Chapter 5
On the Dynamics of the Electron

Introduction

On first consideration it seemed that the aberration of light and the optical and electrical phenomena associated with it were going to provide us a means for determining the absolute motion of the Earth or more accurately its motion, not with respect to the other stars, but with respect to the ether. Fresnel[1] had already tried it, but he soon recognized that the motion of the Earth did not change the laws of refraction and reflection. Analogous experiments, like that of the water-filled telescope and all those where only first-order terms in the aberration were considered were to give only negative results; the explanation for this was soon found. But, Michelson, who had imagined an experiment sensitive to the terms depending on the square of the aberration, failed in turn.

It seems that this impossibility of showing the absolute motion of the Earth experimentally could be a general law of Nature; we are naturally led to accept this law, that we will call the *Relativity Postulate* and to allow it without restriction. Should this postulate, until now in agreement with experiment, later be confirmed or rejected by more precise experiments, it is in any case interesting to look at what its consequences might be.

An explanation was proposed by Lorentz and Fitz Gerald, who introduced the hypothesis of a contraction experienced by all bodies in the direction of motion of the Earth and proportional to the square of the aberration; this contraction, which we will call the *Lorentz contraction*, took into account the Michelson experiment and all those which had been done until now. The hypothesis would become insufficient, however, if the relativity postulate were to be accepted in its full generality.

H. Poincaré (Paris)
Session of July 23, 1905
Poincaré, H. (1906). Sur la dynamique de l'électron. *Rendiconti del circolo matematico di Palermo, 21*, 129–176.

B. D. Popp, *Henri Poincaré: Electrons to Special Relativity*,
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Lorentz sought to supplement it and amend it so as to bring it into full agreement with this postulate. This is what he succeeded in doing in his article entitled *Electromagnetic Phenomena in a System Moving with Any Velocity Smaller than that of Light (*Proceedings of the Amsterdam Academy, May 27, 1904)[2].

The importance of the question led me to take it up again; the results that I obtained are in agreement with those of Lorentz on all important points; I was only led to amend and supplement them in some points of detail. The differences, which are of secondary importance, will be seen later.

Lorentz's idea can be summarized as follows: if one can, without any visible phenomenon being modified, give any system a shared translation, it is because the equations of the electromagnetic environment are not altered by certain transformations, which we will call *Lorentz transformations*; two systems, the one stationary and the other in translation, thus become the exact image of each other.

Langevin[1] had sought to modify Lorentz's idea; for both authors, the moving electron takes the form of a flattened ellipsoid, but for Lorentz two of the axes of the ellipsoid remain constant and in contrast for Langevin it is the volume of the ellipsoid which remains constant. Both authors additionally showed that these two hypotheses agree with Kaufmann's experiments and also with Abraham's primitive hypothesis (undeformable spherical electron)[3].

The advantage of Langevin's theory is that it does not call on electromagnetic forces and binding forces; but it is incompatible with the relativity postulate. That is what Lorentz had shown; it is what I found in turn by another route by calling on the principles of group theory.

That means it's necessary to go back to Lorentz's theory; but to keep it and avoid intolerable contradictions, a special force has to be assumed which explains both the contraction and the two constant axes. I sought to determine this force, and I found that *it could be compared to a constant external pressure acting on the deformable and compressible electron and its work is proportional to the variations in the volume of this electron.*

If the inertia of matter were then exclusively of electromagnetic origin, as is generally accepted since Kaufmann's experiment, and if all the forces are of electromagnetic origin other than this constant pressure that I just spoke of, then the relativity postulate can be established with full rigor. That is what I show by a very simple calculation based on the principle of least action.

But that isn't all. Lorentz, in the work cited, thought it necessary to supplement his hypothesis such that the postulate is still true when there are forces other than electromagnetic forces. According to him, all forces, whatever their origin, are affected by the Lorentz transformation (and consequently by a translation) in the same way as the electromagnetic forces.

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[1]Langevin had been anticipated by Bucherer from Bonn, who came out with the same idea before him. (*See*: Bucherer, *Mathematische Einführung in die Elektronentheorie*; August 1904. Teubner, Leipzig).

It is important to examine this hypothesis more closely and in particular to seek what modifications it would force us to make to the laws of gravitation.

First it is found that it would force us to assume that the propagation of gravitation is not instantaneous but occurs at the speed of light. One could think that this is a sufficient reason to reject the hypothesis, since Laplace had proven that it could not be so. But in reality, the effect of this propagation is in large part compensated by a different cause, such that there is no contradiction between the proposed law and astronomical observations.

Would it be possible to find a law, which satisfies the condition imposed by Lorentz and at the same time reduced to Newton's law any time that the speeds of the stars are small enough that their squares can be neglected (as well as the product of the accelerations by the distances) compared to the square of the speed of light?

The answer to this question must be affirmative as will be seen later.

Is the law thus modified compatible with astronomical observations?

At first glance, it seems so, but the question will only be settled by an in-depth discussion.

But even if we accept that this discussion is settled in favor of the new hypothesis, what will we have to conclude from it? If the attraction propagates with the speed of light, that cannot be because of a fortuitous occurrence, that must be because it is a function of the ether; and then it will be necessary to look into the nature of this function and associate other functions of the fluid with it.

We cannot be satisfied with formulas that are simply juxtaposed and which only happen to agree by lucky chance; said another way, it has to happen because these formulas are mutually involved. The mind would only be satisfied when it believes that it sees the reason for this agreement to such an extent that it has the illusion that it could have anticipated it.

But the question can also be presented from another point of view so that a comparison will be better understood. Let us imagine an astronomer before Copernicus who was thinking about the Ptolemaic system; he would notice that for all the planets one of the two circles, epicycle or deferent, is traversed in the same time. That cannot be by chance; there is therefore some unknown mysterious link between all the planets.

Copernicus, by simply changing the coordinate system regarded as fixed, made this appearance disappear; each planet now describes only one circle and the periods of revolution become independent (until Kepler reestablished the link between them that was thought to have been destroyed).

It is possible that there is something analogous here; if we were to accept the relativity postulate, we would find in the law of gravitation and in the electromagnetic laws a common number which would be the speed of light. We would find it again in other forces of arbitrary origin which can only be explained in two ways:

Either there is nothing in the world that is not of electromagnetic origin.

Or else, this part which would be, to state it that way, shared by all physical phenomena would only be an appearance, something which would arise from our measurement methods. How do we make our measurements? We would start to say, by transporting one or another of the objects regarded as invariable solids; but that is no longer true in the current theory, if the Lorentz contraction is accepted. In this theory, two equal lengths are, by definition, two lengths that light takes the same time to traverse.

Perhaps it would suffice to renounce this definition so that Lorentz's theory was as completely overthrown as was the Ptolemaic system by the intervention of Copernicus. If that were to happen one day, that would not prove that the effort made by Lorentz was pointless, because Ptolemy, whatever we might think of it, was not useless to Copernicus.

I too have not hesitated to publish these few partial results even though at this moment the whole theory itself might seem to be in danger from the discovery of magnetocathode rays.

§1 – Lorentz Transformation

Lorentz adopted a specific system of units so as to make the factors of 4π disappear from the formulas. I will do the same and additionally I will choose the units of length and time such that the speed of light is equal to one. Under these conditions, by calling: f, g, h the electric displacement; α, β, γ the magnetic force; F, G, H the vector potential; ψ the scalar potential; ρ the electric charge density; ξ, η, ζ the electron velocity; and u, v, w the current, the fundamental formulas become:[4]

$$\begin{gathered}
u = \frac{\mathrm{d}f}{\mathrm{d}t} + \rho\xi = \frac{\mathrm{d}\gamma}{\mathrm{d}y} - \frac{\mathrm{d}\beta}{\mathrm{d}z}, \quad \alpha = \frac{\mathrm{d}H}{\mathrm{d}y} - \frac{\mathrm{d}G}{\mathrm{d}z}, \quad f = -\frac{\mathrm{d}F}{\mathrm{d}t} - \frac{\mathrm{d}\psi}{\mathrm{d}x}, \\
\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{\mathrm{d}g}{\mathrm{d}z} - \frac{\mathrm{d}h}{\mathrm{d}y}, \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} + \sum \frac{\mathrm{d}\rho\xi}{\mathrm{d}x} = 0, \sum \frac{\mathrm{d}f}{\mathrm{d}x} = \rho, \quad \frac{\mathrm{d}\psi}{\mathrm{d}t} + \sum \frac{\mathrm{d}F}{\mathrm{d}x} = 0, \\
\square = \Delta - \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \sum \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\mathrm{d}^2}{\mathrm{d}t^2}, \quad \square\psi = -\rho, \quad \square F = -\rho\xi.
\end{gathered} \tag{1}$$

An element of matter of volume $\mathrm{d}x\mathrm{d}y\mathrm{d}z$ experiences a mechanical force whose components $X\mathrm{d}x\mathrm{d}y\mathrm{d}z$, $Z\mathrm{d}x\mathrm{d}y\mathrm{d}z$, $Y\mathrm{d}x\mathrm{d}y\mathrm{d}z$ are determined from the formula:

$$X = \rho f + \rho(\eta\gamma - \zeta\beta). \tag{2}$$

These equations are subject to a remarkable transformation discovered by Lorentz and which is of interest because it explains why no experiment is able to let us know the absolute motion of the universe. Let us set:

$$x' = kl(x + \varepsilon t), t' = kl(t + \varepsilon x), y' = ly, z' = lz, \tag{3}$$

where l and ε are arbitrary constants, and where

$$k = \frac{1}{\sqrt{1-\varepsilon^2}}.$$

If we then set:

$$\Box' = \sum \frac{d^2}{dx'^2} - \frac{d^2}{dt'^2},$$

it will follow:

$$\Box' = \Box l^{-2}$$

Now consider a sphere driven in a motion of uniform translation with the electron and let:

$$(x-\xi t)^2 + (y-\eta t)^2 + (z-\zeta t)^2 = r^2$$

be the equation of this mobile sphere whose volume will be $\frac{4}{3}\pi r^3$.

The transformation will change it into an ellipsoid whose equation is easy to find. It is in fact easily deduced from equations (3):

$$x = \frac{k}{l}(x' - \varepsilon t'), \quad t = \frac{k}{l}(t' - \varepsilon x'), \quad y = \frac{y'}{l}, \quad z = \frac{z'}{l}. \tag{3'}$$

The equation for the ellipsoid then becomes:

$$k^2(x' - \varepsilon t' + \varepsilon\xi x')^2 + (y' - \eta k t' + \eta k \varepsilon x')^2 + (z' - \zeta k t' + \zeta k \varepsilon x')^2 = l^2 r^2.$$

This ellipsoid moves with a uniform motion; for $t' = 0$, it reduces to

$$k^2 x'^2 (1 + \varepsilon\xi)^2 + (y' + \eta k \varepsilon x')^2 + (z' + \zeta k \varepsilon x')^2 = l^2 r^2$$

and its volume is:

$$\frac{4}{3}\pi r^3 \frac{l^3}{k(1+\xi\varepsilon)}.$$

If we want the charge of an electron to be unchanged by the transformation and if we call ρ' the new electric charge density, it will follow:

$$\rho' = \frac{k}{l^3}(\rho + \varepsilon\rho\xi). \tag{4}$$

What will the new speeds ξ', η', ζ' be? It will have to be:

$$\xi' = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\mathrm{d}(x+\varepsilon t)}{\mathrm{d}(t+\varepsilon x)} = \frac{\xi+\varepsilon}{1+\varepsilon\xi},$$
$$\eta' = \frac{\mathrm{d}y'}{\mathrm{d}t'} = \frac{\mathrm{d}y}{k\mathrm{d}(t+\varepsilon x)} = \frac{\eta}{k(1+\varepsilon\xi)}, \quad \zeta' = \frac{\zeta}{k(1+\varepsilon\xi)}$$

hence

$$\rho'\xi' = \frac{k}{l^3}(\rho\xi + \varepsilon\rho), \quad \rho'\eta' = \frac{1}{l^3}\rho\eta, \quad \rho'\zeta' = \frac{1}{l^3}\rho\zeta. \tag{4'}$$

Here is where I need to indicate for the first time a divergence from Lorentz. Lorentz set (up to differences in notation; *loc. cit.*, page 813, formulas 7 and 8[5]):

$$\rho' = \frac{1}{kl^3}\rho, \quad \xi' = k^2(\xi+\varepsilon), \quad \eta' = k\eta, \quad \zeta' = k\zeta\,.$$

That way the formulas:

$$\rho'\xi' = \frac{k}{l^3}(\rho\xi + \varepsilon\rho), \quad \rho'\eta' = \frac{1}{l^3}\rho\eta, \quad \rho'\zeta' = \frac{1}{l^3}\rho\zeta;$$

are found, but the value of ρ' is different.

It needs to be noted that the formulas (4) and (4′) satisfy the continuity condition

$$\frac{\mathrm{d}\rho'}{\mathrm{d}t'} + \sum \frac{\mathrm{d}\rho'\xi'}{\mathrm{d}x'} = 0.$$

In fact, let λ be an undetermined quantity and D the functional determinant of

$$t+\lambda\rho, \quad x+\lambda\rho\xi, \quad y+\lambda\rho\eta, \quad z+\lambda\rho\zeta \tag{5}$$

with respect to t, x, y and z. It will follow:

$$D = D_0 + D_1\lambda + D_2\lambda^2 + D_3\lambda^3 + D_4\lambda^4.$$

with $D_0 = 1$ and $D_1 = \mathrm{d}\rho/\mathrm{d}t + \sum \mathrm{d}\rho\xi/\mathrm{d}x = 0$.

Let $\lambda' = l^2\lambda$, we see that the four functions

$$t'+\lambda'\rho', \quad x'+\lambda'\rho'\xi', \quad y'+\lambda'\rho'\eta', \quad z'+\lambda'\rho'\zeta' \tag{5'}$$

are related to the functions (5) by the same linear relations as the former variables to the new variables. If the functional determinant of the functions (5′) with respect to the new variables is therefore designated D', it will follow:

$$D' = D, \quad D' = D'_0 + D'_1\lambda' + D'_2\lambda'^2 + D'_3\lambda'^3 + D'_4\lambda'^4,$$

hence:

$$D'_0 = D_0 = 1, D'_1 = l^{-2}D_1 = 0 = \frac{d\rho'}{dt'} + \sum \frac{d\rho'\xi'}{dx'}.$$

which was to be proven.

With the hypothesis from Lorentz, this condition would not be fulfilled, because ρ' does not have the same value.

We will now define new vector and scalar potentials so as to satisfy the conditions:

$$\square'\psi' = -\rho', \quad \square' F' = -\rho'\xi'. \tag{6}$$

From that we next draw:

$$\psi' = \frac{k}{l}(\psi + \varepsilon F), \quad F' = \frac{k}{l}(F + \varepsilon\psi), \quad G' = \frac{1}{l}G, \quad H' = H. \tag{7}$$

These formulas do differ from those of Lorentz, but in the final analysis the divergence only bears on the definitions.

We will choose new electric and magnetic fields so as to satisfy the equations:

$$f' = -\frac{dF'}{dt'} - \frac{d\psi'}{dx'}, \quad \alpha' = \frac{dH'}{dy'} - \frac{dG'}{dz'}. \tag{8}$$

It is easy to see that:

$$\frac{d}{dt'} = \frac{k}{l}\left(\frac{d}{dt} - \varepsilon\frac{d}{dx}\right), \quad \frac{d}{dx'} = \frac{k}{l}\left(\frac{d}{dx} - \varepsilon\frac{d}{dt}\right), \quad \frac{d}{dy'} = \frac{1}{l}\frac{d}{dy}, \quad \frac{d}{dz'} = \frac{1}{l}\frac{d}{dz}$$

and from that to conclude:

$$\begin{aligned} f' &= \frac{1}{l^2}f, \quad g' = \frac{k}{l^2}(g + \varepsilon\gamma), \quad h' = \frac{k}{l^2}(h - \varepsilon\beta), \\ \alpha' &= \frac{1}{l^2}\alpha, \quad \beta' = \frac{k}{l^2}(\beta - \varepsilon h), \quad \gamma' = \frac{k}{l^2}(\gamma + \varepsilon g). \end{aligned} \tag{9}$$

These formulas are identical to Lorentz's.

Our transformation does not alter equations (1). In fact, the continuity condition, and also equations (6) and (8), already provided us some of equations (1) (except for accenting of the letters).

Equations (6) connected with the continuity condition give us:

$$\frac{\mathrm{d}\psi'}{\mathrm{d}t'} + \sum \frac{\mathrm{d}F'}{\mathrm{d}x'} = 0. \tag{10}$$

It remains to establish that:

$$\frac{\mathrm{d}f'}{\mathrm{d}t'} + \rho'\xi' = \frac{\mathrm{d}\gamma'}{\mathrm{d}y'} - \frac{\mathrm{d}\beta'}{\mathrm{d}z'}, \quad \frac{\mathrm{d}\alpha'}{\mathrm{d}t'} = \frac{\mathrm{d}g'}{\mathrm{d}z'} - \frac{\mathrm{d}h'}{\mathrm{d}y'}, \quad \sum \frac{\mathrm{d}f'}{\mathrm{d}x'} = \rho'$$

and it can be easily seen that these are necessary consequences of equations (6), (8) and (10).

We must now compare the forces before and after the transformation.

Let X, Y, Z be the force before and X', Y', Z' be the force after the transformation; with all of them referred to a unit volume. In order for X' to satisfy the same equations as before the transformation, it must hold that:

$$\begin{aligned} X' &= \rho' f' + \rho'(\eta'\gamma' - \zeta'\beta'), \\ Y' &= \rho' g' + \rho'(\zeta'\alpha' - \xi'\gamma'), \\ Z' &= \rho' h' + \rho'(\xi'\beta' - \eta'\alpha'), \end{aligned}$$

or, by replacing the quantities by their values (4), (4′) and (9) while making use of equations (2):

$$\begin{aligned} X' &= \frac{k}{l^5}\left(X + \varepsilon \sum X\xi\right), \\ Y' &= \frac{1}{l^5} Y, \\ Z' &= \frac{1}{l^5} Z. \end{aligned} \tag{11}$$

If we represent the force referred, no longer to the unit volume, but now to the unit electrical charge of the electron, by X_1, Y_1, Z_1 and the same qualities after the transformation by X_1', Y_1', Z_1' we will have:

$$X_1 = f + \eta\gamma - \zeta\beta, \quad X_1' = f' + \eta'\gamma' - \zeta'\beta', \quad X = \rho X_1, \quad X' = \rho' X_1'$$

and we will have the equations:

$$X_1' = \frac{k}{l^5}\frac{\rho}{\rho'}\left(X_1 + \varepsilon \sum X_1 \xi\right),$$
$$Y_1' = \frac{1}{l^5}\frac{\rho}{\rho'} Y_1, \qquad (11')$$
$$Z_1' = \frac{1}{l^5}\frac{\rho}{\rho'} Z_1.$$

Lorentz had found (within the difference of notation, page 813, formula (10)):

$$X_1 = l^2 X_1' - l^2 \varepsilon(\eta' g' + \zeta' h'),$$
$$Y_1 = \frac{l^2}{k} Y_1' + \frac{l^2 \varepsilon}{k} \xi' g', \qquad (11'')$$
$$Z_1 = \frac{l^2}{k} Z_1' + \frac{l^2 \varepsilon}{k} \xi' h',$$

Before going farther, the cause of this significant divergence must be found. It obviously means that the formulas for ξ', η', ζ' are not the same, even though the formulas for the electric and magnetic fields are the same.

If the inertia of the electrons is exclusively of electromagnetic origin and if additionally they are only subject to forces of electromagnetic origin, then the equilibrium condition requires that inside the electrons it hold:

$$X = Y = Z = 0.$$

Hence, in light of equations (11), these relations are equivalent to

$$X' = Y' = Z' = 0.$$

The equilibrium conditions of the electrons are therefore unchanged by the transformation.

Unfortunately, such a simple assumption is not allowable. If, in fact, one supposes that $\xi = \eta = \zeta = 0$, the conditions $X = Y = Z = 0$ would lead to $f = g = h = 0$, and consequently $\sum \frac{\mathrm{d}f}{\mathrm{d}x} = 0$, meaning $\rho = 0$. One would arrive at analogous results in the most general case. One therefore has to accept that in addition to electromagnetic forces, there are either other forces or binding. One must then look at what conditions these forces or binding must satisfy for the equilibrium of the electrons to be undisturbed by the transformation. This will be taken up in a subsequent section.

§2 – Principle of Least Action

The way Lorentz deduced his equations from the principle of least action is known[6]. Although I have nothing essential to add to it, I will however go back over the question because I prefer to present it in a slightly different form which will be useful for my purpose. I will set:

$$J = \int \left[\frac{\sum f^2}{2} + \frac{\sum \alpha^2}{2} - \sum Fu \right] \mathrm{d}t\mathrm{d}\tau, \tag{1}$$

by assuming that f, α, F, u, etc. are subject to the following conditions and to those which could be deduced from them by symmetry:

$$\sum \frac{\mathrm{d}f}{\mathrm{d}x} = \rho, \quad \alpha = \frac{\mathrm{d}H}{\mathrm{d}y} - \frac{\mathrm{d}G}{\mathrm{d}z}, \quad u = \frac{\mathrm{d}f}{\mathrm{d}t} + \rho\xi\,. \tag{2}$$

As for the integral J, it must be extended to:

1) the entire space with respect to the element of volume, $\mathrm{d}\tau = \mathrm{d}x\mathrm{d}y\mathrm{d}z$;
2) the limits included between $t = t_0$, $t = t_1$ with respect to time, t.

According to the principle of least action, the integral J must be a minimum if the various quantities which appear in it are subject to:

1) conditions (2);
2) the condition that the state of the system is fixed at the two limit epochs $t = t_0$, $t = t_1$.

This last condition allows us to transform our integrals by integration by parts over time. If we in fact have an integral of the form

$$\int A \frac{\mathrm{d}B\delta C}{\mathrm{d}t} \mathrm{d}t\mathrm{d}\tau,$$

where C is one of the quantities which define the state of the system and δC is its variation, it will be equal (by integrating by parts with respect to time) to:

$$\int |AB\delta C|_{t=t_0}^{t=t_1} \mathrm{d}\tau - \int \frac{\mathrm{d}A}{\mathrm{d}t} \mathrm{d}B\delta C.$$

Since the state of the system is determined at the two limit epochs, $\delta C = 0$ for $t = t_0$, $t = t_1$; therefore the first integral which relates to these two epochs is zero; and only the second remains.

We can similarly integrate by parts relative to x, y or z; we have in fact

$$\int A\frac{\mathrm{d}B}{\mathrm{d}x}\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t = \int AB\mathrm{d}y\mathrm{d}z\mathrm{d}t - \int B\frac{\mathrm{d}A}{\mathrm{d}x}\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t.$$

Since our integrals extend to infinity, in the first integral on the right-hand side x must be made equal to $\pm\infty$; therefore, since we always assume that all our functions become zero at infinity, this integral must be zero and it will follow

$$\int A\frac{\mathrm{d}B}{\mathrm{d}x}\mathrm{d}\tau\mathrm{d}t = -\int B\frac{\mathrm{d}A}{\mathrm{d}x}\mathrm{d}\tau\mathrm{d}t.$$

If the system were assumed subject to binding, it would be necessary to add a binding condition to the conditions imposed on the various quantities appearing in the integral J.

First give F, G, H increments δF, δG, δH; hence:

$$\delta\alpha = \frac{\mathrm{d}\delta H}{\mathrm{d}y} - \frac{\mathrm{d}\delta G}{\mathrm{d}z}.$$

One should have:

$$\delta J = \int\left[\sum\alpha\left(\frac{\mathrm{d}\delta H}{\mathrm{d}y} - \frac{\mathrm{d}\delta G}{\mathrm{d}z}\right) - \sum u\delta F\right]\mathrm{d}t\mathrm{d}\tau = 0,$$

or, by integrating by parts,

$$\begin{aligned}\delta J &= \int\left[\sum\left(\delta G\frac{\mathrm{d}\alpha}{\mathrm{d}z} - \delta H\frac{\mathrm{d}\alpha}{\mathrm{d}y}\right) - \sum u\delta F\right]\mathrm{d}t\mathrm{d}\tau \\ &= -\int\sum\delta F\left(u - \frac{\mathrm{d}\gamma}{\mathrm{d}y} + \frac{\mathrm{d}\beta}{\mathrm{d}z}\right)\mathrm{d}t\mathrm{d}\tau = 0,\end{aligned}$$

hence, by equating the coefficient of the arbitrary δF to zero,

$$u = \frac{\mathrm{d}\gamma}{\mathrm{d}y} - \frac{\mathrm{d}\beta}{\mathrm{d}z}. \tag{3}$$

This relation gives us (with an integration by parts):

$$\begin{aligned}\int\sum Fu\mathrm{d}\tau &= \int\sum F\left(\frac{\mathrm{d}\gamma}{\mathrm{d}y} - \frac{\mathrm{d}\beta}{\mathrm{d}z}\right)\mathrm{d}\tau = \int\sum\left(\beta\frac{\mathrm{d}F}{\mathrm{d}z} - \gamma\frac{\mathrm{d}F}{\mathrm{d}y}\right)\mathrm{d}\tau \\ &= \int\sum\alpha\left(\frac{\mathrm{d}H}{\mathrm{d}y} - \frac{\mathrm{d}G}{\mathrm{d}z}\right)\mathrm{d}\tau,\end{aligned}$$

or

$$\int \sum Fu\mathrm{d}\tau = \int \sum \alpha^2 \mathrm{d}\tau$$

hence finally:

$$J = \int \left(\frac{\sum f^2}{2} - \frac{\sum \alpha^2}{2}\right) \mathrm{d}t\mathrm{d}\tau. \tag{4}$$

Now, and because of the relation (3), δJ is independent of δF and consequently of $\delta\alpha$; let us now vary the other variables.

It follows, by returning to the expression (1) for J,

$$\delta J = \int \left(\sum f\delta f - \sum F\delta u\right) \mathrm{d}t\mathrm{d}\tau.$$

But f, g, h are subject to the first of the conditions (2), such that

$$\sum \frac{\mathrm{d}\delta f}{\mathrm{d}x} = \delta\rho, \tag{5}$$

and which it is appropriate to write:

$$\delta J = \int \left[\sum f\delta f - \sum F\delta u - \psi\left(\sum \frac{\mathrm{d}\delta f}{\mathrm{d}x} - \delta\rho\right)\right] \mathrm{d}t\mathrm{d}\tau. \tag{6}$$

From the principles of calculus of variations, we learn that the calculation must be done as if, ψ being an arbitrary function, δJ were represented by the expression (6) and as if the variations were no longer subject to the condition (5).

We will have additionally

$$\delta u = \frac{\mathrm{d}\delta f}{\mathrm{d}t} + \delta\rho\xi,$$

hence, after integration by parts,

$$\delta J = \int \sum \delta f\left(f + \frac{\mathrm{d}F}{\mathrm{d}t} + \frac{\mathrm{d}\psi}{\mathrm{d}x}\right)\mathrm{d}t\mathrm{d}\tau + \int \left(\psi\delta\rho - \sum F\delta\rho\xi\right)\mathrm{d}t\mathrm{d}\tau. \tag{7}$$

If we first assume that the electrons experience no variation, $\delta\rho = \delta\rho\xi = 0$ and the second integral is zero. Since δJ must become zero, it must follow that:

$$f + \frac{\mathrm{d}F}{\mathrm{d}t} + \frac{\mathrm{d}\psi}{\mathrm{d}x} = 0. \tag{8}$$

In the general case, there rests, therefore:

$$\delta J = \int \left(\psi \delta \rho - \sum F \delta \rho \xi\right) \mathrm{d}t \mathrm{d}\tau. \tag{9}$$

The forces which act on the electrons remain to be determined. To do that we will have to assume that a complementary force $-X\mathrm{d}\tau$, $-Y\mathrm{d}\tau$, $-Z\mathrm{d}\tau$ is applied to each element of the electron and write that this force is in equilibrium with the forces of electromagnetic origin. Let U, V, W be the components of the displacements of the element $\mathrm{d}\tau$ of the electron; this displacement is considered from an arbitrary initial position. Let δU, δV, δW be the variations of this displacement; the virtual work corresponding to the complementary force will be:

$$-\int \sum X \delta U \mathrm{d}\tau,$$

such that the equilibrium condition that we just talked about will be written:

$$\delta J = -\int \sum X \delta U \mathrm{d}\tau \mathrm{d}t. \tag{10}$$

This is a matter of transforming δJ. To do that, we start by looking for the continuity equation expressing that the charge of an electron is conserved by the variation.

Let x_0, y_0, z_0 be the initial position of an electron. Its current position will be

$$x = x_0 + U, \quad y = y_0 + V, \quad z = z_0 + W.$$

We will additionally introduce an auxiliary variable ε, which will produce the variations of our various functions, such that for an arbitrary function A, we will have:

$$\delta A = \delta\varepsilon \frac{\mathrm{d}A}{\mathrm{d}\varepsilon}.$$

It will in fact be useful to be able to switch from the notation of calculus of variations to that of ordinary differential calculus, or vice versa.

It will be possible to regard our functions: first as depending on five variables x, y, z, t, ε, such that the position does not change when only t and ε change—we will designate their derivatives by the ordinary d; second as depending on five variables x_0, y_0, z_0, t, ε, such that a single electron is always followed when only t and ε vary—we will then designate their derivatives by round ∂. We will then have:

$$\xi = \frac{\partial U}{\partial t} = \frac{\mathrm{d}U}{\mathrm{d}t} + \xi \frac{\mathrm{d}U}{\mathrm{d}x} + \eta \frac{\mathrm{d}U}{\mathrm{d}y} + \zeta \frac{\mathrm{d}U}{\mathrm{d}z} = \frac{\partial x}{\partial t}. \tag{11}$$

We now designate by Δ the functional determinant of x, y, z, relative to x_0, y_0, z_0:

$$\Delta = \frac{\partial(x, y, z)}{\partial(x_0, y_0, z_0)}.$$

If, with ε, x_0, y_0, z_0 remaining constant, we give an increase ∂t to t, there will result for x, y, z increases ∂x, ∂y, ∂z and for Δ an increase of $\partial\Delta$ and it will hold that:

$$\partial x = \xi\partial t, \quad \partial y = \eta\partial t, \quad \partial z = \zeta\partial t,$$
$$\Delta + \partial\Delta = \frac{\partial(x + \partial x, y + \partial y, z + \partial z)}{\partial(x_0, y_0, z_0)};$$

hence

$$1 + \frac{\partial\Delta}{\Delta} = \frac{\partial(x + \partial x, y + \partial y, z + \partial z)}{\partial(x, y, z)} = \frac{\partial(x + \xi\partial t, y + \eta\partial t, z + \zeta\partial t)}{\partial(x, y, z)}.$$

From which one can deduce:

$$\frac{1}{\Delta}\frac{\partial\Delta}{\partial \mathrm{t}} = \frac{\mathrm{d}\xi}{\mathrm{d}x} + \frac{\mathrm{d}\eta}{\mathrm{d}y} + \frac{\mathrm{d}\zeta}{\mathrm{d}z}. \tag{12}$$

Since the mass[7] of each electron is invariant, we will have:

$$\frac{\partial\rho\Delta}{\partial t} = 0, \tag{13}$$

hence:

$$\frac{\partial\rho}{\partial t} + \sum\rho\frac{\mathrm{d}\xi}{\mathrm{d}x} = 0, \quad \frac{\partial\rho}{\partial t} = \frac{\mathrm{d}\rho}{\mathrm{d}t} + \sum\xi\frac{\mathrm{d}\rho}{\mathrm{d}x}, \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} + \sum\frac{\mathrm{d}\rho\xi}{\mathrm{d}x} = 0.$$

Such are the various forms of the equation of continuity as it relates to the variable t. We find the analogous forms as it relates to the variable ε. Let:

$$\delta U = \frac{\partial U}{\partial\varepsilon}\delta\varepsilon, \quad \delta V = \frac{\partial V}{\partial\varepsilon}\delta\varepsilon, \quad \delta W = \frac{\partial W}{\partial\varepsilon}\delta\varepsilon;$$

it will follow:

$$\delta U = \frac{\mathrm{d}U}{\mathrm{d}\varepsilon}\delta\varepsilon + \delta U\frac{\mathrm{d}U}{\mathrm{d}x} + \delta V\frac{\mathrm{d}U}{\mathrm{d}y} + \delta W\frac{\mathrm{d}U}{\mathrm{d}z}, \tag{11'}$$

$$\frac{1}{\Delta}\frac{\partial\Delta}{\partial\varepsilon} = \sum\frac{\partial U}{\partial\varepsilon}, \quad \frac{\partial\rho\Delta}{\partial\varepsilon} = 0, \tag{12'}$$

$$\delta\varepsilon\frac{\partial\rho}{\partial\varepsilon}+\sum\rho\frac{\mathrm{d}\delta U}{\mathrm{d}x}=0,\quad \frac{\partial\rho}{\partial\varepsilon}=\frac{\mathrm{d}\rho}{\mathrm{d}\varepsilon}+\sum\frac{\delta U}{\delta\varepsilon}\frac{\mathrm{d}\rho}{\mathrm{d}x},\quad \delta\rho+\frac{\mathrm{d}\rho\delta U}{\mathrm{d}x}=0. \tag{13'}$$

The difference between the definition of $\delta U=\frac{\partial U}{\partial\varepsilon}\delta\varepsilon$ and that of $\delta\rho=\frac{\mathrm{d}\rho}{\mathrm{d}\varepsilon}\delta\varepsilon$ will be noted; it will be noted that it is in fact this definition of δU which is appropriate for the formula (10).

That last equation is going to allow us to transform the first term of (9); in fact we find:

$$\int\psi\delta\rho\mathrm{d}t\mathrm{d}\tau=-\int\psi\sum\frac{\mathrm{d}\rho\delta U}{\mathrm{d}x}\mathrm{d}t\mathrm{d}\tau$$

or, by integrating by parts,

$$\int\psi\delta\rho\mathrm{d}t\mathrm{d}\tau=\int\sum\rho\frac{\mathrm{d}\psi}{\mathrm{d}x}\delta U\mathrm{d}t\mathrm{d}\tau. \tag{14}$$

We now propose to determine:

$$\delta(\rho\xi)=\frac{\mathrm{d}(\rho\xi)}{\mathrm{d}\varepsilon}\delta\varepsilon.$$

We observe the $\rho\Delta$ can only depend on x_0, y_0, z_0; in fact, if an element of electron is considered whose initial position is a rectangular parallelepiped whose edges are $\mathrm{d}x_0, \mathrm{d}y_0, \mathrm{d}z_0$, then the charge of this element is

$$\rho\Delta\mathrm{d}x_0\mathrm{d}y_0\mathrm{d}z_0$$

and, since this charge needs to remain constant, it follows that:

$$\frac{\partial\rho\Delta}{\partial t}=\frac{\partial\rho\Delta}{\partial\varepsilon}=0. \tag{15}$$

From that it is deduced:

$$\frac{\partial^2\rho\Delta}{\partial t\partial\varepsilon}=\frac{\partial}{\partial\varepsilon}\left(\rho\Delta\frac{\partial U}{\partial t}\right)=\frac{\partial}{\partial t}\left(\rho\Delta\frac{\partial U}{\partial\varepsilon}\right). \tag{16}$$

For an arbitrary function A it is known from the equation of continuity that,

$$\frac{1}{\Delta}\frac{\partial A\Delta}{\partial t}=\frac{\mathrm{d}A}{\mathrm{d}t}+\sum\frac{\mathrm{d}A\xi}{\mathrm{d}x}$$

and similarly

$$\frac{1}{\Delta}\frac{\partial A\Delta}{\partial \varepsilon} = \frac{\mathrm{d}A}{\mathrm{d}\varepsilon} + \sum \frac{\mathrm{d}A\,\frac{\partial U}{\partial \varepsilon}}{\mathrm{d}x}$$

Therefore it follows:

$$\frac{1}{\Delta}\frac{\partial}{\partial \varepsilon}\left(\rho\Delta\frac{\partial U}{\partial t}\right) = \frac{\mathrm{d}\rho\frac{\partial U}{\partial t}}{\mathrm{d}\varepsilon} + \frac{\mathrm{d}\left(\rho\frac{\partial U}{\partial t}\frac{\partial U}{\partial \varepsilon}\right)}{\mathrm{d}x} + \frac{\mathrm{d}\left(\rho\frac{\partial U}{\partial t}\frac{\partial V}{\partial \varepsilon}\right)}{\mathrm{d}y} + \frac{\mathrm{d}\left(\rho\frac{\partial U}{\partial t}\frac{\partial W}{\partial \varepsilon}\right)}{\mathrm{d}z} \tag{17}$$

$$\frac{1}{\Delta}\frac{\partial}{\partial t}\left(\rho\Delta\frac{\partial U}{\partial \varepsilon}\right) = \frac{\mathrm{d}\rho\frac{\partial U}{\partial \varepsilon}}{\mathrm{d}t} + \frac{\mathrm{d}\left(\rho\frac{\partial U}{\partial t}\frac{\partial U}{\partial \varepsilon}\right)}{\mathrm{d}x} + \frac{\mathrm{d}\left(\rho\frac{\partial V}{\partial t}\frac{\partial U}{\partial \varepsilon}\right)}{\mathrm{d}y} + \frac{\mathrm{d}\left(\rho\frac{\partial W}{\partial t}\frac{\partial U}{\partial \varepsilon}\right)}{\mathrm{d}z} \tag{17'}$$

The right-hand sides of (17) and (17′) must be equal and, recalling that

$$\frac{\partial U}{\partial t} = \xi, \quad \frac{\partial U}{\partial \varepsilon}\delta\varepsilon = \delta U, \quad \frac{\mathrm{d}\rho\xi}{\mathrm{d}\varepsilon}\delta\varepsilon = \delta\rho\xi,$$

it follows that:

$$\begin{aligned}\delta\rho\xi + \frac{\mathrm{d}(\rho\xi\delta U)}{\mathrm{d}x} + \frac{\mathrm{d}(\rho\xi\delta V)}{\mathrm{d}y} + \frac{\mathrm{d}(\rho\xi\delta W)}{\mathrm{d}z} \\ = \frac{\mathrm{d}(\rho\delta U)}{\mathrm{d}t} + \frac{\mathrm{d}(\rho\xi\delta U)}{\mathrm{d}x} + \frac{\mathrm{d}(\rho\eta\delta U)}{\mathrm{d}y} + \frac{\mathrm{d}(\rho\zeta\delta U)}{\mathrm{d}z}\end{aligned} \tag{18}$$

We now transform the second term from (9) and get:

$$\begin{aligned}\int\sum F\delta\rho\xi\mathrm{d}t\mathrm{d}\tau = \int\Big[\sum F\frac{\mathrm{d}(\rho\delta U)}{\mathrm{d}t} + \sum F\frac{\mathrm{d}(\rho\eta\delta U)}{\mathrm{d}y} + \sum F\frac{\mathrm{d}(\rho\zeta\delta U)}{\mathrm{d}z} \\ - \sum F\frac{\mathrm{d}(\rho\xi\delta V)}{\mathrm{d}y} - \sum F\frac{\mathrm{d}(\rho\xi\delta W)}{\mathrm{d}z}\Big]\mathrm{d}t\mathrm{d}\tau.\end{aligned}$$

By integrating by parts, the right-hand side becomes:

$$\int\left[-\sum\rho\delta U\frac{\mathrm{d}F}{\mathrm{d}t} - \sum\rho\eta\delta U\frac{\mathrm{d}F}{\mathrm{d}y} - \sum\rho\zeta\delta U\frac{\mathrm{d}F}{\mathrm{d}z} + \sum\rho\xi\delta V\frac{\mathrm{d}F}{\mathrm{d}y} + \sum\rho\xi\delta W\frac{\mathrm{d}F}{\mathrm{d}z}\right]\mathrm{d}t\mathrm{d}\tau.$$

Now remark that:

$$\sum\rho\xi\delta V\frac{\mathrm{d}F}{\mathrm{d}y} = \sum\rho\zeta\delta U\frac{\mathrm{d}H}{\mathrm{d}x}, \quad \sum\rho\xi\delta W\frac{\mathrm{d}F}{\mathrm{d}z} = \sum\rho\eta\delta U\frac{\mathrm{d}G}{\mathrm{d}x}.$$

If, in fact, in both sides of these relations, the sums are expanded, they become identities; and let us recall that

$$\frac{dH}{dx} - \frac{dF}{dz} = -\beta, \quad \frac{dG}{dx} - \frac{dF}{dy} = \gamma,$$

the right-hand side in question will become:

$$\int \left[-\sum \rho \delta U \frac{dF}{dt} + \sum \rho \gamma \eta \delta U - \sum \rho \beta \zeta \delta U \right] dt d\tau,$$

such that finally:

$$\delta J = \int \sum \rho \delta U \left(\frac{d\psi}{dx} + \frac{dF}{dt} + \beta\zeta - \gamma\eta \right) dt d\tau = \int \sum \rho \delta U (-f + \beta\zeta - \gamma\eta) dt d\tau.$$

By equating the coefficients of δU in both sides of (10), it follows:

$$X = f - \beta\zeta + \gamma\eta$$

This is equation (2) from the previous section.

§3 – Lorentz Transformation and the Principle of Least Action

We are going to see if the principle of least action gives us the reason for the success of the Lorentz transformation. First it needs to be seen what this transformation does to the integral:

$$J = \int \left(\frac{\sum f^2}{2} - \frac{\sum \alpha^2}{2} \right) dt d\tau$$

(formula 4 from §2).

We first find

$$dt' d\tau' = l^4 dt d\tau$$

because x', y', z', t' are related to x, y, z, t by linear relations whose determinant is equal to l^4; it next follows:

$$\begin{aligned} l^4 \sum f'^2 &= f^2 + k^2(g^2 + h^2) + k^2\varepsilon^2(\beta^2 + \gamma^2) + 2k^2\varepsilon(g\gamma - h\beta) \\ l^4 \sum \alpha'^2 &= \alpha^2 + k^2(\beta^2 + \gamma^2) + k^2\varepsilon^2(g^2 + h^2) + 2k^2\varepsilon(g\gamma - h\beta) \end{aligned} \tag{1}$$

(formulas 9 from §1), hence:

$$l^4\left(\sum f'^2 - \sum \alpha'^2\right) = \sum f^2 - \sum \alpha^2;$$

such that if one sets:

$$J' = \int \left(\frac{\sum f'^2}{2} - \frac{\sum \alpha'^2}{2}\right) \mathrm{d}t'\mathrm{d}\tau'$$

it follows:

$$J' = J.$$

For this equality to be justified, it is however necessary that the limits of integration be the same; until now we have allowed t to vary from t_0 to t_1; and x, y, z to vary from $-\infty$ to $+\infty$. As such, the integration limits would be changed by the Lorentz transformation; but nothing prevents us from assuming $t_0 = -\infty$, $t = +\infty$; with these conditions, the limits of the same for J and for J'.

We now need to compare the following two equations analogous to equation (10) from §2:

$$\begin{aligned} \delta J &= -\int \sum X \delta U \mathrm{d}\tau \mathrm{d}t \\ \delta J' &= -\int \sum X' \delta U' \mathrm{d}\tau' \mathrm{d}t'. \end{aligned} \tag{2}$$

To do that, we first need to compare $\delta U'$ to δU.

Consider an electron whose initial coordinates are x_0, y_0, z_0; at the moment t, these coordinates will be:

$$x = x_0 + U, \quad y = y_0 + V, \quad z = z_0 + W.$$

If the corresponding electron is considered after the Lorentz transformation, its coordinates will be

$$x' = kl(x + \varepsilon t), \quad y' = ly, \quad z' = lz,$$

where

$$x' = x_0 + U', \quad y' = y_0 + V', \quad z' = z_0 + W';$$

but it will only reach these coordinates at the moment

$$t' = kl(t + \varepsilon x)$$

If we were to make our variables undergo variations δU, δV, δW and at the same time we were to give t an increase δt, the coordinates x, y, z will undergo a total increase

$$\delta x = \delta U + \xi\delta t, \quad \delta y = \delta V + \eta\delta t, \quad \delta z = \delta W + \zeta\delta t\,.$$

We will also have:

$$\delta x' = \delta U' + \xi'\delta t', \quad \delta y' = \delta V' + \eta'\delta t', \quad \delta z' = \delta W' + \zeta'\delta t'\,,$$

and because of the Lorentz transformation:

$$\delta x' = kl(\delta x + \varepsilon\delta t), \quad \delta y' = l\delta y, \quad \delta z' = l\delta z, \quad \delta t' = kt(\delta t + \varepsilon\delta x),$$

hence, by assuming $\delta t = 0$, the relations:

$$\begin{aligned}
&\delta x' = \delta U' + \xi'\delta t' = kl\delta U,\\
&\delta y' = \delta V' + \eta'\delta t' = l\delta V,\\
&\delta t' = kl\varepsilon\delta U.
\end{aligned}$$

We observe that

$$\xi' = \frac{\xi + \varepsilon}{1 + \xi\varepsilon}, \quad \eta' = \frac{\eta}{k(1 + \xi\varepsilon)};$$

it will follow, by replacing $\delta t'$ with its value,

$$\begin{aligned}
&kl(1 + \xi\varepsilon)\delta U = \delta U'(1 + \xi\varepsilon) + (\xi + \varepsilon)kl\varepsilon\delta U,\\
&l(1 + \xi\varepsilon)\delta V = \delta V'(1 + \xi\varepsilon) + \eta l\varepsilon\delta U.
\end{aligned}$$

If we recall the definition of k, we can draw from it that:

$$\begin{aligned}
&\delta U = \frac{k}{l}\delta U' + \frac{k\varepsilon}{l}\xi\delta U',\\
&\delta V = \frac{1}{l}\delta V' + \frac{k\varepsilon}{l}\eta\delta U',
\end{aligned}$$

and similarly that

$$\delta W = \frac{1}{l}\delta W' + \frac{k\varepsilon}{l}\zeta\delta U';$$

hence

$$\sum X\delta U = \frac{1}{l}(kX\delta U' + Y\delta V' + Z\delta W') + \frac{k\varepsilon}{l}\delta U' \sum X\xi \tag{3}$$

Hence, because of equations (2) it must be that:

$$\int \sum X'\delta U' \mathrm{d}t' \mathrm{d}\tau' = \int \sum X\delta U \mathrm{d}t \mathrm{d}\tau = \frac{1}{l^4}\int \sum X\delta U \mathrm{d}t' \mathrm{d}\tau'$$

By replacing $\sum X\delta U$by its value (3) and identifying, it follows:

$$X' = \frac{k}{l^5}X + \frac{k\varepsilon}{l^5}\sum X\xi \quad Y' = \frac{1}{l^5}Y, \quad Z' = \frac{1}{l^5}Z.$$

These are equations (11) from §1. The principle of least action therefore leads us to the same result as the analysis from §1.

If we refer back to formulas (1), we see that $\sum f^2 - \sum \alpha^2$ is unchanged by the Lorentz transformation, up to a constant factor; it is not the same for the expression $\sum f^2 + \sum \alpha^2$ which appears in the energy. If we limit ourselves to the case where ε is sufficiently small that its square can be neglected such that $k = 1$ and if we also assume $l = 1$, we find:

$$\sum f'^2 = \sum f^2 + 2\varepsilon(g\gamma - h\beta),$$
$$\sum \alpha'^2 = \sum \alpha^2 + 2\varepsilon(g\gamma - h\beta),$$

or, by addition,

$$\sum f'^2 + \sum \alpha'^2 = \sum f^2 + \sum \alpha^2 + 4\varepsilon(g\gamma - h\beta).$$

§4 – The Lorentz Group

It is important to note that the Lorentz transformation do form a group.

In fact, if one sets:

$$x' = kl(x + \varepsilon t), \quad y' = ly, \quad z' = lz, \quad t' = kl(t + \varepsilon x),$$

and additionally

$$x'' = k'l'(x' + \varepsilon' t'), \quad y'' = l'y' \quad z'' = l'z', \quad t'' = k'l'(t' + \varepsilon' x'),$$

with

$$k^{-2} = 1 - \varepsilon^2, \quad k'^{-2} = 1 - \varepsilon'^2$$

it will follow:

$$x'' = k''l''(x + \varepsilon'' t), \quad y'' = l''y \quad z'' = l''z, \quad t'' = k''l''(t + \varepsilon'' x),$$

with

$$\varepsilon'' = \frac{\varepsilon + \varepsilon'}{1 + \varepsilon\varepsilon'}, \quad l'' = ll', \quad k'' = kk'(1 + \varepsilon\varepsilon') = \frac{1}{\sqrt{1 - \varepsilon''^2}}.$$

If we give l the value 1 and we assume that ε is infinitesimal,

$$x' = x + \delta x, \quad y' = y + \delta y, \quad z' = z + \delta z, \quad t' = t + \delta t,$$

it will follow:

$$\delta x = \varepsilon t, \quad \delta y = \delta z = 0, \quad \delta t = \varepsilon x.$$

That is the infinitesimal generating transformation of the group, which I will call the T_1 transformation and which can be written using the Lie notation:

$$t\frac{\mathrm{d}\varphi}{\mathrm{d}x} + x\frac{\mathrm{d}\varphi}{\mathrm{d}t} = T_1.$$

If we assume $\varepsilon = 0$ and $l = 1 + \delta l$, we would in contrast find

$$\delta x = x\delta l, \quad \delta y = y\delta l, \quad \delta z = z\delta l, \quad \delta t = t\delta l$$

and we will have another infinitesimal transformation T_0 of the group (supposing that l and ε are regarded as independent variables) and with the Lie notation it would be:

$$T_0 = x\frac{\mathrm{d}\varphi}{\mathrm{d}x} + y\frac{\mathrm{d}\varphi}{\mathrm{d}y} + z\frac{\mathrm{d}\varphi}{\mathrm{d}z} + t\frac{\mathrm{d}\varphi}{\mathrm{d}t}.$$

But we could give the particular role that we had given to the x-axis to the y-axis or the z-axis; in that way one would have two other infinitesimal transformations:

$$T_2 = t\frac{d\varphi}{dy} + y\frac{d\varphi}{dt}$$
$$T_3 = t\frac{d\varphi}{dz} + z\frac{d\varphi}{dt}$$

which would not alter the Lorentz equations either.

One can form the combinations imagined by Lie, such as

$$[T_1, T_2] = x\frac{d\varphi}{dy} - y\frac{d\varphi}{dx};$$

but it is easy to see that this transformation is equivalent to a change of coordinate axes, the axes turning a very small angle around the z-axis. We shouldn't therefore be surprised if a similar change leaves the form of the Lorentz equations unchanged, since the equations are obviously independent of the choice of axes.

We are therefore led to consider a continuous group that we will call the *Lorentz group* in which will allow as infinitesimal transformations:

1) the transformation T_0 which will be permutable with all the others;
2) the three transformations T_1, T_2, T_3; and
3) the three rotations $[T_1, T_2]$, $[T_2, T_3]$, $[T_3, T_1]$.

An arbitrary transformation of this group could always be broken down into a transformation of the form:

$$x' = lx, \quad y' = ly, \quad z' = lz, \quad t' = lt$$

and a linear transformation which does not change the quadratic form:

$$x^2 + y^2 + z^2 - t^2.$$

We can also generate our group in another way. Any transformation of the group could be regarded as a transformation of the form:

$$x' = kl(x + \varepsilon t), \quad y' = ly, \quad z' = lz, \quad t' = kl(t + \varepsilon x) \tag{1}$$

preceded and followed by a suitable rotation.

But for our purposes, we should only consider a part of the transformations from this group; we should assume that l is a function of ε, and it will be a matter of choosing this function such that this part of the group, which I will call P, again forms a group.

Turning the system 180° around the y-axis, we should find a transformation which will have to again belong to P. Now this amounts to changing the sign of x, x', z and z'; in that way it is found that:

$$x' = kl(x - \varepsilon t), \quad y' = ly, \quad z' = lz, \quad t' = kl(t - \varepsilon x) \tag{2}$$

Thus l is not changed when ε is changed to $-\varepsilon$.

On the other hand, if P is a group, the inverse substitution of (1), which is written:

$$x' = \frac{k}{l}(x - \varepsilon t), \quad y' = \frac{y}{l}, \quad z' = \frac{z}{l}, \quad t' = \frac{k}{l}(t - \varepsilon x), \tag{3}$$

should also belong to P; it will therefore have to be identical to (2), meaning that

$$l = \frac{1}{l}.$$

It will therefore have to be that $l = 1$.

§5 – Langevin Waves

Langevin[8] put the formulas which define the electromagnetic field produced by the motion of a single electron in a particularly elegant form.

Return to the equations

$$\Box\psi = -\rho, \qquad \Box F = -\rho\xi. \tag{1}$$

It is known that they can be integrated by delayed potentials and that one finds:

$$\psi = \frac{1}{4\pi}\int \frac{\rho_1 \mathrm{d}\tau_1}{r}, \quad F = \frac{1}{4\pi}\int \frac{\rho_1 \xi_1 \mathrm{d}\tau_1}{r}. \tag{2}$$

In these formulas one has:

$$\mathrm{d}\tau_1 = \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}z_1, \quad r^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$$

while ρ_1 and ξ_1 are values of ρ and ξ at the point x_1, y_1, z_1 and at the moment

$$t_1 = t - r.$$

Let x_0, y_0, z_0 be the coordinates of a differential element of an electron at the moment t; and

$$x_1 = x_0 + U, \quad y_1 = y_0 + V, \quad z_1 = z_0 + W$$

be its coordinates at the moment t_1.

U, V, W are functions of x_0, y_0, z_0 such that we will be able to write:

$$\mathrm{d}x_1 = \mathrm{d}x_0 + \frac{\mathrm{d}U}{\mathrm{d}x_0}\mathrm{d}x_0 + \frac{\mathrm{d}U}{\mathrm{d}y_0}\mathrm{d}y_0 + \frac{\mathrm{d}U}{\mathrm{d}z_0}\mathrm{d}z_0 + \xi_1 \mathrm{d}t_1;$$

and if one assumes t to be constant, and also x, y and z:

$$\mathrm{d}t_1 = +\sum \frac{x - x_1}{r}\mathrm{d}x_1.$$

We can then write:

$$\mathrm{d}x_1\left(1 + \xi_1 \frac{x_1 - x}{r}\right) + \mathrm{d}y_1 \xi_1 \frac{y_1 - y}{r} + \mathrm{d}z_1 \xi_1 \frac{z_1 - z}{r}$$
$$= \mathrm{d}x_0\left(1 + \frac{\mathrm{d}U}{\mathrm{d}x_0}\right) + \mathrm{d}y_0 \frac{\mathrm{d}U}{\mathrm{d}y_0} + \mathrm{d}z_0 \frac{\mathrm{d}U}{\mathrm{d}z_0}$$

with the two other equations that can be deduced by circular permutation.

We therefore have:

$$\mathrm{d}\tau_1 \left| 1 + \xi_1 \frac{x_1 - x}{r}, \quad \xi_1 \frac{y_1 - y}{r}, \quad \xi_1 \frac{z_1 - z}{r} \right| = \mathrm{d}\tau_0 \left| 1 + \frac{\mathrm{d}U}{\mathrm{d}x_0}, \quad \frac{\mathrm{d}U}{\mathrm{d}y_0}, \quad \frac{\mathrm{d}U}{\mathrm{d}z_0} \right| \tag{3}$$

by setting

$$\mathrm{d}\tau_0 = \mathrm{d}x_0 \mathrm{d}y_0 \mathrm{d}z_0.$$

We will study the determinants which appear on both sides of (3) and start with the left-hand side; on trying to expand it, one sees that the terms of second and third degree in ξ_1, η_1, ζ_1 disappear and that the determinant is equal to

$$1 + \xi_1 \frac{x_1 - x}{r} + \eta_1 \frac{y_1 - y}{r} + \zeta_1 \frac{z_1 - z}{r} = 1 + \omega,$$

where ω designates the radial component of the speed ξ_1, η_1, ζ_1, meaning the component directed along the radius vector going from the point x, y, z to the point x_1, y_1, z_1.

In order to get the second determinant, I consider the coordinates of various molecules of the electron at a moment t'_1 which is the same for all the differential elements, but in such a way that for the differential element that I consider one can have $t_1 = t'_1$. The coordinates of a differential element will then be:

$$x'_1 = x_0 + U', \quad y'_1 = y_0 + V', \quad z'_1 = z_0 + W',$$

where U', V', W' are what U, V, W become when t_1 is replaced in them by t'_1; as t'_1 is the same for all the differential elements, it will hold:

$$\mathrm{d}x_1' = \mathrm{d}x_0\left(1+\frac{\mathrm{d}U'}{\mathrm{d}x_0}\right) + \mathrm{d}y_0\frac{\mathrm{d}U'}{\mathrm{d}y_0} + \mathrm{d}z_0\frac{\mathrm{d}U'}{\mathrm{d}z_0}$$

and consequently

$$\mathrm{d}\tau_1' = \mathrm{d}\tau_0\left| 1+\frac{\mathrm{d}U'}{\mathrm{d}x_0}, \quad \frac{\mathrm{d}U'}{\mathrm{d}y_0}, \quad \frac{\mathrm{d}U'}{\mathrm{d}z_0}\right|,$$

by setting

$$\mathrm{d}\tau_1' = \mathrm{d}x_1'\mathrm{d}y_1'\mathrm{d}z_1'$$

But the element of electric charge is

$$\mathrm{d}\mu_1 = \rho_1\mathrm{d}\tau_1'$$

and additionally *for the differential element considered*, one has $t_1 = t_1'$ and consequently $\frac{\mathrm{d}U'}{\mathrm{d}x_0} = \frac{\mathrm{d}U}{\mathrm{d}x_0}$, etc.; we can therefore write:

$$\mathrm{d}\mu_1 = \rho_1\mathrm{d}\tau_0\left| 1+\frac{\mathrm{d}U'}{\mathrm{d}x_0}, \quad \frac{\mathrm{d}U'}{\mathrm{d}y_0}, \quad \frac{\mathrm{d}U'}{\mathrm{d}z_0}\right|,$$

such that equation (3) will become:

$$\rho_1\mathrm{d}\tau_1(1+\omega) = \mathrm{d}\mu_1$$

and equations (2):

$$\psi = \frac{1}{4\pi}\int\frac{\mathrm{d}\mu_1}{r(1+\omega)}, \quad F = \int\frac{\xi_1\mathrm{d}\mu_1}{r(1+\omega)}.$$

If we are dealing with a single electron, our integrals will reduce to a single element, provided that only points x, y, z are considered that ae sufficiently far away so that r and ω have substantially the same value for all points of the electron. The potentials ψ, F, G, H will depend on the position of this electron and also its speed, because not only do the ξ_1, η_1, ζ_1 appear in the numerator in F, G, H, but the radial component ω appears in the denominator. It is of course its position and its velocity at the moment t_1 that are involved.

The partial derivatives of ψ, F, G, H with respect to t, x, y, z (and consequently the electric and magnetic fields) will furthermore depend on its acceleration. Additionally, they will depend on it linearly, because in these derivatives this acceleration comes in following a single differentiation.

In that way, Langevin was led to distinguish the terms in the electric and magnetic fields that do not depend on the acceleration (which he calls the speed wave) and those that are proportional to the acceleration (which he calls the acceleration wave).

The Lorentz transformation makes the calculation of these two waves easier. We can in fact apply this transformation to the system such that the speed of the single electron under consideration become zero. We will take the direction of this velocity for the x-axis before the transformation, such that, at the moment t,

$$\eta_1 = \zeta_1 = 0,$$

and we will take $\varepsilon = -\xi_1$, such that

$$\xi_1' = \eta_1' = \zeta_1' = 0.$$

We can therefore reduce the calculation of the two waves to the case where the electron velocity is zero. We start with the velocity wave; we can first remark that this wave is the same as if the motion of the electron were uniform.

If the velocity of electron is zero, it follows:

$$\omega = 0, \quad F = G = H = 0, \quad \psi = \frac{\mu_1}{4\pi r},$$

where μ_1 is the electric charge of the electron. The velocity having been brought to zero by the Lorentz transformation, we therefore have:

$$F' = G' = H' = 0, \quad \psi = \frac{\mu_1}{4\pi r'},$$

where r' is the distance from the point x', y', z' to the point x_1', y_1', z_1', and consequently:

$$\begin{aligned} &\alpha' = \beta' = \gamma' = 0, \\ &f' = \frac{\mu_1 (x' - x_1')}{4\pi r'^3}, \quad g' = \frac{\mu_1 (y' - y_1')}{4\pi r'^3}, \quad h' = \frac{\mu_1 (z' - z_1')}{4\pi r'^3}. \end{aligned}$$

We now do the inverse Lorentz transformation to find the actual field corresponding to a velocity $-\varepsilon$, 0, 0. By referring to equations (9) and (3) from §1:

$$\begin{aligned} &\alpha = 0, \quad \beta = \varepsilon h, \quad \gamma = -\varepsilon g, \\ &f = \frac{\mu_1 k l^3}{4\pi r'^3}(x + \varepsilon t - x_1 - \varepsilon t_1), \quad g = \frac{\mu_1 k l^3}{4\pi r'^3}(y - y_1), \quad h = \frac{\mu_1 k l^3}{4\pi r'^3}(z - z_1), \end{aligned} \tag{4}$$

It can be seen that the magnetic field is perpendicular to the x-axis (direction of the velocity) and to the electric field and that the electric field is directed towards the point:

$$x_1 + \varepsilon(t_1 - t), \quad y_1, \quad z_1, \tag{5}$$

If the electron were to continue to move with a straight and uniform motion with the speed that it had at the moment t_1, meaning with the velocity $-\varepsilon$, 0, 0, this point (5) would be the one that it would occupy at the moment t.

Now switch to the acceleration wave; by using the Lorentz transformation, we can refer its determination to the case where the velocity is zero. This is the case which occurs if an electron is imagined to execute very small amplitude oscillations, but very fast, such that the displacements and the velocities are infinitesimal but the accelerations are finite. This brings us back to the field which was studied in the celebrated paper by Hertz, *Die Kräfte elektrischer Schwingungen nach der Maxwell'schen Theorie*, that considered a very distant point. Under these conditions:

1) The electric and magnetic fields are equal to each other.
2) They are perpendicular to each other.
3) They are perpendicular to the normal to the spherical wavefront, meaning to the sphere whose center is at the point x_1, y_1, z_1.

I state that these three properties will still be present when the velocity is not zero, and for that, it is sufficient for me to prove that they are unchanged by the Lorentz transformation.

In fact, let A be the shared strength of the two fields; let:

$$(x - x_1) = r\lambda, \quad (y - y_1) = r\mu, \quad (z - z_1) = r\nu, \quad \lambda^2 + \mu^2 + \nu^2 = 1.$$

These properties will be expressed by the equalities:

$$A^2 = \sum f^2 = \sum \alpha^2, \quad \sum f\alpha = 0, \quad \sum f(x - x_1) = 0, \quad \sum \alpha(x - x_1) = 0,$$
$$\sum f\lambda = 0, \quad \sum \alpha\lambda = 0;$$

which means again that:

$$\begin{matrix} \frac{b}{A}, & \frac{g}{A}, & \frac{h}{A} \\ \frac{\alpha}{A}, & \frac{\beta}{A}, & \frac{\gamma}{A} \\ \lambda, & \mu, & \nu \end{matrix}$$

are the directional cosines of the three rectangular directions and from that the relations are deduced:

$$f = \beta\nu - \gamma\mu, \quad \alpha = h\mu - g\nu,$$

or

$$fr = \beta(z - z_1) - \gamma(y - y_1), \quad \alpha r = h(y - y_1) - g(z - z_1), \tag{6}$$

along with the equations that can be deduced from them by symmetry.

If we take up equations (3) from §1, we find:

$$\begin{aligned} x' - x'_1 &= kl[(x - x_1) + \varepsilon(t - t_1)] = kl[(x - x_1) + \varepsilon r], \\ y' - y'_1 &= l(y - y_1), \\ z' - z'_1 &= l(z - z_1). \end{aligned} \tag{7}$$

Above, in §3, we found:

$$l^4\left(\sum f'^2 - \sum \alpha'^2\right) = \sum f^2 - \sum \alpha^2.$$

Therefore $\sum f^2 = \sum \alpha^2$ leads to $\sum f'^2 = \sum \alpha'^2$.

On the other hand, by starting from equations (9) from §1, it is found:

$$l^4 \sum f'\alpha' = \sum f\alpha,$$

which shows that $\sum f\alpha = 0$ leads to $\sum f'\alpha' = 0$.

I now state that

$$\sum f'\left(x' - x'_1\right) = 0, \quad \sum \alpha'\left(x' - x'_1\right) = 0\,. \tag{8}$$

In fact, because of equations (7) (and also equations 9 from §1) the left-hand sides of the two equations (8) are written respectively:

$$\begin{aligned} &\frac{k}{l}\sum f(x - x_1) + \frac{k\varepsilon}{l}[fr + \gamma(y - y_1) - \beta(z - z_1)], \\ &\frac{k}{l}\sum \alpha(x - x_1) + \frac{k\varepsilon}{l}[\alpha r - h(y - y_1) + g(z - z_1)], \end{aligned}$$

They therefore become zero because of the equations $\sum f(x - x_1) = \sum \alpha(x - x_1) = 0$ and because equations (6). This is precisely what it was a matter of proving.

It is also possible to arrive at the same result by simple considerations of homogeneity.

In fact, ψ, F, G, H are homogeneous functions of $(x - x_1)$, $(y - y_1)$, $(z - z_1)$, $\xi_1 = \mathrm{d}x_1/\mathrm{d}t_1$, $\eta_1 = \mathrm{d}y_1/\mathrm{d}t_1$, $\zeta_1 = \mathrm{d}z_1/\mathrm{d}t_1$, of degree -1 in $x, y, z, t, x_1, y_1, z_1, t_1$ and their derivatives.

The derivatives of, ψ, F, G, H with respect to x, y, z, t (and consequently also to the two fields f, g, h; α, β, γ) will be homogeneous of degree -2 in the same quantities if we additionally recall that the relation

$$t - t_1 = r = \sqrt{\sum (x - x_1)^2}$$

is homogeneous in these quantities.

Now these derivatives or these fields depend on $x - x_1$, speeds dx_1/dt_1 and accelerations $d^2x_1/dt_1{}^2$; they are made up of a term independent of the accelerations (velocity wave) and a linear term in the accelerations (acceleration wave). Hence dx_1/dt_1 is homogeneous of degree 0 and $d^2x_1/dt_1{}^2$ is homogeneous of degree -1; from this it follows that the velocity wave is homogeneous of degree -2 in $(x - x_1)$, $(y - y_1)$, $(z - z_1)$, and the acceleration wave is homogeneous of degree -1. Therefore, at a very distant point, the acceleration wave dominates and can consequently be regarded as being the same as the total wave. Additionally, the law of homogeneity shows us that the acceleration wave is self-similar at a distant point and at an arbitrary point. It is therefore, at an arbitrary point, similar to the total wave at a distant point. Hence at a distant point the perturbation can only propagate by plane waves such that the two fields must be equal, perpendicular to each other and perpendicular to the direction of propagation.

I will limit myself to referring to the article by Langevin in the *Journal de Physique* (1905)[9] for more details.

§6 – Contraction of Electrons

We assume a single electron driven in a motion of straight and uniform translation. Based on what we just saw, the study of the field created by this electron in the case where the electron is immobile can be determined using the Lorentz transformation; the Lorentz transformation therefore replaces the real moving electron by an ideal immobile electron.

Let $\alpha, \beta, \gamma, f, g, h$; be the real field; let $\alpha', \beta', \gamma', f', g', h'$ be what the field becomes after the Lorentz transformation, such that the ideal field α', f' corresponds to the case of an immobile electron; it follows:

$$\alpha' = \beta' = \gamma' = 0, \quad f' = -\frac{d\psi'}{dx'}, \quad g' = -\frac{d\psi'}{dy'}, \quad h' = -\frac{d\psi'}{dz'};$$

and for the real field (because of formulas 9 from §1):

$$\begin{aligned} \alpha &= 0, & \beta &= \varepsilon h, & \gamma &= -\varepsilon g, \\ f &= l^2 f', & g &= kl^2 g', & h &= kl^2 h'. \end{aligned} \tag{1}$$

It is now a matter of determining the total energy due to the motion of the electron, the corresponding action and the electromagnetic moment in order to be able to calculate the electromagnetic masses of the electron. For a distant point, it is sufficient to consider the electron as reduced to a single point; one is then back at the formulas (4) from the previous section which are generally suitable. But here they would not be sufficient, because the energy is principally located in the parts of the ether closest to the electron.

Several hypotheses can be made on this subject.

According to Abraham's hypothesis, the electrons would be spherical and undeformable.

Then, when the Lorentz transformation would be applied, since the real electron would be spherical, the ideal electron would become an ellipsoid. Following §1, the equation of this ellipsoid would be:

$$k^2(x' - \varepsilon t - \xi t' + \varepsilon\xi x')^2 + (y' - \eta k t' + \eta k \varepsilon x')^2 + (z' - \zeta k t' + \zeta k \varepsilon x')^2 = l^2 r^2.$$

But here we have:

$$\xi + \varepsilon = \eta = \zeta = 0, \quad 1 + \varepsilon\xi = 1 - \varepsilon^2 = \frac{1}{k^2},$$

such that the equation of the ellipsoid becomes:

$$\frac{x'^2}{k^2} + y'^2 + z'^2 = l^2 r^2.$$

If the radius of the real electron is r, the axes of the ideal electron would therefore be:

$$klr, \quad lr, \quad lr.$$

In contrast, in Lorentz's hypothesis, the moving electrons would be deformed such that it would be the real electron which would be an ellipsoid whereas the immobile ideal electron would always be a sphere of radius r; the axes of the real electron will then be:

$$\frac{r}{lk}, \quad \frac{r}{l}, \quad \frac{r}{l}.$$

Call

$$A = \frac{1}{2}\int f^2 \mathrm{d}\tau$$

the *longitudinal electrical energy*;

$$B = \frac{1}{2}\int \left(g^2 + h^2\right)\mathrm{d}\tau$$

the *transverse electric energy*; and

$$C = \frac{1}{2}\int \left(\beta^2 + \gamma^2\right)\mathrm{d}\tau$$

the *transverse magnetic energy*. There is no longitudinal magnetic energy because $\alpha = \alpha' = 0$. Designate the corresponding quantities in the ideal system by A', B', C'. It is first found that:

$$C' = 0, \quad C = \varepsilon^2 B.$$

Additionally, we can observe that the real field depends only on $x + \varepsilon t$, y and z, and write:

$$\begin{aligned} \mathrm{d}\tau &= \mathrm{d}(x + \varepsilon t)\mathrm{d}y\mathrm{d}z, \\ \mathrm{d}\tau' &= \mathrm{d}x'\mathrm{d}y'\mathrm{d}z' = kl^3\mathrm{d}\tau; \end{aligned}$$

hence

$$A' = kl^{-1}A, \quad B' = k^{-1}l^{-1}B, \quad A = \frac{lA'}{k}, \quad B = klB' \cdot$$

In Lorentz's hypothesis, $B' = 2A'$, and A', which is inversely proportional to the radius of the electron, is a constant independent of the speed of the real electron; in this way, it is possible to find the total energy:

$$A + B + C = A'lk\left(3 + \varepsilon^2\right)$$

and the action (per unit time):

$$A + B - C = \frac{3A'l}{k}.$$

We now calculate the electromagnetic momentum; we will find:

$$D = \int (g\gamma - h\beta)\mathrm{d}\tau = -\varepsilon\int \left(g^2 + h^2\right)\mathrm{d}\tau = -2\varepsilon B = -4\varepsilon klA'.$$

But there must be some relations between the energy $E = A + B + C$, the action per unit time $H = A + B - C$ and the momentum D. The first of these relations is:

$$E = H - \varepsilon \frac{\mathrm{d}H}{\mathrm{d}\varepsilon},$$

the second is:

$$\frac{\mathrm{d}D}{\mathrm{d}\varepsilon} = -\frac{1}{\varepsilon}\frac{\mathrm{d}E}{\mathrm{d}\varepsilon};$$

hence:

$$D = \frac{\mathrm{d}H}{\mathrm{d}\varepsilon}, \quad E = H - \varepsilon D. \tag{2}$$

The second of equations (2) is always satisfied; but the first is satisfied only if

$$l = \left(1 - \varepsilon^2\right)^{\frac{1}{6}} = k^{-\frac{1}{3}},$$

meaning if the volume of the ideal electron is equal to that of the real electron, or also if the volume of the electron is constant; that is Langevin's hypothesis.

This stands in contradiction with the result from §4 and with the results obtained by Lorentz in another way. This contradiction is what needs to be explained.

Before bringing up this explanation, I observe that, whatever the hypothesis adopted we will have:

$$H = A + B - C = \frac{l}{k}(A' + B'),$$

or, because $C' = 0$,

$$H = \frac{l}{k}H'. \tag{3}$$

We can compare this result for the equation $J = J'$ obtained in §3.

We have in fact:

$$J = \int H\mathrm{d}t, \quad J' = \int H'\mathrm{d}t'.$$

We will observe the state of the system depends only on $x + \varepsilon t$, y and z, meaning x', y', z', and that we have:

$$t' = \frac{l}{k}t + \varepsilon x', \quad \mathrm{d}t' = \frac{l}{k}\mathrm{d}t. \tag{4}$$

By combining equations (3) and (4), it is found that $J = J'$.

We place ourselves in an arbitrary hypothesis which could be either that of Lorentz, Abraham, or Langevin, or an intermediate hypothesis.

Let

$$r, \quad \theta r, \quad \theta r$$

be the three axes of the real electron; those of the ideal electron will be:

$$klr, \quad \theta lr, \quad \theta lr.$$

Then $A' + B'$ will be the electrostatic energy due to an ellipsoid having axes klr, θlr, θlr.

Let us assume that the electricity spreads over the surface of the electron like that of a conductor or spreads uniformly inside this electron. This energy will be of the form:

$$A' + B' = \frac{\varphi\left(\frac{\theta}{k}\right)}{klr},$$

where φ is a known function.

Abraham's hypothesis consists of assuming:

$$r = \text{const.} \quad \theta = 1\,.$$

Lorentz's hypothesis:

$$l = 1, \quad kr = \text{const.} \quad \theta = k.$$

Langevin's hypothesis:

$$l = k^{-1/3}, \quad k = \theta, \quad klr = \text{const.}$$

Next find:

$$H = \frac{\varphi\left(\frac{\theta}{k}\right)}{k^2 r}.$$

Abraham found, up to differences of notation (Göttinger Nachrichten, 1902, p. 37):

$$H = \frac{a}{r}\,\frac{1-\varepsilon^2}{\varepsilon}\log\frac{1+\varepsilon}{1-\varepsilon},$$

where a is a constant. Now, in Abraham's hypothesis, $\theta = 1$; therefore:

$$\varphi\left(\frac{1}{k}\right) = ak^2\frac{1-\varepsilon^2}{\varepsilon}\log\frac{1+\varepsilon}{1-\varepsilon} = \frac{a}{\varepsilon}\log\frac{1+\varepsilon}{1-\varepsilon} \tag{5}$$

which defines the function φ.

Having laid that out, imagine that the electron is subject to a binding force, such that there is a relation between r and θ; under Lorentz's hypothesis, this relation would be $\theta r = \text{const.}$, in Langevin's $\theta^2 r^3 = \text{const.}$. We will assume more generally:

$$r = b\theta^m,$$

where b is a constant; hence:

$$H = \frac{1}{bk^2}\theta^{-m}\varphi\left(\frac{\theta}{k}\right).$$

What shape will the electron take when the velocity becomes $-\varepsilon t$,[10] *if it is assumed that the only forces involved are binding forces?* That shape will be defined by the equality:

$$\frac{\partial H}{\partial \theta} = 0, \tag{6}$$

or

$$-m\theta^{-m-1}\varphi + \theta^{-m}k^{-1}\varphi' = 0,$$

or

$$\frac{\varphi'}{\varphi} = \frac{mk}{\theta}.$$

If we want there to be a balance such that $\theta = k$, it must be that for $\theta/k = 1$, the logarithmic derivative of φ is equal to m.

If we expand $1/k$ and the right-hand side of (5) in powers of ε, equation (5) becomes:

$$\varphi\left(1 - \frac{\varepsilon^2}{2}\right) = a\left(1 + \frac{\varepsilon^2}{3}\right),$$

by neglecting higher powers of ε.

By differentiating, it follows that:

$$-\varepsilon\varphi'\left(1-\frac{\varepsilon^2}{2}\right)=\frac{2}{3}\varepsilon a.$$

For $\varepsilon = 0$, meaning when the argument of φ is equal to 1, these equations become:

$$\varphi = a, \quad \varphi' = -\frac{2}{3}a, \quad \frac{\varphi'}{\varphi} = -\frac{2}{3}. \tag{7}$$

Therefore it must be that $m = -2/3$ as in Langevin's hypothesis.

This result must be compared with the result concerning the first equation (2) and from which, in reality, it is not different. In fact, let us assume that any element $\mathrm{d}\tau$ of the electron is subject to a force $X\mathrm{d}\tau$ parallel to the x-axis, where X is the same for all elements; we will then have, conforming to the definition of the momentum:

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \int X\mathrm{d}\tau.$$

Additionally, the principle of least action gives us:

$$\delta J = \int X\delta U\mathrm{d}\tau\mathrm{d}t, \quad J = \int H\mathrm{d}t, \quad \delta J = \int D\delta U\mathrm{d}t\,,$$

where δU is the displacement of the center of gravity of the electron; H depends on θ and ε, if it is accepted that r is related to θ by the binding equation; it then follows:

$$\delta J = \int\left(\frac{\partial H}{\partial\varepsilon}\delta\varepsilon + \frac{\partial H}{\partial\theta}\delta\theta\right)\mathrm{d}t.$$

Additionally, $\delta\varepsilon = -\frac{\mathrm{d}\delta U}{\mathrm{d}t}$; hence, by integrating by parts:

$$\int D\delta\varepsilon\mathrm{d}t = \int D\delta u\mathrm{d}t,$$

or

$$\int\left(\frac{\partial H}{\partial\varepsilon}\delta\varepsilon + \frac{\partial H}{\partial\theta}\delta\theta\right)\mathrm{d}t = \int D\delta\varepsilon\mathrm{d}t;$$

hence

$$D = \frac{\partial H}{\partial\varepsilon}, \quad \frac{\partial H}{\partial\theta} = 0.$$

But the derivative $\mathrm{d}H/\mathrm{d}\varepsilon$, which appears in the right-hand side of the first equation (2) is the derivative taken by assuming θ is expressed as a function of ε, such that

$$\frac{\mathrm{d}H}{\mathrm{d}\varepsilon}=\frac{\partial H}{\partial \varepsilon}+\frac{\partial H}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}\varepsilon}.$$

Equation (2) is therefore equivalent to equation (6).

The conclusion is that if the electron is subject to a binding between its three axes, *and if no other force is involved apart from the binding forces*, the shape that this electron will take, when driven at a uniform speed, can only be that of the ideal electron corresponding to a sphere, or that in the case where the binding will be such that the volume is constant, as assumed in Langevin's hypothesis.

In that way we are led to state the following problem: what additional forces, other than the binding forces, would need to be involved to incorporate Lorentz's law or, more generally, any law other than that of Langevin?

The simplest hypothesis, and the first that we needed to examine, is that these additional forces derive from a special potential deriving from the three axes of the ellipsoid and consequently from θ and r; let $F(\theta, r)$ be that potential; in that case the expression for the action will be:

$$J=\int [H+F(\theta,r)]\mathrm{d}t$$

and the equilibrium conditions will be written:

$$\frac{\mathrm{d}H}{\mathrm{d}\theta}+\frac{\mathrm{d}F}{\mathrm{d}\theta}=0,\quad \frac{\mathrm{d}H}{\mathrm{d}r}+\frac{\mathrm{d}F}{\mathrm{d}r}=0. \tag{8}$$

If we assume that r and θ are linked by the relationship $r=b\theta^m$, we will be able to regard r as a function of θ, consider F as only depending on θ and retain only the first equation (8) with:

$$H=\frac{\varphi}{bk^2\theta^m},\quad \frac{\mathrm{d}H}{\mathrm{d}\theta}=\frac{-m\varphi}{bk^2\theta^{m+1}}+\frac{\varphi'}{bk^3\theta^m}.$$

It must be, for $k=\theta$, that equation (8) is satisfied, which gives, in light of equations (7):

$$\frac{\mathrm{d}F}{\mathrm{d}\theta}=\frac{ma}{b\theta^{m+3}}+\frac{2}{3}\frac{a}{b\theta^{m+3}},$$

hence:

$$F = \frac{-a}{b\theta^{m+2}} \frac{m+\frac{2}{3}}{m+2}$$

and in the Lorentz hypothesis, where $m = -1$:

$$F = \frac{a}{3b\theta}.$$

Now let us assume that there is *no* binding and, regarding r and θ as two independent variables, we retain the two equations (8); it will follow:

$$H = \frac{\varphi}{k^2 r}, \quad \frac{\mathrm{d}H}{\mathrm{d}\theta} = \frac{\varphi'}{k^3 r}, \quad \frac{\mathrm{d}H}{\mathrm{d}r} = \frac{-\varphi}{k^3 r^2}.$$

Equations (8) will have to be satisfied for $k = \theta$, $r = b\theta^m$; which gives:

$$\frac{\mathrm{d}F}{\mathrm{d}r} = \frac{a}{b^2\theta^{2m+2}}, \quad \frac{\mathrm{d}F}{\mathrm{d}\theta} = \frac{2}{3}\frac{a}{b\theta^{m+3}}. \tag{9}$$

One of the ways to satisfy these conditions is to set:

$$F = Ar^{\alpha}\theta^{\beta}, \tag{10}$$

where A, α and β are constants; equations (9) must be satisfied for $k = \theta$ and $r = b\theta^m$, which gives:

$$A\alpha b^{\alpha-1}\theta^{m\alpha-m+\beta} = \frac{a}{b^2\theta^{2m+2}}, \quad A\beta b^{\alpha}\theta^{m\alpha+\beta-1} = \frac{2}{3}\frac{a}{b\theta^{m+3}}.$$

By identification, it follows:

$$\alpha = 3\gamma, \quad \beta = 2\gamma, \quad \gamma = -\frac{m+2}{3m+2}, \quad A = \frac{a}{\alpha b^{\alpha+1}}. \tag{11}$$

But the volume of the ellipsoid is proportional to $r^3\theta^2$, such that the additional potential is proportional of the volume of the electron to the power γ.

In the Lorentz hypothesis, $m = -1$ and $\gamma = 1$.

This is therefore the Lorentz hypothesis on the condition of adding an additional potential proportional to the volume of the electron.

Langevin's hypothesis corresponds to $\gamma = \infty$.

§7 – Quasi-Stationary Motion

It remains to be seen whether this hypothesis about the contraction of electrons reflects the impossibility of showing absolute motion; I will start by studying quasi-stationary motion of an electron which is isolated or only subject to the action of other distant electrons.

It is known that motion is called quasi-stationary motion when the changes in velocity are sufficiently slow that the magnetic and electrical energies due to the motion of the electron differ slightly from what they would be in uniform motion; it is also known that it is by starting from this concept of quasi-stationary motion that Abraham arrived at the concept of transverse and longitudinal electromagnetic masses.

I believe I have to be more specific. Let H be our action per unit time:

$$H = \frac{1}{2}\int\left(\sum f^2 - \sum \alpha^2\right)\mathrm{d}\tau$$

where for the moment we only consider the electric and magnetic fields due to the motion of an isolated electron. In the previous section, considering the motion to be uniform, we regarded H as dependent on the speed ξ, η, ζ of the center of gravity of the electron (in the previous section, these three components had values $-\varepsilon$, 0, 0) and the parameters r and θ which define the shape of the electron.

But, if the motion is no longer uniform, H will depend not only on the values ξ, η, ζ, r, θ at the moment being considered, but on the values of the same quantities at other moments which will be different from them by quantities of the same order as the time taken by light to go from one point of the electron to another; in other words, H will depend not only on ξ, η, ζ, r, θ but also on their derivatives with respect to time of all orders.

Hence, the motion will be called quasi-stationary when the partial derivatives of H with respect to the successive derivatives of ξ, η, ζ, r, θ will be negligible compared to the partial derivatives of H with respect to the quantities ξ, η, ζ, r, θ themselves.

The equations of a similar motion could be written:

$$\begin{gathered}\frac{\mathrm{d}H}{\mathrm{d}\theta} + \frac{\mathrm{d}F}{\mathrm{d}\theta} = \frac{\mathrm{d}H}{\mathrm{d}r} + \frac{\mathrm{d}F}{\mathrm{d}r} = 0,\\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}H}{\mathrm{d}\xi} = -\int X\mathrm{d}\tau, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}H}{\mathrm{d}\eta} = -\int Y\mathrm{d}\tau, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}H}{\mathrm{d}\zeta} = -\int Z\mathrm{d}\tau.\end{gathered} \tag{1}$$

In these equations, F has the same meaning as in the previous section; X, Y, Z are the components of the force which acts on the electron: this force is solely due to the electric and magnetic fields produced by *other* electrons.

We observed that H depends on ξ, η, ζ only through the combination

$$V = \sqrt{\xi^2 + \eta^2 + \zeta^2},$$

meaning the magnitude of the velocity; by again calling D the momentum, it follows:

$$\frac{\mathrm{d}H}{\mathrm{d}\xi} = \frac{\mathrm{d}H}{\mathrm{d}V}\frac{\xi}{V} = -D\frac{\xi}{V}$$

hence:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}H}{\mathrm{d}\xi} = \frac{D}{V}\frac{\mathrm{d}\xi}{\mathrm{d}t} - D\frac{\xi}{V^2}\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{\mathrm{d}D}{\mathrm{d}V}\frac{\xi}{V}\frac{\mathrm{d}V}{\mathrm{d}t}, \tag{2}$$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}H}{\mathrm{d}\eta} = \frac{D}{V}\frac{\mathrm{d}\eta}{\mathrm{d}t} - D\frac{\eta}{V^2}\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{\mathrm{d}D}{\mathrm{d}V}\frac{\eta}{V}\frac{\mathrm{d}V}{\mathrm{d}t}, \tag{2'}$$

with

$$V\frac{\mathrm{d}V}{\mathrm{d}t} = \sum \xi \frac{\mathrm{d}\xi}{\mathrm{d}t}. \tag{3}$$

If we take the x-axis as the current direction of the velocity, it follows:

$$\xi = V, \quad \eta = \zeta = 0, \quad \frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t};$$

equations (2) and (2′) become:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}H}{\mathrm{d}\xi} = \frac{\mathrm{d}D}{\mathrm{d}V}\frac{\mathrm{d}\xi}{\mathrm{d}t}, \quad -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}H}{\mathrm{d}\eta} = \frac{D}{V}\frac{\mathrm{d}\eta}{\mathrm{d}t}$$

and the three equations (1) become:

$$\frac{\mathrm{d}D}{\mathrm{d}V}\frac{\mathrm{d}\xi}{\mathrm{d}t} = \int X\mathrm{d}\tau, \quad \frac{D}{V}\frac{\mathrm{d}\eta}{\mathrm{d}t} = \int Y\mathrm{d}\tau, \quad \frac{D}{V}\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \int Z\mathrm{d}\tau. \tag{4}$$

This is why Abraham gave $\mathrm{d}D/\mathrm{d}V$ the name *longitudinal mass* and D/V the name *transverse mass*; recall that $D = \mathrm{d}H/\mathrm{d}V$.

In Lorentz's hypothesis, we have:

$$D = -\frac{\mathrm{d}H}{\mathrm{d}V} = -\frac{\partial H}{\partial V},$$

where $\partial H/\partial V$ represents the derivative with respect V, after r and θ have been replaced by their values as a function of V drawn from the first two equations (1); and additionally it follows after this substitution,

$$H = +A\sqrt{1 - V^2}.$$

We will now choose the units such that the constant factor A is equal to 1, and I set $\sqrt{1 - V^2} = h$, hence:

$$H = +h, \quad D = \frac{V}{h}, \quad \frac{\mathrm{d}D}{\mathrm{d}V} = h^{-3}, \quad \frac{\mathrm{d}D}{\mathrm{d}V}\frac{1}{V^2} - \frac{D}{V^3} = h^{-3}.$$

We will also set:

$$M = V\frac{\mathrm{d}V}{\mathrm{d}t} = \sum \xi \frac{\mathrm{d}\xi}{\mathrm{d}t}, \quad X_1 = \int X\mathrm{d}\tau$$

and we will find for the equation of quasi-stationary motion:

$$h^{-1}\frac{\mathrm{d}\xi}{\mathrm{d}t} + h^{-3}\xi M = X_1. \tag{5}$$

Let's look at what becomes of these equations under the Lorentz transformation. We will set $1 + \xi\varepsilon = \mu$, and we will first have:

$$\mu\xi' = \xi + \varepsilon, \quad \mu\eta' = \frac{\eta}{k}, \quad \mu\zeta' = \frac{\zeta}{k},$$

from which it is easy to find

$$\mu h' = \frac{h}{k}.$$

We also have

$$\mathrm{d}t' = k\mu\mathrm{d}t,$$

hence:

$$\frac{\mathrm{d}\xi'}{\mathrm{d}t'} = \frac{\mathrm{d}\xi}{\mathrm{d}t}\frac{1}{k^3\mu^3}, \quad \frac{\mathrm{d}\eta'}{\mathrm{d}t'} = \frac{\mathrm{d}\eta}{\mathrm{d}t}\frac{1}{k^2\mu^2} - \frac{\mathrm{d}\xi}{\mathrm{d}t}\frac{\eta\varepsilon}{k^2\mu^3}, \quad \frac{\mathrm{d}\zeta'}{\mathrm{d}t'} = \frac{\mathrm{d}\zeta}{\mathrm{d}t}\frac{1}{k^2\mu^2} - \frac{\mathrm{d}\xi}{\mathrm{d}t}\frac{\zeta\varepsilon}{k^2\mu^3},$$

and again:

$$M' = \frac{\mathrm{d}\xi}{\mathrm{d}t}\frac{\varepsilon h^2}{k^3\mu^4} + \frac{M}{k^3\mu^3}$$

and

$$h'^{-1}\frac{\mathrm{d}\xi'}{\mathrm{d}t'} + h'^{-3}\xi' M' = \left[h^{-1}\frac{\mathrm{d}\xi}{\mathrm{d}t} + h^{-3}(\xi+\varepsilon)M\right]\mu^{-1}, \tag{6}$$

$$h'^{-1}\frac{\mathrm{d}\eta'}{\mathrm{d}t'} + h'^{-3}\eta' M' = \left(h^{-1}\frac{\mathrm{d}\eta}{\mathrm{d}t} + h^{-3}\eta M\right)\mu^{-1}h^{-1}. \tag{7}$$

Let us now refer to equations (11′) from §1; there X_1, Y_1, Z_1 can be regarded as having the same meaning as in equations (5). Also, we have $l = 1$ and $\rho'/\rho = k\mu$; these equations therefore become:

$$\begin{aligned} X_1' &= \mu^{-1}\left(X_1 + \varepsilon\sum X_1\xi\right), \\ Y_1' &= k^{-1}\mu^{-1}Y_1. \end{aligned} \tag{8}$$

When we calculate $\sum X_1\xi$ using equations (5), we will find:

$$\sum X_1\xi = h^{-3}M,$$

hence:

$$\begin{aligned} X_1' &= \mu^{-1}\left(X_1 + \varepsilon h^{-3}M\right), \\ Y_1' &= k^{-1}\mu^{-1}Y_1. \end{aligned} \tag{9}$$

By comparing equations (5), (6), (7) and (9), we finally find:

$$\begin{aligned} h'^{-1}\frac{\mathrm{d}\xi'}{\mathrm{d}t'} + h'^{-3}\xi' M' &= X_1', \\ h'^{-1}\frac{\mathrm{d}\eta'}{\mathrm{d}t'} + h'^{-3}\eta' M' &= Y_1', \end{aligned} \tag{10}$$

which shows that the equations of quasi-stationary motion are unaltered by the Lorentz transformation, but that does not yet prove that Lorentz's hypothesis is the only one which leads to this result.

To establish that point, we are going to restrict ourselves, as Lorentz did, to some specific cases which will obviously be sufficient for us to prove a negative proposition.

How are we first going to extend the hypothesis on which the previous calculation rests:

1) Instead of assuming $l = 1$ in the Lorentz transformation, we will assume that l is arbitrary.
2) Instead of assuming that F is proportional to the volume, and consequently that H is proportional to h, we are going to assume that F is an arbitrary function of θ and r, such that (after having replaced θ and r by their values as functions of V, drawn from the first two equations (1)) H is an arbitrary function of V.

I first note that, if it is assumed that $H = h$, one should in fact have $l = 1$; and in fact equations (6) and (7) will remain, except that the left-hand side will be multiplied by $1/l$; equations (9) also, except that the right-hand sides will be multiplied by $1/l^2$; and finally equations (10) except that the right-hand side will be multiplied by $1/l$. If one wants the equations of motion to be unaltered by the Lorentz transformation meaning that equations (10) are not different from equations (5) except for the accenting of the letters, it must be assumed that:

$$l = 1.$$

Now assume that $\eta = \zeta = 0$, hence $\xi = V$ and $\frac{d\xi}{dt} = \frac{dV}{dt}$; equations (5) will take the form:

$$-\frac{d}{dt}\frac{dH}{d\xi} = \frac{dD}{dV}\frac{d\xi}{dt} = X_1, \quad -\frac{d}{dt}\frac{dH}{d\eta} = \frac{D}{V}\frac{d\eta}{dt} = Y_1\,. \tag{5'}$$

We can additionally set:

$$\frac{dD}{dV} = f(V) = f(\xi), \quad \frac{D}{V} = \varphi(V) = \varphi(\xi)\,.$$

If the equations of motion are unaltered by the Lorentz transformation, it should be that:

$$\begin{aligned}
&f(\xi)\frac{d\xi}{dt} = X_1,\\
&\varphi(\xi)\frac{d\eta}{dt} = Y_1,\\
&f(\xi')\frac{d\xi'}{dt'} = X_1' = l^{-2}\mu^{-1}\left(X_1 + \varepsilon\sum X_1\xi\right) = l^{-2}\mu^{-1}X_1(1+\varepsilon\xi) = l^{-2}X_1,\\
&\varphi(\xi')\frac{d\eta'}{dt'} = Y_1' = l^{-2}k^{-1}\mu^{-1}Y_1.
\end{aligned}$$

and consequently:

$$\begin{aligned}
f(\xi)\frac{d\xi}{dt} &= l^2 f(\xi')\frac{d\xi'}{dt'}\\
\varphi(\xi)\frac{d\eta}{dt} &= l^2 k\mu\varphi(\xi')\frac{d\eta'}{dt'}
\end{aligned} \tag{11}$$

But we have:

$$\frac{\mathrm{d}\xi'}{\mathrm{d}t'} = \frac{\mathrm{d}\xi}{\mathrm{d}t}\frac{1}{k^3\mu^3}, \quad \frac{\mathrm{d}\eta'}{\mathrm{d}t'} = \frac{\mathrm{d}\eta}{\mathrm{d}t}\frac{1}{k^2\mu^2},$$

hence:

$$f(\xi') = f\left(\frac{\xi+\varepsilon}{1+\xi\varepsilon}\right) = f(\xi)\,\frac{k^3\mu^3}{l^2},$$
$$\varphi(\xi') = \varphi\left(\frac{\xi+\varepsilon}{1+\xi\varepsilon}\right) = \varphi(\xi)\,\frac{k\mu}{l^2};$$

hence, by eliminating l^2, we find the functional equation:

$$k^2\mu^2\,\frac{\varphi\left(\frac{\xi+\varepsilon}{1+\xi\varepsilon}\right)}{\varphi(\xi)} = \frac{f\left(\frac{\xi+\varepsilon}{1+\xi\varepsilon}\right)}{f(\xi)},$$

or, by setting

$$\frac{\varphi(\xi)}{f(\xi)} = \Omega(\xi) = \frac{D}{V\frac{\mathrm{d}D}{\mathrm{d}V}},$$

this:

$$\Omega\left(\frac{\xi+\varepsilon}{1+\xi\varepsilon}\right) = \Omega(\xi)\,\frac{1+\varepsilon^2}{(1+\xi\varepsilon)^2}$$

equation which must be satisfied for all values of ξ and ε. For $\zeta = 0$ one finds:

$$\Omega(\varepsilon) = \Omega(0)\big(1-\varepsilon^2\big),$$

hence:

$$D = A\left(\frac{V}{\sqrt{1-V^2}}\right)^m,$$

where A is a constant, and where I made $\Omega(0) = 1/m$.

One then finds:

$$\varphi(\xi) = \frac{A}{\xi}\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)^m, \quad \varphi(\xi') = \frac{A}{\xi}\left(\frac{\xi+\varepsilon}{\sqrt{1-\xi^2}\sqrt{1-\varepsilon^2}}\right)^m.$$

However $\varphi(\xi') = \varphi(\xi)k\mu/l^2$, so it follows:[11]

$$(\xi+\varepsilon)^{m-1}\left(1-\varepsilon^2\right)^{-\frac{m}{2}} = -\xi^{m-1}\left(1-\varepsilon^2\right)^{-\frac{1}{2}} l^{-2}.$$

As l must only depend on ε (because, if there are several electrons, l must be the same for all electrons whose velocities ξ can be different), this identity can only hold if one has:

$$m = 1, \quad l = 1\,.$$

Thus Lorentz's hypothesis is the only one which is compatible with the impossibility of showing absolute motion; if this impossibility is accepted, it must be accepted that moving electrons contract so as to become ellipsoids of revolution two axes of which remain constant; the existence of an additional potential proportional to the volume of the electron also has to be accepted, as we showed in the previous section.

Lorentz's analysis is therefore found to be fully confirmed, but we can do better by observing the true reason for the fact we are dealing with; this reason must be sought in the considerations from §4. *The transformations which do not change the equations of motion must form a group and that can occur only if* $l = 1$. Since we must not be able to recognize whether an electron is at rest or in absolute motion, it must be that when it is in motion it experiences a deformation which must be precisely that which the corresponding transformation of the group demands of it.

§8 – Arbitrary Motion

The previous results only apply to quasi-stationary motion, but it is easy to extend them to the general case; it is sufficient to apply the principles from §3, meaning to start with the principle of least action.

It is appropriate to add to the expression for the action:

$$J = \int \left(\frac{\sum f^2}{2} - \frac{\sum \alpha^2}{2}\right) \mathrm{d}t\mathrm{d}\tau,$$

a term representing the additional potential F from §6; this term will obviously take the form:

$$J_1 = \int \sum(F)\mathrm{d}t,$$

where $\sum(F)$ represents the sum of the additional potentials due to the various electrons, where each of them is proportional to the volume of the corresponding electron.

I'm writing (F) between parentheses so as not to confuse it with the vector F, G, H.

The total action is then $J + J_1$. We saw in §3 that J is unchanged by the Lorentz transformation; it now needs to be shown that the same is true of J_1.

For one of the electrons, it holds that:

$$(F) = \omega_0 \tau,$$

where ω_0 is a coefficient specific to the electron and τ is its volume; I can then write:

$$\sum (F) = \int \omega_0 \mathrm{d}\tau,$$

where the integral has to extend to all space, but does so in a way that the coefficient ω_0 is zero outside of the electrons and that inside of each electron it is equal to the special coefficient for that electron. It then follows:

$$J_1 = \int \omega_0 \mathrm{d}\tau \mathrm{d}t,$$

and after the Lorentz transformation:

$$J_1' = \int \omega_0' \mathrm{d}\tau' \mathrm{d}t',$$

Hence $\omega_0 = \omega_0'$; because if the point belongs to an electron, the corresponding point after the Lorentz transformation still belongs to the same electron. Further, we found in §3:

$$\mathrm{d}\tau' \mathrm{d}t' = l^4 \mathrm{d}\tau \mathrm{d}t$$

and, because we now assume $l = 1$,

$$\mathrm{d}\tau' \mathrm{d}t' = \mathrm{d}\tau \mathrm{d}t.$$

We then have:

$$J_1 = J_1'.$$

Which was to be proven

The theorem is therefore general; at the same time, it gives us a solution to the question that we asked at the end of §1: to find additional forces unchanged by the Lorentz transformation. The additional potential (F) satisfies that condition.

We can therefore generalize the results stated at the end of §1 and write:

If the inertia of the electrons is exclusively of electromagnetic origin, if they are only subject to forces of electromagnetic origin, or to forces which give rise to the additional potential (F)*, no experiment will be able to show absolute motion.*

What then are these forces which give rise to the potential (F)? They can obviously be compared to a pressure which governs inside the electron; everything happens as if each electron had a hollow capacitor subject to a constant internal pressure (independent of the volume); the work due to such a pressure would obviously be proportional to changes in the volume.

I must again observe that this pressure is negative. Let's go back to equation (10) from §6, which in Lorentz's hypothesis is written:

$$F = Ar^3\theta^2;$$

equations (11) from §6 will give us:

$$A = \frac{a}{3b^4}.$$

Our pressure is equal to A, up to a constant coefficient, which furthermore is negative.

We now evaluate the electron mass; I want to speak of the "experimental mass", meaning the mass for small velocities; we have (see §6):

$$H = \frac{\varphi\left(\frac{\theta}{k}\right)}{k^2 r}, \quad \theta = k, \quad \varphi = a, \quad \theta r = b;$$

hence

$$H = \frac{a}{bk} = \frac{a}{b}\sqrt{1 - V^2}.$$

For very small V, I may write:

$$H = \frac{a}{b}\left(1 - \frac{1}{2}V^2\right),$$

such that the mass, both longitudinal and transverse, will be a/b.

However, a is a numeric constant; this shows that: *the pressure to which our additional potential gives rise is proportional to the fourth power of the experimental mass of the electron.*

Since Newtonian attraction is proportional to this experimental mass, one is tempted to conclude that there is some relation between the cause which gives rise to gravitation and that which gives rise to this additional potential.

§9 – Hypotheses on Gravitation

Thus Lorentz's theory would fully explain the impossibility of showing absolute motion, if all the forces were of electromagnetic origin.

But there are other forces to which an electromagnetic origin cannot be attributed, such as gravitation for example. It can in fact happen that two systems of bodies produce equivalent electromagnetic fields, meaning exerting the same action on charged bodies and on currents and that however these two systems do not exert the same gravitational action on Newtonian masses. The gravitational field is therefore distinct from the electromagnetic field. Lorentz was therefore compelled to extend his hypothesis by assuming that *forces of any origin, and in particular gravitation, are affected by a translation* (or, if you prefer, by the Lorentz transformation) *in the same way as the electromagnetic forces*.

It is now appropriate to go into the details and examine more closely this hypothesis. If we want the Newtonian force to be affected in the same way by the Lorentz transformation, we can no longer allow that this force depends solely on the relative position of the attracting body and the attracted body at the moment under consideration. It will have to additionally depend on the velocity of both bodies. And that is not all: it will be natural to assume that the force which acts on the attracted body at the moment t depends on the position and velocity of this body at this moment t; but it will additionally depend on the position and velocity of the *attracting* body, not at the moment t, but at *an earlier moment*, as if the gravitation had taken some time to propagate.

We therefore consider the position of the attracted body at the moment t_0 and let, at this moment, x_0, y_0, z_0 be its coordinates, and ξ, η, ζ be the components of its velocity; we will additionally consider the attracting body at the corresponding moment $t_0 + t$ and let, at that moment, $x_0 + x$, $y_0 + y$, $z_0 + z$ be its coordinates; and ξ_1, η_1, ζ_1 be the components of its velocity.

We will first have to have a relation

$$\varphi(t, x, y, z, \xi, \eta, \zeta, \xi_1, \eta_1, \zeta_1) = 0 \tag{1}$$

in order to define the time t. This relation will define the laws of propagation of the gravitational action (I am in no way imposing the condition that the propagation occurs with the same speed in all directions).

Now let X_1, Y_1, Z_1 be the three components of the action exerted at the moment t on the attracted body; it is a matter of expressing X_1, Y_1, Z_1 as functions of

$$t, x, y, z, \xi, \eta, \zeta, \xi_1, \eta_1, \zeta_1. \tag{2}$$

What are the conditions to be satisfied?

1) The condition (1) must not be changed by transformations from the Lorentz group.

2) The components X_1, Y_1, Z_1 will have to be affected by the Lorentz transformations in the same way as the electromagnetic forces designated by the same letters, meaning according to equations (11′) from §1.
3) When the two bodies are at rest, the ordinary law of attraction must be restored.

It needs to be remarked that in this last case, the relation (1) would disappear, because time t doesn't play any role if the two bodies are at rest.

The problem thus stated is obviously indeterminate. We will therefore seek to satisfy other additional conditions as much as possible:

4) Since astronomical observations do not seem to show any meaningful deviation from Newton's law, we will choose the solution which deviates the least from this law, for small velocities of the two bodies.
5) We will make every effort to situate ourselves such that t is always negative; if in fact it is thought that the effect of gravitation requires some time to propagate, it would be more difficult to understand how this effect could depend on the position *not yet reached* by the attracting body.

There is a case where the indeterminacy of this problem disappears; it is the one where both bodies are it relative rest with each other, meaning where:

$$\xi = \xi_1, \quad \eta = \eta_1, \quad \zeta = \zeta_1\,;$$

this is therefore the case that we are going to examine first, by assuming that these velocities are constant, such that both bodies are driven in a shared, straight and uniform translational motion.

We can assume that the x-axis was taken parallel to this translation such that $\eta = \zeta = 0$; and we will take $\varepsilon = -\xi$.

If we apply the Lorentz transformation under these conditions, then after the transformation both bodies will be at rest and it will be that:

$$\xi' = \eta' = \zeta' = 0.$$

Then the components will X'_1, Y'_1, Z'_1 will have to be determined by Newton's law and it will hold, up to a constant factor, that:

$$\begin{aligned} &X'_1 = -\frac{x'}{r'^3}, \quad Y'_1 = -\frac{y'}{r'^3}, \quad Z'_1 = -\frac{z'}{r'^3}, \\ &r'^2 = x'^2 + y'^2 + z'^2. \end{aligned} \tag{3}$$

But, according to §1, we have:

$$
\begin{aligned}
&x' = k(x + \varepsilon t), \quad y' = y, \quad z' = z \quad t' = k(t + \varepsilon x),\\
&\frac{\rho'}{\rho} = k(1 + \xi\varepsilon) = k\left(1 - \varepsilon^2\right) = \frac{1}{k}, \quad \sum X_1\xi = -X_1\varepsilon\,,\\
&X_1' = k\frac{\rho}{\rho'}\left(X_1 + \varepsilon\sum X_1\xi\right) = k^2 X_1\left(1 - \varepsilon^2\right) = X_1,\\
&Y_1' = \frac{\rho}{\rho'}Y_1 = kY_1,\\
&Z_1' = kZ_1.
\end{aligned}
$$

Additionally

$$x + \varepsilon t = x - \xi t, \quad r'^2 = k^2(x - \xi t)^2 + y^2 + z^2$$

and

$$X_1 = \frac{-k(x - \xi t)}{r'^3}, \quad Y_1 = \frac{-y}{kr'^3}, \quad Z_1 = \frac{-z}{kr'^3}; \tag{4}$$

which can be written:

$$X_1 = \frac{\mathrm{d}V}{\mathrm{d}x}, \quad Y_1 = \frac{\mathrm{d}V}{\mathrm{d}y}, \quad Z_1 = \frac{\mathrm{d}V}{\mathrm{d}z}, \quad V = \frac{1}{kr'}. \tag{4'}$$

At first it seems that the indeterminacy remains, because we have made no assumption about the value of t, meaning the speed of transmission and additionally that x is a function of t, but it is easy to see that $x - \xi t$, y, z, which alone appear in our formulas, do not depend on t.

It can be seen that if the two bodies are simply driven in a shared translation, the force which acts are the attracted body is normal to an ellipsoid that has the attracting body at its center.

To go farther, we need to seek the *invariants of the Lorentz group*.

We know that the substitutions from this group (with the assumption $l = 1$) are linear substitutions which do not change the quadratic form:

$$x^2 + y^2 + z^2 - t^2.$$

Let us also set:

$$
\begin{aligned}
\xi = \frac{\delta x}{\delta t}, \quad &\eta = \frac{\delta y}{\delta t}, \quad \zeta = \frac{\delta z}{\delta t},\\
\xi_1 = \frac{\delta_1 x}{\delta_1 t}, \quad &\eta_1 = \frac{\delta_1 y}{\delta_1 t} \quad \zeta_1 = \frac{\delta_1 z}{\delta_1 t}
\end{aligned}
$$

we see that the Lorentz transformation will have the effect of making δx, δy, δz, δt and $\delta_1 x$, $\delta_1 y$, $\delta_1 z$, $\delta_1 t$ undergo the same linear substitutions as x, y, z, t.

Let us regard

$$\begin{array}{cccc} x, & y, & z, & t\sqrt{-1}, \\ \delta x, & \delta y, & \delta z, & \delta t\sqrt{-1}, \\ \delta_1 x, & \delta_1 y, & \delta_1 z, & t\sqrt{-1}, \end{array}$$

as the coordinates of three points P, P', P'' in four-dimensional space. We have seen that the Lorentz transformation is solely a rotation of this space around the origin, which is regarded as fixed. We will have no other distinct invariants besides the six distances of the three points P, P', P'' between each other and the origin, or, if one prefers besides the two expressions:

$$x^2 + y^2 + z^2 - t^2, \quad x\delta x + y\delta y + z\delta z - t\delta t\,,$$

or the four expressions of the same form that are deduced by arbitrarily permuting the three points P, P', P''.

But what we are looking for are functions of 10 variables (2) which are invariants; we therefore need to look among the combinations of our six invariants for those which only depend on these 10 variables, meaning those which are homogeneous of zeroth degree both in δx, δy, δz, δt and in $\delta_1 x$, $\delta_1 y$, $\delta_1 z$, $\delta_1 t$. Thus we will be left with four distinct invariants which are:

$$\sum x^2 - t^2, \quad \frac{t - \sum x\xi}{\sqrt{1 - \sum \xi^2}}, \quad \frac{t - \sum x\xi}{\sqrt{1 - \sum \xi_1^2}}, \quad \frac{1 - \sum \xi\xi_1}{\sqrt{\left(1 - \sum \xi^2\right)\left(1 - \sum \xi_1^2\right)}}. \tag{5}$$

Let us now work on the transformations undergone by the components of the force; let us take up equations (11) from §1 which refer not to the force X_1, Y_1, Z_1, that we are considering here, but to the force X, Y, Z referred to the unit volume. Let us additionally set:

$$T = \sum X\xi;$$

we will see that these equations (11) can be written (with $l = 1$) as:

$$\begin{array}{ll} X' = k(X + \varepsilon T), & T' = k(T + \varepsilon X), \\ Y' = Y, & Z' = Z; \end{array} \tag{6}$$

such that X, Y, Z, T undergo the same transformation as x, y, z, t. The invariants of the group will therefore be:

$$\sum X^2 - T^2, \quad \sum Xx - Tt, \quad \sum X\delta x - T\delta t, \quad \sum X\delta_1 x - T\delta_1 t$$

But these are not the X, Y, Z that we need those are X_1, Y_1, Z_1 with

$$T_1 = \sum X_1 \xi.$$

We see that

$$\frac{X_1}{X} = \frac{Y_1}{Y} = \frac{Z_1}{Z} = \frac{1}{\rho}.$$

Therefore the Lorentz transformation will act on X_1, Y_1, Z_1, T_1, in the same way as on X, Y, Z, T with the difference that these expressions will be additionally multiplied by

$$\frac{\rho}{\rho'} = \frac{1}{k(1+\xi\varepsilon)} = \frac{\delta t}{\delta t'}.$$

Similarly, the transformation will act on ξ, η, ζ, 1, in the same way as on δx, δy, δz, δt with the difference that these expressions will be additionally multiplied by the *same* factor:

$$\frac{\delta t}{\delta t'} = \frac{1}{k(1+\xi\varepsilon)}.$$

Let us then consider X, Y, Z, $T\sqrt{-1}$ as coordinates of a fourth point Q;
then the invariants will be functions of the mutual distances between five points

$$0, P, P', P'', Q$$

and among these functions we will have to keep only those which are homogeneous of zeroth degree both in

$$X, Y, Z, T, \delta x, \delta y, \delta z, \delta t$$

(variables that can next be replaced with X_1, Y_1, Z_1, T_1, ξ, η, ζ, 1), and also in

$$\delta_1 x, \delta_1 y, \delta_1 z, 1$$

(variables that can next be replaced by ξ_1, η_1, ζ_1, 1).

We will thus find, beyond the four invariants (5), for new distinct invariants, which are:

$$\frac{\sum X_1^2 - T_1^2}{1-\sum \xi^2}, \quad \frac{\sum X_1 x - T_1 t}{\sqrt{1-\sum \xi^2}}, \quad \frac{\sum X_1 \xi_1 - T_1}{\sqrt{1-\sum \xi^2}\sqrt{1-\sum \xi_1^2}}, \quad \frac{\sum X_1 \xi - T_1}{1-\sum \xi^2}. \tag{7}$$

The last invariant is always zero according to the definition of T_1.
Having set that, what are the conditions to be satisfied?

1) The left-hand side of the relation (1), which defines the propagation velocity, must be a function of the four invariants (5).

One can obviously make a load of hypothesis; we will consider only two of them:

A) One could have:

$$\sum x^2 - t^2 = r^2 - t^2 = 0,$$

where $t = \pm r$, and, because t must be negative, $t = -r$. This is the same as saying the speed of propagation is equal to that of light. At first it seems this hypothesis must be rejected without examination. Laplace in fact showed that the propagation is either instantaneous or much faster than that of light. But Laplace had examined the hypothesis of the finite propagation speed, everything else remaining the same; here, in contrast, this hypothesis is complicated by many others and it could happen that there might be a more or less perfect compensation between them, like those for which the applications of the Lorentz transformation have already given us many examples.

B) One could have:

$$\frac{t - \sum x\xi_1}{\sqrt{1-\sum \xi_1^2}} = 0 \quad t = \sum x\xi_1 .$$

The speed of propagation is then much faster than that of light; but in some cases t could be negative, which, as we already stated, hardly seems admissible. *We will therefore keep hypothesis A.*

2) The four invariants (7) must be a function of the invariants (5).
3) When both bodies are at absolute rest, X, Y, Z must have the value deduced from Newton's law and when they are at relative rest, the value deduced from equations (4).

In the scenario of absolute rest, the first two invariants (7) must reduce to

$$\sum X_1^2, \quad \sum X_1 x,$$

or, by Newton's law, to

$$\frac{1}{r^4}, \quad -\frac{1}{r};$$

on the other hand, in scenario (A), the second and third invariants (5) become:

$$\frac{-r-\sum x\xi}{\sqrt{1-\sum \xi^2}}, \quad \frac{-r-\sum x\xi_1}{\sqrt{1-\sum \xi_1^2}},$$

meaning, for absolute rest:

$$-r, \quad -r\,.$$

We can therefore allow, *for example*, that the first two invariants (4) reduce to

$$\frac{\left(1-\sum\xi_1^2\right)^2}{\left(r+\sum x\xi_1\right)^4}, \quad -\frac{\sqrt{1-\sum \xi^2}}{r+\sum x\xi_1};$$

but other combinations are possible.

The choice must be made between these combinations, and, additionally, in order to define X_1, Y_1, Z_1 a third equation is needed. For such a choice, we need to make an effort to come as close as possible to Newton's law. Let us therefore look at what happens when (still keeping $t = -r$) the squares of the velocities ξ, η, etc. are neglected. The four invariants (5) then become:

$$0, \quad -r-\sum x\xi, \quad -r-\sum x\xi_1, \quad 1$$

and the four invariants (7):

$$\sum X_1^2, \quad \sum X_1(x+\xi r), \quad \sum X_1(\xi_1-\xi), \quad 0\,.$$

But to be able to make a comparison with Newton's law, another transformation is necessary; here $x_0 + x$, $y_0 + y$, $z_0 + z$ represent the coordinates of the attracting body at the moment $t_0 + t$, and $r = \sqrt{\sum x^2}$; in Newton's law, $x_0 + x_1$, $y_0 + y_1$, $z_0 + z_1$ of the attracting body at the moment t_0, and the distance $r = \sqrt{\sum x_1^2}$ need to be considered.

We can neglect the square of the time t needed for propagation and consequently proceed as if the motion were uniform; we then have:

$$x = x_1 + \xi_1 t, \quad y = y_1 + \eta_1 t, \quad z = z_1 + \zeta_1 t,$$
$$r(r-r_1) = \sum x\xi_1 t;$$

or, because $t = -r$,

$$x = x_1 - \xi_1 r, \quad y = y_1 - \eta_1 r, \quad z = z_1 - \zeta_1 r, \quad r = r_1 - \sum x\xi_1 \,;$$

such that our four invariants (5) become:

$$0, \quad -r_1 + \sum x(\xi_1 - \xi), \quad -r_1, \quad 1$$

and our four invariants (7) become:

$$\sum X_1^2, \quad \sum X_1[x_1 + (\xi - \xi_1)r_1], \quad \sum X_1(\xi_1 - \xi) \quad 0.$$

In the second of these expressions, I wrote r_1 instead of r, because r is multiplied by $\xi - \xi_1$ and because I neglected the square of ξ.

On the other hand, Newton's law would give us, for these four invariants (7),

$$\frac{1}{r_1^4}, \quad -\frac{1}{r_1} - \frac{\sum x_1(\xi - \xi_1)}{r_1^2}, \quad \frac{\sum x_1(\xi - \xi_1)}{r_1^3}, \quad 0\,.$$

If therefore we call A and B the second and third of the invariants (5) and M, N, P the first three invariants (7), we will satisfy Newton's law, up to terms of order of the square of the velocities, by making:

$$M = \frac{1}{B^4}, \quad N = \frac{+A}{B^2}, \quad P = \frac{A - B}{B^3}. \tag{8}$$

This solution is not unique. In fact, let C be the fourth invariant (5), $C - 1$ is of order of the square of ξ, and it is the same for $(A - B)^2$.

We could therefore add a term formed from $C - 1$ multiplied by an arbitrary function of A, B, C and a term formed from $(A - B)^2$ also multiplied by a function of A, B, C to the right hand side of each of equations (8).

At first sight, the solution (8) seems the simplest; it cannot however be adopted. In fact, since M, N, P are functions of X_1, Y_1, Z_1 and of $T_1 = \sum X_1\xi$, the values for X_1, Y_1, Z_1 can be drawn from these three equations (8), but in some cases these values could become imaginary.

To avoid this disadvantage, we will work in another way. Let us set:

$$k_0 = \frac{1}{\sqrt{1 - \sum \xi^2}}, \quad k_1 = \frac{1}{\sqrt{1 - \sum \xi_1^2}},$$

which is justified by analogy with the notation

$$k = \frac{1}{\sqrt{1-\varepsilon^2}}$$

which appears in Lorentz's substitution.

In this case and because of the condition $-r = t$, the invariants (5) become:

$$0, \quad A = -k_0\Big(r + \sum x\xi\Big), \quad B = -k_1\Big(r + \sum x\xi_1\Big), \quad C = k_0k_1\Big(1 - \sum \xi\xi_1\Big).$$

On the other hand, we see that the following system of quantities:

$$\begin{array}{cccc} x, & y, & z, & -r = t \\ k_0X_1, & k_0Y_1, & k_0Z_1, & k_0T_1 \\ k_0\xi, & k_0\eta, & k_0\zeta, & k_0 \\ k_1\xi_1, & k_1\eta_1, & k_1\zeta_1, & k_1 \end{array}$$

undergo the *same* linear substitutions when the transformations of the Lorentz group are applied to them. This leads us to set:

$$\begin{aligned} X_1 &= x\frac{\alpha}{k_0} + \xi\beta + \xi_1\frac{k_1}{k_0}\gamma, \\ Y_1 &= y\frac{\alpha}{k_0} + \eta\beta + \eta_1\frac{k_1}{k_0}\gamma, \\ Z_1 &= z\frac{\alpha}{k_0} + \zeta\beta + \zeta_1\frac{k_1}{k_0}\gamma, \\ T_1 &= -r\frac{\alpha}{k_0} + \beta + \frac{k_1}{k_0}\gamma. \end{aligned} \tag{9}$$

It is clear that if α, β, γ are invariants, X_1, Y_1, Z_1, T_1 will satisfy the fundamental condition, meaning will undergo an appropriate linear substitution under the effect of the Lorentz transformation.

But in order for these equations (9) to be compatible, it needs to be that we have:

$$\sum X_1\xi - T_1 = 0,$$

which, by replacing X_1, Y_1, Z_1, T_1 by their values (9) and by multiplying by k_0^2, becomes:

$$-A\alpha - \beta - C\gamma = 0. \tag{10}$$

What we want is that if the squares of the velocities ξ, etc. and also the products of the accelerations by distances are neglected compared to the square the speed of light as we have done above, then the values X_1, Y_1, Z_1 continue to satisfy Newton's laws.

We can take:

$$\beta = 0, \quad \gamma = -\frac{A\alpha}{C}.$$

With the adopted order of approximation, it follows:

$$k_0 = k_1 = 1, \quad C = 1, \quad A = -r_1 + \sum x(\xi_1 - \xi), \quad B = -r_1.$$
$$x = x_1 + \xi_1 t = x_1 - \xi_1 r.$$

The first equation (9) then becomes:

$$X_1 = \alpha(x - A\xi_1)$$

But if the square of ξ is neglected, $A\xi_1$ can be replaced by $-r_1\xi_1$ or even.by – r, which gives:

$$X_1 = \alpha(x + \xi_1 r) = \alpha x_1.$$

Newton's law would give:

$$X_1 = -\frac{x_1}{r_1^3}.$$

For the invariant α, we need to choose the value which reduces to $-1/r_1^3$ for the chosen order of approximation, meaning $1/B^3$. Equations (9) will become:

$$\begin{aligned} X_1 &= \frac{x}{k_0 B^3} - \xi_1 \frac{k_1}{k_0}\frac{A}{B^3 C}, \\ Y_1 &= \frac{y}{k_0 B^3} - \eta_1 \frac{k_1}{k_0}\frac{A}{B^3 C}, \\ Z_1 &= \frac{z}{k_0 B^3} - \zeta_1 \frac{k_1}{k_0}\frac{A}{B^3 C}, \\ T_1 &= -\frac{r}{k_0 B^3} - \frac{k_1}{k_0}\frac{A}{B^3 C}. \end{aligned} \tag{11}$$

We can first see that the corrected attraction is made up of two components: one parallel to the vector which joins the positions of the two bodies and the other parallel to the velocity of the attracting body.

We recall that when we speak of the position or the velocity of the attracting body, it is about its position or its velocity at the moment when the gravitational wave leaves it; for the attracted body it is instead about its position or its velocity at the moment when the gravitational wave reaches it, with the assumption that this wave propagates at the speed of light.

I think that it would be premature to try to move the discussion of these formulas farther; I will therefore limit myself to a few remarks.

1) The solutions (11) are not unique; $1/B^3$, which enters as a factor throughout, can in fact be replaced by

$$\frac{1}{B^3} + (C-1)f_1(A,B,C) + (A-B)^2 f_2(A,B,C),$$

where f_1 and f_2 are arbitrary functions of A, B, C or even β can now be taken as non-zero but some arbitrary terms can be added to α, β, γ provided that they satisfy the condition (10) and that they be of second-order in ξ, as it relates to α, and first-order as it relates to β and γ.

2) The first equation (11) can be written:

$$X_1 = \frac{k_1}{B^3 C}\left[x\left(1 - \sum \xi\xi_1\right) + \xi_1\left(r + \sum x\xi\right)\right] \tag{11'}$$

and the quantity between square brackets can itself be written:

$$(x + r\xi_1) + \eta(\xi_1 y - x\eta_1) + \zeta(\xi_1 z - x\zeta_1), \tag{12}$$

such that the total force can be broken down into three components corresponding to the three parentheses from the expression (12); the first component has a vague analogy with mechanical force due to the electric field and the two others with the mechanical force due to the magnetic field. To complete the analogy, I can, because of the first remark, replace $1/B^3$ in equations (11) with C/B^3, such that X_1, Y_1, Z_1 now only depend linearly on the velocity ξ, η, $\zeta$of the attracted body because C has disappeared from the denominator of (11′).

Then set:

$$\begin{aligned} k_1(x + r\xi_1) &= \lambda, & k_1(y + r\eta_1) &= \mu, & k_1(z + r\zeta_1) &= \nu, \\ k_1(\eta_1 z - \zeta_1 y) &= \lambda', & k_1(\zeta_1 x - \xi_1 z) &= \mu', & k_1(\xi_1 y - \eta_1 x) &= \nu', \end{aligned} \tag{13}$$

it follows, since C has disappeared from the denominators of (11′), that:

$$\begin{aligned} X_1 &= \frac{\lambda}{B^3} + \frac{\eta\nu' - \zeta\mu'}{B^3}, \\ Y_1 &= \frac{\mu}{B^3} + \frac{\zeta\lambda' - \xi\nu'}{B^3}, \\ Z_1 &= \frac{\nu}{B^3} + \frac{\xi\mu' - \eta\lambda'}{B^3}, \end{aligned} \tag{14}$$

and we will additionally have:

$$B^2 = \sum \lambda^2 - \sum \lambda\prime^2 \tag{15}$$

Then λ, μ, ν or λ/B^3, μ/B^3, ν/B^3, is a kind of electric field while λ', μ', ν' or instead λ'/B^3, μ'/B^3, ν'/B^3 is kind of magnetic field.

3) The relativity postulate would compel us to adopt solution (11) or solution (14) or any one of the solutions which could be deduced from them using the first remark. But, the first question which comes up is that of knowing whether they are compatible with astronomical observations. The divergence from Newton's law is of order ξ^2, meaning 10,000 times smaller than if it were of order ξ, meaning if the propagation occurs with the speed of light, everything else being equal; one could therefore hope that it will not be too large. But we will only be able to learn that from an in-depth discussion.

Paris, July 1905.

H. Poincaré

Translator's Notes

1. See Part II, Chapter 13, p. 251–252 for discussion of the pronunciation of this name.
2. For convenience, the content of this article is reformatted and provided in Part III, Chapter 14.
3. The first reference appears to be to Kaufmann, W. (1901). Die magnetische und electrische Ablenkbarkeit der Becquerelstrahlen und die scheinbare Masse der Elektronen. *Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen, 2*, 143–155; the second reference could be to Abraham, M. (1902). Dynamik des Electrons. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen*, 20–41; or to Abraham, M. (1903). Prinzipien der Dynamik des Eleckrons. *Annalen der Physik, Ser. 4 vol. 10 supplement*, 105–179.
4. Part II, Chapter 12 discusses this choice of notation and on page 228 shows how this half-page would look with our vector formalism.
5. These equations are in Part III, page 263.
6. See for example, Lorentz, H. A. (1902) Contributions to the theory of electrons. I, *Proceedings of the KNAW*, vol. 5 (1902), 608–628.
7. Presumably Poincaré meant "charge" not "mass."
8. Presumably this is a reference to Langevin, P. (1905). Sur l'origine des radiations et l'inertie électromagnétique. J. Phys. Theor. Appl., 4(1), 156–183. Poincaré provides a more complete citation on p. 46.
9. Langevin, P. (1905). Sur l'origine des radiations et l'inertie électromagnétique. *J. Phys. Theor. Appl., 4*(1), 156–183.
10. Presumably Poincaré meant "$-\varepsilon$", not "$-\varepsilon t$".
11. In this equation, note that there should not be a minus sign immediately after the equal sign.