# Chapter 2
# Lorentz's Theory and the Conservation of Momentum

Most likely it will seem strange that at a moment raised to the glory of Lorentz I return to the considerations that I previously presented as an objection to his theory. I could state that the following pages instead are of a nature to reduce this objection and not deepen it.

But I'm not going to make use of that excuse because I have one that is 100 times better. *Good theories are flexible*. Those which have a rigid form and which cannot be adapted without collapsing truly have too little vitality. But if the theory shows us some true relations, it can be dressed in a thousand various forms and it will resist all the assaults and what makes up its essence will not change. This is what I explained in the seminar that I gave recently at the *Congrès international de physique* in Paris.

Good theories overcome all objections; objections that are only specious do not get a hold on them and they triumph even over serious objections but they triumph over them by changing.

The objections help them, which is far from harming them, because the objections allow them to develop all the hidden virtues which were in them. Just so, Lorentz's theory is of that kind, and that is the only excuse that I wish to make.

I'm not asking the reader to forgive me for that, but to forgive me for having presented at such length ideas with so little novelty.

## Part 1

Let us first quickly review the calculation by which it was established that in Lorentz's theory, the principal that for every action there is an equal and opposite reaction is no longer true, at least when one wants to apply it only to matter.

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Poincaré, H. (1900). La théorie de Lorentz et le principe de réaction. *Archives néerlandaises des sciences exactes et naturelles, 5*, 252–278.



B. D. Popp, *Henri Poincaré: Electrons to Special Relativity*,
https://doi.org/10.1007/978-3-030-48039-4_2

Let us seek the resultant of all the ponderomotive forces applied to all the electrons located inside a certain volume. This resultant, or instead its projection on the x-axis, is represented by the integral:

$$X = \int \left[ \eta\gamma - \xi\beta + \frac{4\pi f}{K_0} \right] \rho \mathrm{d}\tau$$

where the integration extends over all elements $\mathrm{d}\tau$ of the volume considered, and where the $\xi$, $\eta$ and $\zeta$ represent the components of the electron velocity.

Because of the equations:

$$\rho\eta = -\frac{\mathrm{d}g}{\mathrm{d}t} + \frac{1}{4\pi}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}z} - \frac{\mathrm{d}\gamma}{\mathrm{d}x}\right); \;\; \rho\zeta = -\frac{\mathrm{d}h}{\mathrm{d}t} + \frac{1}{4\pi}\left(\frac{\mathrm{d}\beta}{\mathrm{d}x} - \frac{\mathrm{d}\alpha}{\mathrm{d}y}\right); \;\; \rho = \sum \frac{\mathrm{d}f}{\mathrm{d}x}$$

and by adding and subtracting the term:

$$\frac{\alpha}{4\pi}\frac{\mathrm{d}\alpha}{\mathrm{d}x},$$

I can write:

$$X = X_1 + X_2 + X_3 + X_4,$$

where:

$$\begin{aligned}
X_1 &= \int \left(\beta\frac{\mathrm{d}h}{\mathrm{d}t} - \gamma\frac{\mathrm{d}g}{\mathrm{d}t}\right)\mathrm{d}\tau \\
X_2 &= \frac{1}{4\pi}\int \left(\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}x} + \beta\frac{\mathrm{d}\alpha}{\mathrm{d}y} + \gamma\frac{\mathrm{d}\alpha}{\mathrm{d}z}\right)\mathrm{d}\tau \\
X_3 &= \frac{-1}{4\pi}\int \left(\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}x} + \beta\frac{\mathrm{d}\beta}{\mathrm{d}x} + \gamma\frac{\mathrm{d}\gamma}{\mathrm{d}x}\right)\mathrm{d}\tau \\
X_4 &= \frac{4\pi}{K_0}\int f \sum \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}\tau
\end{aligned}$$

Integration by parts gives

$$\begin{aligned}
X_2 &= \frac{1}{4\pi}\int \alpha(l\alpha + m\beta + u\gamma)\mathrm{d}\omega - \frac{1}{4\pi}\int \alpha\left(\frac{\mathrm{d}\alpha}{\mathrm{d}x} + \frac{\mathrm{d}\beta}{\mathrm{d}y} + \frac{\mathrm{d}\gamma}{\mathrm{d}z}\right)\mathrm{d}\tau \\
X_3 &= \frac{-1}{8\pi}\int l\left(\alpha^2 + \beta^2 + \gamma^2\right)\mathrm{d}\omega
\end{aligned}$$

where the double integrals extend over all the elements $d\omega$ of the surface which limits the volume under consideration, and where $l$, $m$ and $n$ designate the directional cosines of the normal to this element.

Observing that

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0,$$

it can be seen that one can write:

$$X_2+X_3=\frac{1}{8\pi}\int\left[l\left(\alpha^2-\beta-\gamma^2\right)+2m\alpha\beta+2n\alpha\gamma\right]d\omega. \tag{1}$$

Let us now transform $X_4$.

Integration by parts gives:

$$X_4=\frac{4\pi}{K_0}\int\left(lf^2+mfg+nfh\right)d\omega-\frac{4\pi}{K_0}\int\left(f\frac{df}{dx}+g\frac{df}{dy}+h\frac{df}{dz}\right)d\tau.$$

I call $X_4'$ and $X_4''$ the two integrals of the right-hand side, such that

$$X_4=X_4'-X_4''.$$

If one uses the equations:

$$\begin{aligned}\frac{df}{dy}&=\frac{dg}{dx}+\frac{K_0}{4\pi}\frac{d\gamma}{dt}\\ \frac{df}{dz}&=\frac{dh}{dx}-\frac{K_0}{4\pi}\frac{d\beta}{dt}\end{aligned}$$

we can write:

$$X_4''=\mathrm{Y}+\mathrm{Z}$$

where

$$\begin{aligned}Y&=\frac{4\pi}{K_0}\int\left(f\frac{df}{dx}+g\frac{dg}{dx}+h\frac{dh}{dx}\right)d\tau\\ Z&=\int\left(g\frac{d\gamma}{dt}-h\frac{d\beta}{dt}\right)d\tau.\end{aligned}$$

Next one finds:

$$Y = \frac{2\pi}{K_0} \int l\left( f^2 + g^2 + h^2\right) \mathrm{d}\omega$$
$$X_1 - Z = \frac{\mathrm{d}}{\mathrm{d}t} \int (\beta h - \gamma g) \mathrm{d}\tau.$$

Finally one has:

$$X = \frac{\mathrm{d}}{\mathrm{d}t} \int (\beta h - \gamma) \mathrm{d}\tau + (X_2 + X_3) + \left(X_4' - Y\right), \tag{2}$$

where $X_2 + X_3$ is given by formula (1), whereas:

$$X_4' - Y = \frac{2\pi}{K_0} \int \left[ l\left( f^2 - g^2 - h^2\right) + 2mfg + 2nfh \right] \mathrm{d}\omega.$$

This term, $(X_2 + X_3)$, represents the projection on the $x$-axis of a pressure acting on the differential elements $\mathrm{d}\omega$ of the surface delimiting the volume under consideration. It is immediately recognized that this pressure is nothing other than *Maxwell's magnetic pressure*, introduced by this scientist in a well-known theory.

Similarly the term, $\left(X_4' - Y\right)$, represents the effect of *Maxwell's electrostatic pressure*.

In the absence of the first term:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int (\beta h - \gamma) \mathrm{d}\tau$$

the ponderomotive force would therefore be nothing other than the result of the Maxwell pressures.

If our integrals are extended to all space, the double integrals disappear and there only remains:

$$X = \frac{\mathrm{d}}{\mathrm{d}t} \int (\beta h - \gamma) \mathrm{d}\tau$$

If therefore we call $M$ one of the material masses considered and call $V_x$, $V_y$ and $V_z$ the components of its velocity, it should hold, if conservation of momentum were applicable, that:

$$\sum MV_x = \text{const.}; \quad \sum MV_y = \text{const.}; \quad \sum MV_z = \text{const.}$$

In contrast, we have:

$$\sum MV_x + \int (\gamma g - \beta h)\mathrm{d}\tau = \text{const.}$$
$$\sum MV_y + \int (\alpha h - \gamma f)\mathrm{d}\tau = \text{const.}$$
$$\sum MV_z + \int (\beta f - \alpha g)\mathrm{d}\tau = \text{const.}$$

Observe that

$$\gamma g - \beta h, \quad \alpha h - \gamma f, \quad \beta f - \alpha g$$

are the three components of the *Poynting vector*.

If one sets:

$$J = \frac{1}{8\pi}\sum \alpha^2 + \frac{2\pi}{K_0}\sum f^2,$$

Poynting's equation in fact gives us:

$$\int \frac{\mathrm{d}J}{\mathrm{d}t}\mathrm{d}\tau = \frac{1}{K_0}\int \begin{vmatrix} l & m & n \\ \alpha & \beta & \gamma \\ f & g & h \end{vmatrix} \mathrm{d}\omega + \frac{4\pi}{K_0}\int \rho \sum f\xi \mathrm{d}\tau. \tag{3}$$

The first integral of the left-hand side represents, as is known, the quantity of electromagnetic energy which enters the volume in consideration through its surface as radiation and the second term represents the quantity of electromagnetic energy which is created inside the volume by transformation of energy of other kinds. Electromagnetic energy can be regarded as a fictitious fluid whose density is $K_0 J$ and which moves in space according to Poynting's laws. Except, it has to be allowed that this fluid is not indestructible and that in the element of volume $\mathrm{d}\tau$ during a unit time a quantity of it equal to $\frac{4\pi}{K_0}\rho \mathrm{d}\tau \sum f\xi$ is destroyed (or that an equal quantity of opposite sign is created if this expression is negative); this is what prevents us, in our reasoning, from completely comparing our fictitious fluid to a real fluid.

The quantity of this fluid which passes during unit time through a unit surface oriented perpendicularly to the $x$, $y$ or $z$ axis is equal to:

$$K_0 J U_x, \quad K_0 J U_y, \quad K_0 J U_z,$$

Here $U_x$, $U_y$ and $U_z$ are the components of the velocity of the fluid. By comparison with Poynting's formula, it is found that:

$$K_0 J U_x = \gamma g - \beta h$$
$$K_0 J U_y = \alpha h - \gamma f$$
$$K_0 J U_z = \beta f - \alpha g$$

such that our formulas become:

$$\begin{aligned}\sum MV_x + \int K_0 J U_x \mathrm{d}\tau &= \text{const.}\\ \sum MV_y + \int K_0 J U_y \mathrm{d}\tau &= \text{const.}\\ \sum MV_z + \int K_0 J U_z \mathrm{d}\tau &= \text{const.}\end{aligned} \tag{4}$$

They state that the momentum of the matter itself plus that of our fictitious fluid is given by a constant factor.

In Ordinary Mechanics, if the momentum is constant, it is concluded that the motion of the center of gravity is straight and uniform.

But here we do not have the possibility of concluding that the center of gravity of the system formed by the matter and our fictitious fluid has a straight and uniform motion; this is because this fluid is not indestructible.

The position of the center of gravity of the fictitious fluid depends on the integral over all space

$$\int xJ \mathrm{d}\tau.$$

The derivative of this integral is:

$$\int x \frac{\mathrm{d}J}{\mathrm{d}t} \mathrm{d}\tau = -\int x \left( \frac{\mathrm{d}JU_x}{\mathrm{d}x} + \frac{\mathrm{d}JU_y}{\mathrm{d}y} + \frac{\mathrm{d}JU_z}{\mathrm{d}z} \right) \mathrm{d}\tau - \frac{4\pi}{K_0} \int \rho x \sum f\xi \mathrm{d}\tau$$

Now, the first integral on the right-hand side becomes, by integration by parts:

$$\int JU_x \mathrm{d}\tau$$

$$\text{or } \frac{1}{K_0}\left(C - \sum MV_x\right)$$

where $C$ designates the constant from the right-hand side of the first equation (4).

Let us then represent the total mass of the matter by $M_0$, the coordinates of its center of gravity by $X_0$, $Y_0$ and $Z_0$, the total mass of the fictitious fluid by $M_1$, its center of gravity by $X_1$, $Y_1$ and $Z_1$, the total mass of the system (matter plus fictitious fluid) by $M_2$, and its center of gravity by $X_2$, $Y_2$ and $Z_2$, such that one has:

$$M_2 = M_0 + M_1, \quad M_2X_2 = M_0X_0 + M_1X_1,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(M_0X_0) = \sum MV_x, \quad K_0 \int xJ\mathrm{d}\tau = M_1X_1.$$

It then follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}(M_2X_2) = C - 4x \int \rho x \sum f\xi\mathrm{d}\tau \tag{3}$$

Here is how equation (3) could be stated in ordinary language.

If there is in no way any creation or destruction of electromagnetic energy, the last term disappears; hence the center of gravity of the system formed by matter and electromagnetic energy (regarded as a fictitious fluid) has a straight and uniform motion.

Let us now assume that at certain points there has been destruction of electromagnetic energy which was transformed into nonelectric energy. It would be necessary to consider the system formed not only by matter and electromagnetic energy, but by the nonelectric energy coming from the transformation of the electromagnetic energy.

But it must be agreed that this nonelectric energy remains at the point where the transformation occurred and that it is not subsequently carried along by the matter where it is ordinarily located. There is nothing in this convention which should surprise us because it is only a matter of a mathematical fiction. If this convention is adopted, the motion of the center of gravity of the system is still straight and uniform.

To extend the statement to the case where there is not only destruction, but also creation of energy, it is sufficient to assume that at each point there is some amount of nonelectric energy, at the expense of which electromagnetic energy is formed. We will then retain the preceding agreement, meaning that instead of localizing the nonelectric energy as is ordinarily done, we regard it as immobile. Under this condition, the center of gravity will again move in a straight line.

Let us now return to equation (2) by assuming the integrals extend over an infinitesimal volume. It will then mean that the resultant of the Maxwell pressures which are exerted on the surface of the volume is in equilibrium with:

1) The forces of nonelectric origin applied to the matter which is located in this volume;
2) The inertial forces of this matter;
3) The inertial forces of the fictitious fluid enclosed in this volume.

To define this inertia of the fictitious fluid, it is appropriate that the fluid that was created at an arbitrary point by transformation of the energy, first arises without velocity and that it takes on its velocity from the already existing fluid; if therefore, the quantity of fluid increases, but its velocity remains constant, there will nonetheless be some inertia to overcome because the new fluid takes on the velocity of the former fluid; the velocity of the assembly would decrease if an arbitrary cause didn't become involved to keep it constant. Similarly when there is destruction of electromagnetic energy, the fluid must lose its velocity before being destroyed by ceding it to the remaining fluid.

Because the equilibrium holds for an infinitesimal volume, it will hold for a finite volume. If in fact we decompose it into infinitesimal volumes, the equilibrium will hold for each of them. To move to a finite volume, the collection of forces applied to the various infinitesimal volumes has to be considered; solely among the Maxwell pressures only the forces exerted on the surface of the total finite volume will be retained, and those exerted on the surface elements which separate two infinitesimal contiguous volumes will be eliminated. This elimination will in no way change the equilibrium, because the pressures eliminated in that way are pairwise equal and oppositely directed.

The equilibrium will therefore again occur for the finite volume.

It will therefore occur for all space. But in this case, it is not necessary to consider either the Maxwell pressures which are zero at infinity, or the forces of nonelectric origin which are in balance because of the Newton's third law applicable to the force is considered in Ordinary Mechanics.

The two kinds of inertial forces are therefore in equilibrium; hence there is a double consequence:

1) The principle of the conservation of the projections of the momentum applies to the system of matter and fictitious fluid; we also find equation (4) again.
2) The principle of the conservation of the moments of the momentum or in other words, the *conservation of angular momentum* applies to the system of matter and fictitious fluid. This is a new consequence which supplements the data provided by equation (4).

Since, from the perspective which interests us, electromagnetic energy therefore behaves like a fluid that has inertia, it has to be concluded that if an arbitrary device after having produced electromagnetic energy transmits it by radiation in some direction, then this device will have to *recoil* like an artillery piece which fires a projectile.

Of course, this recoil will not occur if the producing device transmits energy equally in all directions; in contrast it will occur if this symmetry does not exist and if the electromagnetic energy produced is sent in a single direction, as happens for example if the device is a Hertz exciter placed at the focus of a parabolic mirror.

It is easy to evaluate numerically the magnitude of this recoil. If the device has a mass of 1 kg and if it sends 3 million J in a single direction with the speed of light, the velocity due to the recoil is 1 cm/s. In other words if the energy produced by a 3000 W machine is sent in a single direction a force of 1 dyne would be needed to keep the machine in place despite the recoil.

It is obvious that such a weak force could not be detected by experiment. But one could imagine that if, however impossible, sufficiently sensitive measurement devices to show it were available, it would in that way be possible to prove that the conservation of momentum is not applicable to matter alone, and that it would confirm Lorentz's theory and condemn other theories.

Things aren't like that; Hertz's theory and in general all the other theories lead to the same calculation as that for Lorentz.

Just now, I used the example of a Hertz exciter whose radiation would be made parallel by a parabolic mirror, I could have taken a simpler example borrowed from optics; a beam of parallel light rays strikes a mirror perpendicularly and after reflection returns in the opposite direction. Energy first propagates from left to right, for example, and is then returned from right to left by the mirror.

The mirror thus must *recoil,* and the recoil is easily calculated by the preceding considerations.

It is easy to recognize the problem which was already handled by Maxwell in sections 792 and 793 of his work. It also calls for the recoil of the mirror just like what we have deduced from Lorentz's theory.

If we go deeper into the study of the mechanism for this recoil, this is what we find. Let us consider an arbitrary volume and apply equation (2); this equation teaches us that the force of electromagnetic origin which acts on electrons—meaning on the matter contained in the volume—is equal to the resultant of the Maxwell pressures increased by a correction factor which is the derivative of the integral:

$$\int (\beta h - \gamma g)\mathrm{d}\tau.$$

If the regime is established, this integral as constant and the correction term is zero.

The recoil called for by Lorentz's theory is that which is due to the Maxwell pressure. Now, all theories call for Maxwell pressure; all theories therefore call for the same recoil.

## Part 2

But then a new question comes up. We called for the recoil in Lorentz's theory because this theory is contrary to the conservation of momentum. Among the other theories, there are some, like Hertz's theory, which do conserve momentum. How is it that they lead to the same recoil?

Let me give the explanation for this paradox right away and leave the justification of this explanation until later. In Lorentz's theory and in Hertz's theory the device which produces the energy and sends it in one direction recoils, but this energy, thus radiated, propagates by passing through some medium, such as air, for example.

In Lorentz's theory, when the air receives the energy thus radiated, it does not undergo any mechanical action; nor does it receive any either when this energy leaves it after having passed through it. In contrast, in Hertz's theory, when the air receives the energy, it is pushed forward and then it recoils when this energy leaves it. The motion of the air that the energy passes through thus balances, from the perspective of conservation of momentum, the motion of the devices which produce this energy. In Lorentz's theory, this compensation does not happen.

In fact, let's go back to Lorentz's theory and our equation (2) and apply it to a homogeneous dielectric. It is known how Lorentz represents a dielectric medium; this medium would contain electrons capable of small motions, and these motions would produce the dielectric polarization to which the effect would be added, from some perspectives, to that of the electric displacement itself.

Let $X$, $Y$ and $Z$ be the components of this polarization. It then holds:

$$\frac{\mathrm{d}X}{\mathrm{d}t}\mathrm{d}\tau = \sum \rho\xi, \quad \frac{\mathrm{d}Y}{\mathrm{d}t}\mathrm{d}\tau = \sum \rho\eta, \quad \frac{\mathrm{d}Z}{\mathrm{d}t}\mathrm{d}\tau = \sum \rho\zeta\,. \tag{5}$$

The summations on the right-hand sides are extended to all electrons contained inside the element $\mathrm{d}\tau$ and these equations can be regarded as the definition of the dielectric polarization itself.

For the expression for the resultant of the ponderomotive forces (which I no longer designate with $X$ in order to avoid any confusion with polarization), we found the integral:

$$\int \rho\left[\eta\gamma - \zeta\beta + \frac{4\pi f}{K_0}\right]\mathrm{d}\tau$$

or

$$\int \rho\eta\gamma\mathrm{d}\tau - \int \rho\zeta\beta\mathrm{d}\tau + \frac{4\pi}{K_0}\int \rho f\mathrm{d}\tau.$$

The first two integrals can be replaced by

$$\int \gamma\frac{\mathrm{d}Y}{\mathrm{d}t}\mathrm{d}\tau, \quad \int \beta\frac{\mathrm{d}Z}{\mathrm{d}t}\mathrm{d}\tau$$

because of equation (5). As for the third integral, it is zero because the total charge of an element of dielectric containing some number of electrons is zero. Our ponderomotive force therefore reduces to:

$$\int\left(\gamma\frac{\mathrm{d}Y}{\mathrm{d}t} - \beta\frac{\mathrm{d}Z}{\mathrm{d}t}\right)\mathrm{d}\tau.$$

If I designate the force due to the various Maxwell pressures by $\Pi$, such that

$$\Pi = (X_2 + X_3) + \left(X_4' - Y\right)$$

then our equation (2) becomes:

$$\Pi = \int \left(\gamma \frac{\mathrm{d}Y}{\mathrm{d}t} - \beta \frac{\mathrm{d}Z}{\mathrm{d}t}\right) \mathrm{d}\tau + \frac{\mathrm{d}}{\mathrm{d}t} \int (\gamma g - \beta h) \mathrm{d}\tau. \tag{2'}$$

Additionally there is a relation like this

$$a \frac{\mathrm{d}^2 X}{\mathrm{d}t^2} + bX = f \tag{A}$$

where $a$ and $b$ are two constants characteristic of the medium; from this it is easily deduced that:

$$X = \left(n^2 - 1\right) f \tag{B}$$

and even

$$Y = \left(n^2 - 1\right) g, \quad Z = \left(n^2 - 1\right) h$$

where $n$ is the index of refraction for the color considered.

One could be led to replace the relation (A) by others more complicated, for example if one needed to assume more complex ions. It doesn't matter, because one would always be led to equation (B).

To go farther, we are going to assume a plane wave propagating in the direction of the $x$ axis towards positive $x$, for example. If the wave is polarized in the $x$-$z$ plane, one will have

$$X = f = \alpha = Z = h = \beta = 0$$

and

$$\gamma = ng \frac{4\pi}{\sqrt{K_0}}.$$

Incorporating all of these relations, (2′) first becomes

$$\Pi = \int \gamma \frac{\mathrm{d}Y}{\mathrm{d}t} \mathrm{d}\tau + \int \gamma \frac{\mathrm{d}g}{\mathrm{d}t} \mathrm{d}\tau + \int g \frac{\mathrm{d}\gamma}{\mathrm{d}t} \mathrm{d}\tau,$$

where the first integral represents the ponderomotive force. But if the proportions,

$$\frac{g}{1}=\frac{Y}{n^2-1}=\frac{\gamma}{n\left(\frac{4\pi}{\sqrt{K_0}}\right)}$$

are considered, our equation becomes

$$\frac{\sqrt{K_0}}{4\pi}\Pi=n\left(n^2-1\right)\int g\frac{\mathrm{d}g}{\mathrm{d}t}\mathrm{d}\tau+n\int g\frac{\mathrm{d}g}{\mathrm{d}t}\mathrm{d}\tau+n\int g\frac{\mathrm{d}g}{\mathrm{d}t}\mathrm{d}\tau. \tag{6}$$

But to draw something from this formula, it has to be seen how the energy is distributed and propagates in the dielectric medium. The energy is divided into three parts: 1) electric energy, 2) magnetic energy, 3) mechanical energy due to the motion of the ions. The expressions for these three parts are respectively:

$$\frac{2\pi}{K_0}\sum f^2,\quad \frac{1}{8\pi}\sum\alpha^2,\quad \frac{2\pi}{K_0}\sum fX$$

and in the case of a plane waves, they are proportional to each other as:

$$1,\quad n^2,\quad n^2-1\,.$$

In the preceding analysis, we had what we called the momentum of the electromagnetic energy play a role. It is clear that the density of our fictitious fluid will be proportional to the sum of the two parts (electric and magnetic) of the total energy and that the third part, which is purely mechanical, will have to be set aside. But what velocity is it appropriate to give to this fluid? At first, one might think that it is the wave propagation velocity, meaning $1/(n\sqrt{K_0})$. But it is not so simple. At each point the electromagnetic energy and the mechanical energy are proportional; if therefore at one point the electromagnetic energy comes to decrease, the mechanical energy will also decrease, meaning that it will partially transform into electromagnetic energy; there will therefore be creation of the fictitious fluid.

For a moment, designate the density of the fictitious fluid by $\rho$ and its velocity, which I assume to be parallel to the x-axis, by $\xi$; I assume that all our functions depend only on $x$ and $t$, since the plane of the wave is perpendicular to the x-axis. The continuity equation is then written

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}+\frac{\mathrm{d}\rho\xi}{\mathrm{d}x}=\frac{\delta\rho}{\mathrm{d}t}$$

where $\delta\rho$ is the quantity of fictitious fluid created during time d$t$. Now, this quantity is equal to the quantity of mechanical energy destroyed, which is to the quantity of electromagnetic energy destroyed, meaning $-\mathrm{d}\rho$, as $n^2 - 1$ is to $n^2 + 1$; hence

$$\frac{\delta\rho}{n^2-1} = -\frac{\mathrm{d}\rho}{n^2+1};$$

such that our equation becomes

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}\frac{2n^2}{n^2+1} + \frac{\mathrm{d}\rho\xi}{\mathrm{d}x} = 0.$$

If $\xi$ is a constant, this equation shows us that the propagation velocity is equal to

$$\xi\frac{n^2+1}{2n^2}.$$

If the propagation velocity is $1/(n\sqrt{K_0})$, it will then hold that

$$\xi = \frac{2n^2}{(n^2+1)\sqrt{K_0}}$$

If the total energy is $J'$, the electromagnetic energy will be $J = \frac{n^2+1}{2n^2}J'$ and the momentum of the fictitious fluid will be:

$$K_0 J\xi = K_0\frac{n^2+1}{2n^2}J'\xi = \frac{J'\sqrt{K_0}}{n} \tag{7}$$

because the density of the fictitious fluid is equal to the energy multiplied by $K_0$.

Hence in equation (6) the first term of the right-hand side represents the ponderomotive force, meaning the derivative of the momentum of the matter of the dielectric, while the last two terms represent the derivative of the momentum of the fictitious fluid. These two momentums are therefore related to each other as $n^2 - 1$ and 2.

So let $\Delta$ be the density of the dielectric material, and $W_x$, $W_y$ and $W_z$ be the components of its velocity. Let's go back to equation (4). The first term $\sum MV_x$ represents the momentum of all the real matter; we will break it down in two parts. The first part, which we will continue to designate by $\sum MV_x$, will represent the momentum of the energy-producing devices; the second part will represent the momentum of the dielectrics; it will be equal to

$$\int \Delta . W_x \mathrm{d}\tau$$

such that equation (4) will become

$$\sum MV_x + \int (\Delta.W_x + K_0 J U_x) \mathrm{d}\tau = \text{const.} \tag{4'}$$

According to what we just saw, it will follow that:

$$\frac{\Delta.W_x}{n^2 - 1} = \frac{K_0 J U_x}{2}.$$

Further, let us designate, as above, the total energy by $J'$; let us also distinguish the real velocity of the fictitious fluid, meaning that which results from Poynting's law and which we have designated by $U_x$, $U_y$ and $U_z$, and the apparent velocity of the energy, meaning what would be deduced from the propagation speed of the waves and that we will designate by $U'_x$, $U'_y$ and $U'_z$. It results from equation (7) that:

$$JU_x = J'U'_x$$

Equation (4′) can be written in the form:

$$\sum MV_x + \int \left(\Delta.W_x + K_0 J' U'_x\right) \mathrm{d}\tau = \text{const.}$$

Equation (4′) shows the following: if the device radiates energy in a single direction *in the vacuum*, it experiences a recoil which is composed solely, from the perspective of the conservation of momentum, by the motion of the fictitious fluid.

But, if the radiating, instead of occurring in the vacuum, is done in a dielectric, this recoil will be compensated in part by the motion of the fictitious fluid and in part by the motion of the dielectric matter, and the fraction of the recoil of the producing device which will thus be compensated by the motion of the dielectric, meaning by the motion of real matter, will be, I state, this fraction $n^2 - 1/n^2 - 1$.

That is what follows from Lorentz's theory; how do we now switch to Hertz's theory?

The content of Mossotti's ideas on the makeup of dielectrics is known.

Dielectrics, other than the vacuum, are formed of small conducting spheres (or more generally small conducting bodies) separated from each other by an un-polarizable insulating medium analogous to the vacuum. How does one go from that to Maxwell's ideas? One imagines that the vacuum itself had the same makeup: it was not un-polarizable, but formed of conducting cells, separated by partitions formed of an ideal, insulating and un-polarizable matter. The specific inductive power of the vacuum was therefore greater than that of un-polarizable ideal matter (likewise in the primitive understanding of Mossotti, the inductive power of the dielectrics was greater than that of the vacuum, and for the same reason). And the ratio of the first of these powers to the second was even larger as the space occupied by the conducting cells was larger compared to the space occupied by the insulating partitions.

Finally, move to the limit; by regarding the inductive power of the insulating matter as infinitesimal, and at the same time the insulating partitions as infinitesimally thin, such that the space occupied by these partitions is infinitesimal, the inductive power of the vacuum remains finite. *This transition to the limit leads us to Maxwell's theory.*

All this is well known and I will limit myself to quickly reviewing it. So, *between Lorentz's theory and Hertz's theory there is the same relation as between Mossotti's theory and Maxwell's theory.*

In fact let us assume that we could attribute the same makeup to the vacuum as Lorentz attributes to ordinary dielectrics; meaning that we consider it as an un-polarizable medium in which electrons can undergo small motions.

Lorentz's formulas will still be applicable, *only $K_0$ will no longer represent the inductive power of the vacuum, but that of ideal un-polarizable medium*. Now move to the limit by assuming $K_0$ is infinitesimal; it will of course be necessary to compensate for this hypothesis by multiplying the number of electrons such that the inductive powers of the vacuum and the other dielectrics remain finite.

The theory to which this transition to the limit leads is none other than Hertz's theory.

Let $V$ be the speed of light in the vacuum. In Lorentz's basic theory, it is equal to $1/\sqrt{K_0}$; but that is no longer so in the modified theory, where it is equal to

$$\frac{1}{n_0\sqrt{K_0}},$$

where $n_0$ is the index of refraction of the vacuum relative to the un-polarizable ideal medium. If $n$ designates the index of refraction of a dielectric relative to the ordinary vacuum, its index relative to this ideal medium will be $nn_0$ and the speed of light in this dielectric will be

$$\frac{V}{n} = \frac{1}{nn_0\sqrt{K_0}}.$$

In Lorentz's formulas, $n$ must therefore be replaced by $nn_0$.

For example, the dragging of waves in Lorentz's theory is represented by the Fresnel formula,

$$v\left(1 - \frac{1}{n^2}\right)$$

In the modified theory it would be

$$v\left(1-\frac{1}{n^2{n_0}^2}\right)$$

If we move to the limit, then we must make $K_0 = 0$, hence $n_0 = \infty$; therefore in Hertz's theory, the drag will be $v$, meaning that it will be total. This consequence, contrary to Fizeau's experiment, is sufficient to condemn Hertz's theory, such that the only interest of these considerations is as a curiosity.

Let us however go back to our equation (4′). It teaches us that the fraction of the recoil which is compensated by the motion of the dielectric matter is equal to

$$\frac{n^2-1}{n^2+1}.$$

In the modification of Lorentz's theory, this fraction will be:

$$\frac{n^2{n_0}^2-1}{n^2{n_0}^2+1}.$$

If we move to the limit by making $n_0 = \infty$, this fraction is equal to 1, such that the recoil is entirely compensated by the motion of the dielectric matter. In other words, in Hertz's theory the principle of reaction is not violated and applies only to matter.

This is what would be seen again using equation (4′); if in the limit $K_0$ is zero, the term $\int K_0 J' U'_x \mathrm{d}\tau$, which represents the momentum of the fictitious fluid, also becomes zero, such that considering the momentum of the real matter is sufficient.

This consequence follows: *to experimentally demonstrate that conservation of momentum is violated in reality as it is in Lorentz's theory, it would not be sufficient to show that the energy-producing devices experience a recoil,* which would be difficult enough, *it would additionally be necessary to show that the recoil is not compensated by the motion of the dielectrics and in particular by the air through which the electromagnetic waves pass.* This would obviously be much more difficult still.

A final remark on this subject. Let us assume that the medium through which the waves pass is magnetic. A portion of the wave energy will be found in mechanical form. If $\mu$ is the magnetic permeability of the medium, the *total* magnetic energy will be:

$$\frac{\mu}{8\pi}\int\sum\alpha^2\mathrm{d}\tau$$

but only a fraction, specifically:

$$\frac{1}{8\pi}\int\sum\alpha^2 \mathrm{d}\tau$$

will strictly speaking be magnetic energy; the other part:

$$\frac{\mu-1}{8\pi}\int\sum\alpha^2 \mathrm{d}\tau$$

will be *mechanical* energy used to bring the particle currents to a shared orientation perpendicular to the field, against the elastic force which tends to bring these currents into the equilibrium orientation that they take in the absence of a magnetic field.

An analysis can be applied to these media just like the preceding analysis and where the mechanical energy would play the same role as the mechanical energy played in the case of dielectrics. It would in that way be recognized that if non-dielectric (I mean whose dielectric power would be the same as the vacuum) magnetic media existed, the matter of these media would undergo a mechanical action subsequent to the passage of the waves such that the recoil of the producing devices would in part be compensated by the motions of these media, as it is by the media of the dielectrics.

To get out of this case that does not occur in nature, we assume a medium that is both dielectric and magnetic where the fraction of the recoil compensated by the motion of the medium would be stronger than for a nonmagnetic medium of the same dielectric power.

## Part 3

Why is the conservation of momentum obvious to our thinking? It is important to consider this in order to see whether the preceding paradoxes can actually be considered as an objection to Lorentz's theory.

If this principle, in most cases, is obvious to us, it is because its negation would lead to perpetual motion; is that the case here?

Let $A$ and $B$ be two arbitrary bodies, acting on each other, but take away any external action; if the action of one were not equal to the reaction of the other, they could be attached to each other by a rod of invariable length such that they behave like *a single* solid body. Since the forces applied to this solid do not produce equilibrium, the system would start in motion and this motion would go on endlessly by accelerating, *on one condition however,* that the mutual action of the two bodies depend only on their *relative* position and *relative* velocity, but is independent of their *absolute* position and *absolute* velocity.

More generally, for an arbitrary conservative system, let $U$ be its potential energy, $m$ be the mass of one of the points of the system, $x'$, $y'$ and $z'$ be the components of its velocity; the equation for the total energy will be:

$$\sum \frac{m}{2}\left(x'^2 + y'^2 + z'^2\right) + U = \text{const.}$$

Now refer the system to moving axes driven with translational velocity $v$ parallel to the x-axis: let $x'$, y$'$ and $z'$ be the components of the relative velocity relative to these axes, it will follow:

$$x' = x_1' + r, \quad y' = y_1', \quad z' = z_1'.$$

and consequently:

$$\sum \frac{m}{2}\left(\left(x_1' + r\right)^2 + y_1'^2 + z_1'^2\right) + U = \text{const.}$$

Because the *principle of relative motion*, $U$ only depends on the *relative* position of the points of the system, the laws of relative motion do not differ from those of absolute motion and the equation for the total energy and the relative motion is written

$$\sum \frac{m}{2}\left(x_1'^2 + y_1'^2 + z_1'^2\right) + U = \text{const.}$$

By subtracting the two equations one from the other, it is found that

$$r\sum m x_1' + \frac{r^2}{2}\sum m = \text{const.} \tag{8}$$

or

$$\sum m x_1' = \text{const.} \tag{9}$$

which is the analytical expression of conservation of momentum.

The conservation of momentum therefore appears to us to be a consequence of the conservation of energy and the principle of relative motion. This last principle is necessarily obvious to our thinking when applied to an isolated system.

But in the case that concerns us, it does not involve an isolated system, because we only consider the matter itself, aside from which there is still the ether. If all material objects are driven in a shared translation, as for example in the translation of the Earth, the phenomena can differ from what they would be if this translation did not exist because the ether might not be dragged in this translation. The principle of relative motion thus understood and applied to matter alone is so lacking in obviousness to our thinking that experiments were done to show the translation of the Earth. These experiments, it is true, gave negative results but one is somewhat surprised by it.

However one question still comes up. These experiments, as I said, have given a negative result and Lorentz's theory explains this negative result. It seems that the principle of relative motion, which was not required *a priori*, is verified *a posteriori* and that the conservation of momentum should follow from it; and however that is not how it is, how did that happen?

It is because in reality, what we have called the principle of relative motion was only imperfectly confirmed as Lorentz's theory shows. It is due to a composition of effects, but:

1) This compensation only occurred by neglecting $v^2$ unless some additional hypothesis is made that I will not discuss for the moment.

It is however not significant for our purpose, because if $v^2$ is neglected, equation (8) will directly yield equation (9), meaning conservation of momentum.

2) For this compensation to happen, the phenomena have to be referred not to real time $t$, but to some *local time* $t'$ defined in the following manner.

I assume that observers placed at different points, set their watches using a light signal; that they seek to correct these signals for the transmission time but neglecting the translational motion driving them and consequently believing that the signals are transmitted equally quickly in both directions, they limit themselves to crossing the observations by sending a signal from $A$ to $B$ and then another signal from $B$ to $A$. The local time $t'$ is the time marked by the watches set in that way.

If then $V = 1/\sqrt{K_0}$ is the speed of light, and $v$ the translation of the Earth that I assume parallel to the positive x-axis, then:

$$t' = t - \frac{vx}{V^2}$$

3) The apparent energy propagates in relative motion according to the same laws as real energy in absolute motion, but the apparent energy is not exactly equal to the corresponding real energy.
4) In relative motion, the bodies producing electromagnetic energy are subject to an additional apparent force which does not exist in absolute motion.

We are going to look at how these various circumstances resolve the contradiction that I just reported.

Let us imagine an electric energy producing device arranged such that the energy produced is sent in a single direction. This will, for example, be a Hertz exciter provided with a parabolic mirror.

First at rest, the exciter sends energy in the direction of the x-axis and this energy is precisely equal to what is expended in the exciter. As we have seen, the device *recoils* and acquires some velocity.

If we refer everything to mobile axes linked to the exciter, the apparent phenomenon will have to be, except for the exceptions made above, the same as if the exciter were at rest; it is therefore going to radiate an *apparent* quantity of energy which will be equal to the energy expended in the exciter.

Next it will again experience an impulse due to the recoil, and as it is no longer at rest, but already has some velocity, this impulse will produce some work and the total energy of the exciter will increase.

If therefore the real radiated electromagnetic energy were equal to the apparent electromagnetic energy—meaning, as I just stated, the energy expended in the exciter—then the total energy increase of the device would have been obtained without any expenditure. This is contrary to the principle of conservation of energy. If therefore a recoil is produced, it is because the apparent energy is not equal to the real energy and the phenomena in relative motion are not exactly the same as in absolute motion.

Let us now look at things a little more closely. Let $v'$ be the velocity of the exciter, $v$ that of the mobile axes, which I no longer assume to be linked to the exciter, and $V$ that of the radiation; all these velocities are parallel to the positive x-axis. For simplification, we will assume that the radiation has the form of a polarized plane waves, which gives us the equations:

$$f = h = \alpha = \beta = 0,$$

$$4\pi\frac{\mathrm{d}g}{\mathrm{d}t} = -\frac{\mathrm{d}\gamma}{\mathrm{d}x}, \quad -\frac{1}{4\pi V^2}\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}g}{\mathrm{d}x}, \quad V\frac{\mathrm{d}\gamma}{\mathrm{d}x} + \frac{\mathrm{d}\gamma}{\mathrm{d}t} = 0$$

hence:

$$\gamma = 4\pi V g.$$

The real energy contained in the unit volume will be:

$$\frac{\gamma^2}{8\pi} + 2\pi V^2 g^2 = 4\pi V^2 g^2.$$

Now let us look at what happens with the apparent motion relative to the mobile axes. The apparent electric and magnetic fields are:

$$g' = g - \frac{v}{4\pi V^2}\gamma, \quad \gamma' = \gamma - 4\pi v g\,.$$

We therefore have for the apparent energy in a unit volume (neglecting $v^2$ but not $vv'$):

$$\frac{\gamma'^2}{8\pi} + 2\pi V^2 g'^2 = \left(\frac{\gamma^2}{8\pi} - vg\gamma\right) + 2\pi V^2\left(g^2 - \frac{vg\gamma}{2\pi V^2}\right)$$

or else

$$4\pi V^2 g^2 - 2vg\gamma = 4\pi V^2 g^2\left(1 - \frac{2v}{V}\right).$$

The equations of apparent motion are additionally written

$$4\pi\frac{\mathrm{d}g'}{\mathrm{d}t'} = -\frac{\mathrm{d}\gamma'}{\mathrm{d}x'}, \quad -\frac{1}{4\pi V^2}\frac{\mathrm{d}\gamma'}{\mathrm{d}t'} = \frac{\mathrm{d}g'}{\mathrm{d}x'}$$

which shows that the apparent speed of propagation is still $V$.

Let $T$ be the length of the emission; what will be the length actually occupied by the disturbance in space?

The leading edge of the disturbance left at time 0 from point 0 and at time $t$ it is located at point $Vt$; the trailing edge left at time $T$, not at point 0, but from point $v'T$, because the exciter from which it emanated has moved during the time $T$ with a velocity $v'$. This trailing edge is therefore at the moment $t$ at point $v'T + V(t - T)$. The actual length of the disturbance is therefore

$$L = Vt - [v'T + V(t - T)] = (V - v')T.$$

What is now the apparent length? The leading edge left at local time 0 from point 0; at local time $t'$ its abscissa relative to the mobile axes will be $Vt'$. The trailing edge left at time $T$ from point $v'T$ whose abscissa relative to the mobile axes is $(v' - v)T$; the corresponding local time is

$$T\left(1 - \frac{vv'}{V^2}\right).$$

At local time $t'$, it is at point $x$, where $x$ is given by the equations:

$$t' = t - \frac{vx}{V^2}, \quad x = v'T + V(t - T)$$

hence, by neglecting $v^2$:

$$x = [v'T + V(t' - T)]\left(1 + \frac{v}{V}\right).$$

The abscissa of this point relative to the mobile axes will be

$$x - vt' = (v'T - VT)\left(1 + \frac{v}{V}\right) + Vt'.$$

The apparent length of the perturbation will therefore be

$$L' = Vt' - (x - vt') = (V - v')T\Big(1 + \frac{v}{V}\Big) = L\Big(1 + \frac{v}{V}\Big).$$

The total real energy (per unit section) is therefore

$$\Big(\frac{\gamma^2}{8\pi} + 2\pi V^2 g^2\Big)L = 4\pi V^2 g^2 L,$$

and the apparent energy is

$$\begin{aligned}\Big(\frac{\gamma'^2}{8\pi} + 2\pi V^2 g'^2\Big)L' =& 4\pi V^2 g^2 L\Big(1 - \frac{2v}{V}\Big)\Big(1 + \frac{v}{V}\Big) = \\ =& 4\pi V^2 g^2 L\Big(1 - \frac{v}{V}\Big).\end{aligned}$$

If therefore $J\mathrm{d}t$ represents the real energy radiated during the time $\mathrm{d}t$, then $J\mathrm{d}t\big(1 - \frac{v}{V}\big)$ will represent the apparent energy.

Let $D\mathrm{d}t$ be the energy expended in the exciter, it is the same in real motion and in apparent motion.

It is still necessary to account for the recoil. The force of the recoil multiplied by $\mathrm{d}t$ is equal to the increase of the momentum of the fictional fluid, meaning equal to

$$\mathrm{d}tK_0 JV = \frac{J}{V}\mathrm{d}t$$

because the quantity of fluid created is $\mathrm{d}tK_0J$ and its velocity is $V$. The work done by the recoil is therefore:

$$-\frac{v'J\mathrm{d}t}{V}.$$

In the apparent motion, $v'$ needs to be replaced by $v' - v$ and $J$ by $J\big(1 - \frac{v}{V}\big)$.

The apparent work due to the recoil is therefore:

$$-\frac{(v' - v)J\mathrm{d}t}{V}\Big(1 - \frac{v}{V}\Big) = J\mathrm{d}t\Big(-\frac{v'}{V} + \frac{v}{V} + \frac{vv'}{V^2}\Big).$$

Finally in the apparent motion, the apparent additional force that I talked about above (4) must be accounted for. This additional force is equal to

$$-\frac{vJ}{V^2}$$

and its work, by neglecting $v^2$ is $-\frac{vv'}{V^2}J\mathrm{d}\tau$.

Having laid that out, the equation for the total energy in the real motion is written:

$$J - D - \frac{v'J}{V} = 0. \tag{10}$$

The first term represents the radiated energy, the second the energy expenditure and the third the work from the recoil.

The equation for the total energy in apparent motion will be written:

$$J\left(1 - \frac{v}{V}\right) - D + J\left(-\frac{v'}{V} + \frac{v}{V} + \frac{vv'}{V^2}\right) - \frac{vv'}{V^2}J = 0 \tag{11}$$

The first term represents the apparent radiated energy, the second the energy expenditure, the third the apparent work from the recoil and the fourth the work from the apparent additional force.

The agreement of equations (10) and (11) removes the appearance of contradiction reported above.

If therefore, in Lorentz's theory, the recoil can take place without violating the principle of conservation of energy, it is because the apparent energy for an observer carried along with the mobile axes is not equal to the actual energy. Let us therefore assume that our exciter undergoes a motion of recoil and that the observer is carried along in this motion ($v' = v < 0$), the exciter would appear immobile to this observer and it would seem to the observer that the exciter radiates as much energy as at rest. But in reality it will radiate less of it and this is what compensates the work of the recoil.

I could have assumed that the mobile axes are invariably linked to the exciter, meaning $v = v'$, but my analysis would not then have been able to show the role of the apparent additional force. To do that, I had to assume $v'$ much larger than $v$ such that I could neglect $v^2$ without neglecting $vv'$.

I could have also shown the need for the apparent additional forces in the following way:

The actual recoil is $J/V$; in the apparent motion, $J$ has to be replaced by $J(1 - {}^{v}\!/_{V})$ such that the apparent recoil is

$$\frac{J}{V} - \frac{Jv}{V^2}$$

To supplement the real recoil, an apparent additional force

$$-\frac{Jv}{V^2}$$

has to be added to the apparent recoil (I put the − because the recoil, as its name indicates, occurs in the negative direction).

*The existence of the apparent additional force is therefore a necessary consequence of the phenomenon of recoil.*

Thus, according to Lorentz's theory, the principle of conservation of momentum must not apply to matter alone; the principle of relative motion must not apply to matter alone either. What needs to be noted is that there is an intimate and necessary connection between these two facts.

*It would therefore be sufficient to experimentally establish one of these two for the other to be established ipso facto.* It would undoubtedly be less difficult to prove the second; but it is already nearly impossible because, for example, Mr. Liénard calculated that with a machine generating 100 kW, the apparent additional force would only be 1/600 dyne.

An important consequence follows from this correlation between these two facts; it is that Fizeau's experiment is itself already contrary to conservation of momentum. If in fact, as this experiment indicates the dragging of waves is only partial, it is because the *relative* propagation of the waves in a moving medium does not follow the same law as the propagation in a medium at rest; meaning that the principle of relative motion does not apply to matter alone and it must have to undergo at least one correction specifically that which I spoke of above (observation 2) and which consists of referring everything to our "local time". If this correction is not compensated by others, one would have to conclude the conservation of momentum is not true either for matter alone.

In that way, all theories which do not respect this principle would be condemned as a group, *unless we agree to profoundly modify all our ideas on electrodynamics*. That is an idea that I have developed at greater length in a previous article (*Éclairage Électrique*, volume V, number 40).