



An Approach to the Herzog-Schönheim Conjecture Using Automata

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Abstract. Let G be a group and H_1, \dots, H_s be subgroups of G of indices d_1, \dots, d_s respectively. In 1974, M. Herzog and J. Schönheim conjectured that if $\{H_i\alpha_i\}_{i=1}^{i=s}, \alpha_i \in G$, is a coset partition of G , then d_1, \dots, d_s cannot be distinct. In this paper, we present a new approach to the Herzog-Schönheim conjecture based on automata and present a translation of the conjecture as a problem on automata.

Keywords: Free groups · Coset partitions · The Herzog-Schönheim conjecture · Automata

1 Introduction

Let G be a group and H_1, \dots, H_s be subgroups of G . If there exist $\alpha_i \in G$ such that $G = \bigcup_{i=1}^{i=s} H_i\alpha_i$, and the sets $H_i\alpha_i, 1 \leq i \leq s$, are pairwise disjoint, then $\{H_i\alpha_i\}_{i=1}^{i=s}$ is a coset partition of G (or a disjoint cover of G). In this case, all the subgroups H_1, \dots, H_s can be assumed to be of finite index in G [17, 21]. We denote by d_1, \dots, d_s the indices of H_1, \dots, H_s respectively [20]. The coset partition $\{H_i\alpha_i\}_{i=1}^{i=s}$ has multiplicity if $d_i = d_j$ for some $i \neq j$.

If G is the infinite cyclic group \mathbb{Z} , a coset partition of \mathbb{Z} is $\{d_i\mathbb{Z} + r_i\}_{i=1}^{i=s}, r_i \in \mathbb{Z}$, with each $d_i\mathbb{Z} + r_i$ the residue class of r_i modulo d_i . These coset partitions of \mathbb{Z} were first introduced by P. Erdős [11] and he conjectured that if $\{d_i\mathbb{Z} + r_i\}_{i=1}^{i=s}, 1 < d_1 \leq \dots \leq d_s, r_i \in \mathbb{Z}$, is a coset partition of \mathbb{Z} , then the largest index d_s appears at least twice. Erdős' conjecture was proved independently by H. Davenport with R. Rado and L. Mirsky with D. Newman using analysis of complex functions [12, 21, 22]. Furthermore, it was proved that the largest index d_s appears at least p times, where p is the smallest prime dividing d_s [21, 22, 34], that each index d_i divides another index $d_j, j \neq i$, and that each index d_k that does not properly divide any other index appears at least twice [22]. We refer also to [25–28, 35] for more details on coset partitions of \mathbb{Z} (also called covers of \mathbb{Z} by arithmetic progressions) and to [13] for a proof of the Erdős' conjecture using group representations [16].

In 1974, M. Herzog and J. Schönheim extended Erdős' conjecture for arbitrary groups and conjectured that if $\{H_i\alpha_i\}_{i=1}^{i=s}, \alpha_i \in G$, is a coset partition

of G , then d_1, \dots, d_s cannot be distinct. In the 1980's, in a series of papers, M.A. Berger, A. Felzenbaum and A.S. Fraenkel studied the Herzog-Schönheim conjecture [2–4] and in [5] they proved the conjecture is true for the pyramidal groups [14], a subclass of the finite solvable groups. Coset partitions of finite groups with additional assumptions on the subgroups of the partition have been extensively studied. We refer to [6, 33, 36, 37]. In [18], the authors very recently proved that the conjecture is true for all groups of order less than 1440.

The common approach to the Herzog-Schönheim (HS) conjecture is to study it in finite groups. Indeed, given any group G , every coset partition of G induces a coset partition of a finite quotient group of G with the same indices [17]. In this paper, we present a completely different approach to the HS conjecture. The idea is to study it in free groups of finite rank and from there to provide answers for every group. This is possible since any finite or finitely generated group is a quotient group of a free group of finite rank and any coset partition of a quotient group F/N induces a coset partition of F with the same indices [7]. In order to study the Herzog-Schönheim conjecture in free groups of finite rank, we use the machinery of covering spaces. A pair (\tilde{X}, p) is a *covering space* of a topological space X if \tilde{X} is a path connected space, $p : \tilde{X} \rightarrow X$ is an open continuous surjection and every $x \in X$ has an open neighborhood U_x such that $p^{-1}(U_x)$ is a disjoint union of open sets in \tilde{X} , each of which is mapped homeomorphically onto U_x by p . For each $x \in X$, the non-empty set $Y_x = p^{-1}(x)$ is called *the fiber over x* and for all $x, x' \in X$, $|Y_x| = |Y_{x'}|$. If the cardinal of a fiber is m , one says that (\tilde{X}, p) is a *m -sheeted covering* (*m -fold cover*) of X [15, 29].

The fundamental group of the bouquet with n leaves (or the wedge sum of n circles), X , is F_n , the free group of finite rank n and for any subgroup H of F_n of finite index d , there exists a d -sheeted covering space (\tilde{X}_H, p) with a fixed basepoint. The underlying graph of \tilde{X}_H is a directed labelled graph, with d vertices, called *the Schreier graph* and it can be seen as a finite complete bi-deterministic automaton; fixing the start and the end state at the basepoint, it recognises the set of elements in H . It is called *the Schreier coset diagram for F_n relative to the subgroup H* [32, p.107] or *the Schreier automaton for F_n relative to the subgroup H* [30, p.102]. The d vertices (or states) correspond to the d right cosets of H , any edge (or transition) has the form $Hg \xrightarrow{a} Hga$, $g \in F_n$, a a generator of F_n , and it describes the right action of F_n on the right cosets of H . If we fix the start state at the basepoint (H), and the end state at another vertex $H\alpha$, where α denotes the label of some path from the start state to the end state, then this automaton recognises the set of elements in $H\alpha$ and we call it *the Schreier automaton of $H\alpha$* and denote it by $\tilde{X}_{H\alpha}$.

In general, for any automaton M , with alphabet Σ , and d states, there exists a square matrix A of order $d \times d$, with a_{ij} equal to the number of directed edges from vertex i to vertex j , $1 \leq i, j \leq d$. This matrix is non-negative and it is called *the transition matrix of M* [10]. If for every $1 \leq i, j \leq d$, there exists $m \in \mathbb{Z}^+$ such that $(A^m)_{ij} > 0$, the matrix is said to be *irreducible*. For A an irreducible non-negative matrix, *the period of A* is the gcd of all $m \in \mathbb{Z}^+$ such that there is i with $(A^m)_{ii} > 0$. If M has a unique start state i and a unique end state j , then the number of words of length k (in the alphabet Σ) accepted by M is

$a_k = (A^k)_{ij}$. The generating function of M is defined by $p(z) = \sum_{k=0}^{k=\infty} a_k z^k$. It is a rational function: the fraction of two polynomials in z with integer coefficients [10], [31, p. 575].

In [8], we study the properties of the transition matrices and generating functions of the Schreier automata in the context of coset partitions of the free group. Let $F_n = \langle \Sigma \rangle$, and Σ^* denote the free group and the free monoid generated by Σ , respectively. We will consider Σ^* as a subset of F_n . Let $\{H_i\alpha_i\}_{i=1}^{i=s}$ be a coset partition of F_n with $H_i < F_n$ of index $d_i > 1$, $\alpha_i \in F_n$, $1 \leq i \leq s$. Let \tilde{X}_i denote the Schreier graph of H_i , with transition matrix A_i of period $h_i \geq 1$ and $\tilde{X}_{H_i\alpha_i}$ the Schreier automaton of $H_i\alpha_i$, with generating function $p_i(z)$, $1 \leq i \leq s$. For each \tilde{X}_i , A_i is a non-negative irreducible matrix and $a_{i,k} = (A_i^k)_{bf}$, $k \geq 0$, counts the number of words of length k that belong to $H_i\alpha_i \cap \Sigma^*$ (with b and f denoting the start and end state of $H_i\alpha_i$ respectively). Since F_n is the disjoint union of the sets $\{H_i\alpha_i\}_{i=1}^{i=s}$, each element in Σ^* belongs to one and exactly one such set, so n^k , the number of words of length k in Σ^* , satisfies $n^k = \sum_{i=1}^{i=s} a_{i,k}$, for every

$k \geq 0$, and moreover $\sum_{k=0}^{k=\infty} n^k z^k = \sum_{i=1}^{i=s} p_i(z)$. By using this kind of counting argument and studying the behaviour of the generating functions at their poles, we prove that if $h = \max\{h_i \mid 1 \leq i \leq s\}$ is greater than 1, then there is a repetition of the maximal period $h > 1$ and that, under certain conditions, the coset partition has multiplicity. Furthermore, we recover the Davenport-Rado result (or Mirsky-Newman result) for the Erdős' conjecture and some of its consequences.

In this paper, we deepen further our study of the transition matrices of the Schreier automata in the context of coset partitions of F_n and give some new conditions that ensure a coset partition of F_n has multiplicity.

Theorem 1. *Let F_n be the free group on $n \geq 2$ generators. Let $\{H_i\alpha_i\}_{i=1}^{i=s}$ be a coset partition of F_n with $H_i < F_n$ of index d_i , $\alpha_i \in F_n$, $1 \leq i \leq s$, and $1 < d_1 \leq \dots \leq d_s$. Let \tilde{X}_i denote the Schreier graph of H_i , with transition matrix A_i , and period $h_i \geq 1$, $1 \leq i \leq s$. Let $H = \{h_j \mid 1 \leq j \leq s, h_j > 1\}$. Assume $H \neq \emptyset$ and different elements in H are pairwise coprime. Let r_h denote the number of repetitions of h . If for some $h \in H$, $h \leq r_h \leq 2(h - 1)$, then $\{H_i\alpha_i\}_{i=1}^{i=s}$ has multiplicity.*

Furthermore, we show the Herzog-Schönheim conjecture in free groups can be translated into a conjecture on automata.

Conjecture 1. *Let Σ be a finite alphabet, and Σ^* be the free monoid generated by Σ . For every $1 \leq i \leq s$, let M_i be a finite, bi-deterministic and complete automaton with strongly-connected underlying graph. Let d_i be the number of states of M_i ($d_i > 1$), and $L_i \subsetneq \Sigma^*$ be the accepted language of M_i . If Σ^* is equal to the disjoint union of the s languages L_1, L_2, \dots, L_s , then there are $1 \leq j, k \leq s$, $j \neq k$, such that $d_j = d_k$.*

Theorem 2. *If Conjecture 1 is true, then the Herzog-Schönheim conjecture is true.*

The paper is organized as follows. In Sect. 2, we give some preliminaries on automata and on irreducible non-negative matrices. In Sect. 3, we present a particular class of automata adapted to the study of the Herzog-Schönheim conjecture in free groups and describe some of their properties. In Sect. 4, we prove Theorem 1 and Theorem 2. We refer to [7] for more preliminaries and examples: Sect. 2, for free groups and covering spaces and Sect. 3.1, for graphs.

2 Automata, Non-negative Irreducible Matrices

2.1 Automata

We refer the reader to [30, p. 96], [9, p. 7], [23, 24], [10]. A *finite state automaton* is a quintuple (S, Σ, μ, Y, s_0) , where S is a finite set, called the *state set*, Σ is a finite set, called the *alphabet*, $\mu : S \times \Sigma \rightarrow S$ is a function, called the *transition function*, Y is a (possibly empty) subset of S called the *accept (or end) states*, and s_0 is called the *start state*. It can be represented by a directed graph with vertices the states and each transition $\mu(s, a) = s'$ is represented by a labelled edge $s \xrightarrow{a} s'$ from s to s' with label $a \in \Sigma$. The *label of a path* p of length n is the product $a_1 a_2 \dots a_n$ of the labels of the edges of p . The finite state automaton $M = (S, \Sigma, \mu, Y, s_0)$ is *deterministic* if there is only one initial state and each state is the source of exactly one arrow with any given label from Σ . In a deterministic automaton, a path is determined by its starting point and its label [30, p. 105]. It is *co-deterministic* if there is only one final state and each state is the target of exactly one arrow with any given label from Σ . The automaton $M = (S, \Sigma, \mu, Y, s_0)$ is *bi-deterministic* if it is both deterministic and co-deterministic. An automaton M is *complete* if for each state $s \in S$ and for each $a \in \Sigma$, there is exactly one edge from s labelled a . We say that an automaton or a graph is *strongly-connected* if there is a directed path from any state to any other state.

Definition 2.1. Let $M = (S, \Sigma, \mu, Y, s_0)$ be a finite state automaton. Let Σ^* be the free monoid generated by Σ . Let $\text{Map}(S, S)$ be the monoid consisting of all maps from S to S . The map $\phi : \Sigma \rightarrow \text{Map}(S, S)$ given by μ (i.e. $\phi(a) : s \mapsto \mu(s, a)$), can be extended in a unique way to a monoid homomorphism $\phi : \Sigma^* \rightarrow \text{Map}(S, S)$. The range of this map is a monoid called *the transition monoid of M* , which is generated by $\{\phi(a) \mid a \in \Sigma\}$. An element $w \in \Sigma^*$ is *accepted* by M if the corresponding element of $\text{Map}(S, S)$, $\phi(w)$, takes s_0 to an element of the accept states set Y . The set $L \subseteq \Sigma^*$ recognized by M is called *the language accepted by M* , denoted by $L(M)$.

For any directed graph with d vertices or any finite state automaton M , with alphabet Σ , and d states, there exists a square matrix A of order $d \times d$, with $a_{ij} = A_{ij}$ equal to the number of directed edges from vertex i to vertex j , $1 \leq i, j \leq d$. This matrix is non-negative (i.e. $a_{ij} \geq 0$) and it is called *the transition matrix* (as in [10]) or *the adjacency matrix* (as in [31, p. 575]). For any $k \geq 1$, $(A^k)_{ij}$ is equal to the number of directed paths of length k from

vertex i to vertex j . So, if M is a bi-deterministic automaton with alphabet Σ , d states, start state i , accept state j and transition matrix A , then $(A^k)_{ij}$ is the number of words of length k in the free monoid Σ^* accepted by M .

2.2 Irreducible Non-negative Matrices

We refer to [1, Ch. 16], [19, Ch. 8]. There is a vast literature on the topic. Let A be a transition matrix of order $d \times d$ of a directed graph or an automaton with d states, as defined in Sect. 2.1. If for every $1 \leq i, j \leq d$, there exists $m \in \mathbb{Z}^+$ such that $(A^m)_{ij} > 0$, the matrix is said to be *irreducible*. For A an irreducible non-negative matrix, *the period of A* is the gcd of all $m \in \mathbb{Z}^+$ such that there is i with $(A^m)_{ii} > 0$. If the period is 1, A is called *aperiodic*. In [19], an irreducible and aperiodic matrix A is called *primitive* and the period h is called the *index of imprimitivity*.

Let A be an irreducible non-negative matrix of order $d \times d$ with period $h \geq 1$ and spectral radius r . Then the Perron-Frobenius theorem states that r is a positive real number and it is a simple eigenvalue of A , λ_{PF} , called the *Perron-Frobenius (PF) eigenvalue*. It satisfies $\min_j \sum_i a_{ij} \leq \lambda_{PF} \leq \max_i \sum_j a_{ij}$.

The matrix A has a right eigenvector v_R with eigenvalue λ_{PF} whose components are all positive and likewise, a left eigenvector v_L with eigenvalue λ_{PF} whose components are all positive. Both right and left eigenspaces associated with λ_{PF} are one-dimensional. The behaviour of irreducible non-negative matrices depends strongly on whether the matrix is aperiodic or not.

Theorem 2.2. [19, Ch. 8] *Let A be a $d \times d$ irreducible non-negative matrix of period $h \geq 1$, with PF eigenvalue λ_{PF} . Let v_L and v_R be left and right eigenvectors of λ_{PF} whose components are all positive, with $v_L v_R = 1$.*

$$\text{If } h = 1, \lim_{k \rightarrow \infty} \frac{A^k}{\lambda_{PF}^k} = P, \text{ and if } h > 1, \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{m=k-1}^{m=k-1} \frac{A^m}{\lambda_{PF}^m} = P; P = v_R v_L.$$

3 A Particular Class of Automaton Adapted to the Study of the HS Conjecture

3.1 The Schreier Automaton of a Coset of a Subgroup

We now introduce the particular class of automata we are interested in, that is a slightly modified version of the *Schreier automaton for F_n relative to the subgroup H* [30, p. 102], [32, p. 107]. We refer to [7] for concrete examples.

Definition 3.1. Let $F_n = \langle \Sigma \rangle$, and Σ^* denote the free group and the free monoid generated by Σ , respectively. Let $H < F_n$ be of index d . Let (\tilde{X}_H, p) be the covering of the n -leaves bouquet with basepoint \tilde{x}_1 and vertices $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_d$. Let $t_i \in \Sigma^*$ denote the label of a directed path of minimal length from \tilde{x}_1 to \tilde{x}_i . Let \tilde{X}_H be the Schreier coset diagram for F_n relative to the subgroup H , with \tilde{x}_1 representing the subgroup H and the other vertices $\tilde{x}_2, \dots, \tilde{x}_d$ representing the cosets Ht_i accordingly. We call \tilde{X}_H *the Schreier graph of H* , with this correspondence between the vertices $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_d$ and the cosets Ht_i accordingly.

From the correspondence between the vertices and the cosets as described in Definition 3.1, there exists a directed path from any vertex \tilde{x}_i to any other vertex \tilde{x}_j in \tilde{X}_H , that is \tilde{X}_H is a strongly-connected graph. Furthermore, it is n -regular. So, its transition matrix A is non-negative and irreducible, with PF eigenvalue n (the sum of the elements at each row and at each column is equal to n).

Definition 3.2. Let $F_n = \langle \Sigma \rangle$, and Σ^* denote the free group and the free monoid generated by Σ , respectively. Let $H < F_n$ be of index d . Let \tilde{X}_H be the Schreier graph of H . Using the notation from Definition 3.1, let \tilde{x}_1 be the start state and \tilde{x}_f be the end state for some $1 \leq f \leq d$. We call the automaton obtained *the Schreier automaton of Ht_f* and denote it by \tilde{X}_{Ht_f} . The language accepted by \tilde{X}_{Ht_f} is the set of elements in Σ^* that belong to Ht_f . We call the elements in $\Sigma^* \cap Ht_f$, *the positive words in Ht_f* . The identity may belong to this set (and in fact, it does, for $f = 1$).

Example 3.3. Let $\Sigma = \{a, b\}$; $F_2 = \langle a, b \rangle$. Let $K \leq F_2$, of index 4.

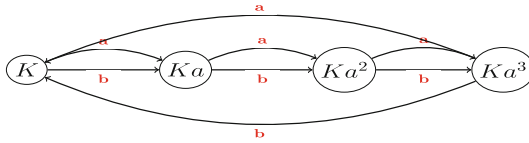


Fig. 1. The Schreier graph \tilde{X}_K of $K = \langle a^4, b^4, ab^{-1}, a^2b^{-2}, a^3b^{-3} \rangle$.

The transition matrix of \tilde{X}_K is $\begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \end{pmatrix}$ with period 4. If K and Ka are

the start and end states, L is the set of positive words in Ka .

3.2 Properties of the Schreier Automata in Coset Partitions

We recall here some results proved in [8].

Theorem 3.4. [8] *Let F_n be the free group on $n \geq 1$ generators. Let $\{H_i \alpha_i\}_{i=1}^s$ be a coset partition of F_n with $H_i < F_n$ of index d_i , $\alpha_i \in F_n$, $1 \leq i \leq s$, and $1 < d_1 \leq \dots \leq d_s$. Let \tilde{X}_i denote the Schreier graph of H_i , with transition matrix A_i , and period $h_i \geq 1$, $1 \leq i \leq s$.*

- (i) *Assume $h_k = \max\{h_i \mid 1 \leq i \leq s\} > 1$. Then there exists $j \neq k$ such that $h_j = h_k$.*
- (ii) *Let $h_\ell > 1$, such that h_ℓ does not properly divide any other period h_i , $1 \leq i \leq s$. Then there exists $j \neq \ell$ such that $h_j = h_\ell$.*

(iii) For every h_i , there exists $j \neq i$ such that either $h_i = h_j$ or $h_i \mid h_j$.

□

If $n = 1$ in Theorem 3.4, $\{H_i \alpha_i\}_{i=1}^{i=s}$ is a coset partition of \mathbb{Z} and we recover the Davenport-Rado result (or Mirsky-Newman result) for the Erdős’ conjecture and some of its consequences. Indeed, for every index d , the Schreier graph of $d\mathbb{Z}$ has a transition matrix with period equal to d , so a repetition of the period is equivalent to a repetition of the index. For the unique subgroup H of \mathbb{Z} of index d , its Schreier graph \tilde{X}_H is a closed directed path of length d (with each edge labelled 1). So, its transition matrix A is the permutation matrix corresponding to the d -cycle $(1, 2, \dots, d)$, and it has period d . In particular, the period of A_s is d_s , and there exists $j \neq s$ such that $d_j = d_s$. Also, if the period (index) d_k of A_k does not properly divide any other period (index), then there exists $j \neq k$ such that $d_j = d_k$. For the free groups in general, we prove that in some cases, the repetition of the period implies the repetition of the index (see [8]).

4 Proof of the Main Results

4.1 Properties of the Transition Matrix of the Schreier Graph

We study the properties of the transition matrix of a Schreier graph.

Lemma 4.1. *Let $H < F_n$ of index d , with Schreier graph \tilde{X}_H and transition matrix A with period $h \geq 1$. Then the following properties hold:*

- (i) *The vectors $v_L = \frac{1}{d}(1, 1, \dots, 1)$, $v_R = (1, 1, \dots, 1)^T$ are left and right eigenvectors of n whose components are all positive, with $v_L v_R = 1$.*
- (ii) *The matrix $P = v_R v_L$ is of order $d \times d$ with all entries equal to $\frac{1}{d}$.*
- (iii) *If $h = 1$, then $\lim_{k \rightarrow \infty} \frac{A^k}{n^k} = P$ and if $h > 1$, then $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=0}^{j=k-1} \frac{A^j}{n^j} = P$.*

Proof. (i), (ii), (iii) As the sum of every row and every column in A is equal to n , $\lambda_{PF} = n$ with right eigenvector $v_R = (1, 1, \dots, 1)^T$ and left eigenvector $(1, 1, \dots, 1)$. Since $(1, 1, \dots, 1)v_R = d$, $v_L = \frac{1}{d}(1, 1, \dots, 1)$ is a left eigenvector that satisfies $v_L v_R = 1$. Computing $v_R v_L$ gives the matrix P of order $d \times d$ with all entries equal to $\frac{1}{d}$. (iii) results from Theorem 2.2. □

The behaviour of exponents of an aperiodic $d \times d$ matrix of a Schreier graph \tilde{X}_H is well known: for every $1 \leq i, j \leq d$, $\lim_{k \rightarrow \infty} \frac{(A^k)_{ij}}{n^k} = \frac{1}{d}$, from Lemma 4.1. It means that the proportion of positive words of every length k (k large enough) that belong to any coset of H tends to the fixed value $\frac{1}{d}$. We turn now to the study of $\lim_{k \rightarrow \infty} \frac{(A^k)_{ij}}{n^k}$, where A is the transition matrix of a Schreier graph \tilde{X}_H of period $h > 1$.

Definition 4.2. For $1 \leq i, j \leq d$, we define m_{ij} , $0 \leq m_{ij} \leq d$, to be the minimal natural number such that $(A^{m_{ij}})_{ij} \neq 0$.

By definition, if $i \neq j$, then m_{ij} is the minimal length of a directed path from i to j in \tilde{X}_H and if $i = j$, then $m_{ij} = 0$. Whenever $h > 1$, only for the exponents $m_{ij} + kh$, $k \geq 0$, $(A^{m_{ij}+kh})_{ij} \neq 0$, that is only positive words of length $m_{ij} + kh$ are accepted by the Schreier automaton, with i and j the start and end states respectively. Note that if H is a subgroup of $\mathbb{Z} = \langle 1 \rangle$ of index d , its transition matrix A is a permutation matrix with period d and $m_{ij} = r$, where $d\mathbb{Z} + r$ is the coset with i and j the start and end states respectively.

Lemma 4.3. *Let $H < F_n$ be of index d , with Schreier graph \tilde{X}_H and transition matrix A with period $h > 1$. Then, the following properties hold:*

- (i) $\frac{(A^k)_{ij}}{n^k} = 0$, whenever $k \not\equiv m_{ij} \pmod{h}$, $1 \leq i, j \leq d$.
- (ii) $\lim_{k \rightarrow \infty} \frac{(A^{m_{ij}+kh})_{ij}}{n^k} = \frac{h}{d}$, $1 \leq i, j \leq d$.
- (iii) for every $0 \leq m \leq h - 1$, there is i such that $m_{1i} \equiv m \pmod{h}$.
- (iv) h divides d .

Proof. (i) By definition, whenever $k \not\equiv m_{ij} \pmod{h}$, $(A^k)_{ij} = 0$.

(ii), (iii), (iv) We define an $d \times h$ matrix B in the following way. Each row i is labelled by a right coset of H in the same order as they appear in the rows and columns of A and each column by $m = 0, 1, 2, \dots, h - 1$, and $(B)_{ij} = \lim_{k \rightarrow \infty} \frac{(A^{j+kh})_{1i}}{n^{j+kh}}$. Roughly, $(B)_{ij}$ is the proportion of positive words of very large length (congruent to $j \pmod{h}$) that belong to the corresponding coset of H . From (i):

$$(B)_{ij} = \begin{cases} 0 & \text{if } m_{1i} \not\equiv j \pmod{h}, \\ \lim_{k \rightarrow \infty} \frac{(A^{m_{1i}+kh})_{1i}}{n^{m_{1i}+kh}} & \text{if } m_{1i} \equiv j \pmod{h}. \end{cases}$$

So, at each row of B , there is a single non-zero entry. As F_n is partitioned by the d cosets of H , all the non-zero elements in B are equal and for every $k \geq 0$ and every $1 \leq i \leq d$, $\sum_{f=1}^{f=d} (A^k)_{if} = n^k$, in particular $\sum_{f=1}^{f=d} \frac{(A^k)_{1f}}{n^k} = 1$.

So, $\sum_{i=1}^{i=d} (B)_{ij} = \sum_{i=1}^{i=d} \lim_{k \rightarrow \infty} \frac{(A^{j+kh})_{1i}}{n^{j+kh}} = \lim_{k \rightarrow \infty} \sum_{i=1}^{i=d} \frac{(A^{j+kh})_{1i}}{n^{j+kh}} = 1$, that is the sum of elements in each column of B is equal to 1. If $h = d$, B is a square matrix and the right cosets can be arranged such that their labels m are in growing order and we have necessarily a diagonal matrix (otherwise there would be a column of zeroes). So, $\lim_{k \rightarrow \infty} \frac{(A^{m_{1i}+kh})_{1i}}{n^{m_{1i}+kh}} = 1$ and (ii), (iii), (iv) hold. Now, assume $d > h$. At each column, there is at least one non-zero entry, so (iv) holds. Furthermore, the number of non-zero entries in each column needs to be the same, so h divides d and for any i , $\frac{d}{h} * (\lim_{k \rightarrow \infty} \frac{(A^{m_{1i}+kh})_{1i}}{n^{m_{1i}+kh}}) = 1$. That is, $\lim_{k \rightarrow \infty} \frac{(A^{m_{1i}+kh})_{1i}}{n^{m_{1i}+kh}} = \frac{h}{d}$.

Furthermore, $\lim_{k \rightarrow \infty} \frac{1}{h} \sum_{j=0}^{j=h-1} \frac{(A^{j+kh})_{1i}}{n^{j+kh}} = \frac{1}{d}$. □

4.2 Conditions that Ensure Multiplicity in a Coset Partition

Let F_n be the free group on $n \geq 1$ generators. Let $\{H_i\alpha_i\}_{i=1}^{i=s}$ be a coset partition of F_n with $H_i < F_n$ of index d_i , $\alpha_i \in F_n$, $1 \leq i \leq s$, and $1 < d_1 \leq \dots \leq d_s$. Let \tilde{X}_i denote the Schreier graph of H_i , with transition matrix A_i , and period $h_i \geq 1$, $1 \leq i \leq s$. In the following lemmas, we prove, under additional assumptions, that there exist conditions that ensure multiplicity.

Lemma 4.4. *Assume that among all periods h_1, \dots, h_s , there exists a unique value $h > 1$. Let r denote the number of repetitions of h . Then, $r \geq h$. Furthermore, if $h \leq r \leq 2(h - 1)$, then $\{H_i\alpha_i\}_{i=1}^{i=s}$ has multiplicity.*

Proof. For every $1 \leq i \leq s$, $(A_i^k)_{1f_i}$ denotes the number of positive words of length k that belong to the coset $H_i\alpha_i$. Let $I = \{1 \leq i \leq s \mid h_i = h\}$. For every $i \in I$, we denote by m_i the minimal natural number such that $(A_i^{m_i})_{1f_i} \neq 0$. We define an $r \times h$ matrix C in the following way. Each row i is labelled by a right coset $H_i\alpha_i$, where $i \in I$ and each column by $m = 0, 1, 2, \dots, h - 1$, and:

$$(C)_{ij} = \begin{cases} 0 & \text{if } m_i \not\equiv j \pmod{h}, \\ \lim_{k \rightarrow \infty} \frac{(A_i^{m_i+hk})_{1f_i}}{n^{m_i+hk}} & \text{if } m_i \equiv j \pmod{h}. \end{cases}$$

Roughly, $(C)_{ij}$ is the proportion of positive words of very large length (congruent to $j \pmod{h}$) that belong to $H_i\alpha_i$, where $i \in I$. At each row of C there is a unique non-zero entry. Since $\{H_i\alpha_i\}_{i=1}^{i=s}$ is a coset partition of F_n , for every k , $\sum_{i=1}^{i=s} (A_i^k)_{1f_i} = n^k$, that is $\sum_{i=1}^{i=s} \frac{(A_i^k)_{1f_i}}{n^k} = 1$. If A_i is aperiodic, then

$$\lim_{k \rightarrow \infty} \frac{(A_i^k)_{1f_i}}{n^k} = \frac{1}{d_i} \text{ from Lemma 4.1. So, } 1 = \lim_{k \rightarrow \infty} \sum_{i=1}^{i=s} \frac{(A_i^k)_{1f_i}}{n^k} = \sum_{i=1}^{i=s} \lim_{k \rightarrow \infty} \frac{(A_i^k)_{1f_i}}{n^k} = \sum_{i \notin I} \frac{1}{d_i} + \sum_{i \in I} \lim_{k \rightarrow \infty} \frac{(A_i^k)_{1f_i}}{n^k}. \text{ That is, } \sum_{i \in I} \lim_{k \rightarrow \infty} \frac{(A_i^k)_{1f_i}}{n^k} = 1 - \sum_{i \notin I} \frac{1}{d_i} = \sum_{i \in I} \frac{1}{d_i}, \text{ since } \sum_{i=1}^{i=s} \frac{1}{d_i} = 1. \text{ So, the sum of elements in each column of } C \text{ is equal to } \sum_{i \in I} \frac{1}{d_i} \text{ and}$$

from Lemma 4.3, the non-zero entries in C have the form $\frac{h}{d_i}$. If $r < h$, then there is necessarily a column of zeroes, so $r \geq h$. If $r = h$, then C is a square matrix and the right cosets can be arranged such that their labels m are in growing order and we have necessarily a diagonal matrix (otherwise there would be a column of zeroes). So, for every $i \in I$, $\frac{h}{d_i} = \sum_{i \in I} \frac{1}{d_i}$. That is, the coset partition $\{H_i\alpha_i\}_{i=1}^{i=s}$

has multiplicity with all the d_i equal for $i \in I$. Now, assume $r > h$. At each column, there is at least one non-zero entry and there are necessarily columns with several non-zero entries. By a simple combinatorial argument, the number n_0 of columns with a single non-zero entry satisfies $h - (r - h) \leq n_0 \leq h - 1$, that is $2h - r \leq n_0 \leq h - 1$. If we assume $r \leq 2(h - 1)$, then $n_0 \geq 2h - r - 2(h - 1) \geq 2$, that is the number of columns with a single non-zero entry is at least 2, so there are at least two $i \in I$, such that $\frac{h}{d_i} = \sum_{i \in I} \frac{1}{d_i}$, and the coset partition

$\{H_i\alpha_i\}_{i=1}^{i=s}$ has multiplicity. Note that for every $0 \leq m \leq h - 1$, there is i such that $m_i \equiv m \pmod{h}$. □

Lemma 4.5. *Assume there exist at least two coprime values $h, h' > 1$ of periods h_1, \dots, h_s . Let r and r' denote the number of repetitions of h and h' respectively. If $r = h$ or $r \leq 2(h-1)$ or $r' = h'$ or $r' \leq 2(h'-1)$, then $\{H_i \alpha_i\}_{i=1}^{i=s}$ has multiplicity.*

Proof. Let $I = \{1 \leq i \leq s \mid h_i = h\}$ and $I' = \{1 \leq i \leq s \mid h_i = h'\}$. Assume with no loss of generality that $h' < h$. From the same argument as in the proof of Lemma 4.4, $\sum_{i \in I \cup I'} \lim_{k \rightarrow \infty} \frac{(A_i^k)_{1f_i}}{n^k} = \sum_{i \in I \cup I'} \frac{1}{d_i}$. We show that each period can be considered independently, that is each period has its own matrix C as defined in the proof of Lemma 4.4. We define an $(r' + r) \times L$ matrix D , where $L = 2hh'$, in the following way. The first r' rows are labelled by right cosets $H_i \alpha_i$, where $i \in I'$, the last r rows are labelled by right cosets $H_i \alpha_i$, where $i \in I$ and each column by $m = 0, 1, 2, \dots, h' - 1, \dots, h - 1, h, \dots, L - 1$, and:

$$(D)_{ij} = \begin{cases} 0 & \text{if } i \in I', m_i \not\equiv j \pmod{h'}, \\ \lim_{k \rightarrow \infty} \frac{(A_i^{m_i+h'k})_{1f_i}}{n^{m_i+h'k}} & \text{if } i \in I', m_i \equiv j \pmod{h'} \\ 0 & \text{if } i \in I, m_i \not\equiv j \pmod{h}, \\ \lim_{k \rightarrow \infty} \frac{(A_i^{m_i+hk})_{1f_i}}{n^{m_i+hk}} & \text{if } i \in I, m_i \equiv j \pmod{h}. \end{cases}$$

So, the sum of elements in each column of D is equal to $\sum_{i \in I \cup I'} \frac{1}{d_i}$ and from

Lemma 4.3, the non-zero entries in D have the form $\frac{h}{d_i}$ for $i \in I$ and $\frac{h'}{d_i}$, for $i \in I'$. Let $0 \leq m \leq h' - 1$, be the minimal number such that the sum of entries of the m -th column is $\sum_{i \in J_0} \frac{h}{d_i} + \sum_{i \in J'_0} \frac{h'}{d_i}$, where $\emptyset \neq J_0 \subset I$ and $\emptyset \neq J'_0 \subset I'$.

So, for every $0 \leq k \leq h' - 1$, the sum of entries of the $(m + kh)$ -th column is $\sum_{i \in J_0} \frac{h}{d_i} + \sum_{i \in J'_k} \frac{h'}{d_i}$, and this implies necessarily $\sum_{i \in J'_0} \frac{h'}{d_i} = \sum_{i \in J'_1} \frac{h'}{d_i} = \dots = \sum_{i \in J'_{h'-1}} \frac{h'}{d_i}$.

We show that $\{\frac{h'}{d_i} \mid i \in J'_0\}$, $\{\frac{h'}{d_i} \mid i \in J'_1\}$, ..., $\{\frac{h'}{d_i} \mid i \in J'_{h'-1}\}$ appear in the first h' columns of D (not necessarily in this order). Let $0 \leq k, l \leq h' - 1$, $k \neq l$. Assume by contradiction that $m + kh \equiv m + lh \pmod{h'}$. So, h' divides $h(k - l)$. As h and h' are coprime, h' divides $k - l$, a contradiction. So, for every $0 \leq k, l \leq h' - 1$, $k \neq l$, $m + kh \not\equiv m + lh \pmod{h'}$. As there are exactly h' values, these correspond to $0, 1, \dots, h' - 1 \pmod{h'}$, and $\{\frac{h'}{d_i} \mid i \in J'_0\}$, $\{\frac{h'}{d_i} \mid i \in J'_1\}$, ..., $\{\frac{h'}{d_i} \mid i \in J'_{h'-1}\}$ appear in the first h' columns of D with $\sum_{i \in J'_0} \frac{h'}{d_i} = \dots = \sum_{i \in J'_{h'-1}} \frac{h'}{d_i}$.

Furthermore, $\sum_{i \in J'_0} \frac{h'}{d_i} = \dots = \sum_{i \in J'_{h'-1}} \frac{h'}{d_i} = \sum_{i \in I'} \frac{1}{d_i}$. Indeed, on one hand, the sum

of elements in the first r' rows and h' columns is equal to $h' \sum_{i \in J'_0} \frac{h'}{d_i}$ and on

the second hand, it is equal to $\sum_{i \in I'} \frac{h'}{d_i}$. Using the same argument, for every $0 \leq$

$k \leq h - 1$, the sum of entries of the $(m + kh')$ -th column is $\sum_{i \in J_k} \frac{h}{d_i} + \sum_{i \in J'_0} \frac{h'}{d_i}$,

and this implies necessarily $\sum_{i \in J_0} \frac{h}{d_i} = \sum_{i \in J_1} \frac{h}{d_i} = \dots = \sum_{i \in J_{h-1}} \frac{h}{d_i}$. We show that $\{\frac{h}{d_i} \mid i \in J_0\}, \{\frac{h}{d_i} \mid i \in J_1\}, \dots, \{\frac{h}{d_i} \mid i \in J_{h-1}\}$ appear in the first h columns of D (not necessarily in this order). Let $0 \leq k, l \leq h-1, k \neq l$. Assume by contradiction that $m + kh' \equiv m + lh' \pmod{h}$. So, h divides $h'(k-l)$. As h and h' are coprime, h divides $k-l$, a contradiction. So, for every $0 \leq k, l \leq h-1, k \neq l, m + kh' \not\equiv m + lh' \pmod{h}$. As there are exactly h values, these correspond to $0, 1, \dots, h-1 \pmod{h}$, and $\{\frac{h}{d_i} \mid i \in J_0\}, \{\frac{h}{d_i} \mid i \in J_1\}, \dots, \{\frac{h}{d_i} \mid i \in J_{h-1}\}$ appear in the first h columns of D , with $\sum_{i \in J_0} \frac{h}{d_i} = \sum_{i \in J_1} \frac{h}{d_i} = \dots = \sum_{i \in J_{h-1}} \frac{h}{d_i}$. Furthermore, $\sum_{i \in J_0} \frac{h}{d_i} = \dots = \sum_{i \in J_{h-1}} \frac{h}{d_i} = \sum_{i \in I} \frac{1}{d_i}$. So, each period has its own matrix C and we apply the results of Lemma 4.3. □

From Lemma 4.5, coprime periods h and h' can be considered independently. But, if h and h' are not coprime, then the situation is different. Indeed, consider the following coset partition of F_2 : $F_2 = H \cup Ka \cup Ka^3$, where K is the subgroup described in Example 3.3 and $H = \langle a^2, b^2, ab \rangle < F_2$ of index 2. The period of the transition matrix of \tilde{X}_H is $h' = 2$ and the period of the transition matrix of \tilde{X}_K is $h = 4$ and the corresponding matrix D as defined in the proof of Lemma 4.5

is $D = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, with the first row labelled H , the second row Ka , the third

row Ka^3 and at each column $0 \leq m \leq 3$. So, if h' divides h , each period cannot have its own matrix C . Yet, using the same kind of arguments as before, it is not difficult to prove that $r \geq h - \frac{h}{h'}r'$ and that if $r \leq 2(h - \frac{h}{h'}r' - 1)$ then the coset partition has multiplicity. We now turn to the proof of Theorem 1.

Proof of Theorem 1. We assume that H , the set of periods greater than 1, is not empty and that different elements in H are pairwise coprime. Let r_h denote the number of repetitions of h . From the proof of Lemma 4.5, each period has its own matrix C and we apply the results of Lemma 4.3. That is, if for some $h \in H, h \leq r_h \leq 2(h-1)$, then $\{H_i \alpha_i\}_{i=1}^{i=s}$ has multiplicity. □

4.3 Translation of the HS Conjecture in Terms of Automata

Let $F_n = \langle \Sigma \rangle$, and Σ^* denote the free group and the free monoid generated by Σ , respectively. Let $\{H_i \alpha_i\}_{i=1}^{i=s}$ be a coset partition of F_n with $H_i < F_n$ of index $d_i > 1, \alpha_i \in F_n, 1 \leq i \leq s$. Let \tilde{X}_i be the Schreier automaton of $H_i \alpha_i$, with language $L_i = \Sigma^* \cap H_i \alpha_i$.

Proof of Theorem 2. Assume Conjecture 1 is true. For every $1 \leq i \leq s$, the Schreier automaton \tilde{X}_i is a finite, bi-deterministic and complete automaton with strongly-connected underlying graph and alphabet Σ . Since F_n is the disjoint union of the sets $\{H_i \alpha_i\}_{i=1}^{i=s}$, each word in Σ^* belongs to one and exactly one such language, so Σ^* is the disjoint union of the s languages L_1, L_2, \dots, L_s . Since Conjecture 1 is true, there is a repetition of the number of states and this implies the coset partition $\{H_i \alpha_i\}_{i=1}^{i=s}$ has multiplicity, that is the HS conjecture in free

groups of finite rank is true. From [7, Thm. 6], this implies the HS conjecture is true for all finitely generated groups, in particular for all finite groups. So, the HS conjecture is true for all groups. \square

A question that arises naturally is whether the HS Conjecture implies Conjecture 1, that is do the conditions of Conjecture 1 imply necessarily the existence of a coset partition of a free group. First, we note that any finite, bi-deterministic, complete automaton M with d states, finite alphabet Σ , and a strongly-connected underlying graph can represent an automaton with accepted language all the words that belong to some coset of a subgroup H of index d in F_Σ , the free group generated by Σ . Indeed, the start state is replaced by H and each state is replaced by a right coset $H\alpha$, where $\alpha \in \Sigma^*$ is the label of a directed path from the start to it. As M is complete and bi-deterministic, at each vertex v , there are $|\Sigma|$ directed edges into v with each such edge labelled by a different label $a \in \Sigma$, and $|\Sigma|$ directed edges out of v , with each such edge labelled by a different label $a \in \Sigma$. For each $a \in \Sigma$, there exists $a^{-1} \in F_\Sigma$ and for each directed edge $H\alpha \xrightarrow{a} H\alpha a$, $\alpha \in \Sigma^*$, in the underlying graph of M , there exists another directed edge $H\alpha \xleftarrow{a^{-1}} H\alpha a$, which is implicit and not drawn. This fact is crucial for the construction of an automaton with accepted language all the words that belong to some coset $H\alpha$ and not only the positive words that belong to this coset. In fact, this is how the Schreier automaton for a free group relative to a subgroup H is defined initially (see [30, p. 102], [32, p. 107]).

So, the existence of the s automata M_1, \dots, M_s , satisfying the conditions of Conjecture 1, with accepted languages L_1, \dots, L_s respectively, leads to the existence of s automata M'_1, \dots, M'_s with accepted language L'_1, \dots, L'_s , where L'_i denotes the set of words that belong to the coset $H_i\alpha_i$, and $H_i < F_\Sigma$, $1 \leq i \leq s$. The question is now: does the assumption that Σ^* is equal to the disjoint union of the s languages L_1, L_2, \dots, L_s imply necessarily that $\{H_i\alpha_i\}_{i=1}^{i=s}$ is a coset partition of F_Σ . If all the s automata M_1, \dots, M_s satisfy the following additional conditions: Σ is equal to the disjoint union of two sets S and S^- , where $S^- = \{a^- \mid a \in S\}$ and a^- is such that for each directed edge $H\alpha \xrightarrow{a} H\alpha a$, $\alpha \in S^*$, in the underlying graph, there exists another directed edge $H\alpha \xleftarrow{a^-} H\alpha a$ in the underlying graph (and it is drawn), then clearly $\Sigma^* = \bigsqcup_{i=1}^{i=s} L_i$ implies

$$F_\Sigma = \bigsqcup_{i=1}^{i=s} H_i\alpha_i \text{ (since } F_\Sigma = \Sigma^* \text{ and } L_i \text{ is the set of words that belong to } H_i\alpha_i \text{).}$$

But, for automata that do not satisfy these additional conditions, it is not clear at all if this is still true. Indeed, it does not seem that if the languages L_1, \dots, L_s are mutually disjoint, then the languages L'_1, \dots, L'_s are also mutually disjoint and moreover, that $\Sigma^* = \bigsqcup_{i=1}^{i=s} L_i$ implies necessarily $F_\Sigma = \bigsqcup_{i=1}^{i=s} L'_i$.

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