Individual Efficient Frontiers in Performance Analysis

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Abstract We propose a new approach for performance comparisons with a goal similar to the DEA or efficiency analysis based on stochastic frontiers. Our approach accounts for varying environmental factors and human resources among the units under consideration by assuming individual production possibility sets (PPS). In a partial equilibrium framework we assume that the observed netputs represent an equilibrium. Thus, each DMU is efficient with respect to its individual PPS. The netputs and estimated prices common for all units reveal characteristics of the individual PPSs and assess the units' relative performance. To obtain such prices from scarce data we assume that the observed netput vectors represent a random sample of netput vectors. We use prices which render the realizations of individual profits or returns of the DMUs most likely. We compare the DEA based efficiency rankings with our performance rankings. Strong rank correlation is observed between the two. The discriminatory power of our ranking is superior to conventional DEA methods.

Keywords Performance analysis · Partial equilibrium · Production analysis · Evolutionary computation

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1 Introduction

Financial accounting figures, such as profit, return on assets, etc., remain widely used and easily understandable performance measures of firms, for instance, in annual and quarterly reports. They are commonly used for performance comparisons of individual firms as well. On the other hand, since the introduction of DEA by Charnes et al. (1978), we have witnessed the success of efficiency analysis both in academic and in field studies. DEA provides a simple framework to compare the efficiency of units with multiple inputs and outputs. Commonly, a production possibility set (PPS) is defined by feasible combinations of input and output vectors, and using some distance function, the efficiency score of a DMU is based on how far its netput vector is from the efficient frontier of the PPS. A number of articles involve a stochastic production frontier which may be parametric or non-parametric; see e.g., Kumbhakar and Lovell (2000), and Kumbhakar et al. (2015).

The approach we put forward does not fall in the domain of DEA or stochastic frontier approaches but it has common goals with them: to produce for units under consideration scores ranking their performance. It includes two important advantages that are not present in the simple original DEA models: first, our approach takes care of different environments and human resources of the units and, second, has superior discriminatory power. Additional elements have been suggested for taking care of the varying environments and lack of discriminations in DEA models.

When productivity analysis is carried out the assumption of units functioning in similar environments is rarely close to the true situation. In the DEA several additions have been suggested (e.g. Ruggiero 1998; Fried et al. 2002; Banker and Natarajan 2008) as a remedy, while our approach deals with different environments assuming individual production possibility sets (PPS). The discriminatory power of DEA related to the scores of the units can been increased by the inclusion of preference information (weight restrictions or benchmarks, see Pedjara-Chaparro et al. 1997; Halme et al. 1999), or by e.g. second stage DEA (e.g. Ramalho et al. 2010). Our approach considers value (profit) or return efficiency (for corresponding DEA formulations see Halme et al. 1999; Kuosmanen et al. 2010, Eskelinen et al. 2014) instead of dealing with technical efficiency. The approach uses the same prices for all units.

One major factor that apparently increases the variety of the units is the quality of the management. Personnel economics research provides strong evidence that a firm's productivity and its production possibility set (PPS) can be strongly influenced by human resources, such as management skills; for an extensive survey, see Bloom and Van Reenen (2011). Furthermore, there are other DMU-specific environmental factors, such as those determined by location. A single PPS may not be entirely feasible for any DMU. Motivated by the above, we assume an individual (possibly unobservable) PPS^j for each DMU^j, $j = 1, \ldots, n$, and propose an approach where performance scores are not based on some common efficient frontier. To avoid confusion, our methodology is introduced as *performance analysis (PA)* to distinguish it from frontier based *efficiency analysis (EA)*.

In a partial equilibrium framework, given prices of inputs and outputs, we assume that each DMU^j chooses the best feasible netput vector; i.e., given the resources and environment of DMU^j, the management and employers do the best within their skills. Noting that each PPS^j is assumed to account for human resource capabilities and other differing factors of the environment for each DMU^j, we assume that the observed choices of the DMUs are equilibrium netput vectors. To obtain estimates for equilibrium prices from the scarce data of netput vectors, we assume that the observed netput vectors represent a random sample of netput vectors.

We use profit or return as a performance measure, which depends on the prices of inputs and outputs. From an admissible set we look for a price vector which renders the realizations of individual performance measures of the DMUs most likely. Such prices are used as estimates for equilibrium prices. Optimality conditions together with such prices and the netput vectors yield an estimated PPS for each DMU^j individually. The generally non-convex likelihood maximization problem for price estimates is solved using an evolutionary algorithm of Deb et al. (2002).

In our performance measurement—unlike typically in DEA approaches—the prices used for evaluation of the DMUs are common for each unit. Profit or return is used as a performance measure. The fact that market conditions are present today everywhere, also in public organizations, supports the one-price-for-all choice as an approximation of real world.

Our approach suffers neither from the lack of discriminatory power often encountered by DEA applications nor from the problems related to economies of scale (DEA can use some tests for diagnosing the returns to scale assumption such as suggested by Kneip et al. 2016). For instance, in the field study discussed in this article, 28–32% of the $DMUs$ are found efficient by DEA.

Since both the frontier based methods and our approach provide a basis of ranking for the DMUs, we compare the rankings of a field study whose results qualitatively represent well numerous other cases we have considered. Despite the differences our test results of the two approaches show a strong correlation of rankings; however, a stronger discriminatory power is achieved by PA.

The rest of the article proceeds as follows. In Sect. 2 we introduce performance analysis (PA). Section 3 reviews traditional efficiency analysis (EA) methods to be used for comparison with PA in Sect. 4. Section 5 concludes. Supplementary material is in the Appendix: an evolutionary optimization procedure for price estimation is presented in Appendix A illustrative simulated examples of PA are in Appendix B; data and results of a field study are shown in the Appendix C.

2 Performance Analysis

We begin by introducing the economic basis of PA in Sect. 2.1. The principle of estimating the price vector is introduced in Sect. 2.2. Thereafter we define PA scores in Sect. 2.3, propose density estimates of profit and return in Sect. 2.4, and discuss computational considerations in Sect. 2.5.

2.1 Economic Foundation

Consider firms or other decision-making units DMU^j for $j = 1, 2, \ldots, n$. Because of differing availability of resources (including human resources) and environmental considerations, we assume a specific production possibility set PPS^j for each DMU^j. In a partial equilibrium framework, consider profit maximizing producers DMU^j, $j = 1, \ldots, n$. For each DMU^j, there are m inputs and k outputs. Let $\xi^j \leq 0$ denote the input vector and $\eta^j \geq 0$ the output vector of DMU^j. For all j, let $g^j(\xi^j, \eta^j)$ be a multi-input multi-output transformation function of DMU^j such that PPS^j is defined by $g^j(\xi^j, \eta^j) \leq 0$. Transformation function $g^j(\xi^j, \eta^j)$ may represent, for instance, CET-GD technology (e.g., Kumbhakar et al. 2015). Let $p(\eta)$ be an integrable price function (inverse demand function) facing aggregate output supply $\eta = \sum_j \eta^j$ and let $c(\xi)$ be an integrable marginal cost function (supply function) facing aggregate input demand $\xi = \sum_j \xi^j$.

Assuming price taking behavior[1] for each DMU^j, consider a competitive equilibrium. For each DMU^j, the *observed inputs* $\xi^j = -x^j \in R^m_+$ and *outputs* $\eta^j = y^j \in R^k_+$ represent equilibrium choices lying on the efficient frontier of PPS^j. For a non-negative input vector $x \in R^m_+$ and a non-negative output vector $y \in R^k_+$, the netput vector z is defined by

$$z^t = (-x^t, y^t), \tag{1}$$

where superscript t refers to a transpose. For all j, z^j is the observed equilibrium netput vector of DMU^j with input vector x^j and output vector y^j.

Given an equilibrium price vector μ^*_x for inputs and μ^*_y for outputs with $\mu^* = (\mu^*_x, \mu^*_y)$, the performance of DMU^k in terms of profit or return may appear superior to DMU^j because of the differences in PPS^k and PPS^j. Using optimality conditions of each DMU^j, we note that price estimates for μ^* together with inputs x^j and outputs y^j imply the individual transformation functions—provided that the number of parameters of each transformation function is not excessive—and thereby the production possibility sets PPS^j are revealed. For numerical examples, see Appendix B.

2.2 Estimating Prices

The price function for outputs, the cost function for inputs, and transformation functions for the DMUs are not known; in addition to observed inputs and outputs, we may only have partial price information which imposes some conditions for

[1]If prices of some products or services are not observable in the market, we interpret the prices resulting from rational expectations equilibrium.

price relationships and possibly takes into account some price observations, for instance. Therefore, to estimate the prices we assume that observed netput vectors z^j represent a random sample from netput vector $\tilde{z}$ with a multivariate pdf $\Phi(z)$. While an efficient production frontier characterizes each PPS^j, we need not assume a bounded support for $\tilde{z}$.

Let row vector $\mu = (\mu_x, \mu_y) \in R^{m+k}$ denote the vector of prices with input prices $\mu_x \in R^m$ and output prices $\mu_y \in R^k$. The prices are expressed in monetary units per unit of product. Partial price information is given by the *admissible set of prices* P. We require $\mu \geq \epsilon$, for some $\epsilon \geq 0$. Prices are restricted by other means as well. For scaling the prices, we may fix the value of some cost and/or revenue component. Some prices may be fixed or restricted to some interval and price ratios may be bounded. We may also employ subjective judgment. For instance, if z^j is seen superior to z^k in terms of profit in a pair-wise comparison among two netput vectors, we may include such judgmental information in the analysis. In this case we require $\mu(z^j - z^k) \geq 0$. We assume that the set of admissible prices P is a non-empty compact and convex set defined by linear equations and linear inequalities.

We now turn to an estimate $\hat{\mu}$ of μ^* to be used in PA. For netput vector $z^t = (-x^t, y^t)$ with $x \geq 0$ and $y \geq 0$, given a price vector $\mu = (\mu_x, \mu_y) \in P$ we determine a performance measure $\kappa = \kappa(\mu, z)$. Subsequently κ stands for profit $\pi = r - c$ or return $\rho = r/c$ with revenue $r = \mu_y y$ and cost $c = \mu_x x$. Given pdf $\Phi(z)$, price vector μ, and the definition of κ, a pdf $\psi(\kappa; \mu)$ of κ is implied for each μ. Of course, $\psi(\kappa; \mu)$ may not have an analytical expression even if $\Phi(z)$ has one. An estimate of $\psi(\kappa; \mu)$ is denoted by $\hat{\psi}(\kappa; \mu)$ and it will be discussed in Sect. 2.4. Prices are parameters of such a pdf and we look for prices which make the individual performance figures of the DMUs most likely. For DMU^j, the performance measure $\kappa_j = \kappa_j(\mu, z^j)$ depends on prices $\mu \in P$ whose values we determine by log-likelihood maximization:

$$\max_{\mu \in P} \sum_{j=1}^{n} \log \hat{\psi}(\kappa_j(\mu, z^j); \mu). \tag{2}$$

An optimal price vector in (2) is denoted by $\hat{\mu}$ and it is used to evaluate the return and value performance scores defined in Sect. 2.3.

2.3 Return and Value Performance Scores

Given an estimate $\hat{\mu}$ of the equilibrium price vector and the netput vector z^j we can evaluate return and profit. Thereby we may state alternative scores for return and value performance.

For *return performance analysis (RPA)*, return ρ plays the role of performance measure κ. Given estimate $\hat{\mu}$ for the equilibrium price vector with components $\hat{\mu}_x$ for inputs and $\hat{\mu}_y$ for outputs, the random return is $\tilde{\rho} = \hat{\mu}_y \tilde{y} / \hat{\mu}_x \tilde{x}$ and we calculate the return $\hat{\rho}_j$ of each DMU^j. Then the *return performance* (RP) score of DMU^j is the probability of $\tilde{\rho} \leq \hat{\rho}_j$. A score 0.68 of DMU^j means that 68% of the realizations of $\tilde{z}$ are inferior or as good as DMU^j or that DMU^j is ranked among top 32%; see Fig. 1.

For *value performance analysis (VPA)* measure κ is profit π. Given price vector estimate $\hat{\mu}$, we obtain the random profit $\tilde{\pi} = \hat{\mu}\tilde{z}$ and we calculate profit $\hat{\pi}_j$ of each DMU^j. Then the *value performance* (VP) score of DMU^j is the probability of $\tilde{\pi} \leq \hat{\pi}_j$.

2.4 Density Estimates of Profit and Return

Consider three cases for the distribution of netput vector $\tilde{z}$: Case 1, $\tilde{z}$ is multivariate normal; Case 2, no distributional assumption is made; Case 3, a parametric family of multivariate distributions is adopted. Case 1 in Sect. 2.4.1 applies to VPA but not for RPA. In Sect. 2.4.2 of Case 2, a kernel density estimate is employed for pdf $\hat{\psi}(\kappa; \mu)$ of the performance measure κ. In Sect. 2.4.3 of Case 3, parameters of pdf $\Phi(z)$ are estimated first to obtain $\hat{\Phi}(z)$ and $\hat{\psi}(\kappa; \mu)$ is derived thereafter. At the first reading, one may proceed directly to Sect. 2.5.

2.4.1 Multivariate Normal Distribution of Netput Vectors

In this section we assume $\tilde{z}$ has a multivariate normal pdf $\Phi(z)$.[2] Maximum likelihood estimates $\bar{z}$ and V for the expected value and the covariance matrix of $\tilde{z}$ are

$$\bar{z} = \frac{1}{n} \sum_j z^j$$

$$V = \frac{1}{n} \sum_j (z^j - \bar{z})(z^j - \bar{z})^t.$$

Hence pdf $\hat{\Phi}(z)$, the estimate of Φ, is the pdf $N(\bar{z}, V)$, and given a price vector $\mu \in P$, the random profit $\pi = \mu\tilde{z}$ has the pdf $N(\bar{\pi}, \sigma^2)$, where $\bar{\pi} = \mu\bar{z}$ and $\sigma^2 = \mu V \mu^t$. Therefore, in case of VPA, $\hat{\psi}(\pi; \mu)$ has a normal distribution. For each DMU^j, price vector $\mu \in P$ and netput vector z^j yield profit $\pi_j = \mu z^j$. Thus the log-likelihood function in (2) for profits π_j (omitting constant terms) is $-(n/2)\log(\sigma^2)$. Hence, the estimate for price vector μ is obtained by minimizing the variance σ^2; i.e. our problem is to find price vector μ to

$$\min_{\mu \in P} \mu V \mu^t. \tag{3}$$

[2] In this case we expect that the likelihood for $x \ngeq 0$ and $y \ngeq 0$ is small.

Given optimal price vector $\hat{\mu}$ in (3), we obtain the normal pdf for the random profit $\tilde{\pi} = \hat{\mu} z$, whose expected value is $\hat{\mu}\bar{z}$ and variance is the optimal objective function value in (3).

2.4.2 Kernel Density Estimate of $\psi(\kappa; \mu)$

Kernel density estimate with Gaussian kernel and bandwidth δ is a standard approach which may be adopted for estimating univariate distribution ψ; see e.g., Rosenblatt (1956) and Silverman (1998). Given price vector μ and netput vectors z^j, with $\kappa_j = \kappa(\mu, z^j)$ we define

$$\hat{\psi}(\kappa; \mu) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\sqrt{2\pi}\delta} \exp[-\frac{(\kappa - \kappa_j)^2}{2\delta^2}]. \tag{4}$$

We employ the following result in Silverman (1998): if the pdf to be estimated is normal with variance σ^2, then an approximate optimal bandwidth δ minimizing the mean integrated square error is

$$\delta = \sigma(4/3n)^{1/5}. \tag{5}$$

Figure 1 shows the kernel density estimate $\hat{\psi}(\rho; \hat{\mu})$ with bandwidth $\delta = 0.088$ in the grocery stores case study of Sect. 4.

We use (5) where σ^2 is replaced with the variance $\hat{\sigma}^2$ of the sample $\{\kappa_j\}$. Since $\hat{\sigma}$ depends on prices, we need to search for a suitable bandwidth δ to satisfy (5) with sample variance associated with the estimate $\hat{\mu}$ of equilibrium prices. In the case studies in Sect. 4 such values of δ range from 0.063 to 0.158.

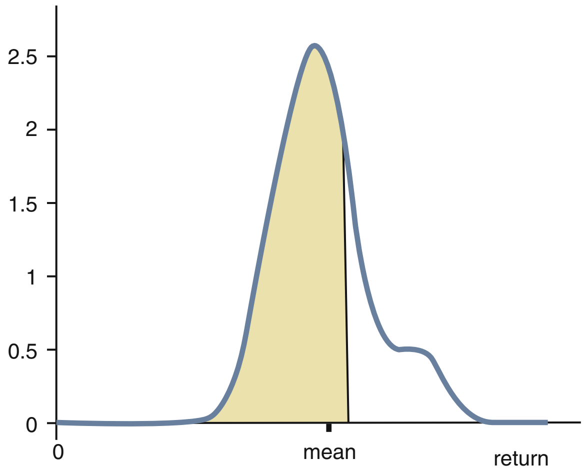

Fig. 1 Kernel density estimate of probability density function $\hat{\psi}(\rho; \hat{\mu})$ of return for RPA in the grocery stores case study of Sect. 4. The shaded area is the return performance (RP) score 0.68 of the DMU ranking eighth among the 25 DMUs

2.4.3 Parametric Distribution of Netput Vectors

Next, consider a family of multivariate pdfs for $\Phi(z)$ with some set of parameters (a multivariate log-normal distribution, for example). The observations z^j, $j = 1, \ldots, n$, are used for parameter estimation and $\hat{\Phi}(z)$ denotes the estimated pdf of $\tilde{z}$. Given pdf $\hat{\Phi}(z)$, price vector μ and the definition of κ, let $\phi(\kappa; \mu)$ denote the associated pdf of the measure κ given price vector μ.

Typically an analytical expression for $\phi(\kappa; \mu)$ is not available, wherefore we employ an approximation $\hat{\psi}$ of ϕ. To derive $\hat{\psi}$, consider a family of normal pdfs $f(\kappa; \kappa', \delta^2)$ of κ with expected values κ' and variance δ^2. In this family, let $\phi(\kappa'; \mu)$ be the pdf of expected values κ'. Then expected pdf at κ is

$$E(\kappa, \delta) \equiv E[f(\kappa; \kappa', \delta^2)] = \int_{\kappa'} f(\kappa; \kappa', \delta^2)\phi(\kappa'; \mu)d\kappa'. \tag{6}$$

As δ approaches zero, $f(\kappa; \kappa', \delta^2)$ approaches the Dirac delta function, and therefore

$$\lim_{\delta \to 0} E(\kappa, \delta) = \phi(\kappa; \mu). \tag{7}$$

We approximate the integral in (6) by a sample average. Using a random sample $\{z^s\}$ of S independent draws from $\hat{\Phi}(z)$, define $\kappa_s = \kappa(\mu, z^s)$. Then $\{\kappa_s\}$ is a random sample of S draws from $\phi(\kappa; \mu)$ and the sample average pdf is

$$\hat{\psi}(\kappa; \mu) = \frac{1}{S}\sum_s f(\kappa; \kappa_s, \delta^2) = \frac{1}{S}\sum_s \frac{1}{\sqrt{2\pi}\delta} \exp[-\frac{(\kappa - \kappa_s)^2}{2\delta^2}]. \tag{8}$$

By (6)–(8), for large S and small $\delta > 0$ we have

$$\hat{\psi}(\kappa; \mu) \approx E(\kappa, \delta) \approx \phi(\kappa; \mu). \tag{9}$$

Equation (8) is in fact a Gaussian kernel density estimate of $\phi(\kappa; \mu)$ based on the sample. However, an advantage compared with (4) is that we now are better informed in choosing the bandwidth δ. Based on pdf $\hat{\Phi}(z)$, the true pdf $\phi(\kappa; \mu)$ is known in principle but not necessarily its analytic expression. However, sample estimates for its moments can be evaluated. Therefore, we employ approximation (8) choosing the bandwidth in such a way that the first few moments of $\phi(\kappa; \mu)$ and $\hat{\psi}(\kappa; \mu)$ are approximately the same.

To get an idea of the precision of this approximation, we compare the moments of κ based on the sample from $\phi(\kappa; \mu)$ and on the approximation $\hat{\psi}(\kappa; \mu)$. For integers $l > 0$, $\hat{m}_l = (1/S)\sum_s \kappa_s^l$ is the sample mean of κ^l and m_l denotes the lth moment of κ with respect to $\hat{\psi}(\kappa; \mu)$. Using (8) and the moments of $N(\kappa_s, \delta^2)$ we obtain (Cook 2012)

$$m_l = \frac{1}{S}\sum_s \sum_{i=0}^{[l/2]} \binom{l}{2i} (2i-1)!!\,\delta^{2i}\,\kappa_s^{(l-2i)} = \sum_{i=0}^{[l/2]} \binom{l}{2i} (2i-1)!!\,\delta^{2i}\,\hat{m}_{(l-2i)}$$
$$= \hat{m}_l + O(\delta^2), \qquad (10)$$

where $[\cdot]$ denotes rounding down and $(\cdot)!!$ denotes double factorial.[3] The residual term $O(\delta^2)$ is of the order of δ^2. For example, $m_1 = \hat{m}_1$, $m_2 = \hat{m}_2 + \delta^2$, $m_3 = \hat{m}_3 + 3\hat{m}_1\delta^2$ $m_4 = \hat{m}_4 + 6\hat{m}_2\delta^2 + 3\delta^4$, etc. For large S, the sample means $\hat{m}_l$ approach the respective moments based on $\phi(\kappa;\mu)$, and for small δ, the moments m_l are close to respective moments $\hat{m}_l$. Silverman's rule (5) here matches the moments unsatisfactory.

Note that for the first moments, $m_1 = \hat{m}_1$. Let $\hat{\sigma}^2 = \hat{m}_2 - \hat{m}_1^2$ denote the sample variance of κ and $\sigma^2 = m_2 - m_1^2$ the variance based on $\hat{\psi}(\kappa;\mu)$. Their relative difference is $\delta^2/\hat{\sigma}^2$. For computations in Sect. 4, we use sample size $S = 1000$ and $1/2\delta^2 = 10^5$. For these choices the relative difference $\delta^2/\hat{\sigma}^2$ of the variances is less than 0.03% in all cases considered. Furthermore, in Sect. 2.4.1

A test of approximation (8) is as follows. In the multivariate normal case for VPA an approximation is not needed but can be used; an optimal price estimate is obtained from (3), while near optimal prices are obtained using the sample approximation (8) in (2). With sample size $S = 1000$ and $1/2\delta^2 = 10^5$ we solve (2) in two cases of Sect. 4 where the distribution of netput vectors most closely resembles a multivariate normal distribution. These cases refer to bank branches and grocery stores. Based on the results we rank the DMUs according to VP scores. Then the ranking is done based on the scores obtained from the "exact" problem (3). The Spearman rank correlation (of approximate vs. "exact") is 1.00 both for bank branches and grocery stores.

2.5 Price Computations

Finally, we discuss computations for obtaining a price vector estimate $\hat{\mu}$ from the likelihood problem (2). In the special and simple case of VPA assuming the netput vector $\tilde{z}$ is multivariate normal an optimal solution for (2) is obtained solving the convex problem (3). For other cases we use evolutionary optimization. Using approximation (8) for pdf $\hat{\psi}$ in (2) the objective function may become highly nonlinear with plenty of local optima; for an illustration of RPA, see Fig. 2 (right) concerning the grocery stores case in Sect. 4. Instead, using the kernel density estimate (4) the objective can be relatively smooth; see Fig. 2 (left). In both cases we end up with a non-convex problem. For global optimization we employ an implementation of the evolutionary optimization procedure PCX-$G3$ (see Deb et al. 2002). The algorithmic steps are presented in Appendix A including some

[3]For integer $k \geq 1$, $k!!$ is the product of positive integers up to k with the same parity as k, and 0!!=(-1)!!=1.

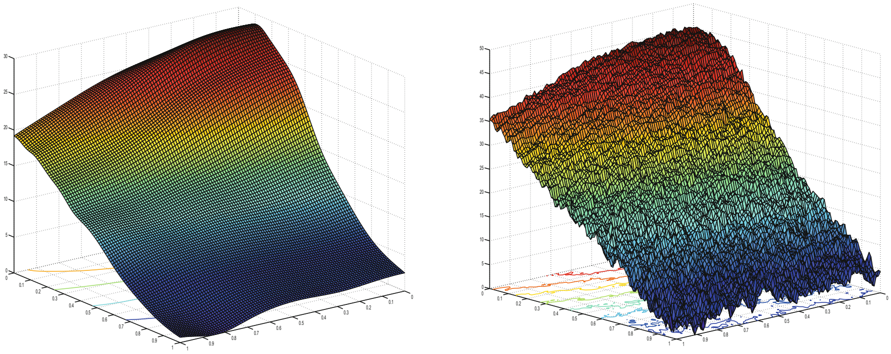

Fig. 2 Log-likelihood functions for RPA in the grocery stores case study with two inputs, two outputs, and price constraints $\mu_1 + \mu_2 = 1$ and $\mu_3 + \mu_4 = 1$. On the left, kernel density estimate (4) with bandwidth $\delta = 0.088$. On the right, multivariate log-normal distribution for netput vectors is employed and approximation (8) with sample size $S = 1000$ and $1/2\delta^2 = 10^5$. Both figures show the log-likelihood in (2) as a function of price vector μ. The horizontal coordinates refer to μ_2 (increasing to the left) and μ_4, both ranging from 0 to 1. Optimal price vector on the left is $\hat{\mu} = (0.914, 0.086, 0.892, 0.108)$ and on the right $\hat{\mu} = (0.912, 0.088, 0.921, 0.079)$

sensitivity analysis for the control parameters of evolutionary optimization. For computations we use $AMPL$ (Fourer et al. 2003) and $MINOS$ (Murtagh and Saunders 1978).

3 Conventional DEA Based Methods

We now review two DEA based approaches for EA, *value (or profit) efficiency analysis* (VEA) based on profit (see e.g., Nerlove 1965, Chambers et al. 1998 and Halme et al. 1999) and *return efficiency analysis* (REA) based on return (see e.g., CCR by Charnes et al. 1978 and BCC by Banker et al. 1984). The rankings based on these methods are used for comparisons with VPA and RPA in Sect. 4 using five real cases of efficiency analysis.

We adopt the presentation of VEA and REA from Kallio and Kallio (2002). We begin by introducing the set of feasible netput vectors (PPS). We judge DMU^r in terms of its netput vector z^r with respect to a production possibility set T of feasible netput vectors z and (as in Sect. 2.2) a set P of admissible price vectors μ. For each DMU^j, we assume that $z^j \in T$.

Consider feasible netput vectors, which are linear combinations of the netput vectors z^j; i.e., for a set $\Lambda \subset R^n$ of weight vectors $\lambda = (\lambda_j)$, we define

$$T = \{z |\ z = \sum_j \lambda_j z^j,\ \lambda \in \Lambda\}. \tag{11}$$

Choices of Λ result in alternative sets T of which one is adopted for efficiency evaluation. In our comparisons of Sect. 4 we use two alternatives. Under a constant returns to scale (CRS) hypothesis,

$$\Lambda = \{\lambda \in R^n \mid \lambda \geq 0\}, \tag{12}$$

and under a variable returns to scale (VRS) hypothesis,

$$\Lambda = \{\lambda \in R^n \mid \sum_j \lambda_j = 1,\ \lambda \geq 0\}. \tag{13}$$

In value efficiency analysis (VEA) the *difference measure of efficiency* of DMU^r is the difference of the best profit achievable by netput vectors in T and the profit of DMU^r and the prices are chosen from the admissible set P to minimize the difference. To test for profit efficiency of DMU^r we solve the problem of finding admissible prices $\mu \in P$ and a scalar θ to

$$\min_{\theta,\mu}\{\theta - \mu z^r \mid \mu \in P \text{ and } \mu z \leq \theta \text{ for all } z \in T\}. \tag{14}$$

At an optimal solution of (14), θ is the maximum profit over T and $\theta - \mu z^r \geq 0$ because $z^r \in T$. If $\theta - \mu z^r = 0$, then z^r maximizes μz over T and DMU^r is profit efficient. The optimal objective function value $\theta - \mu z^r$ in (14) is the difference measure of profit efficiency.

In return efficiency analysis REA, the *ratio measure of return efficiency* of DMU^r is the return (productivity) relative to the best return taking into account all netput vectors in T, and the prices are chosen from the admissible set P to maximize return ratio for DMU^r. To test for return efficiency of netput vector z^r of DMU^r, we solve the problem of finding admissible prices $\mu = (\mu_x, \mu_y) \in P$ and a scalar θ, recalling decomposition of netput vector z in (1), to

$$\max_{\theta,\mu_x,\mu_y}\ \{\frac{\mu_y y^r}{\mu_x x^r}\frac{1}{\theta} \mid \mu \in P \text{ and } \frac{\mu_y y}{\mu_x x} \leq \theta \text{ for all } z \in T\}. \tag{15}$$

At the optimal solution of (15), θ is the maximum return over T and the optimal objective function value in (15) is the ratio measure of return efficiency. This measure is at most one because $z^r \in T$, and it is equal to one if z^r maximizes the return over T in which case DMU^r is return efficient. As usual, LP is applied to solve (14) and (15).

4 Comparison of PA and EA Methods

For comparisons of VEA and REA with VPA and RPA, we used five published field studies concerning (i) bank branches (Eskelinen et al. 2014), (ii) parishes (Halme and Korhonen 2015), (iii) dental care units (Halme and Korhonen 2000), (iv) grocery stores (Korhonen et al. 2002), and (v) power plants (Kuosmanen 2012). Here we only discuss case (i) in some detail; results from the other four cases were very similar.

The bank branch study by Eskelinen et al. (2014) concerns sales performance of branches in the Helsinki OP Bank. The analysis covers the years 2007–2010 in the 25 branches operating in the Helsinki metropolitan area. The bank considers financing and investment services as outputs in the model. The output quantities by bank branch are shown in the Appendix C where both output figures are in average number of aggregated transactions per annum. There are five inputs: total work time in five categories of the sales force. The input figures in average full-time years per annum for each branch are shown as well. For VEA and REA, a constant returns to scale (CRS) hypothesis is adopted for the set T of feasible netput vectors. Hence, T is defined by (11) and (12).

For PA we consider both a multivariate log-normal distribution $\tilde{z}$ and a kernel density estimate for the performance measure (return or profit). We use a set of admissible prices with a lower limit 10^{-6} for all prices and we scale the input prices such that the average cost $\mu_x \bar{x} = 1$, where $\bar{x}$ is the average of input vectors x^j in the sample. Additionally for REA and RPA, we require that the revenue $\mu_y \bar{y} \geq 1$, where $\bar{y}$ is the average of output vectors y^j.[4]

For the bank branch case the Appendix C shows PA and EA based efficiency scores as well as ranking of DMUs based on different methods. Figure 3 (top) shows the comparisons of conventional REA efficiency (horizontal axis in each diagram) vs. return performance of RPA (vertical axis). Figure 3 (bottom) displays a similar comparison of VEA and VPA. In each case, results based on both density estimates (log-Normal/kernel) are depicted.[5] In these figures, one can see the correlation between the pairs of scores. The corresponding Spearman rank correlation ranges from 0.80 to 0.91. The number of efficient DMUs is 9 for both VEA and REA. The ranking based on PA is nearly independent of the distributional assumption of $\tilde{z}$.

[4]For VEA this additional requirement under CRS leads to infeasibility.

[5]Note that in Fig. 3 the REA and RPA scores are positively correlated whereas in Fig. 3 the VEA and VPA scores have negative correlation because high VEA score means poor performance.

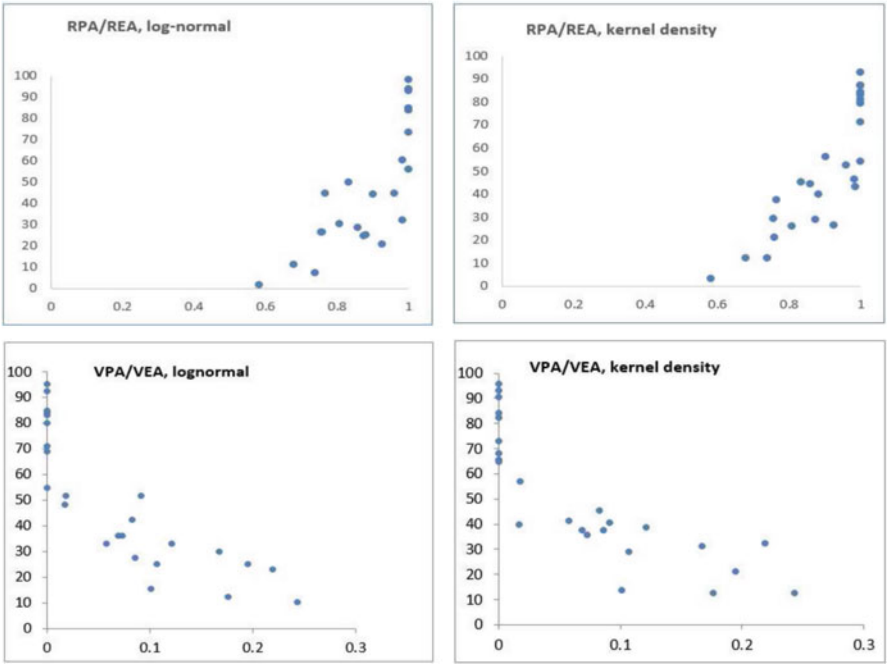

Fig. 3 The bank branches case with 25 units. Top: Correlation diagrams of REA scores (horizontal axis in each diagram) and return performance (vertical axis) of RPA. REA employs the ratio measure of return efficiency. Bottom: Correlation diagrams of VEA scores (horizontal axis in each diagram) and value performance (vertical axis) of VPA. VEA shows the difference measure of profit efficiency

5 Conclusions

We propose a novel approach to measure value (profit) and return performance of decision-making units. The method does not rely on distances from an efficient frontier. Therefore, for the sake of clarity, we discuss performance analysis (PA) instead of frontier based efficiency analysis. Contrary to the assumption made by DEA the units considered typically function in various environments which is why we assume the production possibility sets are individual for each unit. We adopt a partial equilibrium perspective wherefore the observed netput vector of each unit is assumed to be on the efficient frontier of the individual production possibility set. Common prices are calculated for all the units and they represent estimates for equilibrium prices. Our single-price requirement is justified, for instance, by the market forces confronting all kinds of organizations today. Price restrictions can be employed to account for partial price information. The discriminatory power is superior to DEA based methods.

The rankings produced by PA are compared with the rankings based on efficiency analysis of DEA methods. In spite of the significantly different starting points, it turned out that in five published case studies our ranking results compared with conventional DEA based methods of value (profit) and return efficiency were highly correlated. This is an interesting observation as the problem of zero prices is quite common in DEA.

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References

Banker, R. D., Charnes, A., & Cooper, W. W. (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Management Science, 30*, 1078–1092.

Banker, R. D., & Natarajan, R. (2008). Evaluating contextual variables affecting productivity using data envelopment analysis. *Operations Research, 56*, 48–58.

Bloom, N., & Van Reenen, J. (2011). *Handbook of labor economics* (Vol. 4b). Amsterdam: Elsevier. https://doi.org/10.1016/S0169-7218(11)02417-8

Chambers, R. G., Chung, Y., & Färe, R. (1998). Profit, directional distance functions, and Nerlovian efficiency. *Journal of Optimization Theory and Applications, 98*, 351–364.

Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring efficiency of decision making units. *European Journal of Operational Research, 2*(6), 429–444.

Cook, J. (2012). *General formula for normal moments*. http://www.johndcook.com/blog/2012/11/06/general-formula-for-normal-moments/

Deb, K., Anand, A., & Joshi, D. (2002). A computationally efficient evolutionary algorithm for real-parameter optimization. *Journal of Evolutionary Computation, 10*(4), 371–395.

Eskelinen, J., Halme, M., & Kallio, M. (2014). Bank branch sales evaluation using extended value efficiency analysis. *European Journal of Operational Research, 232*, 654–663.

Fourer, R., Gay, D., & Kernighan, B.W. (2003). *AMPL, a modeling language for mathematical programming* (2nd ed.). Pacific Grove: Brooks/Cole Thomson Learning.

Fried, H. O., Lovell, C. A. K., Schmidt, S. S., & Yaisawarng, S. (2002). Accounting for environmental effects and statistical noise in data envelopment analysis. *Journal of Productivity Analysis, 17*, 157–174.

Halme, M., Joro, T., Korhonen, P., Salo, S., & Wallenius, J. (1999). A value efficiency approach to incorporating preference information in data envelopment analysis. *Management Science, 45*(1), 103–115.

Halme, M., & Korhonen, P. (2000). Restricting weights in value efficiency analysis. *European Journal of Operational Research, 126*, 175–188.

Halme, M., & Korhonen, P. (2015). Using value efficiency analysis to benchmark non-homogeneous units. *International Journal of Information Technology & Decision Making, 14*, 727–745. https://doi.org/10.1142/S0219622014500916

Kallio, A. M. I., & Kallio, M. J. (2002). Nonparametric models for evaluating economic efficiency and imperfect competition. *Journal of Productivity Analysis, 18*, 171–189.

Kneip, A., Simar, L., & Wilson, P. W. (2016). Testing hypotheses in nonparametric models of production. *Journal of Business & Economic Statistics, 34*(3), 435–456.

Korhonen, P., Soismaa, M., & Siljamäki, A. (2002). On the use of value efficiency analysis and some further developments. *Journal of Productivity Analysis, 17*, 49–65.

Kumbhakar, S. C., & Lovell, C. A. K. (2000). *Stochastic frontier analysis*. Cambridge: Cambridge University Press.

Kumbhakar, S. C., Wang, H.-J., & Horncastle, A. (2015). *A practitioner's guide to stochastic frontier analysis using stata*. Cambridge: Cambridge University Press.

Kuosmanen, T. (2012). Stochastic semi-nonparametric frontier estimation of electricity distribution networks: Application of the StoNED method in the Finnish regulatory model. *Energy Economics, 34*(6), 2189–2199.

Kuosmanen, T., Kortelainen, M., Sipiläinen, T., & Cherchye, L. (2010). Firm and industry level profit efficiency analysis using absolute and uniform shadow prices. *European Journal of Operational Research, 202*, 584–594.

Murtagh, B., & Saunders, M. (1978). Large-scale linearly constrained optimization. *Mathematical Programming, 14*, 41–72.

Nerlove, K. (1965). *Estimation and identification of Cobb-Douglas production functions*. Chicago: Rand Mcnally.

Pedjara-Chaparro, F., Salinas-Jimenez, J., & Smith, P. (1997). On the role of weight restrictions in data envelopment analysis. *European Journal of Operational Research, 8*, 215–230.

Ramalho, E. A., Ramalho, J. J. S., & Henriques, P. D. (2010). Fractional regression models for second stage DEA efficiency analyses. *Journal of Productivity Analysis, 34*, 239–255.

Rosenblatt, M. (1956). Remarks on some nonparametric estimates of a density function. *The Annals of Mathematical Statistics, 27*(3), 832–837.

Ruggiero, J. (1998). Non-discretionary inputs in data envelopment analysis. *European Journal of Operational Research, 111*(3), 461–469.

Silverman, B. W. (1998). *Density estimation for statistics and data analysis*. Boca Raton: Chapman & Hall.