Chapter 2
Harmonic Forcing of Damped Non-homogeneous Euler-Bernoulli Beams

Arnaldo J. Mazzei Jr. and Richard A. Scott

Abstract This work is an extension of previous studies on vibrations of non-homogeneous structures. It also explores the use of logistic functions. In the studies, frequency response functions (FRFs) were determined for segmented structures, using analytic and numerical approaches. The structures are composed of stacked cells, which are made of different materials and may have different geometric properties. Here the steady state response, due to harmonic forcing, of a segmented damped Euler-Bernoulli beam is investigated. FRFs for the system are sought via two methods. The first uses the displacement differential equations for each segment. Boundary and interface continuity conditions are used to determine the constants involved in the solutions. Then the response, as a function of forcing frequency, can be obtained. This procedure is unwieldy. In addition, determining particular integrals can become cumbersome for arbitrary spatial variations. The second approach uses logistic functions to model the segment discontinuities. The result is a single partial differential equation with variable coefficients. Approaches for numerical solutions are then developed with the aid of MAPLE® software. For free-fixed boundary conditions, spatially constant force and viscous damping, excellent agreement is found between the methods. The numerical approach is then used to obtain the FRF for the case of a spatially varying force.

Keywords Segmented beams · Layered structures · Logistic functions · Resonances of non-homogenous structures

Nomenclature

A	Cross-section area (A_i, cross-section area for i-th material)
a_i	Non-dimensional parameters related to beam properties
B_i	Constants of integration
C_i	Viscous damping coefficient per unit length
CD_i	Non-dimensional damping coefficient
E	Young's modulus (E_i, Young's modulus for i-th material)
f_i	Non-dimensional logistic functions
F_i	Forcing functions (force per unit length)
G_i	Non-dimensional spatial forcing functions
g_i	Non-dimensional forcing functions
I	Area moment of inertia of the beam cross section (I_i, moment of inertia of i-cell)
K	Non-dimensional logistic function parameter
$K_{1,i}$	Non-dimensional ODE parameters related to beam properties
L	Length of beam (L_i, length of i-th cell)
R_i	Non-dimensional spatial functions
t	Time
u	Non-dimensional transverse displacement of the beam, $u = w/L$

A. J. Mazzei Jr. (✉)
Department of Mechanical Engineering, C. S. Mott Engineering and Science Center, Kettering University, Flint, MI, USA
e-mail: amazzei@kettering.edu

R. A. Scott
University of Michigan, Ann Arbor, MI, USA
e-mail: car@umich.edu

D. S. Epp (ed.), *Special Topics in Structural Dynamics & Experimental Techniques, Volume 5*, Conference Proceedings of the Society for Experimental Mechanics Series, https://doi.org/10.1007/978-3-030-47709-7_2

x Longitudinal co-ordinate
w Transverse displacement of the beam
α, β Non-dimensional ODE coefficients
γ Non-dimensional viscosity coefficient
η Non-dimensional structural damping coefficient
λ Complex frequency, $\lambda = (a + bI)$
ξ Non-dimensional spatial co-ordinate, $\xi = x/L$
ρ Mass density (ρ_i, density value for i-th material)
τ Non-dimensional time, $\tau = \Omega_0 t$
Ω_0 Reference frequency

2.1 Introduction

This work adds to a series (see references [1, 2]) on transverse vibrations of layered beams. Here the main interest is the vibration analysis, both theoretical and numerical, of segmented damped beams. The media of interest are structures with different materials and varying cross-sections, which are layered in cells and may be uniform or not.

The objective is the determination of frequency response functions (FRFs). Here, Euler-Bernoulli theory is used for a two-segment configuration under harmonic forcing.

To treat the problem two approaches are discussed. In the first, analytic solutions are derived for the differential equations for each segment. The constants involved are determined using boundary and interface continuity conditions. The response, at a given location, can then be obtained as a function of forcing frequency (FRF). Note that the procedure can become unwieldy for arbitrary spatial variations. In the second, the discrete cell properties are modeled by continuously varying functions, specifically logistic functions. This provides the advantage of working with a single differential equation (albeit one with variable coefficients). The differential equation is then solved numerically by means of MAPLE®[1] software.

Similar analytical and numerical approaches were applied to undamped beams in references [1] and [2] (Euler-Bernoulli and Timoshenko models). Overall results showed that the numerical method worked very well, for both beam models, when compared to the analytical solutions.

In the following a brief literature review is given. For vibrations of layered beams one may refer to the list given in references [1] and [2]. Solids composed by discrete layers are studied in references [3], [4], [5] and [6]. Also, a recent article [7] provides a review of references on this subject. Vibrations of damped Euler-Bernoulli beams are treated in reference [8]. There the model consists of a uniform elastically supported beam, which incorporates different dissipation mechanisms. The oscillatory character of solutions is investigated. In reference [9] estimates for the solutions of an abstract second order evolution equation are given. Applications to models of an elastic beam (possibly non-homogeneous), with a frequency-proportional damping, are discussed. Damping mechanisms on beams are studied in reference [10], where different models of dissipation are presented to account for experimentally observed behavior. Damping effect on beam vibrations is also discussed in reference [11], where the dissipation mechanism is due to air. Minimal influence is observed. The response of internally damped cantilever beams to sinusoidal vibration is given in reference [12]. An expression for the magnitude of the force transmitted to the ends of the beams is derived when both a primary and a secondary force are employed to excite the beams at arbitrary positions. It is demonstrated that the transmitted force can be attenuated significantly through broad ranges of frequency when the primary and secondary forces are of equal magnitude and their location is chosen suitably. Reference [13] treats the spectrum of Euler–Bernoulli beam equation with Kelvin–Voigt damping. Under some assumptions on the coefficients, it is shown that the essential spectrum contains continuous spectrum only, and the point spectrum consists of isolated eigenvalues of finite algebraic multiplicity. The asymptotic behavior of eigenvalues is presented. Damping effects on Timoshenko beams can be found in reference [14]. The numerical analyses provided for outlining the relevant influences on the dynamic response associated to any singular damping mechanism and also the evaluation of the modal critical damping values.

2.2 Basic Structure

The equation of motion for a viscously damped Euler-Bernoulli beam is given below. Figure 2.1 exhibits the underlying variables.

[1] www.maplesoft.com

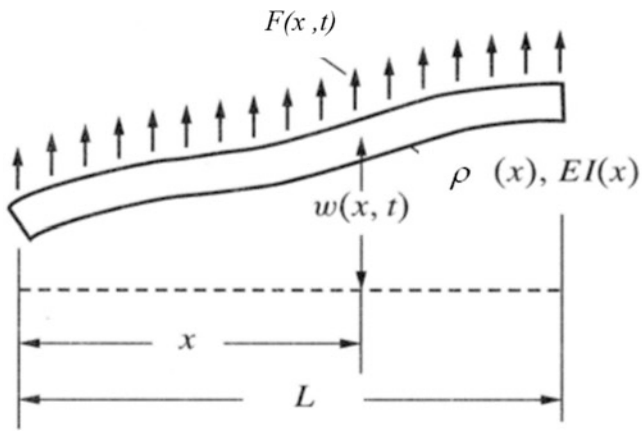


Fig. 2.1 Beam element

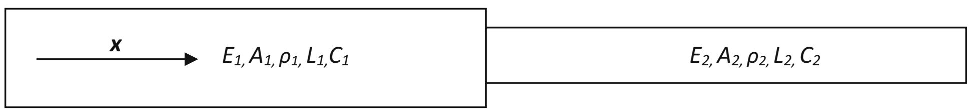


Fig. 2.2 Layered beam

$$\frac{\partial^2}{\partial x^2}\left(E(x)I(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right)+\rho(x)A(x)\frac{\partial^2 w(x,t)}{\partial t^2}+C(x)\frac{\partial w(x,t)}{\partial t}=F(x,t) \tag{2.1}$$

The beam is assumed to be under a damping force, with *C(x)* representing a varying viscous damping coefficient per unit length. Material type can be either homogeneous or a non-homogeneous.

In the following the configuration discussed consists of a beam composed of two cells. Consider the beam shown in Fig. 2.2 which has two cells of different materials. E, ρ, and A may vary in a discontinuous manner. The segments are under transverse loads f_1 and f_2 (force per unit length) and viscous damping forces (per unit length, damping coefficients C_1 and C_2).

Approaches for obtaining the steady state response, due to harmonic forcing, are sought next.

2.3 Solution Approaches

In non-dimensional form, eq. (2.1) can be written as:

$$\frac{\partial^2}{\partial \xi^2}\left(f_1(\xi)\, f_2(\xi)\,\frac{\partial^2 u(\xi,\tau)}{\partial \xi^2}\right)+f_3(\xi)\, f_4(\xi)\,\frac{\partial^2 u(\xi,\tau)}{\partial \tau^2}+(CD)\, f_5(\xi)\,\frac{\partial u(\xi,\tau)}{\partial \tau}=g(\xi,\tau) \tag{2.2}$$

where: $\tau=\Omega_0 t$, $\Omega_0=\sqrt{\frac{E_1 I_1}{\rho_1 A_1 L^4}}$, $\xi=\frac{x}{L}$, $u(\xi,\tau)=\frac{w(x,t)}{L}$, $CD=C_1\sqrt{\frac{L^4}{\rho_1 A_1 E_1 I_1}}$, $g(\xi,\tau)=F(x,t)\,\frac{L^3}{E_1 I_1}$, $E(x)=E_1 f_1(\xi)$, $I(x)=I_1 f_2(\xi)$, $\rho(x)=\rho_1 f_3(\xi)$, $A(x)=A_1 f_4(\xi)$, $C(x)=C_1 f_5(\xi)$. f_i are functions representing the transitions from one cell to another. (For the continuous variation approach logistic functions will be utilized, details are given below.)

For harmonic forcing with frequency λ:

$$g(\xi,\tau)=G(\xi)\, e^{\lambda\tau} \tag{2.3}$$

One can assume solutions of the following form:

$$u(\xi,\tau) = R(\xi)\, e^{\lambda\tau} \tag{2.4}$$

This leads to:

$$\frac{d^2}{d\xi^2}\left(f_1(\xi)\, f_2(\xi)\, \frac{d^2R(\xi)}{d\xi^2}\right) + \lambda^2 f_3(\xi)\, f_4(\xi)\, R(\xi) + \lambda(CD) f_5(\xi)\, R(\xi) = G(\xi) \tag{2.5}$$

Taking $\lambda = (a + bI)$ and separating real and imaginary parts, after some manipulation, gives:

$$\frac{d^2}{d\xi^2}\left(f_1(\xi)\, f_2(\xi)\, \frac{d^2R(\xi)}{d\xi^2}\right) - \left(b^2 f_3(\xi)\, f_4(\xi) + \frac{CD^2(f_5(\xi))^2}{4 f_3(\xi)\, f_4(\xi)}\right) R(\xi) = G(\xi) \tag{2.6}$$

The result, eq. (2.6), is a non-homogeneous ordinary differential equation with variable coefficients. Analytic and numerical solutions are discussed next.

2.3.1 Analytical Approach

For constant properties in each segment, eq. (2.6) can be written as a system of n-equations, where n is the number of cells. For two cells:

$$\frac{d^4R_i(\xi)}{d\xi^4} - \left(K_{1,i}b^2 + K_{2,i}\frac{CD^2}{4}\right) R_i(\xi) = K_{3,i}G_1(\xi)\,,\ i = 1, 2 \tag{2.7}$$

where: $K_{1,1} = 1$, $K_{2,1} = 1$, $K_{3,1} = 1$, $K_{1,2} = \frac{a_3a_4}{a_1a_2}$, $K_{2,2} = \frac{a_5^2}{a_1a_2a_3a_4}$, $K_{3,2} = \frac{a_6}{a_1a_2}$, $a_1 = \frac{E_2}{E_1}$, $a_2 = \frac{I_2}{I_1}$, $a_3 = \frac{\rho_2}{\rho_1}$, $a_4 = \frac{A_2}{A_1}$, $a_5 = \frac{CD_2}{CD_1}$, $a_6 = \frac{G_2}{G_1}$.

General solutions to the linear differential eqs. (2.7) involve solutions to the homogeneous equations and "particular integrals". For arbitrary forcing $G_i(\xi)$, finding tractable particular solutions may pose a problem. Consequently, a constant spatial force is discussed here. (Non-constant spatial forcing is treated later numerically.)

The following forcing is considered: $G_1(\xi) = G_{1,0}$, $G_2(\xi) = G_{2,0}$; $G_{1,0}$, $G_{2,0}$ constants.

Then the general solutions can be written:

$$R_1(\xi) = B_1\cos(\alpha\xi) + B_2e^{\alpha\xi} + B_3\sin(\alpha\xi) + B_4e^{-\alpha\xi} - \frac{G_{1,0}}{\alpha^4} \tag{2.8}$$

$$R_2(\xi) = B_5\cos(\beta\xi) + B_6e^{\beta\xi} + B_7\sin(\beta\xi) + B_8e^{-\beta\xi} - K_{3,2}\frac{G_{1,0}}{\beta^4} \tag{2.9}$$

Where B_i are constants to be determined and $\alpha^4 = \left(K_{1,1}b^2 + K_{2,1}\frac{CD^2}{4}\right)$, $\beta^4 = \left(K_{1,2}b^2 + K_{2,2}\frac{CD^2}{4}\right)$.

The overall analytic solution requires that the boundary conditions be defined. Two sets are considered below.

Free-Fixed Boundary Conditions

For these conditions the moment and shear free end at $\xi = 0$ gives: $\left.\frac{d^2R_1(\xi)}{d\xi^2}\right|_{\xi=0} = 0$ and $\left.\frac{d^3R_1(\xi)}{d\xi^3}\right|_{\xi=0} = 0$. The fixed boundary condition at $\xi = 1$ gives: $R_2(1) = 0$ and $\left.\frac{dR_2(\xi)}{d\xi}\right|_{\xi=1} = 0$. Interface continuity conditions (assuming the cells have the same length) provide: $R_1(\xi) = R_2(\xi)$, $\xi = 0.5$ (displacement continuity), $\frac{dR_1(\xi)}{d\xi} = \frac{dR_2(\xi)}{d\xi}$, $\xi = 0.5$ (slope continuity), $\frac{d^2R_1(\xi)}{d\xi^2} =$

$a_1 a_2 \frac{d^2 R_2(\xi)}{d\xi^2}, \xi = 0.5$ (moment continuity) and $\frac{d^3 R_1(\xi)}{d\xi^3} = a_1 a_2 \frac{d^3 R_2(\xi)}{d\xi^3}, \xi = 0.5$ (shear continuity). These conditions lead to a system of algebraic equations, which in matrix form can be written as:

$$[\mathrm{A}]\{\mathrm{B}\} = \{G\} \tag{2.10}$$

Where:

$$[A] = \begin{bmatrix} -\alpha^2 & \alpha^2 & 0 & \alpha^2 & 0 & 0 & 0 & 0 \\ 0 & \alpha^3 & -\alpha^3 & -\alpha^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\beta) & e^{\beta} & \sin(\beta) & e^{-\beta} \\ 0 & 0 & 0 & 0 & -\beta\ \sin(\beta) & \beta\ e^{\beta} & \beta\ \cos(\beta) & -\beta\ e^{-\beta} \\ \cos\ (0.5\ \alpha) & e^{0.5\ \alpha} & \sin(0.5\ \alpha) & e^{-0.5\ \alpha} & -\cos(0.5\ \beta) & -e^{0.5\ \beta} & -\sin(0.5\ \beta) & -e^{-0.5\ \beta} \\ -\alpha\ \sin(0.5\ \alpha) & \alpha\ e^{0.5\ \alpha} & \alpha\ \cos(0.5\ \alpha) & -\alpha\ e^{-0.5\ \alpha} & \beta\ \sin(0.5\ \beta) & -\beta\ e^{0.5\ \beta} & -\beta\ \cos(0.5\ \beta) & \beta\ e^{-0.5\ \beta} \\ -\alpha^2\ \cos(0.5\ \alpha) & \alpha^2\ e^{0.5\ \alpha} & -\alpha^2\ \sin(0.5\ \alpha) & \alpha^2\ e^{-0.5\ \alpha} & al\ a2\ \beta^2\ \cos(0.5\ \beta) & -al\ a2\ \beta^2\ e^{0.5\ \beta} & al\ a2\ \beta^2\ \sin(0.5\ \beta) & -al\ a2\ \beta^2\ e^{-0.5\ \beta} \\ \alpha^3\ \sin(0.5\ \alpha) & \alpha^3\ e^{0.5\ \alpha} & -\alpha^3\ \cos(0.5\ \alpha) & -\alpha^3\ e^{-0.5\ \alpha} & -al\ a2\ \beta^3\ \sin(0.5\ \beta) & -al\ a2\ \beta^3\ e^{0.5\ \beta} & al\ a2\ \beta^3\ \cos(0.5\ \beta) & al\ a2\ \beta^3\ e^{-0.5\ \beta} \end{bmatrix}$$

$$\{B\} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \end{bmatrix} \{G\} = \begin{bmatrix} 0 \\ 0 \\ \frac{K_{3,2} G_{1,0}}{\beta^4} \\ 0 \\ -\frac{\alpha^4\ G_{1,0}\ K_{3,2} - \beta^4\ G_{1,0}}{\alpha^4\ \beta^4} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Note that natural frequencies can found on setting the determinant of [A] to zero.

Fixed-Fixed Boundary Conditions

For fixed-fixed conditions, $R_1(\xi) = 0, \xi = 0$ and $\left.\frac{dR_1(\xi)}{d\xi}\right|_{\xi=0} = 0$. In this case, the matrices in eq. (2.10) are:

$$[A] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & \alpha & -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\beta) & e^{\beta} & \sin(\beta) & e^{-\beta} \\ 0 & 0 & 0 & 0 & -\beta\ \sin(\beta) & \beta\ e^{\beta} & \beta\ \cos(\beta) & -\beta\ e^{-\beta} \\ \cos\ (0.5\ \alpha) & e^{0.5\ \alpha} & \sin(0.5\ \alpha) & e^{-0.5\ \alpha} & -\cos(0.5\ \beta) & -e^{0.5\ \beta} & -\sin(0.5\ \beta) & -e^{-0.5\ \beta} \\ -\alpha\ \sin(0.5\ \alpha) & \alpha\ e^{0.5\ \alpha} & \alpha\ \cos(0.5\ \alpha) & -\alpha\ e^{-0.5\ \alpha} & \beta\ \sin(0.5\ \beta) & -\beta\ e^{0.5\ \beta} & -\beta\ \cos(0.5\ \beta) & \beta\ e^{-0.5\ \beta} \\ -\alpha^2\ \cos(0.5\ \alpha) & \alpha^2\ e^{0.5\ \alpha} & -\alpha^2\ \sin(0.5\ \alpha) & \alpha^2\ e^{-0.5\ \alpha} & al\ a2\ \beta^2\ \cos(0.5\ \beta) & -al\ a2\ \beta^2\ e^{0.5\ \beta} & al\ a2\ \beta^2\ \sin(0.5\ \beta) & -al\ a2\beta^2\ e^{-0.5\ \beta} \\ \alpha^3\ \sin(0.5\ \alpha) & \alpha^3\ e^{0.5\ \alpha} & -\alpha^3\ \cos(0.5\ \alpha) & -\alpha^3\ e^{-0.5\ \alpha} & -al\ a2\ \beta^3\ \sin(0.5\ \beta) & -al\ a2\ \beta^3\ e^{0.5\ \beta} & al\ a2\ \beta^3\ \cos(0.5\ \beta) & al\ a2\ \beta^3\ e^{-0.5\ \beta} \end{bmatrix}$$

$$\{B\} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \end{bmatrix} \{G\} = \begin{bmatrix} \frac{G_{1,0}}{\alpha^4} \\ 0 \\ \frac{K_{3,2} G_{1,0}}{\beta^4} \\ 0 \\ -\frac{\alpha^4\ G_{1,0}\ K_{3,2} - \beta^4\ G_{1,0}}{\alpha^4\ \beta^4} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

2.3.2 Numerical Approach

For the numerical approach, a continuous variation model is used. With this model, transitions from one cell to another are modeled via logistic functions. Here these functions, f_i, in non-dimensional form, are taken to be:

$$f_i(\xi) = 1 + \left(\frac{\delta_2 - \delta_1}{\delta_1}\right)\left(\frac{1}{2} + \frac{1}{2}\tanh\left(K\left(\xi - \frac{1}{2}\right)\right)\right), i = 1, 2, 3, 4, 5 \tag{2.11}$$

δ_j represents a material property, geometric property or damping (E, I, ρ, A or C). K controls the sharpness of the transition from one cell to another in the function. A large value corresponding to a sharper transition at $\xi = \frac{1}{2}$.

Substituting eqs. (2.11) into (6) leads to a differential equation, with variable coefficients, which may not have analytic solutions. Given the material layout and cross section variation, i.e., the corresponding logistic functions, a MAPLE® routine can be used to obtain numerical approximations to the FRF of the system. This is done by monitoring the response for different values of the frequency b. Resonances can also be obtained via a forced-motion approach (see reference [15]). It consists of using MAPLE®'s two-point boundary value solver to solve a forced motion problem. A constant value for the forcing function G is assumed and the frequency b is varied. By observing the mid-span deflection of the beam, resonant frequencies can be found on noting where changes in sign occur.

The approaches are illustrated in the following numerical examples.

2.4 Numerical Examples

Consider the beam shown in Fig. 2.2 and assume the following materials: Aluminum ($E_1 = 71\ GPa,\ \rho_1 = 2710\ Kg/m^3$) and Silicon Carbide ($E_2 = 210\ GPa,\ \rho_2 = 3100\ Kg/m^3$). These values are taken from a paper in the field [16].

2.4.1 Free-Fixed Boundary Conditions

For the free-fixed case, the determinant of *[A]* in eq. (2.10) leads to the following values for the first two non-dimensional natural frequencies: $b = 5.7228$ and $b = 27.3930$. (The following parameters apply: $a_1 = 2.9577$, $a_2 = a_4 = a_5 = a_6 = 1.0000$, $a_3 = 1.1439$, $G_{1,0} = 1.0000$, $CD = 1.0000$).

Using eq. (2.10) to determine the values of the constants $B_{i,}$ and eq. (2.8), allows for the calculation of the FRF for the system. Setting $\xi = 0.50$ (beam mid-span), amplitudes can be calculated for different values of the non-dimensional frequency b.

The frequency response function, spanning the first two natural frequencies, for the mid-point of the beam is shown in Fig. 2.3.

For the continuous variation model and using the numerical values given above, the continuously varying functions are shown in Fig. 2.4 (note: $K = 500$).

Assuming a value of *1* for the external forcing $G(\xi)$ and using the forced-motion approach ([15]), the resultant deflections are plotted bellow for two distinct values of the frequency b.

The resonance frequencies are taken to occur at $b = 5.9$ and $b = 27.5$, as seen in Fig. 2.5.

Amplitudes for the response at the center of the beam can be monitored from eq. (2.6). The approach leads to the numerical FRF shown in Fig. 2.6 (dots on the plot).

The figure shows an overlap of the numerical results and the results from the analytical approach (eq. (2.8)). It is seen that very good agreement is obtained, the first two resonances and amplitude values correspond well.

From the numerical FRF, the damping ratio of the system, corresponding to the non-dimensional value of damping $CD = 1.0000$, can be estimated. The method used here is the half-power bandwidth [17], applied to the first mode, which, although only applicable to lightly damped single degree of freedom systems, is frequently applied to well-separated modes of multi-degree of freedom systems. It leads to a ratio of approximately 2%.

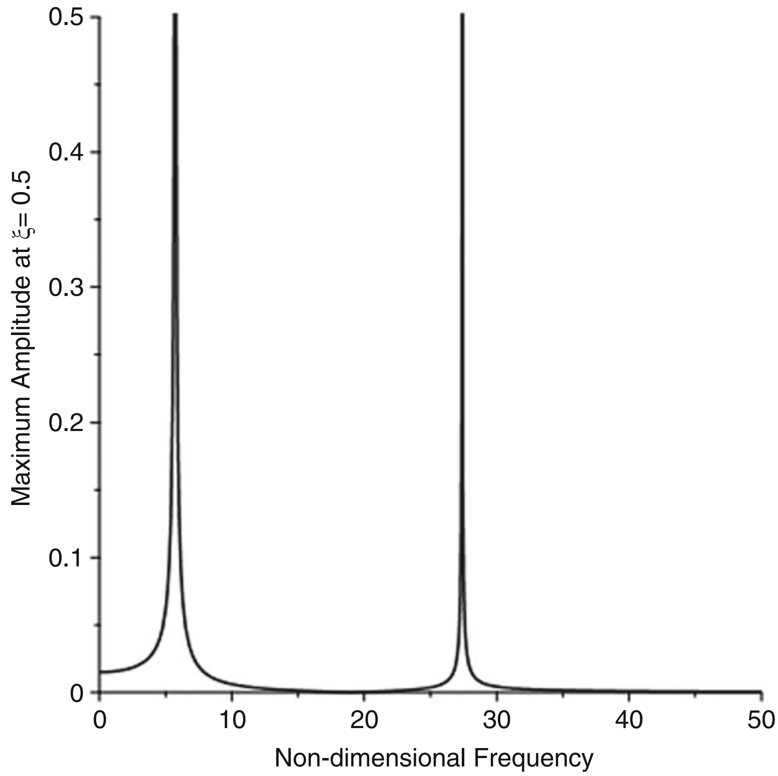


Fig. 2.3 FRF for non-homogeneous beam at mid-point – Free/Fixed

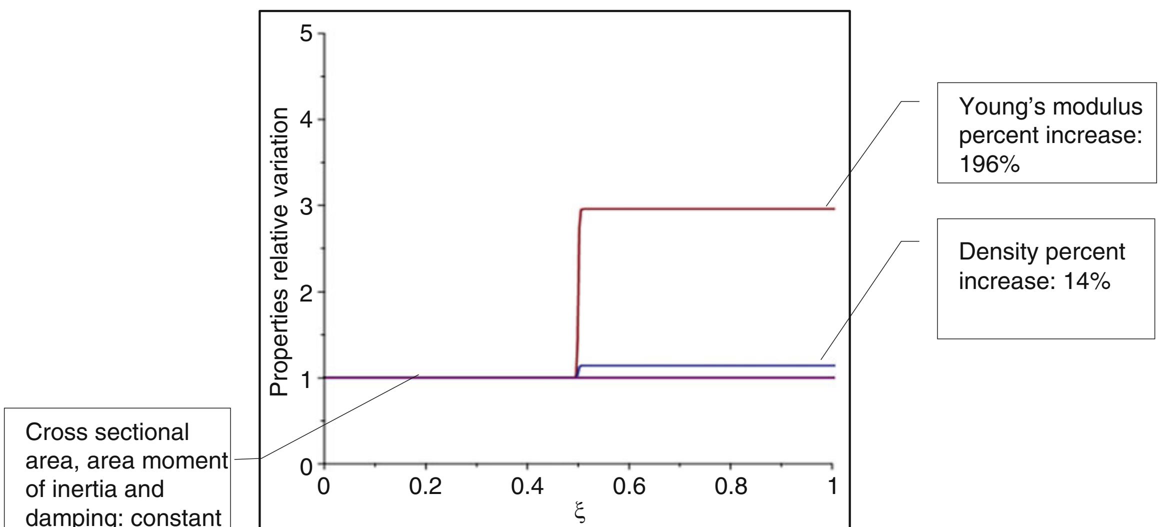
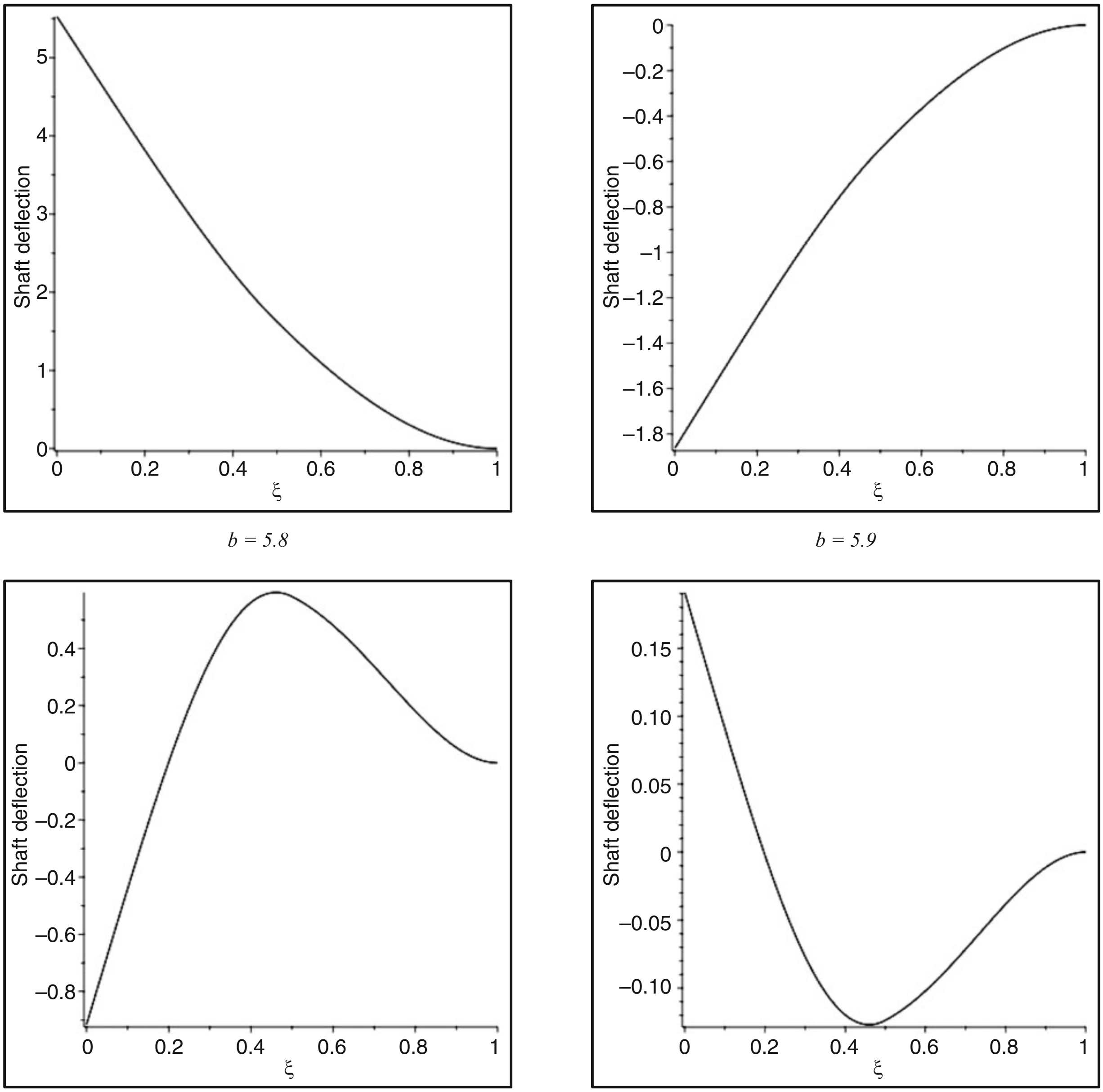


Fig. 2.4 Relative properties variation for two-cell beam

2.4.2 Fixed-Fixed Boundary Conditions

For the fixed-fixed case, the determinant of *[A]* in eq. (2.10) gives the natural frequencies: $b = 27.40$ and $b = 79.49$. Eqs. (2.10) and (2.8) lead to the FRF for the system, which is shown in Fig. 2.7.

The numerical FRF for this case is seen in Fig. 2.8. The overlap of the numerical results and the results from the analytical approach show that good agreement is obtained for the first frequency and amplitude values, with the second one not quite



b = 5.8 *b = 5.9*

b = 27.4 *b = 27.5*

Fig. 2.5 Beam deflections for distinct values of b – Free-Fixed: first and second resonance

given the amplitudes seen in the analytical approach (similar results were obtained for the non-damped case, see reference [1]).

An example is now given in which the spatial force is non-constant. Consider a variable force given by the exponential function: $G(\xi) = e^{-\xi^2}$. The results can be found using the continuous variation model. The FRFs for this case are seen in Fig. 2.9.

2.5 PDE Direct Numerical Approach

Equation (2.2) could be tackled directly via a PDE solver. To this end, here an attempt is made using MAPLE® software.

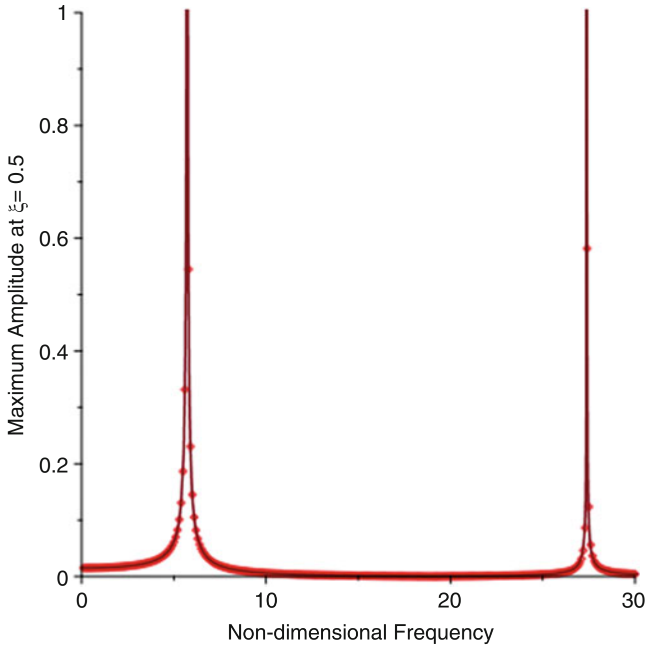


Fig. 2.6 Results comparison – numerical and analytical approaches – Free/Fixed

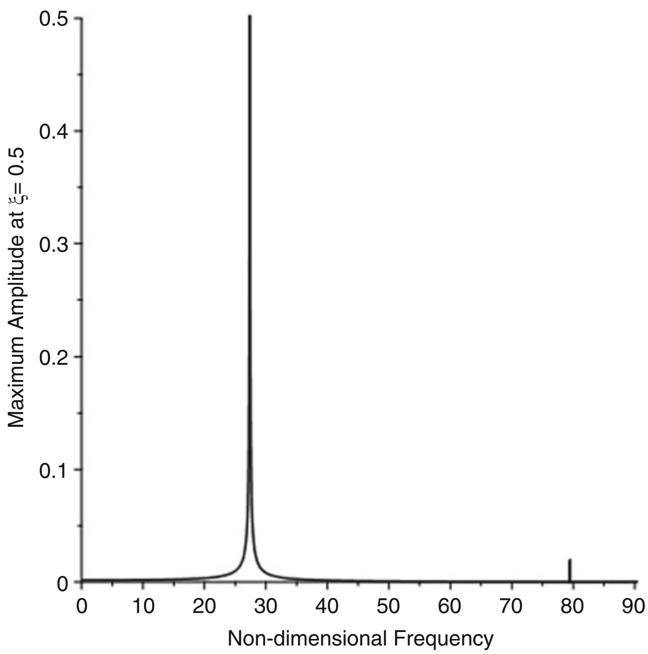


Fig. 2.7 FRF for non-homogeneous beam at mid-point – Fixed/Fixed

On running the numerical routine, it was noticed that convergence of the PDE solver is better achieved with less abrupt changes on the logistic functions. Therefore, here, these functions will be used with a value of $K = 100$. In addition, the beam is subjected to harmonic forcing: $g(\xi, \tau) = \sin(\nu\tau)$ and $CD = 0.1000$ (approximately a damping ratio of 0.2%).

Figure 2.10 shows overlaps for the FRFs obtained using the ODE and PDE solutions. Both cases are illustrated. Note that good agreement is seen for the resonances with some variation on the amplitudes.

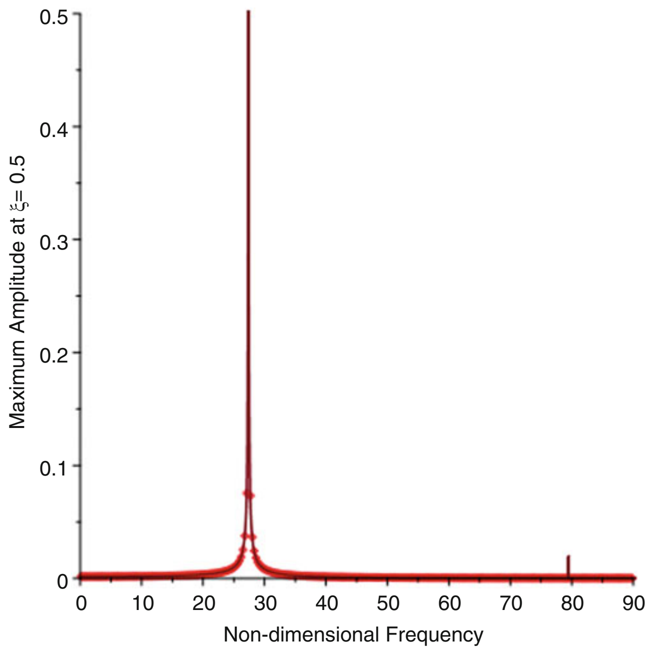


Fig. 2.8 Results comparison – numerical and analytical approaches – Fixed/Fixed

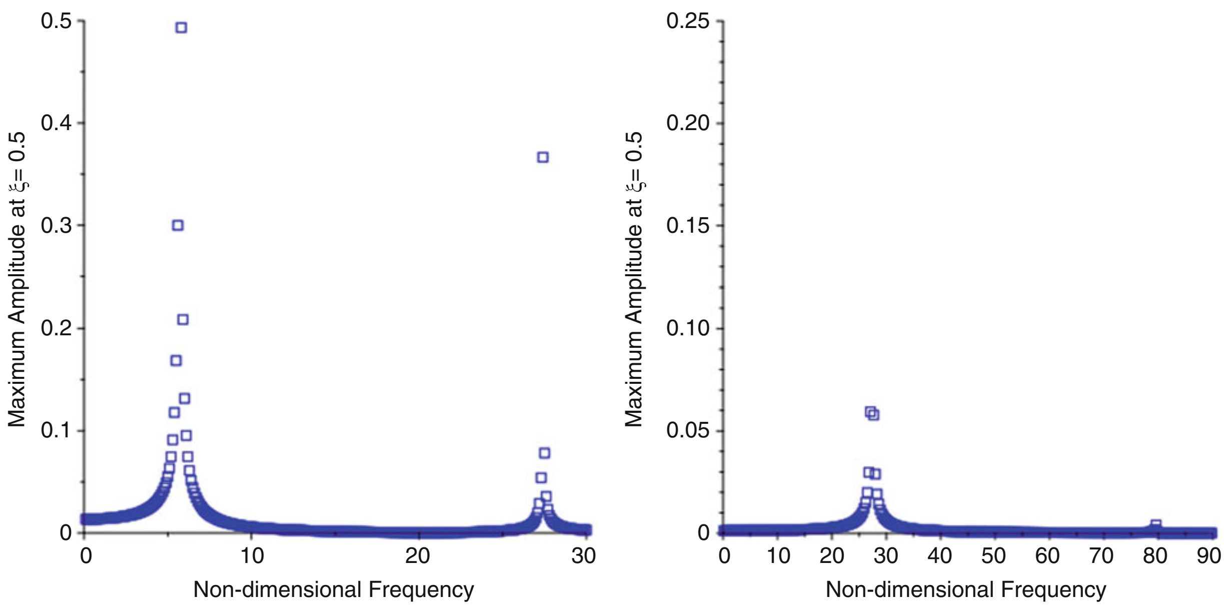


Fig. 2.9 FRFs for exponential force – Free/Fixed (**a**), Fixed/Fixed (**b**)

2.6 A More Complete Model

For cases involving lighter, stronger and more flexible beams, subjected to vibration problems, energy losses due to damping effects become very important. When dealing with such cases, the viscous damping approach discussed above may not be adequate to fully model the behavior of such structures. The model leads to uniform damping rates, which are generally not observed, since damping rates in beams tend to increase with frequency (see, for instance, [18]).

In order to address this issue, a more complete model for the damped beam is presented in the following. The basic viscously damped Euler-Bernoulli equation is modified to include extra energy dissipation mechanisms. The first one includes a Kelvin-Voigt (internal) mechanism and the second one a structural damping. These mechanisms are discussed

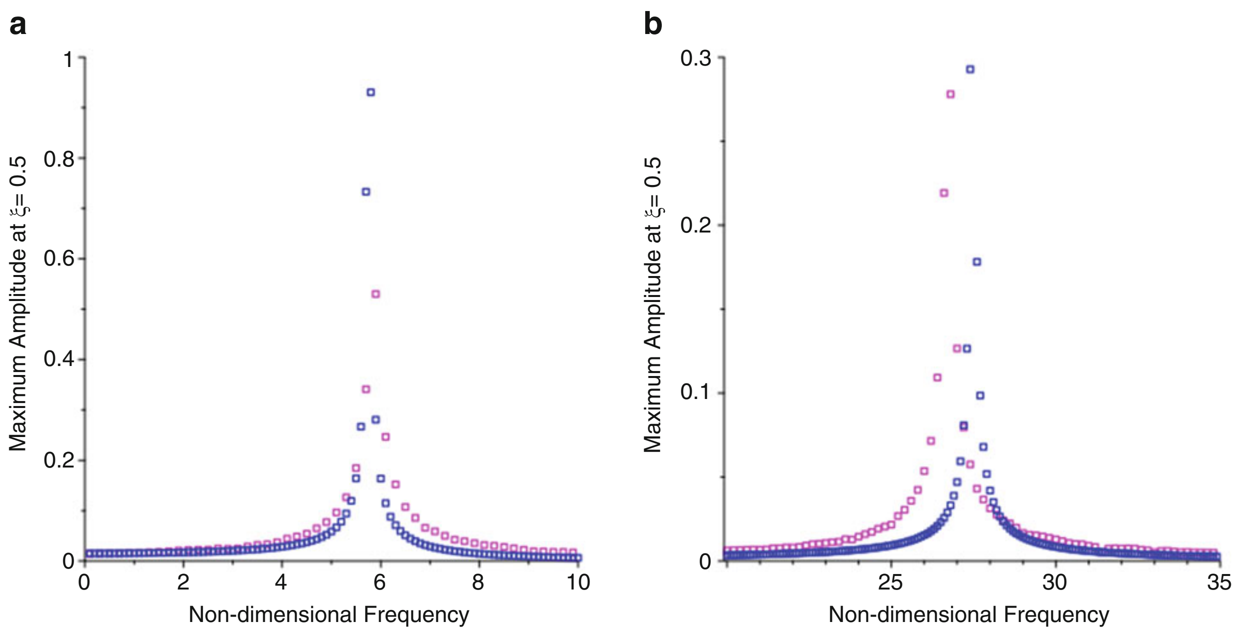


Fig. 2.10 FRFs comparison: ODE versus PDE solutions – Free/Fixed (**a**), Fixed/Fixed (**b**)

in details in reference [19]. Note that in the study it was observed, via experiments, that a predominantly linear relationship between the damping rate and the frequency exists; a behavior which was described as "structural damping".

The modified equation, in non-dimensional form, then becomes:

$$\frac{\partial^2}{\partial \xi^2}\left(f_1(\xi)\, f_2(\xi)\, \frac{\partial^2 u(\xi,\tau)}{\partial \xi^2}\right) + f_3(\xi)\, f_4(\xi)\, \frac{\partial^2 u(\xi,\tau)}{\partial \tau^2} + (CD)\, f_5(\xi)\, \frac{\partial u(\xi,\tau)}{\partial \tau} - \gamma f_6(\xi)\, \frac{\partial^3 u(\xi,\tau)}{\partial \tau\, \partial \xi^2} + \eta \frac{\partial^2}{\partial \xi^2}\left(f_2(\xi)\, f_7(\xi)\, \frac{\partial^3 u(\xi,\tau)}{\partial \tau\, \partial \xi^2}\right) = g(\xi,\tau) \tag{2.12}$$

Where η is a non-dimensional viscosity coefficient, γ is a non-dimensional structural damping coefficient and $f_6(\xi)$ and $f_7(\xi)$ are non-dimensional logistic functions describing variations of these damping mechanisms from one cell to another.

Inclusion of Kelvin-Voigt damping requires that the boundary conditions for the problem be revisited [10]. Here the displacement solutions are assumed to be sufficiently smooth, so that the conditions utilized for the previous model still apply (see [20]).

As discussed above, for the case of stacked beams, analytic solutions for eq. (2.12) may not be feasible.

In this section, the continuous variation approach is employed to obtain direct numerical solutions to the equation. MAPLE® PDE solver is employed.

Consider the same 2-cell beam given above and the following parameters: $g(\xi,\tau) = \sin(\nu\tau)$, $K = 100$, $CD = 0.1000$ (damping ratio of 0.2%), $\gamma = 0.1000$, $\eta = 0.1000$. (For simplicity, damping coefficients are taken to have the same values in both cells.) The logistic functions assumed are: $f_6(\xi) = f_7(\xi) = 1.0000$ (A) and $f_6(\xi) = f_7(\xi) = 1 + \left(1 + \tanh\left(K\left(\xi - \frac{1}{2}\right)\right)\right)$ (B).

Figure 2.11 shows the FRFs for these cases with free-fixed boundary conditions. Only the first mode is shown, since the procedure did not capture large amplitudes at the expected second frequency. It is conjectured that the damping caused the amplitudes to be small and not significantly distinct to stand out in the numerical routine. (Decreasing damping values showed increased amplitudes around expected resonance.)

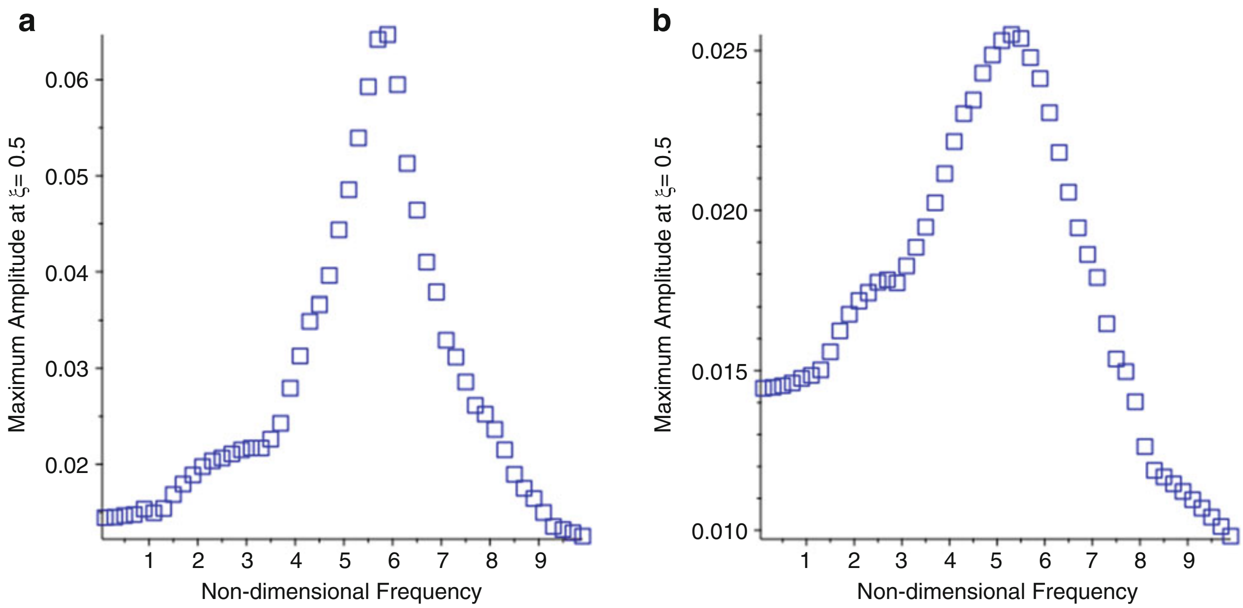


Fig. 2.11 FRFs for expanded model – Free/Fixed

2.7 Conclusions

Modeling discrete property variations via continuously varying functions, in conjunction with numerical solutions, has been shown to lead to good results for resonant frequencies and FRFs of viscously damped layered beams subject to harmonic excitation.

A numerical approach was conducted using MAPLE® software, which shows to lead to accurate solutions based on a comparison to analytical results for a specific case.

Two sets of boundary conditions were studied. Namely, free-fixed and fixed-fixed for a uniform two-cell beam made of aluminum and silicon-carbide.

Very good agreement was observed for both cases, with some variation on the amplitudes for the FRFs.

Comparing numerical solutions from two approaches, using ODE and PDE solvers and continuously varying functions, also showed good agreement.

Lastly, a model involving internal and structural damping, in addition to viscous damping was introduced. Numerical results were shown to be feasible for specific cases, using the numerical PDE solver provided by MAPLE® software.

References

1. Mazzei, A. J., & Scott, R. A.: Harmonic forcing of a two-segment Euler-Bernoulli Beam. Special Topics in Structural Dynamics, Volume 6: Proceedings of the 35th IMAC, A Conference and Exposition on Structural Dynamics 2017, N. Dervilis, ed., Springer International Publishing, Cham, pp. 1–15, (2017)
2. Mazzei, A. J., & Scott, R. A.: Harmonic forcing of a two-segment Timoshenko Beam. Special Topics in Structural Dynamics, Volume 5, N. Dervilis, ed., Springer International Publishing, pp. 1–15, (2019)
3. Lee, E.H., Yang, W.H.: On waves in composite materials with periodic structure. SIAM J. Appl. Math. **25**(3), 492–499 (1973)
4. Hussein, M.I., Hulbert, G.M., Scott, R.A.: Dispersive Elastodynamics of 1D banded materials and structures: analysis. J. Sound Vib. **289**(4–5), 779–806 (July 2)
5. Hussein, M.I., Hulbert, G.M., Scott, R.A.: Dispersive Elastodynamics of 1D banded materials and structures: design. J. Sound Vib. **307**(3–5), 865–893 (June 11)
6. Vasseur, J. O., Deymier, P., Sukhovich, A., Merheb, B., Hladky-Hennion, A. C., & Hussein, M. I.: Phononic band structures and transmission coefficients: methods and approaches. Acoustic Metamaterials and Phononic Crystals, P.A. Deymier, ed., Springer Berlin Heidelberg, pp. 329–372, (2013)
7. Hussein, M.I., Leamy, M.J., Ruzzene, M.: Dynamics of Phononic materials and structures: historical origins, recent Progress, and future outlook. Appl. Mech. Rev. **66**(4), 040802–040802–38 (2014)

8. Leopold, H.: Vibration of the Euler-Bernoulli Beam with Allowance for Dampings. Proceedings of the World Congress on Engineering, London, England, (2008)
9. Di Blasio, G., Kunisch, K., Sinestrari, E.: Mathematical models for the elastic beam with structural damping. Appl. Anal. **48**(1–4), 133–156 (1993)
10. Banks, H. T., & Inman, D.: On Damping Mechanisms in Beams, 383904, NASA, Hampton, VA, United States, (1989)
11. Filipiak, J., Solarz, L., Zubko, K.: Analysis of damping effect on beam vibration. Mol. Quantum Acoust. **27**, (2006)
12. Snowdon, C.: Response of internally damped cantilever beams to sinusoidal vibration. J. Acoust. Soc. Am. **38**, (1965)
13. Zhang, G.-D., Guo, B.-Z.: On the Spectrum of Euler–Bernoulli beam equation with Kelvin–Voigt damping. J. Math. Anal. Appl. **374**(1), 210–229 (2011)
14. Capsoni, A., Maria Viganò, G., Bani-Hani, K.: On damping effects in Timoshenko beams. Int. J. Mech. Sci. **73**, 27–39 (2013)
15. Mazzei, A.J., Scott, R.A.: On the effects of non-homogeneous materials on the vibrations and static stability of tapered shafts. J. Vib. Control. **19**(5), 771–786 (2013)
16. Chiu, T.C., Erdogan, F.: One-dimensional wave propagation in a functionally graded elastic medium. J. Sound Vib. **222**(3), 453–487 (1999)
17. Chopra, A. K.: Dynamics of Structures: Theory and Applications to Earthquake Engineering, Pearson, Hoboken, NJ, (2017)
18. Bishop, R.E.D.: The treatment of damping forces in vibration theory. J. R. Aeronaut. Soc. **59**(539), 738–742 (1955)
19. Russell, D.: 4. On mathematical models for the elastic beam with frequency-proportional damping. Control and Estimation in Distributed Parameter Systems, Society for Industrial and Applied Maths, pp. 125–169, (1992)
20. Lang, H., & Leyendecker, S.: Complex frequency response for linear beams with Kelvin-Voigt viscoelastic material. 4th Joint International Conference on Multibody System Dynamics, Montréal, Québec, Canada, (2016)

Arnaldo J. Mazzei Jr. is a Prof. of Mechanical Engineering at Kettering University. He received his Ph.D. in Mechanical Eng. from the University of Michigan. Currently he teaches courses in Vehicle Dynamics, Chassis Systems, Vibrations and CAE. Research interests: Automotive Engineering, System Dynamics and Vibrations. He is an active member of SAE and SEM.