



# Equivalence of Linear Tree Transducers with Output in the Free Group

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**Abstract.** We show that equivalence of deterministic linear tree transducers can be decided in polynomial time when their outputs are interpreted over the free group. Due to the cancellation properties offered by the free group, the required constructions are not only more general, but also simpler than the corresponding constructions for proving equivalence of deterministic linear tree-to-word transducers.

**Keywords:** Linear tree transducer · Free group · Equivalence problem · Polynomial time

## 1 Introduction

In 2009, Staworko and Niehren observed that equivalence for *sequential* tree-to-word transducers [13] can be reduced to the morphism equivalence problem for context-free languages. Since the latter problem is decidable in polynomial time [10], they thus proved that equivalence of sequential tree-to-word transducers is decidable in polynomial time. This decision procedure was later accompanied by a canonical normal form which can be applied to learning [3,4]. Sequentiality of transducers means that subtrees must always be processed from left to right. This restriction was lifted by Boiret who provided a canonical normal form for unrestricted linear tree-to-word transducers [1]. Construction of that normal form, however, may require exponential time implying that the corresponding decision procedure requires exponential time as well. In order to improve on that, Palenta and Boiret provided a polynomial time procedure which just normalizes the *order* in which an unrestricted linear tree-to-word transducer processes the subtrees of its input [2]. They proved that after that normalization, equivalent transducers are necessarily *same-ordered*. As a consequence, equivalence of linear tree-to-word transducers can thus also be reduced to the morphism equivalence problem for context-free languages and thus can be decided in polynomial time. Independently of that, Seidl, Maneth and Kemper showed by algebraic means, that equivalence of general (possibly non-linear) tree-to-word transducers is decidable [11]. Their techniques are also applicable if the outputs of transducers are not just in a free monoid of words, but also if outputs are in a free group. The latter means that output words are considered as equivalent not just when they are literally equal, but also when they

become equal after cancellation of matching positive and negative occurrences of letters. For the special case of *linear* tree transducers with outputs either in a free monoid or a free group, Seidl et al. provided a *randomized* polynomial time procedure for in-equivalence. The question remained open whether for outputs in a free group, randomization can be omitted. Here, we answer this question to the affirmative. In fact, we follow the approach of [2] to normalize the order in which tree transducers produce their outputs. For that normalization, we heavily rely on commutation laws as provided for the free group. Due to these laws, the construction as well as the arguments for its correctness, are not only more general but also much cleaner than in the case of outputs in a free monoid only. The observation that reasoning over the free group may simplify arguments has also been made, e.g., by Tomita and Seino and later by Senizergues when dealing with the equivalence problem for deterministic pushdown transducers [12, 14]. As morphism equivalence on context-free languages is decidable in polynomial time—even if the morphism outputs are in a free group [10], we obtain a polynomial time algorithm for equivalence of tree transducers with output in the free group.

Missing proofs can be found in the extended version of this paper [5].

## 2 Preliminaries

We use  $\Sigma$  to denote a finite *ranked* alphabet, while  $A$  is used for an *unranked* alphabet.  $\mathcal{T}_\Sigma$  denotes the set of all trees (or terms) over  $\Sigma$ . The *depth*  $\text{depth}(t)$  of a tree  $t \in \mathcal{T}_\Sigma$  equals 0, if  $t = f()$  for some  $f \in \Sigma$  of rank 0, and otherwise,  $\text{depth}(t) = 1 + \max\{\text{depth}(t_i) \mid i = 1, \dots, m\}$  for  $t = f(t_1, \dots, t_m)$ . We denote by  $\mathcal{F}_A$  the representation of the free group generated by  $A$  where the carrier is the set of *reduced* words instead of the usual quotient construction: For each  $a \in A$ , we introduce its *inverse*  $a^-$ . The set of elements of  $\mathcal{F}_A$  then consists of all words over the alphabet  $\{a, a^- \mid a \in A\}$  which do not contain  $aa^-$  or  $a^-a$  as factors. These words are also called *reduced*. In particular,  $A^* \subseteq \mathcal{F}_A$ . The group operation “ $\cdot$ ” of  $\mathcal{F}_A$  is concatenation, followed by *reduction*, i.e., repeated cancellation of subwords  $aa^-$  or  $a^-a$ . Thus,  $abc^- \cdot cb^-a =_{\mathcal{F}_A} aa$ . The *neutral element* w.r.t. this operation is the empty word  $\varepsilon$ , while the inverse  $w^-$  of some element  $w \in \mathcal{F}_A$  is obtained by reverting the order of the letters in  $w$  while replacing each letter  $a$  with  $a^-$  and each  $a^-$  with  $a$ . Thus, e.g.,  $(abc^-)^- = cb^-a^-$ .

In light of the inverse operation  $(\cdot)^-$ , we have that  $v \cdot w =_{\mathcal{F}_A} v'w'$  where  $v = v'u$  (as words) for a maximal suffix  $u$  so that  $u^-$  is a prefix of  $w$  with  $w = u^-w'$ . For an element  $w \in \mathcal{F}_A$ ,  $\langle w \rangle = \{w^l \mid l \in \mathbb{Z}\}$  denotes the cyclic subgroup of  $\mathcal{F}_A$  generated from  $w$ . As usual, we use the convention that  $w^0 = \varepsilon$ , and  $w^{-l} = (w^-)^l$  for  $l > 0$ . An element  $p \in \mathcal{F}_A$  different from  $\varepsilon$ , is called *primitive* if  $w^l =_{\mathcal{F}_A} p$  for some  $w \in \mathcal{F}_A$  and  $l \in \mathbb{Z}$  implies that  $w =_{\mathcal{F}_A} p$  or  $w =_{\mathcal{F}_A} p^-$ , i.e.,  $p$  and  $p^-$  are the only (trivial) roots of  $p$ . Thus, primitive elements generate *maximal* cyclic subgroups of  $\mathcal{F}_A$ . We state two crucial technical lemmas.

**Lemma 1.** *Assume that  $y^m =_{\mathcal{F}_A} \beta \cdot y^n \cdot \beta^-$  with  $y \in \mathcal{F}_A$  primitive. Then  $m = n$ , and  $\beta =_{\mathcal{F}_A} y^k$  for some  $k \in \mathbb{Z}$ .*

*Proof.* Since  $\beta \cdot y^n \cdot \beta^- =_{\mathcal{F}_A} (\beta \cdot y \cdot \beta^-)^n$ , we find by [8, Proposition 2.17] a primitive element  $c$  such that  $y$  and  $\beta \cdot y \cdot \beta^-$  are powers of  $c$ . As  $y$  is primitive,  $c$  can be chosen as  $y$ . Accordingly,

$$y^j =_{\mathcal{F}_A} \beta \cdot y \cdot \beta^- \tag{1}$$

holds for some  $j$ . If  $\beta$  is a power of  $y$ , then  $\beta \cdot y \cdot \beta^- =_{\mathcal{F}_A} y$ , and the assertion of the lemma holds. Likewise if  $j = 1$ , then  $\beta$  and  $y$  commute. Since  $y$  is primitive, then  $\beta$  necessarily must be a power of  $y$ .

For a contradiction, therefore now assume that  $\beta$  is not a power of  $y$  and  $j \neq 1$ . W.l.o.g., we can assume that  $j > 1$ . First, assume now that  $y$  is *cyclically reduced*, i.e., the first and last letters,  $a$  and  $b$ , respectively, of  $y$  are not mutually inverse. Then for each  $n > 0$ ,  $y^n$  is obtained from  $y$  by  $n$ -concatenation of  $y$  as a word (no reduction taking place). Likewise, either the last letter of  $\beta$  is different  $a^-$  or the first letter of  $\beta^-$  is different from  $b^-$  because these two letters are mutually inverse. Assume that the former is the case. Then  $\beta \cdot y$  is obtained by concatenation of  $\beta$  and  $y$  as words (no reduction taking place). By (1),  $\beta \cdot y^n =_{\mathcal{F}_A} y^{j \cdot n} \cdot \beta$ . for every  $n \geq 1$ . Let  $m > 0$  denote the length of  $\beta$  as a word. Since  $\beta$  can cancel only a suffix of  $y^{j \cdot n}$  of length at most  $m$ , it follows, that the word  $\beta y$  must a prefix of the word  $y^{m+1}$ . Since  $\beta$  is not a power of  $y$ , the word  $y$  can be factored into  $y = y'c$  for some non-empty suffix  $c$  such that  $\beta = y^j y'$ , implying that  $yc = cy$  holds. As a consequence,  $y = c^l$  for some  $l > 1$ —in contradiction to the irreducibility of  $y$ .

If on the other hand, the first letter of  $\beta^-$  is not the inverse of the last letter of  $y$ , then  $y \cdot \beta^-$  is obtained as the concatenation of  $y$  and  $\beta^-$  as words. As a consequence,  $y\beta^-$  is a suffix of  $y^{m+1}$ , and we arrive at a contradiction.

We conclude that the statement of the lemma holds whenever  $y$  is cyclically reduced. Now assume that  $y$  is not yet cyclically reduced. Then we can find a maximal suffix  $r$  of  $y$  (considered as a word) such that  $y = r^-sr$  holds and  $s$  is cyclically reduced. Then  $s$  is also necessarily primitive. (If  $s =_{\mathcal{F}_A} c^n$ , then  $y =_{\mathcal{F}_A} (r^-cr)^n$ ). Then assertion (1) can be equivalently formulated as

$$s^j =_{\mathcal{F}_A} (r \cdot \beta \cdot r^-) \cdot y \cdot (r \cdot \beta \cdot r^-)^-$$

We conclude that  $r \cdot \beta \cdot r^- =_{\mathcal{F}_A} s^l$  for some  $l \in \mathbb{Z}$ . But then  $\beta =_{\mathcal{F}_A} (r^- \cdot s \cdot r)^l =_{\mathcal{F}_A} y^l$ , and the claim of the lemma follows.

**Lemma 2.** *Assume that  $x_1, x_2$  and  $y_1, y_2$  are distinct elements in  $\mathcal{F}_A$  and that*

$$x_i \cdot \alpha \cdot y_j \cdot \beta =_{\mathcal{F}_A} \gamma \cdot y'_j \cdot \alpha' \cdot x'_i \cdot \beta' \tag{2}$$

*holds for  $i = 1, 2$  and  $j = 1, 2$ . Then there is some primitive element  $p$  and exponents  $r, s \in \mathbb{Z}$  such that  $x_1 \cdot \alpha =_{\mathcal{F}_A} x_2 \cdot \alpha \cdot p^r$  and  $y_1 =_{\mathcal{F}_A} p^s \cdot y_2$ .*

*Proof.* By the assumption (2),

$$\begin{aligned} \gamma &=_{\mathcal{F}_A} (x_1 \cdot \alpha \cdot y_j \cdot \beta) \cdot (y'_j \cdot \alpha' \cdot x'_1 \cdot \beta')^- \\ &=_{\mathcal{F}_A} (x_2 \cdot \alpha \cdot y_j \cdot \beta) \cdot (y'_j \cdot \alpha' \cdot x'_2 \cdot \beta')^- \end{aligned}$$

for all  $j = 1, 2$ . Thus,

$$\begin{aligned} x_1 \cdot \alpha \cdot y_j \cdot \beta \beta'^{-} x_1'^{-} \alpha'^{-} y_j'^{-} &=_{\mathcal{F}_A} x_2 \cdot \alpha \cdot y_j \cdot \beta \cdot \beta'^{-} \cdot x_2'^{-} \cdot \alpha'^{-} \cdot y_j'^{-} && \text{implying} \\ y_j \cdot \alpha \cdot x_2 \cdot x_1 \cdot \alpha \cdot y_j &=_{\mathcal{F}_A} \beta \cdot \beta'^{-} \cdot x_2'^{-} \cdot x_1' \cdot \beta' \cdot \beta^{-} \end{aligned}$$

for  $j = 1, 2$ . Hence,

$$\begin{aligned} y_1 \cdot \alpha \cdot x_2 \cdot x_1 \cdot \alpha \cdot y_1 &=_{\mathcal{F}_A} y_2 \cdot \alpha \cdot x_2 \cdot x_1 \cdot \alpha \cdot y_2 && \text{implying} \\ (x_2 \cdot \alpha)^{-} x_1 \cdot \alpha &=_{\mathcal{F}_A} (y_1 \cdot y_2^{-}) \cdot ((x_2 \cdot \alpha)^{-} x_1 \cdot \alpha) \cdot (y_1 \cdot y_2^{-})^{-} \end{aligned}$$

Since  $x_1$  is different from  $x_2$ , also  $x_1 \cdot \alpha$  is different from  $x_2 \cdot \alpha$ . Let  $p$  denote a primitive root of  $(x_2 \cdot \alpha)^{-} x_1 \cdot \alpha$ . Then by Lemma 1,

$$\begin{aligned} x_1 \cdot \alpha &=_{\mathcal{F}_A} x_2 \cdot \alpha \cdot p^r \\ y_1 &=_{\mathcal{F}_A} p^s \cdot y_2 \end{aligned}$$

for suitable exponents  $r, s \in \mathbb{Z}$ .

As the elements of  $\mathcal{F}_A$  are words, they can be represented by *straight-line* programs (SLPs). An SLP is a context-free grammar where each non-terminal occurs as the left-hand side of exactly one rule. We briefly recall basic complexity results for operations on elements of  $\mathcal{F}_A$  when represented as SLPs [7].

**Lemma 3.** *Let  $U, V$  be SLPs representing words  $w_1, w_2 \in \{a, a^{-} \mid a \in A\}$ , respectively. Then the following computations/decision problems can be realized in polynomial time*

- compute an SLP for  $w_1^{-}$ ;
- compute the primitive root of  $w_1$  if  $w_1 \neq \varepsilon$ ;
- compute an SLP for  $w =_{\mathcal{F}_A} w_1$  with  $w$  reduced;
- decide whether  $w_1 =_{\mathcal{F}_A} w_2$ ;
- decide whether it exists  $g \in \mathcal{F}_A$ , such that  $w_1 \in g \cdot \langle w_2 \rangle$  and compute an SLP for such  $g$ .

In the following, we introduce *deterministic linear tree transducers* which produce outputs in the free group  $\mathcal{F}_A$ . For convenience, we follow the approach in [11] where only *total* deterministic transducers are considered—but equivalence is relative w.r.t. some *top-down deterministic* domain automaton  $B$ . A top-down deterministic automaton (DTA)  $B$  is a tuple  $(H, \Sigma, \delta_B, h_0)$  where  $H$  is a finite set of states,  $\Sigma$  is a finite ranked alphabet,  $\delta_B : H \times \Sigma \rightarrow H^*$  is a partial function where  $\delta_B(h, f) \in H^k$  if the the rank of  $f$  equals  $k$ , and  $h_0$  is the start state of  $B$ . For every  $h \in H$ , we define the set  $\text{dom}(h) \subseteq \mathcal{T}_\Sigma$  by  $f(t_1, \dots, t_m) \in \text{dom}(h)$  iff  $\delta_B(h, f) = h_1 \dots h_m$  and  $t_i \in \text{dom}(h_i)$  for all  $i = 1, \dots, m$ .  $B$  is called *reduced* if  $\text{dom}(h) \neq \emptyset$  for all  $h \in H$ . The *language*  $\mathcal{L}(B)$  accepted by  $B$  is the set  $\text{dom}(h_0)$ . We remark that for every DTA  $B$  with  $\mathcal{L}(B) \neq \emptyset$ , a reduced DTA  $B'$  can be constructed in polynomial time with  $\mathcal{L}(B) = \mathcal{L}(B')$ . Therefore, we subsequently assume w.l.o.g. that each DTA  $B$  is reduced.

A (total deterministic) *linear tree transducer* with output in  $\mathcal{F}_A$  ( $\text{LT}_A$  for short) is a tuple  $M = (\Sigma, A, Q, S, R)$  where  $\Sigma$  is the ranked alphabet for the

input trees,  $\mathbf{A}$  is the finite (unranked) output alphabet,  $Q$  is the set of *states*,  $S$  is the *axiom* of the form  $u_0$  or  $u_0 \cdot q_0(x_0) \cdot u_1$  with  $u_0, u_1 \in \mathcal{F}_{\mathbf{A}}$  and  $q_0 \in Q$ , and  $R$  is the set of *rules* which contains for each state  $q \in Q$  and each input symbol  $f \in \Sigma$ , one rule of the form

$$q(f(x_1, \dots, x_m)) \rightarrow u_0 \cdot q_1(x_{\sigma(1)}) \cdot \dots \cdot u_{n-1} \cdot q_n(x_{\sigma(n)}) \cdot u_n \quad (3)$$

Here,  $m$  is the rank of  $f$ ,  $n \leq m$ ,  $u_0, \dots, u_n \in \mathcal{F}_{\mathbf{A}}$  and  $\sigma$  is an injective mapping from  $\{1, \dots, n\}$  to  $\{1, \dots, m\}$ . The *semantics* of a state  $q$  is the function  $\llbracket q \rrbracket : \mathcal{T}_{\Sigma} \rightarrow \mathcal{F}_{\mathbf{A}}$  defined by

$$\llbracket q \rrbracket(f(t_1, \dots, t_m)) =_{\mathcal{F}_{\mathbf{A}}} u_0 \cdot \llbracket q_1 \rrbracket(t_{\sigma(1)}) \cdot \dots \cdot u_{n-1} \cdot \llbracket q_n \rrbracket(t_{\sigma(n)}) \cdot u_n$$

if there is a rule of the form (3) in  $R$ . Then the *translation* of  $M$  is the function  $\llbracket M \rrbracket : \mathcal{T}_{\Sigma} \rightarrow \mathcal{F}_{\mathbf{A}}$  defined by  $\llbracket M \rrbracket(t) =_{\mathcal{F}_{\mathbf{A}}} u_0$  if the axiom of  $M$  equals  $u_0$ , and  $\llbracket M \rrbracket(t) =_{\mathcal{F}_{\mathbf{A}}} u_0 \cdot \llbracket q \rrbracket(t) \cdot u_1$  if the axiom of  $M$  is given by  $u_0 \cdot q(x_0) \cdot u_1$ .

*Example 1.* Let  $\mathbf{A} = \{a, b\}$ . As a running example we consider the  $\text{LT}_{\mathbf{A}} M$  with input alphabet  $\Sigma = \{f^2, g^1, k^0\}$  where the superscripts indicate the rank of the input symbols.  $M$  has axiom  $q_0(x_0)$  and the following rules

$$\begin{array}{lll} q_0(f(x_1, x_2)) \rightarrow q_1(x_2) b q_2(x_1) & q_0(g(x_1)) \rightarrow q_0(x_1) & q_0(k) \rightarrow \varepsilon \\ q_1(f(x_1, x_2)) \rightarrow q_0(x_1) q_0(x_2) & q_1(g(x_1)) \rightarrow a b q_1(x_1) & q_1(k) \rightarrow a \\ q_2(f(x_1, x_2)) \rightarrow q_0(x_1) q_0(x_2) & q_2(g(x_1)) \rightarrow a b q_2(x_1) & q_2(k) \rightarrow a b \end{array}$$

Two  $\text{LT}_{\mathbf{A}} S$ ,  $M$ ,  $M'$  are *equivalent* relative to the DTA  $B$  iff their translations coincide on all input trees accepted by  $B$ , i.e.,  $\llbracket M \rrbracket(t) =_{\mathcal{F}_{\mathbf{A}}} \llbracket M' \rrbracket(t)$  for all  $t \in \mathcal{L}(B)$ .

To relate the computations of the  $\text{LT}_{\mathbf{A}} M$  and the domain automaton  $B$ , we introduce the following notion. A mapping  $\iota : Q \rightarrow H$  from the set of states of  $M$  to the set of states of  $B$  is called *compatible* if either the set of states of  $M$  is empty (and thus the axiom of  $M$  consists of an element of  $\mathcal{F}_{\mathbf{A}}$  only), or the following holds:

1.  $\iota(q_0) = h_0$ ;
2. If  $\iota(q) = h$ ,  $\delta_B(h, f) = h_1 \dots h_m$ , and there is a rule in  $M$  of the form (3) then  $\iota(q_i) = h_{\sigma(i)}$  for all  $i = 1, \dots, n$ ;
3. If  $\iota(q) = h$  and  $\delta_B(h, f)$  is undefined for some  $f \in \Sigma$  of rank  $m \geq 0$ , then  $M$  has the rule  $q(f(x_1, \dots, x_m)) \rightarrow \perp$  for some dedicated symbol  $\perp$  which does not belong to  $\mathbf{A}$ .

**Lemma 4.** *For an  $\text{LT}_{\mathbf{A}} M$  and a DTA  $B = (H, \Sigma, \delta_B, h_0)$ , an  $\text{LT}_{\mathbf{A}} M'$  with a set of states  $Q'$  together with a mapping  $\iota : Q' \rightarrow H$  can be constructed in polynomial time such that the following holds:*

1.  $M$  and  $M'$  are equivalent relative to  $B$ ;
2.  $\iota$  is compatible.

*Example 2.* Let  $\text{LT}_A M$  be defined as in Example 1. Consider DTA  $B$  with start state  $h_0$  and the transition function  $\delta_B = \{(h_0, f) \mapsto h_1 h_1, (h_1, g) \mapsto h_1, (h_1, h) \rightarrow \varepsilon\}$ . According to Lemma 4,  $\text{LT}_A M'$  for  $M$  then is defined as follows.  $M'$  has axiom  $\langle q_0, h_0 \rangle(x_0)$  and the rules

$$\begin{aligned} \langle q_0, h_0 \rangle(f(x_1, x_2)) &\rightarrow \langle q_1, h_1 \rangle(x_2) b \langle q_2, h_1 \rangle(x_1) \\ \langle q_1, h_1 \rangle(g(x_1)) &\rightarrow ab \langle q_1, h_1 \rangle(x_1) \quad \langle q_1, h_1 \rangle(k) \rightarrow a \\ \langle q_2, h_1 \rangle(g(x_1)) &\rightarrow ab \langle q_2, h_1 \rangle(x_1) \quad \langle q_2, h_1 \rangle(k) \rightarrow ab \end{aligned}$$

where the rules with left-hand sides  $\langle q_0, h_0 \rangle(g(x_1))$ ,  $\langle q_0, h_0 \rangle(h)$ ,  $\langle q_1, h_1 \rangle(f(x_1, x_2))$ ,  $\langle q_2, h_1 \rangle(f(x_1, x_2))$ , all have right-hand-sides  $\perp$ . The compatible map  $\iota$  is then given by  $\iota = \{\langle q_0, h_0 \rangle \mapsto h_0, \langle q_1, h_1 \rangle \mapsto h_1, \langle q_2, h_1 \rangle \mapsto h_1\}$ . For convenience, we again denote the pairs  $\langle q_0, h_0 \rangle$ ,  $\langle q_1, h_1 \rangle$ ,  $\langle q_2, h_1 \rangle$  with  $q_0, q_1, q_2$ , respectively.

Subsequently, we w.l.o.g. assume that each  $\text{LT}_A M$  with corresponding DTA  $B$  for its domain, comes with a compatible map  $\iota$ . Moreover, we define for each state  $q$  of  $M$ , the set  $\mathcal{L}(q) = \{\llbracket q \rrbracket(t) \mid t \in \text{dom}(\iota(q))\}$  of all outputs produced by state  $q$  (on inputs in  $\text{dom}(\iota(q))$ ), and  $\mathcal{L}^{(i)}(q) = \{\llbracket q \rrbracket(t) \mid t \in \text{dom}(\iota(q)), \text{depth}(t) < i\}$  for  $i \geq 0$ .

Beyond the availability of a compatible map, we also require that all states of  $M$  are *non-trivial* (relative to  $B$ ). Here, a state  $q$  of  $M$  is called *trivial* if  $\mathcal{L}(q)$  contains a single element only. Otherwise, it is called *non-trivial*. This property will be established in Theorem 1.

### 3 Deciding Equivalence

In the first step, we show that equivalence relative to the DTA  $B$  of *same-ordered*  $\text{LT}_A$ s is decidable. For a DTA  $B$ , consider the  $\text{LT}_A$ s  $M$  and  $M'$  with compatible mappings  $\iota$  and  $\iota'$ , respectively.  $M$  and  $M'$  are *same-ordered* relative to  $B$  if they process their input trees in the same order. We define set of pairs  $\langle q, q' \rangle$  of *co-reachable* states of  $M$  and  $M'$ . Let  $u_0 \cdot q_0(x_1) \cdot u_1$  and  $u'_0 \cdot q'_0(x_1) \cdot u'_1$  be the axioms of  $M$  and  $M'$ , respectively, where  $\iota(q_0) = \iota'(q'_0)$  is the start state of  $B$ . Then the pair  $\langle q_0, q'_0 \rangle$  is co-reachable. Let  $\langle q, q' \rangle$  be a pair of co-reachable states. Then  $\iota(q) = \iota'(q')$  should hold. For  $f \in \Sigma$ , assume that  $\delta_B(\iota(q), f)$  is defined. Let

$$\begin{aligned} q(f(x_1, \dots, x_m)) &\rightarrow u_0 q_1(x_{\sigma(1)}) u_1 \dots u_{n-1} q_n(x_{\sigma(n)}) u_n \\ q'(f(x_1, \dots, x_m)) &\rightarrow u'_0 q'_1(x_{\sigma'(1)}) u'_1 \dots u'_{n-1} q'_n(x_{\sigma'(n)}) u'_n \end{aligned} \tag{4}$$

be the rules of  $q, q'$  for  $f$ , respectively. Then  $\langle q_j, q'_{j'} \rangle$  is co-reachable whenever  $\sigma(j) = \sigma'(j')$  holds. In particular, we then have  $\iota(q_j) = \iota'(q'_{j'})$ .

The pair  $\langle q, q' \rangle$  of co-reachable states is called *same-ordered*, if for each corresponding pair of rules (4),  $n = n'$  and  $\sigma = \sigma'$ . Finally,  $M$  and  $M'$  are *same-ordered* if for every co-reachable pair  $\langle q, q' \rangle$  of states of  $M, M'$ , and every  $f \in \Sigma$ , each pair of rules (4) is same-ordered whenever  $\delta_B(\iota(q), f)$  is defined.

Given that the  $\text{LT}_A$ s  $M$  and  $M'$  are same-ordered relative to  $B$ , we can represent the set of pairs of runs of  $M$  and  $M'$  on input trees by means of

a single context-free grammar  $G$ . The set of nonterminals of  $G$  consists of a distinct start nonterminal  $S$  together with all co-reachable pairs  $\langle q, q' \rangle$  of states  $q, q'$  of  $M, M'$ , respectively. The set of terminal symbols  $T$  of  $G$  is given by  $\{a, a^-, \bar{a}, \bar{a}^- \mid a \in \mathbf{A}\}$  for fresh distinct symbols  $\bar{a}, \bar{a}^-, a \in \mathbf{A}$ . Let  $\langle q, q' \rangle$  be a co-reachable pair of states of  $M, M'$ , and  $f \in \Sigma$  such that  $\delta_B(\iota(q), f)$  is defined. For each corresponding pair of rules (4),  $G$  receives the rule

$$\langle q, q' \rangle \rightarrow u_0 \bar{u}'_0 \langle q_1, q'_1 \rangle u_1 \bar{u}'_1 \dots u_{n-1} \bar{u}'_{n-1} \langle q_n, q'_n \rangle u_n \bar{u}'_n$$

where  $\bar{u}'_i$  is obtained from  $u'_i$  by replacing each output symbol  $a \in \mathbf{A}$  with its barred copy  $\bar{a}$  as well as each inverse  $a^-$  with its barred copy  $\bar{a}^-$ . For the axioms  $u_0 q(x_1) u_1$  and  $u'_0 q'(x_1) u'_1$  of  $M, M'$ , respectively, we introduce the rule  $S \rightarrow u_0 \bar{u}'_0 \langle q, q' \rangle u_1 \bar{u}'_1$  where again  $\bar{u}'_i$  are the barred copies of  $u'_i$ . We define morphisms  $f, g : T^* \rightarrow \mathcal{F}_{\mathbf{A}}$  by

$$\begin{aligned} f(a) &= a & f(a^-) &= a^- & f(\bar{a}) &= f(\bar{a}^-) = \varepsilon \\ g(\bar{a}) &= a & g(\bar{a}^-) &= a^- & g(a) &= g(a^-) = \varepsilon \end{aligned}$$

for  $a \in \mathbf{A}$ . Then  $M$  and  $M'$  are equivalent relative to  $B$  iff  $g(w) =_{\mathcal{F}_{\mathbf{A}}} f(w)$  for all  $w \in \mathcal{L}(G)$ . Combining Plandowski's polynomial construction of a test set for a context-free language to check morphism equivalence over finitely generated free groups [10, Theorem 6], with Lohrey's polynomial algorithm for checking equivalence of SLPs over the free group [6], we deduce that the equivalence of the morphisms  $f$  and  $g$  on all words generated by the context-free grammar  $G$ , is decidable in polynomial time. Consequently, we obtain:

**Corollary 1.** *Equivalence of same-ordered  $\text{LT}_{\mathbf{A}}$ s relative to a DTA  $B$  is decidable in polynomial time.*  $\square$

Next, we observe that for every  $\text{LT}_{\mathbf{A}}$   $M$  with compatible map  $\iota$  and non-trivial states only, a *canonical* ordering can be established. We call  $M$  *ordered* (relative to  $B$ ) if for all rules of the form (3), with  $\mathcal{L}(q_i) \cdot u_i \cdot \dots \cdot u_{j-1} \cdot \mathcal{L}(q_j) \subseteq v \cdot \langle p \rangle$ ,  $p \in \mathcal{F}_{\mathbf{A}}$  the ordering  $\sigma(i) < \dots < \sigma(j)$  holds. Here we have naturally extended the operation “ $\cdot$ ” to sets of elements.

We show that two ordered  $\text{LT}_{\mathbf{A}}$ s, when they are equivalent, are necessarily *same-ordered*. The proof of this claim is split in two parts. First, we prove that the *set* of indices of subtrees processed by equivalent co-reachable states are identical and second, that the order is the same.

**Lemma 5.** *Let  $M, M'$  be  $\text{LT}_{\mathbf{A}}$ s with compatible maps  $\iota$  and  $\iota'$ , respectively, and non-trivial states only so that  $M$  and  $M'$  are equivalent relative to the DTA  $B$ . Let  $\langle q, q' \rangle$  be a pair of co-reachable states of  $M$  and  $M'$ . Assume that  $\delta_B(\iota(q), f)$  is defined for some  $f \in \Sigma$  and consider the corresponding pair of rules (4). Then the following holds:*

1.  $\{\sigma(1), \dots, \sigma(n)\} = \{\sigma'(1), \dots, \sigma'(n')\}$ ;
2.  $\sigma = \sigma'$ .

*Proof.* Since  $\langle q, q' \rangle$  is a co-reachable pair of states, there are elements  $\alpha, \alpha', \beta, \beta' \in \mathcal{F}_A$  such that

$$\alpha \cdot \llbracket q \rrbracket(t) \cdot \beta =_{\mathcal{F}_A} \alpha' \cdot \llbracket q' \rrbracket(t) \cdot \beta'$$

holds for all  $t \in \text{dom}(\iota(q))$ . Consider the first statement. Assume for a contradiction that  $q_k(x_j)$  occurs on the right-hand side of the rule for  $q$  but  $x_j$  does not occur on the right-hand side of the rule for  $q'$ . Then, there are input trees  $t = f(t_1, \dots, t_m)$  and  $t' = f(t'_1, \dots, t'_m)$ , both in  $\text{dom}(\iota(q))$ , such that  $\llbracket q_k \rrbracket(t_j) \neq_{\mathcal{F}_A} \llbracket q_k \rrbracket(t'_j)$  and  $t_i = t'_i$  for all  $i \neq j$ . Moreover, there are  $\mu_1, \mu_2 \in \mathcal{F}_A$  s.t.

$$\alpha \cdot \llbracket q \rrbracket(t) \cdot \beta =_{\mathcal{F}_A} \alpha \cdot \mu_1 \cdot \llbracket q_k \rrbracket(t_j) \cdot \mu_2 \cdot \beta \neq_{\mathcal{F}_A} \alpha \cdot \mu_1 \cdot \llbracket q_k \rrbracket(t'_j) \cdot \mu_2 \cdot \beta =_{\mathcal{F}_A} \alpha \cdot \llbracket q \rrbracket(t') \cdot \beta$$

But then,

$$\alpha \cdot \llbracket q \rrbracket(t) \cdot \beta =_{\mathcal{F}_A} \alpha' \cdot \llbracket q' \rrbracket(t) \cdot \beta' =_{\mathcal{F}_A} \alpha' \cdot \llbracket q' \rrbracket(t') \cdot \beta' =_{\mathcal{F}_A} \alpha \cdot \llbracket q \rrbracket(t') \cdot \beta$$

—a contradiction. By an analogous argument for some  $x_j$  only occurring in the right-hand side of the rule for  $q'$  the first statement follows.

Assume for contradiction that the mappings  $\sigma$  and  $\sigma'$  in the corresponding rules (4) differ. Let  $k$  denote the minimal index so that  $\sigma(k) \neq \sigma'(k)$ . W.l.o.g., we assume that  $\sigma'(k) < \sigma(k)$ . By the first statement,  $n = n'$  and  $\{\sigma(1), \dots, \sigma(n)\} = \{\sigma'(1), \dots, \sigma'(n)\}$ . Then there are  $\ell, \ell' > k$  such that

$$\sigma'(k) = \sigma(\ell) < \sigma(k) = \sigma'(\ell')$$

Let  $t = f(t_1, \dots, t_n) \in \text{dom}(\iota(q))$  be an input tree. For that we obtain

$$\begin{aligned} \mu_0 &:= u_0 \cdot \llbracket q_1 \rrbracket(t_{\sigma(1)}) \cdot \dots \cdot u_{k-1} & \mu'_0 &:= u'_0 \cdot \llbracket q'_1 \rrbracket(t_{\sigma'(1)}) \cdot \dots \cdot u'_{k-1} \\ \mu_1 &:= u_k \cdot \llbracket q_{k+1} \rrbracket(t_{\sigma(k+1)}) \cdot \dots \cdot u_{\ell-1} & \mu'_1 &:= u'_k \cdot \llbracket q'_{k+1} \rrbracket(t_{\sigma'(k+1)}) \cdot \dots \cdot u'_{\ell'-1} \\ \mu_2 &:= u_\ell \cdot \llbracket q_\ell \rrbracket(t_{\sigma(\ell)}) \cdot \dots \cdot u_n & \mu'_2 &:= u'_{\ell'} \cdot \llbracket q'_{\ell'} \rrbracket(t_{\sigma'(\ell')}) \cdot \dots \cdot u'_n \end{aligned}$$

Then for all input trees  $t' \in \text{dom}(\iota(q_k))$ ,  $t'' \in \mathcal{L}(\text{dom}(\iota(q'_k)))$ ,

$$\alpha \cdot \mu_0 \cdot \llbracket q_k \rrbracket(t') \cdot \mu_1 \cdot \llbracket q_\ell \rrbracket(t'') \cdot \mu_2 \cdot \beta =_{\mathcal{F}_A} \alpha' \cdot \mu'_0 \cdot \llbracket q'_k \rrbracket(t'') \cdot \mu'_1 \cdot \llbracket q'_{\ell'} \rrbracket(t') \cdot \mu'_2 \cdot \beta'$$

Let  $\gamma' = \mu_0^- \alpha^- \alpha' \mu'_0$ . Then

$$\llbracket q_k \rrbracket(t') \cdot \mu_1 \cdot \llbracket q_\ell \rrbracket(t'') \cdot \mu_2 \cdot \beta =_{\mathcal{F}_A} \gamma' \cdot \llbracket q'_k \rrbracket(t'') \cdot \mu'_1 \cdot \llbracket q'_{\ell'} \rrbracket(t') \cdot \mu'_2 \cdot \beta'$$

By Lemma 2, we obtain that for all  $w_1, w_2 \in \mathcal{L}(q_k)$  and  $v_1, v_2 \in \mathcal{L}(q_\ell)$ ,  $w_2^- \cdot w_1 \in \mu_1 \cdot \langle p \rangle \cdot \mu_1^-$  and  $v_1 \cdot v_2^- \in \langle p \rangle$  for some primitive  $p$ .

If  $\ell = k + 1$ , i.e., there is no further state between  $q_k(x_{\sigma(k)})$  and  $q_\ell(x_{\sigma(\ell)})$ , then  $\mu_1 =_{\mathcal{F}_A} u_k$ ,  $\mathcal{L}(q_k) \subseteq w \cdot u_k \cdot \langle p \rangle \cdot u_k^-$  and  $\mathcal{L}(q_\ell) \subseteq \langle p \rangle \cdot v$  for some fixed  $w \in \mathcal{L}(q_k)$  and  $v \in \mathcal{L}(q_\ell)$ . As  $\sigma(k) > \sigma'(k) = \sigma(\ell)$ , this contradicts  $M$  being ordered.

For the case that there is at least one occurrence of a state between  $q_k(x_{\sigma(k)})$  and  $q_\ell(x_{\sigma(\ell)})$ , we show that for all  $\alpha_1, \alpha_2 \in u_k \cdot \mathcal{L}(q_{k+1}) \cdot \dots \cdot u_{\ell-1} =_{\mathcal{F}_A} \hat{L}$ ,



$\alpha_1^- \alpha_2 \in \langle p \rangle$  holds. We fix  $w_1, w_2 \in \mathcal{L}(q_k)$  and  $v_1, v_2 \in \mathcal{L}(q_\ell)$  with  $w_1 \neq w_2$  and  $v_1 \neq v_2$ . For every  $\alpha \in \hat{L}$ , we find by Lemma 2, primitive  $p_\alpha$  and exponent  $r_\alpha \in \mathbb{Z}$  such that  $v_1 \cdot v_2^- =_{\mathcal{F}_A} p_\alpha^{r_\alpha}$  holds. Since  $p_\alpha$  is primitive, this means that  $p_\alpha =_{\mathcal{F}_A} p$  or  $p_\alpha =_{\mathcal{F}_A} p^-$ . Furthermore, there must be some exponent  $r'_\alpha$  such that  $w_1^- \cdot w_2 =_{\mathcal{F}_A} \alpha \cdot p^{r'_\alpha} \cdot \alpha^-$ . For  $\alpha_1, \alpha_2 \in \hat{L}$ , we therefore have that

$$p^{r_{\alpha_1}} =_{\mathcal{F}_A} (\alpha_1^- \cdot \alpha_2) \cdot p^{r_{\alpha_2}} \cdot (\alpha_1^- \cdot \alpha_2)^-$$

Therefore by Lemma 1,  $\alpha_1^- \cdot \alpha_2 \in \langle p \rangle$ . Let us fix some  $w_k \in \mathcal{L}(q_k)$ ,  $\alpha \in \hat{L} =_{\mathcal{F}_A} u_k \cdot \mathcal{L}(q_{k+1}) \cdot \dots \cdot u_{\ell-1}$ , and  $w_\ell \in \mathcal{L}(q_\ell)$ . Then  $\mathcal{L}(q_k) \subseteq w_k \cdot \alpha \cdot \langle p \rangle \cdot \alpha^-$ ,  $\hat{L} \subseteq \alpha \cdot \langle p \rangle$  and  $\mathcal{L}(q_\ell) \subseteq \langle p \rangle \cdot w_\ell$ . Therefore,

$$\mathcal{L}(q_k) \cdot u_k \cdot \dots \cdot \mathcal{L}(q_\ell) \subseteq w_k \cdot \alpha \cdot \langle p \rangle \cdot \alpha^- \cdot \alpha \cdot \langle p \rangle \cdot \langle p \rangle \cdot w_\ell =_{\mathcal{F}_A} w_k \cdot \alpha \cdot \langle p \rangle \cdot w_\ell$$

As  $\sigma(k) > \sigma'(k) = \sigma(\ell)$ , this again contradicts  $M$  being ordered.

It remains to show that every  $\text{LT}_A$  can be ordered in polynomial time. For that, we rely on the following characterization.

**Lemma 6.** *Assume that  $L_1, \dots, L_n$  are neither empty nor singleton subsets of  $\mathcal{F}_A$  and  $u_1, \dots, u_{n-1} \in \mathcal{F}_A$ . Then there are  $v_1, \dots, v_n \in \mathcal{F}_A$  such that*

$$L_1 \cdot u_1 \cdot \dots \cdot L_{n-1} \cdot u_{n-1} \cdot L_n \subseteq v \cdot \langle p \rangle \quad (5)$$

holds if and only if for  $i = 1, \dots, n$ ,  $L_i \subseteq v_i \cdot \langle p_i \rangle$  with

$$\begin{aligned} p_n &=_{\mathcal{F}_A} p \\ p_i &=_{\mathcal{F}_A} (u_i \cdot v_{i+1}) \cdot p_{i+1} \cdot (u_i \cdot v_{i+1})^- \quad \text{for } i < n \end{aligned}$$

and

$$v^- \cdot v_1 \cdot u_1 \cdot \dots \cdot v_{n-1} \cdot u_{n-1} \cdot v_n \in \langle p \rangle \quad (6)$$

*Proof.* Let  $s_1 = \varepsilon$ . For  $i = 2, \dots, n$  we fix some word  $s_i \in L_1 \cdot u_1 \cdot L_2 \cdot \dots \cdot L_{i-1} \cdot u_{i-1}$ . Likewise, let  $t_n = \varepsilon$  and for  $i = 1, \dots, n-1$  fix some word  $t_i \in u_i \cdot L_{i+1} \cdot \dots \cdot L_n$ , and define  $v_i =_{\mathcal{F}_A} s_i^- \cdot v \cdot t_i^-$ .

First assume that the inclusion (5) holds. Let  $p'_i =_{\mathcal{F}_A} t_i \cdot p \cdot t_i^-$ . Then for all  $i$ ,  $s_i \cdot L_i \cdot t_i \subseteq v \cdot \langle p \rangle$ , and therefore

$$L_i \subseteq s_i^- \cdot v \cdot \langle p \rangle \cdot t_i^- =_{\mathcal{F}_A} s_i^- \cdot v \cdot t_i^- \cdot t_i \cdot \langle p \rangle \cdot t_i^- =_{\mathcal{F}_A} v_i \langle p'_i \rangle$$

We claim that  $p'_i = p_i$  for all  $i = 1, \dots, n$ . We proceed by induction on  $n-i$ . As  $t_n = \varepsilon$ , we have that  $p'_n = p = p_n$ . For  $i < n$ , we can rewrite  $t_i =_{\mathcal{F}_A} u_i \cdot w_{i+1} \cdot t_{i+1}$  where  $w_{i+1} \in L_{i+1}$  and thus is of the form  $v_{i+1} \cdot p_{i+1}^{k_{i+1}}$  for some exponent  $k_{i+1}$ .

$$\begin{aligned} p'_i &=_{\mathcal{F}_A} t_i \cdot p \cdot t_i^- \\ &=_{\mathcal{F}_A} u_i \cdot w_{i+1} \cdot t_{i+1} \cdot p \cdot t_{i+1}^- \cdot w_{i+1}^- \cdot u_i^- \\ &=_{\mathcal{F}_A} u_i \cdot w_{i+1} \cdot p_{i+1} \cdot w_{i+1}^- \cdot u_i^- && \text{by I.H.} \\ &=_{\mathcal{F}_A} u_i \cdot v_{i+1} \cdot p_{i+1} \cdot v_{i+1}^- \cdot u_i^- \\ &=_{\mathcal{F}_A} p_i \end{aligned}$$

It remains to prove the inclusion (6). Since  $w_i \in L_i$ , we have by (5) that  $v^- w_1 \cdot u_1 \cdot \dots \cdot w_n \cdot u_n \in \langle p \rangle$  holds. Now we calculate:

$$\begin{aligned} v^- \cdot w_1 \cdot u_1 \cdot \dots \cdot u_{n-1} \cdot w_n &=_{\mathcal{F}_A} v^- \cdot v_1 \cdot p_1^{k_1} \cdot u_1 \cdot \dots \cdot u_{n-1} \cdot v_n \cdot p_n^{k_n} \\ &=_{\mathcal{F}_A} v^- \cdot v_1 \cdot u_1 \cdot v_2 \cdot p_2^{k_1+k_2} \cdot u_2 \cdot \dots \cdot u_{n-1} \cdot v_n \cdot p_n^{k_n} \\ &\dots \\ &=_{\mathcal{F}_A} v^- \cdot v_1 \cdot u_1 \cdot \dots \cdot v_{n-1} \cdot u_{n-1} \cdot v_n \cdot p_n^k \end{aligned}$$

where  $k = k_1 + \dots + k_n$ . Since  $p_n = p$ , the claim follows.

The other direction of the claim of the lemma follows directly:

$$\begin{aligned} L_1 u_1 \dots L_{n-1} u_{n-1} L_n &\subseteq v_1 \cdot \langle p_1 \rangle \cdot u_1 \cdot \dots \cdot v_{n-1} \cdot \langle p_{n-1} \rangle \cdot u_{n-1} \cdot v_n \cdot \langle p_n \rangle \\ &=_{\mathcal{F}_A} v_1 \cdot u_1 \cdot v_2 \cdot \langle p_2 \rangle \cdot \langle p_2 \rangle \cdot u_2 \cdot \dots \cdot v_{n-1} \cdot \langle p_{n-1} \rangle \cdot u_{n-1} \cdot v_n \cdot \langle p_n \rangle \\ &=_{\mathcal{F}_A} v_1 \cdot u_1 \cdot v_2 \cdot \langle p_2 \rangle \cdot u_2 \cdot \dots \cdot v_{n-1} \cdot \langle p_{n-1} \rangle \cdot u_{n-1} \cdot v_n \cdot \langle p_n \rangle \\ &\dots \\ &=_{\mathcal{F}_A} v_1 \cdot u_1 \cdot v_2 \cdot \dots \cdot u_{n-1} \cdot v_n \cdot \langle p_n \rangle \\ &=_{\mathcal{F}_A} v_1 \cdot u_1 \cdot v_2 \cdot \dots \cdot u_{n-1} \cdot v_n \cdot \langle p \rangle \\ &\subseteq v \cdot \langle p \rangle \end{aligned}$$

where the last inclusion follows from (6).

Let us call a non-empty, non-singleton language  $L \subseteq \mathcal{F}_A$  *periodic*, if  $L \subseteq v \cdot \langle p \rangle$  for some  $v, p \in \mathcal{F}_A$ . Lemma 6 then implies that if a concatenation of languages and elements from  $\mathcal{F}_A$  is periodic, then so must be all non-singleton component languages. In fact, the languages in the composition can then be arbitrarily permuted.

**Corollary 2.** *Assume for non-empty, nonsingleton languages  $L_1, \dots, L_n \subseteq \mathcal{F}_A$  and  $u_1, \dots, u_{n-1} \in \mathcal{F}_A$  that property (5) holds. Then for every permutation  $\pi$ , there are elements  $u_{\pi,0}, \dots, u_{\pi,n} \in \mathcal{F}_A$  such that*

$$L_1 \cdot u_1 \cdot \dots \cdot L_{n-1} \cdot u_{n-1} \cdot L_n = u_{\pi,0} \cdot L_{\pi(1)} \cdot u_{\pi,1} \cdot \dots \cdot u_{\pi,n-1} \cdot L_{\pi(n)} \cdot u_{\pi,n}$$

*Example 3.* We reconsider  $\text{LT}_A M'$  and  $\text{DTA } B$  from Example 2. We observe that  $\mathcal{L}(q_1) \subseteq a \cdot \langle ba \rangle$ ,  $\mathcal{L}(q_2) \subseteq \langle ab \rangle$ , and thus  $\mathcal{L}(q_0) = \mathcal{L}(q_1) \cdot b \cdot \mathcal{L}(q_2) \subseteq \langle ab \rangle$ . Accordingly, the rule for state  $q_0$  and input symbol  $f$  is not ordered. Following the notation of Corollary 2, we find  $v_1 = a$ ,  $u_1 = b$  and  $v_2 = \varepsilon$ , and the rule for  $q_0$  and  $f$  can be reordered to

$$q_0(f(x_1, x_2)) \rightarrow ab \cdot q_2(x_1) \cdot b^- a^- \cdot q_1(x_2)b$$

This example shows major improvements compared to the construction in [2]. Since we have inverses at hand, only *local* changes must be applied to the sub-sequence  $q_1(x_2) \cdot b \cdot q_2(x_1)$ . In contrast to the construction in [2], neither auxiliary states nor further changes to the rules of  $q_1$  and  $q_2$  are required.

By Corollary 2, the order of occurrences of terms  $q_k(x_{\sigma(k)})$  can be permuted in every sub-sequence  $q_i(x_{\sigma(i)}) \cdot u_i \cdot \dots \cdot u_{j-1} q_j(x_{\sigma(j)})$  where  $\mathcal{L}(q_i) \cdot u_i \cdot \dots \cdot u_{j-1} \cdot \mathcal{L}(q_j) \in u \cdot \langle p \rangle$  is periodic, to satisfy the requirements of an ordered  $\text{LT}_A$ . A sufficient

condition for that is, according to Lemma 6, that  $\mathcal{L}(q_k)$  is periodic for each  $q_k$  occurring in that sub-sequence. Therefore we will determine the subset of *all* states  $q$  where  $\mathcal{L}(q)$  is periodic, and if so elements  $v_q, p_q$  such that  $\mathcal{L}(q) \subseteq v_q \cdot \langle p_q \rangle$ . In order to do so we compute an *abstraction* of the sets  $\mathcal{L}(q)$  by means of a complete lattice which both reports constant values and also captures periodicity.

Let  $\mathcal{D} = 2^{\mathcal{F}_A}$  denote the complete lattice of subsets of the free group  $\mathcal{F}_A$ . We define a *projection*  $\alpha : \mathcal{D} \rightarrow \mathcal{D}$  by  $\alpha(\emptyset) = \emptyset$ ,  $\alpha(\{g\}) = \{g\}$ , and for languages  $L$  with at least two elements,

$$\alpha(L) = \begin{cases} g\langle p \rangle & \text{if } L \subseteq g\langle p \rangle \text{ and } p \text{ is primitive} \\ \mathcal{F}_A & \text{otherwise} \end{cases}$$

The projection  $\alpha$  is a *closure* operator, i.e., is a monotonic function with  $L \subseteq \alpha(L)$ , and  $\alpha(\alpha(L)) = \alpha(L)$ . The image of  $\alpha$  can be considered as an *abstract* complete lattice  $\mathcal{D}^\sharp$ , partially ordered by subset inclusion. Thereby, the abstraction  $\alpha$  commutes with least upper bounds as well as with the group operation. For that, we define *abstract* versions  $\sqcup, \star : (\mathcal{D}^\sharp)^2 \rightarrow \mathcal{D}^\sharp$  of set union and the group operation by

$$A_1 \sqcup A_2 = \alpha(A_1 \cup A_2) \quad A_1 \star A_2 = \alpha(A_1 \cdot A_2)$$

In fact, “ $\sqcup$ ” is the least upper bound operation for  $\mathcal{D}^\sharp$ . The two abstract operators can also be more explicitly defined by:

$$\begin{aligned} \emptyset \sqcup L &= L \sqcup \emptyset &= L \\ \mathcal{F}_A \sqcup L &= L \sqcup \mathcal{F}_A &= \mathcal{F}_A \\ \{g_1\} \sqcup \{g_2\} &= \begin{cases} \{g_1\} & \text{if } g_1 = g_2 \\ g_1 \cdot \langle p \rangle & \text{if } g_1 \neq g_2, p \text{ primitive root of } g_1^{-1} \cdot g_2 \end{cases} \\ \{g_1\} \sqcup g_2 \cdot \langle p \rangle &= g_2 \cdot \langle p \rangle \sqcup \{g_1\} = \begin{cases} g_2 \cdot \langle p \rangle & \text{if } g_1 \in g_2 \cdot \langle p \rangle \\ \mathcal{F}_A & \text{otherwise} \end{cases} \\ g_1 \cdot \langle p_1 \rangle \sqcup g_2 \cdot \langle p_2 \rangle &= \begin{cases} g_1 \cdot \langle p_1 \rangle & \text{if } p_2 \in \langle p_1 \rangle \text{ and } g_2^{-1} \cdot g_1 \in \langle p_1 \rangle \\ \mathcal{F}_A & \text{otherwise} \end{cases} \end{aligned}$$

$$\begin{aligned} \emptyset \star L &= L \star \emptyset = \emptyset \\ \mathcal{F}_A \star L &= L \star \mathcal{F}_A = \mathcal{F}_A \quad \text{for } L \neq \emptyset \\ \{g_1\} \star \{g_2\} &= \{g_1 \cdot g_2\} \\ \{g_1\} \star g_2 \cdot \langle p \rangle &= (g_1 \cdot g_2) \cdot \langle p \rangle \\ g_1 \cdot \langle p \rangle \star \{g_2\} &= (g_1 \cdot g_2) \cdot \langle g_2^{-1} \cdot p \cdot g_2 \rangle \\ g_1 \cdot \langle p_1 \rangle \star g_2 \cdot \langle p_2 \rangle &= \begin{cases} (g_1 \cdot g_2) \cdot \langle p_2 \rangle & \text{if } g_2^{-1} \cdot p_1 \cdot g_2 \in \langle p_2 \rangle \\ \mathcal{F}_A & \text{otherwise} \end{cases} \end{aligned}$$

**Lemma 7.** *For all subsets  $L_1, L_2 \subseteq \mathcal{F}_A$ ,  $\alpha(L_1 \cup L_2) = \alpha(L_1) \sqcup \alpha(L_2)$  and  $\alpha(L_1 \cdot L_2) = \alpha(L_1) \star \alpha(L_2)$ .*

We conclude that  $\alpha$  in fact represents a *precise* abstract interpretation in the sense of [9]. Accordingly, we obtain:

**Lemma 8.** *For every  $\text{LT}_A M$  and DTA  $B$  with compatible map  $\iota$ , the sets  $\alpha(\mathcal{L}(q))$ ,  $q$  state of  $M$ , can be computed in polynomial time.*

*Proof.* We introduce one unknown  $X_q$  for every state  $q$  of  $M$ , and one constraint for each rule of  $M$  of the form (3) where  $\delta(\iota(q), f)$  is defined in  $B$ . This constraint is given by:

$$X_q \sqsupseteq u_0 \star X_{q_1} \star \dots \star u_{n-1} \star X_{q_n} \star u_n \tag{7}$$

As the right-hand sides of the constraints (7) all represent monotonic functions, the given system of constraints has a *least* solution. In order to obtain this solution, we consider for each state  $q$  of  $M$ , the sequence  $X_q^{(i)}, i \geq 0$  of values in  $\mathcal{D}^\sharp$  where  $X_q^{(0)} = \emptyset$ , and for  $i > 0$ , we set  $X_q^{(i)}$  as the least upper bound of the values obtained from the constraints with left-hand side  $X_q$  of the form (7) by replacing the unknowns  $X_{q_j}$  on the right-hand side with the values  $X_{q_j}^{(i-1)}$ . By induction on  $i \geq 0$ , we verify that for all states  $q$  of  $M$ ,

$$X_q^{(i)} = \alpha(\mathcal{L}^{(i)}(q))$$

holds. Note that the induction step thereby, relies on Lemma 7.

As each strictly increasing chain of elements in  $\mathcal{D}^\sharp$  consists of at most four elements, we have that the least solution of the constraint system is attained after at most  $3 \cdot N$  iterations, if  $N$  is the number of states of  $M$ , i.e., for each state  $q$  of  $M$ ,  $X_q^{(3N)} = X_q^{(i)}$  for all  $i \geq 3N$ . The elements of  $\mathcal{D}^\sharp$  can be represented by SLPs where the operations  $\star$  and  $\sqcup$  run in polynomial time, cf. Lemma 3. Since each iteration requires only a polynomial number of operations  $\star$  and  $\sqcup$ , the statement of the lemma follows.

We now exploit the information provided by the  $\alpha(\mathcal{L}(q))$  to remove trivial states as well as order subsequences of right-hand sides which are periodic.

**Theorem 1.** *Let  $B$  be a DTA such that  $\mathcal{L}(B) \neq \emptyset$ . For every  $\text{LT}_A M$  with compatible map  $\iota$ , an  $\text{LT}_A M'$  with compatible map  $\iota'$  can be constructed in polynomial time such that*

1.  $M$  and  $M'$  are equivalent relative to  $B$ ;
2.  $M'$  has no trivial states;
3.  $M'$  is ordered.

*Proof.* By Lemma 8, we can, in polynomial time, determine for every state  $q$  of  $M$ , the value  $\alpha(\mathcal{L}(q))$ . We use this information to remove from  $M$  all trivial states. W.l.o.g., assume that the axiom of  $M$  is given by  $u_0 \cdot q_0(x_0) \cdot u_1$ . If the state  $q_0$  occurring in the axiom of  $M$  is trivial with  $\mathcal{L}(q_0) = \{v\}$ , then  $M_1$  has no states or rules, but the axiom  $u_0 \cdot v \cdot u_1$ .

Therefore now assume that  $q_0$  is non-trivial. We then construct an  $\text{LT}_A M_1$  whose set of states  $Q_1$  consists of all *non-trivial* states  $q$  of  $M$  where the compatible map  $\iota_1$  of  $M_1$  is obtained from  $\iota$  by restriction to  $Q_1$ . Since  $\mathcal{L}(M) \neq \emptyset$ ,

the state of  $M$  occurring in the axiom is non-trivial. Accordingly, the axiom of  $M$  is also used as axiom for  $M_1$ . Consider a non-trivial state  $q$  of  $M$  and  $f \in \Sigma$ . If  $\delta(\iota(q), f)$  is not defined  $M_1$  has the rule  $q(f(x_1, \dots, x_m)) \rightarrow \perp$ . Assume that  $\delta(\iota(q), f)$  is defined and  $M$  has a rule of the form (3). Then  $M_1$  has the rule

$$q(f(x_1, \dots, x_m)) \rightarrow u_0 \cdot g_1 \cdot \dots \cdot u_{n-1} \cdot g_n \cdot u_n$$

where for  $i = 1, \dots, n$ ,  $g_i$  equals  $q_i(x_{\sigma(i)})$  if  $q_i$  is non-trivial, and equals the single word in  $\mathcal{L}(q_i)$  otherwise. Obviously,  $M$  and  $M_1$  are equivalent relative to  $B$  where  $M_1$  now has no trivial states, while for every non-trivial state  $q$ , the semantics of  $q$  in  $M$  and  $M_1$  are the same relative to  $B$ . Our goal now is to equivalently rewrite the right-hand side of each rule of  $M_1$  so that the result is ordered. For each state  $q$  of the  $\text{LT}_A$  we determine whether there are  $v, p \in \mathbb{B}^*$  such that  $\mathcal{L}(q) \subseteq v\langle p \rangle$ , cf. Lemma 8. So consider a rule of  $M_1$  of the form (3). By means of the values  $\alpha(\mathcal{L}(q_i))$ ,  $i = 1, \dots, n$ , together with the abstract operation “ $\star$ ”, we can determine maximal intervals  $[i, j]$  such that  $\mathcal{L}(q_i) \cdot u_i \cdot \dots \cdot u_{j-1} \cdot \mathcal{L}(q_j)$  is periodic, i.e.,  $\alpha(\mathcal{L}(q_i)) \star u_i \cdot \dots \star u_{j-1} \star \alpha(\mathcal{L}(q_j)) \subseteq v \cdot \langle p \rangle$  for some  $v, p \in \mathcal{F}_A$ . We remark that these maximal intervals are necessarily disjoint. By Corollary 2, for every permutation  $\pi : [i, j] \rightarrow [i, j]$ , elements  $u', u'_i, \dots, u'_j, u'' \in \mathcal{F}_A$  can be found so that  $q_i(x_{\sigma(i)}) \cdot u_i \cdot \dots \cdot u_{j-1} \cdot q_j(x_{\sigma(j)})$  is equivalent to  $u' \cdot q_{\pi(i)}(x_{\sigma(\pi(i))}) \cdot u'_i \cdot \dots \cdot u'_{j-1} \cdot q_{\pi(j)}(x_{\sigma(\pi(j))}) \cdot u''$ .

In particular, this is true for the permutation  $\pi$  with  $\sigma(\pi(i)) < \dots < \sigma(\pi(j))$ . Assuming that all group elements are represented as SLPs, the overall construction runs in polynomial time.

In summary, we arrive at the main theorem of this paper.

**Theorem 2.** *The equivalence of  $\text{LT}_A$ s relative to some DTA  $B$  can be decided in polynomial time.*

*Proof.* Assume we are given  $\text{LT}_A$ s  $M, M'$  with compatible maps (relative to  $B$ ). By Theorem 1, we may w.l.o.g. assume that  $M$  and  $M'$  both have no trivial states and are ordered. It can be checked in polynomial time whether or not  $M$  and  $M'$  are same-ordered. If they are not, then by Lemma 5, they cannot be equivalent relative to  $B$ . Therefore now assume that  $M$  and  $M'$  are same-ordered. Then their equivalence relative to  $B$  is decidable in polynomial time by Corollary 1. Altogether we thus obtain a polynomial decision procedure for equivalence of  $\text{LT}_A$ s relative to some DTA  $B$ .

## 4 Conclusion

We have shown that equivalence of  $\text{LT}_A$ s relative to a given DTA  $B$  can be decided in polynomial time. For that, we considered *total* transducers only, but defined the domain of allowed input trees separately by means of the DTA. This does not impose any restriction of generality, since any (possibly partial) linear deterministic top-down tree transducer can be translated in polynomial time to

a corresponding *total*  $LT_A$  together with a corresponding DTA (see, e.g., [11]). The required constructions for  $LT_A$ s which we have presented here, turn out to be more general than the constructions provided in [2] since they apply to transducers which may not only output symbols  $a \in A$ , but also their inverses  $a^-$ . At the same time, they are *simpler* and easier to be proven correct due to the combinatorial and algebraic properties provided by the free group.

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## References

1. Boiret, A.: Normal form on linear tree-to-word transducers. In: Dediu, A.-H., Janoušek, J., Martín-Vide, C., Truthe, B. (eds.) LATA 2016. LNCS, vol. 9618, pp. 439–451. Springer, Cham (2016). [https://doi.org/10.1007/978-3-319-30000-9\\_34](https://doi.org/10.1007/978-3-319-30000-9_34)
2. Boiret, A., Palenta, R.: Deciding equivalence of linear tree-to-word transducers in polynomial time. In: Brlek, S., Reutenauer, C. (eds.) DLT 2016. LNCS, vol. 9840, pp. 355–367. Springer, Heidelberg (2016). [https://doi.org/10.1007/978-3-662-53132-7\\_29](https://doi.org/10.1007/978-3-662-53132-7_29)
3. Laurence, G., Lemay, A., Niehren, J., Staworko, S., Tommasi, M.: Normalization of sequential top-down tree-to-word transducers. In: Dediu, A.-H., Inenaga, S., Martín-Vide, C. (eds.) LATA 2011. LNCS, vol. 6638, pp. 354–365. Springer, Heidelberg (2011). [https://doi.org/10.1007/978-3-642-21254-3\\_28](https://doi.org/10.1007/978-3-642-21254-3_28)
4. Laurence, G., Lemay, A., Niehren, J., Staworko, S., Tommasi, M.: Learning sequential tree-to-word transducers. In: Dediu, A.-H., Martín-Vide, C., Sierra-Rodríguez, J.-L., Truthe, B. (eds.) LATA 2014. LNCS, vol. 8370, pp. 490–502. Springer, Cham (2014). [https://doi.org/10.1007/978-3-319-04921-2\\_40](https://doi.org/10.1007/978-3-319-04921-2_40)
5. Löbel, R., Luttenberger, M., Seidl, H.: Equivalence of linear tree transducers with output in the free group. CoRR abs/2001.03480 (2020). <http://arxiv.org/abs/2001.03480>
6. Lohrey, M.: Word problems on compressed words. In: Díaz, J., Karhumäki, J., Lepistö, A., Sannella, D. (eds.) ICALP 2004. LNCS, vol. 3142, pp. 906–918. Springer, Heidelberg (2004). [https://doi.org/10.1007/978-3-540-27836-8\\_76](https://doi.org/10.1007/978-3-540-27836-8_76)
7. Lohrey, M.: The Compressed Word Problem for Groups. SM. Springer, New York (2014). <https://doi.org/10.1007/978-1-4939-0748-9>
8. Lyndon, R.C., Schupp, P.E.: Combinatorial Group Theory. CM, vol. 89. Springer, Heidelberg (2001). <https://doi.org/10.1007/978-3-642-61896-3>
9. Müller-Olm, M.: Variations on Constants. LNCS, vol. 3800. Springer, Heidelberg (2006). <https://doi.org/10.1007/11871743>
10. Plandowski, W.: Testing equivalence of morphisms on context-free languages. In: van Leeuwen, J. (ed.) ESA 1994. LNCS, vol. 855, pp. 460–470. Springer, Heidelberg (1994). <https://doi.org/10.1007/BFb0049431>
11. Seidl, H., Maneth, S., Kemper, G.: Equivalence of deterministic top-down tree-to-string transducers is decidable. J. ACM **65**(4), 21:1–21:30 (2018). <https://doi.org/10.1145/3182653>
12. Sénizergues, G.:  $T(A) = T(B)$ ? In: Wiedermann, J., van Emde Boas, P., Nielsen, M. (eds.) ICALP 1999. LNCS, vol. 1644, pp. 665–675. Springer, Heidelberg (1999). [https://doi.org/10.1007/3-540-48523-6\\_63](https://doi.org/10.1007/3-540-48523-6_63)

13. Staworko, S., Laurence, G., Lemay, A., Niehren, J.: Equivalence of deterministic nested word to word transducers. In: Kutylowski, M., Charatonik, W., Gębala, M. (eds.) FCT 2009. LNCS, vol. 5699, pp. 310–322. Springer, Heidelberg (2009). [https://doi.org/10.1007/978-3-642-03409-1\\_28](https://doi.org/10.1007/978-3-642-03409-1_28)
14. Tomita, E., Seino, K.: A direct branching algorithm for checking the equivalence of two deterministic pushdown transducers, one of which is real-time strict. *Theory Comput. Sci.* **64**(1), 39–53 (1989). [https://doi.org/10.1016/0304-3975\(89\)90096-0](https://doi.org/10.1016/0304-3975(89)90096-0)