Chapter 12
Adoption of Vector Notation for Classical Electrodynamics

Considerations

Soon after starting to prepare these translations, I was faced with a choice about how to deal with Poincaré's vector and differential notation. Poincaré followed conventions in this matter which have not stood the test of time. The choice was therefore whether to retain the notational conventions used by Poincaré or to rewrite his equations (and affected text) to use notation more familiar and comfortable for me and my readers. Having reached this point, you're certainly aware that I retained the original notational conventions.

When I made that choice, I recognized that I would need to provide readers of these translations with some guidance; that is one purpose of this chapter.

Beyond providing guidance on understanding the notation that Poincaré used, this chapter looks at the notation used in (Lorentz, Electromagnetic phenomena in a system moving with any velocity smaller than that of, 1904), since Poincaré was certainly familiar with that notation in 1905, and then steps back over 15 years earlier to look at the notational choices made and developed by Joshua Willard Gibbs, Oliver Heaviside and Henri Poincaré in their efforts to understand James Clerk Maxwell's work on electricity and magnetism. All three resist the use of quaternions; only Poincaré continued Maxwell's use of Cartesian coordinates and explicit derivatives. In 1890 the right question is, why hadn't Poincaré seen and adopted Heaviside's notation? The answer may be related to a larger question of how readily scientific and technical knowledge passed across the English Channel at that time. By 1905 that question is no longer relevant and is replaced by the question, why did Poincaré continue to use the same notation after having seen a more compact and effective alternative? Now the answer lies somewhere in the range of preference, comfort and personal choice.

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Poincaré's Notation

While Poincaré is familiar with the difference between full and partial differentials, this is not reflected in the notation where d is used for both. In general (and there may be exceptions) it is safe to assume that when Poincaré writes $\frac{\mathrm{d}}{\mathrm{d}t}$ he in fact means $\frac{\partial}{\partial t}$ (in our notation). Please be aware that there are occasions when he does use the symbol ∂ with a different meaning which he defines on its first use.

If that were the only, or even the main, difference in notation a footnote would suffice and this chapter would be unnecessary. Poincaré writes out vectors and differentials by their individual, Cartesian components. In order to have a specific example, refer to (Poincaré, Sur la dynamique de l'électron, 1906, p. 132; p. 48–49)[1] written with Poincaré's notation.

Lorentz adopted a specific system of units so as to make the factors of 4π disappear in the formulas. I will do the same and additionally I will choose the units of length and time such that the speed of light is equal to one. Under these conditions, by calling: f, g, h the electric displacement; α, β, γ the magnetic force; F, G, H the vector potential; ψ the scalar potential; ρ the electric charge density; ξ, η, ζ the electron velocity; and u, v, w the current, then the fundamental formulas become:

$$\begin{gathered}
u=\frac{\mathrm{d}f}{\mathrm{d}t}+\rho\xi\frac{\mathrm{d}\gamma}{\mathrm{d}y}-\frac{\mathrm{d}\beta}{\mathrm{d}z},\alpha=\frac{\mathrm{d}H}{\mathrm{d}y}-\frac{\mathrm{d}G}{\mathrm{d}z},f=-\frac{\mathrm{d}F}{\mathrm{d}t}-\frac{\mathrm{d}\psi}{\mathrm{d}x}\\
\frac{\mathrm{d}\alpha}{\mathrm{d}t}=\frac{\mathrm{d}g}{\mathrm{d}z}-\frac{\mathrm{d}h}{\mathrm{d}y},\frac{\mathrm{d}\rho}{\mathrm{d}t}+\sum\frac{\mathrm{d}\rho\xi}{\mathrm{d}x}=0,\sum\frac{\mathrm{d}f}{\mathrm{d}x}=\rho,\frac{\mathrm{d}\psi}{\mathrm{d}t}+\sum\frac{\mathrm{d}F}{\mathrm{d}x}=0,\\
\square=\Delta-\frac{\mathrm{d}^2}{\mathrm{d}t^2}=\sum\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\frac{\mathrm{d}^2}{\mathrm{d}t^2},\square\psi=-\rho,\square F=-\rho\xi.
\end{gathered}\tag{1}$$

An element of matter of volume $\mathrm{d}x\mathrm{d}y\mathrm{d}z$ experiences a mechanical force whose components $X\mathrm{d}x\mathrm{d}y\mathrm{d}z$, $Z\mathrm{d}x\mathrm{d}y\mathrm{d}z$, $Y\mathrm{d}x\mathrm{d}y\mathrm{d}z$ are determined from the formula:

$$X=\rho f+\rho(\eta\gamma-\zeta\beta).\tag{2}$$

These equations are subject to a remarkable transformation discovered by Lorentz and which are of interest because they explain why no experiment is able to let us know the absolute motion of the universe. Let us set:

(continued)

[1]Here and elsewhere, the first page number refers to the original publication and the second page number (following the semicolon) refers to the page number in Part I of this book.

$$x' = kl(x + \varepsilon t), t' = kl(t + \varepsilon x), y' = ly, z' = lz, \tag{3}$$

where l and ε are arbitrary constants, and where

$$\mathrm{k} = \frac{1}{\sqrt{1 - \epsilon^2}}.$$

As we read through the first paragraph, we first see that Poincaré adopts electrostatic units for charge and takes $c = 1$. Next, he tells us that the components of the electric field and magnetic field are respectively: f, g, h and α, β, γ. We would normally write these as $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively. He provides the components of the vector potential and also the scalar potential, although these go unused. There are the components of the electron velocity ξ, η, ζ or $\boldsymbol{v}$. And finally, below equation (1) are the components of the electromagnetic force on the electron X, Y, Z or $\boldsymbol{F}$.

This isn't quite enough to allow us to make sense of equations (1) as two choices for notational compactness need to be pointed out. First, note that each of the first four equations refers to only the first component of three vector equations. This means that the first equation (dropping the current density and using the partial derivative symbol) needs to be expanded from one component

$$\frac{\partial f}{\partial t} + \rho\xi = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}$$

to include the other two components

$$\frac{\partial g}{\partial t} + \rho\eta = -\frac{\partial \gamma}{\partial x} + \frac{\partial \alpha}{\partial z},$$
$$\frac{\partial h}{\partial t} + \rho\zeta = \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}.$$

There is a vector cross product hidden in here.

Second, the next three equations are all scalar equations, but in the summations there is no indication that the sum should be done over the three components of the vectors. This means that, correctly understood, the second of these vector equations should be written:

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = \rho.$$

Here there is a vector dot product. Applying this understanding, the Laplacian (Δ) in the first equation in the last row will be understood as:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The last step is then to introduce the vector differential operator del (or nabla):

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Using del and replacing the components with vectors, we can write these three examples as:

$$\begin{aligned}
&\frac{\partial \boldsymbol{E}}{\partial t} + \rho \boldsymbol{v} = \nabla \times \boldsymbol{B}, \\
&\nabla \cdot \boldsymbol{E} = \rho, \\
&\nabla^2
\end{aligned}$$

Pulling all this together, we can rewrite this example in familiar notation:

Lorentz adopted a specific system of units so as to make the factors of 4π disappear in the formulas. I will do the same and additionally I will choose the units of length and time such that the speed of light is equal to one. Under these conditions, by calling: $\boldsymbol{E}$, the electric field; $\boldsymbol{B}$, the magnetic field; $\boldsymbol{A}$, the vector potential; ψ, the scalar potential; ρ, the electric charge density; $\boldsymbol{v} = (v_x, v_y, v_z)$, the electron velocity; and $\boldsymbol{J}$, the current density, then the fundamental formulas become:

$$\begin{gathered}
\boldsymbol{J} = \frac{\partial \boldsymbol{E}}{\partial t} + \rho \boldsymbol{v} = \nabla \times \boldsymbol{B}, \quad \boldsymbol{B} = -\nabla \times \boldsymbol{A}, \quad \boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla \psi, \\
\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}, \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \quad \nabla \cdot \boldsymbol{E} = \rho, \quad \frac{\partial \psi}{\partial t} + \nabla \cdot \boldsymbol{A} = 0 \\
\Box = \nabla^2 - \frac{\partial^2}{\partial t^2} = \sum \frac{\partial^2}{\partial {x_i}^2} - \frac{\partial^2}{\partial t^2}, \qquad \Box \psi = -\rho, \qquad \Box \boldsymbol{A} = -\rho \boldsymbol{v}.
\end{gathered} \tag{1}$$

(continued)

An element of matter of volume dxdydz experiences a mechanical force $\boldsymbol{F}$dxdydz determined from the formula:

$$\boldsymbol{F} = \rho \boldsymbol{E} + \rho(\boldsymbol{v} \times \boldsymbol{B}). \tag{2}$$

These equations are subject to a remarkable transformation discovered by Lorentz and which are of interest because they explain why no experiment is able to let us know the absolute motion of the universe. Let us set:

$$x' = kl(x + \varepsilon t), \quad t' = kl(t + \varepsilon x), \quad y' = ly, \quad z' = lz, \tag{3}$$

where l and ε are arbitrary constants, and where

$$k = \frac{1}{\sqrt{1 - \epsilon^2}}.$$

Note that the vector and scalar potentials ($\boldsymbol{A}$ and ψ) can be eliminated resulting in the familiar four Maxwell's equations for the electric and magnetic fields and a continuity equation for the charge density.

Lorentz's Notation

The previous subsection describes Poincaré's choice of notation that involves writing the components of vectors and derivatives individually, only writing one component of vector equations and using an ambiguous summation for scalar product of two vectors.

In writing (Poincaré, Sur la dynamique de l'électron, 1905) and (Poincaré, Sur la dynamique de l'électron, 1906), Poincaré heavily references Lorentz's paper from the previous year (Lorentz, Electromagnetic phenomena in a system moving with any velocity smaller than that of, 1904). We can therefore be confident that Poincaré fully understood the notation that Lorentz had used even though he did not adopt it. It is therefore worth looking at the notation in Lorentz's paper.

There, Maxwell's equations appear on page 811 (in this book it is page 261; the equation numbers are unchanged) as equations (2) together with the formula for electromagnetic force per unit charge ($\mathfrak{f}$). In Lorentz's notation the equations are:

$$\begin{aligned}
&\mathrm{div}\mathfrak{d} = \varrho, \quad \mathrm{div}\mathfrak{h} = 0,\\
&\mathrm{rot}\mathfrak{h} = \frac{1}{c}\left(\dot{\mathfrak{d}} + \varrho\mathfrak{v}\right),\\
&\mathrm{rot}\mathfrak{d} = -\frac{1}{c}\dot{\mathfrak{h}},\\
&\mathfrak{f} = \mathfrak{d} + \frac{1}{c}[\mathfrak{v}\cdot\mathfrak{h}].
\end{aligned}$$

Two notational clues are provided by Lorentz on the previous page. He states, "a vector will be denoted by a German letter" and that the notation $[\mathfrak{v}\cdot\mathfrak{h}]$ is the "vector [cross] product." rot (for rotation) is clearly curl. Replacing fraktur with bold uppercase Roman letters, a dot over a quantity with $\frac{\partial}{\partial t}$ and using curl and $\times$, these equations become:

$$\begin{aligned}
&\mathrm{div}\ \boldsymbol{D} = \rho, \quad \mathrm{div}\ \boldsymbol{H} = 0,\\
&\mathrm{curl}\ \boldsymbol{H} = \frac{1}{c}\left(\frac{\partial \boldsymbol{D}}{\partial t} + \rho\boldsymbol{v}\right),\\
&\mathrm{curl}\ \boldsymbol{D} = -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t},\\
&\boldsymbol{F} = \boldsymbol{D} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{H}.
\end{aligned}$$

Because of a difference in their definitions, $\mathfrak{f} = X/\rho$. Since both Poincaré and Lorentz are working in units where the vacuum permittivity and permeability are by definition 1, $\boldsymbol{D} = \boldsymbol{E}$ and $\boldsymbol{H} = \boldsymbol{B}$ in vacuum.

Lorentz introduces the vector and scalar potentials on page 813 (locally, page 264) in equations (11) and (12); the Laplacian and grad are defined on page 814 (also, page 264) below equation (14).

Unlike Poincaré, Lorentz's notation here shows Heaviside's influence although it has acquired a German accent. The accent comes through most clearly in the use of fraktur for vectors. As will be seen below, Augustus Föppl (Föppl, 1894) could be a plausible source. There are a few differences: Heaviside and Föppl used a large V for the vector product (instead of the square bracket notation) and used curl, not rot; Heaviside disliked the use of fraktur because it was difficult to read and write.

This would seem to be the right point to step back to about 1890.

Understanding J. C. Maxwell

Situating our attention in 1890 places us amidst several endeavors to understand *A Treatise on Electricity and Magnetism* (Maxwell, 1873) then over 15 years old. The effort appears to have had three main components: finding the focus by reducing the number of equations for the electric and magnetic fields to four; organizing and clarifying the exposition of the ideas; and providing a notational machinery driven by the demands of the physics. The result of this effort was several major works on electricity and magnetism including: (Poincaré, Électricité et optique, I Les théories de Maxwell, 1890), (Heaviside, 1893) and (Föppl, 1894). This list is not comprehensive but does include major works from three different countries in as many languages. This was not exclusively the province of English scholars.

The effort to understand Maxwell was seen as challenging. Poincaré introduces his discussion of Maxwell's theory (Poincaré, Électricité et optique, I Les théories de Maxwell, 1890, p. v) in 1890 with the statement "The first time a French reader opens Maxwell's book, a rising unease and often even distrust initially mixes with their admiration. It is only after a prolonged exchange and at the cost of great effort that this feeling passes. Some eminent minds still have it even now." Although they are not French, it is easy to imagine Gibbs, American, and Heaviside, English, agreeing with this sentiment. In the case of Heinrich Hertz in Germany, there is no need to speculate since he wrote (Hertz, Electric Waves, 1893, p. 27), "If we read Maxwell's equations and always interpret the meaning of the word 'electricity' in a suitable way, nearly all the contradictions which at first are so surprising can be made to disappear. Nevertheless, I must admit that I have not succeeded in doing this completely, or to my entire satisfaction; otherwise instead of hesitating, I would speak more definitely." This is followed by a footnote where Hertz, apparently in reference to the above quotation from Poincaré or to its spirit, writes, "Poincaré . . . expresses a similar opinion."

Returning to the focus of this chapter, the following sections look at the different notations used by J. W. Gibbs, O. Heaviside, A. Föppl and H. Poincaré, and secondarily at notation used by others whose work Poincaré likely read. The point here is to survey the notational choices at that time in order to compare them with the notation subsequently used by Poincaré in 1905.[2]

[2]For a full and comprehensive history of vector analysis, (Crowe, 1985) is an essential source.

J. W. Gibbs

Of the authors in these sections, J. W. Gibbs alone did not write a book inspired by his efforts to understand and explain Maxwell's work, even though that was the context of his work on vector notation. However, he and Heaviside were the first using vectors and vector analysis publicly in 1879 and 1882[3] respectively. In 1884 Gibbs had a book (Gibbs, 1884) on vector analysis printed privately, but copies were somewhat widely circulated. There is no positive indication that Poincaré had a copy of the book. For many years running Gibbs taught a course on this subject at Yale University; his course material forms the basis for a book (Wilson, 1901) by a former student that was published and released through normal channels.

In (Gibbs, 1884) on pages 16 and 17, Gibbs defines the *derivative* (we would say *gradient*) of a scalar u (∇u), and the *divergence* and *curl* of a vector ω (respectively $\nabla \cdot \omega$ and $\nabla \times \omega$) in terms of Cartesian unit vectors and derivatives with respect to Cartesian variables. When writing equations, he consistently uses dell (∇) and not abbreviations like grad, div, curl or rot, and uses $\cdot$ and $\times$ for the vector dot product and cross product respectively. Also notice that Gibbs (as we have also seen with Poincaré) does not notationally distinguish full and partial derivatives.

This choice of notation matches what an upper-level undergraduate physics major or graduate student during the last 50 years would have seen in (Jackson, Classical Electrodynamics, 1999). In that book Jackson's definitions corresponding to the ones from Gibbs presented in the previous paragraph appear inside the back cover.

For a second comparison, consider the older classic (Morse & Feshbach, 1953). On page 31 (equation 1.4.1) they define the gradient of the scalar ψ and use both gradψ and $\nabla\psi$. Similarly, the divergence of a vector $\boldsymbol{F}$ is defined on page 35, equation 1.4.5 with both div$\boldsymbol{F}$ and $\nabla \cdot \boldsymbol{F}$ and on page 41 with both curl$\boldsymbol{F}$ and $\nabla \times \boldsymbol{F}$. The notation using ∇ appears to have been provided for the information of readers who might encounter it somewhere else. For example, when it comes to writing Maxwell's equations, Morse and Feshbach on page 205 (equation 2.5.11) use the notation curl$\boldsymbol{E}$ and div$\boldsymbol{B}$. As will be seen next curl and div were introduced by Heaviside.

O. Heaviside

In 1882, Oliver Heaviside began a long series of papers in the English trade journal *The Electrician*[4]. It was in this series of papers that he began his use of vector notation in public writing. In a paper appearing in the *Philosophical Magazine* in

[3] These dates are from (Nahin, 2002, pp. 194-6).

[4] (Nahin, 2002) in a section of the same title on page 101 and following has an interesting description of the journal *The Electrician*. It seems unlikely that Poincaré would have seen anything written there by Heaviside, unless someone specifically brought it to his attention.

1885 Heaviside (quoted in (Nahin, 2002, p. 196)) wrote, “Owing to the extraordinary complexity of the investigation when written out in Cartesian form (which I began doing, but gave up aghast), some abbreviated method of expression becomes desirable. . . I therefore adopt with some simplification, the method of vectors, which seems indeed the only proper method.”

When Heaviside’s book on electromagnetic theory was published in 1893, he devoted an entire chapter to the presentation of vector analysis (Heaviside, 1893, Chapter 3). Early in the chapter (page 138), Heaviside references “Prof. Gibbs’s pamphlet” (Gibbs, 1884) and adds it is “an able and in some respects original little treatise on vector analysis, though too condensed and also too advanced for learners’ use.” From this quote we can also see that even when Heaviside is offering praise, his words can have a sharp edge. He also ends the paragraph sharply: “As regards his notation, however, I do not like it.” This would seem to refer to the vector product, div and curl as discussed shortly.

Proceeding into this chapter, we find first his definition of the scalar product of two vectors (Heaviside, 1893, p. 149, eqn. 12) in terms of the magnitudes of the two vectors and the cosine of the angle between them. At this point, Heaviside does not introduce the components of the vectors going into his definition. He does however apply the product to unit vectors and prove the various properties. Then, seven pages later, he shows the expansion of a vector in terms of components and orthogonal unit vectors. This is clear and without unexpected turns, except perhaps for Heaviside’s recommendation to leave the dot out of the product as one might do in scalar algebra.

Next, Heaviside defines the vector cross product, the vector product of two vectors, first in terms of the magnitude of the two vectors, the sine of the angle between them and the direction along the line perpendicular to the plane defined by the two vectors (Heaviside, 1893, p. 157, eqn. 34). He then gives the definition of the vector product in terms of the components of the two vectors and the unit vectors (Heaviside, 1893, p. 159, eqn. 41). Heaviside represented the scalar product of two vectors, the dot product, with a dot centered between the two vectors. In contrast, for the vector product, he does not place a symbol between the two vectors—like a cross, $\times$, for example—and places a V in front of the two vectors. In that way, Heaviside writes the vector product of vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ as V$\boldsymbol{AB}$.[5] This is of course not what we are familiar with for the notation for vector product. First, it isn’t suggestive of a cross product and second one can easily imagine the potential for confusion between the symbol V for vector product and V used for some other magnitude. Föppl, as we’ll discuss below, uses a typographically distinctive variant of V; that would offer one possibility for reducing confusion. Following the choice made by Gibbs and using the cross-product notation we’re familiar with is a much better choice.

[5]This appears suggestive of the notation for the vector component of a quaternion product.

The next definition provided by Heaviside is the differential operator ∇ (Heaviside, 1893, p. 178, eqn 119):

$$\nabla = i\frac{\mathrm{d}}{\mathrm{d}x} + j\frac{\mathrm{d}}{\mathrm{d}y} + k\frac{\mathrm{d}}{\mathrm{d}z}.$$

which he notes, "is a fictitious vector, inasmuch as … its components are not magnitudes but differentiators."

With that notational machinery defined, Heaviside is then in the position to define first the divergence (Heaviside, 1893, p. 188, eqn. 143) and then the curl (Heaviside, 1893, p. 191, eqn. 149). Each definition is provided first in terms of the scalar and vector products (respectively) of ∇ and a vector. In that way, his definition of curl starts with $\mathrm{V}\,\nabla\,\boldsymbol{E}$, continues with the components and unit vectors, and ends with curl$\boldsymbol{E}$.

Heaviside explains his choice to use div and curl on page 194, writing, "The scalar product of ∇ and $\boldsymbol{D}$ conveys no such distinct idea as does divergence; nor does the vector product of ∇ and $\boldsymbol{E}$ speak so plainly as the curl or rotation of *E*." We can take this as Heaviside's explanation for what he did not like about Gibbs's notation, which used $\nabla \cdot \boldsymbol{D}$ and $\nabla \times \boldsymbol{E}$. The div and curl notation used in Morse and Feschbach, described above, would therefore meet with Heaviside's approval.

A. Föppl

Following our look at Heaviside's notation, we turn our attention to Föppl's book (Föppl, 1894). This order is appropriate because Föppl's work benefits from Heaviside's work. In his Foreword (Föppl, 1894, S. VII), Föppl wrote, "In presenting calculation with vectors, and in many other respects, I followed most closely the pattern provided by O. Heaviside in his treatises—which have recently been made available in bookstores as a collection. My presentation is altogether more influenced by the work of this master than by that of any other physicist—excepting Maxwell himself, of course. I consider Heaviside to be the outstanding successor to Maxwell in speculative-critical respects, just as undoubtedly Hertz—whom we unfortunately lost so young—was his successor in experimentally and confirming respects."[6] I think we should assume that Föppl is referring to the published collection of papers that Heaviside had written for *The Electrician*, and not the 1893 book that we were just discussing. Either way, in light of this testimony and the notation used in the book we can conclude that this is not a wholly independent effort to understand and present Maxwell's work and is all the same a worthy effort to reach a larger audience. That audience appears to have included

[6]Translation from German provided by Ilse Andrews, personal communication, January 8, 2018.

Albert Einstein as this book is widely cited as the textbook from which he learned electricity and magnetism.

To our point in this chapter, Augustus Föppl is clear in emphasizing the significant advantage of using vector notation (Heaviside's in his case) instead of writing out Cartesian coordinates.

H. Hertz

Of secondary importance, there are three other physicists whose work on electricity and magnetism Poincaré certainly read. Let us look first at the notation of Heinrich Hertz. In various places Poincaré discusses Hertz's theory of electromagnetic radiation. Most likely, Poincaré would have encountered this in (Hertz, Die Kräft electrischer Schwingungen behandelt nach der Maxwell'schen Theorie, 1889). This article is also available in translation as part of a collection in (Hertz, Electric Waves, 1893). In that article, and in others in the collection, Hertz uses notation based on Cartesian coordinates similar to Poincaré and does not use vector analysis and vector differential operators.

J. Larmor

Next, look at (Larmor, 1893). Poincaré wrote a series of commentaries on this paper, however I haven't discussed them elsewhere in this book. There are very few mathematical equations in Larmor's work and none involve vectors. This would not have led Poincaré to read or consider a different notation

P. Langevin

Finally look at (Langevin, 1905). Poincaré for example summarizes this paper in (Poincaré, Sur la dynamique de l'électron, 1906 §5). Langevin makes limited use of vectors, although he does write out the vector potential and position, velocity and acceleration of a body in Cartesian coordinates.

H. Poincaré: *Électricité et optique*

In the 1880s, Poincaré took up the study of Maxwell's work and in 1890 he published a book with his lectures from the second semester of 1888–89 on Maxwell's theory (Poincaré, Électricité et optique, I Les théories de Maxwell, 1890); he did not use vectors and vector analysis, instead sticking with Cartesian coordinates. Gibbs's book, *Elements of Vector Analysis*, was printed in 1881. Heaviside's book, *Electromagnetic Theory*, with a chapter on his independent vector analysis was published in 1893. And, Föppl's book in which he had two chapters on vector analysis based on Heaviside's work was published in 1894. The four—publishing books within a few years of each other, faced with understanding Maxwell's theory and working with the mathematical notation for electrodynamics—abandoned the use of W. Hamilton's quaternions that Maxwell had used in 1873 along with Cartesian components. In contrast to the other three Poincaré did not develop vector analysis independently or adopt it then or later.

First Edition

In these published lectures, Poincaré uses the same notation as (Poincaré, Sur la dynamique de l'électron, 1906, p. 132) 15 years later. In fact by using this notation (repeated here on page 229), it is easy to recognize that equation (3) (Poincaré, 1890, p. 15): $\frac{\mathrm{d}f}{\mathrm{d}x}+\frac{\mathrm{d}g}{\mathrm{d}y}+\frac{\mathrm{d}h}{\mathrm{d}z}=\rho$ states that divergence of the electric field is due to the charge density. Likewise (Poincaré, 1890, p. 146):

$$\frac{\mathrm{d}\alpha}{\mathrm{d}x}+\frac{\mathrm{d}\beta}{\mathrm{d}y}+\frac{\mathrm{d}\gamma}{\mathrm{d}z}=0$$

states that the divergence of the magnetic field is zero.

Similarly, one can read in equation (3) on page 144:

$$\begin{aligned}\alpha &= \frac{\mathrm{d}H}{\mathrm{d}y}-\frac{\mathrm{d}G}{\mathrm{d}z}\\ \beta &= \frac{\mathrm{d}F}{\mathrm{d}z}-\frac{\mathrm{d}H}{\mathrm{d}x}\\ \gamma &= \frac{\mathrm{d}G}{\mathrm{d}x}-\frac{\mathrm{d}F}{\mathrm{d}y}\end{aligned}$$

meaning that the magnetic field is given by the curl of the vector potential. (Note that in the above equations all derivatives are partial derivatives.)

In these examples and in general skimming (Poincaré, 1890), while the notation appears bulky, it is not an impediment after some practice. In fact, written this way, one can by inspection see that the divergence of the curl of the vector potential is identically zero. Written as $\nabla\cdot\nabla\times\boldsymbol{A}$, such a verification by inspection is not possible. It is understandable that Poincaré could have felt comfortable with this notation using Cartesian coordinates and explicit derivatives in 1890 and therefore not been motivated to independently develop vector analysis as Gibbs and Heaviside did.

Second Edition

In 1901 a second edition of *Électricité et optique* was published (Poincaré, Électricité et optique, 1901), now including lectures from 1899. In a *Notice before the Introduction*, Poincaré states that the lecture notes from his courses in 1888 in 1890 are reprinted "with some reworking and modification" and some material deleted because it was superseded by the lectures from 1899. The new material is presented in the second part.

A comparison of the tables of contents from the two editions shows no changes in the subsection titles, the title of Chapter II is changed, a new subsection is added to Chapter III and the original Chapter XIII has been deleted. The deleted chapter is clearly the material Poincaré referred to in the Notice. During casual comparison of the other chapters, I found a couple of added sentences that clarified what was already there and did not add new content. The reworking is not an overhaul of the previous content and may only be responses to particular questions or requests for clarification from readers or students. The new subsection in Chapter III is a Remark providing a proof of an assumption made in the preceding calculations.

The equations reproduced and discussed above from pages 15, 146 and 144 (Poincaré, Électricité et optique, I Les théories de Maxwell, 1890) now appear on pages 15, 118 and 117 (Poincaré, Électricité et optique, 1901). These equations have not been changed from the first to second edition of *Électricité et optique*. Since there is no other indication that Poincaré made more than spot changes, the absence of changes in these equations only suggests that he was satisfied with the notation involving explicit Cartesian coordinates that he used 10 years later.

The second part containing the new material in the second edition (Poincaré, Électricité et optique, 1901) starts on page 229. Scanning a few pages suffices to show that he is continuing to use the same terminology. For example, at the top of page 241 there is the formula:

$$\frac{\mathrm{d}F}{\mathrm{d}x}+\frac{\mathrm{d}G}{\mathrm{d}y}+\frac{\mathrm{d}H}{\mathrm{d}z}=-\int\left(\frac{\mathrm{d}f}{\mathrm{d}x'}\mathrm{d}x'+\frac{\mathrm{d}f}{\mathrm{d}y'}\mathrm{d}y'+\frac{\mathrm{d}f}{\mathrm{d}z'}\mathrm{d}z'\right)$$

(where F, G and H are components of the vector potential and f is a function of the potential energy and separation of two current loops). It is more compact to write this as:

$$\nabla\cdot\boldsymbol{A}=-\int\nabla f\cdot\mathrm{d}\boldsymbol{x}.$$

In the first instance of its use with this meaning, on page 292, a summation sign ($\sum$) is used as shorthand for vector products. As an explanation, Poincaré writes with reference to equation (16 *bis*), "the sign indicates a cyclic permutation to be done on the letters α, β, γ; x, y, z, and F, G, H." This is the same convention for dot products as used in equation 1 (Poincaré, Sur la dynamique de l'électron, 1906) and repeated above on page 228; in particular it is discussed in the second observation on notational compactness.

The first line of equation (16 *bis*) as published is:

$$T = \frac{1}{8\pi} \int \sum \left(\frac{\mathrm{d}\gamma}{\mathrm{d}y} - \frac{\mathrm{d}\beta}{\mathrm{d}z} \right) F \mathrm{d}\tau$$

and following Poincaré's explanation, it should be expanded to:

$$T = \frac{1}{8\pi} \int \left[\left(\frac{\mathrm{d}\gamma}{\mathrm{d}y} - \frac{\mathrm{d}\beta}{\mathrm{d}z} \right) F + \left(\frac{\mathrm{d}\alpha}{\mathrm{d}z} - \frac{\mathrm{d}\gamma}{\mathrm{d}x} \right) G + \left(\frac{\mathrm{d}\beta}{\mathrm{d}x} - \frac{\mathrm{d}\alpha}{\mathrm{d}y} \right) H \right] \mathrm{d}\tau$$

where $\mathrm{d}\tau$ is an infinitesimal volume element.

In our familiar notation this is:

$$T = \frac{1}{8\pi} \int \nabla \times \boldsymbol{B} \cdot \boldsymbol{A} \mathrm{d}\tau$$

Poincaré uses integration by parts to show that this is equal to:

$$T = \frac{1}{8\pi} \int \boldsymbol{B} \cdot \nabla \times \boldsymbol{A} \mathrm{d}\tau = \frac{1}{8\pi} \int \boldsymbol{B} \cdot \boldsymbol{B} \mathrm{d}\tau$$

Explicitly writing the Cartesian coordinates, as Poincaré does, certainly makes it easier to see and verify the integration by parts. Perhaps Poincaré did see a practical advantage to staying with the familiar notation.

Poincaré's Notation, Again

If Poincaré had not seen vector analysis with differential operators earlier, he did see it when he studied (Lorentz, Electromagnetic phenomena in a system moving with any velocity smaller than that of, 1904), since Lorentz had used vectors and differential operators. Poincaré therefore did understand the notation. Even after this exposure, Poincaré continued to use in his writing largely the same notation for the partial differential equations of electrodynamics that he had used in 1890.

This consistency in his choice of notation makes it difficult to identify other occasions between 1890 and 1904 when Poincaré might have been exposed to vector analysis and differential operators. In his writing, Poincaré is sparing in his use of references and when they are provided, they appear in-line in brief form. Notation is clearly an area where Poincaré is conservative in his choices.

It is not only the notation that makes (Poincaré, 1890) seem somewhat unsatisfactory. Earlier I indicated that the effort to understand Maxwell required effort and three main components (above, page 233). Notation, just discussed, was the third of these components.

The first of the three components was focus. Relating to the focus in both editions of *Électricité et optique*, the most obvious issue is the use of the scalar and vector potentials (whose components are F, G, H in the above equations). Heaviside explicitly abandons the vector potential (Heaviside, 1893, p. 46), "[Maxwell] however, makes use of an auxiliary function, the vector potential of the electric current, and this rather complicates the matter, especially as regards the physical meaning of the process. It is always desirable when possible to keep as near as one can to first principles."

Another issue with focus is the use of auxiliary variables for "displacement velocity" and other quantities which conceal the time derivatives of electric and magnetic field components and also results in confusion with current density.

The strong point of Poincaré's work is the exposition; he has done a good job in these lectures of organizing and presenting Maxwell's theory for his students.

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