Chapter 11
Nonlinear Deformations of an Elastic Sphere with Couple Stresses and Distributed Dislocations

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Abstract The problem of nonlinear moment theory of elasticity about the equilibrium of a hollow sphere with distributed dislocations is considered. For an arbitrary isotropic micropolar elastic material and a spherically symmetric distribution of screw and edge dislocations, the problem is reduced to a system of nonlinear ordinary differential equations. In the case of a physically linear micropolar body model, exact solutions are found for the eigenstresses in the sphere due to the spherically symmetric distribution of edge dislocations.

Keywords Micropolar medium · Nonlinear elasticity · Spherical symmetry · Dislocation density · Eigenstresses · Exact solution

11.1 Introduction

In the present paper, we consider the spherically symmetric problems of the nonlinear theory of dislocations taking into account couple stresses, i.e., in the framework of the micropolar theory of elasticity. This model is also called the Cosserat continuum. The model of a micropolar medium is used to describe granular polycrystalline bodies, polymer composites, suspensions, liquid crystals, geophysical structures, biological tissues, metamaterials, nanostructured materials, etc. [1–5]. The basics of the nonlinearly theory of the Cosserat elastic continuum had been given in [6–10].

An important point of the microstructure of solids is the defects of the crystal lattice such as dislocations and disclinations [11–13]. The linear theory of continuously distributed dislocations and disclinations in micropolar media is developed in [14–16], and the nonlinear theory in [17, 18]. At present, only a very limited number of exact solutions in nonlinear theory of dislocations and disclinations for micropolar elastic media are known in the literature. Several solutions for isolated dislocations and disclinations in the nonlinear elastic Cosserat continuum are found in [8]. The

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planar axisymmetric problem of large deformations of a micropolar medium with distributed wedge disclinations is solved in [18].

Below in the paper, spherically symmetric solutions for large deformations of a micropolar medium with distributed dislocations were found. Spherically symmetric solutions of the nonlinear theory of dislocations without taking into account couple stresses were found earlier [19, 20]. In the framework of the linear micropolar theory of elasticity, spherically symmetric deformations of a hollow sphere with distributed dislocations and disclinations were studied in [21].

11.2 Basic Relations of Nonlinear Micropolar Continuum

Deformation of an elastic medium is described by mapping from a reference configuration to an actual one. In the case of micropolar continua, it is defined by two kinematically independent fields of translations and rotations

$$\mathbf{r} = \mathbf{r}(\mathbf{R}) = \mathbf{R} + \mathbf{u}(\mathbf{R}), \quad \mathbf{H} = \mathbf{H}(\mathbf{R}),$$

where $\mathbf{R} = X_k \mathbf{i}_k$ and X_k, $\mathbf{r} = x_s \mathbf{i}_s$ and x_s are the position vectors and Cartesian coordinates in the reference and actual configurations, respectively, $\mathbf{i}_k$ are corresponding constant base vectors ($k = 1, 2, 3$), $\mathbf{u}$ is the translation vector, and $\mathbf{H}$ is the proper orthogonal tensor describing the rotational degrees of freedom of micropolar continua cold often microrotation tensor [6–8].

In what follows we use the following definitions of operators of gradient, divergence, and rotor (curl) in the coordinates of the reference configuration:

$$\begin{aligned} \text{Grad}\, \boldsymbol{\Psi} &= \mathbf{R}^N \otimes \frac{\partial \boldsymbol{\Psi}}{\partial Q^N}, \quad \text{Div}\ \boldsymbol{\Psi} = \mathbf{R}^N \cdot \frac{\partial \boldsymbol{\Psi}}{\partial Q^N}, \\ \text{Rot}\ \boldsymbol{\Psi} &= \mathbf{R}^N \times \frac{\partial \boldsymbol{\Psi}}{\partial Q^N}, \quad \mathbf{R}^N = \mathbf{i}_k \frac{\partial Q^N}{\partial X_k}, \end{aligned} \tag{11.1}$$

where $\boldsymbol{\Psi}$ is an arbitrary differentiable tensor field of any order, $Q^N = Q^N(X_1, X_2, X_3)$, $N = 1, 2, 3$, are Lagrangian curvilinear coordinates, $\otimes$ denotes the tensor product, and cross and dot stand for vector and scalar products, respectively.

The governing equations of a micropolar elastic continuum are given by the following equations, see [6–10]:

Equilibrium equations:

$$\text{Div}\ \mathbf{D} + \rho \mathbf{f} = 0, \quad \text{Div}\ \mathbf{G} + \left(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{D}\right)_\times + \rho \mathbf{l} = 0. \tag{11.2}$$

Constitutive relations:

$$
\begin{gathered}
\mathbf{D} = \mathbf{P} \cdot \mathbf{H}, \quad \mathbf{G} = \mathbf{K} \cdot \mathbf{H}, \\
\mathbf{P} = \frac{\partial W}{\partial \mathbf{E}}, \quad \mathbf{K} = \frac{\partial W}{\partial \mathbf{L}}, \quad W = W(\mathbf{E}, \mathbf{L}).
\end{gathered}
\tag{11.3}
$$

Geometric relations:

$$
\begin{gathered}
\mathbf{E} = \mathbf{F} \cdot \mathbf{H}^{\mathrm{T}}, \\
\mathbf{L} = \frac{1}{2}\mathbf{R}^{N} \otimes \left(\frac{\partial \mathbf{H}}{\partial Q^{N}} \cdot \mathbf{H}^{\mathrm{T}} \right)_{\times} = \frac{1}{2}\mathbf{I}\,\mathrm{tr}\big[\mathbf{H} \cdot (\mathrm{Rot}\,\mathbf{H})^{\mathrm{T}}\big] - \mathbf{H} \cdot (\mathrm{Rot}\,\mathbf{H})^{\mathrm{T}}, \\
\mathbf{F} = \mathrm{Grad}\,\mathbf{r}.
\end{gathered}
\tag{11.4}
$$

Here $\mathbf{D}$ and $\mathbf{G}$ are the stress and couples stress tensors of the first Piola–Kirchhoff type, respectively, while $\mathbf{P}$ and $\mathbf{K}$ are these of the second Piola–Kirchhoff type, $\mathbf{F}$ is deformation gradient, $\mathbf{E}$ and $\mathbf{L}$ are the strain tensors in the nonlinear micropolar continuum called stretch and wryness tensors, respectively (see [6–10]), $\mathbf{I}$ is unit tensor, ρ is material density in the reference configuration, $\mathbf{f}$ is the external distributed mass load, $\mathbf{l}$ is the external distributed mass couple load, W is the strain energy density, and $\boldsymbol{\Phi}_{\times}$ denotes the vector invariant of the second-order tensor $\boldsymbol{\Phi}$

$$
\boldsymbol{\Phi}_{\times} = \left(\Phi_{MN}\mathbf{R}^{M} \otimes \mathbf{R}^{N}\right)_{\times} \stackrel{\Delta}{=} \Phi_{MN}\mathbf{R}^{M} \times \mathbf{R}^{N}.
$$

For isotropic micropolar continua strain energy W, stress and couples stressed tensors $\mathbf{P}$ and $\mathbf{K}$ are isotropic functions of strain tensors $\mathbf{E}$ and $\mathbf{L}$ [9]. This implies of the following relations:

$$
\begin{gathered}
W\left(\mathbf{Q}^{\mathrm{T}} \cdot \mathbf{E} \cdot \mathbf{Q}, (\det \mathbf{Q})\mathbf{Q}^{\mathrm{T}} \cdot \mathbf{L} \cdot \mathbf{Q}\right) = W(\mathbf{E}, \mathbf{L}) \\
\mathbf{P}\left(\mathbf{Q}^{\mathrm{T}} \cdot \mathbf{E} \cdot \mathbf{Q}, (\det \mathbf{Q})\mathbf{Q}^{\mathrm{T}} \cdot \mathbf{L} \cdot \mathbf{Q}\right) = \mathbf{Q}^{\mathrm{T}} \cdot \mathbf{P}(\mathbf{E}, \mathbf{L}) \cdot \mathbf{Q} \\
\mathbf{K}\left(\mathbf{Q}^{\mathrm{T}} \cdot \mathbf{E} \cdot \mathbf{Q}, (\det \mathbf{Q})\mathbf{Q}^{\mathrm{T}} \cdot \mathbf{L} \cdot \mathbf{Q}\right) = (\det \mathbf{Q})\mathbf{Q}^{\mathrm{T}} \cdot \mathbf{K}(\mathbf{E}, \mathbf{L}) \cdot \mathbf{Q}
\end{gathered}
\tag{11.5}
$$

for any arbitrary orthogonal tensor $\mathbf{Q}$. In Eqs. (11.5) we take into account that $\mathbf{E}$ and $\mathbf{P}$ are the true second-order tensors, while $\mathbf{L}$ and $\mathbf{K}$ are the second-order pseudotensors.

11.3 Continuously Distributed Dislocations

Let V be the area occupied by the elastic medium in the reference configuration. In order to introduce the dislocation density in a micropolar medium, we consider the problem of determining the displacement field $\mathbf{u}(\mathbf{R})$ of the medium from the tensor fields $\mathbf{E}$ and $\mathbf{H}$ specified in the multiply connected area V. These tensor fields are assumed to be differentiable and unambiguous in V. Considering that according to (11.4)

$$\mathrm{Grad}\,\mathbf{u} = \mathbf{F} - \mathbf{I} = \mathbf{E} \cdot \mathbf{H} - \mathbf{I} \tag{11.6}$$

we see that in the case of a multiply connected area, the vector field $\mathbf{u}(\mathbf{R})$, and therefore the vector field $\mathbf{r}(\mathbf{R})$, are not uniquely defined, in general. This means the presence of isolated translational dislocations in the body [8], each of which is characterized by Burgers vector $\mathbf{b}_n$, by virtue of (11.6)

$$\mathbf{b}_n = \oint_{\Gamma_n} d\mathbf{R} \cdot (\mathbf{E} \cdot \mathbf{H} - \mathbf{I}) = \oint_{\Gamma_n} d\mathbf{R} \cdot \mathbf{F}, \quad n = 1, 2, \ldots, n_0 \tag{11.7}$$

Here Γ_n is a simple closed contour enclosing the line of the nth dislocation only. The total Burgers vector of a discrete set of n_0 dislocations according to (11.7) is defined by the formula

$$\mathbf{B} = \sum_{n=1}^{n_0} \mathbf{b}_n = \sum_{n=1}^{n_0} \oint_{\Gamma_n} d\mathbf{R} \cdot \mathbf{F}. \tag{11.8}$$

Due to known properties of curvilinear integrals, the sum of integrals in (11.8) can be replaced by one integral along the closed contour Γ_0 enclosing the lines of all n_0 dislocations

$$\mathbf{B} = \oint_{\Gamma_0} d\mathbf{R} \cdot \mathbf{F}. \tag{11.9}$$

According to [17, 18], we passed from a discrete set of dislocations to their continuous distribution, transforming the curvilinear integral (11.9) to a surface integral using Stokes' formula

$$\mathbf{B} = \iint_{\Sigma_0} \mathbf{N} \cdot \mathrm{Rot}\ \mathbf{F}\, d\Sigma. \tag{11.10}$$

Here Σ_0 is the surface drawn over the contour Γ_0, $\mathbf{N}$ is the unit normal to Σ_0. The expression (11.10) allows to introduce the density of continuously distributed dislocations as a second-order tensor $\boldsymbol{\alpha} \overset{\Delta}{=} \mathrm{Rot}\ \mathbf{F}$. The flux of $\boldsymbol{\alpha}$ through any surface is equal to the total Burgers vector of dislocations crossing this surface. Hereinafter, the dislocation density tensor is considered a given function of Lagrangian coordinates Q^N, which must satisfy the solenoidity condition

$$\mathrm{Div}\ \boldsymbol{\alpha} = 0. \tag{11.11}$$

If dislocations with a given tensor density are distributed in the body, then the displacement field and the vector field $\mathbf{r}$ do not exist. In this case, the third equation (11.4) is replaced by the incompatibility equation

$$\mathrm{Rot}\ \mathbf{F} = \boldsymbol{\alpha} \tag{11.12}$$

and the tensor $\mathbf{F}$ is called the distortion tensor.

The complete system of equilibrium equations for a micropolar elastic body with distributed dislocations contains tensor fields of distortion $\mathbf{F}$ and microrotation $\mathbf{H}$ as unknown functions and consists of the equilibrium equation (11.2), the incompatibility equation (11.12), the constitutive equations (11.3), and the geometric relations $(11.4)_1$, $(11.4)_2$.

11.4 Spherically Symmetric State

We consider an elastic body in the form of a hollow sphere with an outer radius of R_0 and an inner radius of R_1. We introduce the spherical coordinates $Q^1 = R$, $Q^2 = \Phi$ (longitude), $Q^3 = \Theta$ (latitude) using the formulas

$$X_1 = R\cos\Phi\cos\Theta, \quad X_2 = R\sin\Phi\cos\Theta, \quad X_3 = R\sin\Theta.$$

In what follows, we will use an orthonormal vector basis $\mathbf{e}_R$, $\mathbf{e}_\Phi$, $\mathbf{e}_\Theta$ consisting of unit vectors tangent to coordinate lines

$$\begin{aligned}
\mathbf{e}_R &= (\mathbf{i}_1\cos\Phi + \mathbf{i}_2\sin\Phi)\cos\Theta + \mathbf{i}_3\sin\Theta,\\
\mathbf{e}_\Phi &= -\,\mathbf{i}_1\sin\Phi + \mathbf{i}_2\cos\Phi,\\
\mathbf{e}_\Theta &= -\,(\mathbf{i}_1\cos\Phi + \mathbf{i}_2\sin\Phi)\sin\Theta + \mathbf{i}_3\cos\Theta.
\end{aligned} \tag{11.13}$$

According to [19] a second-order tensor field $\mathbf{S}$ will be called spherically symmetric, if its components in the basis $\mathbf{e}_R$, $\mathbf{e}_\Phi$, $\mathbf{e}_\Theta$ on each spherical surface $R = const$ are identical in all points of the surface and the tensor $\mathbf{S}$ is invariant with respect to rotations around the radial axis, i.e., around the vector $\mathbf{e}_R$. A general representation of the spherically symmetric second-order tensor field has the form [19]

$$\mathbf{S} = S_1(R)\mathbf{g} + S_2(R)\mathbf{d} + S_3(R)\mathbf{e}_R\otimes\mathbf{e}_R \tag{11.14}$$
$$\mathbf{g} = \mathbf{e}_\Phi\otimes\mathbf{e}_\Phi + \mathbf{e}_\Theta\otimes\mathbf{e}_\Theta, \quad \mathbf{d} = \mathbf{e}_\Phi\otimes\mathbf{e}_\Theta - \mathbf{e}_\Theta\otimes\mathbf{e}_\Phi.$$

Using Eqs. (11.13)–(11.14), the following relations are proved:

$$\begin{aligned}\text{Rot }\mathbf{S} &= \frac{1}{R}\frac{d}{dR}\Big(RS_2\Big)\mathbf{g} + \left(\frac{S_3 - S_1}{R} - \frac{dS_1}{dR}\right)\mathbf{d} + \frac{2}{R}S_2\mathbf{e}_R \otimes \mathbf{e}_R \\ \text{Div }\mathbf{S} &= \left[\frac{dS_3}{dR} + \frac{2}{R}\Big(S_3 - S_1\Big)\right]\mathbf{e}_R.\end{aligned} \tag{11.15}$$

We assume that there are distributed dislocations in the sphere and their tensor density is

$$\boldsymbol{\alpha} = \alpha_1(R)\mathbf{g} + \alpha_2(R)\mathbf{d} + \alpha_3(R)\mathbf{e}_R \otimes \mathbf{e}_R. \tag{11.16}$$

The first summand in Eq. (11.16) describes the distribution of screw dislocations whose lines coincide with meridians and parallels. The second summand corresponds to a spherically symmetric distribution of edge dislocations. And the third one describes the distribution of screw dislocations with a radial axis.

Considering the equilibrium problem of an elastic sphere with distributed dislocations, unknown functions $\mathbf{F}$ and $\mathbf{H}$ will be sought in the form of spherically symmetric tensor fields

$$\mathbf{F} = F_1(R)\mathbf{g} + F_2(R)\mathbf{d} + F_3(R)\mathbf{e}_R \otimes \mathbf{e}_R \tag{11.17}$$

$$\mathbf{H} = H_1(R)\mathbf{g} + H_2(R)\mathbf{d} + H_3(R)\mathbf{e}_R \otimes \mathbf{e}_R \tag{11.18}$$

The requirements $\mathbf{H} \cdot \mathbf{H}^{\mathrm{T}} = \mathbf{I}$, $\det \mathbf{H} = 1$, meaning that the rotation tensor is proper orthogonal, lead to the equalities $H_1^2 + H_2^2 = 1$, $H_3 = 1$. The expression (11.18) takes the form

$$\mathbf{H} = \cos\chi(R)\,\mathbf{g} + \sin\chi(R)\,\mathbf{d} + \mathbf{e}_R \otimes \mathbf{e}_R. \tag{11.19}$$

The geometrical meaning of Eq. (11.19) is that each elementary volume of the elastic sphere rotates around a radial axis by an angle $\chi(R)$.

Using Eqs. (11.4), (11.17), (11.19), we find the strain and wryness tensors

$$\begin{aligned}\mathbf{E} &= (F_1\cos\chi + F_2\sin\chi)\,\mathbf{g} + (F_2\cos\chi - F_1\sin\chi)\,\mathbf{d} + F_3\mathbf{e}_R \otimes \mathbf{e}_R \\ \mathbf{L} &= \frac{\sin\chi}{R}\mathbf{g} + \frac{\cos\chi - 1}{R}\mathbf{d} + \frac{d\chi}{dR}\mathbf{e}_R \otimes \mathbf{e}_R.\end{aligned} \tag{11.20}$$

Considering a micropolar medium to be isotropic, let $\mathbf{Q} = \mathbf{Q}_1 = 2\mathbf{e}_R \otimes \mathbf{e}_R - \mathbf{I}$, $\det \mathbf{Q}_1 = 1$ in relations (11.5). In accordance with (11.20) we obtain

$$\mathbf{Q}_1^{\mathrm{T}} \cdot \mathbf{E} \cdot \mathbf{Q}_1 = \mathbf{E}, \quad \mathbf{Q}_1^{\mathrm{T}} \cdot \mathbf{L} \cdot \mathbf{Q}_1 = \mathbf{L} \tag{11.21}$$

From Eqs. (11.5) and (11.21) follow the equalities

$$\mathbf{Q}_1 \cdot \mathbf{P} = \mathbf{P} \cdot \mathbf{Q}_1, \quad \mathbf{Q}_1 \cdot \mathbf{K} = \mathbf{K} \cdot \mathbf{Q}_1 \tag{11.22}$$

whence follows

$$\begin{aligned}\mathbf{P} &= P_{RR}\,\mathbf{e}_R\otimes\mathbf{e}_R + P_{\Phi\Phi}\,\mathbf{e}_\Phi\otimes\mathbf{e}_\Phi \\ &\quad + P_{\Phi\Theta}\,\mathbf{e}_\Phi\otimes\mathbf{e}_\Theta + P_{\Theta\Phi}\,\mathbf{e}_\Theta\otimes\mathbf{e}_\Phi + P_{\Theta\Theta}\,\mathbf{e}_\Theta\otimes\mathbf{e}_\Theta \\ \mathbf{K} &= K_{RR}\,\mathbf{e}_R\otimes\mathbf{e}_R + K_{\Phi\Phi}\,\mathbf{e}_\Phi\otimes\mathbf{e}_\Phi \\ &\quad + K_{\Phi\Theta}\,\mathbf{e}_\Phi\otimes\mathbf{e}_\Theta + K_{\Theta\Phi}\,\mathbf{e}_\Theta\otimes\mathbf{e}_\Phi + K_{\Theta\Theta}\,\mathbf{e}_\Theta\otimes\mathbf{e}_\Theta.\end{aligned} \tag{11.23}$$

Relations similar to Eqs. (11.21), (11.22) are also satisfied for the tensor $\mathbf{Q} = \mathbf{Q}_2 = \mathbf{e}_R\otimes\mathbf{e}_R + \mathbf{d}$, $\det\mathbf{Q}_2 = 1$, whence we have

$$P_{\Phi\Phi} = P_{\Theta\Theta}, \quad P_{\Phi\Theta} = -P_{\Theta\Phi}, \quad K_{\Phi\Phi} = K_{\Theta\Theta}, \quad K_{\Phi\Theta} = -K_{\Theta\Phi} \tag{11.24}$$

Equations (11.23), (11.24) mean that upon deformation in the form of (11.17) and (11.18) the stress and couple stress tensors for any isotropic homogeneous material are spherically symmetrical

$$\begin{aligned}\mathbf{P} &= P_1(R)\mathbf{g} + P_2(R)\mathbf{d} + P_3(R)\mathbf{e}_R\otimes\mathbf{e}_R \\ \mathbf{K} &= K_1(R)\mathbf{g} + K_2(R)\mathbf{d} + K_3(R)\mathbf{e}_R\otimes\mathbf{e}_R.\end{aligned} \tag{11.25}$$

In accordance with Eqs. (11.3), (11.17), (11.19) the stress and couple stress tensors of the first Piola–Kirchhoff type **D** and **G**, respectively, are also spherically symmetric and take the form

$$\begin{aligned}\mathbf{D} &= D_1(R)\mathbf{g} + D_2(R)\mathbf{d} + D_3(R)\mathbf{e}_R\otimes\mathbf{e}_R \\ \mathbf{G} &= G_1(R)\mathbf{g} + G_2(R)\mathbf{d} + G_3(R)\mathbf{e}_R\otimes\mathbf{e}_R\end{aligned} \tag{11.26}$$

$$D_1 = P_1\cos\chi - P_2\sin\chi, \quad D_2 = P_1\sin\chi + P_2\cos\chi, \quad D_3 = P_3$$
$$G_1 = K_1\cos\chi - K_2\sin\chi, \quad G_2 = K_1\sin\chi + K_2\cos\chi, \quad G_3 = K_3$$

We suppose that mass loads are defined by spherically symmetric vector fields

$$\mathbf{f} = f(R)\mathbf{e}_R, \quad \mathbf{l} = l(R)\mathbf{e}_R \tag{11.27}$$

According to Eqs. (11.15), (11.17), (11.26) and (11.27), vector equilibrium equations (11.2) are reduced to two scalar equations

$$\frac{dD_3}{dR} + \frac{2(D_3 - D_1)}{R} + \rho f(R) = 0 \tag{11.28}$$

$$\frac{dG_3}{dR} + \frac{2(G_3 - G_1)}{R} + 2F_1 D_2 + \rho\, l(R) = 0 \tag{11.29}$$

Tensor incompatibility equation (11.12) in the case of a spherically symmetric deformation, by virtue of Eqs. (11.15)–(11.17), is equivalent to three scalar equations

$$2F_2 = R\alpha_3, \quad \frac{d}{dR}\Big(RF_2\Big) = R\alpha_1, \quad F_3 = \frac{d}{dR}\Big(RF_1\Big) + R\alpha_2 \tag{11.30}$$

The solenoidity condition (11.11) of the dislocation density tensor according to Eq. (11.15) leads to the following restriction on components of the tensor $\boldsymbol{\alpha}$

$$\alpha_1 = \alpha_3 + \frac{1}{2}R\frac{d\alpha_3}{dR}. \tag{11.31}$$

Note that the density of edge dislocations α_2 is not included in Eq. (11.31). This means that $\alpha_2(R)$ can be an arbitrary function, including the Dirac delta function. Since the dislocation densities $\alpha_m(R)$, $m = 1, 2, 3$ are considered to be given functions, the first Eq. (11.30) defines the function $F_2(R)$; the second Eq. (11.30) is the result of the first one and Eq. (11.31). The third Eq. (11.30) expresses the function $F_3(R)$ through the function $F_1(R)$. The latter remains an unknown function. The values D_m, G_n, $(m, n = 1, 2, 3)$ are expressed through this function as well as through another unknown function $\chi(R)$ using the constitutive equations for isotropic material. Hence the equilibrium equations (11.28), (11.29) form a system of nonlinear ordinary differential equations with respect to $F_1(R)$, $\chi(R)$. Mass loads $f(R)$, $l(R)$ are considered to be given functions.

We assume that an elastic sphere is loaded with uniformly distributed pressure: p_0 on the outer surface $R = R_0$ and p_1 on the inner surface $R = R_1$. In addition, the surfaces of the sphere is loaded with uniformly distributed torque with intensities m_0 and m_1 per unit surface area of the deformed body. Therefore the boundary conditions for the specified system of equations are as follows:

$$\begin{aligned} D_3 &= -p_0F_1^2, \quad G_3 = m_0F_1^2 \quad \text{at } R = R_0 \\ D_3 &= -p_1F_1^2, \quad G_3 = m_1F_1^2 \quad \text{at } R = R_1. \end{aligned} \tag{11.32}$$

Thus, the problem of large deformations of an elastic sphere with a spherically symmetric dislocation distribution is reduced to a nonlinear boundary value problem for a system of ordinary differential equations in the case of an arbitrary isotropic micropolar material.

11.5 Physically Linear Material

As a specific model of an isotropic micropolar body, we consider a physically linear material [2, 7, 8] for which the specific strain energy is the quadratic form of tensors $\mathbf{E} - \mathbf{I}$ and $\mathbf{L}$, while tensors $\mathbf{P}$ and $\mathbf{K}$ are linear functions of these tensors.

$$\begin{gathered}2W=\lambda\,\mathrm{tr}^2\,(\mathbf{E}-\mathbf{I})+(\mu+\varkappa)\,\mathrm{tr}\left[(\mathbf{E}-\mathbf{I})\cdot\left(\mathbf{E}^{\mathrm{T}}-\mathbf{I}\right)\right]\\ +(\mu-\varkappa)\,\mathrm{tr}\,(\mathbf{E}-\mathbf{I})^2+\gamma_1\,\mathrm{tr}^2\,\mathbf{L}+\gamma_2\,\mathrm{tr}\left(\mathbf{L}\cdot\mathbf{L}^{\mathrm{T}}\right)+\gamma_3\,\mathrm{tr}\,\mathbf{L}^2.\\ \mathbf{P}=\lambda\mathbf{I}\,\mathrm{tr}\,(\mathbf{E}-\mathbf{I})+(\mu+\varkappa)\,(\mathbf{E}-\mathbf{I})+(\mu-\varkappa)\left(\mathbf{E}^{\mathrm{T}}-\mathbf{I}\right),\\ \mathbf{K}=\gamma_1\mathbf{I}\,\mathrm{tr}\,\mathbf{L}+\gamma_2\mathbf{L}+\gamma_3\mathbf{L}^{\mathrm{T}}.\end{gathered}\tag{11.33}$$

Here λ, μ, $\varkappa$, γ_1, γ_2, γ_3 are material constants. For a spherically symmetric state based on Eqs. (11.3), (11.17)–(11.20), (11.26), (11.33) we get

$$\begin{aligned}D_1&=\lambda(F_3+2F_1\cos\chi-3)\cos\chi+2\mu\,(F_1\cos\chi-1)\cos\chi\\&\quad+2\varkappa F_1\sin^2\chi+(\lambda+\mu-\varkappa)\,F_2\sin 2\chi\\ D_2&=\lambda(F_3+2F_1\cos\chi-3)\sin\chi+2\mu\,(F_1\cos\chi-1)\sin\chi\\&\quad-\varkappa F_1\sin 2\chi+2\lambda F_2\sin^2\chi+(\mu+\varkappa)F_2-(\mu-\varkappa)F_2\cos 2\chi\\ D_3&=\lambda(F_3+2F_1\cos\chi-3)+2\mu\,(F_3-1)+2\lambda F_2\sin\chi\\ G_1&=\frac{1}{R}\left[\gamma_1\left(R\cos\chi\frac{d\chi}{dR}+\sin 2\chi\right)+\gamma_2\sin\chi+\gamma_3\,(\sin 2\chi-\sin\chi)\right]\\ G_2&=\frac{1}{R}\left[\gamma_1\sin\chi\left(R\frac{d\chi}{dR}+2\sin\chi\right)+\gamma_2\Big(1-\cos\chi\Big)+\gamma_3\,(\cos\chi-\cos 2\chi)\right]\\ G_3&=\Big(\gamma_1+\gamma_2+\gamma_3\Big)\frac{d\chi}{dR}+\frac{2\gamma}{R}\sin\chi\end{aligned}\tag{11.34}$$

Relations (11.28)–(11.32), (11.34) form a nonlinear boundary value problem of the equilibrium of a sphere with distributed dislocations in the case of a physically linear micropolar material.

11.6 Exact Solution of the Eigenstress Problem

Eigenstresses in the theory of dislocations are called stresses in an elastic body that are caused only by dislocations but with no any of external loads. Assuming $f=0$, $l=0$, $p_0=0$, $p_1=0$, $m_0=0$, $m_1=0$ in Eqs. (11.28), (11.29), (11.32), (11.34), we obtain the formulation of the boundary spherically symmetric eigenstress problem for physically linear micropolar material.

We consider a special case about the distribution of dislocations when $\alpha_1(R)=0$, $\alpha_3(R)=0$, and $\alpha_2(R)$ is an arbitrary function. Then, by virtue of (11.30), we obtain $F_2(R)=0$. The couple equilibrium equation (11.29) is satisfied if $\chi(R)=0$. Indeed, then according to Eq. (11.34) we get $D_2=0$, $G_1=G_2=G_3=0$, and stress is expressed as follows:

$$D_1 = \frac{2\mu}{1-2\nu}\big(F_1 + \nu F_3 - 1 - \nu\big),$$
$$D_3 = \frac{2\mu}{1-2\nu}\big[2\nu F_1 + (\nu - 1)F_3 - 1 - \nu\big] \tag{11.35}$$
$$\nu = \frac{\lambda}{2(\lambda+\mu)}$$

Inverting Eq. (11.35) we get

$$F_1 = \frac{(1-\nu)D_1 - \nu D_3}{2\mu(1+\nu)} + 1, \quad F_3 = \frac{D_3 - 2\nu D_1}{2\mu(1+\nu)} + 1. \tag{11.36}$$

The equilibrium equation (11.28), the incompatibility equation $(11.30)_3$, and expressions (11.36) give the differential equation for the function $D_3(R)$

$$\frac{d^2 D_3}{dR^2} + \frac{4}{R}\frac{dD_3}{dR} = \beta(R), \quad \beta(R) \triangleq \frac{4\mu(1+\nu)\alpha_2(R)}{(\nu-1)R}. \tag{11.37}$$

Equation (11.37) has the following general solution

$$D_3(R) = -\frac{1}{3R^3}\int_{R_1}^{R} r^4\beta(r)\,dr + \frac{1}{3}\int_{R_1}^{R} r\beta(r)\,dr + C_1 + C_2 R^{-3}. \tag{11.38}$$

Constants C_1 and C_2 are determined from the boundary conditions $D_3(R_0) = 0$, $D_3(R_1) = 0$ and have the form

$$C_1 = -\frac{C_2}{R_1^3}, \quad C_2 = \frac{R_0^3 R_1^3}{3(R_0^3 - R_1^3)}\left[\int_{R_1}^{R_0} r\beta(r)\,dr - \frac{1}{R_0^3}\cdot\int_{R_1}^{R_0} r^4\beta(r)\,dr\right] \tag{11.39}$$

The stress $D_1(R)$ is expressed via solution (11.38), (11.39) using Eq. (11.28), and then the distortion components are found on the basis of Eq. (11.36).

The obtained exact solution of the nonlinear eigenstress problem is characterized by the absence of couple stresses. It turns out that there is another exact solution of this eigenstress problem in which the couple stresses are not identically equal to zero. In this solution, the microrotation field is a half-turn (i.e., a 180-degree rotation) around the radial axis. This field of microrotations corresponds to the equality $\chi(R) = \pm\pi$. As it follows from Eq. (11.34), with $\cos\chi = -1$ we obtain $G_1 = 0, G_2 \neq 0, G_3 = 0$, $D_2 = 0$ and the couple equilibrium equation (11.29) for $l = 0$ is satisfied, as well as the couple boundary conditions (11.32) for $m_0 = m_1 = 0$. The stress D_1 and D_3 at $\cos\chi = -1$ on the basis of Eq. (11.34) are written as follows:

$$
\begin{aligned}
D_1 &= \frac{2\mu\nu}{1-2\nu}\big(F_1 - \nu F_3 + 1 + \nu\big),\\
D_3 &= \frac{2\mu\nu}{1-2\nu}\big[(1-\nu)F_3 - 2\nu F_1 - 1 - \nu\big]
\end{aligned}
\tag{11.40}
$$

From Eq. (11.40) we get

$$
F_1 = \frac{(1-\nu)D_1 + \nu D_3}{2\mu(1+\nu)} - 1, \quad F_3 = \frac{2\nu D_1 + D_3}{2\mu(1+\nu)} + 1. \tag{11.41}
$$

The equation for $D_3(R)$ arising from Eq. (11.28), the third of Eq. (11.30), and Eq. (11.41) will take the form

$$
\begin{gathered}
\frac{d^2 D_3}{dR^2} + \frac{4}{R}\frac{dD_3}{dR} - \frac{4\nu D_3}{(1-\nu)R^2} = \gamma(R),\\
\gamma(R) \triangleq \frac{4\mu(1+\nu)}{(1-\nu)R^2}\Big[2 - R\alpha_2(R)\Big]
\end{gathered}
\tag{11.42}
$$

Equation (11.42) has the following solution

$$
\begin{aligned}
D_3(R) = \sqrt{\frac{1-\nu}{9+7\nu}}\Bigg[& R^{\lambda_1}\int_{R_1}^{R} r^{\lambda_2+4}\gamma(r)\,dr\\
& -R^{\lambda_2}\int_{R_1}^{R} r^{\lambda_1+4}\gamma(r)\,dr + B_1 R^{\lambda_1} + B_2 R^{\lambda_2}\Bigg]\\
\lambda_1 = -\frac{3}{2} + \frac{1}{2}\sqrt{\frac{9+7\nu}{1-\nu}}, & \quad \lambda_2 = -\frac{3}{2} - \frac{1}{2}\sqrt{\frac{9+7\nu}{1-\nu}}
\end{aligned}
\tag{11.43}
$$

Constants B_1, B_2 are determined from the boundary conditions $D_3(R_1) = D_3(R_0) = 0$.

Thus, within the framework of a physically linear micropolar material model, two exact solutions are found for eigenstresses in a nonlinearly elastic sphere with boundary dislocations distributed with density $\boldsymbol{\alpha} = \alpha_2(R)(\mathbf{e}_\Phi \otimes \mathbf{e}_\Theta - \mathbf{e}_\Theta \otimes \mathbf{e}_\Phi)$ where $\alpha_2(R)$ is an arbitrary function. One of these solutions is characterized by the absence of couple stresses. In the other solution, couple stresses are non-zero and are described by the formula

$$
\mathbf{G} = \frac{2(\gamma_2 - \gamma_3)}{R}\Big(\mathbf{e}_\Phi \otimes \mathbf{e}_\Theta - \mathbf{e}_\Theta \otimes \mathbf{e}_\Phi\Big) \tag{11.44}
$$

11.7 Conclusion

In this paper, the stress state at large spherically symmetric deformations of an elastic sphere made of a micropolar material is studied. A spherically symmetric distribution of screw and edge dislocations is specified in the sphere. Hydrostatic pressures and distributed couple loads are applied on the outer and inner surfaces of the sphere. Using the theory of spherically symmetric tensor fields and the properties of isotropic tensor functions, we reduce the original three-dimensional problem to a boundary value problem for a nonlinear system of ordinary differential equations. The problem is reduced to ordinary differential equations for an arbitrary isotropic micropolar elastic material. As a special case, the eigenstress problem is studied, that is, the problem of a self-balanced stress state that exists with no external force and couple loads. Within the framework of the physically linear micropolar medium model, exact analytical solutions to the eigenstresses problem in a sphere caused by a spherically symmetric distribution of edge dislocations are found. It is established that for the same dislocation density, there are two self-balanced spherical symmetric states of the sphere. In one of them, couple stresses are identical to zero, and in the other there is a non-trivial field of couple stresses.

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