Chapter 15
Polylog Dimensional Subspaces of ℓ_∞^N

Gideon Schechtman and Nicole Tomczak-Jaegermann

In memory of Jean Bourgain, the brightest mathematical mind we have ever encountered

Abstract We show that a subspace of ℓ_∞^N of dimension $n > (\log N \log\log N)^2$ contains 2-isomorphic copies of ℓ_∞^k where k tends to infinity with $n/(\log N \log\log N)^2$. More precisely, for every $\eta > 0$, we show that any subspace of ℓ_∞^N of dimension n contains a subspace of dimension $m = c(\eta)\sqrt{n}/(\log N \log\log N)$ of distance at most $1 + \eta$ from ℓ_∞^m.

15.1 Introduction

The dichotomy problem of Pisier asks whether a Banach space X either contains, for every n, a subspace K-isomorphic to ℓ_∞^n, for some (equivalently all) $K > 1$, or, for every n, every n-dimensional subspace of X 2-embeds in ℓ_∞^N only if N is exponential in n. This is equivalent to the question of whether for some (equivalently

This research was completed in Fall 2017 while both authors where members of the Geometric Functional Analysis and Application program at MSRI, supported by the National Science Foundation under Grant No. 1440140.
The author "Gideon Schechtman" was supported in part by the Israel Science Foundation.

2010 AMS Subject Classification 46B07, 46B20

G. Schechtman (✉)
Department of Mathematics, Weizmann Institute of Science, Rehovot, Israel
e-mail: gideon@weizmann.ac.il

N. Tomczak-Jaegermann
Department of Mathematics, University of Alberta, Edmonton, AB, Canada
e-mail: nicole.tomczak@ualberta.ca

B. Klartag, E. Milman (eds.), *Geometric Aspects of Functional Analysis*, Lecture Notes in Mathematics 2266,
https://doi.org/10.1007/978-3-030-46762-3_15

all) absolute $K > 1$ and any sequence $n_N \leq N$ with $n_N/\log N \to \infty$ when $N \to \infty$, every subspace of ℓ_∞^N of dimension n_N contains a subspace of dimension m_N K-isomorphic to $\ell_\infty^{m_N}$ where $m_N \to \infty$ when $N \to \infty$.

We remark in passing that the equivalence between the two versions of the problem ("some $K > 1$" versus "all $K > 1$") is due to the fact proved by R.C. James that, for all $1 < \kappa < K < \infty$, a space which is K isomorphic to ℓ_∞^n contains a subspace κ isomorphic to ℓ_∞^m where $m \to \infty$ as $n \to \infty$. (James proof is essentially included in [5]. A somewhat more precise statement and proof, still due to James, can be read e.g. in [8, p. 283].)

As is exposed in [7], Maurey proved that if X^* has non-trivial type (Equivalently does not contain uniformly isomorphic copies of ℓ_1^n-s. This is a condition stronger than X has non-trivial cotype; equivalently, does not contain uniformly isomorphic copies of ℓ_∞^n-s), then we get the required conclusion: For every n, every n-dimensional subspace of X 2-embeds in ℓ_∞^N only if N is exponential in n.

Another partial result was obtained by Bourgain in [1] where he showed in particular that the conclusion holds if $n_N > (\log N)^4$.

Here we show some improvement over this result of Bourgain: The conclusion holds if $n_N/(\log N \log\log N)^2$ tends to ∞.

Theorem 15.1 *Let n, and N be integers such that $n > (\log N \log\log N)^2$. Then, for some absolute constant $c > 0$ and for every $0 < \eta < 1$, any subspace of ℓ_∞^N of dimension n contains a subspace of dimension $m = c\eta^2\sqrt{n}/(\log N \log\log N)$ of distance at most $1 + \eta$ from ℓ_∞^m.*

Note that we get some specific estimates for the dimension of the contained subspace $(1 + \eta)$-isomorphic to an ℓ_∞ space of its dimension. Although we are interested in small n-s, the result gives some estimate in the whole range. This is also the case in Bourgain's result: He proved that if $n \geq N^\delta$ than any subspace of ℓ_∞^N of dimension n contains a subspace $(1 + \eta)$-isomorphic to an ℓ_∞ of dimension $m \geq c\eta^5\delta^2\sqrt{n}/\log(1/\delta)$. Comparing the two, our result gives better estimates for m when $n \lesssim e^{c(\eta)\sqrt{\log N}}$ and worse when n is larger. Recall also that for n proportional to N, Figiel and Johnson [3] proved earlier that m can be taken of order $\sqrt{N}$ (and no better). This is not recovered by our result.

The general idea of the proof of Theorem 15.1 is the same as in [1] but the technical details are somewhat different. At the end of this note we also speculate that, up to the $(\log\log N)^2$ factor, our result may be best possible.

Our result was essentially achieved a long time ago, circa 1990. Since several people showed interest in it lately we decided to write it up with the hope that more modern methods (and younger minds) may be able to improve it farther.

15.2 Proofs

The main technical tool in the proof of Theorem 15.1 is the following proposition

Proposition 15.1 *Let n, and N be integers such that* $n > (\log N)^{3/2} \log\log N$. *Let* $[a_i(j)]$ *be an* $n \times N$ *matrix with* $a_i(j) \geq 0$ *for* $i = 1, \ldots, n$ *and* $j = 1, \ldots, N$. *Assume that*

$$\sum_{i=1}^{n} a_i(j)^2 \leq 1 \ \textit{for} \ \ j = 1, \ldots, N$$

and

$$\sum_{i=1}^{n} a_i(j) \leq 3\sqrt{logN} \ \ \textit{for} \ \ j = 1, \ldots, N.$$

Moreover, assume that, for some $\gamma > 0$, *for every* $i = 1, \ldots, n$ *there exists* $1 \leq j \leq N$ *such that* $a_i(j) \geq \gamma$. *Denote by* a_i *the* i*-th row of the matrix. Then, for some positive constants,* $c(\gamma), K(\gamma)$ *depending only on* γ *and for every* $0 < \eta < 1$, *there are disjoint subsets* $\sigma_1, \ldots, \sigma_m$ *of* $\{1, \ldots, n\}$ *with* $m \geq c(\gamma)\eta^2 n/(\log N)^{3/2} \log\log N$, *Such that*

$$\|\sum_{r=1}^{m} \sum_{i\in\sigma_r} a_i\|_\infty / \min_{1\leq r\leq m} \|\sum_{i\in\sigma_r} a_i\|_\infty \leq (1 + K(\gamma)\eta).$$

We first show how to deduce Theorem 15.1 from the proposition above.

Proof of Theorem 15.1 Let X be an n dimensional subspace of ℓ_∞^N. The π_2 norm of the identity on X is equal to $\sqrt{n}$ [4, 9] and by the main theorem of [10] (see [11] for the constant $\sqrt{2}$) this quantity can be computed, up to constant $\sqrt{2}$ on n vectors. This means that there are n vectors $a_i = (a_i(1), \ldots, a_i(N))$, $i = 1, \ldots, n$, in X satisfying

$$\sum_{i=1}^{n} a_i(j)^2 \leq 1, \ \ \text{for all} \ \ j = 1, \ldots, N$$

and

$$\sum_{i=1}^{n} \|a_i\|_\infty^2 \geq n/2.$$

The first condition implies in particular that $\|a_i\|_\infty^2 \leq 1$ for each i so necessarily for a subset σ' of $\{1, \ldots, n\}$ of cardinality at least $n/4$, $\|a_i\|_\infty \geq 1/2$ for all $i \in \sigma'$. The existence of a subset σ' of $\{1,\dots,n\}$ of cardinality at least $n/4$ satisfying the two conditions

$$\sum_{i\in\sigma'} a_i(j)^2 \le 1, \quad \text{for all } \ j=1,\dots,N, \quad \text{and } \ \|a_i\|_\infty \ge 1/2 \ \text{ for all } \ i\in\sigma' \tag{15.1}$$

is all that we shall use from now on. In Remark 15.1 below we'll show another way to obtain this.

Next we would like to choose a subset σ of σ' of cardinality of order $\sqrt{n\log N}$ such that the matrix $[|a_i(j)|]$, $i\in\sigma$, $j=1,\dots,N$, will satisfy the assumptions of Proposition 15.1. So let ξ_i, $i\in\sigma'$, be independent $\{0,1\}$ valued random variables with $\mathrm{Prob}(\xi_i=1)=\sqrt{(\log N)/n}$. Since for all j $\sum_{u\in\sigma'}|a_i(j)|\le\sqrt{n}$, $\mathbb{E}\sum_{u\in\sigma'}|a_i(j)|\xi_i\le\sqrt{\log N}$. By the most basic concentration inequality, using the fact that $\sum_{i\in\sigma'}a_i(j)^2\le 1$, for all j,

$$\begin{aligned}\mathrm{Prob}(&\sum_{i\in\sigma'}|a_i(j)|\xi_i>3\sqrt{\log N})\\ &\le \mathrm{Prob}(\sum_{i\in\sigma'}|a_i(j)|(\xi_i-\mathbb{E}\xi_i)>2\sqrt{\log N})\le e^{-2\log N}=1/N^2.\end{aligned}$$

It follows that with probability larger than $1-1/N$

$$\sum_{i\in\sigma'}|a_i(j)|\xi_i\le 3\sqrt{\log N}$$

for all j. Since by a similar argument also $\sum_{i\in\sigma'}\xi_i\ge\frac{\sqrt{n\log N}}{16}$ with probability tending to 1 when $N\to\infty$ we get a subset σ of cardinality $n'\ge\frac{\sqrt{n\log N}}{16}$ satisfying

$$\sum_{i\in\sigma}|a_i(j)|\le 3\sqrt{\log N} \ \text{ for all } \ j=1,\dots,N.$$

Note that the condition $n\ge 256(\log N\log\log N)^2$ implies that $n'\ge(\log N)^{3/2}\log\log N$. It follows that the matrix $[|a_i(j)|]$, $i\in\sigma'$, $j=1,\dots,N$ satisfies the conditions of Proposition 15.1 with n' replacing n and $\gamma=1/2$. We thus get that, for some absolute positive constants c,K, there are disjoint subsets $\sigma_1,\dots,\sigma_m$ of $\{1,\dots,n\}$ with

$$m\ge 16c\eta^2 n'/(\log N)^{3/2}\log\log N\ge c\eta^2\sqrt{n}/\log N\log\log N,$$

such that

$$\|\sum_{r=1}^m\sum_{i\in\sigma_r}|a_i|\|_\infty/\min_{1\le r\le m}\|\sum_{i\in\sigma_r}|a_i|\|_\infty\le(1+K\eta).$$

Rescaling, we may assume that $\min_{1\le r\le m} \|\sum_{i\in\sigma_r} |a_i|\|_\infty = 1$. Let j_r denote the label of (one of) the largest coordinates of $\sum_{i\in\sigma_r} |a_i|$. Assume as we may that $\eta < 1/K$. Then no two r's can share the same j_r. Changing the labelling we can also assume $j_r = r$.

Put $x_r = \sum_{i\in\sigma_r} sign(a_i(r))a_i$. Then for all r, $\|x_r\|_\infty \ge 1$ and for all $j = 1,\ldots,N$,

$$\sum_{r=1}^{m} |x_r(j)| \le 1 + K\eta. \tag{15.2}$$

So the sequence x_r, $r = 1,\ldots,m$, is $(1+K\eta)$-dominated by the ℓ_∞^m basis; i.e.,

$$\|\sum_{r=1}^{m} \alpha_r x_r\|_\infty \le (1+K\eta) \max_{1\le r\le m} |\alpha_r| \quad \text{for all } \{\alpha_r\}_{r=1}^m.$$

The lower estimate is achieved similarly: Assume $\max_{1\le r\le m} |\alpha_r| = |\alpha_{r_0}|$ and note that

$$\|\sum_{r=1, r\ne r_0}^{m} \sum_{i\in\sigma_r} |a_i(r_0)|\|_\infty \le K\eta.$$

Then,

$$\begin{aligned}\|\sum_{r=1}^{m} \alpha_r x_r\|_\infty &\ge |\sum_{r=1}^{m} \alpha_r x_r(r_0)| \\ &\ge |\alpha_{r_0}| \sum_{i\in\sigma_{r_0}} |a_i(r_0)| - \sum_{r=1,r\ne r_0}^{m} |\alpha_r| \sum_{i\in\sigma_r} |a_i(r_0)| \\ &\ge ((1-K\eta) \max_{1\le r\le m} |\alpha_r|.\end{aligned}$$

We have thus found a subspace of x of dimension $m \ge c\eta\sqrt{n}/(\log N \log\log N)$ whose distance to ℓ_∞^m is at most $(1+K\eta)/(1-K\eta)$. Changing the last quantity to $1+\eta$, paying by changing c to another absolute constant, is standard. ■

In the proof of Proposition 15.1 we shall use the following Lemma which follows immediately from Lemma 2 in [2] (but, following the proof of that lemma from [2], is a bit easier to conclude).

Lemma 15.1 *Let ξ_i, $i \in \{1, \dots, n\}$, be independent $\{0, 1\}$ valued random variables with Prob$(\xi_i = 1) = \delta$. Then for all $q \geq 1$,*

$$(\mathbb{E}(\sum_{i=1}^{n} \xi_i)^q)^{1/q} \leq C(\delta n + q).$$

C is a universal constant.

We now pass to the

Proof of Proposition 15.1 We shall assume as we may that $\eta < \gamma$. We first deal with the small $a_i(j)$-s. Fix $\varepsilon > 0$ to be defined later. Let

$$b_i(j) = \begin{cases} a_i(j) & \text{if } a_i(j) \leq \varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

We will show that for any $\delta > 0$, and with high probability for a random subset $\sigma \subset \{1, \dots, n\}$ of cardinality $|\sigma| \sim \delta n$

$$\sum_{i \in \sigma} b_i(j) \leq C(\delta\sqrt{\log N} + \varepsilon \log N) \ \text{ for } \ j = 1, \dots, N, \tag{15.3}$$

where C is an absolute constant.

Indeed, set $p = \log N$. Fix $\delta > 0$ and let ξ_i denote selectors with mean δ as in Lemma 15.1. By Chebyshev inequality, (15.3) follows from the estimate

$$\sup_j \left(\mathbb{E}(\sum_{i=1}^{n} \xi_i(\omega) b_i(j))^p \right)^{1/p} \leq C(\delta\sqrt{\log N} + \varepsilon \log N). \tag{15.4}$$

Indeed,

$$\begin{aligned} &\left(\mathbb{E} \sum_{j=1}^{N} (\sum_{i=1}^{n} \xi_i(\omega) b_i(j))^p \right)^{1/p} \\ &\qquad \leq N^{1/\log N} \sup_j \left(\mathbb{E}(\sum_{i=1}^{n} \xi_i(\omega) b_i(j))^p \right)^{1/p} \leq eC(\delta\sqrt{\log N} + \varepsilon \log N). \end{aligned}$$

Now apply Chebyshev's inequality.

Fix $1 \le j \le N$ and denote $(b_i(j))_i \in \mathbb{R}^n$ by b. Considering the level sets of b we may assume without loss of generality that b is of the form

$$b = \sum_{k=2\log(1/\varepsilon)}^{\infty} 2^{-k/2} \chi_{D_k},$$

(log is $\log_2$) where the sets $D_k \subset \{1, \dots, n\}$ are mutually disjoint and χ_{D_k} denotes the characteristic function of the set D_k, for $k = 2\log(1/\varepsilon), \dots$. Thus,

$$\begin{aligned}
&\left(\mathbb{E}(\sum_{j=1}^{n} \xi_j(\omega) b_i(j))^p\right)^{1/p} \\
&\le \sum_{k=2\log(1/\varepsilon)}^{\infty} 2^{-k/2} \left(\mathbb{E}(\sum_{j\in D_k} \xi_j(\omega))^p\right)^{1/p} \\
&\le C \sum_{k=2\log(1/\varepsilon)}^{\infty} 2^{-k/2} (\delta |D_k| + p)) \quad \text{by Lemma 15.1} \\
&\le C\delta \sum_{k=2\log(1/\varepsilon)}^{\infty} 2^{-k/2} |D_k| + Cp \sum_{k=2\log(1/\varepsilon)}^{\infty} 2^{-k/2}.
\end{aligned} \tag{15.5}$$

To estimate the first term in (15.5) note that

$$\sum_{k=2\log(1/\varepsilon)}^{\infty} 2^{-k/2} |D_k| = \|b\|_1 \le 3\sqrt{\log N}.$$

The second term is clearly smaller than an absolute constant times εp.

Combining the latter two estimates with (15.5) we get (15.4) and hence also (15.3).

To deal with the large coordinates, set, for $j = 1, \dots, N$,

$$A_j = \{1 \le i \le n;\ a_i(j) \ge \varepsilon\}.$$

Since $\sum_{i=1}^{n} a_i(j) \le 3\sqrt{\log N}$,

$$|A_j| \le 3\sqrt{\log N}/\varepsilon \ \text{ for } \ j = 1, \dots, N. \tag{15.6}$$

An argument similar to the one that proved (15.4) also shows that a random set $\sigma \subset \{1, \dots, n\}$ of cardinality $|\sigma| \sim \delta n$ satisfies

$$|\sigma \cap A_j| \le C(\delta\sqrt{\log N}/\varepsilon + \log N). \tag{15.7}$$

Indeed, this follows easily by applying the following inequality with $p = \log N$,

$$\left(\mathbb{E}\left(\sum_{i=1}^{\sqrt{\log N}/\varepsilon} \xi_i\right)^p\right)^{1/p} \leq C(\delta\sqrt{\log N}/\varepsilon + \log N).$$

Moreover, Chebyshev's inequality implies that we can find a set $\sigma \subset \{1, \ldots, n\}$ of cardinality at least $\frac{1}{2}\delta n$ which satisfies (15.3) and (15.7) simultaneously (say, with the same absolute constant C).

Choose now $\delta = 2\eta/\sqrt{\log N}$ and $\varepsilon = \eta/\log N$. Then we get a set $\sigma \subset \{1, \ldots, n\}$ of cardinality at least $\eta n/\sqrt{\log N}$. such that

$$\sum_{i\in\sigma} b_i(j) \leq 3C\eta \ \text{ for } \ j = 1, \ldots, N, \tag{15.8}$$

and

$$|\sigma \cap A_j| \leq 3C\log N \ \text{ for } \ j = 1, \ldots, N. \tag{15.9}$$

Define $j_1 \in \{1, \ldots, N\}$ and s_1 by

$$s_1 = \sum_{i\in\sigma\cap A_{j_1}} a_i(j_1) = \max_j \sum_{i\in\sigma\cap A_j} a_i(j).$$

For $r > 1$ define $S_{r-1} = \sigma \setminus (A_{j_1} \cup \cdots \cup A_{j_{r-1}})$ and j_r and s_r by

$$s_r = \sum_{i\in S_{r-1}\cap A_{j_r}} a_i(j_r) = \max_j \sum_{i\in S_{r-1}\cap A_j} a_i(j).$$

By rearranging the columns we may assume $j_r = r$ for all r. Now, (15.9) implies that $|S_r| \geq |\sigma| - 3Cr\log N$ so S_r is not empty for $1 \leq r \leq \frac{\eta n}{3C(\log N)^{3/2}}$. Also,

$$\gamma \leq s_r \leq 3\sqrt{\log N} \leq 3\log N \ \text{ for } \ 1 \leq r \leq \frac{\eta n}{3C(\log N)^{3/2}}.$$

The sequence s_r is non-increasing, divide it into $(\log((3\log N)/\gamma))/\log(1+\eta)$ intervals such that in each interval $\max s_r/\min s_r$ is at most $1+\eta$. There is an interval R with $|R| \geq \frac{(\log(1+\eta))\eta n}{3C(\log N)^{3/2}\log((3\log N)/\gamma)} \geq \frac{\eta^2 n}{6C(\log N)^{3/2}\log((3\log N)/\gamma)}$ such that

$$\max_{r\in R} s_r / \min_{r\in R} s_r \leq 1+\eta.$$

Put $\sigma_r = S_{r-1} \cap A_r$. Since $\min_{r\in R} s_r \geq \gamma > \eta$ we are done in view of (15.8) and the fact that $s_r \geq \sum_{S_{r-1}\cap A_s} a_i(s)$ for $r < s$. ■

15.3 Remarks

Remark 15.1 Here is an alternative way to get (15.1):

Let X be an n-dimensional normed space which, without loss of generality we assume is in John's position, i.e., the maximal volume ellipsoid inscribed in the unit ball of X is the canonical sphere S^{n-1}. A weak form of the Dvoretzky–Rogers lemma asserts that there are orthonormal vectors $x_1, \ldots, x_n$ such that $\|x_i\|_X \geq c$ for some universal positive constant c. This is proved by a simple volume argument, see for example Theorem 3.4 in [6]. (There it is shown that there are $[n/2]$ such vectors. This is enough for us but it's also easy and well known how to use these $n/2$ orthonormal vectors to get n orthonormal vectors with a somewhat worse lower bound on their norms.)

The map $T : \ell_2^n \to X$ defined by $Te_i = x_i$ is norm one. Note that

$$1 = \|T\| = \sup_{\|x^*\|_{X^*}\leq 1} (\sum_{i=1}^{n} (x^*(x_i))^2)^{1/2}.$$

When X is isometric to a subspace of ℓ_∞^N there are N elements $x_j^* \in B_{X^*}$ such that, for all $x \in X$, $\|x\| = \max_{1\leq j\leq N} x_j^*(x)$. From this it is easy to deduce that

$$\sup_{\|x^*\|_{X^*}\leq 1} (\sum (x^*(x_i))^2)^{1/2} = \max_{1\leq j\leq N} (\sum_{i=1}^{n} (x_j^*(x_i))^2)^{1/2}.$$

Denoting $a_i(j) = x_j^*(x_i)$ we get (15.1).

Remark 15.2 Here we would like to suggest an approach toward showing that the dichotomy conjecture fails and maybe even that one can't get below the estimate $n > (\log N)^2$ in Theorem 15.1.

Let X and Y be two l dimensional normed spaces. Put $n = l^2$ and $N = 36^l$. Let $\{x_i\}_{i=1}^{6^l}$ be a $1/2$ net in the sphere of X and $\{y_i^*\}_{i=1}^{6^l}$ be a $1/2$ net in the sphere of Y^*. Note that for every $T : X \to Y$,

$$\max_{1\leq i,j\leq 6^l} y_i^*(Tx_j) \leq \|T\| \leq 4 \max_{1\leq i,j\leq 6^l} y_i^*(Tx_j).$$

Consequently, $B(X, Y)$, the space of operators from X to Y with the operator norm, 4-embeds into ℓ_∞^N. Note that $\dim(B(X, Y)) = n \sim (\log N)^2$.

(Un)fortunately, $B(X, Y)$ cannot serve as a negative example since it always contains ℓ_∞-s with dimension going to infinity with N. This was pointed out to us by Bill Johnson. Indeed, by Dvoretzky's theorem, ℓ_2^k 2-embeds into Y and into X^*, for some k tending to infinity with n. Let I denote the first embedding and Q be the adjoint of the second embedding. It is then easy to see that $T \to ITQ$ is a 4-embedding of $B(\ell_2^k, \ell_2^k)$ into $B(X, Y)$. Finally, $B(\ell_2^k, \ell_2^k)$ contains isometrically ℓ_∞^k.

However, to get a negative answer to the dichotomy problem, it is enough to find n dimensional X and Y and a subspace Z of $B(X, Y)$ of dimension m with m/n tending to infinity with n which has good cotype, i.e., if Z contains a 2-isomorph of ℓ_∞^k then k is bounded by a universal constant. If one can find such an example with $m \geq cn^2$ for some universal positive constant c then it will even show that one can't get below the estimate $n > (\log N)^2$ in Theorem 15.1.

References

1. J. Bourgain, Subspaces of L_N^∞, arithmetical diameter and Sidon sets, in *Probability in Banach Spaces, V* (Medford, 1984). Lecture Notes in Mathematics, vol. 1153 (Springer, Berlin, 1985), pp. 96–127
2. J. Bourgain, Bounded orthogonal systems and the $\Lambda(p)$-set problem. Acta Math. **162**(3–4), 227–245 (1989)
3. T. Figiel, W.B. Johnson, Large subspaces of l_∞^n and estimates of Gordon-Lewis constant. Israel J. Math. **37**(1–2), 92–112 (1980)
4. D.J.H. Garling, Y. Gordon, Relations between some constants associated with finite dimensional Banach spaces. Israel J. Math. **9**, 346–361 (1971)
5. R.C. James, Uniformly non-square Banach spaces. Ann. Math. **80**(2), 542–550 (1964)
6. V.D. Milman, G. Schechtman, *Asymptotic theory of finite-dimensional normed spaces*. With an appendix by M. Gromov. Lecture Notes in Mathematics, vol. 1200 (Springer, Berlin, 1986), pp. viii+156
7. G. Pisier, Remarques sur un résultat non publié de B. Maurey, (French) [[Remarks on an unpublished result of B. Maurey]] in *Seminar on Functional Analysis*, 1980–1981. (École Polytech., Palaiseau, 1981), Exp. No. V, p. 13
8. G. Schechtman, Euclidean sections of convex bodies, in *Asymptotic Geometric Analysis*. Fields Institute Communications, vol. 68 (Springer, New York, 2013), pp. 271–288
9. M.G. Snobar, p-absolutely summing constants. (Russian) Teor. Funkciĭ Funkcional. Anal. i Priložen. **216**(16), 38–41, (1972)
10. N. Tomczak-Jaegermann, Computing 2-summing norm with few vectors. Ark. Mat. **17**(2), 273–277 (1979)
11. N. Tomczak-Jaegermann, *Banach-Mazur Distances and Finite-Dimensional Operator Ideals*. Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 38. (Longman Scientific & Technical, Harlow; Wiley, New York, 1989)