



Avoidability of Additive Cubes over Alphabets of Four Numbers

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Abstract. Let $\mathcal{A} \subset \mathbb{N}$ be a set of size 4 such that \mathcal{A} cannot be obtained by applying the same affine function to all of the elements of $\{0, 1, 2, 3\}$. We show that there is an infinite sequence of elements of \mathcal{A} that contains no three consecutive blocks of same size and same sum (additive cubes). Moreover, it is possible to replace \mathbb{N} by \mathbb{C} in the statement.

Keywords: Abelian/additive equivalence · Abelian/additive powers · Combinatorics on words

1 Introduction

Let $k \geq 2$ be an integer and $(G, +)$ a semigroup. An *additive k th power* is a non-empty word $w_1 \cdots w_k$ over $\mathcal{A} \subseteq G$ such for every $i \in \{2, \dots, k\}$, $|w_i| = |w_1|$ and $\sum w_i = \sum w_1$ (where $\sum v$ denotes the sum of the letters in v seen as numbers). It is a longstanding question whether there exists an infinite word w over a finite subset of \mathbb{N} that avoids additive squares (additive 2nd powers) [3, 4, 6]. One motivation for studying this problem is that a positive answer to this question would imply that additive squares are avoidable over any semigroup that contains some finitely generated infinite semigroup [6] (an application of van der Waerden's theorem shows that additive powers are not avoidable over any other semigroup [4]). Cassaigne et al. [1] showed that there exists an infinite word over the finite alphabet $\{0, 1, 3, 4\} \subset \mathbb{Z}$ without additive cubes (additive 3rd powers). Rao [7] used this result to show that there exist infinite words avoiding additive cubes over any alphabet $\{0, i, j\} \subset \mathbb{N}^3$ with i and j coprime, $i < j$ and $6 \leq j \leq 9$ (and he conjectured that the second condition can be replaced by $6 \leq j$). This motivates the following more general problem:

Problem 1. Characterize the finite subsets of \mathbb{N} over which additive cubes are avoidable.

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It seems restrictive to use \mathbb{N} instead of \mathbb{R} (or \mathbb{C}), but solving Problem 1 for alphabets of the form $\{0, a_1, \dots, a_m\} \in \mathbb{N}$ with the a_i 's being coprime completely solves the problem for any finite alphabet over \mathbb{C} (if the a_i 's are given in increasing order one can additionally assume a_1 be smaller than $a_m - a_{m-1}$). For the sake of completeness, we give a short proof of this fact in Sect. 2.

If Rao's conjecture were true then the only remaining 3-letter alphabets over \mathbb{C} to characterize would be $\{0, 1, 2\}$, $\{0, 1, 3\}$, $\{0, 1, 4\}$ and $\{0, 2, 5\}$ (see [9, Section 2.2.2] for details). However, this conjecture is known to be true for only finitely many such alphabets (up to a trivial equivalence relation defined in Sect. 2.1). In the present paper we propose a twist on previously used ideas to show our main theorem (see Corollary 1).

Main Theorem. *Let $\mathcal{A} \subset \mathbb{C}$ be an alphabet with $|\mathcal{A}| \geq 4$. If \mathcal{A} is not equivalent to $\{0, 1, 2, 3\}$ then additive cubes are avoidable over \mathcal{A} .*

This also implies that additive cubes are avoidable over any alphabet of complex numbers of size at least 5. Rao used the fact that additive cubes are avoidable over $\{0, 1, 3, 4\}$ to show that they are avoidable over some 3-letter alphabets [7, Section 3.2], so our result might also be of importance for tackling Problem 1 for alphabets of size 3.

The present paper is organized as follows. We first recall some notation and we define the equivalence between two alphabets. Equipped with this equivalence relation we explain why it is enough to study alphabets of integers or alphabets of the form $\{0, 1, a_2, a_3, \dots, a_m\}$ with $m \in \mathbb{N}$ and $a_2, \dots, a_m \in \mathbb{Q}$. Then we introduce the word $\mathbf{W}_{a,b,c,d}$, based on the construction of [1], and we show that for all but finitely (up to our equivalence relation) many values of a, b, c , and d , the word $\mathbf{W}_{a,b,c,d}$ avoids additive cubes. Finally, using the literature for the remaining alphabets, we conclude that additive cubes are avoidable over all the remaining alphabets of size 4, with the sole exception of $\{0, 1, 2, 3\}$. We leave the case of $\{0, 1, 2, 3\}$ open, and comment on our calculations regarding this case in the last section.

2 Preliminaries

We use the standard notation introduced in Chapter 1 of [5]. In the rest of the present article all of our alphabets are finite sets of complex numbers. For the rest of this section, let $\mathcal{A} \subset \mathbb{C}$ be such an alphabet. We denote by ε the empty word and by $|\mathcal{A}|$ the cardinality of the alphabet \mathcal{A} . Given a word $w \in \mathcal{A}^*$, we denote by $|w|$ the length of w and by $|w|_\alpha$ the number of occurrences of the letter $\alpha \in \mathcal{A}$ in w . Two words u and v are *abelian equivalent*, denoted by $u \simeq_{ab} v$ if u and v are permutations of each other. They are *additively equivalent*, denoted by $u \simeq_{ad} v$, if $|u| = |v|$ and $\sum u = \sum v$, where $\sum v$ denotes the sum of the letters in v (this make sense since the letters are complex numbers). A word $uvw \in \mathcal{A}^*$ is an *abelian cube* (respectively, an *additive cube*) if $u \simeq_{ab} v \simeq_{ab} w$ (respectively, if $u \simeq_{ad} v \simeq_{ad} w$).

2.1 Alphabets in \mathbb{N}

For any function $h : \mathcal{A} \rightarrow \mathbb{C}$ and words w over $\mathcal{A} \subset \mathbb{C}$, the word $h(w)$ is obtained by replacing each letter of w by its image under h . We say that two alphabets $\mathcal{A}, \mathcal{A}' \subset \mathbb{C}$ of same size are *equivalent* if there is a function $h : \mathcal{A} \rightarrow \mathcal{A}'$ such that for all $u, v \in \mathcal{A}^*$,

$$u \simeq_{ad} v \iff h(u) \simeq_{ad} h(v).$$

Let us now show that for any alphabet of complex numbers, we either already know that additive cubes are avoidable or the alphabet is equivalent to an alphabet of integers. We start by giving sufficient conditions for two alphabets to be equivalent.

Lemma 1. *Let $u, v \in \mathcal{A}^*$ be two finite words, let $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$ and $f : \mathbb{C} \rightarrow \mathbb{C}$, $x \mapsto ax + b$. Then $u \simeq_{ad} v$ if and only if $f(u) \simeq_{ad} f(v)$.*

The proof is left to the reader. Recall that two complex numbers a and b are said to be rationally independent if $k_1a + k_2b = 0$ for $(k_1, k_2) \in \mathbb{Z}^2$ implies $k_1 = k_2 = 0$.

Lemma 2. *Let $\mathcal{A} \subset \mathbb{C}$.*

- (i) *If $|\mathcal{A}| \leq 2$ then additive cubes are not avoidable over \mathcal{A} .*
- (ii) *If $|\mathcal{A}| > 2$ and if there are $a, b, c \in \mathcal{A}$, such that $b - a$ and $c - a$ are rationally independent, then additive cubes are avoidable over \mathcal{A} .*
- (iii) *If $|\mathcal{A}| > 2$ and if for any pairwise different $a, b, c \in \mathcal{A}$, the differences $b - a$ and $c - a$ are rationally dependent, then there exists an alphabet $\mathcal{A}' = \{0, a_1, \dots, a_m\} \subset \mathbb{N}$ with $\gcd(a_1, \dots, a_m) = 1$ such that \mathcal{A} and \mathcal{A}' are equivalent.*

Proof.

(i) This statement follows from the fact that abelian cubes are not avoidable over two letters [2].

(ii) Since $b - a$ and $c - a$ are rationally independent, for any $k_1, k_2, k_3 \in \mathbb{Z}$, if $0k_1 + (b - a)k_2 + (c - a)k_3 = 0$ then $k_2 = k_3 = 0$. Thus for any words $u, v \in \{0, b - a, c - a\}^*$, if $\sum u = \sum v$ then u has the same number of occurrences of $b - a$ (resp., $c - a$) as v ; moreover, if $|u| = |v|$ then u and v also have the same number of occurrences of 0. Thus, for any word $u, v \in \{0, b - a, c - a\}^*$, if $u \simeq_{ad} v$ then $u \simeq_{ab} v$. From Lemma 1 (with $f : x \mapsto x + a$), for any $u, v \in \{a, b, c\}^*$, if $u \simeq_{ad} v$ then $u \simeq_{ab} v$. Since abelian cubes are avoidable over 3 letters [2], we deduce that additive cubes are avoidable over \mathcal{A} .

(iii) Let $\{b_1, \dots, b_m\} = \mathcal{A}$. For any i , $b_i - b_1$ and $b_2 - b_1$ are rationally dependent which implies $\frac{b_i - b_1}{b_2 - b_1} \in \mathbb{Q}$. Thus there exists a $q \in \mathbb{Z}$ such that for all i , $q \frac{b_i - b_1}{b_2 - b_1} \in \mathbb{Z}$ and $\gcd\left(q \frac{b_2 - b_1}{b_2 - b_1}, q \frac{b_3 - b_1}{b_2 - b_1}, \dots, q \frac{b_m - b_1}{b_2 - b_1}\right) = 1$. Let $s = \min_{1 \leq i \leq m} \left(q \frac{b_i - b_1}{b_2 - b_1}\right)$. Finally, we apply Lemma 1 with $f : x \mapsto q \frac{x - b_1}{b_2 - b_1} - s$ and we get that the alphabet $\{q \frac{b_1 - b_1}{b_2 - b_1} - s, q \frac{b_2 - b_1}{b_2 - b_1} - s, q \frac{b_3 - b_1}{b_2 - b_1} - s, \dots, q \frac{b_m - b_1}{b_2 - b_1} - s\}$ satisfies all the required conditions. This concludes the proof.

Thus solving Problem 1 for alphabets of the form $\{0, a_1, \dots, a_m\} \subset \mathbb{N}$ with coprime a_i 's completely solves the problem for any finite alphabet over \mathbb{C} . Notice that, in case (iii), one can add the condition that $a_1 < a_m - a_{m-1}$, (otherwise apply $f : x \mapsto a_m - x$ to this alphabet). One could also add that in the case $|\mathcal{A}| = 2$, one can avoid additive 4th powers (with an argument similar to (ii) and the fact that abelian 4th powers are avoidable over 2 letters [2]).

Remark 1. Every alphabet $\{a_0, a_1, \dots, a_m\} \subset \mathbb{N}$ is equivalent to the alphabet $\{0, 1, f(a_2), \dots, f(a_m)\} \subset \mathbb{Q}$, where $f : x \mapsto \frac{x-a_0}{a_1-a_0}$. Therefore, in Sects. 3 and 4, instead of considering alphabets of four integers we consider alphabets of the form $\{0, 1, c, d\} \subset \mathbb{Q}$.

3 The Infinite Word $\mathbf{W}_{a,b,c,d}$

Let $a, b, c, d \in \mathbb{R}$ and let $\varphi_{a,b,c,d} : \{a, b, c, d\}^* \rightarrow \{a, b, c, d\}^*$ be the following morphism:

$$\varphi_{a,b,c,d}(a) = ac \quad ; \quad \varphi_{a,b,c,d}(b) = dc \quad ; \quad \varphi_{a,b,c,d}(c) = b \quad ; \quad \varphi_{a,b,c,d}(d) = ab.$$

Let $\mathbf{W}_{a,b,c,d} := \lim_{n \rightarrow +\infty} \varphi_{a,b,c,d}^n(a)$ be the infinite fixed point of $\varphi_{a,b,c,d}$. Cassaigne et al. [1] showed in 2014 that $\mathbf{W}_{0,1,3,4}$ avoids additive cubes. In particular, this implies that $\mathbf{W}_{0,1,3,4}$ avoids abelian cubes. This property does not depend on the choice of a, b, c, d , therefore we deduce the following lemma.

Lemma 3. *For any pairwise distinct a, b, c, d , the word $\mathbf{W}_{a,b,c,d}$ avoids abelian cubes.*

We define the *Parikh vector* Ψ as the map

$$\begin{aligned} \Psi : \{a, b, c, d\}^* &\longrightarrow \mathbb{Z}^4 \\ w &\longmapsto {}^t(|w|_a \quad |w|_b \quad |w|_c \quad |w|_d). \end{aligned}$$

Let $M_\varphi = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ be the adjacency matrix of $\varphi_{a,b,c,d}$ and τ be the vector corresponding to the numerical approximation¹ $\tau \doteq \begin{pmatrix} 0.5788-0.5749i \\ -0.3219+0.2183i \\ -0.0690+0.6165i \\ -0.1662-0.6810i \end{pmatrix}$, which is

¹ We stress the fact that this is not an issue to use numerical approximation. Indeed, all our computations are numerically stable (additions, multiplications and no divisions by numbers close to zero) and if we start with sufficiently accurate approximations, we get sufficiently accurate approximations at the end (see footnote 2 for the only case where it matters that a coefficient is exactly 0). Moreover, there is an algebraic extension of \mathbb{Q} of degree 24 that contains all the eigenvalues of the matrices (according to mathematica) and thus we could use the original proof of [1, Theorem 8] to get an exact value for C and only use exact computation in our article. However, one might think that this is convenient to use the fact that these roots can be expressed with radicals, but maintaining exact expressions involving radicals is much more inefficient and would lead to even more unreadable computations.

related to the eigenvalue $0.4074 + 0.4766i$ of M_φ and precisely defined in Sect. 2.1 of [1]. For the sake of conciseness, the definition is omitted here. We recall the following result from [1].

Theorem 1 ([1, Theorem 3.1]). *There exists a positive real constant C such that for any two factors of $\mathbf{W}_{a,b,c,d}$ (not necessarily adjacent) u and v*

$$|\tau \cdot (\Psi(u) - \Psi(v))| < C,$$

where $2.175816 < C < 2.175817$.

Let us summarize the main idea behind Theorem 1. The asymptotic behavior of the Parikh vectors of factors is closely related to the asymptotic behavior of the iterations of the matrix M_φ (since $\Psi(\varphi(u)) = M_\varphi(\Psi(u))$). Moreover, the eigenvalue corresponding to this eigenvector is of norm less than 1 and thus the associated subspace is contracting. We deduce that $\tau(\Psi(u))$ is bounded for any factor u . Theorem 1 provides good bounds in this particular case. Equipped with Lemma 3 and Theorem 1 we deduce the following one.

Lemma 4. *For any pairwise distinct $a, b, c, d \in \mathbb{R}$, let $M_{a,b,c,d} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \end{pmatrix}$. Suppose that $\mathbf{W}_{a,b,c,d}$ contains an additive cube, then there exists a vector $x \in \ker(M_{a,b,c,d}) \cap \mathbb{Z}^4 \setminus \{0\}$ such that $|\tau \cdot x| < C$, where C is given in Theorem 1.*

Proof. Let uvw be an additive cube factor of $\mathbf{W}_{a,b,c,d}$. By Lemma 3, uvw cannot be an abelian cube. Thus either $\Psi(u) \neq \Psi(v)$ or $\Psi(v) \neq \Psi(w)$. Without loss of generality, $\Psi(u) \neq \Psi(v)$. In this case, let $x = \Psi(u) - \Psi(v) \neq 0$. Since x is the difference of two Parikh vectors we get $x \in \mathbb{Z}^4$. Since uvw is an additive cube, $|u| = |v|$ and $|u|_a a + |u|_b b + |u|_c c + |u|_d d = |v|_a a + |v|_b b + |v|_c c + |v|_d d$. This implies that $M_{a,b,c,d}(\Psi(u) - \Psi(v)) = 0$ which can be rewritten as $x \in \ker(M_{a,b,c,d})$. Therefore, $x \in \ker(M_{a,b,c,d}) \cap \mathbb{Z}^4 \setminus \{0\}$. By assumption u and v are two factors of $\mathbf{W}_{a,b,c,d}$ and by Theorem 1 we get $|\tau \cdot x| < C$, which concludes the proof.

This Lemma contains the main idea of the present work. If we want to know for which choices of a, b, c and d , the word $\mathbf{W}_{a,b,c,d}$ avoids additive cubes, it is sufficient to study the behavior of the lattice $\ker(M_{a,b,c,d}) \cap \mathbb{Z}^4 \setminus \{0\}$.

4 The Case of $\mathbf{W}_{0,1,c,d}$

Let us first study the lattice $\ker(M_{0,1,c,d}) \cap \mathbb{Z}^4 \setminus \{0\}$ for $c, d \in \mathbb{R}$. We show that in many cases additive cubes are avoidable over $\{0, 1, c, d\}$.

Theorem 2. *Let $c, d \in \mathbb{R}$. Suppose $d > c > 1$, $c \notin \{5/4, 4/3, 3/2, 2\}$ and $d \notin \{6 - 4c, 5 - 3c, 4 - 2c, 3 - c, 2c - 3, 2c - 2, 2c - 1, 3c - 3, 2\}$. Then $\mathbf{W}_{0,1,c,d}$ avoids additive cubes.*

Proof. From Lemma 4, it is sufficient to show that under the assumptions on c and d we get $|\tau \cdot x| \geq C$ for any $x \in \ker(M_{0,1,c,d}) \cap \mathbb{Z}^4 \setminus \{0\}$. Let us first express this set of vectors in a more convenient way. It is straightforward to check that if $\alpha = (c-1, -c, 1, 0)$ and $\beta = (d-1, -d, 0, 1)$, then $\{\alpha, \beta\}$ is a basis of $\ker(M_{0,1,c,d})$. For any reals m and n , if $m\alpha + n\beta$ is an integral vector then $m \in \mathbb{N}$ (resp., $n \in \mathbb{N}$) because otherwise its third (resp., its fourth) coordinate is not an integer and $mc + nd \in \mathbb{Z}$, otherwise the first and second coordinates are not integers. We deduce that

$$\ker(M_{0,1,c,d}) \cap \mathbb{Z}^4 = \{m\alpha + n\beta \mid m, n \in \mathbb{Z}, mc + nd \in \mathbb{Z}\}.$$

Thus, we only need to show that, under the assumptions, for any $m, n \in \mathbb{Z}$ with $mc + nd \in \mathbb{Z}$ and $(m, n) \neq (0, 0)$, we get

$$|\tau \cdot (m\alpha + n\beta)| \geq C. \quad (1)$$

Let us show that (1) holds if $n = 0$. In this case, $m \neq 0$, $|\tau \cdot m\alpha| = |m||\tau \cdot \alpha|$ and $mc \in \mathbb{Z}$. Numerical computation gives $f_0(c) := |\tau \cdot \alpha| \doteq \sqrt{1.83908 + c(-3.05698 + 1.44043c)}$. The minimum of f_0 is reached at $c \doteq \frac{3.05698}{2 \times 1.44043} \doteq 1.06114$. Thus for any numbers $x, y \in \mathbb{R}$ with $x < y$ and $1.06114 < y$ the minimum of f_0 over the interval $[x, y]$ is given by $f_0(\max(1.06114, x))$. We distinguish several cases depending on the value of c .

- If $c > 2.85$ a straightforward computation gives $|\tau \cdot \alpha| > C$ and $|\tau \cdot m\alpha| > C$.
- If $c \in [1.9, 2.9] \setminus \{2\}$, a computation gives $|\tau \cdot \alpha| > \frac{C}{2}$. Moreover, in this case $m \in \mathbb{Z}$ and $mc \in \mathbb{Z}$ imply $|m| \geq 2$ (since $c \notin \mathbb{Z}$) and $|\tau \cdot m\alpha| > C$.
- If $c \in [1.55, 1.95]$, a computation gives $|\tau \cdot \alpha| > \frac{C}{3}$. Moreover, in this case $m \in \mathbb{Z}$ and $mc \in \mathbb{Z}$ imply $|m| \geq 3$ (since $2c \notin \mathbb{Z}$) and we get $|\tau \cdot m\alpha| > C$.
- If $c \in [1.3, 1.65] \setminus \{4/3, 3/2\}$, a computation gives $|\tau \cdot \alpha| > \frac{C}{4}$. Moreover, in this case $m \in \mathbb{Z}$ and $mc \in \mathbb{Z}$ imply $|m| \geq 4$ (since $3c, 2c \notin \mathbb{Z}$) and we get $|\tau \cdot m\alpha| > C$.
- If $c \in]1, 1.35] \setminus \{5/4, 4/3\}$, a computation gives $|\tau \cdot \alpha| > \frac{C}{5}$. Moreover, in this case $m \in \mathbb{Z}$ and $mc \in \mathbb{Z}$ imply $|m| \geq 5$ (since $4c, 3c, 2c \notin \mathbb{Z}$) and we get $|\tau \cdot m\alpha| > C$.

Let us show that (1) is true if $|n| \geq 4$ and $m \in \mathbb{Z}$. We have

$$|m\tau \cdot \alpha + n\tau \cdot \beta| = |n||\tau \cdot \alpha| \left| \frac{m}{n} + \frac{\tau \cdot \beta}{\tau \cdot \alpha} \right| \geq |n||\tau \cdot \alpha| \left| \operatorname{Im} \left(\frac{m}{n} + \frac{\tau \cdot \beta}{\tau \cdot \alpha} \right) \right| \geq k|n|, \quad (2)$$

where $k = |\tau \cdot \alpha| \left| \operatorname{Im} \left(\frac{\tau \cdot \beta}{\tau \cdot \alpha} \right) \right|$. Numerical computations give:

$$k^2 \doteq \frac{1}{c^2 - 2.12228c + 1.27676} \left(0.217137d^2 + 0.533079dc + 0.327181c^2 + 0.217127d - 0.911556c + 0.634921 \right),$$

$$k^2 - \left(\frac{C}{4} \right)^2 \doteq \frac{1}{c^2 - 2.12228c + 1.27676} \left(0.257151 + 0.0312991c^2 + c(-0.283614 + 0.533079d) + (-0.742604 + 0.217137d)d \right).$$

The denominator $c^2 - 2.12228c + 1.27676$ is positive for any real c . Thus the sign of $k^2 - (\frac{C}{4})^2$ is the same as the sign of the numerator. For a given d , the minimum of the numerator is reached for $c \doteq 0.00443843 - 0.00834245d < 0$ (since $d > 1$). Thus the numerator is an increasing function of c for $c > 0$ and in particular for fixed d and $1 \leq c < d$ the minimum is reached at $c = 1$ and is given by $0.00483619 + (-0.209525 + 0.217137d)d$ which is positive since $d > 1$. We conclude that $k > \frac{C}{4}$. We use Eq. (2) to get that if $|n| \geq 4$, then $|m\tau \cdot \alpha + n\tau \cdot \beta| > C$.

It remains to deal with the cases $|n| \in \{1, 2, 3\}$. It is enough in (1) to consider the cases $n \in \{1, 2, 3\}$. We treat each case in a similar way. Let us start with the case $n = 1$. We get numerically

$$\begin{aligned} P_{c,d,1}(m) &:= |\tau \cdot (m\alpha + \beta)|^2 - C^2 \\ &\doteq -4.16782 + 0.712407m - 1.17373cm + 1.83908m^2 - 3.05698cm^2 \\ &\quad + 1.44043c^2m^2 + (-1.17373 - 3.05698m + 2.88085cm)d + 1.44043d^2. \end{aligned}$$

$P_{c,d,1}(m)$ is a quadratic polynomial in d . Computing the discriminant yields $P_{c,d,1}(m) > 0$, for all $c \in \mathbb{R}$ if and only if $\Delta_c(d) \doteq 25.3914 + 3.07144m - 1.25108m^2 < 0$. This is a quadratic inequality² in m and solving it yields

$$m \notin [-3.44178, 5.89681] \implies |\tau \cdot (m\alpha + \beta)| > C.$$

Thus we only need to check that for every $m \in \{5, 4, 3, 2, 1, 0, -1, -2, -3\}$ such that $mc + d \in \mathbb{Z}$, $P_{c,d,1}(m) > 0$. Let us detail the cases $m = -3$ and $m = 4$. Numerically, we get $P_{c,d,1}(-3) \doteq 10.2467 + 12.9638c^2 + c(-23.9917 - 8.64256d) + d(7.99723 + 1.44043d)$. This is a quadratic polynomial in d and we deduce³ that

$$P_{c,d,1}(-3) > 0 \iff d \in]-\infty, 3c - 3.54573[\cup]3c - 2.00625, \infty[.$$

Thus, in particular, since by hypothesis $d \neq 3c - 3$ then either $P_{c,d,1}(-3) > 0$ or $d \in [3c - 3.54573, 3c - 2.00625]$ and then $-3c + d \notin \mathbb{Z}$. The condition $P_{c,d,1}(4) > 0$ is equivalent to $d \in]6.1107 - 4c, \infty[$. Since $d > c > 1$ and $d \neq 6 - 4c$ then either $P_{c,d,1}(4) > 0$ or $d + 4c \notin \mathbb{Z}$. The other cases are similar. We give, in Table 1, for each of them the condition on the reals and the assumptions that allow us to conclude.

² We remark that it is no numerical coincidence that c does not appear in the expression of $\Delta_c(d)$. It follows from $P_{c,d,1}(m) = (x + ym + z(d + cm))^2 + (x' + y'm + z'(d + cm))^2 - C^2$ with $x, y, z \in \mathbb{R}$.

³ As for the previous note, it is no numerical coincidence that there is no complicated square root involving c since c does not appear in the discriminant.

Table 1. Study of $P_{c,d,1}(m)$ for $m \in \{5, 4, 3, 2, 1, 0, -1, -2, -3\}$.

(A): an equivalent condition for d	A sufficient condition to get (A)
$P_{c,d,1}(5) > 0 \Leftrightarrow d \in]6.78141 - 5c, \infty[$	$d > c > 1$
$P_{c,d,1}(4) > 0 \Leftrightarrow d \in]6.1107 - 4c, \infty[$	$d > c > 1$ and $d \neq 6 - 4c$
$P_{c,d,1}(3) > 0 \Leftrightarrow d \in]5.26804 - 3c, \infty[$	$d > c > 1$ and $d \neq 5 - 3c$
$P_{c,d,1}(2) > 0 \Leftrightarrow d \in]4.31762 - 2c, \infty[$	$d > c > 1$ and $d \neq 4 - 2c$
$P_{c,d,1}(1) > 0 \Leftrightarrow d \in]3.27931 - c, \infty[$	$d > c > 1$ and $d \neq 3 - c$
$P_{c,d,1}(0) > 0 \Leftrightarrow d \notin [-1.34171, 2.15655]$	$d > c > 1$ and $d \neq 2$
$P_{c,d,1}(-1) > 0 \Leftrightarrow d \in]c + 0.939592, \infty[$	$d > c$
$P_{c,d,1}(-2) > 0$ $\Leftrightarrow d \notin [2c - 3, 2c - 2, 2c - 1]$	$d \notin \{2c - 3, 2c - 2, 2c - 1\}$
$P_{c,d,1}(-3) > 0$ $\Leftrightarrow d \notin [3c - 3.54573, 3c - 2.00625]$	$d \neq 3c - 3$

The next case is $n = 2$ and we treat it in a similar fashion. We get numerically

$$\begin{aligned} P_{c,d,2}(m) &:= |\tau \cdot (m\alpha + 2\beta)|^2 - C^2 \\ &\quad \pm -2.46898 + 1.42481m - 2.34745cm + 1.83908m^2 - 3.05698cm^2 \\ &\quad + 1.44043c^2m^2 + (-4.6949 - 6.11397m + 5.76171cm)d + 5.76171d^2. \end{aligned}$$

Computing the discriminant yields $P_{c,d,2}(m) > 0$, for all $d \in \mathbb{R}$ if and only if $\Delta_c(d) := 78.9442 + (24.5715 - 5.00433m)m < 0$. This is a quadratic inequality in m and solving it yields $m \notin [-2.21427, 7.12433] \implies |\tau \cdot (m\alpha + \beta)| > C$. Thus we only need to check that, under the assumptions, for every $m \in \{7, 6, 5, 4, 3, 2, 1, 0, -1, -2\}$ such that $mc + 2d \in \mathbb{Z}$, $P_{c,d,2}(m) > 0$. Each case is similar to the cases with $n = 1$. We give, in Table 2, for each of them the condition on the reals and the assumptions that allow us to conclude. The only remaining case is $n = 3$ and we treat it in a similar fashion. We get numerically

$$\begin{aligned} P_{c,d,3}(m) &:= |\tau \cdot (m\alpha + 2\beta)|^2 - C^2 \\ &\quad \pm 0.362434 + 2.13722m - 3.52118bm + 1.83908m^2 - 3.05698cm^2 \\ &\quad + 1.44043c^2m^2 + (-10.5635 - 9.17095m + 8.64256cm)d + 12.9638d^2. \end{aligned}$$

Computing the discriminant yields $P_{c,d,3}(m) > 0$, for all $d \in \mathbb{R}$ if and only if $\Delta_c(d) := 92.7941 + 82.929m - 11.2597m^2 < 0$. This is a quadratic inequality in m and solving it yields $m \notin [-0.986756, 8.35184] \implies |\tau \cdot (m\alpha + \beta)| > C$. Thus we only need to check that, under the assumptions, for every $m \in \{8, 7, 6, 5, 4, 3, 2, 1, 0\}$ such that $mc + 3d \in \mathbb{Z}$, $P_{c,d,3}(m) > 0$. Each case is similar to the cases $n = 1, 2$. We give, in Table 3, for each of them the condition on the reals and the assumptions that allow us to conclude. This finishes the proof of Theorem 2.

Table 2. Theorem 2: Study of $P_{c,d,2}(m)$ for $m \in \{7, 6, 5, 4, 3, 2, 1, 0, -1, -2\}$.

(B_1) : an equivalent condition for d	A sufficient condition to get (B_1)
$P_{c,d,2}(7) > 0 \Leftrightarrow d \notin]3.91363 - 3.5c, 4.32919 - 3.5c[$	$d > c > 1$
$P_{c,d,2}(6) > 0 \Leftrightarrow d \notin]3.00088 - 3c, 4.1808 - 3c[$	$d > c > 1$
$P_{c,d,2}(5) > 0 \Leftrightarrow d \notin]2.30029 - 2.5c, 3.82024 - 2.5c[$	$d > c > 1$
$P_{c,d,2}(4) > 0 \Leftrightarrow d \notin]1.67431 - 2c, 3.38509 - 2c[$	$d > c > 1$
$P_{c,d,2}(3) > 0 \Leftrightarrow d \notin]1.09888 - 1.5c, 2.89938 - 1.5c[$	$d > c > 1$
$P_{c,d,2}(2) > 0 \Leftrightarrow d \notin]0.566427 - c, 2.3707 - c[$	$d > c > 1$
$P_{c,d,2}(1) > 0 \Leftrightarrow d \notin]0.0766769 - 0.5c, 1.79931 - 0.5c[$	$d > c > 1$
$P_{c,d,2}(0) > 0 \Leftrightarrow d \notin]-0.363621, 1.17847[$	$d > c > 1$
$P_{c,d,2}(-1) > 0 \Leftrightarrow d \notin]-0.732884 + 0.5c, 0.486592 + 0.5c[$	$d > c$
$P_{c,d,2}(-2) > 0 \Leftrightarrow d \notin]-0.925154 + c, -0.382276 + c[$	$d > c$

In fact, using a symmetry argument we can improve the previous result.

Theorem 3. For any $(c, d) \in \mathbb{R}^2 \setminus \mathcal{F}$ additive cubes are avoidable over $\{0, 1, c, d\}$ where

$$\mathcal{F} = \left\{ \left(\frac{10}{9}, \frac{14}{9} \right), \left(\frac{9}{8}, \frac{3}{2} \right), \left(\frac{9}{8}, \frac{13}{8} \right), \left(\frac{8}{7}, \frac{10}{7} \right), \left(\frac{8}{7}, \frac{11}{7} \right), \left(\frac{8}{7}, \frac{12}{7} \right), \left(\frac{7}{6}, \frac{11}{6} \right), \right. \\ \left(\frac{7}{6}, \frac{3}{2} \right), \left(\frac{7}{6}, \frac{5}{3} \right), \left(\frac{6}{5}, \frac{8}{5} \right), \left(\frac{6}{5}, \frac{9}{5} \right), \left(\frac{6}{5}, 2 \right), \left(\frac{5}{4}, \frac{7}{4} \right), \left(\frac{5}{4}, 2 \right), \left(\frac{5}{4}, \frac{9}{4} \right), \\ \left(\frac{5}{4}, \frac{5}{2} \right), \left(\frac{5}{4}, \frac{11}{4} \right), \left(\frac{5}{4}, 3 \right), \left(\frac{5}{4}, \frac{13}{4} \right), \left(\frac{5}{4}, \frac{7}{2} \right), \left(\frac{4}{3}, 2 \right), \left(\frac{4}{3}, \frac{7}{3} \right), \left(\frac{4}{3}, \frac{8}{3} \right), \\ \left(\frac{4}{3}, 3 \right), \left(\frac{4}{3}, \frac{10}{3} \right), \left(\frac{4}{3}, \frac{11}{3} \right), \left(\frac{4}{3}, 4 \right), \left(\frac{3}{2}, \frac{5}{2} \right), \left(\frac{3}{2}, 3 \right), \left(\frac{3}{2}, \frac{7}{2} \right), \left(\frac{3}{2}, 4 \right), \\ \left. \left(\frac{3}{2}, \frac{9}{2} \right), \left(\frac{3}{2}, 5 \right), (4, 5) \right\} \\ \cup \{(2, t), (t, 2t - 2), (t, 2t - 1), (t, 3t - 3) : t \in \mathbb{R}\} \cap \{(c, d) : d > c > 1\}.$$

Proof. Let \mathcal{X} be the following set of pairs of parametric equations:

$$\mathcal{X} = \{(5/4, t), (4/3, t), (3/2, t), (2, t), (t, 6 - 4t), (t, 5 - 3t), (t, 4 - 2t), \\ (t, 3 - t), (t, 2t - 3), (t, 2t - 2), (t, 2t - 1), (t, 3t - 3), (t, 2) : t \in \mathbb{R}\}.$$

For any pair $e = (x(t), y(t))$ of parametric equations, we denote by $\mathcal{C}(e)$ the associated parametric curve (that is the set of points defined by $\{(x(t), y(t)) : t \in \mathbb{R}\}$). By the Theorem 2 for any $c, d \in \mathbb{R}$ with $c > d > 1$ and $(c, d) \notin \bigcup_{e \in \mathcal{X}} \mathcal{C}(e)$ additive cubes are avoidable over $\{0, 1, c, d\}$. Moreover, for any $c, d \in \mathbb{R}$ with $d > c > 1$, the alphabet $\{0, 1, c, d\}$ is equivalent to the alphabet $\{0, 1, \frac{d-1}{d-c}, \frac{d}{d-c}\}$ (via the affine map $x \mapsto \frac{d-x}{d-c}$). Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, (x, y) \mapsto \left(\frac{y-1}{y-x}, \frac{y}{y-x} \right)$. We

Table 3. Theorem 2: Study of $P_{c,d,3}(m)$ for $m \in \{8, 7, 6, 5, 4, 3, 2, 1, 0\}$.

(B_2) : an equivalent condition for d	A sufficient condition to get (B_2)
$P_{c,d,3}(8) > 0 \Leftrightarrow d \notin]3.00699 - 2.66667c, 3.46726 - 2.66667c[$	$d > c > 1$
$P_{c,d,3}(7) > 0 \Leftrightarrow d \notin]2.45816 - 2.33333c, 3.30867 - 2.33333c[$	$d > c > 1$
$P_{c,d,3}(6) > 0 \Leftrightarrow d \notin]2.00508 - 2c, 3.05432 - 2c[$	$d > c > 1$
$P_{c,d,3}(5) > 0$ $\Leftrightarrow d \notin]1.59624 - 1.66667c, 2.75573 - 1.66667c[$	$d > c > 1$
$P_{c,d,3}(4) > 0$ $\Leftrightarrow d \notin]1.21937 - 1.33333c, 2.42518 - 1.33333c[$	$d > c > 1$
$P_{c,d,3}(3) > 0 \Leftrightarrow d \notin]0.870753 - c, 2.06637 - c[$	$d > c > 1$
$P_{c,d,3}(2) > 0$ $\Leftrightarrow d \notin]0.551146 - 0.666667c, 1.67855 - 0.666667c[$	$d > c > 1$
$P_{c,d,3}(1) > 0$ $\Leftrightarrow d \notin]0.266517 - 0.333333c, 1.25575 - 0.333333c[$	$d > c > 1$
$P_{c,d,3}(0) > 0 \Leftrightarrow d \notin]0.0358908, 0.778955[$	$d > c > 1$

deduce that for any $c, d \in \mathbb{R}$ with $d > c > 1$ and $(c, d) \notin \bigcup_{e \in \mathcal{X}} \mathcal{C}(f \circ e)$ additive cubes are avoidable over $\{0, 1, c, d\}$. Let

$$\mathcal{F} = \left(\bigcup_{e \in \mathcal{X}} \mathcal{C}(f \circ e) \right) \cap \left(\bigcup_{e \in \mathcal{X}} \mathcal{C}(e) \right) \cap \{(c, d) : d > c > 1\}. \tag{3}$$

Then, for any $c, d \in \mathbb{R}$ with $d > c > 1$ and $(c, d) \notin \mathcal{F}$ additive cubes are avoidable over $\{0, 1, c, d\}$. Let us now compute \mathcal{F} . First, one computes

$$\begin{aligned} \mathcal{C}(\{f \circ e : e \in \mathcal{X}\}) = \mathcal{C} \left(\left\{ (t, 6t - 4), (t, 5t - 3), (t, 4t - 2), (t, 3t - 1), (t, \frac{3}{2}t - 1), \right. \right. \\ \left. \left. (t, 2(t - 1)), (2, t), (t, 3(t - 1)), (t, 2t), (t, 5t - 4), \right. \right. \\ \left. \left. (t, 4t - 3), (t, 3t - 2), (t, 2t - 1) \right\} \right). \end{aligned}$$

We get the set from Theorem 3 by simply computing the intersection of the two sets in (3) (this is done by solving the 169 equations).

5 The Case of $\mathbf{W}_{1,0,c,d}$

We show the next result by using a similar procedure as the one in the proof of Theorem 2 in Sect. 4.

Theorem 4. *Let $c, d \in \mathbb{R}$. Suppose we have $d > c > 1$, $d \notin \{2, c + 1, c + 2, 2c + 2, 2c + 1, 2c, 3c, 3c + 1, 1 + \frac{c}{2}, \frac{1}{2} + c\}$. Then $\mathbf{W}_{1,0,c,d}$ avoids additive cubes.*

Proof. Following the proof of Theorem 2, we only need to show that, under the assumptions, for any $m, n \in \mathbb{Z}$ with $mc + nd \in \mathbb{Z}$, we have $|\tau \cdot (m\alpha + n\beta)| > C$, where $\alpha = (-c, c - 1, 1, 0)$ and $\beta = (-d, d - 1, 0, 1)$.

Let us first show that this is the case if $n = 0$. Two subcases occur:

- If $c > 1.71$ a computation gives $|\tau \cdot \alpha| > C$ and $|\tau \cdot m\alpha| > C$.
- If $c \in]1, 2[$, a computation gives $|\tau \cdot \alpha| > \frac{C}{2}$. Moreover, in this case $m \in \mathbb{Z}$ and $mc \in \mathbb{Z}$ imply $|m| \geq 2$ (since $c \notin \mathbb{Z}$) and we get $|\tau \cdot m\alpha| \geq C$.

Let us now show that $|\tau \cdot (m\alpha + n\beta)| > C$ if $|n| \geq 4$ and $m \in \mathbb{Z}$. The same computation as (2) gives:

$$|m\tau \cdot \alpha + n\tau \cdot \beta| \geq k|n|, \quad \text{where } k = |\tau \cdot \alpha| \left| \operatorname{Im} \left(\frac{\tau \cdot \beta}{\tau \cdot \alpha} \right) \right|. \quad (4)$$

The same approach as in proof of Theorem 2 can be used to verify that $k^2 - (\frac{C}{4})^2 > 0$ for any $d > c > 1$. This gives with inequality (4) that if $|n| \geq 4$, then $|m\tau \cdot \alpha + n\tau \cdot \beta| > C$. It remains to treat the cases $|n| \in \{1, 2, 3\}$ but it is enough to consider the cases $n \in \{1, 2, 3\}$, as previously. We start with the case $n = 1$. Once again $P_{c,d,1}(m) := |\tau \cdot (m\alpha + \beta)|^2 - C^2$ is a quadratic polynomial in d . Computing its discriminant yields $P_{c,d,1}(m) > 0$, for all $c \in \mathbb{R}$ if and only if $\Delta_c(d) := 25.3914 + 3.07144m - 1.25108m^2 < 0$. This is a quadratic inequality in m and solving it yields $m \notin [-3.44178, 5.89681] \implies |\tau \cdot (m\alpha + \beta)| > C$ (the conditions on m happen to be exactly the same as in Sect. 4). Thus we only need to check that for every $m \in \{5, 4, 3, 2, 1, 0, -1, -2, -3\}$ such that $mc + d \in \mathbb{Z}$, we have $P_{c,d,1}(m) > 0$.

All the cases are similar to what we did in the previous proof. We give, in Table 4, for each of them the condition on the reals and the assumptions that allow us to conclude. The next case is $n = 2$ and we treat it in a similar fashion.

We verify that the only interesting cases are $m \notin [-2.21427, 7.12433]$. Thus we only need to check that for every $m \in \{7, 6, 5, 4, 3, 2, 1, 0, -1, -2\}$ such that $mc + 2d \in \mathbb{Z}$, we get $P_{c,d,2}(m) > 0$. Each case is similar to the cases with $n = 1$. We omit here, because of the lack of space, the table that give for each of them the condition on the reals and the assumptions that allow us to conclude. This table can be found at <https://members.loria.fr/FLietard/tables-of-values/>.

The only remaining case is $n = 3$. We once again compute the discriminant of $P_{c,d,3}(m)$ seen as a polynomial in d . We deduce that $m \notin [-0.986756, 8.35184] \implies |\tau \cdot (m\alpha + \beta)| > C$.

Summing up, we only need to check that for every $m \in \{8, 7, 6, 5, 4, 3, 2, 1, 0\}$ such that $mc + 3d \in \mathbb{Z}$, $P_{c,d,3}(m) > 0$. We prove this statement by solving each of the corresponding 9 equations and this concludes the proof of Theorem 4.

We could improve this result with the same approach as the one we used in the proof of Theorem 3, but we already have a strong enough result for our purpose.

Table 4. Theorem 4: Study of $P_{c,d,1}(m)$ for $m \in \{5, 4, 3, 2, 1, 0, -1, -2, -3\}$.

(C_1) : an equivalent condition for d	A sufficient condition to get (C_1)
$P_{c,d,1}(5) > 0 \Leftrightarrow d \notin]-0.781405 - 5c, 1.35518 - 5c[$	$d > c > 1$
$P_{c,d,1}(4) > 0 \Leftrightarrow d \notin]-1.1107 - 4c, 1.80675 - 4c[$	$d > c > 1$
$P_{c,d,1}(3) > 0 \Leftrightarrow d \notin]-1.26804 - 3c, 2.08636 - 3c[$	$d > c > 1$
$P_{c,d,1}(2) > 0 \Leftrightarrow d \notin]-1.31762 - 2c, 2.25822 - 2c[$	$d > c > 1$
$P_{c,d,1}(1) > 0 \Leftrightarrow d \notin]-1.27931 - c, d > 2.34218 - c[$	$d > c > 1$
$P_{c,d,1}(0) > 0 \Leftrightarrow d \notin [-1.15655, 2.34171]$	$d > c > 1$ and $d \neq 2$
$P_{c,d,1}(-1) > 0 \Leftrightarrow d \notin]-0.939592 + c, 2.24702 + c[$	$d > c$ and $d \notin \{c + 1, c + 2\}$
$P_{c,d,1}(-2) > 0 \Leftrightarrow d \notin]-0.595226 + 2c, 2.02493 + 2c[$	$d \notin \{2c + 2, 2c + 1, 2c\}$
$P_{c,d,1}(-3) > 0 \Leftrightarrow d \notin]0.00625218 + 3c, 1.54573 + 3c[$	$d \notin \{3c, 3c + 1\}$

6 The Remaining Alphabets

We use Theorem 3 and Theorem 4 to get the following result:

Theorem 5. *Let*

$$\mathcal{F} = \left\{ \left(\frac{10}{9}, \frac{14}{9} \right), \left(\frac{9}{8}, \frac{13}{8} \right), \left(\frac{8}{7}, \frac{11}{7} \right), \left(\frac{7}{6}, \frac{5}{3} \right), \left(\frac{6}{5}, \frac{8}{5} \right), \left(\frac{6}{5}, 2 \right), \left(\frac{5}{4}, \frac{7}{4} \right), \right. \\ \left. \left(\frac{5}{4}, 2 \right), \left(\frac{5}{4}, \frac{9}{4} \right), \left(\frac{5}{4}, \frac{5}{2} \right), \left(\frac{5}{4}, \frac{13}{4} \right), \left(\frac{5}{4}, \frac{7}{2} \right), \left(\frac{4}{3}, 2 \right), \left(\frac{4}{3}, \frac{7}{3} \right), \right. \\ \left. \left(\frac{4}{3}, \frac{8}{3} \right), \left(\frac{4}{3}, \frac{10}{3} \right), \left(\frac{4}{3}, \frac{11}{3} \right), \left(\frac{3}{2}, \frac{5}{2} \right), \left(\frac{3}{2}, 3 \right), \left(\frac{3}{2}, \frac{7}{2} \right), (4, 5), \left(\frac{4}{3}, \frac{5}{3} \right), \right. \\ \left. \left(\frac{3}{2}, 2 \right), \left(\frac{8}{5}, \frac{9}{5} \right), \left(\frac{5}{3}, 2 \right), \left(\frac{7}{4}, \frac{9}{4} \right), \left(2, \frac{5}{2} \right), (2, 3), (2, 4), (2, 5), \left(\frac{5}{2}, 3 \right), \right. \\ \left. \left(\frac{5}{2}, \frac{9}{2} \right), (3, 4), (3, 5), (3, 6), (4, 6), (4, 9) \right\}.$$

and $(c, d) \in \mathbb{R}^2 \setminus \mathcal{F}$. Then additive cubes are avoidable over $\{0, 1, c, d\}$.

Proof. This set is obtained by taking the intersection of the sets of forbidden pairs from Theorem 3 and Theorem 4.

In order to study the remaining alphabets (those of the form $\{0, 1, c, d\}$ with $c, d \in \mathcal{F}$) let us recall the following results from the literature.

Theorem 6 ([7, Section 3.2]). *Additive cubes are avoidable over the following alphabets: $\{0, 1, 5\}, \{0, 1, 6\}, \{0, 1, 7\}, \{0, 2, 7\}, \{0, 3, 7\}, \{0, 1, 8\}, \{0, 3, 8\}, \{0, 1, 9\}, \{0, 2, 9\}, \{0, 4, 9\}$.*

Theorem 7 ([8, Theorem 9]). *Additive cubes are avoidable over the following alphabets: $\{0, 2, 3, 6\}$, $\{0, 1, 2, 4\}$, $\{0, 2, 3, 5\}$.*

We use the fact that all but one of the remaining alphabets contain an alphabet equivalent to an alphabet from Theorem 6 or Theorem 7. Our main result after this reduction is the following:

Theorem 8. *For any rational numbers c and d with $c < d$ and $(c, d) \neq (2, 3)$ additive cubes are avoidable over $\{0, 1, c, d\}$.*

Proof. $\{0, 1, \frac{10}{9}, \frac{14}{9}\}$ contains an alphabet equivalent to $\{0, 1, 5\}$ (apply $x \mapsto 9x - 9$ to $\{1, \frac{10}{9}, \frac{14}{9}\}$). We deduce from Theorem 6 that additive cubes are avoidable over both alphabets. We proceed in a same way for the other alphabets and we provide for each of them the alphabet from Theorem 6 or from Theorem 7 in Table 5. This concludes the proof.

Table 5. Each remaining alphabet, with exception of $\{0, 1, 2, 3\}$, contains an alphabet equivalent to an alphabet from Theorems 6 or 7.

$(\frac{10}{9}, \frac{14}{9}), (\frac{5}{4}, \frac{7}{2}), (\frac{4}{3}, \frac{5}{3}), (4, 5), (2, \frac{5}{2}), (2, 5), (3, 5)$	$\{0, 1, 5\}$
$(\frac{6}{5}, \frac{8}{5}), (\frac{6}{5}, 2), (\frac{5}{4}, \frac{5}{2}), (\frac{5}{3}, 2), (\frac{5}{2}, 3), (3, 6), (4, 6)$	$\{0, 1, 6\}$
$(\frac{7}{6}, \frac{5}{3}), (\frac{4}{3}, \frac{10}{3})$	$\{0, 1, 7\}$
$(\frac{3}{2}, \frac{7}{2})$	$\{0, 2, 7\}$
$(\frac{5}{4}, \frac{7}{4}), (\frac{4}{3}, \frac{7}{3})$	$\{0, 3, 7\}$
$(\frac{8}{7}, \frac{11}{7}), (\frac{4}{3}, \frac{11}{3}), (\frac{3}{2}, \frac{5}{2})$	$\{0, 1, 8\}$
$(\frac{5}{4}, 2), (\frac{4}{3}, \frac{8}{3})$	$\{0, 3, 8\}$
$(\frac{9}{8}, \frac{13}{8}), (\frac{5}{4}, \frac{13}{4}), (\frac{8}{5}, \frac{9}{5})$	$\{0, 1, 9\}$
$(\frac{5}{2}, \frac{9}{2})$	$\{0, 2, 9\}$
$(\frac{5}{4}, \frac{9}{4}), (\frac{7}{4}, \frac{9}{4}), (4, 9)$	$\{0, 4, 9\}$
$(\frac{4}{3}, 2), (\frac{3}{2}, 3)$	$\{0, 2, 3, 6\}$
$(\frac{3}{2}, 2), (2, 4)$	$\{0, 1, 2, 4\}$
$(3, 4)$	$\{0, 1, 3, 4\}$

We reformulate this result in terms of Problem 1.

Corollary 1. *Let $\mathcal{A} \subset \mathbb{C}$ be an alphabet with $|\mathcal{A}| \geq 4$. If \mathcal{A} is not equivalent to $\{0, 1, 2, 3\}$ then additive cubes are avoidable over \mathcal{A} . In particular, if $|\mathcal{A}| \geq 5$, then additive cubes are avoidable over \mathcal{A} .*

Note that we have shown that for all but finitely many integral alphabets of size 4 (up to the equivalence relation given in Sect. 2.1) the word $\mathbf{W}_{a,b,c,d}$ can be used to avoid additive cubes. This is probably not the only fixed point of a morphism with this property. Indeed, as long as the adjacency matrix of a

morphism has at most two eigenvalues of norm at least 1, we can deduce an inequality similar to that of Theorem 1 (see Proposition 7 in [8] for details). If the word also avoids abelian cubes, we can show an inequality similar to this in Lemma 4. The conditions of this Lemma should be strong enough to study the lattice in a similar way to what we did.

Let us conclude by restating two remaining related open questions. First it is natural to ask whether additive cubes are avoidable over the only remaining alphabet (see Problem 7 in [8]).

Question 1. Are additive cubes avoidable over $\{0, 1, 2, 3\}$?

On the one hand, Rao [7] claims that he got a word of length 1.4×10^5 over the alphabet $\{0, 1, 2, 3\}$ without additive cubes. Damien Jamet, the first author and Thomas Stoll constructed over this alphabet several words of length greater than 10^7 without additive cubes (see <https://members.loria.fr/FLietard/un-mot-sur-0123/> for such a word). Therefore, it seems to be reasonable to believe that there exists an infinite word without additive cubes over $\{0, 1, 2, 3\}$. On the other hand, for every alphabet $\{a, b, c, d\}$ different from $\{0, 1, 2, 3\}$ it is possible to provide a short morphism with the same eigenvalues as those of $\varphi_{a,b,c,d}$ or $\varphi_{a,b,c,d}^2$ with an infinite fixed point avoiding additive cubes. An exhaustive research shows, however, that every morphism over $\{0, 1, 2, 3\}$ with images of size at most 7 fails to provide an infinite fixed point without additive cubes. We do not dare to conjecture whether or not a morphism providing such an infinite word exists.

It seems that additive cubes are avoidable over most alphabets of size 3. Our result might stimulate research to treat the following question.

Question 2. Can we characterize the sets of integers of size 3 over which additive cubes are avoidable?

In fact, with the exception of $\{0, 1, 2, 3\}$, the alphabets of size three are the only remaining case of Problem 1 due to Lemma 2 and Theorem 8.

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