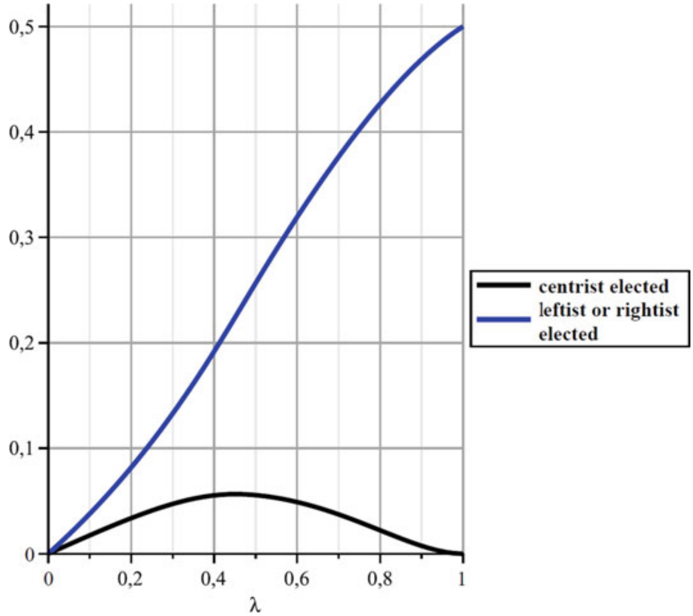

Fig. 2 Single-peakedness: vulnerability of one-shot rules to the truncation paradox

Further aspects of the behavior of one-shot scoring rules are perhaps more interesting. Note that in the general case, the probability that a one-shot scoring rule exhibits the truncation paradox, given that the winner is a given candidate, is the same from one candidate to another. But when preferences are single-peaked, results from computations provided for the next proposition and sketched in Fig. 2 show that a one-shot scoring rule is more vulnerable to the truncation paradox when the centrist candidate c is elected than when the leftist candidate a or the rightist candidate b is elected.

In the next propositions, we report global probabilities we obtained by performing the probability computation over all the possible scenarios for the truncation paradox on the single-peaked domain. As expected, these probabilities are lower than those we observe on the universal domain. However, these probabilities remain significantly high.

Proposition 7 *Consider the one-shot rule associated with the scoring vector* $w_\lambda = (1, \lambda, 0)$ *with* $0 < \lambda \leq 1$ *. As the total number n of voters tends to infinity, the limit* *probability $P_{TP}(F_\lambda, SP)$ of observing a voting situation in which the truncation paradox may occur is as follows:*

$$\text{If } 0 \le \lambda \le \frac{1}{2}, P_{TP}(F_\lambda, SP) = \frac{\lambda\left(-94\lambda^5 - 441\lambda^4 - 30\lambda^7 + 108\lambda^6 + 1602\lambda^3 - 2456\lambda^2 + 1812\lambda - 513 + 4\lambda^8\right)}{9(2\lambda-3)(\lambda-1)^2(-2+\lambda)^2\left(3+\lambda^2-2\lambda\right)(\lambda+3)}$$

$$\text{If } \frac{1}{2} \le \lambda \le 1, P_{TP}(F_\lambda, SP) = \frac{-65\lambda^2 - 6\lambda^4 + 38\lambda^3 + 43\lambda - 4\lambda^5 + 2\lambda^6 - 4}{(\lambda-2)^2(\lambda+3)\left(\lambda^2-2\lambda+3\right)}$$

Proposition 8 *Consider the runoff rule associated with the scoring vector $w_\lambda = (1, \lambda, 0)$ with $0 < \lambda \le 1$. As the total number n of voters tends to infinity, the limit probability $P_{TP}\left(F'_\lambda, SP\right)$ of observing a voting situation in which the truncation paradox may occur is as follows:*

$$\text{If } 0 \le \lambda \le \frac{1}{2}, P_{TP}\left(F'_\lambda, SP\right) = \frac{\lambda\left(-110\lambda^6 + 322\lambda^5 - 64\lambda^7 + 4374\lambda^2 + 365\lambda^4 + 22\lambda^8 + 657 - 2684\lambda^3 - 2850\lambda\right)}{24(2-\lambda)(-3+\lambda)(-3+2\lambda)(\lambda-1)^2\left(3+\lambda^2-2\lambda\right)(\lambda+3)}$$

$$\text{If } \frac{1}{2} \le \lambda \le 1, P_{TP}\left(F'_\lambda, SP\right) = \frac{(\lambda-1)^2\left(-\lambda^3+2\lambda^2+7\lambda-15+\lambda^4\right)}{4(2-\lambda)\left(\lambda^2-2\lambda+3\right)(-3+2\lambda)(\lambda+3)}$$

The vulnerabilities to the truncation paradox reported in Propositions 7 and 8 are computed using very similar arguments to the proofs of Proposition 3 and Proposition 4 respectively. One simply needs to consider the possible scenarios described in Propositions 5 and 6; the details are omitted.

Table 7 reports the figures we get from Propositions 7 and 8.

With runoff scoring rules the effect of single-peaked preferences is indeed remarkable: it reduces the probability of truncation to less than 0.035 for all the runoff scoring rules. Moreover, when the centrist candidate c is the winner, the probability that a runoff scoring rule is very low and even null when the weight λ lies between 0.5 and 1. What is also surprising is that the runoff version of the Antiplurality rule is now immune to the truncation paradox. To see this, note that when $\lambda = 1$ and candidate c is ranked last by no voter, candidate is always qualified for the second round whether preferences are truncated by voters of type 2 (or type 4) or not. Figure 3 shows how single-peakedness imposes a downwards curve on the vulnerability of runoff scoring rules to the truncation paradox.

Figure 4 is a comparative visualization of the vulnerability of both one-shot scoring rules and runoff scoring rules to the truncation paradox when preferences are single-peaked. It obviously highlights the fact that each one-shot scoring rule is still more vulnerable to the truncation paradox than its runoff version, even with single-peaked preferences.

Table 7 Values of $P_{TP}(F_\lambda, SP)$ and $P_{TP}(F'_\lambda, SP)$

λ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$P_{TP}(F_\lambda, SP)$	–	0.0553	0.1155	0.1801	0.2472	0.3122	0.3683	0.4137	0.4489	0.4762	0.5000
$P_{TP}(F'_\lambda, SP)$	0	0.0158	0.0284	0.0357	0.0361	0.0293	0.0207	0.0129	0.0063	0.0017	0

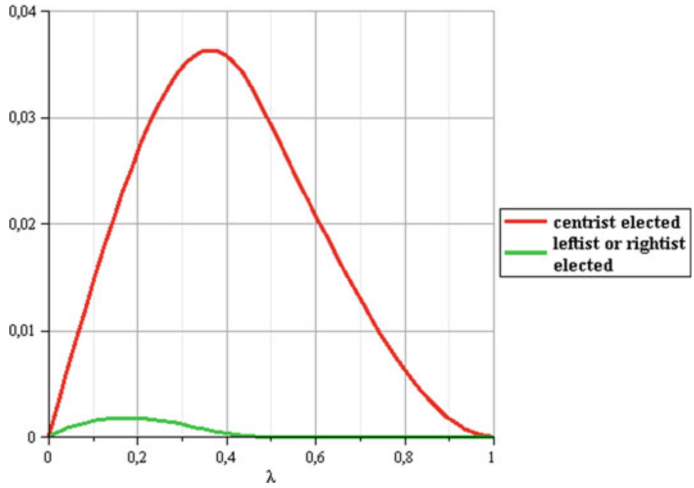

Fig. 3 Single-peakedness: vulnerability of runoff rules to the truncation paradox

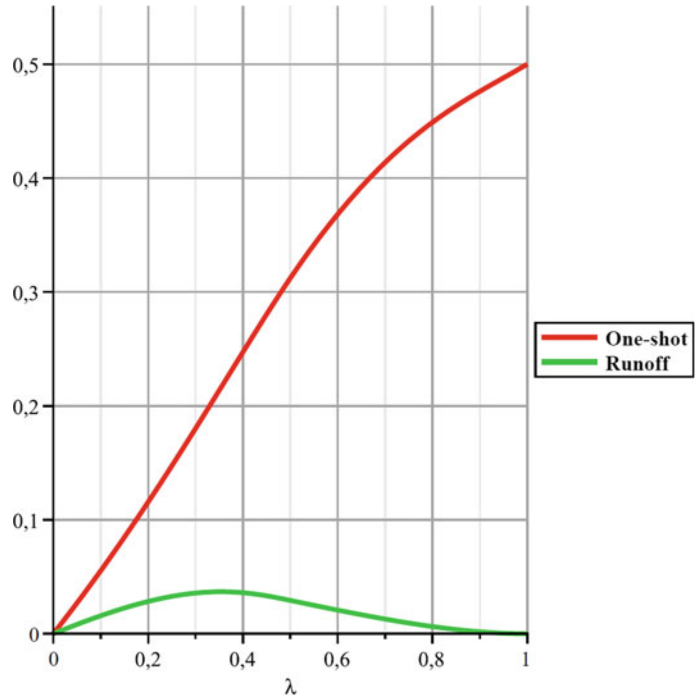

Fig. 4 Single-peakedness: vulnerability of one-shot rules and runoff rules to the truncation paradox

5 Concluding Remarks

Since Fishburn and Brams (1984), we have known that almost all the well-known voting rules are vulnerable to the truncation paradox except the Plurality rule, Plurality runoff, and Approval voting. In this paper, we have characterized all the three-candidate voting situations under which the truncation paradox can occur for scoring rules and scoring runoff rules under both the universal and the single-peaked domains. Then we computed the limiting probability of the truncation paradox. By comparing our results to those obtained by Lepelley and Merlin (2001) and Kamwa et al. (2018) concerning the likelihood of the Abstention paradox, we concluded that the Abstention paradox is less likely to occur than the truncation paradox. Hence, making voting compulsory in order to counter the paradoxical outcomes caused by abstention behavior seems not to be a good choice at all.

With single-peaked preferences, we found that the occurrence of the truncation paradox depends on the configuration of the second run: if the never-bottom-ranked candidate loses the second run versus one of the other candidates, the truncation paradox never occurs; if this candidate is rushed out at the first stage, the truncation paradox never occurs with all the scoring runoff rules located between the Borda runoff and the Antiplurality runoff.

Acknowledgements The authors are grateful to two anonymous reviewers for their helpful remarks and comments.

Appendices

A. Proof of Proposition

Consider a voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ on $A = \{a, b, c\}$, the one-shot rule associated with $0 < \lambda \leq 1$ and a pair $\{x, y\}$ of candidates. Let z be the third candidate.

Necessity. Assume that x is the election winner at π, and that the truncation paradox is liable to occur in π in favor of y. Then by truncating their true preferences, a coalition of voters, say S , favors the election of y. Moreover each voter in S strictly prefers y to x. Since the truncation operation only affects the second-ranked candidates of each voter in S, then the preferences of each voter in S is yxz or yzx. At the new voting situation π', y wins. Without loss of generality, we denote by $n_{xyz}(\pi)$ the total number of voters in π who rank x first, y second and z last at π. Note that $|S| \leq n_{yxz}(\pi) + n_{yzx}(\pi)$. Then from π' to $\pi\left(\left[yxz\right]\right)$, the score of y increases, the scores of both x and z decrease. Hence y also wins in $\pi\left(\left[yxz, yzx\right]\right)$.

Sufficiency. Assume that x is the election winner at π while y wins in $\pi\left(\left[yxz, yzx\right]\right)$. Clearly, the truncation paradox is liable to occur in π in favor of y since all voters who truncate their preferences in $\pi\left(\left[yxz, yzx\right]\right)$ prefers y to x.

B. Computation Details for Proposition

Let T_x denote the set of all voting situations in which x is the election winner while the truncation paradox is liable to occur; and T_{xy} the subset of T_x that consists of all voting situations in which truncating preferences may favor the election of y. Note for example that

$$T_a = T_{ab} \cup T_{ac} \text{ and } |T_a| = |T_{ab}| + |T_{ac}| - |T_{ab} \cap T_{ac}|.$$

By Proposition 1, $\pi \in T_{ab}$ if and only if $S(\pi, \lambda, a) \geq S(\pi, \lambda, b)$, $S(\pi, \lambda, a) \geq S(\pi, \lambda, c)$, $S(\pi\,[bac, bca]\,, \lambda, b) > S(\pi\,[bac, bca]\,, \lambda, a)$ and $S(\pi\,[bac, bca]\,, \lambda, b) \geq S(\pi\,[bac, bca]\,, \lambda, c)$. Equivalently,

$$\pi \in T_{ab} \Longleftrightarrow \begin{cases} (\lambda-1)n_1 - n_2 + (1-\lambda)n_3 + n_4 - \lambda n_5 + \lambda n_6 \leq 0 \\ -n_1 + (\lambda-1)n_2 - \lambda n_3 + \lambda n_4 + (1-\lambda)n_5 + n_6 \leq 0 \\ (1-\lambda)n_1 + n_2 - n_3 - n_4 + \lambda n_5 - \lambda n_6 < 0 \\ -\lambda n_1 + \lambda n_2 - n_3 - n_4 + n_5 + (1-\lambda)\, n_6 \leq 0 \end{cases}$$

Clearly, each of the six possible sets T_{xy} with $x, y \in A$ can be similarly described by a set of four linear constraints as with T_{ab} above. As n tends to infinity, $vol\left(P_{xy}\right)$ is the 5-dimensional volume of the polytope P_{xy} obtained from the characterization of T_{xy} by replacing each n_j by $p_j = \frac{n_j}{n}$. Note that some inequalities in the characterization of P_{xy} may be strict. We simply ignore this while evaluating $vol\left(P_{xy}\right)$ by considering the closure of P_{xy} obtained from the characterization of P_{xy} by turning each strict inequality ($<$) to its larger form ($\leq$); by doing so, we simply move from P_{xy} to its closure without changing the volume. Taking into account that T_a, T_b and T_c are disjoint sets of voting situations, and since by symmetries, all the six possible T_{xy} generates polytopes of equal volume, the limit probability $P\,(F_\lambda, TP, IAC)$ under the IAC assumption, of observing a voting situation in which the truncation paradox may occur is

$$P_{\mathrm{TP}}(F) = \frac{vol(P_a) + vol(P_b) + vol(P_c)}{vol(P)} = 720 vol\,(P_{ab}) - 360 vol\,(P_{ab} \cap P_{ac})$$

where P is the simplex $P = \{(p_1, p_2, \ldots, p_6) : \sum_{t=1}^{6} p_j = 1 \text{ with } p_j \geq 0, j = 1, 2, \ldots, 6\}$. Given $0 < \lambda \leq 1$, computing $vol\,(P_{ab})$ and $vol\,(P_{ab} \cap P_{ac})$, one obtains the result of Proposition 1. All volume computations performed in this paper use the same technique as in Cervone et al. (2005).[4] Roughly, one needs for example to determine all vertices of the given polytope and then triangulate the set of those vertices into simplices. More details are presented in Moyouwou and Tchantcho (2015) and Gehrlein and Lepelley (2011); further illustrations are available in Gehrlein et

[4]This technique has recently been used in many research papers, such as Diss and Gehrlein (2015; 2012), Gehrlein et al. (2015), Moyouwou and Tchantcho (2015), Kamwa and Valognes (2017), Kamwa et al. (2018) and Kamwa (2019) among others.

al. (2015) or more recently in Lepelley et al. (2018). A Maple procedure is also available from authors upon request. Of course, there is an abundant literature on volume computations with very efficient algorithms and packages such as Büeler et al. (2000) and Lawrence (1991) for Maple users or Bruns and Ichim (2010) and Bruns et al. (2018, 2019).

C. Proof of Proposition

Consider a voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ on $A = \{a, b, c\}$ and the runoff rule associated with $0 < \lambda \leq 1$. Assume that x is the winner, y is the challenger and z is the first-round loser.

1. *Necessity.* First assume that the truncation paradox is liable to occur at π in favor of y. Then by truncating their true preferences, a coalition of voters, say S, diminishes the score of x in such a way that x is now ruled out in the first round and y wins against z at the second round. Each voter in S strictly prefers y to x. The truncation operation by such a voter is only intended to diminish the score of x in the first round. Thus the preferences of each voter in S is yxz. In the new voting situation π', y wins. Since from π' to $\pi\left(\left[yxz\right]\right)$, the score of y does not decrease, the score of x does not increase, the score of z is unchanged and the second round duel is not affected by the truncation operation, then y also wins in $\pi\left(\left[yxz\right]\right)$ against z in the second round. *Sufficiency.* Assume that y wins the majority duel against z and x is the first-round loser at $\pi\left(\left[yxz\right]\right)$. Then under the corresponding runoff rule, y wins in $\pi\left(\left[yxz\right]\right)$ against z at the second round. Hence, the truncation paradox occurs.
2. *Necessity.* Assume that the truncation paradox is liable to occur at π in favor of z. By truncating their true preferences, members of some coalition, say S, favor the election of z whom they strictly prefer to x. In the new voting situation π', z wins the majority duel against x or against y. First suppose that z wins in π' against x at the second round. Then y is the first-round loser at π'. Moreover, voters in S all strictly prefer z to x; and the truncation operation is intended, at the first round in π', to diminish the score of y. Thus the preference of each voter in S is zyx. Hence $|S| \leq n_{zyx}(\pi)$. Therefore, in $\pi\left(\left[zyx\right]\right)$, z also wins against x and y is the first round loser. Finally, suppose that z wins in π' against y at the second round. Then x is the first-round loser at π'. Voters in S all strictly prefer z to y; and the truncation operation is intended, at the first round in π', to diminish the score of x. The preference of each voter in S is then zxy. This implies that $|S| \leq n_{zxy}(\pi)$. In $\pi\left(\left[zyx\right]\right)$, z also wins against y and x is the first-round loser. *Sufficiency.* Assume that z wins the majority duel against y and x is the first-round loser in $\pi\left(\left[zxy\right]\right)$. Then under the corresponding runoff rule, z wins in $\pi\left(\left[zxy\right]\right)$ against y at the second round. In the same way, suppose that z wins the majority duel against x and y is the first-round loser in $\pi\left(\left[zyx\right]\right)$. Then under the corresponding runoff rule, z wins in $\pi\left(\left[zyx\right]\right)$ against x at the second round. In both cases, the truncation paradox occurs.

D. Computations Details for Proposition

Given $0 < \lambda \leq 1$, let R_{xy} denote the set of all voting situations in which the truncation paradox is liable to occur in favor of some candidate u under the runoff rule associated with the weight λ while x and y are respectively the election winner and the challenger. Let z be the first-round loser in each voting situation in R_{xy}. Denote by R_{xyy} the subset of R_{xy} that consists of all voting situations in which truncating preferences may favor the election of y; by R_{xyz} the subset of R_{xy} that consists of all voting situations in which truncating preferences may favor the election of z against x at the second round; and by R'_{xyz} the subset of R_{xy} that consists of all voting situations at which truncating preferences may favor the election of z against y at the second round. Then by Proposition 3

$$R_{ab} = R_{abb} \cup R_{abc} \cup R'_{abc}.$$

Note that R_{abb} and R_{abc} are disjoint sets of voting situations since y wins the majority duel against z in each voting situation in R_{abb} while the converse holds in each voting situation in R_{abc}. Therefore

$$|R_{ab}| = |R_{abb}| + |R_{abc}| + \left|R'_{abc}\right| - \left|R_{abb} \cap R'_{abc}\right| - \left|R_{abc} \cap R'_{abc}\right|.$$

Note that by Proposition 3, R_{abb}, R_{abc} and R'_{abc} are each defined by some set of linear constraints. Therefore the probability that the corresponding runoff rule exhibits the truncation paradox is derived by computing the volume of the polytopes P_{abb}, P_{abc} and P'_{abc} associated to R_{abb}, R_{abc} and R'_{abc} respectively. More precisely, by considering the six possible sets R_{xy} for all two ordered pairs (x, y) from $\{a, b, c\}$ and taking into account possible symmetries, the limit probability $P\left(F_\lambda, TP, IAC\right)$, under the IAC assumption, of observing a voting situation with three candidates in which the truncation paradox may occur is

$$P_{\mathrm{TP}}(F') = 720\left[vol\left(P_{abb}\right) + vol\left(P_{abc}\right) + vol\left(P'_{abc}\right) - vol\left(P_{abb} \cap P'_{abc}\right) - vol\left(P_{abc} \cap P'_{abc}\right)\right]$$

E. Proof of Proposition

Assume that preferences are single-peaked in such a way that candidate c is bottom-ranked by no voters. When candidate a is elected, voters who prefer b to a are of type 4 or type 6. But these voters do not affect the score of candidate a by a sincere truncation of their preferences. Thus candidate b cannot be elected by sincere truncation of preferences. Similarly, when candidate b is elected, there is no way for voters who strictly prefer a to b to favor the election of a by simply truncating their rankings. Therefore, the truncation paradox may only occur in favor of c when a (or b) is the winner of a one-shot scoring rule and preferences are single-peaked.

F. Proof of Proposition

(i) Let us assume that candidate a wins versus candidate b [5] with $\lambda \in [\frac{1}{2}, 1]$. Note we should have $S(\pi, \lambda, c) \leq S(\pi, \lambda, a)$ and $S(\pi, \lambda, c) \leq S(\pi, \lambda, b)$. It follows that

$$S(\pi, \lambda, c) - \frac{S(\pi, \lambda, a) + S(\pi, \lambda, a)}{2} = \frac{2-\lambda}{2}(x_5 + x_6) + \frac{2\lambda - 1}{2}(x_4 + x_2) \leq 0.$$

This occurs if and only if $\lambda = \frac{1}{2}$ and $x_5 = x_6 = 0$. In this case, the three candidates all tie and there is no route for the profitable truncation of preferences.

Now suppose that $\lambda \in]0, \frac{1}{2}[$ and show that the paradox can occur for all λ in this interval. Let us assume a voting situation where $n_2 = n_5 = \alpha$ and $n_3 = n_4 + 1 = z$ with $z = \lceil \frac{2}{\lambda} - 3 \rceil + 1$ and $\alpha = \lfloor \frac{2z-1}{1-2\lambda} \rfloor - 1$ for $\lambda \in]0, \frac{1}{2}[$. The scores are: $S(\pi, \lambda, a) = \alpha + \lambda z$, $S(\pi, \lambda, b) = \alpha + \lambda(z-1)$ and $S(\pi, \lambda, c) = 2z - 1 + 2\lambda\alpha$. It follows that $S(\pi, \lambda, a) > S(\pi, \lambda, b)$, $S(\pi, \lambda, b) > S(\pi, \lambda, c)$. Candidate c is eliminated and candidate a wins the second run since $aM(\pi)b$. Assume that all the voters of type 3 truncate. The new scores are: $S(\pi', \lambda, a) = \alpha$, $S(\pi', \lambda, b) = S(\pi, \lambda, b)$ and $S(\pi', \lambda, c) = S(\pi, \lambda, c)$. We still have $S(\pi, \lambda, b) > S(\pi, \lambda, c)$. Let us show that $S(\pi, \lambda, c) - S(\pi', \lambda, a)$.

$$\begin{aligned} S(\pi, \lambda, c) - S(\pi', \lambda, a) &= 2z - 1 - (1-2\lambda)x \\ &= 2z - 1 - (1-2\lambda)\left(\left\lfloor \frac{2z-1}{1-2\lambda} \right\rfloor - 1\right) \\ &= 2z - 1 - (1-2\lambda)\left\lfloor \frac{2z-1}{1-2\lambda} \right\rfloor + (1-2\lambda) \end{aligned}$$

For all $\lambda \in]0, \frac{1}{2}[$, we have $z > 1$; so, $2z > 1$ and $2z - 1 > 0$. Also, $(1-2\lambda) > 0$ and we know that $(1-2\lambda)\lfloor \frac{2z-1}{1-2\lambda} \rfloor \leq 2z - 1$. Thus, $S(\pi, \lambda, c) - S(\pi', \lambda, a) > 0$. So, candidate a is eliminated. Since $x + 2z - 1 > \alpha$, $cM(\pi')b$: candidate c is the new winner. Thus, by sincere truncation of their rankings, voters of type 3 have favored their best candidate.

(ii) Assume that candidate a wins the second stage versus c.[6] This means that aMc through $x_2 \geq x_4 + x_5 + x_6$ and thus $x_2 \geq \frac{1}{2}$. Note that only voters of type 5 have an incentive to manipulate and the possibility to affect the score of a by sincere truncation of their rankings in favor of c. But by any truncation from π to a new

[5] The symmetric to the case "candidate b wins the second stage versus a" is handled in a similar way.

[6] This is symmetric to the case "candidate b wins the second stage versus c".

voting situation π', we still have $S(\pi', \lambda, a) \geq \frac{1}{2}$ and $S(\pi, \lambda, b) = x_4 + \lambda x_6 \leq x_4 + x_5 + x_6 \leq \frac{1}{2}$. Therefore candidate c still gets through to the second round and wins the election against c.

(iii) Let us assume that candidate c wins the second stage versus a. Let us first consider $\lambda = 1$. Evidently, candidate c wins. If she wins versus candidate a, this means that $S(\pi, \lambda, a) > S(\pi, \lambda, b)$ which is equivalent to (i) $n_2 + n_5 > n_4 + n_6$. Candidate b will become the new winner after voters of type 4 truncate if (ii) $S(\pi, \lambda, b) > S(\pi', \lambda, c)$; and that (iii) $bM(\pi')a$. This last requirement is equivalent to $n_4 + n_6 > n_2 + n_5$ which contradicts (i): voters of type 4 cannot manipulate for $\lambda = 1$. If voters of type 2 truncate, nothing will happen since the new score of candidate c, although diminished by λn_2, will still be greater than that of candidate b: voters of type 2 cannot manipulate for $\lambda = 1$. Thus, the truncation paradox is not possible for $\lambda = 1$. To prove that it can happen for $\lambda \in]0\ 1[$, one can consider a profile such that $n_2 = 3$, $n_4 = 2$ $n_5 = 1$ and $n_6 = 1$.

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Game Theory

Dummy Players and the Quota in Weighted Voting Games: Some Further Results

Fabrice Barthélémy and Mathieu Martin

1 Introduction

In this chapter, we compute the probability of having a dummy player in a weighted voting game. A dummy player is a player who has no power or no influence in a collective decision. The most famous example is probably the well-known case of Luxembourg in the Council of Ministers of the EU between 1958 and 1973. It proves that it is possible to have a weight different from 0 (Luxembourg had one vote) and absolutely no power in the decisions. Of course, such decisions are undesirable but we should not worry about them if it could be shown that their occurrence is rare. In the classical case of majority games, Barthélémy et al. (2013) show that the probability of having a dummy player is unfortunately far from 0. To illustrate, this probability can reach about 50% for 4, 5 or 6 players; for more than 6 players, the probability decreases but we have to consider more than 15 players for obtaining results lower than 1%. Do these negative results hold when other quotas are considered?

Our goal is to determine the quota values which minimize this probability and then to answer to the previous question.

We know that under unanimity rule, each player has a veto power and hence is not a dummy. Can we find other quotas for which the risk of having a dummy player is zero? We give a negative answer to this question. Consequently, the major part of this study is devoted to the determination of quota values that minimize (or maximize) the probability of having a dummy player. To the best of our knowledge, this issue has never been tackled in the literature, with the exception of Barthélémy et al. (2019),

F. Barthélémy (✉)
Department of Social Sciences, Université de Versailles Saint-Quentin en Yvelines, CEMOTEV, Guyancourt Cedex 78047, France
e-mail: fabrice.barthelemy@uvsq.fr

M. Martin
Department of Economics, CY Cergy Paris Université THEMA, Cergy-Pontoise Cedex 95011, France
e-mail: mathieu.martin@u-cergy.fr

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a companion paper. The results we obtain in the second part of the current paper complete their results, where some analytical results are given for 3, 4 and 5 players. Some general formulas are proposed typically in the spirit of a huge literature on this subject (see, for example Gerhlein and Lepelley 2017). Even if this approach is the most interesting, it is limited because the calculus are too complicated when the number of players increases. Moreover, for 4 and 5 players, contrary to the case with only 3 players, only limiting representations are obtained, when the sum of the weights (the number of seats for example) tends to infinity. It is why we proceed here by computer enumeration or simulations, in function of the size of the variables.

The chapter is organized as follows: in the Sect. 2, our notation, definitions and assumptions are introduced. In the Sect. 3, we present general preliminary results which show that, from a theoretical point of view, avoiding a dummy player seems to be complicated. Of course, it does not mean that the probability of such an event is high, it only means that it is very easy to find a game such that there is a dummy player. Some numerical results for the probability of having at least one dummy player are then obtained in Sect. 4. The main conclusions of our study are summarized and discussed in the Sect. 5.

2 Notation, Definitions and Assumptions

The formal framework of this study is the same as in Barthélémy et al. (2019). The reader is referred to this companion paper for further (but here unnecessary) details on this framework.

A voting game is a pair (N, W) where $N = \{1, 2, \ldots, n\}$ is the set of n players (or voters) and W the set of winning coalitions, that is the set of groups of players which can enforce their decision. We consider the class of weighted voting games $[q; w_1, w_2, \ldots, w_n]$, where q is the quota needed to form a winning coalition and w_i is the number of votes (weight) of the ith player; we assume that q and w_i are integers. A coalition S is winning if and only if $\sum_{i \in S} w_i \geq q$. The total number of votes, $\sum_{i \in N} w_i$, is denoted by w. A particular case is the majority game where $q_{maj} = \frac{w}{2} + 1$ if w is even and $q_{maj} = \frac{w+1}{2}$ if w is odd. We assume that the game is proper, that is $q \geq q_{maj}$. When $q = w$, we get the unanimity rule: each player has a veto power and is not a dummy. Our study will focus on the weighted voting games such that $q \leq w - 1$. The relative quota, denoted Q with $Q = q/w$, is used in the tables. We assume, without loss of generality, that $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$.

A player i is a *dummy* player in a voting game (N, W) if $S \in W$ implies $S \backslash \{i\} \in W$ for every $S \in W$. In words, player i is never decisive in every winning coalition: the coalition wins with or without him (her). To illustrate, player 3 is a dummy in the weighted voting game $[5; 3, 2, 1]$. In voting power theory (see Straffin 1994; Felsenthal and Machover 1998 for a presentation), it means that this player has no power.

A weighted voting game is said to be *admissible* if each player has at least one vote ($w_n \geq 1$) and never more than $q - 1$ votes (there is no dictator). Of course, if player

j is a dummy player, then player k with $k > j$ is also a dummy. Also notice that, as $w_1 \le q - 1$ in an admissible weighted voting game, player 2 is never a dummy and, consequently, the maximum number of dummy players is equal to $n - 2$.[1]

The purpose of this paper is to compute the probability of obtaining a dummy player, given n, w and q (or Q), and to derive the quota which minimizes this probability (denoted by $\underline{Q}$) and the quota which maximizes this probability (denoted by $\overline{Q}$). The probability of having at least one dummy player is denoted by $P(w, n, q)$ or $P(w, n, Q)$ when w is finite and $P(n, Q)$ when w is infinite.

In order to compute $P(w, n, q)$ (or $P(w, n, Q)$ or $P(n, Q)$), we consider a particular probabilistic model called IAC (Impartial Anonymous Culture) which is one of the most often used in such problems where the likelihood of a voting event is to be calculated (see, for instance, Lepelley et al. 2008, Diss et al. 2012, or Courtin et al. 2014).

In the current context, using this model is tantamount to assume that, n, w and q being given, all the admissible distributions of the w_i's, i.e. all the distributions such that ($q - 1 \ge w_1 \ge w_2 \ge \cdots \ge w_n \ge 1$ and $\sum_{i \in N} w_i = w$) are equally likely to occur.

3 Preliminary Results

We present preliminary results concerning the general possibility of obtaining a dummy player in a weighted voting game. We know that there is no dummy player when $q = w$. We also know from Leech (2002) (see also Barthélémy et al. 2013) that, in the three-player case, there is no dummy player when $q = q_{maj}$. Can we find some other values of q and n (the number of players) for which the "dummy paradox" never occurs? The following propositions give a negative answer to this question (as soon as w, the total number of votes, is not very small).

Proposition 1 *For all n, q and w such that $n \ge 4$, $q_{maj} \le q \le w - 1$ and $w \ge 6n - 12$, there exists a game $[q; w_1, w_2, \ldots, w_n]$ where player n is a dummy player.*

This result is a consequence of the four following lemmas:

Lemma 1 *For all n and w such that $n \ge 4$, $w \ge 6n - 12$ and $q = \frac{w}{2} + 1$, there exists a game $[\frac{w}{2} + 1; w_1, w_2, \ldots, w_n]$ where player n is a dummy.*

Proof of Lemma 1 Consider the following game: $[\frac{w}{2} + 1; \frac{w}{2} - 2, \frac{w-6n+26+a}{4}, \frac{w-6n+26-a}{4}, 3, 3, \ldots, 3, 1]$ with $a = 0$ if n is odd (even) and $\frac{w}{2}$ is even (odd) and $a = 2$ if n is odd (even) and $\frac{w}{2}$ is odd (even). Furthermore, the number of individuals whose weight is 3 is equal to $n - 4$.

[1] In the particular case where $q = q_{maj}$, not only player 2 but also player 3 cannot be a dummy; see Proposition 1 in Barthélémy et al. (2013). In this case, the maximum number of dummy players is $n - 3$.

We have to show that (1) $q > w_1 \geq w_2 \geq \cdots \geq w_n$ and, since $w_n = 1$, we have to show that (2) there is no coalition $S \subseteq N \backslash \{n\}$ such that $\sum_{i \in S} w_i = q - 1$. Indeed, this is the only situation for which player n, with one vote, is not a dummy player.

(1) Clearly $q > w_1$ and $w_2 \geq w_3$, thus we have just to show that $w_1 \geq w_2$ and $w_3 \geq 3$ (if $n = 4$ we have to show that $w_3 \geq 1$ which is obvous). Assume that $w_2 > w_1$ then we obtain $w < 34 - 6n + a$. By hypothesis, we have $w \geq 6n - 12$ thus $w_2 > w_1$ if $34 - 6n + a > 6n - 12$ or $n < \frac{46+a}{12} \leq 4$, a contradiction of $n \geq 4$. QED Assume now that $w_3 < 3$, it means that $\frac{w-6n+26-a}{4} < 3$ or $w < 6n - 14 + a$. By hypothesis, $w \geq 6n - 12$ thus $6n - 14 + a > w \geq 6n - 12$ and then $a > 2$ which is impossible. Thus $w_3 \geq 3$.

(2) Consider a coalition $S \subseteq N \backslash \{n\}$ with $1 \in S$. Since $w_3 \geq 3$ we have $\sum_{i \in S} w_i \geq q$ except if $S = \{1\}$. Since $w_1 = q - 3$, it is never possible to obtain a total weight equal to $q - 1$ with player 1 belonging to S. Assume now that player 1 does not belong to S: we have $\sum_{i=2}^{n-1} w_i = q$. Since $w_i \geq 3$, $i = 2, 3, \ldots, n - 1$, it is not possible to obtain a total weight equal to $q - 1$ in S. QED

Lemma 2 *For all n, q and w such that* $n \geq 4$, $w > 4n - 8$, $\frac{w}{2} + 1 < q \leq w - 2n + 5$, *there exists a game* $[q; w_1, w_2, \ldots, w_n]$ *where player n is a dummy.*

Proof of Lemma 2 Consider the following game: $[q; q - 2, w - q - 2n + 7, 2, 2, \ldots, 2, 1]$. We have to show that (1) $q > w_1 \geq w_2 \geq \cdots \geq w_n$ and, since $w_n = 1$, we have to show that (2) there is no coalition $S \subseteq N \backslash \{n\}$ such that $\sum_{i \in S} w_i = q - 1$.

(1) We have just to show that $w_1 \geq w_2$ and $w_2 \geq 2$. Assume that $w_2 > w_1$. Thus $w - q - 2n + 7 > q - 2$ or $n < \frac{w-2q+9}{2}$. Since $n \geq 4$,we have $4 \leq n < \frac{w-2q+9}{2}$ and then $q \leq \frac{w+1}{2}$, a contradiction. QED Furthermore, $w_2 \geq 2$ if $w - q - 2n + 7 \geq 2$ or $q \leq w - 2n + 5$, which is true by hypothesis.

(2) Consider a coalition $S \subseteq N \backslash \{n\}$ with $1 \in S$. Since $w_2 \geq 2$ we have $\sum_{i \in S} w_i \geq q$ except if $S = \{1\}$. Since $w_1 = q - 2$, it is never possible to obtain a total weight equal to $q - 1$ with player 1 belonging to S. Assume now that $1 \notin S$, thus we have $\sum_{i=2}^{n-1} w_i = w - q + 1$. But $w - q + 1 = q - 1$ implies $q = \frac{w}{2} + 1$, a contradiction. Therefore there is no $\bar{S} \subset S$ such that $\sum_{i \in \bar{S}} w_i = q - 1$. QED

Remark that the condition $w > 4n - 8$ guarantees that $w - 2n + 5 > \frac{w}{2} + 1$.

Lemma 3 *For all n, q and w such that* $n \geq 4$, $w \geq 5n - 12$ *and* $w - 2n + 6 \leq q \leq w - n + 2$, *there exists a game* $[q; w_1, w_2, \ldots, w_n]$ *where player n is a dummy.*

Proof of Lemma 3 Consider the following game: $[q; q - n + 1, w - q + 1, 1, \ldots, 1]$. We have to show that (1) $q > w_1 \geq w_2 \geq \cdots \geq w_n$ and, since $w_n = 1$, we have to show that (2) there is no coalition $S \subseteq N \backslash \{n\}$ such that $\sum_{i \in S} w_i = q - 1$.

(1) We have just to show that $w_1 \geq w_2$ and $w_2 \geq 1$. Assume that $w_2 > w_1$, thus $q - n + 1 < w - q + 1$ or $q < \frac{w+n}{2}$. We know that $q \geq w - 2n + 6$ thus $\frac{w+n}{2} > q \geq w - 2n + 6$, that is to say $w < 5n - 12$, a contradiction. It is obvious that $w_2 \geq 1$. QED

(2) Remark that $w_1 + w_3 + w_4 + \cdots + w_{n-1} = q - 2$. Thus players 1 and 2 must belong to the coalition $S \subseteq N \backslash \{n\}$ if we want $\sum_{i \in S} w_i \geq q - 1$. We have $w_1 + w_2 = w - n + 2$ and by hypothesis, $q \leq w - n + 2$, thus $w_1 + w_2 \geq q$. It is not possible to obtain a total weight equal to $q - 1$. QED

Lemma 4 *For all n, q and w such that* $n \geq 4$, $w \geq 4n - 9$ *and* $w - n + 3 \leq q \leq w - 1$, *there exists a game* $[q; w_1, w_2, \ldots, w_n]$ *where player n is a dummy.*

Proof of Lemma 4 Consider the game $[q; q - (n - w + q - 1)(w - q + 1), w - q + 1, ..., w - q + 1, 1 \ldots, 1]$, knowing that the number of players with a weight equal to $w - q + 1$ is equal to $n - w + q - 1$ and the number of players with a weight equal to 1 is equal to $w - q$. We have to show that (1) $q > w_1 \geq w_2 \geq \cdots \geq w_n$ and, since $w_n = 1$, we have to show that (2) there is no coalition $S \subseteq N \backslash \{n\}$ such that $\sum_{i \in S} w_i = q - 1$.

(1) Assume $w_1 \geq q$. Thus we have $q - (n - w + q - 1)(w - q + 1) \geq q$ or $(n - w + q - 1)(w - q + 1) \leq 0$ which is impossible since $(n - w + q - 1) > 0$ and $(w - q + 1) > 0$. QED Let us show now that $w_1 \geq w_2$. w_2 is maximized when q is minimized, that is to say when $q = w - n + 3$. Thus $\bar{w}_2 = n - 2$ ($\bar{w}_2$ is the highest value of w_2). We then obtain the lowest value of w_1, denoted $\underline{w}_1$ such that $\underline{w}_1 = w - 3n + 7$ (q is replaced by $w - n + 3$ in w_1). We have always $w_1 \geq w_2$ if $\underline{w}_1 \geq \bar{w}_2$ that is to say $w - 3n + 7 \geq n - 2$ or $w \geq 4n - 9$ which is true by hypothesis.

(2) Assume that there exists a coalition S such that $\sum_{i \in S} w_i = q - 1$ knowing that player n does not belong to S. Let $S_1 = \{1, 2, \ldots, n - w + q\}$ be a coalition in which any player i is such that $w_i \neq 1$. We have $\sum_{i \in S_1} w_i = q \neq q - 1$ thus $S_1 \neq S$. It means that player 1 or player 2 (w.l.o.g) does not belong to S. Let $S_2 = \{1, 3, \ldots, n - 1\}$ be the coalition with all the players except 2 and n. We have $\sum_{i \in S_2} w_i = w - (w - q + 1) - 1$ or $\sum_{i \in S_2} w_i = q - 2 \neq q - 1$. Thus $S_2 \neq S$ which implies that 2 belongs to S. Therefore any i such that $w_i = w - q + 1$ belongs to S and since $w_1 \geq w_2$, 1 belongs to S as well. It means that $\sum_{i \in S} w_i \geq q$, a contradiction. QED

Proof of proposition 1 The different constraints given by the Lemmas 1–4 are the following: $w \geq 6n - 12$ and $q = \frac{w}{2} + 1$ (Lemma 1), $w > 4n - 8$ and $\frac{w}{2} + 1 < q \leq w - 2n + 5$ (Lemma 2), $w \geq 5n - 12$ and $w - 2n + 6 \leq q \leq w - n + 2$ (lemma 3) and $w \geq 4n - 9$ and $w - n + 3 \leq q \leq w - 1$ (Lemma 4). It is easy to verify that all possible quotas for admissible games (from $\frac{w}{2} + 1$ to $w - 1$) are considered. Furthermore, a very simple calculus shows that the more restrictive constraint is $w \geq 6n - 12$. QED

Proposition 1 only deals with at least 4 players. Proposition 2 presents a similar result for the 3-player case for $q > q_{maj}$.

Proposition 2 *In the three-player case, there exists a game* $[q; w_1, w_2, w_3]$ *such that player* 3 *is a dummy for each value of q,* $q \leq w - 1$ *and* $q \neq q_{maj}$.

Proof of proposition 2 Consider the game $[q; q-2, w-q+1, 1]$. We have to show that (1) $w_1 \geq w_2 \geq w_3$ and (2) player 3 is a dummy player *i.e.* $w_1 + w_2 \neq q - 1$.

(1) $w_2 \geq w_3$ if $w - q + 1 \geq 1$ or $w \geq q$ which is always true. Furthermore, $w_1 \geq w_2$ if $q - 2 \geq w - q + 1$ or $q \geq \frac{w+3}{2}$ which is always true since $q \neq q_{maj}$. QED

(2) We have $w_1 + w_2 = w - 1$ which is different from $q - 1$ since $q < w$. QED

4 Numerical Results

From Barthélémy et al. (2019), we know that, in weighed voting games with 3, 4 or 5 players, the probability of having a dummy player is very sensitive to the choice of the quota and can be very high; in addition, it turns out that, except for quotas very close to 1, increasing the quota does not decrease the probability of dummy players. To what extent can we generalize these conclusions to voting games with a larger number of players? It is the question we investigate in this section.

Our results, presented in tables and graphs, are obtained with one of the two following approaches. The first one gives exact results whereas the second one is based on simulations and provides estimated probabilities. Note that for small numbers of players, these results corroborate what we have obtained analytically in Barthélémy et al. (2019).

Exact computations are done by considering the exhaustive list of all possible vectors of weights for a given number w of votes. For all these vectors $(w_1, \ldots, w_n)$, we check whether or not the last player is decisive (remember that $w_1 \geq w_2 \geq \ldots \geq w_n$). To do this, we use the classical Banzhaf power index[2] since a player, by construction, is a dummy if his (her) index is equal to zero. We compute this index using a generating functions approach which leads to exact values (this point is fundamental because we are looking for an index with a zero value, which prohibits the use of approximation methods). Finally, the exact probability of having at least one dummy player is the ratio between the number of times the last player is never decisive and the number of vectors $(w_1, \ldots, w_n)$ considered as admissible (in accordance with the assumed uniform distribution of weight vectors). Unfortunately, enumerating all these distributions is highly time consuming when the number of players and w become large (see, for example, Barthélémy et al. (2011)). It is the reason why we also resort to simulations.

Our simulations are based on random vectors of weights. The estimated probability of having at least one dummy player is then obtained by dividing the number of times the last player is never decisive by the number of vectors $(w_1, \ldots, w_n)$ randomly generated. In order to simulate the probability of a dummy player, two steps have to be considered. First, we have to simulate a vector of weights for a given w and a given number of players n. This can be done by using for instance the `Rancom`

[2] For a clear and simple presentation, see Straffin (1994).

algorithm proposed by Nijenhuis and Wilf (1978). Second, we have to check whether there is at least one dummy player in the weighted game associated to these weights. This is done as mentioned above by using the Banzhaf power index. Then repeating these two steps k times gives the estimated probability which corresponds to the proportion of weighted games leading to a dummy player.

We analyze first the results for a finite total number of votes w. In a second step, we will extend our study to the case where this total number of votes tends to infinity.

4.1 Finite Case

We compute both exact and simulated probabilities. Tables 1, 2 and 3 give the probability of having at least one dummy player according to given values of the quota Q, for weighted games with 3, 5 and 10 players.[3]

The probabilities $P(n, w, Q)$ are not monotonic with respect to parameters Q, w and n. However, we observe that the probability tends to 0 when n increases, which is a particular illustration of the so-called Penrose's law (or Penrose's limit theorem). Penrose (1946, 1952) argues (without rigorous proof) that, if the number of players is large while the quota is fixed at half of the total weight, then the ratio between the voting powers of any two players, measured by their Banzhaf index, tends to the ratio between their weights. Lindner and Machover (2003) have shown that, if the Penrose's law is not always true, "experience suggests that counter-examples are atypical" and they conjecture that the theorem holds under rather general conditions, for large classes of weighted voting games, other values of the quota and other measures of voting power. Using simulations, Chang et al. (2006) conclude that if the result holds only for a quota of 50% when the Banzhaf index is considered, the Penrose's law remains valid for all values of the quota when power is measured by the Shapley-Shubik index.[4] Let us notice that a dummy player with the Banzhaf index is a dummy player with the Shapley-Shubik index as well (and reciprocally). As there are no weights equal to zero by construction in our study, each player tends to get a positive power when n tends to infinity and the probability of having a dummy player tends to zero. Figure 1 illustrates this result for $w = 60$ and $Q = 2/3, 0.75, 0.90, 0.95$.[5]

Tables 1, 2 and 3 report as well the optimal probabilities $P(n, w, \underline{Q})$, $P(n, w, \overline{Q})$ (denoted $P_{\min}$ and $P_{\max}$) and the corresponding quotas $\underline{Q}$ and $\overline{Q}$. For a given w, we compute all the quotas Q running from majority to unanimity. More precisely, we consider all the quotas from $Q = 0.50$ (corresponding to either $q = w/2 + 1$ or $q = (w + 1)/2$), to $Q = (w - 1)/w$ (the closest quota to unanimity, $q = w - 1$).

[3]This is an arbitrary choice. Any number of players, reasonably large, can be studied.

[4]See Straffin (1994) for a presentation of this power index.

[5]Obviously, other values of w or Q lead to the same kind of curves.

Table 1 Probability of having at least one dummy for 3 players (in%)

	Value of Q												
w	0.5	2/3	0.75	0.8	5/6	0.9	0.95	0.98	0.99	$P_{\min}$	$\underline{Q}$	$P_{\max}$	$\overline{Q}$
10	0	33.33	57.14	57.14	37.50	37.50	–	–	–	0	(0.50)	57.14	(0.8000)
15	0	30.77	58.82	58.82	50.00	31.58	–	–	–	0	(0.50)	58.82	(0.8000)
20	0	50.00	59.26	62.07	54.84	43.75	24.24	–	–	0	(0.50)	62.07	(0.8000)
25	0	44.44	62.79	63.04	58.33	37.25	21.15	–	–	0	(0.50)	63.04	(0.8000)
30	0	40.00	63.49	63.64	59.42	43.84	17.33	–	–	0	(0.50)	63.64	(0.8000)
35	0	50.00	65.12	64.44	56.25	40.00	15.69	–	–	0	(0.50)	65.12	(0.7714)
40	0	46.15	64.81	64.96	58.06	44.96	25.76	–	–	0	(0.50)	65.49	(0.7750)
45	0	43.36	65.47	65.10	59.24	41.21	23.21	–	–	0	(0.50)	65.97	(0.7778)
50	0	50.00	66.28	65.57	60.42	45.05	21.26	11.06	–	0	(0.50)	66.29	(0.7800)
55	0	47.37	66.67	65.77	61.21	42.28	19.52	10.32	–	0	(0.50)	66.67	(0.7636)
60	0	45.00	65.98	65.91	61.82	45.36	25.84	9.333	–	0	(0.50)	66.93	(0.7667)
65	0	50.00	66.67	66.13	59.63	42.86	24.29	8.807	–	0	(0.50)	67.23	(0.7692)
70	0	47.83	66.96	66.30	60.32	45.45	22.66	8.088	–	0	(0.50)	67.44	(0.7714)
75	0	46.01	67.27	66.34	60.97	43.33	21.41	7.676	–	0	(0.50)	67.51	(0.7733)
80	0	50.00	66.97	66.52	61.51	45.65	26.09	7.129	–	0	(0.50)	67.70	(0.7750)
85	0	48.28	67.28	66.60	62.03	43.69	24.75	6.811	–	0	(0.50)	67.77	(0.7765)
90	0	46.67	67.51	66.67	62.36	45.65	23.55	6.370	–	0	(0.50)	67.83	(0.7778)
95	0	50.00	67.74	66.77	60.92	43.99	22.46	6.117	–	0	(0.50)	67.99	(0.7684)
100	0	48.48	67.36	66.85	61.38	45.79	26.12	11.30	5.762	0	(0.50)	68.05	(0.7700)

Table 2 Probability of having at least one dummy for 5 players (in%)

	Value of Q												
w	0.5	2/3	0.75	0.8	5/6	0.9	0.95	0.98	0.99	$P_{\min}$	$\underline{Q}$	$P_{\max}$	$\overline{Q}$
10	0	0	0	0	14.29	14.29	–	–	–	0	(0.5000)	14.29	(0.9000)
15	26.09	21.43	26.67	26.67	23.33	30.00	–	–	–	7.69	(0.6000)	30.00	(0.9333)
20	10.61	25.00	29.27	26.51	33.33	38.10	32.14	–	–	10.61	(0.5000)	38.10	(0.9000)
25	33.09	27.22	31.38	35.26	35.60	44.27	33.33	–	–	22.42	(0.5600)	44.27	(0.9200)
30	20.14	30.29	36.76	36.46	39.20	46.95	31.83	–	–	20.14	(0.5000)	46.95	(0.9000)
35	40.42	36.95	38.07	40.18	44.49	50.15	30.56	–	–	19.46	(0.5429)	50.15	(0.9186)
40	25.61	35.92	39.43	40.80	44.82	51.44	46.28	–	–	23.34	(0.5500)	51.44	(0.9000)
45	42.13	36.72	41.90	43.26	46.15	53.55	44.99	–	–	24.09	(0.5333)	53.55	(0.9111)
50	30.20	39.45	42.42	43.58	46.29	53.81	43.51	25.74	0	24.76	(0.5400)	54.67	(0.9200)
55	44.59	38.82	44.13	45.13	47.32	55.35	41.86	24.44	–	24.44	(0.9818)	55.35	(0.9091)
60	32.91	39.09	44.31	45.46	47.43	55.88	49.39	23.10	–	23.10	(0.9833)	57.14	(0.9167)
65	45.52	42.11	44.69	46.51	49.47	57.05	48.26	21.98	–	21.98	(0.9846)	57.05	(0.9077)
70	35.60	41.37	45.72	46.73	49.85	56.99	46.95	20.87	–	20.86	(0.9895)	57.79	(0.9143)
75	46.78	41.28	46.04	47.57	49.91	57.96	45.69	19.92	–	19.92	(0.9867)	58.15	(0.9200)
80	37.23	43.14	46.23	47.71	50.23	58.06	51.86	18.99	–	18.99	(0.9875)	58.53	(0.9125)
85	47.26	42.52	46.95	48.34	50.25	58.85	50.80	18.19	–	18.19	(0.9882)	58.90	(0.9176)
90	38.91	42.34	47.15	48.49	50.56	58.75	49.71	17.41	–	17.41	(0.9889)	59.40	(0.9111)
95	48.06	44.27	47.74	48.98	51.77	59.44	48.60	16.72	–	16.72	(0.9895)	59.73	(0.9158)
100	40.03	43.67	47.73	49.09	51.70	59.42	52.71	29.66	16.05	16.05	(0.9900)	59.89	(0.9100)

Table 3 Probability of having at least one dummy for 10 players (in%)

	Value of Q												
w	0.5	2/3	0.75	0.8	5/6	0.9	0.95	0.98	0.99	$P_{\min}$	$\underline{Q}$	$P_{\max}$	$\overline{Q}$
10	0	0	0	0	0	0	0	–	–	0	(0.5000)	0	(0.5000)
15	0	0	0	0	0	0	0	–	–	0	(0.5000)	0	(0.5000)
20	0	0	0	0	0	0	2.38	–	–	0	(0.5000)	2.38	(0.9500)
25	0	0	0	1.83	0	0	6.71	–	–	0	(0.5000)	6.71	(0.9600)
30	0	0.57	1.13	0	1.70	2.64	10.19	–	–	0	(0.5000)	10.19	(0.9667)
35	2.59	2.96	1.10	2.41	3.16	4.19	13.81	–	–	0.13	(0.6571)	13.81	(0.9714)
40	0.29	1.09	1.00	1.09	0.61	1.64	6.77	–	–	0.20	(0.5550)	16.66	(0.9750)
45	1.85	1.62	2.42	2.26	3.56	2.81	9.97	–	–	0.38	(0.5556)	19.19	(0.9778)
50	0.30	1.19	0.91	1.50	1.95	6.51	13.29	21.20	–	0.30	(0.5000)	21.20	(0.9800)
55	2.95	0.79	2.43	3.06	3.06	8.14	16.46	22.92	–	0.56	(0.5636)	22.92	(0.9818)
60	0.84	1.01	1.71	1.40	2.35	4.17	15.43	24.27	–	0.48	(0.5000)	24.27	(0.9833)
65	2.82	2.04	1.57	3.01	2.95	5.54	18.02	25.38	–	0.72	(0.6308)	25.38	(0.9846)
65*	2.85	1.95	1.79	2.97	2.82	5.87	18.17	25.45	–				
70*	0.67	1.40	2.25	2.52	4.46	9.02	20.74	25.87	–				
75*	3.45	2.07	1.82	2.80	2.55	10.42	23.38	26.79	–				
80*	0.96	1.88	1.76	2.61	4.30	7.74	17.18	27.09	–				
85*	3.45	1.25	2.30	3.42	3.93	8.79	18.94	27.96	–				
90*	0.92	1.44	2.26	2.78	4.34	9.82	21.90	27.66	–				
95*	4.87	2.33	3.30	3.42	5.04	10.68	22.68	28.44	–				
100*	1.01	1.49	2.41	3.17	5.07	9.67	24.83	35.55	27.64				

*Simulated values with 10 000 replications and $w = 9\,999$

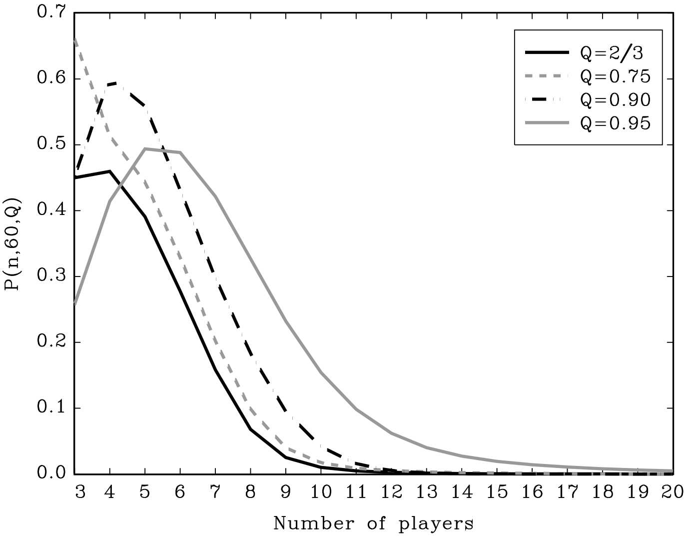

Fig. 1 Illustration of the Penrose's law for $w = 60$

For instance, for $w = 10$, we calculate the probability of having at least one dummy player for $q = 6, 7, 8$ and 9, leading to relative quotas equal to 0.5, 0.6, 0.7, 0.8 and 0.9.[6] In this case, we get unanimity as soon as q is greater than 9 and this implies that for all $Q > 0.9$, $P(10, n, Q) = 0\%$.

Similarly, when considering $w = 20$, the unanimity case is obtained as soon as $Q > 0.95$ (corresponding to $q > 19$). In Tables 1, 2 and 3, the symbol '–' is represented for cases where the relative quota Q is equivalent to the unanimity case ($q = w$). Remark that we do not take into account the unanimity case in the computation of $P_{\min}$ (as mentioned above, unanimity leads to a zero probability).

As the number of different quotas may be high (49 possible quotas for $w = 100$), only selected values of Q are reported in Tables 1, 2 and 3, with $Q \in \{0.5, 2/3, 0.75, 0.8, 5/6, 0.9, 0.95, 0.98, 0.99\}$. But $P_{\min}$ and $P_{\max}$ are computed using all the possible quotas. For instance, with $w = 15$ and $n = 5$, $P_{\min} = 7.69\%$ with a quota $\underline{Q} = 0.60$ (not reported in Table 2). If more than one quota q lead to the smallest probability $P_{\min}$, $\underline{q}$ is the smallest value, and $\underline{Q}$ is the corresponding relative value. For the above example, each $Q \in [0.60, 2/3[$ (corresponding to $[9/15, 10/15[$) leads to the same probability $P_{\min} = 7.69\%$, and our convention gives $\underline{Q} = 0.60$.

Concerning $\underline{Q}$, a strange phenomenon is worth noticing. For small values of w, $\underline{Q}$ is close to the majority and when w increases, $\underline{Q}$ suddenly tends to unanimity. Figure 2 illustrates this phenomenon in the 5-player case. The value $\tilde{w}$ of the total number of votes for which $\underline{Q}$ becomes (almost) the unanimity instead of (almost) the majority increases with n, $n \geq 4$, the number of players. For instance $\tilde{w} = 35$ in the

[6]Note that $Q = 0.5$ and $Q = 0.6$ correspond to the same quota $q = 6$.

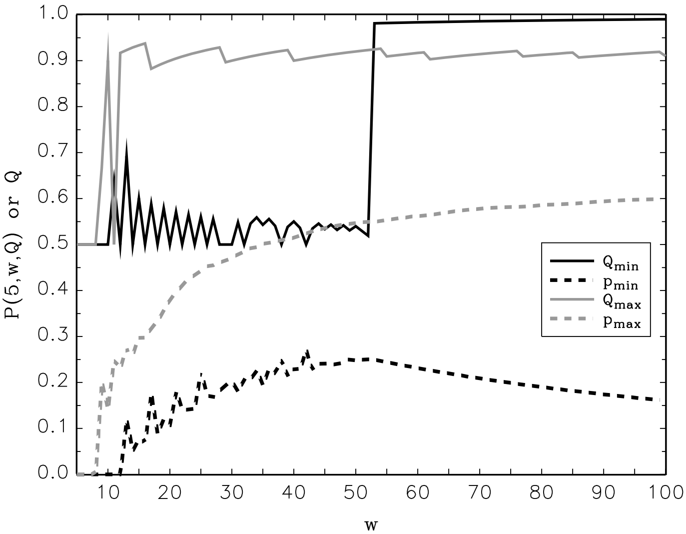

Fig. 2 5-player case, smallest and highest probabilities of a dummy player according to *w*

4-player case, $\tilde{w} = 53$ in the 5-player case, $\tilde{w} = 87$ in the 6-player case and $\tilde{w} > 100$ when $n \geq 6$.

4.2 *Infinite Case*

Limiting probabilities are simulated probabilities computed with high values of the total number of votes w.

Table 4 reports the estimated limiting probabilities from 3 to 15 players. Each row illustrates the Penrose's law: the probabilities tend to decrease when n increases, as previously mentioned for the finite case.[7] Figure 3 is given as an illustration of this remark with four values of Q.

[7]Note however that for high values of Q, the convergence is not clear and more players are needed in order to recover Penrose's law.

Table 4 Simulated limiting probability of having at least one dummy for 3–15 players, $w = 9\,999$, 10 000 simulations (in%)

	Number of players												
Quota Q	3	4	5	6	7	8	9	10	11	12	13	14	15
0.50	0	49.34	53.20	49.54	42.54	34.13	26.16	17.71	11.43	7.00	4.13	1.99	0.68
0.51	0.51	39.91	40.01	34.30	27.27	21.01	15.72	10.15	6.64	3.48	1.95	0.63	0.13
0.52	1.62	35.03	36.05	33.68	28.97	21.42	16.16	10.44	6.35	3.57	1.80	0.59	0.16
0.55	8.48	34.12	41.20	37.04	30.40	22.69	16.72	10.42	6.20	3.45	1.66	0.59	0.18
0.60	23.27	47.16	46.39	39.97	31.96	23.76	16.60	10.89	6.25	3.61	1.87	0.82	0.12
0.65	43.18	51.19	48.62	42.55	33.46	25.04	18.16	11.23	7.17	4.01	2.07	0.77	0.20
2/3	50.08	52.85	49.78	42.22	34.80	25.39	17.86	11.78	6.92	4.01	2.19	0.85	0.25
0.70	62.12	55.67	52.85	44.26	35.14	26.27	19.54	11.86	7.72	4.48	2.15	0.89	0.32
0.75	68.81	56.43	53.86	45.90	37.47	28.71	20.87	13.96	9.27	5.57	3.17	1.60	0.72
0.80	68.03	63.21	55.58	48.67	40.73	31.54	24.35	17.08	11.83	7.60	4.77	2.53	1.17
0.85	68.38	68.27	58.57	50.64	43.83	35.72	28.75	21.83	15.93	11.03	6.98	4.64	2.73
0.90	46 08	65.00	65.59	57.47	49.49	43.55	36.23	29.28	23.35	17.57	13.13	9.92	6.96
0.95	26.58	44.93	58.37	62.52	61.63	55.29	50.30	43.73	38.62	33.45	28.73	23.39	19.79
0.98	11.08	21.56	32.61	42.10	50.58	35.54	58.57	59.28	57.09	53.90	50.32	45.76	42.89
0.99	5.50	11.40	18.54	25.89	33.14	39.61	46.69	51.43	54.52	56.44	58.16	57.86	57.63

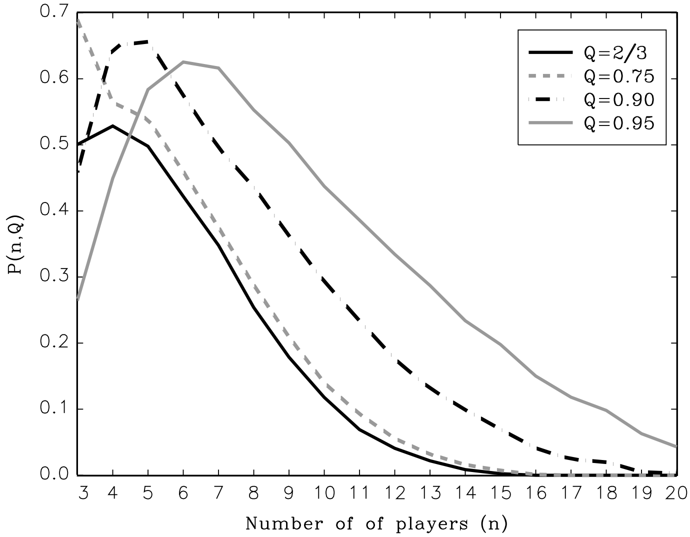

Fig. 3 Penrose's law when w tends to infinity

Figure 4 presents the limiting probabilities plotted for 4, 6, 8, 10 and 15 players as a function of the quota Q. This Figure illustrates that fixing the quota at values higher than 50% (as 2/3 or even 3/4) does not lead to a smaller probability of having at least one dummy player. Moreover, in general, increasing the quota tends to increase this probability, except for values of Q very close to 1. This is particularly clear when the number of players is high.

5 Conclusion and Final Remarks

The main conclusions of our study can be summarized as follows.

(1) Barthélémy et al. (2019) have shown that the probability of having a dummy player can be surprisingly high and is very sensitive to the choice of the quota in voting games with a small number of players. Our results confirm this conclusion for voting games with a larger number of players.
(2) As suggested in Barthélémy et al. (2019), we conclude that the choice of a quota close to 1 (*e.g.* 0.95), that could be suggested by the observation that there is no dummy player for $Q = 1$, would be a serious mistake: for Q close to 1, the risk of a dummy player would be more likely maximized rather than minimized.
(3) In order to minimize the probability of having dummy players, it is advisable to choose a quota between 0.50 and 0.55.

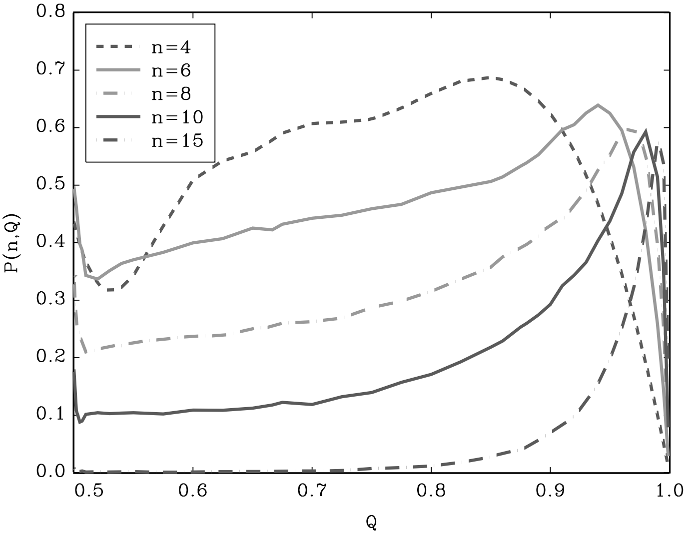

Fig. 4 Limiting probability of having at least one dummy player for 4–15 players

(4) The probability of having a dummy first increases then decreases with the number of players, *whatever the quota value*, suggesting that the Penrose's law holds for $Q > 1/2$. It is worth noticing, however, that this convergence towards 0 is much slower with high values of Q than with small values. To illustrate, with 15 players, the probability of a dummy player is less than 0.25% for $0.5 < Q < 0.66$ and is still about 43% for $Q = 0.98$.

(5) Our conclusions have been obtained by assuming that the weighted voting games are generated via the IAC probabilistic model. Of course, this model does not constitute the only possibility for generating random games. It is therefore logical to ask whether the main phenomena we have observed arise with other distributions of the weighted voting games. To obtain some insights on this issue, we have run some further simulations based on an alternative probabilistic model. With this model, that we call NORM, the weight vectors $(w_1, w_2, \ldots, w_n)$ are generated by picking n numbers in a uniform distribution on [0, 1] and then normalizing these numbers. We then proceed exactly as we did with the IAC model. The results we obtain via this method for the "infinite" case ($w = 9\ 999$)[8] are displayed in Table 5. We obtain the same qualitative conclusions as before, which suggests that our conclusions could be relatively robust.

[8]The results we have obtained for smaller values of w are available from the authors upon request.

Table 5 Simulated limiting probability of having at least one dummy for 3–15 players $w = 9\,999$, 10 000 simulations (in%) (NORM)

	Number of players												
Quota Q	3	4	5	6	7	8	9	10	11	12	13	14	15
0.50	0	61.68	43.47	39.00	28.05	20.70	13.26	8.86	4.92	2.26	1.39	0.46	0.18
0.51	0.28	51.98	33.15	26.00	16.11	11.27	7.37	4.51	2.56	0.92	0.53	0.14	0.03
0.52	1.03	44.54	28.22	23.45	16.82	11.14	7.05	4.67	2.57	1.09	0.50	0.12	0.03
0.55	5.96	33.04	28.36	23.12	17.10	11.09	7.22	4.45	2.47	1.11	0.54	0.23	0.03
0.60	21.61	34.32	35.71	25.35	18.48	12.56	7.84	4.67	2.82	1.23	0.72	0.19	0.02
0.65	47.16	34.79	35.23	26.17	19.22	13.32	8.92	5.64	3.30	1.68	0.82	0.37	0.11
2/3	58.05	38.12	33.69	26.96	18.93	13.37	8.98	6.00	3.51	1.76	0.94	0.35	0.14
0.70	71.97	40.80	32.92	31.20	21.30	15.07	10.24	6.90	4.12	2.18	1.15	0.69	0.11
0.75	70.93	47.69	37.28	29.62	25.32	18.37	12.87	8.97	5.559	3.31	1.90	0.81	0.50
0.80	60.45	65.55	42.43	35.11	27.94	21.81	16.66	12.95	8.259	5.32	3.56	1.97	1.16
0.85	46.35	65.94	59.22	43.09	35.37	29.08	22.32	17.66	13.94	9.38	6.67	4.48	2.77
0.90	30.19	51.98	63.09	60.80	50.87	40.97	34.20	28.63	23.34	18.05	14.07	10.66	7.73
0.95	15.00	28.34	41.28	53.08	59.73	60.33	57.41	52.12	45.30	39.25	34.72	30.36	25.50
0.98	5.91	12.12	18.84	26.44	34.05	41.92	48.39	53.41	56.83	57.32	58.35	56.32	52.82
0.99	2.95	5.95	9.49	13.90	18.60	23.73	29.39	35.07	40.21	43.75	48.29	50.75	54.27

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Who Wins and Loses Under Approval Voting? An Analysis of Large Elections

Sébastien Courtin and Matías Núñez

1 Introduction

Approval Voting (AV) is the method of election according to which a voter can vote for as many candidates as he wishes, the elected candidate(s) being the one(s) who receive(s) the most votes. This simple voting rule has attracted interest from scholars in political science and economics (see Laslier 2010) due to its flexibility: voters approve of each candidate independently of the rest of the candidates. As far as preference aggregation is concerned (on which we focus), one of the main features of equilibria is presented in Laslier (2009). It provides a strong argument for the use of AV in a model of large elections: AV selects the Condorcet Winner (CW) as long as the voters expect that no pair of candidates gets exactly the same number of votes.

Our main contribution is to fully characterize the set of equilibrium winners under Approval voting following Myerson (1993) model. This characterization is stated provided that the electorate is "large enough". By "large enough", we consider the benchmark for the study of large elections in which: (i) each voter does not affect the pivotal probabilities since his influence becomes negligible and (ii) yet his probability of affecting the outcome is strictly positive so that a rational voter selects the ballot that gives him the highest expected utility. Our results contrast with the previous ones

S. Courtin (✉)
Normandie Université, Unicaen, CREM, UMR CNRS, Caen 6211, France
e-mail: sebastien.courtin@unicaen.fr

M. Núñez
CREST, Ecole Polytechnique & CNRS, Palaiseau, France
e-mail: matias.nunez@ensae.fr

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in the strategic voting literature in its generality.[1] Most of the previous works focus on precise examples (for some given preference profile) and compute the set of equilibria under some voting rules. Few general results (that is that apply to any preference profile) are available. Among these results, one deals with Plurality and the other with Approval Voting. Under Plurality voting, one can construct equilibria in which any candidate who is not the Condorcet loser can win the election [see Myerson (2002) for instance]. Under Approval voting, the existence of an equilibrium in which the Condorcet winner wins, as previously described, is a salient feature. Yet, little is known about the rest of equilibria in elections with this rule. Informally, one might expect that Approval voting should reduce, when compared with other voting rules, the set of voting equilibria (and of equilibrium winners) and hence be more robust to the concept of focal manipulation as described by Myerson (1993). However, the previous results do not focus on the whole set of equilibrium winners under Approval as we do. Moreover, note that Brams (2016) perform a related analysis assuming that voters are sincere. Our work can hence be thought as an extension of their work to a strategic environment. Our approach is based on the candidates who are viable and unviable. A candidate a is unviable if, in the election, the number of voters who do not rank a last is smaller than the number of voters who rank some other candidate first. A viable candidate is one who is not unviable; as will be shown, only viable candidates can win when voters play best responses. Our notion of viability is somewhat related to the underlying idea of critical strategies as presented by Brams (2016). The equilibrium winners are as follows.

If there are at most two viable candidates, then the unique equilibrium winner is the Condorcet Winner (Theorem 1). Furthermore, we prove that if the unique equilibrium winner is the Condorcet winner for every utility representation of an ordinal preference profile, then there are at most two viable candidates. We hence derive necessary and sufficient conditions for implementing the Condorcet Winner as the unique equilibrium winner in terms of the number of viable candidates. To prove such a result, we need to impose two mild restrictions in the preference profile: the Simple Asymmetry (*SA*) and the Inverse Asymmetry (*IA*). According to *SA*, for any pair of candidates x, y the number of voters who prefer x to y are different from the number of voters who prefer y to x. The role of *SA* is simple: it removes equilibria with two winners. *IA* states that for any triple of candidates x, y, z, the number of voters who prefer x to y and y to z is different from the number of voters who prefer z to y and y to x. The role of *IA* is subtler as will be shown by Example 2, which proves that the Condorcet loser can be the only winner in equilibrium when *IA* fails to hold.

[1] In a recent computer science literature the use of probabilistic models in contexts with human participants has received quite a lot of criticism. Especially, they argue that the common approach to handle uncertainty is by maximizing expected utility, which requires a cardinal utility function as well as detailed probabilistic information. However, often such probabilities are not easy to estimate or apply. Therefore a number of alternative frameworks for modelling uncertainty (including for voting settings) have been proposed. For an up-to-date coverage of this literature, see Meir (2014) and Lev (2019).

On the contrary with at least three viable candidates, the situation is much more nuanced. We prove that for any such preference profile, we can build an equilibrium in which all viable candidates are tied for victory (Theorem 2). Note that this equilibrium exists for some set of cardinal utilities but need not exist for all utility profiles representing an ordinal preference profile.

We then go on to derive some implications of the previous results.

The first consequence is that when many candidates are viable, taking into account equilibria with more than one winner seems unavoidable. We believe that these equilibria are not degenerate but, on the contrary, are inherent to the voting system. One possible interpretation is that, unless the electorate is polarized over two candidates (there are at most two viable candidates included), the rule is unable to make a clear choice.

Secondly, we obtain a description of equilibrium winners in elections with three candidates. In these elections (on which most of the literature focus), we prove that either the CW wins or the three candidates are tied in any equilibrium. This reinforces our claim according to which equilibria with ties cannot be ignored. Indeed, doing so, leads us to conclude that some elections *do not admit an equilibrium at all*. For instance, any election without a CW only admits equilibria with ties (Theorem 1).

This work is structured as follows. After briefly reviewing the literature on models of large elections, Sect. 2 introduces the general framework and Sect. 3 describes the strategic behavior of the voters. Section 4 analyzes the relation between AV and the Condorcet Winner whereas Sect. 5 focus on elections with many viable candidates. Section 6 concludes the paper.

1.1 Related Literature

One common feature of the models dealing with the study of large elections with strategic voters are the pivotal probabilities. It is often assumed that a voter anticipates that with some small probability (even though strictly positive) his vote is relevant to modify the outcome of the election. Determining and comparing the magnitude of these probabilities is hence key to describe the voters' strategic behavior. Our model is no exception.

Yet, there are different approaches that have been taken to incorporate this assumption. One may either, as in the present model or as in Myerson (1993), make some simple assumptions about the pivotal probabilities, without explicitly incorporating a mechanism that actually leads to positive pivotal probabilities. Or one may add a certain uncertainty to the model that generates positive pivotal probabilities. Myerson (2000, 2002), among others, assumes that the actual number of voters is uncertain and follows a Poisson distribution (Poisson voting game). Laslier (2009) assumes that the actual number of voters is given, but that each voter's vote has a certain small probability of being wrongly recorded ("Florida-tremble"). Other than implicit versus explicit mechanism generating pivotal probabilities, Laslier (2009) and the present model are essentially the same.[2]

The previously discussed modelling approaches can be divided into two groups: those in which the voters' anticipations follow some intuitive behavioral assumption and those who do not. Our work belongs to the first group together with Myerson (1993) and Laslier (2009), whereas, in general, preference aggregation Poisson games belong to the second group.[3] Informally intuitive behavioral assumption entails that a voter believes that in case of being pivotal, it is way more probable to break a tie in which (at least) one of the winners is involved than a tie in which no winner is involved. In Laslier (2009), they prove that situation arises endogenously when the scores of the candidates are treated as independent random variables and the number of voters is large enough. In our model, we follow this behavioral assumption.

Two remarks can be made on this behavioral assumption. First, it is true that Myerson (2002) finds that a Poisson voting game is inconsistent with the model of "reduced form"": due to Myerson (1993). Moreover, Nunez (2010) has an example in which preferences satisfy the Simple Asymmetry and the Inverse Asymmetry, there exists a Condorcet winner, but it is not selected in an equilibrium with a unique winner. This is a mathematical objection to the current model. However, when using the ordering condition (which is quite intuitive), we interpret it as a behavioral one. We do not claim that it can be derived under very general conditions from a model. On the contrary, our claim is that, when the condition fails, the strategic reasoning might be highly unreasonable. For instance, in the example in Nunez (2010), the voters anticipate that the most probable pivot event takes place between the first and the third ranked candidate rather than between the two candidates with the most votes. This seems to be hardly sustainable with experimental data [see Forsythe (1993) among others]. Moreover, Lachat (2019) test one model of strategic voting in which the ordering condition holds on data of an AV election in the Zurich cantons (with several winners). They find substantial evidence that these models correctly predict strategic behavioral both at the individual and at the aggregate level.

2 Elections

The finite set of voters and candidates are respectively denoted by $\mathcal{N} = \{1, \ldots, n\}$ and $\mathcal{X} = \{a, b, \ldots, k\}$. Note that n is supposed to be large. The strict preferences of a voter are defined by a utility function $u : \mathcal{X} \to \mathbb{R}$, in which $u(x)$ denotes the utility a voter gets if candidate x wins the election. In other words, for each $i \in \mathcal{N}$ and for any pair of candidates $x, y \in \mathcal{X}$, x is strictly preferred to y, denoted $x \succ_i y$, if

[2] In these models when the size of the electorate becomes large, the voter becomes almost certain of the distribution of the voters' preferences. In order to tackle this features, a recent strand of the literature is focusing on models of aggregate uncertainty such as the works of Fisher (2014) and Bouton (2016).

[3] So is the case if one focuses on classic equilibrium refinements such as perfection or Mertens' stability as proved by De Sinopoli (2006, 2014).

and only if $u_i(x) > u_i(y)$. Given that our focus is on whether the Condorcet Winner is selected, we consider exclusively strict preferences over alternatives under which this concept is unambigously defined.

Election $E := (\mathcal{X}, \mathcal{N}, u)$ is then characterized by its set of candidates $\mathcal{X}$, its set of voters $\mathcal{N}$ and the utility vector $u = (u_i)_{i \in \mathcal{N}}$ that depicts the utility function of each voter.

For any pair of candidates $x, y \in \mathcal{X}$, the majority relation M is defined as follows. We say that x is M-preferred to y, denoted xMy, if and only if $N(x, y) > N(y, x)$, with $N(x, y) = \#\{i \in \mathcal{N} \mid x \succ_i y\}$. The majority relation allows us to introduce the notion of Condorcet Winner (CW), that is a candidate who is majority preferred in any pairwise comparison. This concept will be of special importance in this work and its formal definition is as follows.

Definition 1 An election E admits a Condorcet winner if there exists some candidate $x \in \mathcal{X}$ such that:

$$xMy \quad \text{for any } y \in \mathcal{X} \setminus \{x\}.$$

Throughout the work, we make two (slight) assumptions that ensure that social preferences are asymmetric: Simple and Inverse Asymmetry. Note that both conditions are quite mild.

The first one concerns the preferences of the electorate over any pair of candidates, as follows.

Definition 2 An election E satisfies Simple Asymmetry (SA) if:

$$\text{for any } \ x, y \in \mathcal{X}, \ N(x, y) \neq N(y, x).$$

The assumption *SA* is rather weak. Its role is to remove knife-edge cases in which the electorate is divided in two exact halves: the ones who prefer x to y and the ones who prefer y to x. When the population is large, the probability of these knife-edge cases is typically very small.

The second one concerns preferences over triples of candidates and is defined as follows. For any triple of candidates $x, y, z \in \mathcal{X}$, we let $N(x, y, z)$ denote the number of voters who prefer x to y and y to z; formally, $N(x, y, z) = \#\{i \in \mathcal{N} \mid x \succ_i y \succ_i z\}$.

Definition 3 An election E satisfies Inverse Asymmetry (IA) if:

$$\text{for any } x, y, z \in \mathcal{X}, \ N(x, y, z) \neq N(z, y, x).$$

The role of *IA* and *SA* is to avoid non-generic situations in which the number of players of certain type exactly coincide with the number of players of a different type. This goes in line with many models of incomplete information where the number of voters of each of the different types are drawn from a common distribution. Indeed, in such models, when the number of voters goes to infinity, the probability that two types have exactly the same number of voters becomes negligible. Both our assumptions, *IA* and *SA* mimic these vanishingly small probabilities, in a setting where (*i*) there is a large number of voters and (*ii*) voters have complete information over the preferences of the rest of voters.

The next definition concerns some sort of candidates in an election. A candidate y in election E is unviable if and only if there exists some other candidate x such that the number of voters who rank x first (denoted $N(x, \ldots)$) is higher than the number of voters who do not rank y last (denoted $n - N(\ldots, y)$) so that

Definition 4 A candidate y in election E is unviable if:

$$\exists\, x \in \mathcal{X} \quad \text{with}\, N(x, \ldots) > n - N(\ldots, y).$$

with $N(\ldots, y)$ the number of voters who rank y last.

Any candidate who is not unviable is viable. The set of viable and unviable candidates are respectively denoted by $\mathcal{X}^v$ and $\mathcal{X}^{uv}$ so that

$$\mathcal{X} = \mathcal{X}^v \cup \mathcal{X}^{uv}.$$

Note that (Courtin, 2017) show that if an election E is such that $\mathcal{X}^v \leq 2$ and *SA* holds, then E admits a Condorcet Winner.

3 The Electoral Game

As previously discussed, we assume that the voters are strategic and vote simultaneously through the Approval voting method. In other words, each voter can approve of as many candidates as he wishes by choosing a ballot $v = (v_a, \ldots, v_k)$ where $v_x \in \{0, 1\}$ denotes the number of points given to candidate x by a voter. In the following the set of all possible ballots will be denoted V. We follow Myerson (1993) by assuming that each voter maximizes his expected utility to determine which ballot in the set V he will cast. In this model, his vote has an impact in his payoff if it changes the winner of the election. Therefore, a voter needs to estimate the probability of these situations: the *pivot* events. We say that two candidates are tied if their vote totals are equal. Furthermore, let H denote the set of all unordered pairs of candidates. We denote a pair $\{x, y\}$ in H as xy with $xy = yx$.

For each pair of candidates x and y, the xy-pivot probability p_{xy} is the probability of the outcome perceived by the voters that candidates x and y will be tied for first place in the election. A voter perceives that the probability that he will change the winner of the election from candidate x to candidate y by casting ballot v with $v_x \geq v_y$

to be linearly proportional to $\mid v_x - v_y \mid$. Moreover, the perceived chance of changing the winner from y to x is identical to the one of changing the winner from x to y. [4]

A pivot vector p is a vector listing the pivot probabilities for all pairs of candidates is denoted by $p = (p_{xy})_{xy \in H}$.

This vector p is assumed to be identical and common knowledge for all voters in the election. A voter with xy-pivot probability p_{xy} anticipates that submitting the ballot v can change the winner of the election from candidate x to candidate y with a probability of $p_{xy} \times \max\{v_x - v_y, 0\}$.

We let $U_i(v; p)$ denote the expected utility gain of voter i from casting ballot v given the pivot vector p with:

$$U_i(v; p) = \sum_{xy \in H} (v_x - v_y) \cdot p_{xy} \cdot [u_i(x) - u_i(y)]. \qquad (U)$$

A strategy profile $\sigma = (\sigma_i, \sigma_{-i})$ is any mapping from $\mathcal{N}$ into the set of probability distributions over V. That is, each σ_i describes the probability with which voter i chooses each ballot v in the set V. The expected utility gain of a voter when he plays the strategy σ_i equals $U_i(\sigma_i; p) = \sum_{v \in V} \sigma_i(v) U_i(v; p)$.
Given the strategy profile σ, the size of the electorate who casts ballot v is denoted by $\tau(v) = \sum_{i \in \mathcal{N}} \sigma_i(v)$. Therefore, the score of candidate x equals $S(x; \sigma) = \sum_{v \in V} v_x \tau(v)$ given the strategy profile σ.

Definition 5 For any strategy profile σ, the set of *winners* at σ, $W(\sigma) \subseteq \mathcal{X}$, contains the candidates whose score $S(x; \sigma)$ is maximal given σ.

Given a pivot vector p, the set of pure best responses of a voter equals $\mathrm{BR}_i(p) = \{v \in V \mid v \in \arg\max_{v' \in V} U_i(v'; p)\}$. Given the strategy σ_i of a voter i, its support denotes the set of pure strategies played with positive probability according to σ_i: $\mathrm{Supp}(\sigma_i) = \{v \in V \mid \sigma_i(v) > 0\}$;

For any candidate y, let $v^y = (v^y_x)_{x \in \mathcal{X}}$ represent the ballot that assigns 1 point to candidate y and zero to the rest of them ($v^y_y = 1$ and $v^y_x = 0$ if $x \neq y$). The following lemma will simplify the voter's expected utility and hence helps to understand his best responses.

Lemma 1 *For any ballot $v \in V$, any pivot vector p and any voter $i \in \mathcal{N}$,*

$$U_i(v\,;\,p) = \sum_{\{y:\, v_y = 1\}} U_i(v^y\,;\,p),$$

[4]This is roughly equivalent to assume that the probability of candidates x and y being tied for first place is the same as the probability of candidate x being in first place one point ahead of candidate y (and both candidates above the rest of the candidates), which is in turn the same one as the probability of candidate y being in first place one vote ahead of candidate x. (Myerson, 1993) justify this assumption by arguing that it seems reasonable when the electorate is large enough.

To see why Lemma 1 is correct, for any ballot $v = (v_x)_{x \in \mathcal{X}}$ which assigns no points to candidate x (i.e. $v_x = 0$), we let $v \cup \{x\}$ denote the ballot that assigns one point to x and v_y points to any candidate $y \neq x$. The linear expected utility of the voters given by (U) implies that for any ballot $v \in V$, $U_i(v \cup \{x\}\ ;\ p) - U_i(v\ ;\ p) = U_i(v^x; p)$. In other words, $U_i(v \cup \{x\}\ ;\ p) = U_i(v\ ;\ p) + U_i(v^x\ ;\ p)$ which implies the claim.

This lemma says that the expected utility of approving of a set of candidates equals the sum of the expected utilities of voting *independently for each of them*. Thus, any best response consists on approving of the candidates for which approving them marginally improves the voter's expected utility. Those candidates with null expected utility might be included but need not. The following lemma presents the structure of voters' best responses.

Lemma 2 *(Best Responses) For any pivot vector p and any voter $i \in \mathcal{N}$, the voter's set of best responses is as follows:*

(i) if $U_i(v^x\ ;\ p) > 0$ then $v_x = 1$ for any $v \in BR_i(p)$.
(ii) if $U_i(v^x\ ;\ p) < 0$ then $v_x = 0$ for any $v \in BR_i(p)$.
(iii) if $U_i(v^x\ ;\ p) = 0$, then there is some $v \in BR_i(p)$ with $v_x = 1$.

As depicted by the above lemma, the structure of voters' best responses is particularly simple since one only needs to compute the expected utility of each of the different candidates that are to be included in the ballot. Moreover, one can easily check that, in any best response, a voter always approves his most preferred candidate and never approves his least preferred one if the pivot vector p is such that $p_{xy} > 0$ for some $xy \in H$.

Given the pivot vector p, one can choose a best response σ such that the score of each candidate x can take any value in $[\min S(x;\sigma), \max S(x;\sigma)]$ with:

$$\min S(x;\sigma) = \#\{i \in N \mid U_i(v^x; p) > 0\}$$

and

$$\max S(x;\sigma) = \#\{i \in N \mid U_i(v^x; p) \geq 0\}.$$

Note that the minimal score of candidate x corresponds to the situation in which only the voters who get a *strictly* positive expected utility of voting x do vote for him. On the contrary, the maximal score is reached when every voter with a non-negative expected utility of voting x votes for x.

Proposition 1 *In any election E and any strategy profile σ in which voters use best responses, this defines an equilibrium state in which the set of winners only contains viable candidates.*

Proof Assume by contradiction that there is some unviable candidate y in some election E such that $y \in W(\sigma)$ for some strategy profile σ. By definition, since y is unviable in E then $\exists\, x \in \mathcal{X}$ with $N(x,\ldots) > n - N(\ldots, y)$. Moreover, if the voters play a best response, they always vote for their preferred candidate and never for their worst preferred one as shown by Lemma 2. Hence, $\min S(x;\sigma) > \max S(y;\sigma)$ so that $y \notin W(\sigma)$, as required. □

We now move to the equilibrium concept that we will follow. Following Myerson (1993), we assume that voters expect candidates with lower scores to be less likely serious contenders for first place than candidates with higher scores. In other words, if the score for some candidate x is strictly higher than the score for some candidate y, then the voters would perceive that candidate x's being tied for winning with any third candidate z is much more likely than candidate y's being tied for first place with candidate z.

Definition 6 Given any strategy profile σ and any candidate z, the pivot vector satisfies the *ordering condition* with respect to any $\varepsilon \in (0, 1)$ if

$$S(x;\sigma) > S(y;\sigma) \Longrightarrow \varepsilon p_{xz} \geq p_{yz},$$

for any two candidates x, y.

This implies that pivot probabilities involving candidates with low vote shares are zero in a similar fashion to the definition of proper equilibrium. We follow this assumption.[5]

Moreover, we also assume that the probability of three (or more) candidates being tied for first place is very small in comparison to the probability of a two-candidate tie.

As will be shown, as long as the election does not admit a Condorcet winner, the model prescribes that in any equilibrium, there are at least three winners. This might, at first glance, seem counterintuitive with the previous assumption according to which ties with more than two candidates are negligible, which is not the case of pivots with exactly two candidates. However, note that in Poisson games in which this assumption does not hold, Myerson (2002) writes that "[j]ust because all candidates have equal expected scores per voter in the limit does not imply that they have equal chance of winning in large equilibria" . Myerson (2002) then proves that the strategic reasoning in an equilibrium with three tied winners deals just with the two-candidate pivots. Our assumption is hence not unduly restrictive.

Given any strategy profile σ, a sequence of pivot vectors $\{p^\varepsilon\}_{\varepsilon\to 0}$ satisfies the ordering condition if, for each $\varepsilon > 0$, p^ε is a positive pivot vector that satisfies the ordering condition.

Definition 7 The strategy profile σ is an *equilibrium* of election E if and only if, there exists a sequence of pivot vectors p^ε with $p^\varepsilon_{xy} > 0$ for every $xy \in H$ that satisfies the ordering condition given σ and such that, for each ballot v and for each voter i,

$$v \in \text{Supp}_i(\sigma) \implies v \in \text{BR}_i(p^\varepsilon) \text{ for each } \varepsilon > 0.$$

[5] The reader also can refer to the recent contribution by Kavai (2013) for an empirical test of strategic voting in a model in which different weakening of the ordering condition is proposed. See also the recent experimental work by Bouton (2016).

As shown by Myerson (1993), an equilibrium exists for any possible distribution of the voters' utilities which makes the model very attractive for the study of voting rules. It should be stressed that, in this definition, the pivot probabilities p_{xy} are supposed to be constant when the voter contemplates casting one ballot or the other. More specifically, these pivot probabilities, from each voter's perspective, should be the probabilities of ties or 1-vote differences among all other voters' ballots, before his own ballot is counted. But then the independence of pivot probability on the perceiving voter can be justified if the true stochastic model treats all voters symmetrically. This is why the literature tends to use models where voters have independent identically-distributed types.

The focus of the paper is on the equilibrium winners under Approval voting. For any election E, the set of equilibrium winners is denoted by $W_E(\sigma)$.

In order to illustrate the model, we conclude this section by an example of an election E that gives an excellent description of the equilibria with ties that we describe throughout.

Example 1 Consider an election E with three alternatives $\mathcal{X} = \{a, b, c\}$ and three groups of voters. The first one is endowed with the utility profile[6] $u_A = (10, x, 0)$, the second one with $u_B = (0, 10, y)$ and the final one with $u_C = (z, 0, 10)$ with $0 < x, y, z < 10$. The shares of the different groups are, respectively, 40%, 35% and 25%. Hence, this election does not admit a Condorcet Winner. Therefore, at first sight, one can imagine that a will be the winner in some equilibrium. Indeed, the voters focus on the pair $\{a, b\}$ in the sense that these candidates are the ones with the two highest expected scores. This implies that the most likely pivot event occurs between a and b so that the voters in groups 1 and 3 approve of a and the ones in group 2 approve of b so that, a gets a higher score than b, namely $S(a; \sigma) = 65\% > S(b; \sigma) = 35\%$, and a is the winner. However, the logic of the model is more complex. Indeed, one still needs to consider the pivot events in which alternative c is involved since every possible pivot event occurs with positive probability. However, since a has a higher score than b, then it is infinitely more likely that c is involved in a pivot with a than with b (as described by the ordering condition). Hence, the voters in groups 2 and 3 approve of c since they all prefer c to a. However, this implies that $S(a; \sigma){=}65\%{>}S(c; \sigma){=}60\%{>}S(b; \sigma){=}35\%$ so that it is not anymore rational that voters focus on the pair $\{a, b\}$ but rather on the pair $\{a, c\}$; in other words, there is no equilibrium in which the voters focus on the pair $\{a, b\}$. This reasoning applies to any pair of candidates so that there is no equilibrium σ in which the voters just focus on a pair of candidates in the sense that these candidates are the ones with the two highest expected scores. Moreover, one can prove that any equilibrium in this election leads to a tie among the three candidates so that $W_E(\sigma) = \{a, b, c\}$ for any equilibrium σ in E. As we will see in the next section, this inevitably generates ties in equilibrium.

[6]The utility values in each vector are the utilities of alternative a, b and c respectively.

4 Approval Voting and the Condorcet Winner

This section describes the conditions that ensure that the unique equilibrium winner is the Condorcet Winner under Approval voting. Moreover, it shows that when the election admits no Condorcet Winner, there are at least three equilibrium winners at any equilibrium. These results make more precise the relation between Approval voting and the Condorcet Winner.

The main characteristic of these results is that they do not depend *explicitly* on the voters' best responses. In other words, we do not need to completely define how the voters vote in order to predict who the equilibrium winners are. The main logic is driven by the voters' anticipations to the possible scores of the candidates, greatly simplifying the task at hand. As far as scenarios with a few number of viable candidates are considered, the main implication is summarized in the following theorem: the Condorcet Winner is the unique winner in equilibrium.

Within the proofs, we write $S(x)$ rather than $S(x; \sigma)$ to simplify notations. That is, we remove the explicit reference to the strategy profile.

Theorem 1 *If the election E satisfies both IA and SA, then:*

1. *If there are at most two viable candidates, then the unique equilibrium winner is the Condorcet Winner.*
2. *If there is no Condorcet winner, the set of equilibrium winners $W_E(\sigma)$ contains at least three candidates for any equilibrium σ.*

Theorem 1 is the main result of this section.

The two assumptions about the society, *IA* and *SA*, play a key role in the proof, although they do not have the same role.

Proposition 2 *If the election E satisfies SA, then there is no equilibrium with two winners.*

Proof Assume, by contradiction, that there is an equilibrium with two winners. W.l.o.g. we let x and y be this pair of candidates. Due to the ordering condition, the most probable pivot outcome in which x (resp. y) is involved is against y (resp. x). Therefore, the voters who strictly prefer x over y vote for x and the ones who strictly prefer y over x vote for y. Hence, the score of x equals $N(x, y)$ whereas the one of y equals $N(y, x)$. However, since *SA* holds, the scores of such candidates are different, contradicting the assumption that both x and y are both equilibrium winners. □

Consider now the role of *IA*. While the role of *SA* in selecting equilibria is intuitive, the role of *IA* is subtler. We first prove that it ensures that if there is an equilibrium with a unique winner, then this candidate is the Condorcet Winner. However, in order to see that this condition is necessary and important, Example 2 demonstrates that the Condorcet Loser (a candidate who is never M-preferred to any other candidate in the election) might be the unique winner when *IA* does not hold.

Example 2 Let $\mathcal{X} = \{a, b, c\}$, and consider a society with four possible utility functions: $u_A = (10, \mu, 0)$, $u_B = (10, 0, \mu)$, $u_C = (\mu, 10, 0)$ and $u_D = (0, \mu, 10)$ with $10 > \mu > 5$. The proportion of voters with each utility profile equals respectively 0.2, 0.35, 0.25 and 0.2. Therefore, b is the Condorcet Loser since aMb and cMb. Moreover *IA* does not hold since there is the same number of voters with utility vectors u_A and u_D so that $N(a, b, c) = N(c, b, a)$.

It is easy to see that the strategy profile σ with

$$\sigma_A \to \{a, b\},\ \sigma_B \to \{a, c\},\ \sigma_C \to \{b\} \quad \text{and} \quad \sigma_D \to \{b, c\},$$

leads to the victory of b. Moreover, the strategy profile σ is justified by the pivot vector $p^\varepsilon = (p^\varepsilon_{ab}, p^\varepsilon_{ac}, p^\varepsilon_{bc}) = (1/2 - \varepsilon, 2\varepsilon, 1/2 - \varepsilon)$ and hence is an equilibrium. Therefore, the Condorcet loser b is the unique equilibrium winner at σ. As the next result shows, this bad outcome does not occur when the election satisfies *IA*.

Proposition 3 *If the election E satisfies IA and there is an equilibrium with a unique winner, then this candidate is the Condorcet Winner.*

Proof Assume that there is a unique winner in equilibrium, denoted a. Due to the ordering condition, every voter knows that, when $\varepsilon \to 0$, the pivot outcome in which any candidate $x \neq a$ is involved against a becomes infinitely more likely than the rest of pivot events.

We have two cases: either there is a tie in the scores of two candidates (who are not the winners) or there is no tie.

Case 1: Assume first that, given σ, there is a tie in the score of two candidates who are not the winners. We denote them b and c w.l.o.g. As the most likely pivot outcome in which both are involved is against a, we know that the unique voters who vote for b (resp. c) are the ones who prefer b (resp. c) to a.

Therefore, the scores of both candidates are the following ones:

$$S(b) = N(b, a, c) + N(b, c, a) + N(c, b, a),$$

and

$$S(c) = N(c, a, b) + N(c, b, a) + N(b, c, a).$$

Since the condition *IA* holds, it follows that the scores of b and c cannot be equal, a contradiction.

In other words, when *IA* holds, there is not an equilibrium with a unique winner in which two candidates have the same score. So that, if there is a unique winner in equilibrium, the only possible case is that there is no tie in the scores, to be analyzed in the *Case* 2.

Case 2: Assume now that there are no ties in the scores. Note first that $N(x, a) \neq N(y, a)$ for any pair $x, y \in \mathcal{X}$. To prove this, it suffices to see that $N(x, a) = N(x, a, y) + N(x, y, a) + N(y, x, a)$ and $N(y, a) = N(y, a, x) + N(y, x, a) + N(x,$

$y, a)$. The condition *IA* implies that $N(x, a, y) \neq N(y, a, x)$. Therefore, $N(x, a) \neq N(y, a)$ for any pair $x, y \in \mathcal{X}$.

W.l.o.g. we assume that $N(b, a) > N(c, a) > \cdots > N(k, a)\ \forall\ b, c, \ldots, k \in \mathcal{X}$.

Since every voter anticipates that the most likely pivot outcome involving any candidate $x \neq a$ is against a, it follows that the score of each candidate $x \neq a$ equals $N(x, a)$ the share of voters who strictly prefer x to a whereas the one of a equals $N(a, b)$. Hence, the scores of the candidates satisfy $S(a) > S(b) > \cdots > S(k)$.

Assume that a is not the CW so that there is some candidate y with yMa. If $y = b$, then $N(b, a) > N(a, b)$ so that the score of b is higher than the score of a, a contradiction with a being the winner. If $y \neq b$, then $N(y, a) > 1/2$ so that $S(y) = N(y, a) > 1/2 > N(b, a) = S(b)$. Therefore, y is ranked second. In this case, the score of a equals $N(a, y) < 1/2$, a contradiction with a being the winner. Hence, it can only be the case that a is M-preferred to the rest of the candidates: for any $x \in \mathcal{X} \setminus \{a\}$, aMx. In other words, a is the Condorcet winner. □

Finally, *IA* entails that if there is a CW in the profile, there exists an equilibrium in which this candidate is the unique winner.

Proposition 4 *If the election E satisfies IA and admits a Condorcet winner, then there exists an equilibrium that uniquely selects this candidate.*

Proof Take a society in which there is a CW (denoted a) and in which *IA* holds. Since *IA* holds, we can assume w.l.o.g. that $N(b, a) > N(c, a) > \cdots > N(k, a)$. Indeed, as shown in the proof of Proposition 3 (case 2), if *IA* holds, then $N(x, a) \neq N(y, a)$ $\forall x, y \in \mathcal{X}$. Assume that the scores satisfy $S(a) > S(b) > \cdots > S(k)$. Due to the ordering condition, it follows that the most likely pivot in which a is involved is against b whereas the most likely pivot outcome in which any other candidate x is against a. Thus, the score of a equals $N(a, b)$ whereas the score of x ($x \neq a$) equals $N(x, a)$. As a is the CW, it follows that $N(a, b) > 1/2$ and that $N(x, a) < 1/2$ for any $x \neq a$. Finally, since $N(b, a) > N(c, a) > \cdots > N(k, a)$, the scores satisfy $S(a) > S(b) > \cdots > S(k)$ as wanted. Thus we have proved that there exists an equilibrium in which the CW is the unique winner, concluding the proof. □

Proof of Theorem 1 As previously mentioned, if $\#\mathcal{X}^v \leq 2$, then the election admits a CW. By Proposition 4 we have shown that if there is a CW, there exists an equilibrium in which he is the unique winner. Moreover, there is no other equilibrium with a unique winner as ensured by Proposition 3. As shown by Proposition 2, there is no equilibrium with two winners since *SA* is satisfied. Therefore, the only type of equilibrium that might exist is the one in which at least three candidates are tied. However, the candidates who are unviable cannot be in the set of winners. Hence, when $\#\mathcal{X}^v \leq 2$, there is no equilibrium in which at least three candidates wins, which concludes the proof of part 1 of the Theorem 1. The part 2 of the Theorem 1 is a direct implication of the different results of this section. □

One main implication of Theorem 1 is that in elections with three candidates, the equilibrium winners are as follows.

Corollary 1 *If the election E satisfies both IA and SA and there are three candidates in the election, there are at most two sets of equilibrium winners:*

1. *the Condorcet Winner,*
2. *the three candidates belong to the winning set.*

5 On the Indeterminacy of Approval Voting

We now focus on the elections that admit at least three viable candidates.

We first focus on elections with three candidates and then explain how to extend the results to elections with at least four candidates.

5.1 Three Candidates

The next theorem gives a simple condition for the existence of a tie among viable candidates: if three candidates are viable, such an equilibrium exists. Let $\mathcal{U}$ a set of utilities, then we have

Theorem 2 *Assume that the election has three candidates. If all candidates are viable, then there is a closed set of utilities* $\hat{\mathcal{U}} \subseteq \mathcal{U}$ *such that for any election with utility vector* $u \in \hat{\mathcal{U}}$*, there is an equilibrium in which all viable candidates are tied for victory.*

This subsection presents the proof of Theorem 2. In the second part of this subsection, we prove that the equilibrium described by Theorem 2 need not exist for every utility representation. Indeed, Example 3 discusses an election with three candidates that admits no tie among equilibrium winners for some set of utilities.

We first present one proposition and one technical lemmata that will be useful for proving Theorem 2.

Proposition 5 *For each election E, there exists some* $m \in \mathbb{N}^+$ *such that*

$$n - N(\ldots, a) \geq m \Longleftrightarrow a \in \mathcal{X}^v.$$

Proof of Theorem 1 Take any election E with $\mathcal{X}^{uv} = \emptyset$. Thus the result trivially follows. Consider now any election that $\mathcal{X}^{uv} \neq \emptyset$ and let a, b be two candidates such that $a = \arg\min_{x \in \mathcal{X}^v} n - N(\ldots, x)$ and $b = \arg\max_{x \in \mathcal{X}^{uv}} n - N(\ldots, x)$. Note that if we prove that $n - N(\ldots, a) > n - N(\ldots, b)$, then the result follows.

Thus, let us assume by contradiction that $n - N(\ldots, a) \leq n - N(\ldots, b)$. It follows that $a \in \mathcal{X}^v$ and $b \in \mathcal{X}^{uv}$. Thus, there exists some $y \in \mathcal{X}$ with $N(y, \ldots) > n - N(\ldots, b)$. Since we have assumed that $n - N(\ldots, a) \leq n - N(\ldots, b)$, it follows that $N(y, \ldots) > n - N(\ldots, a)$ and hence $a \in \mathcal{X}^{uv}$, showing the desired contradiction. □

Lemma 3 *For any candidate a, there exists some sequence of pivot probabilities* p^{ε} *that induces when* $\varepsilon \to 0$*, the following boundaries for the score of candidate a:*

$$\min S(a) = N(a, \ldots) \text{ and } \max S(a) = n - N(\ldots, a).$$

Proof of Theorem 1 Consider any candidate d and assume that

$$\lim_{\varepsilon \to 0} \frac{p^{\varepsilon}_{ad}}{p^{\varepsilon}_{ab}} = \lim_{\varepsilon \to 0} \frac{p^{\varepsilon}_{ad}}{p^{\varepsilon}_{ac}} = 0 \text{ for any } ad \neq ab, ac \quad (g).$$

Note that, given (g), when $\varepsilon \to 0$, the voter's decision concerning whether to cast a vote for a only depends on the pivotal events in which candidates b and c are involved, the rest of them becoming infinitely less likely.

We divide the voters into the usual six groups according their ordinal preference over a, b and c (as described in the primer of the proof).

Note that the N_1 and N_2 voters always vote for a and the N_4 and N_6 voters never vote for a independently of the pivot vector.

Moreover, we let $R_3 R_5 = 1$ and we assume that $U_3(a) = 0$ and $U_5(a) = 0$.

It follows that given p^{ε},

$$\min S(a) = N(a, \ldots), \quad \text{and} \quad \max S(a) = n - N(\ldots, a)$$

Note that the proof is done with homogenous cardinal utilities but a similar argument applies with heterogeneous cardinal utilities. □

We can now present the proof of Theorem 2.

Proof of Theorem 1 Voters' preferences are strict so that we divide the voters into six groups as follows:

N_1	N_2	N_3	N_4	N_5	N_6
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a
n_1	n_2	n_3	n_4	n_5	n_6

with for example N_1 being the set of voters i with preference ordering $a \succ_i b \succ_i c$, with $\#N_1 = n_1$. A set of voters sharing the same preference ordering is denoted N_l with $l = 1, \ldots, 6$.

We first assume that the voters in the same group (i.e. sharing the same ordinal preferences) have the same cardinal utilities. This assumption simplifies the proof and will be relaxed in the second part of the proof.

Part 1. Homogenous cardinal utilities within each group

The proof proceeds as follows. It builds for any type distribution in which $\mathcal{X}^{uv}$ is empty (i.e. $\#\mathcal{X}^v = 3$), a set of utilities and a strategy profile such that the three candidates are tied. Moreover, it builds a pivot vector that justifies the strategy profile proving that in equilibrium the three candidates are tied.

W.l.o.g. we let $n - N(\ldots, a) \geq n - N(\ldots, b) \geq n - N(\ldots, c)$.

For each voter i, we let t_i, m_i and b_i respectively denote his top, middle and bottom-ranked candidate over a, b, c. Moreover, for each $i \in \mathcal{N}$ we let R_i denote the following ratio $\frac{u_i(t_i)-u_i(m_i)}{u_i(m_i)-u_i(b_i)}$. Since all the voters sharing the same ordinal preferences have the same cardinal utilities, it follows that for any $i, j \in N_l$, $R_i = R_j$.

Therefore w.l.o.g. R_l stands for the ratio R_i for each $i \in N_l$.

We consider the set of utilities $\hat{\mathcal{U}}$ defined as follows:

$$\hat{\mathcal{U}} = \{u_{\mathcal{N}} \in \mathcal{U} \mid R_1R_6 = 1,\ R_3R_5 = 1,\ R_1 = R_2R_3 \text{ and } R_3 = R_1R_4\}.$$

The set $\hat{\mathcal{U}}$ is closed with an empty interior since it is the intersection of lower dimensional hyperplanes. Note that this set is not empty since we can independently choose each R_l. Moreover, we implicitly assume that $n_l > 0$ for each $l = 1, \ldots, 6$. A similar argument applies if $n_l \geq 0$ for $l = 1, \ldots, 6$.

We set $p^\varepsilon = (\varepsilon p_{ab}, \varepsilon p_{ac}, \varepsilon p_{bc})$ with

$$p_{ab} = \frac{1}{1 + R_1 + R_3}, \quad p_{ac} = \frac{R_3}{1 + R_1 + R_3} \quad \text{and} \quad p_{bc} = \frac{R_1}{1 + R_1 + R_3}.$$

One can check that the previous pivot probabilities imply that:

$$p_{m_ib_i} = R_i p_{t_im_i} \quad \text{for each } \ i \in \mathcal{N},$$

which is equivalent to

$$U_i(v^{m_i}; p^\varepsilon) = 0 \quad \text{for each } i \in \mathcal{N},$$

where v^{m_i} stands for the ballot that assigns one point to m_i (the middle-ranked candidate of voter i) and zero points to the rest of the candidates.

Given the description of the best responses given by Lemma 2, we know that the previous equality implies that every voter i is indifferent between voting for his top candidate (t_i) and for his top-two candidates (t_i, m_i). Hence, given p^ε one can choose a best response σ such that the score of each candidate x can take any value in $[N(x, \ldots), n - N(\ldots, x)]$.

Since $\mathcal{X}^{uv}$ is empty, it follows that $n - N(\ldots, c) > N(a, \ldots), N(b, \ldots)$. Moreover, by assumption, $n - N(\ldots, a) \geq n - N(\ldots, b) \geq n - N(\ldots, c)$. Thus, one can choose the three scores equal to $n - N(\ldots, c)$.

So far we have proved that for each vector $u \in \hat{U}$ and given p^ε, there exists a best response σ that leads to three tied winners. Moreover the pivot probability vector p^ε satisfies the ordering condition since the three candidates are tied given σ. Therefore, σ is an equilibrium, concluding the proof with homogeneous utilities.

Part 2. Heterogenous cardinal utilities within each group

We now allow the voters to have different cardinal utilities while having the same ordinal preferences. Therefore, there is not anymore a unique R_i for each voter i in each group N_l.

For each pivot vector p, we can divide the voters in each group N_l in three possible categories: those for which $U_i(v^{m_i}; p) > 0$, those for which $U_i(v^{m_i}; p) = 0$ and finally those for which $U_i(v^{m_i}; p) < 0$.

For each group N_l, we denote by N_l^* the group of voters such that for each $i \in N_l^*$, $U_i(m_i) = 0$. We let $R_l^* = R_i$ for each $i \in N_l^*$ and for each l.

Consider a voter i in N_1^* with middle-ranked candidate b. Therefore,

$$U_i(v^b; p) = 0 \iff p_{bc} = R_1^* p_{ab}.$$

Any voter j in N_1 with $R_j > R_1^*$ is such that $U_j(v^b; p) < 0$, whereas if $R_j < R_1^*$, it is the case that $U_j(v^b; p) > 0$. The same reasoning applies for each voter in any of the N_l groups. Therefore, R_l^* determines the best responses of the other voters in the group N_l. Moreover, the number of voters in N_l who vote for their middle-ranked candidate can vary from 0 to n_l, since one can set R_l^* to be equal to any R_i for each $i \in N_l$.

We consider the set of utilities $\hat{\mathcal{U}}^*$ defined as follows:

$$\hat{\mathcal{U}}^* = \{u \in \mathcal{U} \mid R_1^* R_6^* = 1,\ R_3^* R_5^* = 1,\ R_1^* = R_2^* R_3^* \text{ and } R_3^* = R_1^* R_4^*\},$$

and $p^\varepsilon = (\varepsilon p_{ab}, \varepsilon p_{ac}, \varepsilon p_{bc})$ with

$$p_{ab} = \frac{1}{1 + R_1^* + R_3^*},\quad p_{ac} = \frac{R_3^*}{1 + R_1^* + R_3^*} \text{ and } p_{bc} = \frac{R_1^*}{1 + R_1^* + R_3^*}.$$

Given p^ε, a similar reasoning to the one in Part 1 proves that each N_l^* is non-empty. It follows that given p^ε one can choose a best response σ such that the score of each candidate x can take any value in $[N(x, \ldots), n - N(\ldots, x)]$. Therefore, since p^ε satisfies the ordering condition, this proves that σ is an equilibrium, concluding the proof for heterogenous preferences. □

Theorem 2 proves that for some set of utilities, there is an equilibrium in which all the candidates in the race are tied. However, it should be noted that this sort of equilibria need not exist for *every utility* representation of the election. The following example illustrates this point with just three candidates.

Example 3 Let $\mathcal{X} = \{a, b, c\}$ and consider a society with the following proportions with $0 < \mu < 10$: $\frac{1}{9}$ of the voters with $u_A = (10, \mu, 0)$; $\frac{2}{9}$ of the voters with $u_B = (10, 0, \mu)$; $\frac{4}{9}$ of the voters with $u_C = (10 - \mu, 10, 0)$ and $\frac{2}{9}$ of the voters with $u_D = (10 - \mu, 0, 10)$. The candidate a is the CW and $\mathcal{X}^{uv} = \emptyset$. Note that both SA and IA

hold. Since $\mathcal{X}^{uv} = \emptyset$, every candidate is viable. Hence, there is a strategy profile under which the candidate wins with positive probability. However, whether this occurs in equilibrium depends on the intensity of the voters' utilities as will be shown in the next lines.

Indeed, Proposition 2 implies that there is no equilibrium with two winners. Moreover, since there is a CW, Proposition 4 ensures that there exists an equilibrium in which a is the unique winner. Finally, there is no other equilibrium with a unique winner as ensured by Proposition 3. In other words, neither b or c can win alone.

One question remains to be answered: is there an equilibrium with the three candidates tied for victory? These equilibria might or not exist as a function of the voters' intensities of preferences.

There is no equilibrium in this election in which the three candidates have the same score with no voter being indifferent between single and double voting. Indeed, when no voter is indifferent between single and double voting, it follows that all the voters with the same utility vector vote in the same way. One can check that in any strategy profile in which the voters play best responses (each voter voting for his top candidate or for his two top candidates), there is no equality between the scores of the candidates. Hence, there is no such equilibrium with a three-way tie.

Thus, in order to have such an outcome, some type of voters are indifferent between single and double voting. In equilibrium, voters always approve of their most preferred candidate and never approve of their worst preferred one.

If just one type of voters play a mixed strategy, then it is not possible to obtain a three-way tie. If at least two types play in mixed strategies, then either C or D voters vote also for their middle ranked candidate so that a has the highest score.

Indeed, assume first that a C voter plays a mixed strategy over his two best responses so that $U_C(0, 1, 0) = U_C(1, 1, 0)$. Due to (U), the previous equality is equivalent to $U_C(1, 0, 0) = 0$ so that

$$p^{\varepsilon}_{13}(10 - \mu) - p^{\varepsilon}_{12}\mu = 0. \qquad (*)$$

However, when $(*)$ holds, we have that $U_D(1, 0, 1) > U_D(0, 0, 1)$. To see why, note first that $U_D(1, 0, 1) > U_D(0, 0, 1) \iff U_D(1, 0, 0) > 0$. Moreover, remark that $U_D(1, 0, 0) = (10 - \mu)p^{\varepsilon}_{12} - \mu p^{\varepsilon}_{13}$ so that, when $(*)$ holds,

$$U_D(1, 0, 0) = \frac{10(10 - 2\mu)}{\mu} p^{\varepsilon}_{13} > 0.$$

which holds since $\mu < 5$.

Therefore, if a C voter plays a mixed strategy, D voters vote for their second ranked candidate a, leading to its victory. A symmetric argument applies when a D voter plays a mixed strategy. Therefore, in any mixed strategy profile in which either C or D voters play a mixed strategy between their two best responses, a is the sole winner of the election.

Hence, the only possibility for the existence of an equilibrium in which the three candidates get the same score is to assume that A and B voters both play a mixed strategy. However, this implies that

$$U_A(0, 1, 0) = 0 \iff -p^{\varepsilon}_{12}(10 - \mu) + p^{\varepsilon}_{23}\mu = 0,$$

and

$$U_B(0, 0, 1) = 0 \iff -p^{\varepsilon}_{13}(10 - \mu) + p^{\varepsilon}_{23}\mu = 0.$$

The previous two equalities imply that the unique pivot probability vector justifying such best responses equals $p^{\varepsilon} = (\frac{\mu}{10+\mu}\varepsilon, \frac{\mu}{10+\mu}\varepsilon, \frac{10-\mu}{10+\mu}\varepsilon)$. However, as previously noted, $U_C(1, 0, 0) = p^{\varepsilon}_{13}(10 - \mu) - p^{\varepsilon}_{12}\mu$ which is strictly positive given p^{ε} since $\mu < 5$. Hence, as in the previous case, if both A and B voters play a mixed strategy, C voters give one point to a, leading to its victory. Therefore, there is no equilibrium with three winners. Moreover, by Proposition 4, we know that there exists an equilibrium in which a is the unique winner. Furthermore, Proposition 2 implies that there is no equilibrium with two winners. Hence, in any equilibrium, a is the unique winner as long as $\mu < 5$. Hence, the CW is the unique equilibrium winner.

This example illustrates then that that for some set of utilities, when there is a CW and $\mathcal{X}^{uv} = \emptyset$, the unique equilibrium winner is the CW. However, for a different utility representation, we can find an equilibrium in which the three candidates get the same score. For example, if we set $\mu = 6$, there is an equilibrium in which the three candidates are tied for victory with a score of 4/9 as long as $p^{\varepsilon} = (p^{\varepsilon}_{ab}, p^{\varepsilon}_{ac}, p^{\varepsilon}_{bc}) = (3/7\varepsilon, 2/7\varepsilon, 2/7\varepsilon)$.

5.2 Viable* Candidates and Many Candidates

We now move on to describe elections with at least four candidates and the sufficient conditions for the presence of ties among viable candidates.

To see why Theorem 2 does not hold with more candidates, we present now the following example.

Example 4 Let $\mathcal{X} = \{a, b, c, d\}$ and consider an election with $\frac{6}{10}$ of the voters with $u_A = (10, 9, 8, 0)$ and $\frac{4}{10}$ of the voters with $u_B = (9, 10, 0, 8)$. This election has 3 viable candidates: a, b, and c. In particular, the latter is viable because the number of voters who rank a first is equal to the number of candidates who do not rank c last. Candidate c cannot win in any equilibrium. This is because c only receives approval votes if there is a high enough probability pivot event where it is facing d. But d always receives fewer approval votes than a as long as voters use best responses. So, by the ordering condition, no voter approves of c. In fact, the only equilibrium winner in this election is the CW.

The previous example shows that one needs to introduce a stronger condition than viability to ensure the existence of ties among viable candidates. In particular, this condition takes into account the iterative reasoning described in the previous example. The rest of this section described the notion of $k-$viable and $k-$unviable which are needed to derive a version of Theorem 2 with at least four candidates.

We first introduce some notation that will be useful throughout concerning degrees of viability and then prove how these definitions help to describe the strategic behavior under the Approval rule.

For notational purposes, we respectively relabel the set of viable and unviable candidates by $\mathcal{X}_0^v$ and $\mathcal{X}_0^{uv}$ (rather than $\mathcal{X}^v$ and $\mathcal{X}^{uv}$). Indeed, we say that a candidate in $\mathcal{X}_0^v$ (resp. in $\mathcal{X}_0^{uv}$) is $0-$viable (res. $0-$unviable). It follows that

$$\mathcal{X} = \mathcal{X}_0^v \cup \mathcal{X}_0^{uv}.$$

The main reason for this relabelling is the introduction of viability degrees for the different candidates as follows.

Definition 8 For any integer $k \geq 1$, a candidate y in election E is $k-$unviable if:

$$\exists\, x \in \mathcal{X}_{k-1}^v \quad \text{with}\, N(x, \ldots) > n - N^k(\ldots, y).$$

with

$$N^k(\ldots, y) := \#\{i \in \mathcal{N} \mid x \succ_i y \quad \text{for any}\, x \in \mathcal{X}_{k-1}^v\},$$

where $\mathcal{X}_k^{uv}$ stands for the set of $k-$ unviable candidates and $\mathcal{X}_k^v = \mathcal{X} \setminus \bigcup_{j=1}^k \mathcal{X}_j^{uv}$ the set of $k-$viable candidates.

Definition 8 is hence defining, recursively, the sets of $k-$viable and $k-$unviable candidates. It should be remarked that, for any integer $k \geq 0$,

$$\mathcal{X}_k^v \subset \mathcal{X}_{k-1}^v \quad \text{since}\ \mathcal{X} \setminus \bigcup_{j=1}^{k} \mathcal{X}_j^{uv} \subset \mathcal{X} \setminus \bigcup_{j=1}^{k-1} \mathcal{X}_j^{uv}$$

so that any $k-$viable candidate is also $(k-1)-$viable. The converse does not hold. Building on the previous set inclusion, it is easy to see that for any non-negative integer k such that $\mathcal{X}_k^v \neq \emptyset$, $N^k(\ldots, z) > N^{k-1}(\ldots, z)$.

The next definition deals with the candidates which are viable for every degree.

Definition 9 A candidate x in election E is viable* if x is $k-$viable for any positive integer k.

The set of viable* candidates is denoted $\mathcal{X}^*$. Such a set is non-empty by construction.

Theorem 3 *For any non-negative integer k, any election E and any equilibrium* σ*,*

1. *a voter never approves of his least preferred* $k-$*viable candidate.*
2. *no* $k-$*unviable candidate belongs to the set of equilibrium winners* $W(\sigma)$*.*

3. *the set of equilibrium winners $W(\sigma)$ contains only viable* candidates.*

Proof of Theorem 1 The proof proceeds by induction. Steps A and B prove that (1) and (2) hold. Step A proves the claim for $k = 0$ and Step B proves how to iterate the same reasoning. The result (3) is an immediate consequence of both (1) and (2).

Step A: $k = 0$.

Step A. is divided in two parts. We first prove that no voter approves of his least preferred $0-$viable candidate (in A.1) and then show that this implies no $1-$unviable candidate belongs to the set of equilibrium winners for any equilibrium (in A.2).

Step A. 1.

Assume that there is some equilibrium σ in which some voter approves of his least preferred $0-$viable candidate z. It follows that, there exists a sequence of pivot vectors p^{ε} with

$$U_i(v^z\,;\,p^{\varepsilon}) \geq 0 \quad \text{for any } \varepsilon \in (0, 1),$$

where $U_i(v^z; p^{\varepsilon}) = \sum_{xy \in H}(v^z_x - v^z_y) \cdot p_{xy} \cdot [u_i(x) - u_i(y)]$, with $v^z_x = 0$ for any $x \neq z$ and $v^z_z = 1$. Note that that for any pair xy in which z is not involved, $v^z_x - v^z_y = 0$. Hence the expected utility for voter i can be rewritten as:

$$U_i(v^z; p^{\varepsilon}) = \sum_{x \neq z}(0 - 1) \cdot p_{xz} \cdot [u_i(x) - u_i(z)].$$

Take some equilibrium σ. The ordering condition implies that for any $\varepsilon \in (0, 1)$, any $x \in W(\sigma)$ and $y \notin W(\sigma)$, $\varepsilon p^{\varepsilon}_{xz} \geq p^{\varepsilon}_{yz}$. Therefore,

$$\lim_{\varepsilon \to 0} \frac{p^{\varepsilon}_{yz}}{p^{\varepsilon}_{xz}} = 0. \quad (a)$$

Moreover, Proposition 1 implies that only $0-$viable candidates are in $W(\sigma)$. Therefore, since z is the least preferred $0-$viable candidate for voter i, this implies that

$$u_i(x) - u_i(z) > 0 \;\text{ for any } x \in W(\sigma) \subset X^v_0. \quad (b)$$

Combining (a) with (b), it follows that

$$\lim_{\varepsilon \to 0} U_i(v^z\,;\,p^{\varepsilon}) < 0,$$

which proves that voter i does not approve of z in equilibrium. This concludes the proof of Step A.1.

Step A. 2. Assume that there is some $1-$unviable candidate y so that $\exists\, x \in \mathcal{X}$ with $N(x, \ldots) > n - N^1(\ldots, y)$. Step A.1 proves that no voter votes for his least preferred $0-$viable candidate in equilibrium. Moreover, Lemma 2 proves that no vote for his least preferred candidate in equilibrium. It follows that for any equilibrium

σ, $S(y;\sigma) \leq n - N^1(\ldots, y)$ since $N^1(\ldots, y)$ denotes the number of voters who rank y last among all candidates and last among the 0−viable candidates. Since, in equilibrium, all voters vote for their most preferred candidate, it follows that:

$$S(x;\sigma) \geq N(x,\ldots) > n - N^1(\ldots, y) > S(y;\sigma),$$

so that y is not in the winning set. Hence, no 1−unviable candidate is in the set of equilibrium winners, as wanted.

Hence, Step A. has proved that no voter votes for his least preferred 0−viable candidate and that this implies that no 1−unviable candidate is among the winners in equilibrium. We now move to Step B. that proves the induction argument.

Step B: Induction Argument.

Assume now that no voter approves of his least preferred $j-$viable candidate with $j \in \{0, \ldots, k-1\}$ and an $h-$unviable candidate does not belong to the set of equilibrium winners for any $h \in \{0, \ldots, k\}$. This Step proves that this implies that no voter approves of his least preferred $k-$viable candidate and that that an $(k+1)-$unviable candidate does not belong to the set of equilibrium winners.

Step B.1. We first prove that a voter never approves of his least preferred $k-$viable candidate.

Since no $j-$unviable candidate belongs to the set of equilibrium winners for any $j \in \{0, \ldots, k\}$, it follows that just $k-$viable candidates are in $W(\sigma)$ since $\mathcal{X}_k^v = \mathcal{X} \setminus \bigcup_{j=1}^k \mathcal{X}_j^{uv}$.

Denote by z the least preferred $k-$viable candidate of some voter i. This implies that just $k-$viable candidates are in $W(\sigma)$, it follows that $\lim_{\varepsilon\to 0} U_i(v^z\,;\, p^\varepsilon) < 0$ for any sequence of pivot vectors p^ε satisfying the ordering condition. Hence, it is not a best response to approve of z, as wanted.

Step B.2. We now prove that no $(k+1)-$unviable candidate belongs to the set of equilibrium winners. Assume, by contradiction that there is some $(k+1)-$unviable candidate y in some set $W(\sigma)$ of equilibrium winners. Since y is $(k+1)-$unviable, it follows that:

$$\exists\, x \in \mathcal{X} \quad \text{with}\, N(x,\ldots) > n - N^k(\ldots, y).$$

However, in Step B.1., we have proved that no voter approves his least preferred $k-$viable candidate y. Moreover, we have assumed that no voter approves of his least preferred $j-$viable candidate with $j \in \{0, \ldots, k-1\}$.

It follows that for any equilibrium σ, $S(y;\sigma) \leq n - N^k(\ldots, y)$ since $N^k(\ldots, y)$ denotes the number of voters who rank y last among the set of candidates and the set of $j-$viable candidates for any $j = 0, \ldots, k$.

Furthermore, in equilibrium, all voters approve of their first ranked candidate which implies that:

$$S(x;\sigma) \geq N(x,\ldots) > n - N^k(\ldots, y) > S(y;\sigma),$$

so that y is not in the winning set. Hence, no $(k+1)-$unviable candidate belongs to the set of equilibrium winners, entailing a contradiction and finishing the proof. □

Theorem 4 *Assume that the election has at least four candidates. If there are at least three viable* candidates, then there is a closed set of utilities $\hat{\mathcal{U}} \subseteq \mathcal{U}$ such that for any election with utility vector $u \in \hat{\mathcal{U}}$, there is an equilibrium in which all viable* candidates are tied for victory.*

Proof of Theorem 1 We prove that, for any profile with at least three viable* candidates, there exists some strategy profile σ and some sequence of pivot vectors $p^{\varepsilon} = (p^{\varepsilon}_{xy})_{xy \in H}$ that constitutes an equilibrium in which all candidates in $\mathcal{X}^*$ are tied for victory so that $W(\sigma) = \mathcal{X}^*$.

The proof is divided in three sections: the preferences, the pivot probabilities and the conclusion.

Section I: The voters' preferences

Take a preference profile with $\#\mathcal{X}^* \geq 3$. Moreover, take some candidate $c \in \mathcal{X}^*$ and such that $n - N(\ldots, c) = \min_{x \in \mathcal{X}^*} n - N(\ldots, x)$.

Due to Lemma 5, any $d \in \mathcal{X}^*$ satisfies $n - N(\ldots, d) \geq n - N(\ldots, c)$ whereas if $d \in \mathcal{X} \setminus \mathcal{X}^*$ then $n - N(\ldots, d) < n - N(\ldots, c)$.

Moreover, let $a, b \in \mathcal{X}^*$ with

$$a = \arg\max_{x \in \mathcal{X}^*} n - N(\ldots, x) \text{ and}$$

$$b = \arg\max_{x \in \mathcal{X}^* \setminus \{a\}} n - N(\ldots, x).$$

Consider the voters' preferences restricted to the set of candidates $M = \{a, b, c\}$. Moreover, for each $i \in N$ we recall that $R_i = \frac{u_i(t_i) - u_i(m_i)}{u_i(m_i) - u_i(b_i)}$.

Section II: The Pivot Probabilities

We assume that the sequence of pivot probabilities p^{ε} satisfies for any $xy \neq ab, ac, bc$,

$$\lim_{\varepsilon \to 0} \frac{p^{\varepsilon}_{xy}}{p^{\varepsilon}_{xa}} = 0, \ \lim_{\varepsilon \to 0} \frac{p^{\varepsilon}_{xy}}{p^{\varepsilon}_{xb}} = 0 \ \text{ and } \lim_{\varepsilon \to 0} \frac{p^{\varepsilon}_{xy}}{p^{\varepsilon}_{xc}} = 0. \ (f)$$

The condition (g) implies that when $\varepsilon \to 0$, the voter's decision concerning whether to cast a vote for $x \neq a, b, c$, only depends on the pivotal events in which candidates b and c are involved, the rest of them becoming infinitely less likely.

Given these assumptions, we have two implications concerning the voters' decisions. These implications are different if one considers the decision over a, b and c or a different candidate.

Section $II.1$: Votes for a,b,c.

Consider the expected utility for a voter i of casting ballot v^a which consists of a vote just for candidate a (and no points for the rest of the candidates):

$$U_i(v^a; p^\varepsilon) = \sum_{ax \in H} p^\varepsilon_{ax}(u_i(a) - u_i(x)).$$

However, since (f) applies, it follows that

$$\lim_{\varepsilon \to 0} \frac{p^\varepsilon_{xy}}{p^\varepsilon_{ab} + p^\varepsilon_{ac} + p^\varepsilon_{bc}} = 0,$$

whenever $xy \neq ab, ac, bc$. Therefore, the following limit equality holds

$$\lim_{\varepsilon \to 0} \frac{U_i(v^a; p^\varepsilon)}{p^\varepsilon_{ab} + p^\varepsilon_{ac} + p^\varepsilon_{bc}} = \frac{p^\varepsilon_{ab}}{p^\varepsilon_{ab} + p^\varepsilon_{ac} + p^\varepsilon_{bc}}(u_i(a) - u_i(b)) + \frac{p^\varepsilon_{ac}}{p^\varepsilon_{ab} + p^\varepsilon_{ac} + p^\varepsilon_{bc}}(u_i(a) - u_i(c)).$$

Hence, writing $q^\varepsilon_{xy} = \frac{p^\varepsilon_{xy}}{p^\varepsilon_{ab} + p^\varepsilon_{ac} + p^\varepsilon_{bc}}$, it follows that

$$\lim_{\varepsilon \to 0} \text{Sign}(U_i(v^a; p^\varepsilon)) = \text{Sign}\left(q^\varepsilon_{ab}(u_i(a) - u_i(b)) + q^\varepsilon_{ac}(u_i(a) - u_i(c))\right).$$

Note that the sign of the utility is the only information needed to determine the voter's best response (since under AV, no constraints are given on the number of dis/approved candidates). Therefore, following a similar reasoning, it can be deduced that:

$$\lim_{\varepsilon \to 0} \text{Sign}(U_i(v^b; p^\varepsilon)) = \text{Sign}\left(-q^\varepsilon_{ab}(u_i(a) - u_i(b)) + q^\varepsilon_{bc}(u_i(b) - u_i(c))\right),$$

and

$$\lim_{\varepsilon \to 0} \text{Sign}(U_i(v^c; p^\varepsilon)) = \text{Sign}\left(-q^\varepsilon_{ac}(u_i(a) - u_i(c)) - q^\varepsilon_{bc}(u_i(b) - u_i(c))\right).$$

Therefore, the decision of the voters over these candidates is equivalent to the one with just three candidates (a, b and c). As discussed in the primer of the proof, we can choose a set of utilities and conditions on p^ε_{ab}, p^ε_{ac} and p^ε_{bc} such that the three candidates are tied for victory for some best response σ. The score of the three candidates equals $n - N(\ldots, c)$.

Section $II.2$: Votes for the rest of the viable candidates.

Consider any candidate $d \in \mathcal{X}^v$. By assumption, note that $n - N(\ldots, c) \leq n - N(\ldots, d)$.

Moreover, it is the case that $N(d, \ldots) < n - N(\ldots, c)$. Indeed, assume by contradiction that $N(d, \ldots) > n - N(\ldots, c)$. Then, c is unviable, a contradiction since c in $\mathcal{X}^{in}$.

As proved by Lemma 3, we can choose a best response σ such that the score of candidate d can take any value in $[N(d, \ldots), n - N(\ldots, d)]$.

Moreover, since $N(d, \ldots) < n - N(\ldots, c) \leq n - N(\ldots, d)$, we can set $S(d) = n - N(\ldots, c)$.

In other words, for each candidate $d \in \mathcal{X}^v$, we can find pivot probabilities, cardinal utilities and voters' best responses such that $S(d) = n - N(\ldots, c)$.

Section $II.3$: **Votes for the rest of the unviable candidates.**

The votes for these candidates do not affect the pivot probabilities. Indeed, since an unviable candidate cannot win (by definition), this candidate is not in the winning set of the equilibrium.

Section III: **Conclusion.**

It follows that given p^ε we can choose a strategy profile σ such that every voter chooses among his best responses and $W(\sigma) = \mathcal{X}^v$. Since p^ε follows the ordering condition, σ is an equilibrium in which all the viable candidates are tied as wanted. □

6 Concluding Comments

This work focuses on the implications of allowing strategic voters to vote for as many candidates as they want. We divide the elections into two categories: the ones in which at most two candidates are viable (a) and the ones in which at least there are three viable candidates (b). Our work fully characterizes the set of equilibrium winners for each election.

The results are surprisingly different in both scenarios. It can be argued that the first scenario is much more plausible than the second one at an empirical level. However, note that this intuition holds for plurality elections in which the Duverger's law tends to hold. Within the model, the election takes place under approval voting so that defining a priori what is more plausible seems elusive.

In scenario (a), our model uniquely predicts that the unique equilibrium winner is the Condorcet Winner. Moreover, note that the existence of two viable candidates is a necessary and sufficient condition for the uniqueness of this equilibrium. This is a strong argument for the use of this rule since it coincides with the recommendation made by different fairness theories (i.e. tournament solutions) that entitle that such a candidate should win if it exists.

In contrast, in scenario (b), we prove that there is some equilibrium in which the set of viable candidate coincides with the set of equilibrium winners. More specifically, we show that for any preference profile that admits at least three viable candidates, we can build an equilibrium in which all these candidates are tied for victory. More precisely, our result states that, given the ordinal preferences, there are some cardinal preferences that would admit a voting equilibrium in which every viable candidate is a likely winner. Note that the set of cardinal preferences that admit such large sets of winners may be very small. Finally, our result suggests that this rule may exhibit some indecisiveness when many candidates might win.

A potentially interesting venue for the current model would be to test our model of strategic voting on experimental data or real data. Clearly, the main testable prediction that can be derived from our contribution is the presence of ties among winners with Approval voting when (*i*) there is no Condorcet Winner or (*ii*) when there are at least three viable candidates. Moreover, pushing further the notion of viable candidate and understanding how this concept can be adapted to other theoretical and empirical models seems also very pertinent.

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Techniques for Probability Computations

Combinatorics of Election Scores

Alexander Karpov

1 Introduction

There are many electoral systems that determine the winner by computing a score of each candidate and a candidate with the best score is declared elected. Positional scoring rules, the Young rule, and the Dodgson rule are among these rules. A problem of calculating a score of a candidate (election score problem) is of special interest in computational social choice. Election score problem can be harder than winner determination problem (Fitzsimmons and Hemaspaandra 2018).

Some ranking aggregation methods determine the winner ranking by computing a score of each ranking and a ranking with the best score becomes a winner. The Kemeny rule is the most important example of such methods.

Election scores vector contains some information about preference profile. In general, preference profiles cannot be reconstructed from election scores vector, but it is possible to find a class of preference profiles, which lead to the same election scores vector. The higher the number of possible election scores vectors, the more information election score vector possibly contains.

Intuitively, the plurality scores contain less information about preference profile than the Borda scores. In some sense, the plurality rule needs less information form preference profile, than the Borda rule. The celebrated Fishburn (1977) classification of voting rules does not represent this relation. According to Fishburn's classification, the Borda rule belongs to C2—voting rules that only depend on weighted pairwise majority comparisons (De 2000); but the plurality rule belongs to C3—voting rules that require more information than weighted pairwise majority comparisons.

A. Karpov (✉)
National Research University Higher School of Economics,
20 Myasnitskaya str., Moscow 10100, Russia
e-mail: akarpov@hse.ru

Russian Academy of Science, Institute of Control Science, 65 Profsoyuznaya str.,
Moscow 117997, Russia

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The aim of our study is to find the number of different election scores, generated by the most popular rules. If a particular scoring rule (in the most general sense) leads to a higher number of election scores, then it utilizes more information from preference profiles.

Starting from an impartial, anonymous, and neutral culture (IANC) model (Egecioglu and Giritgil 2013) and enumerative combinatorics of anonymous and neutral equivalence classes of preference profiles (ANECs), we introduce reverse invariant ANECs (RIANECs) and self-symmetric ANECs (SSANECs). These classes of ANECs exploit the reversal symmetry of preference profiles. Note that we do not introduce any probability distribution over preference profiles. We consider the IANC model only as a combinatorial object. For a three-alternatives case, we design a multigraph representation of ANECs and a bracelet representation of RIANECs. These objects are well-studied in combinatorics.

An election scores vector may not allow for all preference profiles to be separately identified. Some preference profiles from different ANECs are indiscernible (they lead to the same election scores vector), e.g., preference profiles with the same structure of the top alternatives under the plurality rule. The class of preference profiles that lead to the same election scores vector under a voting rule is called a voting situation.

Voting situations induce a partition of the set of ANECs. A finer partition means that a rule in some sense utilizes more information from preference profile. For a perfectly discernible voting rule, which distinguishes all ANECs, the number of voting situations equals the number of ANECs.

To illustrate the general framework, we analyze three rules with candidates scores: the plurality rule, the Borda rule, and the scoring rules in the extreme case, and the Kemeny rule as an example of rankings scores. The main results are obtained for the case of three alternatives. Our framework can be applied to other rules and a higher number of alternatives.

There are two close approaches for evaluation of information complexity in social choice literature [see Boutilier and Rosenschein (2016) for survey of related literature]. The first one is the communication complexity theory of Conitzer and Sandholm (2005). They found bounds of communication required to elicit election results. They studied the worst-case number of bits that must be communicated to execute a given voting rule, when nothing is known in advance about the voters' preferences. The second approach is informational requirements of social choice rules of Sato (2009, 2016). In these papers complexity is studied in terms of message functions and message profile, i.e., each preference order is translated to the message space according to a voting rule. Our model differs from both approaches presented in the literature. There is no concept of the worst-case scenario in our paper. In contrast to Sato (2009, 2016) we have information derived from preference profile, but not from preference orders. Surely our approach is more general.

Combinatorics of preference profiles has a long tradition in social choice theory. May (1948), Guilbaud (1952) were the first who estimated the likelihood of Condorcet cycle and related events. Gehrlein and Fishburn (1976, 1979) further developed this field. It became important area of research with own problems and own line of development. The latest papers in this direction contain counting results related to single-peaked domain (Lackner and Lackner 2017) and group-separable domain (Karpov 2019).

The structure of the paper is as follows: Sect. 2 describes a mathematical model of IANC and the basic combinatorial theory; Sect. 3 presents the main result about voting rules; and Sect. 4 compares voting rules and concludes the paper. Proofs of propositions are given in Appendix.

2 Framework

Let a finite set $X = \{1, \ldots, m\}$, $m \geq 2$ be the set of alternatives and a finite set $N = \{1, \ldots, n\}$, $n \geq 2$ be the set of agents (voters). Each agent $i \in N$ has a strict preference order P_i over X (linear order). Let $\mathcal{L}(X)$ be the set of all possible linear orders over X. An n-tuple of preference orders generates a preference profile $\mathcal{P} = (P_1, \ldots, P_n) \in \mathcal{L}(X)^n$.

Within this model, the names of voters (anonymity) and names of alternatives (neutrality) do not matter. An anonymous and neutral equivalence class (ANEC) is a set of preference profiles that can be obtained from each other by permuting preference orders and renaming alternatives. Let a set S^k be the set of all permutation of k-elements set. The permutation of voters is denoted by $\sigma \in S^n$, and the permutation of alternatives is denoted by $\tau \in S^m$. Performing a change of the names of alternatives and a change on the names of individuals leads to an action of the group $S^n \times S^m$ on the set of preference profiles. The image of profile $\mathcal{P}$ under permutations σ,τ is denoted by $\mathcal{P}^{(\sigma,\tau)}$. Preference profiles $\mathcal{P}$,$\mathcal{P}'$ belong to the same ANEC if and only if there are permutations $\sigma \in S^n$,$\tau \in S^m$, such that $\mathcal{P}^{(\sigma,\tau)} = \mathcal{P}'$. This relation, which is symmetric, is denoted as $\mathcal{P} \sim_{ANEC} \mathcal{P}'$. The complementary binary relation is denoted as $\nsim_{ANEC}$. The function $pos(P_i, j) = |\{x \in X | xP_ij\}| + 1$ indicates the position of candidate j in preference profile P_i.

We use the following rounding functions: $\lfloor x \rfloor$ is rounding down to the nearest integer $\lceil x \rceil$ is rounding up to the nearest integer, and $Round[x]$ is rounding down if the fractional part is less than 0.5 and rounding up if the fractional part is greater than or equal to 0.5.

2.1 ANEC Enumeration Problem

The ANEC enumeration problem is a starting point of the paper. It was solved (Egecioglu and Giritgil, 2013). To state the authors' results, we need some notation from their papers. A partition λ of an integer n is a weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ with $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. Each of the integers $\lambda_i > 0$ is called a part of n. For example, $\lambda = (3, 2, 2)$ is a partition of $n = 7$ into three parts. It has two parts of size two and one part of size three. If λ is a partition of n, then this is denoted by $\lambda \vdash n$. Each partition of n has a type denoted by the symbol $1^{\alpha_1}2^{\alpha_2}\dots n^{\alpha_n}$, which signifies that $\lambda = (3, 2, 2)$ is $1^0 2^2 3^1 4^0 5^0 6^0 7^0$. We can omit the zeros that appear as exponents and write the type of λ as $2^2 3^1$.

Let $p_k(n)$ be the number of partitions of n with exactly k parts. It is also the number of partitions of n in which the largest part has size k (Stanley 2012). Let $p_{k,l}(n)$ be the number of partitions with k parts, each of which does not exceed l. Separating the greatest part, we obtain

$$p_{k,l}(n) = \sum_{i=1}^{\lfloor \frac{n}{k} \rfloor} p_{k-1,\min\{l,n-2i\}}(n-i).$$

For $k = 2$ and $n - 1 \geq l \geq \frac{n}{2}$, we have:

$$p_{2,l} = l - \lfloor \frac{n-1}{2} \rfloor.$$

A permutation σ of n defines a partition of n where the parts of the partition are the cycle lengths in the cycle decomposition of σ. The cycle type of σ is defined as the type of the resulting partition. For example, $\sigma = (142)(35)(67)$ has cycle type $2^2 3^1$. For any $\lambda \vdash n$ of type $1^{\alpha_1}2^{\alpha_2}\dots n^{\alpha_n}$, define a number:

$$Z_\lambda = 1^{\alpha_1}2^{\alpha_2}\dots n^{\alpha_n}\alpha_1!\alpha_2!\dots\alpha_n!.$$

The number of permutations of cycle type $1^{\alpha_1}2^{\alpha_2}\dots n^{\alpha_n}$ is given by $z_\lambda^{-1}n!$ where λ is the partition of cycle lengths of σ.

Proposition 1 *(Egecioglu and Giritgil 2013) For n voters and m alternatives, the number of ANECs is equal to*

$$\#ANEC(m,n) = \sum_{\lambda \vdash m} z_\lambda^{-1} \binom{\frac{n}{d} + \frac{m!}{d} - 1}{\frac{m!}{d} - 1},$$

where $d = d(\lambda) =$ Least common multiple of λ; z_λ is is defined above and

$$\binom{x}{k} = \begin{cases} \frac{x!}{k!(x-k)!} & \text{if } x \text{ is integer,} \\ 0 & \text{otherwise.} \end{cases}$$

$\#ANEC(m,n)$ is a polynomial in n of degree $m! - 1$.

Veselova (2016) compared $\#ANEC(m,n)$ with $\#AEC(m,n)$ (anonymous equivalence classes) and $\#EC(m,n)$ (equivalence classes) and explored their asymptotic properties. Asymptotically, the IANC, IAC, and IC models lead to the same results. IAC and IC models are applicable for the equiprobable generation of preference profiles and simulation studies (Gehrlein and Lepelley 2011, 2017). In our model, we do not make simulations. The neutrality property is needed for defining voting situations. Neutrality also clarifies combinatorial structures, which arise in this paper.

A preference profile $\mathcal{P}'$ is the **reversal of preference profile** $\mathcal{P}$ if $\forall x \in X$, $\forall i \in \mathcal{N}$, $pos(P_i, x) = m + 1 - pos(P'_i, x)$. This type of symmetry was studied by Saari and Barney (2003), Crisman (2014), Bubboloni and Gori (2015, 2016).

An ANEC is **self-symmetric (SSANEC)** if for every $\mathcal{P}$ from the ANEC, the reverse profile $\mathcal{P}'$ belongs to the same ANEC. A pair of ANECs is **reverse symmetric** if for every $\mathcal{P}$ from one ANEC, the reverse profile $\mathcal{P}'$ belongs to the other ANEC.

Considering reverse symmetric ANECs as equivalent, we obtain a set of **reverse invariant ANECs (RIANECs)** and binary relation $\sim_{RIANEC}$, which contains relation $\sim_{ANEC}$.

2.2 *Three Alternatives Case*

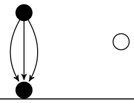
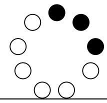
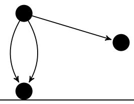
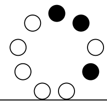
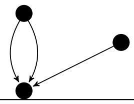
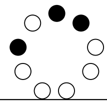
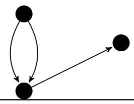
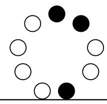
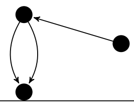
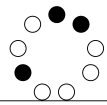
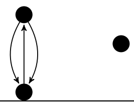
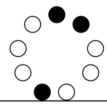
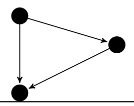
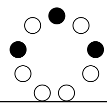
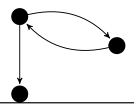
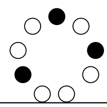
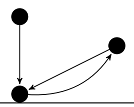
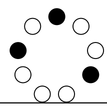
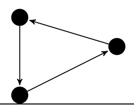
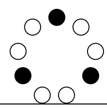
This section introduces two new representations of preference profiles. These representations reveal the internal structure of preference profiles and enable us to calculate the number of RIANECs and SSANECs.

Definition 1 Multigraph representation of preference profile. Having 3 alternatives as vertices of a graph, for each preference order in the profile, we define an arc from the best alternative to the worst alternative.

Multigraph representation is anonymous and neutral. The renaming of alternatives leads to graph isomorphism. The corresponding multigraph uniquely represents a preference profile. Table 1 contains several examples of the multigraph representation of preference profiles. $\#ANEC(3, n)$ is also the number of multigraphs with 3 nodes and n arcs (it is the A037240 sequence in the on-line encyclopedia of integer sequences, published electronically at http://oeis.org; henceforth OEIS). In the 3-alternatives case, formula from Proposition 1 leads to:

$$\#ANEC(3,n) = \begin{cases} \frac{1}{6}\binom{n+5}{5} + \frac{1}{16}(n+4)(n+2) + \frac{1}{9}(n+3), \text{if } n \equiv 0 \pmod 6; \\ \frac{1}{6}\binom{n+5}{5}, \text{if } n \equiv (1 \text{or } 5) \pmod 6; \\ \frac{1}{6}\binom{n+5}{5} + \frac{1}{16}(n+4)(n+2), \text{if } n \equiv (2 \text{or } 4) \pmod 6; \\ \frac{1}{6}\binom{n+5}{5} + \frac{1}{9}(n+3), \text{if } n \equiv 3 \pmod 6. \end{cases}$$

Table 1 3 agents and 3 alternatives case. List of ANECs. Multigraph and bracelet representations of preference profiles

Number	ANEC	Multigraph representation	Bracelet representation
$\mathcal{P}_1$	*a a a* *b b b* *c c c*		
$\mathcal{P}_2$	*a a a* *b b c* *c c b*		
$\mathcal{P}_3$	*a a b* *b b a* *c c c*		
$\mathcal{P}_4$	*a a c* *b b a* *c c b*		
$\mathcal{P}_5$	*a a c* *b b c* *c c a*		
$\mathcal{P}_6$	*a a c* *b b b* *c c a*		
$\mathcal{P}_7$	*a a b* *b c a* *c b c*		
$\mathcal{P}_8$	*a a b* *b c c* *c b a*		
$\mathcal{P}_9$	*a b c* *b a a* *c c b*		
$\mathcal{P}_{10}$	*a b c* *b c a* *c a b*		

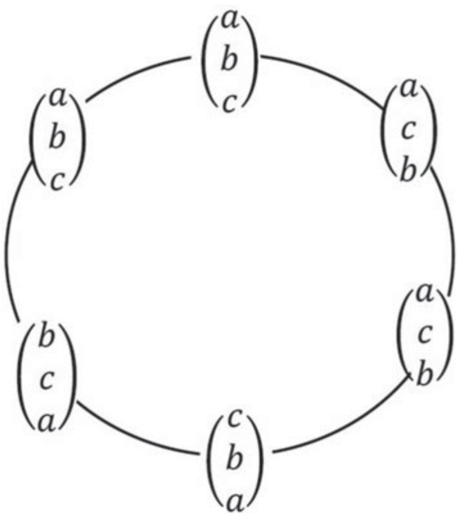


Fig. 1 Circle representation of preference orders

For $m = 3$, there are six different preference orders. Defining residues modulo 6, we numerate preference orders from 0 to 5 in the clockwise manner:

$$P_0 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, P_1 = \begin{pmatrix} a \\ c \\ b \end{pmatrix}, P_2 = \begin{pmatrix} c \\ a \\ b \end{pmatrix}, P_3 = \begin{pmatrix} c \\ b \\ a \end{pmatrix}, P_4 = \begin{pmatrix} b \\ c \\ a \end{pmatrix}, P_5 = \begin{pmatrix} b \\ a \\ c \end{pmatrix}.$$

Each subsequent preference order is obtained from the previous preference order by one pairwise swap of consecutive alternatives. The next nearest preference order is obtained by two swaps. The highest number of swaps is three, which leads to preference order reversal. Putting preference orders on a loop we obtain a circle representation of preference orders, which is presented in Fig. 1.

A preference profile is a string of n preference orders, each of 6 possible types $P_0, P_1, P_2, P_3, P_4, P_5$. By anonymity, the order of preference orders does not matter. Only the number of different preference orders matters. Neutrality links different preference orders in the cycle structure depicted in Fig. 1. Circle from Fig. 1 is also a permutahedron. For $m = 4$, the corresponding permutahedron becomes much more complicated and does not have all symmetries, which are observed in case of $m = 3$.

Permutating (renaming) one pair of alternatives (one or three swaps in a preference order) leads to a preference order circle turnover. There are 3 possible pairs and 3 axes that divide the circle into halves. Two possibilities of permutating (renaming) three alternatives (two swaps in a preference order) lead to the preference order circle rotating on 2 preference orders in a clockwise or counterclockwise manner. The circle representation of preference orders is applied for the Kemeny rule analysis further in the paper.

Neutrality, anonymity, and reverse invariance lead to a simpler representation of preference profiles. Instead of using 6 types of preference orders (letters) in the standard representation of preference profiles, we can use only 2 types of letters. Because we apply the cycle structure of a string, taking all rotations and turnovers as equivalent, we call the new object a bracelet (term is borrowed from combinatorics theory). For demonstration purposes, we provide a definition using beads of two colors. It is possible to rewrite definition in terms of string with two types of letters.

Definition 2 **Bracelet representation of preference profiles.** According to the circle representation of preference orders (Fig. 1), we numerate preference orders from 0 to 5 in a clockwise manner. Every preference order in a profile is represented by a black bead. The space between preference orders in the circle representation of preference orders is represented by a white bead. A preference profile is a bracelet with $n + 6$ beads, where exactly 6 beads are white. We take n_0 black beads, where n_0 is the number of type 0 preference orders, then one white bead, then n_1 black beads, where n_1 is the number of type 1 preference orders, etc. Adding a white bead between n_5 black beads, where n_5 is the number of type 5 preference orders, and n_0 black beads, where n_0 is the number of type 0 preference orders, we complete the bracelet.

Table 1 contains examples of the bracelet representation of preference profiles. The starting point of the circle representation of preference orders and the way of numbering (clockwise or counter-clockwise) do not matter. Bracelets are equivalent up to rotating and turnover. The bracelet representation of preference profiles is anonymous, neutral, and reverse invariant.

Permutating one pair of alternatives (one or three swaps in a preference order) leads to bracelet turnover. Two possibilities of permutating three alternatives (two swaps in a preference order) leads to preference order bracelet rotation. Reversing a preference profile leads to a rotation on three preference orders in a clockwise manner.

For $n = m = 3$, we have $\#ANEC(3, 3) = 10$. According to Table 1, in the case of 3 alternatives and 3 agents, there are 10 different multigraphs and 7 different bracelets representing 10 ANECs [(preference profiles represented different ANECS and numbering of ANECs are borrowed from (Karpov 2017)]. Some ANECs have equivalent bracelet representations. These ANECs belong to the same RIANEC. For example, Table 1 presents preference profiles $\mathcal{P}_2$ and $\mathcal{P}_3$, which are reverse symmetric and bracelets generated by these preference profiles are equivalent.

Every $(n + 6)$—beads bracelet (turnover invariant) with 6 white beads corresponds to a RIANEC. Shevelev (A005513 in OEIS) proved the formula for the number of such bracelets. Thus, we have Proposition 2.

Proposition 2 *(Shevelev, A005513 in OEIS). For $m = 3$, the number of reverse invariant ANECs is equal to*

$$\#RIANEC(3,n) = \begin{cases} \frac{1}{12}\binom{n+5}{5} + \frac{1}{96}(n+7)(n+4)(n+2) + \frac{1}{18}(n+6), \textit{ if } n \equiv 0 \textit{ (mod 6);} \\ \frac{1}{12}\binom{n+5}{5} + \frac{1}{96}(n+5)(n+3)(n+1), \textit{ if } n \equiv (1 \textit{ or } 5) \textit{ (mod 6);} \\ \frac{1}{12}\binom{n+5}{5} + \frac{1}{96}(n+7)(n+4)(n+2), \textit{ if } n \equiv (2 \textit{ or } 4) \textit{ (mod 6);} \\ \lfloor \frac{1}{12}\binom{n+5}{5} + \frac{1}{96}(n+5)(n+3)(n+1) + \frac{1}{18}(n+6) \rfloor, \textit{ if } n \equiv 3 \textit{ (mod 6).} \end{cases}$$

This sequence arises in several enumeration problems. To the best of the author's knowledge, Hoskins and Penfold (1982), who studied the geometry of fabrics, were the first with this sequence.

All ANECs are either self-symmetric or have reverse symmetric ANECs. Thus, we have $\#ANEC(3,n) = 2\#RIANEC(3,n) - \#SSANEC(3,n)$. From this, we obtain the number of SSANECs, which leads to proposition 3.

Proposition 3 *For $m = 3$, the number of self-symmetric ANECs is equal to*

$$\#SSANEC(3,n) = \begin{cases} \lceil \frac{1}{48}(n+4)^2(n+2), \textit{ if } n \textit{ is even}; \\ \frac{1}{48}(n+5)(n+3)(n+1), \textit{ if } n \textit{ is odd}. \end{cases}$$

The number of SSANECs is relatively small, with $\lim\limits_{n\to\infty} \frac{\#SSANEC(3,n)}{\#ANEC(3,n)} = 0$ and $\lim\limits_{n\to\infty} \frac{\#RIANEC(3,n)}{\#ANEC(3,n)} = \frac{1}{2}$. Almost all ANECs have their own symmetric ANEC. Appendix 2 contains a table with $\#ANEC(3,n)$, $\#RIANEC(3,n)$ and $\#SSANEC(3,n)$.

3 Voting Situations Induced by Voting Rules

We consider voting rules, which can be represented as a procedure of finding the highest/lowest score in some vector. Here, we consider positional scoring rules and the Kemeny rule. In the anonymous and neutral model election results are equivalent if one can be obtained from another by renaming alternatives.

A voting situation is a set of all ANECs, which lead to the same anonymous and neutral election results.

Let a voting rule $\alpha : \mathcal{L}(X)^n \to 2^X$ has the following representation:

$$\alpha = \arg\max_{i \in X} cs_i,$$

where $cs \in \mathbb{R}^m$ is a candidates scores vector. Preference profiles $\mathcal{P}$,$\mathcal{P}'$ belong to the same voting situation if and only if there are permutations $\sigma \in S^n$,$\tau \in S^m$, such that $cs((\mathcal{P}^\sigma)^\tau) = cs(\mathcal{P}')$.

Let a ranking rule $\alpha : \mathcal{L}(X)^n \to 2^X$ has the following representation:

$$\beta = \arg\max_{i \in X} cs_i,$$

where $cs \in \mathbb{R}^{m!}$ is a candidates scores vector. Preference profiles $\mathcal{P}$,$\mathcal{P}'$ belong to the same voting situation if and only if there are permutations $\sigma \in S^n$,$\tau \in S^m$, such that $cs((\mathcal{P}^\sigma)^\tau) = cs(\mathcal{P}')$.

It is important to note that we do not distinguish preference profiles and voting rules by the final choice.

For example, for the plurality rule, only the top alternative in each preference order is elicited. For the case of 3 agents and 3 alternatives, we have only three types of voting situations described by the top alternatives up to renaming voters and alternatives: (a, a, a), (a, a, b), and (a, b, c). By neutrality, we do not distinguish (a, a, a) and (b, b, b). For the plurality rule we have the same problem for the both situations: What is the winner in unanimous case? From information that is utilized by the plurality rule, we do not distinguish preference profiles $\mathcal{P}_1$ and $\mathcal{P}_2$. For the case of 3 agents and 3 alternatives, the plurality rule partitions the set of ANECs into three parts:$(\mathcal{P}_1, \mathcal{P}_2)$,$(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5, \mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)$, and $(\mathcal{P}_9, \mathcal{P}_{10})$. We have three voting situations generated by the plurality rule. For the case of 4 agents and 3 alternatives, we have only four types of voting situations described by top alternatives: (a, a, a, a), (a, a, a, b), (a, a, b, b), and (a, a, b, c).

$\#Rule(m, n)$ is the number of voting situations induced by a rule. For the case of 3 agents and 3 alternatives, we have $\#Plurality(3, 3) = 3$. $\#Rule(m, n)$ does not exceed $\#ANEC(m, n)$. In the case of $\#Rule(m, n) = \#ANEC(m, n)$, we can unambiguously reconstruct the ANEC from a voting situation (this property is called strong discernibility). In other cases, that reconstruction is impossible. Strong discernibility is an important property in preference diversity measurement. It has‘ been proven that in some classes of preference diversity indices, there is no function that satisfies strong discernibility (Hashemi and Endriss 2014) and reverse invariant discernibility (Karpov 2017). Some voting rules utilize the same information as preference diversity indices and fail to achieve strong or reverse invariant discernibility.

In the following subsections, the number of voting situations induced by the plurality rule, the Kemeny rule, the Borda rule, and the scoring rules in extreme cases are calculated.

3.1 The Plurality Rule

The plurality rule compares alternatives by the number of preference orders where an alternative occupies the top position. The plurality rule utilizes information about the partition of top choices. This partition has from 1 to m parts, which leads to proposition 4.

Proposition 4 *The number of voting situations induced by the plurality rule is equal to*

$$\#Plurality(m, n) = \sum_{i=1}^{m} p_i(n).$$

Because for all $k > n$, we have $p_k(n) = 0$, then for each n the sequence $\#Plurality(3, n)$ has an upper bound.

$\#Plurality(3, n)$ is also the number of multigraphs with 3 nodes and n edges. The corresponding multigraph is defined in the following way. For each preference order in the profile, we define an edge connecting the worst alternative and the second-worst alternative.

Corollary 1 ***(A001399, OEIS).*** *For $m = 3$, the number of equivalent classes generated by the plurality rule is equal to*

$$\#Plurality(3, n) = round\left[\frac{1}{12}(n+3)^2\right].$$

Note that ties in rounding in this formula never arise.

The values of $\#Plurality(3, n)$ and the number of voting situations induced by other rules are given in conclusion.

3.2 The Borda Rule

The Borda rule is a scoring rule in which the worst alternative has a score of 0, and the best alternative has a score of m-1. The Borda rule utilizes information only about the sum of scores for each alternative.

For the 3 alternatives and 3 agents case, we have the following voting situations (ANECs and corresponding sums of scores vector in decreasing order):

$$\mathcal{P}_1 : \begin{pmatrix}6\\3\\0\end{pmatrix}; \mathcal{P}_2 : \begin{pmatrix}6\\2\\1\end{pmatrix}; \mathcal{P}_3 : \begin{pmatrix}5\\4\\0\end{pmatrix}; \mathcal{P}_4 : \begin{pmatrix}5\\2\\2\end{pmatrix}; \mathcal{P}_5 : \begin{pmatrix}4\\4\\1\end{pmatrix}; \mathcal{P}_{6,8,9} : \begin{pmatrix}4\\3\\2\end{pmatrix}; \mathcal{P}_7 : \begin{pmatrix}5\\3\\1\end{pmatrix}; \mathcal{P}_{10} : \begin{pmatrix}3\\3\\3\end{pmatrix}.$$

Voting situations correspond to different partitions of the sum of the scores.

Proposition 5 *For $m = 3$ and $n \geq 2$, the number of voting situations induced by the Borda rule is equal to*

$$\#Borda(3, n) = \lceil \frac{1}{2}(n+1)^2 \rceil.$$

The proof for Proposition 5, and subsequent propositions are given in Appendix.

3.3 The Discernibility Potential of the Scoring Rules

The scoring rules utilize information only about the sum of scores for each alternative. In an extreme case of irrational scores (e.g., for $m = 3$ scores 0,1,$\sqrt{2}$), each combination of the scores (e.g., for $m = 3$, $n = 5$ scores 0,0,1,$\sqrt{2}$, $\sqrt{2}$) leads to a unique sum of scores ($1 + 2\sqrt{2}$ in our example). In this case, we can unambiguously derive the combination of scores from the sum of the score vector. In other words, we can derive the scoring matrix [(the scorix in the terminology of Perez-Fernandez and De Baets (2017)]. The element at the ith row and jth column of this matrix equals the number of times that the i-th candidate is ranked at the j-th position

$$a_{ij} = |\{k|pos(P_k, i) = j\}|.$$

The scoring matrix is used not only for scoring rules but also for other voting rules, e.g., for the threshold rule (Aleskerovetal 2010), and the rank-dependent scoring rules (Goldsmith et al. 2016). This matrix was investigated in pure mathematics in MacMahon (1918), where we find 10 scoring matrices for the 3 alternatives and 3 agents case, which corresponds to the voting situations (ANECs and corresponding scoring matrices):

$$\mathcal{P}_1\colon \begin{pmatrix} 3&0&0\\0&3&0\\0&0&3 \end{pmatrix};\ \mathcal{P}_2\colon \begin{pmatrix} 3&0&0\\0&2&1\\0&1&2 \end{pmatrix};\ \mathcal{P}_3\colon \begin{pmatrix} 2&1&0\\1&2&0\\0&0&3 \end{pmatrix};\ \mathcal{P}_4\colon \begin{pmatrix} 2&1&0\\0&2&1\\1&0&2 \end{pmatrix};\ \mathcal{P}_5\colon \begin{pmatrix} 2&0&1\\1&2&0\\0&1&2 \end{pmatrix};$$

$$\mathcal{P}_6\colon \begin{pmatrix} 2&0&1\\0&3&0\\1&0&2 \end{pmatrix};\ \mathcal{P}_7\colon \begin{pmatrix} 2&1&0\\1&1&1\\0&1&2 \end{pmatrix};\ \mathcal{P}_8\colon \begin{pmatrix} 2&0&1\\1&1&1\\0&2&1 \end{pmatrix};\ \mathcal{P}_9\colon \begin{pmatrix} 1&2&0\\1&1&1\\1&0&2 \end{pmatrix};\ \mathcal{P}_{10}\colon \begin{pmatrix} 1&1&1\\1&1&1\\1&1&1 \end{pmatrix}.$$

From the generating function in (A257464 in OEIS) and the author's calculations, we derive Proposition 6.

Proposition 6 *For $m = 3$, the number of voting situations induced by the scoring rules in the extreme case is equal to*

$$\#Scor(3,n) = \begin{cases} \lceil \frac{1}{48}(n^4 + 6n^3 + 18n^2 + 36n + 32), & \textit{if } n \textit{ is even};\\ \frac{1}{48}(n^4 + 6n^3 + 18n^2 + 18n + 5)\rceil, & \textit{if } n \textit{ is odd}. \end{cases}$$

MacMahon (1918) contains series for a higher m. In the extreme case, the scoring rule has a higher degree polynomial on n than the other scoring rules considered in this paper (the plurality rule and the Borda rule). For $m = 3$, the simplest example of indiscernibility is:

$$\hat{\mathcal{P}} = \begin{pmatrix} a & a & b & c \\ b & b & c & a \\ c & c & a & b \end{pmatrix}; \tilde{\mathcal{P}} = \begin{pmatrix} a & a & b & c \\ b & c & a & b \\ c & b & c & a \end{pmatrix}.$$

$\hat{\mathcal{P}}$ and $\tilde{\mathcal{P}}$ belong to different ANECs. The last three preference orders are different in different preference profiles, but they have the same scoring matrix. These preference orders represent different versions of the Condorcet cycle. Having the scoring matrix, it is impossible to reconstruct ANEC. For small n, there is a small number of such situations, but for big n the vast majority of ANECs have such indiscernibility.

3.4 The Kemeny Rule

The Kemeny rule uses the swap distance between preference orders (the number of pairwise swaps of consecutive alternatives that is needed to transform one order into another). Having the bracelet representation of preference profiles (Fig. 1), we obtain a circle of distances from order P_0 (Fig. 2). The circles of distances from other preference orders can be obtained by rotating the circle of distances from Fig. 2.

Permutating one pair of alternatives (one or three swaps in a preference order) lead to the turnover of the circle in Fig. 2. Two possibilities of permutating three alternatives (two swaps in the preference order) leads to the circle rotating. Reversing the preference profile leads to the circle rotating on 3 preference orders in a clockwise manner.

The Kemeny rule finds the order with the lowest sum of swap distances from the order to all orders in the preference profile. The Kemeny rule utilizes information only about the sums of swap distances between the preference profile and different preference orders.

For the 3 alternatives and 3 agents case, we have the following voting situations (ANECs and the corresponding sums of distances in the circle form, which is invariant up to rotating and turnover):

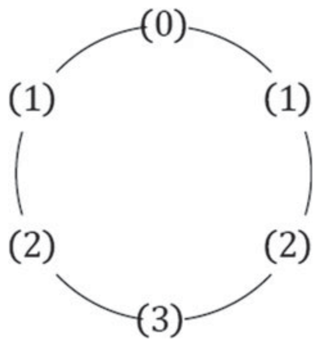


Fig. 2 Circle of distances from order P_0

$$\mathcal{P}_1:\ -0-3-6-9-6-3-;$$

$$\mathcal{P}_2, P_3:\ -1-2-5-8-7-4-;$$

$$\mathcal{P}_4, \mathcal{P}_5:\ -2-3-4-7-6-5-;$$

$$\mathcal{P}_6, \mathcal{P}_8, P_9:\ -3-4-5-6-5-4-;$$

$$\mathcal{P}_7:\ -2-3-6-8-6-3-;$$

$$\mathcal{P}_{10}:\ -4-5-4-5-4-5-.$$

Proposition 7 *For $m = 3$, the number of voting situations induced by the Kemeny rule is equal to*

$$\#Kemeny(3, n) = \begin{cases} \lceil \frac{1}{72}(4n^3 + 21n^2 + 54n + 45) \rceil, \textit{ if } n \textit{ is even}; \\ \lceil \frac{1}{72}(4n^3 + 21n^2 + 36n + 11) \rceil, \textit{ if } n \textit{ is odd}. \end{cases}$$

The Kemeny rule is more complicated and has a higher number of voting situations than the rules above.

4 Conclusion

In this paper, we found the number of voting situations associated with voting rules. Table 2 contains a table with the number of ANECs, the number of voting situations, the polynomial degree for the above-mentioned rules for the 3 alternatives case

Table 2 Number of voting situations, $m = 3$

	The number of voters							Polynomial degree
	2	3	4	5	6	7	8	
#ANEC	5	10	24	42	83	132	222	5
#RIANEC	4	7	16	26	50	76	126	5
#SSANEC	3	4	8	10	17	20	30	3
Plurality	2	3	4	5	7	8	10	2
Borda	5	8	13	18	25	32	41	2
Scoring	5	10	23	40	73	114	180	4
Kemeny	4	6	12	17	28	37	54	3

The plurality rule has the lowest number of voting situations. The scoring rules in the extreme case have the highest number of voting situations.

The number of voting situations induced by the plurality and Borda rules is represented by a polynomial in n of degree 2. At the limit, the number of voting situations induced by the Borda rule is 6 times higher than the number of voting situations induced by the plurality rule. The number of voting situations induced by Kemeny rule is represented by a polynomial in n of degree 3.

Strong discernibility arises only three times in the class of scoring rules: $I(Scoring, 3, 2) = I(Scoring, 3, 3) = I(Borda, 3, 2) = 1$. In other cases, it is impossible to reconstruct the ANEC from the voting situation. A general comparison of a higher number of voting rules is the goal for the future research.

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5 Appendix

Proof of proposition 5. For $m = 3$, a homogenous preference profile (n preference orders P_0) has vector of rank sums $\begin{pmatrix} 2n \\ n \\ 0 \end{pmatrix}$.Any preference profile has vector of ranks sums either $\begin{pmatrix} 2n \\ n \\ 0 \end{pmatrix}$ or vectors α,β,γ,δ,ϵ, defined here:

$$\alpha = \begin{pmatrix} 2n - x \\ n + x \\ 0 \end{pmatrix}; \ \beta = \begin{pmatrix} 2n \\ n - x \\ x \end{pmatrix}; \ \gamma = \begin{pmatrix} 2n - y \\ n - (x - y) \\ x \end{pmatrix}; \ \delta = \begin{pmatrix} 2n - x \\ n + y \\ x - y \end{pmatrix}; \ \epsilon = \begin{pmatrix} 2n - x \\ n \\ x \end{pmatrix}.$$

For all vectors α,β,γ,δ,ϵ, we have that the first component is not less than the second; the second component is not less than the third. Substituting x preference orders from P_0 to P_5, we obtain type α preference profile. There are $\lfloor \frac{n}{2} \rfloor$ preference profiles of type α.Similarly, we have $\lfloor \frac{n}{2} \rfloor$ voting situations of type β.

Substituting $\frac{y}{2}$ preference orders from P_0 to P_3 and $x-y$ preference orders from P_0 to P_1, we obtain type γ voting situations with even y.Substituting $\frac{(y-1)}{2}$ preference orders from P_0 to P_3, $x-y-1$ preference orders from P_0 to P_1, and one preference order from P_0 to P_2, we obtain type γ voting situations with odd y.

We can construct all possible type γ voting situations using these two design methods. There are two natural restrictions on x,y for saving the order of alternatives:

$$2x - y \leq n,$$

$$2y - x \leq n.$$

From this, we find the number of type γ voting situations (if $n \geq 4$):
for $y \leq \frac{x}{2}$, it is $\sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} p_2(i) + \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{\lfloor \frac{2n}{3} \rfloor} p_{2,n-i}(i)$;
for $y \geq \frac{x}{2}$, it is $\sum_{i=2}^{\lfloor \frac{2n}{3} \rfloor} p_2(i) + \sum_{i=\lfloor \frac{2n}{3} \rfloor+1}^{n-1} \left[p_2(i) - p_{2,2i-n-1}(i)\right]$;
for $y = \frac{x}{2}$, it is $\lfloor \frac{\lfloor \frac{2n}{3} \rfloor}{2} \rfloor$.
Thus, the number of type γ voting situations is equal to

$$\sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} p_2(i) + \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{\lfloor \frac{2n}{3} \rfloor} p_{2,n-i}(i) + \sum_{i=2}^{\lfloor \frac{2n}{3} \rfloor} p_2(i) + \sum_{i=\lfloor \frac{2n}{3} \rfloor+1}^{n-1} \left[p_2(i) - p_{2,2i-n-1}(i)\right] - \lfloor \frac{\lfloor \frac{2n}{3} \rfloor}{2} \rfloor$$

It is also the number of type δ voting situations.

Substituting $\frac{x}{2}$ preference orders from P_0 to P_3, we obtain type ϵ voting situations with even x. Substituting $\frac{(x-1)}{2}$ preference orders from P_0 to P_3, one preference order from P_0 to P_1, and one preference order from P_0 to P_5, we obtain type γ voting situations with odd x.

We can construct all possible type ϵ voting situations using these two design methods. There is one restriction on x for saving the order of alternatives:

$$x \leq n.$$

If $n \geq 2$ then the number of type ϵ voting situations is equal to n.

Summing over all types, we have (if $n \geq 4$)

$$1 + 2\left(\lfloor \frac{n}{2} \rfloor + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} p_2(i) + \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{\lfloor \frac{2n}{3} \rfloor} p_{2,n-i}(i) + \sum_{i=2}^{n-1} p_2(i) \right.$$
$$\left. - \sum_{i=\lfloor \frac{2n}{3} \rfloor+1}^{n-1} p_{2,2i-n-1}(i) - \lfloor \frac{\lfloor \frac{2n}{3} \rfloor}{2} \rfloor \right) + n$$

Modifying this, we obtain

$$3n - 1 + \lfloor \frac{2n}{3} \rfloor^2 - \lfloor \frac{2n}{3} \rfloor + \lfloor \frac{n}{2} \rfloor \left(\lfloor \frac{n}{2} \rfloor + 3 - 2n \right)$$
$$+ 2 \left(\sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} \lfloor \frac{i}{2} \rfloor + \sum_{i=2}^{n-1} \lfloor \frac{i}{2} \rfloor - \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{\lfloor \frac{2n}{3} \rfloor} \lfloor \frac{i-1}{2} \rfloor \right.$$
$$\left. + \sum_{i=\lfloor \frac{2n}{3}+1 \rfloor + 1}^{n-1} \left[\lfloor \frac{i-1}{2} \rfloor \right] - \lfloor \frac{\lfloor \frac{2n}{3} \rfloor}{2} \rfloor \right).$$

Calculating the sums, we obtain the result, which is also correct for $n = 2$, and $n = 3$.

Proof of proposition 7 Let $f_0(k)$: $\mathbb{N}_0 \to \mathbb{N}_0$ be the sum of distances between preference order P_0 and preference order P_i, where $i \equiv k \pmod 6$. Let $g_0(k) = f_0(k) + f_0(k+2)$ and $h_0(k) = (2g_0(k) - g_0(k+3))/6 = (2f_0(k) + 2f_0(k+2) - f_0(k+3) - f_0(k+5))/6$. This transformation is presented in Fig. 3a. The first circle is the circle of distances from order P_0 to all six preference orders. After transformation, each preference order is presented by three subsequent ones. This transformation has the inversion presented in Fig. 3b. We design one-to-one correspondence between $f_0(k)$ and $h_0(k)$.

In the same fashion, we define functions $f_j(k)$,$g_j(k)$,$h_j(k)$, $j = \overline{0,5}$. Instead of summing distances from preference orders $\sum_{j=0}^{5} f_j(k)n_j$, we sum transformed values $\sum_{j=0}^{5} h_j(k)n_j$. We have bisection between these sums. Any sum of transformed values $\sum_{j=0}^{5} h_j(k)n_j$ can be represented by the circle presented in Fig. 4.

Permutating one pair of alternatives (one or three swaps in a preference order) leads to Fig. 4 circle turnover. Two possibilities of permutating three alternatives (two swaps in a preference order) lead to the circle rotating. Reversing the preference profile leads to a rotating on 3 preference orders in a clockwise manner. By these operations we can always construct a circle, such that $x + y + z \leq 3n - x - y - z$ and $x \geq y \geq z$. From this definition, we have the following additional restrictions on x,y,z

$$x + y \geq n - z;$$

$$n - y + n - z \geq x;$$

$$z + n - y + n - z + y \geq 2x + 2(n - x).$$

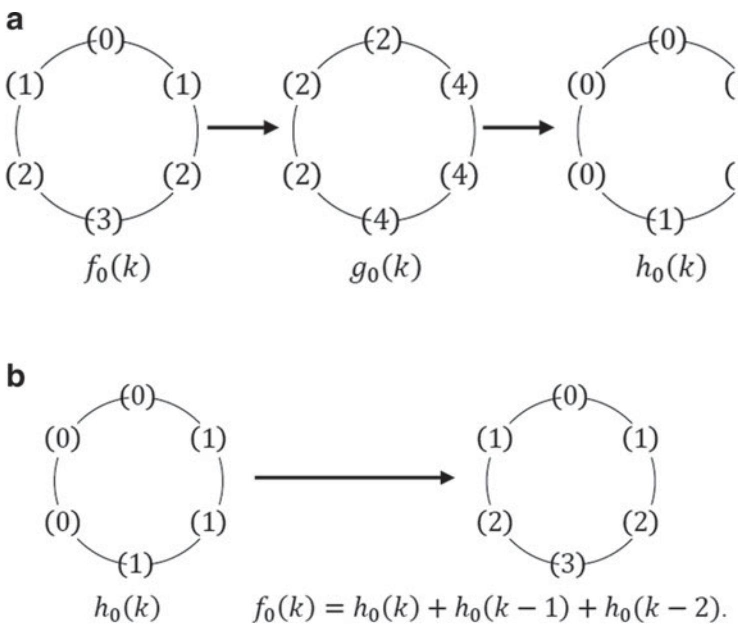
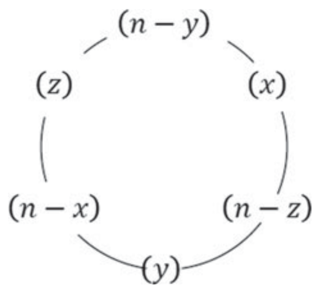


Fig. 3 Correspondence between $f_0(k)$ and $h_0(k)$

Fig. 4 Circular partition of preference orders



From these inequalities, we have $n \le x+y+z \le \lfloor \frac{3n}{2} \rfloor$. We will calculate the number of partitions of $i = n, \lfloor \frac{3n}{2} \rfloor$ with 1,2, or 3 parts, such that each part does not exceed n (and the special case of $i = \frac{3n}{2}$).A complementary partition has sum $3n - i$.

If $x+y+z = \frac{3n}{2}$, then the partitions of $i = \frac{3n}{2}$ and $3n - i = \frac{3n}{2}$ can be the same. Partitions with $y = \frac{n}{2}$ are are symmetric ($n-x = z, n-z = x, n-y = y$). The number of such partitions is equal to

$$\frac{1}{2}\left(p_{2,n}\left(\frac{3}{2}\right)+p_{3,n}\left(\frac{3}{2}\right)+\frac{n}{2}+1\right)=$$

$$=\frac{1}{2}\left(1+n-\lfloor\frac{n}{4}\rfloor\left(\frac{n}{2}-1\right)+\frac{n^2}{2}+\lfloor\frac{n}{4}\rfloor^2-\lfloor\frac{3n-2}{4}\rfloor-\sum_{j=1}^{\frac{n}{2}}\lfloor\frac{3n-2j-2}{4}\rfloor\right). \qquad (*)$$

For all other i, we have

$$1+\sum_{i=n}^{\lfloor\frac{3n-1}{2}\rfloor}p_{2,n}(i)+\sum_{i=n}^{\lfloor\frac{3n-1}{2}\rfloor}p_{3,n}(i).$$

For $n\geq 3$, we have

$$1+\lfloor\frac{n}{2}\rfloor+\sum_{i=n+1}^{\lfloor\frac{3n-1}{2}\rfloor}\left(n-\lfloor\frac{i-1}{2}\rfloor\right)+\sum_{i=n}^{\lfloor\frac{3n-1}{2}\rfloor}\sum_{j=1}^{\lfloor\frac{i}{3}\rfloor}\left(\min i-2j,n-\lfloor\frac{i-j-1}{2}\rfloor\right). \qquad (**)$$

Summing (*) for even n and (**) for all n, we obtain the result, which is also correct for $n=1$ and $n=2$.

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From Gehrlein-Fishburn's Method on Frequency Representation to a Direct Proof of Ehrhart's extended Conjecture

Issofa Moyouwou, Nicolas Gabriel Andjiga, and Boniface Mbih

1 Introduction

Many authors from various fields have been interested in a theoretical or an event-specific analysis of the following question: what is the exact number of integer to a finite set of linear inequalities involving integer coefficients of bounded integer free variables and integer parameters as a function of the parameters (a closed-form representation)? This is exactly the challenge in social choice theory when one aims at evaluating how frequent a voting event is. Ehrhart (1962, 1967, 1977) has conducted the first study of this problem in the case of a unique parameter. Combining some geometric considerations on the set of vertices of a polytope and periodic numbers, Ehrhart described closed-form representations by pseudo-polynomial functions, these are piecewise defined functions with polynomial expressions over each parameter class modulo a positive integer period. Moreover, Ehrhart (1962, p. 139) suggested that pseudo-polynomials still model the case of several parameters as follows:

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I. Moyouwou (✉) · N. G. Andjiga
Ecole Normale Supérieure - Département de Mathématiques,
Université de Yaoundé1, BP 47 Yaoundé, Cameroun
e-mail: imoyouwou2@yahoo.fr

N. G. Andjiga
e-mail: andjiga2002@yahoo.fr

B. Mbih
CREM UMR CNRS 6211, Université de Caen Normandie, Esplanade de la Paix, 14032 Caen, France
e-mail: boniface.mbih@unicaen.fr

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Conjecture 1 (Ehrhart's conjecture) For any significant diophantine linear system of any dimension, linearly dependent on several positive integral parameters, the symbolic number of integer solutions as a function of those parameters is expressed at different subdomains of the parameter space by different pseudo-polynomials.

Proofs have been provided to Ehrhart's conjecture. For instance, Clauss et al. (1997) established an inductive proof assuming the result for a unique parameter. Moreover with additional works on the set of vertices of a parameterized polyhedron by Loechner and Wilde (1996), the authors provided an algorithmic method for computing the closed-form formula for the exact number of integer solutions to a finite set of linear constraints depending linearly on a set of parameters. Another important contribution to this problem comes from Barvinok (1993) and Barvinok and Pommersheim (1999) who develop a polynomial time algorithm for counting integral points in polyhedra. The Barvinok's algorithm allows powerful implementations as in Verdoolaege et al. (2004). More recent implementations and software packages are available from Bruns et al. (2017, 2019). The common aspect of those approaches is the intensive connection of computations with geometric properties of polyhedra and rational generating functions.

In the field of social choice theory, Gehrlein and Fishburn (1976) when investigating the issue of transitivity in majority voting in three-alternative elections, came out with a set of linear inequalities that depend on the one hand on a collection of integer variables each corresponding to the number of voters with the same preferences; and on the other hand on an integer parameter n which is the size of the electorate. By an appropriate rearrangement of the constraints instead of geometric considerations, Gehrlein and Fishburn (1976) found, through known relations for sums of powers of integers, the closed-form representation as a polynomial according to even and odd values of n. Following this pioneering example, many authors have applied this case-by-case analysis to evaluate how frequent some voting phenomena are; for a very short selection, see Lepelley and Merlin (2001), Gehrlein and Lepelley (1999), Lepelley (1993), Lepelley and Mbih (1987), or Gehrlein (1982). Roughly speaking, Gehrlein–Fishburn's method consists in an *appropriate rearrangement* of variables according to a *judicious partition* of the parameter domain in order to bring out *summation index limits* ready for summation tools.

Huang et al. (2000) attempted to generalize Gehrlein-Fishburn's method, but their suggested algorithm (Huang et al. 2000, p. 152) failed to be general even in the case of a unique parameter. Gehrlein (2002,2005) enriches the work of Huang and Chua with two computer programs, namely EUPIA and EUPIA2, to compute quasi-polynomials for the specific case of one or two parameters using interpolation to find out coefficients. In this paper, our main contribution is to give a straightforward proof of Ehrhart's conjecture by using only basic rearrangements of variables and known relations for sums of powers of integers as pioneered by Gehrlein and Fishburn (1976). We even state and prove a more general version as follows: for any significant diophantine linear system of any dimension, linearly dependent on several positive integer parameters, the sum of any multivariate polynomial over the set of the integer solutions as a function of those parameters is expressed at different subdomains of the parameter space by different pseudo-polynomials. Clearly, the Ehrhart's conjecture follows as the particular case of a constant polynomial. All these are achieved using a systematic way to split a given domain in terms of parameters and to bring out feasible summation index limits, which then allow direct computerized evaluations instead of manual partitioning of spaces that has been used frequently in the above mentioned contributions.

The remainder of this paper is organized as follows. In Sect. 2, we first give some illustrations of the Gehrlein-Fishburn procedure and then, give a more formal statement of the results together with proofs. Algorithms for computerized evaluations are described in Sect. 3 with the compact form of each algorithm relegated to the appendix. Section 4 contains some further illustrations and applications. In Sect. 5, we consider the continuous case; that is when variables and parameters are real numbers. Sect. 6 concludes the paper.

2 A Direct Proof of the Ehrhart's Conjecture

2.1 How Does the Gehrlein–Fishburn's Method Work?

Our overview of the Gehrlein–Fishburn's method to compute the total number of integer solutions to a system of linear inequalities consists of three major steps as follows :

- **Step 1**. Choose a variable and fit its constraints in such a way that its values are those of a fixed interval, by so doing the initial system is split in many subsystems. Reiterate the same procedure on each subsystem in the process as many times as there are variables, constraints only on parameters prescribe the validity domain of the corresponding subsystem.
- **Step 2**. For each subsystem from Step 1, verify that lower bounds and upper bounds of variables are both integers; otherwise judicious congruences on variables or parameters are to be considered in order to rule out non integer bounds. Thereafter use summation tools to derive the number of integer solutions to each subsystem as a polynomial on parameters.
- **Step 3**. Partition the parameter domain relatively to the collection of validity conditions associated with subdomains from Step 1 and add polynomials from Step 2 defined on the same domain.

Before general technicalities presented through proofs in the next section, let us consider an example to outline the main stages of the procedure.

Example 1 Given two non negative integers n and m, let us consider the following system of linear constraints

$$\begin{cases} 2x + y \leq n \\ x + y \geq m \\ x \geq 0, y \geq 0 \\ x, y \text{ integers} \end{cases} \tag{1}$$

Question: given two integers n and m, what is the total number of ordered pairs (x, y) *of integers that satisfy* (1)? *The three steps below show how to derive this result.*

Step 1. We first choose to reorganize constraints on y as follows[1]

$$\begin{cases} 2x + y \leq n \\ x + y \geq m \\ x \geq 0, y \geq 0 \end{cases} \Leftrightarrow \begin{cases} y \leq n - 2x \\ y \geq m - x \\ y \geq 0, x \geq 0 \end{cases} \Leftrightarrow \begin{cases} \max(0, m - x) \leq y \leq n - 2x \\ x \geq 0 \end{cases}$$

Now we must rewrite our constraints without the predicate ‘max’. This simply amounts to finding conditions which lead to $\max(0, m - x) = 0$ or to $\max(0, m - x) = m - x$ otherwise. Two disjoint cases arise as follows:

$$\begin{cases} \max(0, m - x) = 0 \\ \text{with } m - x \leq 0 \end{cases} \quad \text{or} \quad \begin{cases} \max(0, m - x) = m - x \\ \text{with } m - x \geq 1 \end{cases}$$

As a consequence, integer solutions to (1) are collected from

$$\begin{cases} 0 \leq y \leq n - 2x \\ \text{with} \begin{cases} 0 \leq n - 2x \\ m - x \leq 0 \\ x \geq 0 \end{cases} \end{cases} \quad \text{or} \quad \begin{cases} m - x \leq y \leq n - 2x \\ \text{with} \begin{cases} m - x \leq n - 2x \\ m - x \geq 1 \\ x \geq 0 \end{cases} \end{cases} \tag{2}$$

In each of the two subsystems above and in addition to constraints that precise the interval of y’s variations, the remaining inequalities involved x and parameters only. Attention is then turned on x’s constraints. From the first system at (2), constraints that involved only x can be rearranged as follows

$$\begin{cases} 0 \leq n - 2x \\ m - x \leq 0 \\ x \geq 0 \end{cases} \Leftrightarrow \begin{cases} 2x \leq n \\ x \geq m \\ x \geq 0 \end{cases} \Leftrightarrow \begin{cases} 2m \leq 2x \leq n \\ m \leq n \end{cases}$$

[1] Given two numbers a and b, $\max(a, b)$ and $\min(a, b)$ refer to the greatest and the smallest numbers between a and b, respectively.

and from the second system at (2), a similar basic operation leads to

$$\begin{cases} m-x \le n-2x \\ m-x \ge 1 \\ x \ge 0 \end{cases} \Leftrightarrow 0 \le x \le \min(n-m, m-1)$$

$$\Leftrightarrow \begin{cases} 0 \le x \le n-m \\ n-m \le m-1 \\ 0 \le n-m \end{cases} \text{ or } \begin{cases} 0 \le x \le m-1 \\ n-m \ge m \\ 0 \le m-1 \end{cases}$$

Thus the system at (1) has been split into three disjoint subsystems labeled below

$$\overset{(S_1)}{\begin{cases} m-x \le y \le n-2x \\ 0 \le x \le n-m \\ m \le n \le 2m-1 \end{cases}} \quad \overset{(S_2)}{\begin{cases} m-x \le y \le n-2x \\ 0 \le x \le m-1 \\ n \ge 2m, m \ge 1 \end{cases}} \quad \overset{(S_3)}{\begin{cases} 0 \le y \le n-2x \\ 2m \le 2x \le n \\ 2m \le n \end{cases}} \tag{3}$$

Step 2. In (S_1) and (S_2), variables have integer bounds; but the right bound of x in (S_3) depends on n mod 2 to be integer. Clearly (S_3) becomes

$$\begin{cases} 0 \le y \le n-2x \\ m \le x \le \frac{n}{2} \\ 2m \le n, n \text{ even} \end{cases} \text{ or } \begin{cases} 0 \le y \le n-2x \\ m \le x \le \frac{n-1}{2} \\ 2m \le n, n \text{ odd} \end{cases}$$

Let $F_i(n, m)$ be the total number of integer solutions to (S_i), $i = 1, 2, 3$.

$$F_1(n, m) = \begin{cases} \sum_{x=0}^{n-m} \sum_{y=m-x}^{n-2x} 1 \text{ if } m \le n \le 2m-1 \\ 0 \text{ otherwise} \end{cases}$$
$$= \begin{cases} \frac{(n-m+1)(n-m+2)}{2} \text{ if } m \le n \le 2m-1 \\ 0 \text{ otherwise} \end{cases}$$

$$F_2(n, m) = \begin{cases} \sum_{x=0}^{m-1} \sum_{y=m-x}^{n-2x} 1 \text{ if } n \ge 2m, m \ge 1 \\ 0 \text{ otherwise} \end{cases}$$
$$= \begin{cases} \frac{m(2n-3m+3)}{2} \text{ if } n \ge 2m \\ 0 \text{ otherwise} \end{cases}$$

$$F_3(n,m) = \begin{cases} \sum\limits_{x=m}^{n/2} \sum_{y=0}^{n-2x} 1 \text{ if } n \geq 2m, n \text{ even} \\ \sum\limits_{x=m}^{(n-1)/2} \sum_{y=0}^{n-2x} 1 \text{ if } n \geq 2m, n \text{ even} \\ 0 \text{ otherwise} \end{cases}$$

$$= \begin{cases} \frac{(n-2m+2)^2}{4} \text{ if } n \geq 2m, n \text{ even} \\ \frac{(n-2m+1)(n-2m+3)}{4} \text{ if } n \geq 2m, n \text{ odd} \\ 0 \text{ otherwise} \end{cases}$$

Step 3. Let $F(n, m)$ be the number of integer solutions to the system at (1).

$$F(n,m) = \begin{cases} F_1(n,m) \text{ if } m \leq n \leq 2m-1 \\ F_2(n,m) + F_3(n,m) \text{ if } n \geq 2m \\ 0 \text{ otherwise} \end{cases}$$

$$= \begin{cases} \frac{(n-m+1)(n-m+2)}{2} \text{ if } m \leq n \leq 2m-1 \\ \frac{4n-2m-2m^2+n^2+4}{4} \text{ if } n \geq 2m, n \text{ even} \\ \frac{4n-2m-2m^2+n^2+3}{4} \text{ if } n \geq 2m, n \text{ odd} \\ 0 \text{ otherwise} \end{cases}$$

In summary $F(n, m)$ is a piecewise defined polynomial in n and m. Especially, for n and m such that $n \geq 2m$, $F(n, m)$ is a polynomial according to even or odd values of n. Such polynomials are sometimes called pseudo-polynomials, quasi-polynomials or Ehrhart polynomials; those are polynomials whose coefficients depend on some congruences defined on arguments.

2.2 General Framework

Consider a system (S) of *linear inequalities*

$$(S)\colon \alpha_i + \sum_{j=1}^{q+h} \alpha_{ij} x_j \leq 0; \quad i \in \{1, \ldots, I\} \tag{4}$$

where α_{ij}, x_j and α_i are integers; x_j with $1 \leq j \leq q$ are variables; x_j with $q < j \leq q+h$ are given parameters. Given parameters $x_{q+1}, \ldots, x_{q+h}$, an integer solution to (4) is any q-tuple $(x_1, \ldots, x_q)$ of integers such that $(x_1, \ldots, x_{q+h})$ simultaneously satisfies each of the I constraints at (4). The set of all such solutions or lattice points is denoted by $LP(S, x_{q+1}, \ldots, x_{q+h})$ while the set of all $(q+h)$-tuples $(x_1, \ldots, x_{q+h})$ that meet all constraints from (S) is denoted by $LP(S)$. Gehrlein–Fishburn's procedure of computing the total number $\left|LP(S, x_{q+1}, \ldots, x_{q+h})\right|$ of lattice points in $LP(S, x_{q+1}, \ldots, x_{q+h})$ uses known relations for sums of powers of integers after partitioning $LP(S)$ in terms of subsets, each one defined by a system of hierarchical linear inequalities which suggest summation index limits, that is a system of linear inequalities fit in the form

$$(H)\colon a_j + \sum_{k=j+1}^{q+h} a_{jk}x_k \le c_j x_j \le b_j + \sum_{k=j+1}^{q+h} b_{jk}x_k; \quad j \in \{1, \ldots, q\} \tag{5}$$

where a_{jk}, b_{jk}, x_j, a_j, b_j and c_j are integers; $c_j > 0$; x_j with $1 \le j \le q$ are variables; x_j with $q < j \le q + h$ are given parameters. The system (H) of hierarchical linear inequalities defined by (5) is *feasible* (or has *feasible bounds*) if it happens that

$$a_q + \sum_{k=q+1}^{q+h} a_{qk}x_k \le b_q + \sum_{k=q+1}^{q+h} b_{qk}x_k \tag{6}$$

and for all p in $\{1, \ldots, q-1\}$ and for all integers $x_{p+1}, \ldots, x_q$

$$\begin{aligned} &\text{if } a_j + \sum_{k=j+1}^{q+h} a_{jk}x_k \le c_j x_j \le b_j + \sum_{k=j+1}^{q+h} b_{jk}x_k, \ldots j = p+1, \ldots, q \\ &\text{then } a_p + \sum_{k=p+1}^{q+h} a_{pk}x_k \le b_p + \sum_{k=p+1}^{q+h} b_{pk}x_k \end{aligned} \tag{7}$$

To illustrate these, one can observe that the system at (1) is not hierarchical and that each of the three systems at (3) is a feasible hierarchical system. Now let us consider the example below:

Example 2 Consider $(S) : \begin{cases} n \le y \le 2x \\ 1 \le x \le n \\ x, y, n \text{ integers} \end{cases}$

This system is a hierarchical system. Now suppose that x and n are such that $1 \le x < \frac{n}{2}$. Then $1 \le x \le n$ holds but $n \le 2x$ is false. Therefore the condition $n \le 2x$ cannot be seen as a consequence of $1 \le x \le n$ and (S) is not feasible.[2]

For finiteness, we consider only bounded sets $LP(S, x_{q+1}, \ldots, x_{q+h})$ of lattice points, meaning that there are two integers M_1 and M_2, depending on parameters, such that ,

$$M_1 \le x_j \le M_2, \quad \text{for all } (x_1, \ldots, x_q) \in LP(S) \text{ and for all } j = 1, 2, \ldots, q.$$

Remark 1 Note that an inequality of the form $\alpha + \sum_{J=1}^{p} \alpha_j x_j < 0, \alpha + \sum_{J=1}^{p} \alpha_j x_j > 0$ or $\alpha + \sum_{J=1}^{p} \alpha_j x_j \ge 0$ with rational coefficients α and α_j and integer variables x_j is equivalent to an inequality $\alpha' + \sum_{J=1}^{p} \alpha'_j x_j \le 0$ with integer coefficients α' and α'_j.

[2]Feasibility refers here to a specific relation between bounds in a hierarchical system rather than to the existence of a solution as normally.

Lemma 1 *The set $LP(E)$ of all integers x, solutions to the following system*

$$(E):\ \max\{\alpha_i, i = 1, 2, \ldots, \alpha\} \leq x \leq \min\{\beta_j, j = 1, 2, \ldots, \beta\}$$

with α_i, α, β_j and β integers, can be split into $\alpha \times \beta$ subsets $LP(E_{uv})$, possibly empty, defined by

$$(E_{uv}) : \begin{cases} \alpha_u \leq x \leq \beta_v \\ \alpha_u \leq \beta_v \\ \alpha_i + 1 \leq \alpha_u \text{ for } i < u \\ \alpha_i \leq \alpha_u \text{ for } i > u \\ \beta_v + 1 \leq \beta_j \text{ for } j < v \\ \beta_v \leq \beta_j \text{ for } j > v \end{cases}$$

Proof Since $\max\{\alpha_i, i = 1, 2, \ldots, \alpha\}$ picks up its value from $\{\alpha_i, i = 1, 2, \ldots, \alpha\}$ and $\min\{\beta_j, j = 1, 2, \ldots, \beta\}$ its value from $\{\beta_j, j = 1, 2, \ldots, \beta\}$, there are exactly $\alpha \times \beta$ disjoint possible cases to describe (E) as follows

$$(E_{uv}): \begin{cases} \alpha_u \leq x \leq \beta_v \\ \max(\alpha_i, i = 1, 2, \ldots, \alpha) \neq \alpha_i \text{ if } i < u \\ \max(\alpha_i, i = 1, 2, \ldots, \alpha) = \alpha_u \\ \min\left(\beta_j, j = 1, 2, \ldots, \beta\right) \neq \beta_j \text{ if } j < v \\ \min\left(\beta_j, j = 1, 2, \ldots, \beta\right) = \beta_v \end{cases}$$

which are equivalent to

$$(E_{uv}) : \begin{cases} \alpha_u \leq x \leq \beta_v \\ \alpha_i < \alpha_u \text{ if } i < u \\ \alpha_i \leq \alpha_u \text{ if } i > u \\ \beta_j > \beta_v \text{ if } j < v \\ \beta_j \geq \beta_v \text{ if } j > v \end{cases}$$

As $\alpha_u - \beta_v \leq 0$ is a consequence of $\alpha_u \leq x \leq \beta_v$, the result follows after replacing $\alpha_i < \alpha_u$ by $\alpha_i + 1 \leq \alpha_u$ and $\beta_j > \beta_v$ by $\beta_j \geq 1 + \beta_v$. □

Theorem 1 *A bounded set of lattice points defined by a system of linear inequalities with integer variables, integer parameters and integer coefficients can be partitioned in subsets defined by systems of hierarchical linear inequalities feasible under linear constraints on parameters.*

Proof Consider a system (S) of linear inequalities as stated in (4) with bounded set of lattice points $LP(S)$. We focus our attention on x_1. By reorganizing inequalities in terms of positive, negative and null coefficients of x_1 respectively, (S) can be rewritten as follows:

$$\begin{cases} \alpha_{i1}x_1 \leq -\alpha_i + \sum_{j=2}^{q+h}(-\alpha_{ij})x_j, \, i \in I_1 \\ -\alpha_{i1}x_1 \geq \alpha_i + \sum_{j=2}^{q+h}\alpha_{ij}x_j, \, i \in I_2 \\ \alpha_i + \sum_{j=2}^{q+h}\alpha_{ij}x_j \leq 0, \, i \in I_3 \end{cases} \tag{8}$$

where $\alpha_{i1} > 0, i \in I_1, \alpha_{i1} < 0, i \in I_2$ and $\alpha_{i1} = 0, i \in I_3$. As we assume $LP(S)$ to be bounded, I_1 and I_2 are nonempty. By relabeling subscripts and adjusting coefficients, we can now write (S) as follows

$$\begin{cases} c_1x_1 \leq \min\left\{b_i + \sum_{j=2}^{q+1}b_{ij}x_j, \, i = 1, \ldots, b\right\} \\ c_1x_1 \geq \max\left\{a_i + \sum_{j=2}^{q+1}a_{ij}x_j, \, i = 1, \ldots, a\right\} \\ \alpha_i + \sum_{j=2}^{q+1}\alpha_{ij}x_j \leq 0, \, i \in I_3 \end{cases} \tag{9}$$

Note that $a = |I_1|, b = |I_2|, c_1 = lcm\{\alpha_{i1}, i \in I_1 \cup I_2\}$ and $c_1 = t_i\alpha_{i1}, i \in I_1 \cup I_2$ have been used to introduce new coefficients. By Lemma 1 and according to the latest system above, $LP(S)$ can be partitioned into $a \times b$ subsets $LP(S_{uv})$ below

$$(S_{uv}): \begin{cases} a_u + \sum_{j=2}^{q+h}a_{uj}x_j \leq c_1x_1 \leq b_v + \sum_{j=2}^{q+1}b_{vj}x_j \\ (a_u - b_v) + \sum_{j=2}^{q+h}(a_{uj} - b_{vj})x_j \leq 0 \\ (a_i - a_u) + \varepsilon_i + \sum_{j=2}^{q+h}(a_{ij} - a_{uj})x_j \leq 0, \, i = 1, \ldots, a \\ (b_v - b_i) + \delta_i + \sum_{j=2}^{q+h}(b_{vj} - b_{ij})x_j \leq 0, \, i = 1, \ldots, b \\ \alpha_i + \sum_{j=2}^{q+h}\alpha_{ij}x_j \leq 0, \, i \in I_3 \end{cases} \tag{10}$$

where $\varepsilon_i = 1$ if $i < u$ and $\varepsilon_i = 0$ if $i > u$; $\delta_i = 1$ if $i < v$ and $\delta_i = 0$ if $i > v$. Inequalities involving x_1 in each subsystem (S_{uv}) have been fit in the hierarchical form; and the remaining inequalities on variables $x_2, \ldots, x_q$ are defined on integers variables and parameters with integer coefficients. The same procedure, we use to partition (S) into subsystems with inequalities on x_1 fit in the hierarchical form, can be reiterated on each subsystem to fit inequalities on x_2 in the hierarchical form. After q iterations, (S) will be split in terms of subsystems of hierarchical linear inequalities;

the possibly remaining inequalities on parameters are feasibility constraints for each subsystem. □

There exists an abundant literature on sums of powers of integers. Among other related topics in this field, the problem is the following one: given an integer $k \geq 0$, what is the closed-form representation of the sum of the kth powers of the first n positive integers? It is well-known from Hardy et al. (1979), or Nunemacher and Young (1987) that this sum is a polynomial P_k of degree $k+1$ in n. Very simple ways of getting P_k are available; see for example Bloom (1993).

Computing $\left|LP(S, x_{q+1}, \ldots, x_{q+h})\right|$ using known relations for sums of powers of integers requires summation index limits which are not straightforward with a system (S) that fit form (4); but they are only suggested with a system in form (5). For example, the total number of lattice points for a feasible system (H) in form (5) with $c_j = 1$ for $j = 1, \ldots, q$, is clearly a q-summation of the unit which then yields a polynomial function of the parameters $x_{q+1}, \ldots, x_{q+h}$ by using known sums of powers of integers. But coming out with only such particular feasible systems after partitioning a system is not assured through Theorem 1. Huang and Chua (2000) have provided a way to split a system of hierarchical linear inequalities in order to have only integer bounds of variables. But additional conditions are required to make those bounds ready for summation tools as shown in the following examples.

Example 3 Consider (H) and (H') below with integers variables x and y :

$$(H)\text{: } \begin{cases} 1+y \leq x \leq 3+y \\ 1 \leq y \leq 2 \end{cases} \quad \text{and} \quad (H')\text{: } \begin{cases} 1+y \leq x \leq 3-y \\ 1 \leq y \leq 2 \end{cases}$$

Clearly $|LP(H)| = 6$ and $\left|LP(H')\right| = 1$. But through an abusive summation

$$\sum_{y=1}^{2} \sum_{x=1+y}^{3+y} (1) = \sum_{y=1}^{2} 3 = 6 \quad \text{and} \quad \sum_{y=1}^{2} \sum_{x=1+y}^{3-y} (1) = \sum_{y=1}^{2} (3-2y) = 0.$$

The second summation fails because (S') lacks feasibility conditions.

To circumvent erroneous summations due to non feasible bounds, feasibility conditions should be included together with some appropriate congruence relations on parameters as those provided by Huang and Chua (2000, p. 148). We extend here such congruence relations to the case of multiple parameters with further considerations on summations of polynomials.

Consider a multivariate polynomial u

$$u\left(x_1, x_2, \ldots, x_{q+h}\right) = \sum_{k_1 \leq \eth_1, \ldots, k_{q+h} \leq \eth_{q+h}} C_{k_1, \ldots, k_{q+h}} x^{k_1} \ldots x^{k_{q+h}}$$

on $x_1, x_2, \ldots, x_{q+h}$ of degree $\eth_j$ in x_j for $j = 1, \ldots, q+h$. Given parameters $x_q, \ldots, x_{q+h}$, denote by $LP\,(S, u)$ the sum of polynomial u over the set $LP\left(S, x_q, \ldots, x_{q+h}\right)$

of all integer solutions to the set (S) of linear constraints; that is

$$LP\,(S,u) = \sum_{(x_1,x_2,\ldots,x_q)\in LP(S,x_q,\ldots,x_{q+h})} u\,(x)\,.$$

The following lemma is useful for the upcoming proofs.

Lemma 2 *Consider a polynomial P in x of degree m, some integer parameters $x_{q+1},\ldots,x_{q+h}$, two affine transformations $a = a_0 + a_{q+1}x_{q+1} + \cdots + a_{q+h}x_{q+h}$ and $b = b_0 + b_{q+1}x_{q+1} + \cdots + b_{q+h}x_{q+h}$ of the parameters such that a and b are two integers and $a \le b$. Then the sum*

$$\sum_{n=a}^{b} P(n)$$

is a multivariate polynomial of degree at most $m+1$ in each parameter $x_j, j = q+1,\ldots,q+h$.

Proof Noting that P and P_k, $k = 0, 1, \ldots, m$ can be set in the form

$$P(x) = \sum_{k=0}^{m} c_k x^k \text{ and } n^k = P_k(n) - P_k(n-1),$$

it follows that

$$\sum_{n=a}^{b} P(n) = \sum_{k=0}^{m} c_k\,(P_k(b) - P_k(a-1))$$

Since P_k for $k = 0, 1, \ldots, m$, is of degree $k+1 \le m+1$, the term $P_k(a) - P_k(b-1)$ is of degree at most $m+1$ in each parameter.

Remark 2 Note in the previous proof that

$$\sum_{n=a}^{b} P(n) = P^*(b) - P^*(a-1) \quad \text{with} \quad P^* = \sum_{k=0}^{m} c_k P_k.$$

This will be useful in computations.

Theorem 2 *Let u be a multivariate polynomial on x_1, x_2, …, x_{q+h} of degree $\mathfrak{d}_j$ in x_j for $j = 1,\ldots,q+h$ and (H) a feasible hierarchical system as defined by (5), (6) and (7). Let $\mathfrak{d} = \mathfrak{d}_1 + \cdots + \mathfrak{d}_q$, d_1=1 and*

$$d_t = lcm\left\{\frac{c_j d_j}{gcd\,(c_j d_j, a_{jt})}, \frac{c_j d_j}{gcd\,(c_j d_j, b_{jt})}, j = 1,\ldots,\min(q,t-1)\right\}$$

for t=2,…,q+h and choose r_t in $\{0, 1, \ldots, d_t - 1\}$, $t = q+1,\ldots,q+h$.

Then for all integer parameters $x_{q+1}, \ldots, x_{q+h}$ *such that*

$$x_{q+1} \equiv r_{q+1} \text{ mod } d_{q+1}, \ldots, x_{q+h} \equiv r_{q+h} \text{ mod } d_{q+h}$$

the sum $LP(H, u, x_{q+1}, \ldots, x_{q+h})$ *of* u *over* $LP(H, x_{q+1}, \ldots, x_{q+h})$ *is a multivariate polynomial function on* $x_{q+1}, \ldots, x_{q+h}$ *with at most degree* $\mathfrak{d} + \mathfrak{d}_j + q$ *in each parameter* x_j, $j = q+1, \ldots, q+h$.

Proof Given parameters $x_{q+1}, \ldots, x_{q+h}$ such $x_k \equiv r_t \text{ mod } d_t$, $t = q+1, \ldots, q+h$, pose

$$\begin{aligned}&f(x_{q+1}, \ldots, x_{q+h}) = LP(H, u) \text{ and}\\ &g(r_1, \ldots, r_q, x_{q+1}, \ldots, x_{q+h})\\ &= f(x_{q+1}, \ldots, x_{q+h} | \text{assuming } x_1 \equiv r_1 \text{ mod } d_1, \ldots, x_q \equiv r_q \text{ mod } d_q)\end{aligned}$$

Then it is clear that

$$f(x_{q+1}, \ldots, x_{q+h}) = \sum_{r \in \Theta(d)} g(r_1, \ldots, r_q, x_{q+1}, \ldots, x_{q+h})$$

where

$$\Theta(d) = \left\{(r_1, \ldots, r_q) | r_j \in \{0, 1, \ldots, d_j - 1\}, j = 1, \ldots, q\right\}. \tag{11}$$

Since $d_1 = 1$, then $r_1 = 0$ (r_1 has been maintained just for uniformity reasons). By definition, $f(x_{q+1}, \ldots, x_{q+h})$ is a finite sum of $g(r_1, \ldots, r_q, x_{q+1}, \ldots, x_{q+h})$ over $(r_1, \ldots, r_q)$. Thus, we only have to show that $g(r_1, \ldots, r_q, x_{q+1}, \ldots, x_{q+h})$ is, for each $r = (r_1, \ldots, r_q)$ in Θ, a multivariate polynomial function on $x_{q+1}, \ldots, x_{q+h}$ with at most degree $\mathfrak{d} + \mathfrak{d}_t + q$ in each parameter x_t. For this purpose, consider y_j such that $x_j = d_j y_j + r_j$, $j = 1, \ldots, q+h$. After replacing each x_j by $d_j y_j + r_j$ per inequality in (5) and after collecting terms with y_j variables, we obtain

$$(H_r)\colon A_j + \sum_{t=j+1}^{q+h} A_{jt} y_t \leq y_j \leq B_j + \sum_{t=j+1}^{q+h} B_{jt} y_t; \quad j \in \{1, \ldots, q\} \tag{12}$$

where[3]

$$A_{jt} = \frac{a_{jt} d_t}{c_j d_j},\ B_{jt} = \frac{b_{jt} d_t}{c_j d_j},$$

$$A_j = \left\lceil \left(\sum_{t=j+1}^{q} \frac{r_t}{c_j d_j}\right) + \left(\frac{a_j}{c_j d_j} - \frac{r_j}{d_j}\right) \right\rceil \text{ and}$$

[3] Given a real number x, $\lceil x \rceil$ is the smallest integer greater than or equal to x while $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

$$B_j = \left\lfloor \left(\sum_{t=j+1}^{q} \frac{r_t}{c_j d_j} \right) + \left(\frac{b_j}{c_j d_j} - \frac{r_j}{d_j} \right) \right\rfloor$$

By definition of coefficients d_k, the new coefficients A_{jt} and B_{jt} are both integers. Moreover, feasibility constraints (6) and (7) make relations (12) ready for summation. Therefore $g(r_1, \ldots, r_q, x_{q+1}, \ldots, x_{q+h}) = LP(H_r, u)$ is a q-summation over variables $y_1, \ldots, y_q$. The bounds of each summation are affine transformations of the parameters and the remaining variables at each step. Thus by Lemma 2, $g(r_1, \ldots, r_q, x_{q+1}, \ldots, x_{q+h})$ is a multivariate polynomial function in $y_{q+1}, \ldots, y_{q+h}$ of degree at most $\mathfrak{d} + \mathfrak{d}_j + q$ in $y_j = \dfrac{x_j - r_j}{d_j}$, $j = q+1 \ldots, q+h$. Hence $g(r_1, \ldots, r_q, x_{q+1}, \ldots, x_{q+h})$ is a multivariate polynomial function on $x_{q+1}, \ldots, x_{q+h}$ with at most degree$\mathfrak{d} + \mathfrak{d}_j + q$ in each x_j. □

Definition 1 A function F defined on $\mathbb{Z}^h$ is a pseudo-polynomial if there exists a h-tuple $d = (d_1, d_2, \ldots, d_h)$ of positive integers such that for all $r_j \in \{0, 1, \ldots, d_j - 1\}, j = 1, \ldots, h$,

$$F(y_1, \ldots, y_h | \text{assuming } y_1 \equiv r_1 \bmod d_1, \ldots, y_h \equiv r_h \bmod d_h)$$

is a polynomial function on $y_1, \ldots, y_h$. In this case, the h-tuple d is a pseudo-period of F. The degree of F with respect to y$_j$ is the maximal degree of $F(y_1, \ldots, y_h|\text{assuming } y_1 \equiv r_1 \bmod d_1, \ldots, y_h \equiv r_h \bmod d_h)$ with respect to y_h over $(r_1, \ldots, r_h) \in \{0, 1, \ldots, d_1 - 1\} \times \cdots \times \{0, 1, \ldots, d_h - 1\}$.

The following result is a restatement of Theorem 2 in terms of pseudo-polynomials.

Theorem 3 *The sum $LP(H, u)$ of a multivariate polynomial $u(y_1, \ldots, y_{q+h})$ of degree $\mathfrak{d}_j$ in y_j over the set $LP(H, y_{q+1}, \ldots, y_{q+h})$ of all lattice points $(y_1, \ldots, y_q)$ defined by a feasible hierarchical system (H) of linear inequalities with q integer variables $y_1, \ldots, y_q$, h integer parameters $y_{q+1}, \ldots, y_{q+h}$ with integer coefficients is a pseudo-polynomial on parameters $y_{q+1}, \ldots, y_{q+h}$ with degree at most $q + \mathfrak{d}_j + \mathfrak{d}_1 + \cdots + \mathfrak{d}_q$ in y_j.*

From Theorem 1, all bounded sets $LP(S)$ of lattice points defined by a finite set of linear inequalities can be partitioned into a finite number of subsets $LP(H_k)$, each defined by a system of hierarchical linear inequalities feasible under some (linear) constraints C_k, on parameters. By Theorem 3, $|LP(H_k, u)|$ is a pseudo-polynomial on parameters . Therefore the following results hold.

Corollary 1 *Given a bounded set $LP(S, y_{q+1}, \ldots, y_{q+h})$ of lattice points defined by a finite set of linear inequalities with q integer variables $y_1, \ldots, y_q$, h integer parameters $y_{q+1}, \ldots, y_{q+h}$ with integer coefficients, and a multivariate polynomial $u(y_1, \ldots, y_{q+h})$ of degree $\mathfrak{d}_j$ in y_j, there exist some linear constraints C_1, ..., C_K on parameters, some pseudo-polynomials F_1, ..., F_K on parameters with at most degree $q + \mathfrak{d}_j + \mathfrak{d}_1 + \cdots + \mathfrak{d}_q$ on y_j such that*

$$LP(S,u) = \sum_{k=1}^{K} 1_{C_k}(y_1,\ldots,y_h)F_k(y_1,\ldots,y_h)$$

where $1_{C_k}(y_{q+1},\ldots,y_{q+h}) = 1$ *if the collection of parameters* $(y_{q+1},\ldots,y_{q+h})$ *satisfies* C_k*; and* $1_{C_k}(y_{q+1},\ldots,y_{q+h}) = 0$ *otherwise.*

In Corollary 1, constraints $C_1,\ldots,C_K$ do not necessarily define disjoint domains on parameters $y_{q+1},\ldots,y_{q+h}$. Since there exists a finite number of those constraints, one can split them into a finite partition. As consequence, the following holds.

Corollary 2 *Given a bounded set* $LP(S, y_{q+1},\ldots,y_{q+h})$ *of lattice points defined by a finite set of linear inequalities with* q *integer variables* $y_1,\ldots,y_q$*,* h *integer parameters* $y_{q+1},\ldots,y_{q+h}$ *with integer coefficients; and a multivariate polynomial* $u(y_1,\ldots,y_{q+h})$ *of degree* $\mathfrak{d}_j$ *in* y_j*, there exists a finite partition* $D_1,\ldots,D_m$ *of parameters such that on each domain* D_j*,* $LP(S,u)$ *is a pseudo-polynomial on parameters with degree at most* $q+\mathfrak{d}_j+\mathfrak{d}_1+\cdots+\mathfrak{d}_q$ *in* y_j*.*

When $u(y_1,\ldots,y_{q+h}) = 1$, it follows that $LP(S,u) = |LP(S)|$ is the total number of lattice points in $LP(S, y_{q+1},\ldots,y_{q+h})$. Thus $|LP(S)|$ is a pseudo-polynomial on parameters. Ehrhart's conjecture then follows by observing that $\mathfrak{d}_j+\mathfrak{d}_1+\cdots+\mathfrak{d}_q = 0$ for $u(y_1,\ldots,y_{q+h}) = 1$.

3 Algorithms

Proofs of Theorem 1 and Theorem 2 guideline the procedure in three stages.

3.1 The Subdomain-Search Procedure

At this stage, consider a system (S) of linear constraints

$$(S)\colon \alpha_i + \sum_{j=1}^{q+h} \alpha_{ij}x_j \leq 0; \quad i \in \{1,\ldots,I\} \tag{13}$$

where α_{ij}, x_j and α_i are integers; x_j with $1 \leq j \leq q$ are integer variables; x_j with $q < j \leq q+h$ are integer parameters. For computations, the system (S), as input, is the list

$$\mathbf{cst} = [\alpha_i + \sum_{j=1}^{q+h} \alpha_{ij}x_j; \quad i \in \{1,\ldots,I\}]$$

of all left-hand-side expressions obtained from (S) by omitting "≤ 0" for each constraint. The aim is to split these constraints into hierarchical subsystems. The overall functioning is as follows. Choose a variable and fit its constraints in hierarchical form as suggested in (8), (9) and (10). By so doing the initial system is split into many subsystems. Reiterate the same procedure on each subsystem in the process to obtain only hierarchical subsystems with feasibility constraints on parameters. All these can be handled by some procedures.

First, the routine, called **MoveOne** (see Algorithm 1), splits the system **cst** into subsystems where a given variable **var** has constraints of the form (10). In each algorithm, **nops**(L) denotes the number of elements in list L; and given two distinct integers x and y, $\varepsilon(x, y) = 1$ if $x < y$ and $\varepsilon(x, y) = 0$ otherwise. A list from **MoveOne** has the format

$$[lb, c, ub, cond] \tag{14}$$

which means that variable **var** satisfies $lb \leq c * \mathbf{var} \leq ub$; both lb and ub do not depend on **var**; and the variables in **cst** other than **var** satisfy each constraints in *cond*. To summarize, note that the inputs in **MoveOne** are a list of constraints **cst** and a variable **var** in **cst**; and the output is a collection of lists each being of form (14).

Now, the subdomain-search procedure used to find subdomains is called **MoveAll** (see Algorithm 2) and simply iterates **MoveOne** over all variables to split the initial domain **cst** into hierarchical and disjoint subdomains. The final result is a set **rescst** of tables and a congruence list **dd** for parameters. Each table, which represents a subdomain, has five columns and $1 + q$ rows; q is the number of variables $x_1, x_2, \ldots, x_q$ other than the h parameters $x_{q+1}, x_{q+2}, \ldots, x_h$. Each of the q first rows corresponds to a variable x_t in the format

$$[lb_t, c_t, ub_t, x_t, d_t]$$

which, means that, in the corresponding subdomain, the variable x_t satisfies $lb_t \leq c_t * x_t \leq ub_t$; d_t is the congruence on x_t provided by Theorem 2.

In the last row, the first cell contains feasibility conditions; the fourth cell contains the list of parameters and the fifth cell contains the list of congruences for the parameters. A typical subdomain, say **subdo**, is a table in the form (15).

$lb_1(x_2, \ldots, x_q)$	c_1	$ub_1(x_2, \ldots, x_q)$	x_1	d_1
$lb_2(x_3, \ldots, x_q)$	c_2	$ub_2(x_3, \ldots, x_q)$	x_2	d_2
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$lb_{q-1}\left(x_q\right)$	c_{q-1}	$ub_{q-1}(x_q)$	x_{q-1}	d_{q-1}
lb_q	c_q	ub_q	x_q	d_q
feasibility		q	$x_{q+1}, \ldots, x_{q+h}$	$d_{q+1}, \ldots, d_{q+h}$

(15)

In the algorithm **MoveAll**, a_{jt} and $b_{jt}, j = 1, \ldots, t-1$, are the coefficients of x_t in lb_j and ub_j respectively. Briefly, the inputs in **MoveAll** are a list of constraints **cst**, a list of variable $x_1, \ldots, x_q$ and a list of parameters $x_{q+1}, \ldots, x_{q+h}$; and the outputs are a collection of tables each been of the form (15) and a congruence list **dd** for parameters.

3.2 Summing a Polynomial over a Feasible Hierarchical System

Each table **subdo** in the output of **MoveAll** has format (15) where all functions lb_t and ub_t that give bounds are affine functions of their variables. Summing a multivariate polynomial $u(x_1, \ldots, x_{q+h})$ over **subdo**, is done using appropriate congruences provided in Theorem 2 and computed in **MoveAll**. The routine called **SubdoCom** (see Algorithm 3) is used for this purpose and is based on the procedure in the proof of Theorem 2 to obtain (12). The inputs in **SubdoCom** are a feasible hierarchical system **subdo** in format (15), a multivariate polynomial $u(x_1, \ldots, x_{q+h})$, the congruence list **dd** for parameters and an h-tuple $\theta = (\theta_1, \ldots, \theta_h) \in \Theta(\mathbf{dd})$ that defines a class of parameters such that $x_{q+j} \equiv \theta_j \bmod \mathbf{dd}_j$ for all $j = 1, \ldots, h$; that is $x_{q+j} = \mathbf{dd}_j y_{q+j} + \theta_j$ for some integer $y_{q+j} \geq 0$. The outputs of **SubdoCom** is a polynomial **Poly** that gives the sum of $u(x_1, \ldots, x_{q+h})$ over **subdo** assuming that $x_{q+j} \equiv \theta_j$; mod $\mathbf{dd}_j$ for all $j = 1, \ldots, h$.

3.3 The Sum of a Polynomial over a Polytope

Note that the set of lattice points that satisfy a given set **cst** of linear constraints can be split, using **MoveAll**, into disjoint subsystems. Each such subsystem is a hierarchical system feasible under some constraints on the parameters. Denote by $S_1, S_2, \ldots, S_p$ all the subsystems obtained from **MoveAll** and stored all as distinct tables in format (15). The feasibility condition C_j for the subsystem S_j is stored in the first cell of the last row of S_j. As it was the case in Example 1, see (3), two distinct domains of parameters associated with two distinct feasibility constraints C_k and C_l may overlap. Denote by D the set of all h-tuples $\left(x_{q+1}, \ldots, x_{q+h}\right)$ that satisfy at least one feasibility condition $C_j, j = 1, \ldots, p$. Let $\{D_1, \ldots, D_m\}$ be a partition of D. Given $j \in \{1, \ldots, m\}$, let $\left(C_j\right)_{j \in I_k}$, with $I_k \subseteq \{1, 2, \ldots, p\}$, be the collection of all feasibility constraints C_j satisfied by each $\left(x_{q+1}, \ldots, x_{q+h}\right) \in D_k$. Given $j \in I_k$, $\theta \in \Theta(\mathbf{dd})$ and a multivariate polynomial $u(x_1, \ldots, x_{q+h})$, denote by $LP(S_j, u, \theta, x_{q+1}, \ldots, x_{q+h})$ the sum of $u\left(x_1, \ldots, x_{q+h}\right)$ over the set of all lattice points $(x_1, \ldots, x_q)$ in S_j assuming that the parameters $(x_{q+1}, \ldots, x_{q+h})$ satisfy C_j with $x_{q+j} \equiv \theta_j$; mod $\mathbf{dd}_j$ for all $j = 1, \ldots, h$. Note that $LP(S_j, u, \theta, x_{q+1}, \ldots, x_{q+h})$ is a polynomial that can be computed using **SubdoCom**. Therefore the sum $LP(D_k, u, x_{q+1}, \ldots, x_{q+h})$ of $u\left(x_1, \ldots, x_{q+h}\right)$ over the set of all lattice points $(x_1, \ldots, x_q) \in \cup_{j \in I_k} S_j$ assuming that the parameters belongs to D_k is a pseudo-polynomial completely determined for all $\left(x_{q+1}, \ldots, x_{q+h}\right) \in D_k$ and for all $\theta \in \Theta(\mathbf{dd})$ by the sum

$$LP(D_k, u, \theta, x_{q+1}, \ldots, x_{q+h}) = \sum_{j \in I_k} LP(S_j, u, \theta, x_{q+1}, \ldots, x_{q+h}) \tag{16}$$

where $LP(D_k, u, \theta, x_{q+1}, \ldots, x_{q+h})$ is the value of $LP(D_k, u, x_{q+1}, \ldots, x_{q+h})$ assuming that $x_{q+j} \equiv \theta_j$; mod $\mathbf{dd}_j$ for all $j = 1, \ldots, h$. Clearly, to fully determine $LP(D_k, u, \theta, x_{q+1}, \ldots, x_{q+h})$, we only combine in a routine called **Reparti** (see Algorithm 4) the three previous routines as soon as a partition of the parameters is obtained from the collection of feasibility constraints.

Remark 3 Note that the ordering of variables $x_1, \ldots, x_q$ does not matter, but judicious choices lead to less number of subsystems. For instance, one can order variables with respect to their respective frequencies (total number of occurrences in constraints). A test of compatibility among constraints is useful to cancel subsystems with no solution. Also note that out of feasibility conditions attached to each subsystem, no solution to the initial system exists.

4 Further Illustrations and Applications

4.1 Further Illustrations

We provide here some further illustrations to the attention of the readers who are interested in the algorithms provided in this paper. The first example is about the importance of the feasibility of a hierarchical system: not all hierarchical systems, even with integer bounds, are ready for summation using known sums of powers of integers.

Example 4 Consider the example below due to Clauss et al. (1997)

$$1 \le i \le n \text{ and } i \le j \le m \tag{17}$$

where i and j are integer variables, n and m are integer parameters. *What is the total number $f(n, m)$ of solutions (i, j) to (17) as a function of the parameters n and m*?

Stage 1. We first fit j's constraints in a hierarchical (17) form

$$\Leftrightarrow i \le j \le m,\ 1 \le i \le n \text{ and } i \le m$$

The same action is reiterated on constraints involving only variable (17) i:

$$\begin{aligned}
&\Leftrightarrow i \le j \le m \text{ and } 1 \le i \le \min(n, m) \\
&\Leftrightarrow \begin{cases} i \le j \le m \text{ and } 1 \le i \le n \\ 1 \le n \le m \end{cases} \text{or} \begin{cases} i \le j \le m \text{ and } 1 \le i \le m \\ n > m \ge 1 \end{cases}
\end{aligned}$$

Stage 2. Each coefficient d_k is equal to 1 and there is no need to split each of the two subsystems.

Stage 3. The parameter domain has already been partitioned into two subdomains defined by $1 \le n \le m$ and $n > m \ge 1$. Hence the total number of solutions to (17) is

$$f(n,m) = \begin{cases} \sum_{i=1}^{n} \sum_{j=i}^{m} 1 = \frac{n(2m-n+1)}{2} \text{ if } 1 \le n \le m \\ \sum_{i=1}^{m} \sum_{j=i}^{m} 1 = \frac{m(m+1)}{2} \text{ if } n > m \ge 1 \end{cases}$$

Note that the system at (17) is hierarchical with integer bounds; but not ready for summation without feasibility constraints on parameters n and m. One would have wrongly obtained a single polynomial by a direct summation $\sum_{i=1}^{n} \sum_{j=i}^{m} 1 = \frac{n(2m-n+1)}{2}$ without looking for feasibility constraints.

The next example is about a step by step illustration of the functioning of our routines: how inputs and outputs look like.

Example 5 Consider the following system reported by Clauss et al. (1997):

$$1 \le i,\ j \le n \text{ and } 2i \le 3j \ (n \text{ is the unique parameter}) \tag{18}$$

where i and j are integer variables, n is an integer parameter. *What is the total number $f(n)$ of solutions (i, j) to (18) as a function of parameter n?*

Stage 1. We first fit the system in feasible hierarchical form(18) as follows :

$$\Leftrightarrow 2 \le 2i \le 3j,\ 2 \le 3j \text{ and } j \le n$$
$$\Leftrightarrow 2 \le 2i \le 3j, 2 \le 3j \le 3n \text{ and } 2 \le 3n$$

Stage 2. To simplify $i's$ and $j's$ coefficients, we compute a pseudo-period as stated in Theorem 2 as follow $d_1 = 1$ for i, $d_2 = 2$ for j and $d_3 = 2$ for n. Now solutions of (18) are collected from appropriate subsystems as follows :

$$\begin{array}{c||c} \text{for } n = 2n' & \text{for } n = 2n'+1 \\ \begin{cases} 1 \le i \le 3j' \\ 1 \le j' \le n' \\ j = 2j' \end{cases} \text{ or } \begin{cases} 1 \le i \le 3j'+1 \\ 0 \le j' \le n'-1 \\ j = 2j'+1 \end{cases} & \begin{cases} 1 \le i \le 3j' \\ 1 \le j' \le n' \\ j = 2j' \end{cases} \text{ or } \begin{cases} 1 \le i \le 3j'+1 \\ 0 \le j' \le n' \\ j = 2j'+1 \end{cases} \end{array} \tag{19}$$

These two stages are covered by **MoveAll**. The inputs are the list $\left[i - j, j - n, 2 * i - 3 * j\right]$ of constraints, the list $\left[j, i\right]$ of variables and the list $[n]$ containing the single parameter n. The outputs are a list of a single domain and the congruence list (with only one element) for the parameter n, represented as

$$\left[\left[\begin{array}{ccccc} 2 & 2 & 3j & i & 1 \\ 2 & 3 & 3n & j & 2 \\ [2-3n] & [\ ] & 2 & [n] & [2] \end{array}\right]\right], [2]$$

Stage 3. The result, according to the two disjoint domains (19), is :

$$f(n) = \begin{cases} \sum_{j'=1}^{\frac{n}{2}} \sum_{i=1}^{3j'} 1 + \sum_{j'=0}^{\frac{n}{2}} \sum_{i=1}^{3j'+1} 1 = \frac{1}{2}n + \frac{3}{4}n^2 \text{ if } n \equiv 0 \bmod 2 \\ \sum_{j'=1}^{\frac{n-1}{2}} \sum_{i=1}^{3j'} 1 + \sum_{j'=0}^{\frac{n-1}{2}} \sum_{i=1}^{3j'+1} 1 = \frac{3}{4}n^2 + \frac{1}{2}n - \frac{1}{4} \text{ if } n \equiv 1 \bmod 2 \end{cases}$$

This result is obtained by using the procedure **Reparti**. The inputs are the list $\left[i - j, j - n, 2 * i - 3 * j\right]$ of constraints, the polynomial $u(n) = 1$, the list $\left[j, i\right]$ of variables and the list $[n]$. The procedure **Reparti** then outputs the pseudo-polynomial $f(n)$ above assuming that $n \equiv a \bmod 2$ for each $a \in \Theta(2) = \{0, 1\}$. The result is printed on the screen. An alternative consists in storing all in a single table as follows:

$$\begin{bmatrix} [0] & [1/4\, n\, (3\, n + 2)] \\ [1] & [(1/4\, n + 1/4)\, (3\, n - 1)] \end{bmatrix}$$

4.2 Some Applications

In the next examples, some applications of an implementation[4] of the algorithms we built using *Maple* are presented. The aim is to point out some advantages the current algorithms offer, and some of their limits. In the first application, we consider a system due to Gehrlein and Lepelley (1999) who were studying the Condorcet efficiency of the negative plurality rule under the Maximal Culture assumption (a probability distribution on voting situations). The example illustrates how tedious it was to perform all the calculations manually. It also tells us that the way the procedure **MoveAll** splits a given set of constraints may not be optimal; the pseudo periods for parameters may also be a strict multiple of the actual period. Although duplications (that appear with a strict multiple of the period on a parameter) can be easily canceled, higher periods are time consuming. We add to the work by Gehrlein and Lepelley (1999), the missing part of the pseudo-polynomial for odd L.

Example 6 Let $g(L)$ be total number of solutions to the following system :

$$\begin{cases} n_1 + n_2 + n_3 > n_4 + n_5 + n_6 \\ n_1 + n_2 + n_4 > n_3 + n_5 + n_6 \\ n_1 + n_3 > n_5 + n_6 \\ n_2 + n_4 > n_5 + n_6 \\ n_i \leq L, i = 1, \ldots, 6 \end{cases} \tag{20}$$

[4] A copie is available at https://github.com/imoyouwou/EhrhartPolynome.

Gehrlein and Lepelley (1999) reorganized constraints at (20) into 36 subsystems to compute $g(L)$ for even values of L. Our Maple procedure returns 101 subdomains and a period of 24 in L; canceling all duplications gives[5]:

$$g(L) = \begin{cases} \frac{L(661L^5+3216L^4+6640L^3+7200L^2+4264L+1104)}{2880} & \text{if } L \text{ is even and } L > 4 \\ \frac{(L+1)^2(661L^4+1894L^3+2191L^2+924L+90)}{2880} & \text{if } L \text{ is odd and } L > 4 \end{cases}$$

Example 7 (**Application 1**) We revisit in this application the probability that a Condorcet winner exists in a three-candidate election with n voters. We assume that each ranking of candidates is the preferences of at least m voters. This may be the case when each candidate is acclaimed by some unwavering supporters who disagree on the ranking of the two other candidates. Without ties, there are six types of individual rankings over the set $\{a_1, a_2, a_3\}$ of candidates, each defined by one of the following rankings of candidates :

$$\begin{array}{llll} \text{Type 1} & a_1 \succ a_2 \succ a_3 & \text{Type 4} & a_2 \succ a_3 \succ a_1 \\ \text{Type 2} & a_1 \succ a_3 \succ a_2 & \text{Type 5} & a_3 \succ a_1 \succ a_2 \\ \text{Type 3} & a_2 \succ a_1 \succ a_3 & \text{Type 6} & a_3 \succ a_2 \succ a_1 \end{array} \tag{21}$$

For $j = 1, \ldots, 6$, denote by n_j the total number of voters of Type j. A voting situation is any 6-tuple $(n_1, n_2, \ldots, n_6)$ of non negative integers that sum to n.

The following system gives a complete characterization of voting situations $n = (n_1, n_2, n_3, n_4, n_5, n_6)$ at which a_1 is the Condorcet winner (a_1 receives more votes in a pairwise majority voting than any of the two other candidates) assuming that there are at least m voters of each type:

$$\begin{cases} n_1 + n_2 + n_3 > n_4 + n_5 + n_6 \\ n_1 + n_2 + n_4 > n_3 + n_5 + n_6 \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \\ n_i \geq m, i = 1, \ldots, 6 \end{cases} \tag{22}$$

The domain defined by (22) is split using **MoveAll** into four subdomains with respect to parameters n and m to obtain, via **Reparti**, the total number of voting situations that meet (22); and by multiplying the corresponding result by 3, one gets the total number $h(n, m)$ of voting situations at which a Condorcet winner exists:

[5]Compared to the 36 subdomains presented in Gehrlein and Lepelley (1999), the decomposition in the procedure we use with an arbitrary ordering of variables was obviously not optimal. But one may still recover the optimal ordering by performing the decomposition for all the $6! = 120$ possible orderings of variables. But finding the optimal decomposition may be time consuming and remains an exciting topic.

$$h(n,m) = \begin{cases} \frac{(n-6m)(n-6m+4)^2(n-6m+2)^2}{128} & \text{if } n \text{ is even and } 0 \leq m \leq \frac{n}{6} \\ \frac{(n-6m+5)(n-6m+1)(n-6m+3)^3}{128} & \text{if } n \text{ is odd and } 0 \leq m \leq \frac{n}{6} \\ 0 & \text{otherwise} \end{cases}$$

The set of all voting situations in our voting context is characterized by

$$\begin{cases} n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \\ n_j \geq m, j = 1, 2, \ldots, 6 \end{cases}$$

and therefore contains $h_0(n,m)$ voting situations with

$$h_0(n,m) = \begin{cases} \frac{(n-6m+5)(n-6m+4)(n-6m+3)(n-6m+2)(n-6m+1)}{120} & \text{if } 0 \leq m \leq \frac{n}{6} \\ 0 & \text{otherwise} \end{cases}$$

Now assume that all voting situations are equally likely to be observed. This probability distribution known as the Impartial Anonymous Culture (IAC) assumption was presented in Kuga and Nagatani (1974) and popularized in social choice theory by Gehrlein and Fishburn (1976). Combining the results on $h(n,m)$ and $h_0(n,m)$, one gets the following result,

Proposition 1 *Consider a three-candidate election with n voters and assume that there are at least m voters with each of the six types of preferences listed at (21). Then, under the IAC assumption, the conditional probability Prob(n, m) that a Condorcet winner exists assuming that each ranking is reported by at least m voters, is*

$$Prob(n,m) = \begin{cases} \frac{15(n-6m)(n-6m+2)(n-6m+4)}{16(n-6m+3)(n-6m+5)(n-6m+1)} & \text{if } n \text{ is even and } 0 \leq m \leq \frac{n}{6} \\ \frac{15(n-6m+3)^2}{16(n-6m+2)(n-6m+4)} & \text{if } n \text{ is odd and } 0 \leq m \leq \frac{n}{6} \end{cases}$$

Given a value of n, as m increases from 0 to $n/6$, $Prob(n,m)$ decreases if n is odd but increases if n is even. Thus the classical

$$\text{Prob}(n,0) = \begin{cases} \frac{15(n)(n+2)(n+4)}{16(n+3)(n+5)(n+1)} & \text{if } n \text{ is even} \\ \frac{15(n+3)^2}{16(n+2)(n+4)} & \text{if } n \text{ is odd} \end{cases}$$

respectively underestimates and overestimates $\text{Prob}(n,m)$ for even n and odd n. Moreover let $\text{Prob}(\alpha)$ be the limit of $\text{Prob}(n,\alpha n)$ as n tends to infinity. Intuitively $\text{Prob}(\alpha)$ is the limit of $\text{Prob}(n,m)$ as n tends to infinity assuming that the ratio m/n of voters with each type of preferences is at least equal to α. Then the following is a direct consequence of Proposition 1.

Corollary 3 *Consider a three-candidate election and assume that there are at least a ratio α of the electorate with each of the six types of preferences listed at (21).*

Then, as the total number of voters tends o infinity, the probability $Prob(\alpha)$ *that a Condorcet winner exists is*

$$Prob(\alpha) = \begin{cases} \frac{15}{16} \text{ if } 0 \leq \alpha < \frac{1}{6} \\ 0 \quad \text{otherwise} \end{cases}$$

Clearly, Prob(α) is exactly the limit probability that a Condorcet winner exists as n tends to infinity. Thus the assumption that each possible type of preferences is reported by at least a given ratio of the electorate does not at all alter the probability that a Condorcet winner exists (as the total number of voters tends to infinity); see Gehrlein et al. (2013) for further analysis of voting events when one assumes that not all preference types are represented.

Example 8 (**Application 2**) All the examples provided above deal only with counting lattice points. As stated in Theorem 3, the full strength of the Gehrlein-Fishburn method allows to address further problems with even polynomial summand. This is the case in this application. *The question is, what is the impact of the existence of a Condorcet winner on the Kendall's coefficient of concordance?* Since the Kendall's coefficient of concordance is a multivariate polynomial, this question is out of the scope of almost all existing procedures. Still with the notation of Example 7, the Kendall's coefficient of concordance in a voting situation $(n_1, n_2, \ldots, n_6)$ with three candidates and n voters is denoted by $C\,(n_1, \ldots, n_6)$ and is equal to

$$\frac{(n_1 + n_2 - n_4 - n_6)^2 + (n_3 + n_4 - n_2 - n_5)^2 + (n_5 + n_6 - n_1 - n_3)^2}{2n^2}$$

The Kendall's coefficient of concordance in a voting situation is a measure of agreement in the n rankings of the three candidates; see Kendall and Smith (1939) or Gehrlein et al. (2015). The higher $C\,(n_1, \ldots, n_6)$, the stronger the agreement in individual rankings is. Put another way, our question is, *what is the expected degree of concordance* $\overline{C}(n)$ *across all voting situations*? Note that the set S of all voting situations with n voters is characterized by

$$(S) : \begin{cases} n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \\ n_j \geq 0, j = 1, 2, \ldots, 6 \end{cases}$$

and the expected degree of concordance over S is $\overline{C}(n) = LP\,(S, C) \,/\, |S|$. Computations via **Reparti** with $u(n_1, \ldots, n_6) = C(n_1, \ldots, n_6)$ give

$$LP\,(S, C) = \frac{(n+1)\,(n+2)\,(n+3)\,(n+4)\,(n+5)\,(n+6)}{840n}$$

Since $|S| = (n+5)\,(n+4)\,(n+3)\,(n+2)\,(n+1)\,/120$, it follows that the expected degree of concordance over all voting situations with n voters is

$$\overline{C}(n) = \frac{n+6}{7n}.$$

To measure the impact of the existence of a Condorcet winner on the Kendall's degree of concordance, we simply compute the conditional expected degree of concordance $\overline{C}\ (n|CW)$ assuming that a_1 is a Condorcet winner. This is $LP\ (S_1, C)\ /\ |S_1|$ where S_1 is the set of all voting situations at which a_1 is the Condorcet winner:

$$(S_1) : \begin{cases} n_1 + n_2 + n_5 > n_3 + n_4 + n_6 \\ n_1 + n_2 + n_3 > n_4 + n_5 + n_6 \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \\ n_j \geq 0, j = 1, 2, \ldots, 6 \end{cases}$$

Note that $|S_1| = h(n, 0)$ as in Example 7; computing $LP\ (S_1, C)$ as outlined above gives

$$LP\ (S_1, C) = \begin{cases} \frac{(n+4)(n+2)(n+6)\left(127\,n^3+968\,n^2+1892\,n+568\right)}{322560n} & \text{if } n \text{ is even} \\ \frac{(n+5)(n+3)(n+1)\left(127\,n^4+1524\,n^3+5630\,n^2+6348\,n-189\right)}{322560n^2} & \text{if } n \text{ is odd} \end{cases}$$

Therefore,

$$\overline{C}\ (n|CW) = \begin{cases} \overline{C}\ (n) + \frac{(n+6)\left(7n^3+248n^2+932n+568\right)}{840n^2(n+4)(n+2)} & \text{if } n \text{ is even} \\ \overline{C}\ (n) + \frac{(n-1)(n+7)\left(7n^2+42n+27\right)}{840n^2(n+3)^2} & \text{if} n \text{ is odd} \end{cases}$$

Clearly the existence of a Condorcet winner increases the degree of concordance among individual preferences. However the corresponding increment is relatively small and tends to $1/120$ as n tends to infinity.

5 The Continuous Case

Due to large periods that may induce a considerable number of congruence classes to be explored or possible non integer parameters, there are numerous investigations in social choice that deal with continuous variables and parameters. This is for example the case in Cervone et al. (2005) or Moyouwou and Tchantcho (2017); a rich panorama on both the finite and the continuous cases are remarkably expounded in Gehrlein and Lepelley (2017), Gehrlein and Lepelley (2010) and Gehrlein (2006). The problem of counting lattice points is now replaced by the problem of computing the volume of a given polytope (on a nonempty and bounded subset of $\mathbb{R}^d$ defined by a finite set of linear constraints). Again the Gehrlein-Fishburn's method can also be applied to compute the volumes of polytopes as illustrated in Lepelley and Merlin (2001). Moreover, the flexibility of real numbers vanishes all problems related to periodicity and replaces summations by integrations.

Lemma 3 *The set $P(E)$ of real number x, solutions to the following system*

$$(E)\text{: } \max\{\alpha_i, i = 1, 2, \dots, \alpha\} \le x \le \min\{\beta_j, j = 1, 2, \dots, \beta\}$$

where α_i, α, β_j and β are real numbers, can be split into $\alpha \times \beta$ subsets $P(E'_{uv})$ defined by

$$(E'_{uv}) : \begin{cases} \alpha_u \le x \le \beta_v \\ \alpha_u \le \beta_v \\ \alpha_i < \alpha_u \text{ if } i < u \\ \alpha_i \le \alpha_u \text{ if } i > u \\ \beta_v < \beta_j \text{ if } j < v \\ \beta_v \le \beta_j \text{ if } j > v \end{cases}$$

Proof Very similar to the one of Lemma 1. □

Definition 2 A feasible hierarchical polytope is a set of points in $\mathbb{R}^d$ which can be described by a set of constraints of type (5) with feasibility constraints (6) and (7); real variables $x_1, \dots, x_q$, real coefficients a_{jk} and b_{jk}; real constants a_j, b_j and c_j and real parameters $x_{q+1}, \dots, x_{q+h}$.

Theorem 4 *(i) A polytope in $\mathbb{R}^q$ can be partitioned into a finite number of feasible hierarchical polytopes over the set of parameters.*

(ii) The integral $P(H, u)$ of a multivariate polynomial $u(x_1, \dots x_{q+h})$ of degree $\mathfrak{d}_j$ in x_j over a polytope $P(H, y_{q+1}, \dots, y_{q+h})$ of all solutions $(y_1, \dots, y_q)$ to a feasible hierarchical system (H) of linear inequalities with q variables $y_1, \dots, y_q$, h parameters $y_{q+1}, \dots, y_{q+h}$ is a piecewise defined polynomial on parameters $y_{q+1}, \dots, y_{q+h}$ with degree at most $q + \mathfrak{d}_j + \mathfrak{d}_1 + \dots + \mathfrak{d}_q$ in x_j.

Proof (i) Very similar to the one in Theorem 1.

(ii) Note that the integral $P(H, u)$ of the polynomial u over a feasible hierarchical polytope P in $\mathbb{R}^q$ as defined by (5), (6) and (7), is a q-integration

$$P(H, u) = \int_{f(q)}^{g(q)} \dots \int_{f(2)}^{g(2)} \int_{f(1)}^{g(1)} u(x_1, \dots x_{q+h}) \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_q$$

where bounds have the form

$$f(j) = a_j + \sum_{k=j+1}^{q+h} a_{jk} x_k \quad \text{and} \quad g(j) = b_j + \sum_{k=j+1}^{q+h} b_{jk} x_k.$$

Part (i) above concludes the proof since the integration over the polytope can be transformed into a finite number of integrations over feasible hierarchical polytopes.

Remark 4 (i) Note that in the case of continuous variables, inequalities of types $\leq$ and $\geq$ can respectively be replaced without any changes in the result by inequalities of types $<$ and $>$.

(ii) For continuous computations,

(iia) at Stage 1, systems and subsystems are split as in Lemma 3;

(iib) at Stage 2, summations are simply replaced by integrations;

(iic) Stage 3 remains unchanged.

Remark 5 Ong et al. (2003) have presented a way to compute the exact volume of a non parameterized polyhedron. The previous algorithm for continuous cases can be viewed as a version of this method for parameterized polytopes and polynomial integrands.

Example 9 (**Application 3**) In this application, we re-evaluate the probability that a Condorcet winner exists assuming that each candidate is supported by a fraction λ of the electorate in any pairwise confrontation. Consider a three-candidate election. There are six types of voters' preferences as described at (21). Now suppose that each of the three candidates has an electoral fief in such a way that each candidate involved in a pairwise majority voting collects at least a proportion of λ favorable votes. Let $x_j = n_j/n, j = 1, 2, \ldots, 6$. The supporting polytope for all voting situations is described by

$$(P):\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1 \\ x_1 + x_2 + x_5 \geq \lambda \\ x_1 + x_2 + x_5 \leq 1 - \lambda \\ x_1 + x_2 + x_3 \geq \lambda \\ x_1 + x_2 + x_3 \leq 1 - \lambda \\ x_1 + x_3 + x_4 \geq \lambda \\ x_1 + x_3 + x_4 \leq 1 - \lambda \\ x_j \geq 0, j = 1, 2, \ldots, 6 \end{cases}$$

The volume $v_0(\lambda)$ of this polytope is the following piecewise defined polynomial in λ :

$$v_0(\lambda) = \begin{cases} \frac{1-60\lambda^3+150\lambda^4-90\lambda^5}{120} & \text{if } \lambda \in \left[0, \frac{1}{3}\right] \\ \frac{\left(24\lambda^2-4\lambda+1\right)(1-2\lambda)^3}{40} & \text{if } \lambda \in \left[\frac{1}{3}, \frac{1}{2}\right] \\ 0 & \text{otherwise} \end{cases}$$

Under these assumptions, the polytope of all voting situations at which a_1 is the Condorcet winner is described by

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1 \\ x_1 + x_2 + x_5 \geq \lambda \\ x_1 + x_2 + x_5 \leq 1 - \lambda \\ x_1 + x_2 + x_3 \geq \lambda \\ x_1 + x_2 + x_3 \leq 1 - \lambda \\ x_1 + x_3 + x_4 \geq \lambda \\ x_1 + x_3 + x_4 \leq 1 - \lambda \\ x_1 + x_2 + x_5 > \frac{1}{2} \\ x_1 + x_2 + x_3 > \frac{1}{2} \\ x_j \geq 0, j = 1, 2, \ldots, 6 \end{cases}$$

the volume $v_1(\lambda)$ of which is given by

$$v_1(\lambda) = \begin{cases} \frac{800\lambda^4 - 320\lambda^3 - 448\lambda^5 + 5}{1920} & \text{if } \lambda \in \left[0, \frac{1}{4}\right] \\ \frac{(8\lambda + 92\lambda^2 + 3)(1 - 2\lambda)^3}{960} & \text{if } \lambda \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ 0 & \text{otherwise} \end{cases}$$

Under the IAC assumption, the probability of observing a voting event described by some linear constraints tends to the ratio $vol(P)/vol(S)$ as the the total number of voters tends to infinity; where P and S are respectively the polytopes that support all occurrences of the corresponding voting event and the polytope of all voting situations.

Proposition 2 *Consider a three-candidate election and assume that a ratio* $\lambda \in [0, \frac{1}{2}[$ *of the electorate supports each candidate involved in a pairwise majority voting. Under the IAC assumption, the limit* $Prob^*(\lambda)$*, as the total number of voters tends to infinity, of the conditional probability that a Condorcet winner exists assuming that each candidate involved in a pairwise majority voting receives at least a proportion of* λ *favorable votes, is*

$$Prob^*(\lambda) = \begin{cases} \frac{3(5 - 448\lambda^5 + 800\lambda^4 - 320\lambda^3)}{16 - 1440\lambda^5 + 2400\lambda^4 - 960\lambda^3} & \text{if } \lambda \in \left[0, \frac{1}{4}\right] \\ \frac{3(3 - 10\lambda + 80\lambda^2 - 480\lambda^3 + 1040\lambda^4 - 736\lambda^5)}{8 - 480\lambda^3 + 1200\lambda^4 - 720\lambda^5} & \text{if } \lambda \in \left[\frac{1}{4}, \frac{1}{3}\right] \\ \frac{3(3 + 8\lambda + 92\lambda^2)}{24 - 96\lambda + 576\lambda^2} & \text{if } \lambda \in \left[\frac{1}{3}, \frac{1}{2}\right[\end{cases}$$

Note that the classical $\text{Prob}^*(0) = \frac{15}{16}$ maximizes $\text{Prob}^*(\lambda)$ and then is an overestimated value in some real election contests where candidates have non negligible electoral fiefs. This is presumably the case in a society with an established political divide. Nevertheless, one can observe that $\text{Prob}^*(\lambda)$ tends to $\frac{3}{4}$ as λ tends to $\frac{1}{2}$ assuming $\lambda < \frac{1}{2}$. This confirms a relatively large probability that a Condorcet winner exists even in the particular case we describe here.

6 Discussion and Conclusion

We have provided a direct proof of Ehrhart's extended conjecture by giving a case independent description of Gehrlein–Fishburn's method. This was the main objective of the paper. The second achievement consists in showing that this approach also allows computerized evaluations with one or more variables and parameters together with polynomial summands.

Because of tedious but elementary summations, applying the Gehrlein–Fishburn's method may sometimes lead to infeasible instances. That is also the case when other techniques based on algebraic and geometric tools are used with a large number of variables and constraints. But only efficient implementations and recent contributions facilitate some hard actions like to find out vertices of a polytope, to count integer points in polyhedra, to factorize or decompose some expressions or solving a system of linear inequalities with several variables. What was important was to present how does Gehrlein–Fishburn method works using basic eliminations of variables. Even in continuous (and parameterized) cases, the method remarkably remains operational. Further investigations are to be made discussing how improvements of some steps are to be achieved, what algebraic or geometric properties do decompositions we have used, etc.

The main difficulty observed is that the number of subspaces generated increases exponentially with the number of variables and parameters. But in some examples, the choice of the ordering of variables when rearrangements are executed reduces or increases drastically the number of subspaces. An open question is then to find the optimal ordering of variables that leads to the smallest number of subspaces. Another observation concerns the periodicity. In some cases, the collection of periods provided in Huang et al. (2000) or in Theorem 2 in this paper may be reduced. How to obtain the smallest period is then another open issue.

Finally, Schürmann (2013) provides a formal approach of how symmetries between variables can be exploited to ease computations. Using symmetries to reduce the total number of variables was in use by social choice theorists as in Gehrlein and Lepelley (1999), Gehrlein (1990), or Lepelley and Mbih (1987). The use of symmetries reduces the total number of subdomains in the Gehrlein–Fishburn procedure and turns the summand into a polynomial. We hope that combining symmetries with sums of polynomials over polytopes will emerge to new procedures, as powerful as those of Bruns et al. (2017, 2019).

7 Appendix

Input: **cst**:: a list of linear constraints; **var**:: a variable in **cst**
Output: **rescst**:: A set of (ordered) lists with four terms each
rescst ←{};**lbound** ←[];**ubound** ←[]; **cstnull** ←[];
j ←**lcm** ([**coeff** (**cst** [i],**var**),$i = 1..$**nops**(**cst**)]);
for $i \leftarrow 1$ **to nops** *(***cst***)* **do** # this loop finds integer bounds of j****var**
  if coeff(**cst**[i], **var**) > 0 **then**
    ubound $\leftarrow \left\lceil \dfrac{-\mathbf{cst}[i] * j}{\mathbf{coeff}(\mathbf{cst}\,[i]\,,\mathbf{var})} + j * \mathbf{var}, op(\mathbf{curcst}) \right\rceil$;
  else if coeff(**cst**[i], **var**) < 0 **then**
    lbound $\leftarrow \left\lfloor \dfrac{\mathbf{cst}[i] * j}{\mathbf{coeff}(\mathbf{cst}\,[i]\,,\mathbf{var})} - j * \mathbf{var}, op(\mathbf{curcst}) \right\rfloor$
  else
    cstnull ← [**cst** [i],op(**curcst**)]
  end
end
foreach $(u, v) \in \{1, \ldots,$ **nops**$(lbound)\} \times \{1, \ldots,$ **nops**$(ubound)\}$ **do**
  varcst ←[$lbound[u] - ubound[v]$,op(**cstnull**)] ; `/* start a domain */`
  forall the $i \in \{1, \ldots,$ **nops**$(lbound)\} \backslash \{u\}$ **do** # write constraints to rule out *max*
    varcst ←[$lbound[i] + \varepsilon(i, u) - lbound[u]$,op(**varcst**)] ;
  end
  forall the $i \in \{1, \ldots,$ **nops**$(lbound)\} \backslash \{v\}$ **do** # write constraints to rule out *min*
    varcst ←[$ubound[i] + 1 - \varepsilon(i, v) - ubound[v]$,op(**varcst**)] ;
  end
  subdo ←[lbound[u],j***var**,ubound[v],**varcst**]; **rescst** ←[**subdo**, *op*(**rescst**)];
end
rescst;

Algorithm 1: Procedure **MoveOne**

Input: **cst**:: list of linear constraints; q variables $x_1, \dots, x_q$; h parameters $x_{q+1}, \dots, x_{q+h}$
Output: **rescst**:: set of subdomains; **dd**:: congruence list for parameters

```
subdos ←MoveOne (cst,x_1) ;            /* bounds of the first variable */
for i ← 1 to nops (subdos) do # store each subdomain as a table
    subdo:=table[1 + q, 5];
    subdo [1,1..4]←[subdos[i][1, 1], subdos[i][1, 2], subdos[i][1, 3], x_1];
    subdo [1 + q, 1]←subdos[i][4];           /* constraints on x_2, ..., x_{q+h} */
    subdo [1 + q, 3]←1;                                /* this is level 1 */
    subdo [1, 5]←1;                         /* the congruence term for x_1 */
    subdos [i]←subdo;
end
rescst ← []; dd ← [1, ..., 1] ;                          /* dd has h terms */
while subdos ≠ ∅ do
    subdo ←subdos [1];
    move:=MoveOne (subdo [1 + nvar, 1],var);
    if subdo[[q + 1, 3] = q − 1 then
        for i ← 1 to nops (move) do
            subdoo ←subdo;
            t←subdoo [1 + q, 3] + 1; subdo [1 + q, 3]← t;     /* level updated */
            subdoo [t,1..4]←[move[i][1, 1], move[i][1, 2], move[i][1, 3], x_t];
            subdoo [1 + q, 1]←move[i][4];
            subdoo [q, 5]←lcm{ c_j d_j / gcd(c_j d_j, a_jq), c_j d_j / gcd(c_j d_j, b_jq), j = 1, ..., q − 1 };
            for t ← q + 1 to q + h do
                dd_t ←lcm{ dd_{t−q}, c_j d_j / gcd(c_j d_j, a_jt), c_j d_j / gcd(c_j d_j, b_jt), j = 1, ..., q };
            end
            subdoo [1 + q, 5]← dd;          /* congruences for parameters */
            rescst ← [subdoo, op(rescst)];
        end
    else
        subdoss ← [];
        for i ← 1 to nops (move) do
            subdoo ←subdo;
            t ←subdoo [1 + q, 3] + 1; subdoo [1 + q, 3]←t;
            subdoo[t, 1..4] ←[move[i][1, 1], move[i][1, 2], move[i][1, 3], x_t];
            subdoo [t, 1]←move[i][4];
            subdoo [t, 5]←lcm{ c_j d_j / gcd(c_j d_j, a_jt), c_j d_j / gcd(c_j d_j, b_jt), j = 1, ..., t − 1 };
            subdoss ←[subdoo,op(subdoss)];
            subdos ←subsop(1=op(subdoss),subdos);      /* replace the current
            subdomain in subdos by the collection of new subdomains
            from MoveOne  */
        end
    end
end
rescst;dd;
```

Algorithm 2: Procedure **MoveAll**

```
Input: subdo::a table of format (15); dd:: the congruence list; θ such that
        θ ∈ Θ(dd); u(x_1, ..., x_{q+h}):: a multivariate polynomial
Output: Poly:: a polynomial in the parameters x_t, t = q+1, ..., q+h
q ← −1 + total number of rows in subdo;
h ← number of parameters from subdo[q+1, 4];
Poly ← 0 ;                                   /* initialization */
d ← [subdo[1, 5], ..., subdo[q, 5], dd_1, ..., dd_h];   /* congruence list
*/
foreach (r_1, ..., r_q) ∈ θ(d) do
    r ← [r_1, ..., r_q, θ_1, ..., θ_h];
    subdoo ← subs(x_1 = dd_1 * y_1 + r_1, ..., x_{q+h} = dd_{q+h} * y_{q+h} + r_{q+h}, subdo) ;
    /* substitute d_t * y_t + r_t to x_t in subdo to obtain
    subdoo  */
    poly ← u(d_1 * y_1 + r_1, ..., d_{q+h} * y_{q+h} + r_{q+h});
    for t ← 1 to q do
        lb_t ← (subdoo[t, 1] − subdoo[t, 2] * r_t) / (d_t * subdoo[t, 2]); lb_t ← lb_t − const(lb_t) + ⌈const(lb_t)⌉;
        /* const(x) =constant in x */
        ub_t ← (subdoo[t, 3] − subdoo[t, 2] * r_t) / (d_t * subdoo[t, 2]); lb_t ← ub_t − const(ub_t) + ⌊const(ub_t)⌋;
        poly ← poly*(ub_t) − poly*(lb_t − 1); /* a summation; see Remark 2 */
    end
    Poly ← Poly + poly;
end
Poly ← subs(y_t = (x_r − r_t) / dd_t, t = q+1, ..., q+h, Poly);
Poly;
```

Algorithm 3: Procedure **SubdoCom**

Input: **cst**:: list of linear constraints; q variables $x_1, \ldots, x_q$; h parameters $x_{q+1}, \ldots, x_{q+h}$; $u(x_1, \ldots, x_{q+h})$:: a multivariate polynomial
Output: **Poly**:: a polynomial in the parameters x_t, $t = q+1, \ldots, q+h$
subdos ← MoveAll(**cst**, $(x_1, \ldots, x_q)$, $(x_{q+1}, \ldots, x_{q+h})$);
Store the first output of **subdos** as a list $S_1, \ldots, S_p$ of tables;
dd ← congruence list for parameters; /* second output of **subdos** */
q ← total number of variables ; h ← total number of parameters ;
C ← [] ; /* empty list */
foreach $j = 1, \ldots, p$ **do**
  $C_j \leftarrow S_j[1+q, 1]$; /* store feasibility constraints from S_j */
end
Partition $C_1, \ldots, C_p$ into disjoint subdomains of parameters $D_1, \ldots, D_m$ such that for each $k = 1, \ldots, m$, all $(x_{q+1}, \ldots, x_{q+h}) \in D_k$ satisfy $C_j, j \in I_k$ for a given $I_k \subseteq \{1, \ldots, p\}$;
foreach $k = 1, \ldots, m$ **do**
  print("for $(x_{q+1}, \ldots, x_{q+h} \in D_k$");
  foreach $\theta \in \Theta$(**dd**) **do**
    Poly ← 0;
    foreach $j \in I_k$ **do**
      Poly ← **Poly** + SubdoCom(S_j, **dd**, θ, $u(x_1, \ldots, x_{q+h})$);
      print(**Poly**); /* simplifying or factorizing are options */
    end
  end
end

Algorithm 4: Procedure **Reparti**

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IAC Probability Calculations in Voting Theory: Progress Report

Abdelhalim El Ouafdi, Issofa Moyouwou, and Hatem Smaoui

1 Introduction

In voting theory, probabilistic analysis aims to assess the frequency with which various electoral outcomes can be observed. The primary motivation is, in one hand, to quantify the potential impact of voting paradoxes on real-world elections and, on the other hand, to compare the alternative voting rules on the basis of their ability to meet certain normative criteria. These quantitative results can of course be obtained, in the form of estimates, by empirical and experimental methods (via actual election data and computer simulations).[1] However, the most significant part of the research on this topic makes use of analytical methods in order to obtain exact results describing the theoretical probabilities of the voting events under investigation. The book by Gehrlein (2006), entirely devoted to the famous Condorcet paradox, and the two books by Gehrlein and Lepelley (2011, 2017), in addition to containing the most complete and essential literature reviews on the subject, constitute an excellent illustration of the richness and dynamism of this line of research. William Gehrlein and Dominique Lepelley are certainly the two most eminent and most prolific authors in this field, and one of the main objectives of this paper is also to pay tribute to

[1] A good summary of empirical and experimental studies can be found in Gehrlein (2006) and Gehrlein and Lepelley (2011). See also Regenwetter et al. (2006), Tideman and Plassmann (2012, 2014), Gehrlein et al. (2016, 2018) and Brandt et al. (2016, 2020).

A. El Ouafdi
University of Lyon, UJM Saint-Etienne, GATE UMR 5824, Saint-Etienne, France
e-mail: abdelhalim.el.ouafdi@univ-st-etienne.fr

I. Moyouwou
ENS-University of Yaounde I, Yaounde, Cameroon
e-mail: imoyouwou2@yahoo.fr

H. Smaoui (✉)
Université de La Réunion, CEMOI EA13, Saint Denis, La Réunion, France
e-mail: hatem.smaoui@univ-reunion.fr

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their fundamental contribution to the probabilistic analysis of voting paradoxes and voting rules.

The analytical approach uses theoretical models based on certain assumptions about the voters' preferences. In the literature, the most often used probabilistic models are the impartial culture condition (IC), introduced by Guilbaud (1952), and the impartial anonymous culture condition (IAC), described initially by Kuga and Nagatani (1974) and formalized by Gehrlein and Fishburn (1976).[2] Over the past two decades, IAC probability calculation techniques have made substantial progress, particularly through methodological studies that have linked these calculations to their appropriate mathematical framework (Huang and Chua 2000; Cervone et al. 2005; Wilson and Pritchard 2007; Lepelley et al. 2008). In this paper, we wish to report on this progress, by a brief description of the methods of calculation used in this field and by reviewing some of the results that the application of these methods made it possible to obtain. For the sake of simplicity, we have chosen to restrict the themes of these representative results to four issues that are among the most often addressed by the probabilistic analysis of electoral outcomes: the election of the Condorcet winner, the election of the Condorcet loser, the (non)monotonicity of voting rules, and finally their manipulability.

In general, these specific voting events are described and studied in a very simple formal framework where individual preferences are represented by linear orderings on the set of candidates. For example, with three candidates (a, b, and c), there are six possible individual preference rankings: *abc*, *acb*, *bac*, *bca*, *cab*, and *cba* (the notation *abc* means that a is preferred to b, b is preferred to c, and, by transitivity, a is preferred to c). With n voters and m candidates, a profile is an ordered list of n individual preferences chosen from the $m!$ possible rankings; a voting situation is an anonymous profile. IC model assumes that each individual preference ranking (and so each profile) is equally likely to be observed. IAC assumes that each possible voting situation is equally likely to be observed. An (anonymous) voting rule is defined as a function that associates a winning candidate with each voting situation.

Most of the studies that will be presented in this brief report deal with voting rules that belong to the class of weighted scoring rules (WSR s) or to the class of scoring elimination rules (SER s). With m candidates, a WSR is defined by a scoring vector $(\lambda_1, \lambda_2, \ldots, \lambda_m)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$, and $\lambda_1 > \lambda_m$ such that each candidate receives λ_k points each time he/she is ranked kth by a voter. The candidate with the most total points wins. The most common WSR s are plurality rule, PR ($\lambda_1 = 1$ and $\lambda_k = 0$ for $k > 1$), negative plurality rule, NPR ($\lambda_m = 0$ and $\lambda_k = 1$ for $k < m$), and Borda rule, BR ($\lambda_k = (m-k)/(m-1)$). In three-candidate election, the scoring vector is of the form $(1, \lambda, 0)$, $0 \leq \lambda \leq 1$, and we have $\lambda = 0$ for PR, $\lambda = 1$ for NPR, and $\lambda = 1/2$ for BR. Scoring elimination rules use WSR s in a multi-stage process of sequential elimination: In each stage, the candidate with the lowest total points is eliminated. With three candidates, a WSR is used in a first round to eliminate the candidate with the lowest total points, and in a second round, the two remaining candidates are confronted and the one who obtains the majority

[2]For a justification of research based on these assumptions, see Gehrlein and Lepelley (2004).

of votes wins. Plurality elimination rule (PER), negative plurality elimination rule ($NPER$), and Borda elimination rule (BER) are the sequential versions of PR, NPR, and BR, respectively.[3]

Given a voting situation, a Condorcet winner (CW) is a candidate who beats each other candidate in pairwise majority comparisons. In the same way, a Condorcet loser (CL) is a candidate who loses against every other candidate in pairwise majority contests. It is well known that the CW and the CL do not always exist. However, it is generally accepted that a "good" voting rule should select the CW when such a candidate exists (CW condition). Voting rules that satisfy this property are called Condorcet consistent.[4] In the same way, it seems reasonable to require the non-election of the CL, when such a candidate exists (CL condition). In this sense, the non-selection of the CW or the selection of the CL can be considered as voting paradoxes (the selection of the CL is known as (strong) Borda paradox). Failure of a given voting rule to meet the CW condition or the CL condition is viewed as a flaw of this rule. The other two imperfections that can affect the voting rules and that we focus on in this paper, are monotonicity failure and vulnerability to strategic manipulation. A monotonicity paradox occurs when an increased support of a candidate who won an election makes him or her a loser (More is Less Paradox, MLP), or when a decreased support of a candidate who lost an election makes him or her a winner (Less is More Paradox, LMP).[5] A strategic manipulation of a voting rule occurs in an election when some voters express insincere preferences in order to obtain a final winner that they prefer to the candidate that would have been elected if they had voted in a sincere way.

In the remainder of this paper, we focus on the exact probabilistic results describing the theoretical frequency of these four voting events under each of the following six voting rules: PR, NPR, BR, PER, $NPER$, and BER (we also mention the results obtained for the entire class of WSR s and for that of SER s). Our main objective being to give a general idea on the evolution of the techniques of probabilities calculation in voting theory, we will not review here the results obtained by assuming IC hypothesis because computation methods used under this model are (almost) the same for twenty years. We therefore begin with a brief description of the general framework of probability calculations under IAC condition and introduce some useful notations (Sect. 2). We then present the different methods used in these calculations, summing up the basic idea of each method and illustrating its scope by a short review of the results it has allowed to obtain (Sects. 3–6). Finally, we conclude with a few remarks on the progress made so far and on the orientations to be considered to push even further the limits of the probabilistic analysis of voting rules.

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[3]Note that the paper only deals with the classical form of elimination process. It is worth noting that other methods of elimination are studied in the literature (see, e.g., Kim and Roush 1996).

[4]Black's procedure, Copeland's rule, and Dodgson's method are examples of methods belonging to this important class of voting rules (see, e.g., Fishburn 1977).

[5]Here, *MLP* and *LMP* are defined for a fixed electorate. These two paradoxes can also be defined with a variable electorate (see Lepelley and Merlin 2001).

2 Probabilities Calculations Under the IAC Condition

In an election with n voters and m candidates, we denote by R_1, …, $R_{m!}$ the $m!$ possible individual preference rankings. A voting situation is then represented by an $m!$-tuple of integers, n_i, that sums to n, where n_i denotes the number of voters having the individual preference R_i. For $m = 3$, voting situations are 6-tuples, $(n_1, \ldots, n_6)$, with the six possible individual rankings labeled as follows: $abc(R_1)$, $acb(R_2)$, $bac(R_3)$, $cab(R_4)$, $bca(R_5)$, and $cba(R_6)$. Note that voting situations are 24-tuples for $m = 4$, 120-tuples for $m = 5$, etc. We denote by $V(n, m)$ the set of all possible voting situations with n voters and m candidates. Under the IAC assumption, the elementary events are the voting situations. Thus, for a voting event E, and for a fixed m, if we denote by $E(n, m)$ the set of elements of $V(n, m)$ in which E occurs, the probability of E is a function of n that is given by:

$$Pr(E, n, m) = |E(n, m)|/|V(n, m)| \tag{1}$$

In this identity, $|V(n, m)|$ and $|E(n, m)|$ denote the cardinalities of sets $V(n, m)$ and $E(n, m)$, respectively. The expression of $|V(n, m)|$ is well known and is given by:

$$|V(n, m)| = \binom{n + m! - 1}{m! - 1} \tag{2}$$

In general, $E(n, m)$ is described by a parametric system, $S(n)$, of linear (in)equalities with integer (or rational) coefficients on the variables n_i and on the parameter n. Therefore, the computation of $Pr(E, n, m)$ is reduced to the enumeration of all the integer solutions of $S(n)$. Note that this is a combinatorial problem that is not always easy to solve and that the transition from three options ($m = 3$) to four options ($m = 4$) is actually a move from a calculation with 6 variables to a calculation with 24 variables (with $m = 5$, we move to much more complex computations, involving 120 variables). Often, especially when the probability of an event is difficult to obtain as a function of n, or when this expression is too cumbersome, one is satisfied to calculate the limiting probability of E, defined by:

$$Pr(E, \infty, m) = \lim_{n \to \infty} Pr(E, n, m) \tag{3}$$

One of the very first events examined by probabilistic studies is the event CW : "there exists a Condorcet winner". Consider the event CW^a: "candidate a is the Condorcet winner". By formula (1), and using the symmetry of IAC with respect to the m candidates, we have:

$$Pr(CW, n, m) = m\left|CW^a(n, m)\right|/|V(n, m)| \tag{4}$$

With three candidates ($m = 3$), the set $CW^a(n, 3)$ is characterized by the following parametric linear system:

$$S^a(n) : \begin{cases} n_1, n_2, n_3, n_4, n_5, n_6 \geq 0 \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \\ n_1 + n_2 + n_4 > n/2 \\ n_1 + n_2 + n_3 > n/2 \end{cases}$$

The two first conditions (the six sign inequalities and the equality) characterize the set of all voting situations with three candidates and n voters, $V(n, 3)$. The two last conditions describe the fact that a is the Condorcet winner (a beats b by a majority of votes and a beats c by a majority of votes).

Knowing the probability that a CW exists, one can consider calculating the Condorcet efficiency of a voting rule F, denoted by $CE(F, n, m)$, and defined as the conditional probability that F elects the CW, given that such a candidate exists. Some studies have also investigated the probability of electing the CL, when such a candidate exists, $Pr(CL - F, n, m)$. The other notations that will be useful later in this paper are the following. For a monotonicity paradox M (MLP or LMP), we denote by $Pr(M - F, n, m)$ the vulnerability of F to M (i.e., the probability that M occurs when using F). The global vulnerability of F to monotonicity paradoxes (i.e., the probability that a voting situation gives rise to MLP or LMP under F) is denoted by $Pr(GMP - F, n, m)$. Finally, the vulnerability of F to coalitional manipulability is denoted by $VM(F, n, m)$.

We close this section with three brief remarks on the IC and IAC models and on the relevance of the theoretical results obtained under these conditions:

- The two models are based on a hypothesis of equiprobability. In both cases, this hypothesis can be justified by the absence of information a priori on the voter preferences. Note that with IC, individual preferences are completely independent. By contrast, IAC implicitly introduces a certain degree of interaction between individuals, which induces less heterogeneous preferences than with IC.
- The probabilities obtained under IAC are in general (slightly) lower than those obtained with IC. As pointed by Berg and Lepelley (1992), this can be explained intuitively by the fact that the homogeneity introduced by IAC makes the occurrence of voting paradoxes less likely (for a paradox to occur, a certain antagonism of individual preferences is required).
- In general, IC and IAC represent scenarios that exaggerate the probability of voting events. Thus, the probabilities computed with these two models should be perceived, not as estimates of the likelihood of these events in real situations, but rather as upper bounds. In particular, when the theoretical probability of a voting event is very small, this event is assuredly very unlikely to be observed in reality (Gehrlein and Lepelley 2004).

3 The Algebraic Approach

The first method for probabilities computation under IAC was developed by Gehrlein and Fishburn (1976). This simple algebraic counting technique, which was used until the early 2000s (and beyond in some cases), is based on the use of multiple summations and their reduction by the formulas on sums of powers of integers (Selby 1965). Let us go back to the example of event CW^a: "a is the Condorcet winner", with three candidates ($m = 3$). If we assume that n is odd, then the last two inequalities in the parametric system $S^a(n)$ can be written as $n_3 + n_5 + n_6 \leq (n-1)/2$ and $n_4 + n_5 + n_6 \leq (n-1)/2$, respectively. As the variables n_i are inter-related, the first step in Gehrlein–Fishburn procedure is to transform $S^a(n)$ into a form that will facilitate the enumeration. From the equality in the second condition in $S^a(n)$, we can replace n_1 by $n - (n_2 + n_3 + n_4 + n_5 + n_6)$. It is then easy to show that the number of integer solutions of $S^a(n)$ is equal to the number of 5-tuples of integers, $(n_2, n_3, n_4, n_5, n_6)$, that meet the following five restrictions:

$$0 \leq n_2 \leq n - n_6 - n_5 - n_4 - n_3, 0 \leq n_3 \leq \frac{n-1}{2} - n_6 - n_5,$$
$$0 \leq n_4 \leq \frac{n-1}{2} - n_6 - n_5, 0 \leq n_5 \leq \frac{n-1}{2} - n_6, 0 \leq n_6 \leq \frac{n-1}{2}$$

With this rearrangement of the conditions on the n_i's, the cardinality of the set $CW^a(n, 3)$ can be computed as:

$$\left|CW^a(n,3)\right| = \sum_{n_6=0}^{\frac{n-1}{2}} \sum_{n_5=0}^{\frac{n-1}{2}-n_6} \sum_{n_4=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_2=0}^{n-n_6-n_5-n_4-n_3} 1 \tag{5}$$

The second step is to algebraically reduce this multiple summation by sequentially using known relations for sums of powers of integers. The process starts by the evaluation of the last summation, $\sum_{n_2=0}^{n-n_6-n_5-n_4-n_3} 1$, which can be obviously replaced by $(n - n_6 - n_5 - n_4 - n_3 + 1)$. Then, the n_3 summation in (5) becomes $\sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5} [(n - n_6 - n_5 - n_4 + 1) - n_3]$ and can be easily calculated (using the formula $\sum_{t=0}^{k} t = k(k+1)/2$). Continuing this way, it can be showed that $|CW^a(n, 3)| = (n+1)(n+3)^3(n+5)/384$. Using formulas (1), (2), and (4) for $m = 3$, the analytic representation of the probability of the event $CW(n, 3)$, for odd n, is obtained as (Gehrlein and Fishburn 1976):

$$Pr(CW, n, 3) = \frac{15(n+1)^2}{16(n+2)(n+4)} \tag{6}$$

The representation for even n, calculated by Lepelley (1989), is given by:

$$Pr(CW, n, 3) = \frac{15n(n+2)(n+3)}{16(n+1)(n+3)(n+5)} \tag{7}$$

As we can see, the two formulas (for odd n and for even n) give the same limit as $n \to \infty$; thus, the value of the limiting probability is given by $Pr(CW, \infty, 3) = 15/16$.

Despite of its apparent simplicity, this algebraic method has made it possible to produce a very large number of results that have significantly contributed to advance the probabilistic analysis of voting rules. It is not an exaggeration to say that, until the early 2000s, almost all the analytical representations of the likelihood of voting events, under IAC, were achieved by using this method. With a few exceptions, all these results deal with the case of three-candidate elections.[6] We limit ourselves here to mentioning only a small part of this abundant bibliography, the one that gives the first results on the probabilities of the four voting events and the six voting rules under consideration.

Representations for the Condorcet efficiency, as a function of n, were obtained by Gehrlein (1982) and Gehrlein and Lepelley (2001), for PR, NPR, BR, PER, and $NPER$ (BER is known to always select the CW when such a candidate exists). The limiting probabilities are given by $CE(PR, \infty, 3) = 88.15\%$, $CE(NPR, \infty, 3) = 62.96\%$, $CE(BR, \infty, 3) = 91.11\%$, $CE(PER, \infty, 3) = 95.85\%$, and $CE(NPER, \infty, 3) = 97.04\%$. The probability of electing the Condorcet loser (when such a candidate exists), with n voters, was calculated by Lepelley (1993) for PR and NPR (the other four voting rules never select the CL). The limiting probabilities are respectively given by $Pr(CL - PR, \infty, 3) = 2.96\%$ and $Pr(CL - NPR, \infty, 3) = 3.15\%$. The first analytical results on the frequency of monotonicity paradoxes were obtained by Lepelley et al. (1996), for PER and $NPER$. The associated limiting values are $Pr(MLP - PER, \infty, 3) = 4.51\%$, $Pr(MLP - NPER, \infty, 3) = 5.56\%$, $Pr(LMP - PER, \infty, 3) = 1.97\%$, and $Pr(LMP - NPER, \infty, 3) = 6.48\%$. Other results, dealing with variable electorate versions of MLP and LMP, were proposed by Lepelley and Merlin (2001). Finally, the exact formulas (as a function of n), describing the vulnerability of PR, NPR, and PER to strategic manipulation (by a coalition of voters) were provided by Lepelley and Mbih (1987, 1994). For a large number of voters, the obtained formulas give: $VM(PR, \infty, 3) = 29.16\%$, $VM(NPR, \infty, 3) = 51.85\%$, and $VM(PER, \infty, 3) = 11.11\%$. For $NPER$, only the limiting probability was possible to compute in Lepelley and Mbih (1994): $VM(NPER, \infty, 3) = 43.05\%$.

This sample of results shows the importance of Gehrlein–Fishburn procedure in probability calculations under the IAC hypothesis. However, the implementation of this method often faces a number of difficulties. First, we must start by rearranging the inequalities in $S(n)$ in a way that allows the use of multiple summations, which

[6] A description of the few studies dealing with the case of four (and more) candidates can be found in (Gehrlein 2006, pp. 107–152).

is not always easy to do, especially because there is no mechanical procedure to accomplish this operation. Second, it often happens that the Min and Max functions appear in certain lower and upper summation bounds, which leads to partitioning the set $E(n, m)$ into several sub-spaces and to further complicate the calculations (see, e.g., Gehrlein and Lepelley 2001). Finally, in the expression of these bounds, when some of the n_i's coefficients are not integer, the Gehrlein–Fishburn procedure may fail to produce the desired result. The scope of this method is therefore rather limited to simple voting events in three-candidate elections. For example, it does not allow to compute the analytical representations of $Pr(MLP - BR, n, 3)$, $Pr(LMP - BR, n, 3)$, $VM(NPER, n, 3)$, and $VM(BR, n, 3)$. As for the results for $m = 4$ and the results depending on a second parameter (other than n) with $m = 3$, these representations seem to be (in general) inaccessible until now with this approach.

4 The Geometric Approach for Limiting Probabilities

Saari (1994) is the first author to introduce tools of geometric analysis in the study of voting rules. His extensive work has significantly contributed to a more complete and deeper understanding of most voting paradoxes and impossibility theorems. His geometric approach has also made it possible to develop a probability calculation technique in the limit case where the number of voters tends to infinity (Saari and Tataru 1994, 1999). This method, based on volume calculations and on results from Schlăfli (1950), was later used, in Merlin and Tataru (1997), Saari and Valognes (1999), and Merlin et al. (2000, 2002), among other studies, to obtain a number of interesting asymptotic results under the IC hypothesis.

With the assumption of IAC, Cervone et al. (2005) developed a very similar method that allows to reduce the problem of computing limiting probabilities, in three-candidate elections, to a problem of pure geometry. They start by transforming each voting situation $(n_1, \ldots, n_6)$ into a normalized (anonymous) profile $(x_1, \ldots, x_6)$ where $x_i = n_i/n$ represents the fraction of voters who favor the preference ranking R_i. Since $x_i \geq 0$, for each i, and $\sum x_i = 1$, the normalized profiles correspond to points (with rational coordinates) in Δ^5, the 5-simplex in $\mathbb{R}^6$. In the same way, for a voting event, E, described by a linear system $S(n)$, we can associate the convex region R described by the linear system $S(1)$ (where the n_i's are replaced by the x_i's and n is replaced by 1). If we respectively denote by $\text{Vol}(\Delta^5)$ and $\text{Vol}(R)$ the 5-volume of Δ^5 and the 5-volume of R, then the limiting probability of E is obtained as:

$$Pr(E, 3, \infty) = \text{Vol}(R)/\text{Vol}(\Delta^5) \tag{8}$$

$\text{Vol}(\Delta^5)$ is easy to obtain and is known to be equal to $\sqrt{6}/120$. To compute $\text{Vol}(R)$, the authors apply a procedure based on the general formula giving the volume of a pyramid and on a recursive technique of triangulation. The volume of a pyramid in dimension d is equal to Vh/d, where V is the $(d-1)$-dimensional volume of the base, and h is the height of the apex above the base. The method begins by determining all the vertices of R and then uses one of them to decompose R into a collection of pyramids having the chosen vertex as their apex and the various faces of R as their bases. The faces of R are 4-dimensional convex regions and are in turn broken in pyramids and so forth. This generates a recursive procedure for computing the volume of the region R; the base case for the recursion is the 2-dimensional case where the "pyramid" is simply a triangle.

With this method, the calculation of the limit probabilities, $Pr(E,\infty,m)$, is reduced to the calculation of the volumes of convex regions (in general, of dimension 5 for $m=3$). It is no longer necessary to obtain the exact expressions of $Pr(E,n,m)$ as a function of n and then calculate their limit when n tends to infinity (this is a significant simplification when $Pr(E,n,m)$ is difficult to obtain). For example, to compute $Pr(CW,\infty,3)$, it suffices to introduce the convex region R^a associated with $CW^a(n,3)$ when $n\to\infty$; then (using formulas (4) and (8)), we have $Pr(CW,\infty,3)=3\text{Vol}(R^a)/\text{Vol}\big(\Delta^5\big)$. We know that $\text{Vol}\big(\Delta^5\big)=\sqrt{6}/120$ and it can be showed, applying the technique we have just outlined, that $\text{Vol}(R^a)=\sqrt{6}/384$ (Cervone et al. 2005). We thus recover the result of Gehrlein and Fishburn (1976) and Lepelley (1989), $Pr(CW,\infty,3)=15/16$.

Thanks to their geometric approach, Cervone et al. (2005) have been able to provide a complete answer to a much more complex problem. They obtained the exact analytical representation of the limiting Condorcet efficiency, $CE(\lambda,\infty,3)$, of all weighted scoring rules $WSR(\lambda)$, $\lambda\in[0,1]$. In particular, they showed that the Borda rule ($\lambda=0.5$) does not maximize $CE(\lambda,\infty,3)$ (the maximum is reached for $\lambda=0.37228$). Other asymptotic results in the form of general formulas (in λ) for all WSR s were obtained by applying the technique of Cervone et al. (2005). For example, Diss and Gehrlein (2012) developed limiting representations for the probability that a Borda paradox will be observed under each WSR. The main conclusion of this paper is that, in realistic voting scenarios, it is very unlikely that a strict Borda paradox[7] would ever be observed for any WSR and that occurrences of a strong Borda paradox (electing the Condorcet loser) should be relatively rare, but not impossible to observe.

Moyouwou (2012) made it more systematic to obtain this type of results (general exact formulas for the whole class of WSR s or SER s), by using the triangulation algorithm of Cohen and Hickey (1979) and by introducing routines using MAPLE codes to undertake the operations involved in this algorithm.[8] This method of calculation was used in a series of articles, including Gehrlein et al. (2013, 2015), Moyouwou and Tchantcho (2017), and Lepelley et al. (2018). In the last article, the authors offer some new exact results describing the vulnerability to monotonicity paradoxes (MLP, LMP, GMP) of the whole class of SER s. In particular, they show that when

[7]A strict Borda paradox occurs when a voting rule completely reverses the rankings (on the set of candidates) that are obtained by the pairwise majority comparisons (in particular, the *CW* becomes the loser and the *CL* becomes the winner under the considered voting rule).

[8]See also Moyouwou and Tchantcho (2017).

three-candidate elections are close, the risk of monotonicity failure is high for PER, $NPER$, and BER (this is especially true under PER, for which the probability of GMP is higher than 32%). In Gehrlein et al. (2013), analytical formulas in term of λ are provided for $VM(\lambda, \infty, 3)$, the asymptotic vulnerability of $SER(\lambda)$ to coalitional manipulation. These analyses were extended by Moyouwou and Tchantcho (2017) who notably showed that the plurality rule minimizes $VM(\lambda, \infty, 3)$ (when the size of the manipulating coalition is unrestricted).

It appears from the studies just cited that the geometric approach developed by Cervone et al. (2005) is very useful when analyzing an entire class of voting rules (typically the WSR s and the SER s) and comparing the rules belonging to this class on the basis of their asymptotic probabilities to meet certain normative criteria. However, this type of simultaneous analysis remains limited to the case of three candidates, and it seems difficult, for the moment, to envisage similar investigations for four-candidate elections with this calculation technique. In fact, the scope of this method essentially depends on the efficiency of the procedure used to find the vertices and perform the triangulations; it is therefore not excluded that future improvements in triangulation algorithms will make it possible to deal with the case of four candidates.

5 Huang–Chua Method and EUPIA Procedure

In the results obtained in the literature applying the Gehrlein–Fishburn procedure, it has been observed that the analytical representations of the probabilities of the voting events always appear in the form of a quotient of two polynomials in n and that these representations are periodic in n (with, in general, a period equal to 2, 6, 9 or 12). Huang and Chua (2000) transformed this observation into a general result that for any voting event E such that $E(n, m)$ is described by a system of linear constraints $S(n)$, the number $|E(n, m)|$, for fixed m, can be described by a periodic polynomial (i.e., a polynomial, $f(n)$, with coefficients depending on a certain period q). Consider, for example, the event CW^a: "a is the Condorcet winner", characterized by the system $S^a(n)$. We have seen in Sect. 3 that $|CW^a(n, 3)| = (n+1)(n+3)^3(n+5)/384$, for odd n. And, using formulas (1), (2) (4), and (7), we can also see that for even n, we have $|CW^a(n, 3)| = (n+2)^3(n+3)(n+4)/384$. Thus, $|CW^a(n, 3)|$ is described by a five-degree periodic polynomial with periodicity $q = 2$.

The Huang–Chua result leads to fundamental simplification in probabilistic calculations under the IAC condition, avoiding in particular to go through the cumbersome (manual) partitioning of the set $E(n, m)$, as it is frequently the case with the Gehrlein–Fishburn procedure. Indeed, knowing the degree of the periodic polynomial expression $f(n)$, it is enough to find the period q and to process by interpolation to determine the periodic coefficients of $f(n)$. Huang and Chua (2000) suggest a simple algorithm that allows to simultaneously identify these unknown values. This algorithm is based on the interpolation technique and an iterative process of computer enumeration of the elements of $E(n, m)$ for initial values of the parameter n. Gehrlein

(2002) has improved this approach by developing EUPIA procedure,[9] which applies to both IAC and MC models,[10] and overcomes a number of technical difficulties that may be encountered when using the Huang–Chua algorithm. An extension of this procedure (EUPIA 2), proposed by Gehrlein (2005), allows to obtain representations for the conditional probability that voting outcomes are observed, given that voting situations are constrained to have some specified values of a measurable parameter (describing, in general, the degree of homogeneity of individual preferences).

The use of these new computational tools has generated a number of results that would have been difficult to obtain with the algebraic approach of Gehrlein and Fishburn. The Huang and Chua algorithm was mainly used in the study of the manipulability of voting rules. Huang and Chua (2000) completed the results of Lepelley and Mbih (1987, 1994), especially by providing the exact expression of $VM(NPER, n, 3)$ as a function of n. Favardin et al. (2002) obtained representations for the vulnerability of the Borda rule to individual manipulation. Favardin and Lepelley (2006) consider various electoral environments in which strategic manipulation can occur and derive some analytical representations for the manipulability of a large number of voting rules. EUPIA was used by Gehrlein (2002) to develop probability representations for a number of different voting outcomes, which are considered to be intractable to obtain with the use of standard algebraic techniques (as, e.g., the probability that all weighted scoring rules on three candidates give the same winner). This procedure was also applied by Gehrlein and Lepelley (2003) to compare the median voting rule with other voting rules, notably on the basis of their manipulability and their Condorcet efficiency. Finally, the two-parameter algorithm EUPIA 2 has been very useful in a number of studies on the impact that different degrees of mutual coherence of individual preferences may have on the probability of certain voting electoral outcomes, such as the existence of the Condorcet winner (Gehrlein 2005), the election of the Condorcet winner (Gehrlein and Lepelley 2009; Gehrlein et al. 2011), and the occurrence of Borda paradox (Gehrlein and Lepelley 2010).

As we have already pointed out, the result of Huang and Chua (2000) corresponds to a crucial change in the methods of calculating probabilities of voting events under IAC. As a consequence of this result, the technical efforts focused on the development of a procedure for the systematic computation of the period and the coefficients of the periodic polynomial representing $|E(n, m)|$. This goal has been partially achieved with the Huang–Chua, EUPIA, and EUPIA 2 algorithms that have been successfully applied to solve problems that lead to calculations involving small periodicities. Unfortunately, the execution time of the interpolation procedure increases exponentially depending on the periodicity, and these algorithms become inoperative when the (unknown) periods are too large. This is the problem encountered, for example, by Favardin and Lepelley (2006) who could not obtain the exact expression of $VM(BR, n, 3)$ and the exact value of $VM(BR, \infty, 3)$. This difficulty

[9]EUPIA: Effectively Unlimited Precision Integer Arithmetic.

[10]Under the maximal culture assumption (MC), all voting situations with at most n voters are assumed to be equally likely to be observed.

severely reduces the efficiency of the Huang–Chua algorithm, the EUPIA procedure, and all methods based on an interpolation technique and prevents them from being used to analyze a large number of voting events with three candidates and (almost) all voting events with four candidates.

6 Ehrhart Theory-Based Methods

We have seen in Sect. 4 that the calculation of the limiting probability of a voting event E can be formulated as a geometric problem. This is also true for the calculation of the probability of E as a function of n (the number of voters). Indeed, when m (the number of candidates) is fixed, the parametric linear system $S(n)$, describing the set $E(n, m)$, defines a (rational) parametric polytope P_n (with a single parameter, n).[11] Computing $Pr(E, n, m)$, i.e., counting the number of integer solutions of $S(n)$, is equivalent to the geometric problem of counting the number of integer points belonging to P_n. Wilson and Pritchard (2007) and Lepelley et al. (2008) drew the attention of voting theorists to the existence of a well-established mathematical approach for performing such a calculation, based on Ehrhart's theory (Ehrhart 1962) and efficient counting algorithms. The basic result of this theory concerns a particular type of parametric polytopes, that of the dilatation of a rational polytope P by a positive integer factor n, denoted by nP. In this case, the number of integer points of nP is a quasi-polynomial on n (i.e., a polynomial on n with periodic coefficients), of degree equal to the dimension of P. For example, in the event CW^a: "a is the Condorcet winner", with $m = 3$, the system $S^a(n)$ defines the dilatation nP where P is the (semi-open) rational polytope, of dimension 5, defined by the system $S^a(1)$ (obtained when n is replaced by 1). We can therefore deduce from Ehrhart's theorem that $|CW^a(n, 3)|$ is a quasi-polynomial on n of degree 5.

As it can be seen, the theoretical result proposed by Huang and Chua (2000) corresponds to the algebraic version of the basic result of Ehrhart theory. However, this theory is more general and more advanced and continues to be enriched by numerous studies in mathematics and computer science. For instance, Ehrhart's theorem has been extended to the general class of parametric polytopes, with one or more parameters (Clauss and Loechner 1998), and algorithms have been proposed to compute the coefficients of the quasi-polynomial describing the number of integer points in parametric polytopes.

The first algorithms, based on Ehrhart theory, that have been introduced in probability calculations under IAC condition are Clauss's method (Clauss 1996), Barvinok's algorithm (Barvinok 1994; [Barvinok]) and LattE (De Loera et al. 2004;

[11] A rational polytope P of dimension d is a bounded subset of $\mathbb{R}^d$, defined by a system of integer linear inequalities. P is said to be semi-open when some of these inequalities are strict. A parametric polytope of dimension d (with a single parameter n) is a d-dimensional rational polytope P_n of the form $P_n = \{x \in \mathbb{R}^d : M_X \geq bn + c\}$, where M is a $t \times d$ integer matrix, b and c are two integer vectors with t components.

[LattE]). The use of these powerful tools greatly facilitates the derivation of probability representations for voting outcomes. In particular, for the four voting events that interest us here, and in the case of three candidates, they made it possible to compute all the probabilities (as a function of n and when $n \to \infty$) that the previous methods failed to obtain. The limiting value of the vulnerability of Borda rule to coalitional manipulation was obtained (separately) by Wilson and Pritchard (2007) and Lepelley et al. (2008): $VM(BR, \infty, 3) = 132, 953/264, 600$. It is worth noticing that this exact result (50.247%) is very close to the approximation given in Favardin and Lepelley (2006), 50.25%. Note also that the quasi-polynomial involved in the expression of $VM(BR, n, 3)$ is of period 210, which explains why this result was not possible to obtain with the Huang–Chua algorithm. For monotonicity paradoxes, the five missing analytical representations were provided by Smaoui et al. (2016): $Pr(MLP - BER, \infty, 3) = 1.12\%$, $Pr(LMP - BER, \infty, 3) = 0.28\%$, $Pr(GMP - PER, \infty, 3) = 1.05\%$, $Pr(GMP - NPER, \infty, 3) = 6.02\%$, and $Pr(GMP - BER, \infty, 3) = 1.40\%$.

Among the three algorithms cited above, Barvinok algorithm is the most used in IAC calculations, in the case of three candidates. Since 2008, the use of the program [Barvinok] has led to many analytical results describing the frequency of various voting events and should allow us to solve most of the probabilistic problems that we could consider in voting theory for three-candidate elections. Recall that in this case ($m = 3$), there are only 6 variables and the quasi-polynomials describing $|E(n, 3)|$ are generally of degree 5. With four candidates ($m = 4$), there are 24 variables, and the quasi-polynomials are of degree 23. In this case, [Barvinok], as well as the other two programs cited, fails to produce the desired quasi-polynomials (the maximum number of variables that they can deal with seems to be about 20). Consequently, it is not possible to analyze four-candidate elections with these three programs. However, we know that for $m = 4$, the periods of the quasi-polynomials can be very large and that the exact formulas for $Pr(E, n, 4)$ can be far too heavy for meaningful analysis. Therefore, the probabilistic calculations, for $m = 4$, must focus on obtaining the limiting probabilities, $Pr(E, \infty, 4)$. This amounts, as we have already seen, to the computation of the volume of the polytope associated with the system $S(n)$ describing the set $E(n, m)$.

The volume of a rational polytope P can be obtained either by a direct use of a volume computation algorithm, or as the leading coefficient of the quasi-polynomial associated with the dilated polytope nP (it is well known that this coefficient is equal to the (normalized) volume of P). Until 2015, computational algorithms (for volumes or quasi-polynomials) could not handle the case of 23-dimentional polytopes. Nevertheless, exact probabilistic results dealing with the case $m = 4$ could be obtained from 2013. Schürmann (2013) proposed a method that enables to reduce the number of variables involved in the volume calculation, by exploiting the possible symmetries in the linear systems describing voting events. Applying this method, and using a new version of LattE (lattE integral, De Loera et al. 2013), he was able to obtain the first exact results (after Gehrlein 2001) giving the exact limiting probability of voting events with four candidates. The Condorcet efficiency of plurality rule is one of three limiting values computed in Schürmann (2013):

$$CE(PR, \infty, 4) = \frac{10658098255011916449318509}{14352135440302080000000000} (74.26\%)$$

This value was recovered by Bruns and Söger (2015) and El Ouafdi et al. (2019). It should be noted that Bruns and Söger performed their calculations by an improved version of Normaliz (Bruns and Söger 2015, [Normaliz]) which became the first algorithm to be able to calculate (most) volumes and quasi-polynomials in dimension 23. For their part, El Ouafdi et al. (2019) have developed a method that combines the use of LattE and Lrs, a program for computing the coordinates of the vertices of a rational polytope (see [Lrs]). The limiting values for the Condorcet efficiency of PER and for the probability of electing the CL for PR and NPR were calculated by Bruns et al. (2019): $CE(PER, \infty, 4) = 91.16\%$, $Pr(CL - PR, \infty, 4) = 2.27\%$ and $Pr(CL - NPR, \infty, 4) = 2.38\%$.[12] These values were also found independently by El Ouafdi et al. (2019), who also calculated the Condorcet efficiency for NPR, BR, $NPER$, and BER : $CE(NPR, \infty, 4) = 55.16\%$, $CE(BR, \infty, 4) = 87.06\%$, $CE(NPER, \infty, 4) = 84.50\%$, and $CE(BER, \infty, 4) = 99.66\%$. By using the LattE-Lrs method (and the latest version of Normaliz, based on a new computation technique called "Descent", see Bruns and Ichim 2018), the last authors were able to obtain the first results on the vulnerability to coalitional manipulation in four-candidate elections: $VM(PR, \infty, 4) = 87.28\%$ and $VM(PER, \infty, 4) = 38.63\%$. The number of digits in the fraction giving the exact value of $VM(PER, \infty, 4)$ gives an indication on the complexity of calculations. We show it here as a comparison with one of the first probabilities calculated under IAC ($Pr(CW, \infty, 3) = 15/16$):

$$\frac{2789407566080353053037581459785742662134938536492206505121233415246691931}{7221170963210711706178813462890519530799725914619904000000000000000000000}$$

We have limited ourselves here to the results concerning the four election outcomes and the six voting procedures that interest us. To our knowledge, the only other studies offering analytical representations for probabilities in four-candidate elections are Brandt et al. (2016, 2020) which deal with certain Condorcet extensions, and Diss and Doghmi (2016), Bubboloni et al. (2018), Diss and Mahajne (2019), and Diss et al. (2019) which analyze multi-winner voting rules in committee elections. It should be mentioned that all analytical findings in Brandt et al. (2016, 2020) were obtained by applying Normaliz and that (almost) all volume computations in Diss and Mahajne (2019) and Diss et al. (2019) were performed by Convex, the second software, after Normaliz, capable of processing 23-dimensional polytopes (see Franz 2016, [Convex]).

[12] It is important to mention that, in Bruns et al. (2019) as well as in El Ouafdi et al. (2019), all results concerning *PER*, *NPER*, and *BER* deal with a truncated version of these three iterative procedures, in which in a first step, the two candidates obtaining the lowest scores are eliminated and the second (and final) step is a majority contest between the two remaining candidates (in this case, *PER* coincides with the so-called plurality runoff rule, often used in political elections).

It is clear from the studies presented in this section, and more generally all the recent literature, that the methods based on Ehrhart theory and on volume computation techniques are today the natural tools for probabilities calculation under IAC condition. The connection with these mathematical themes also made it possible to use specialized software to analyze voting events previously considered as very difficult to tackle. The first results obtained in the case of four candidates are a good illustration of the power and the efficiency of these new tools. They seem to us to be able to answer most of the problems that we may consider in the case $m = 4$, to begin with that of determining the limiting probabilities, not yet calculated, for the four voting events and the six voting rules considered in this paper.

7 Concluding Remarks

We can now consider that computing IAC probabilities for three-candidate elections with linear preferences (implying calculations with 6 variables) has become easy: We can obtain not only a wide variety of probability representations depending on the number of voters (and, of course, the corresponding limiting probabilities), but also some limiting representations depending on other parameters such as the degree of homogeneity of preferences or the value of λ in an election using a scoring rule. A lot of results have been obtained, but we believe that some further studies remain to be conducted: For example, analyzing the impact of group coherence on the manipulability of various voting rules would be of great interest. The most recent software also allows to consider three-candidate elections with preferences that are not necessarily linear, with calculations implying more than 6 variables. For instance, some studies exist that consider dichotomous or trichotomous preferences (implying calculations with 12 or 24 variables).

The case of four-candidate elections can now be addressed (24 variables when preferences are supposed to be linear). About ten papers have already studied this case, and we think that some other papers analyzing four-candidate elections will be published in the next few years. Note however that representations as a function of n, the number of voters, although possible, is too complicated to be useful in this framework. But limiting representations depending on some other parameter can certainly be obtained.

If we except some easy problems where symmetries exist (e.g., the probability of having a Condorcet winner), the move from four to five candidate elections (120 variables) seems to be out of reach with the current techniques. Progress in software (and in mathematics) has to be made if we want to deal with five-alternative elections. Observe however that in this case, for most of the probabilities of interest, the exact fractions associated with the probabilities could be too large to be exhibited! Finally, if we want to know what happens when the number m of candidates increases, a way of doing (in addition to simulations studies) could be to investigate analytically the IAC probabilities of various electoral outcomes as a function on m for a given number of voters.

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