Chapter 11
The Alon–Milman Theorem for Non-symmetric Bodies

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Abstract A classical theorem of Alon and Milman states that any d dimensional centrally symmetric convex body has a projection of dimension $m \geq e^{c\sqrt{\ln d}}$ which is either close to the m-dimensional Euclidean ball or to the m-dimensional cross-polytope. We extended this result to non-symmetric convex bodies.

11.1 Introduction

Some fundamental results from the theory of normed spaces have been shown to hold in the more general setting of non-symmetric convex bodies. Dvoretzky's theorem [3, 7] was extended in [6] and [5]; Milman's Quotient of Subspace theorem [8] and duality of entropy results were extended in [9]. In this note, we extend the Alon–Milman Theorem.

A *convex body* is a compact convex set in $\mathbb{R}^d$ with non-empty interior. We denote the orthogonal projection onto a linear subspace H or $\mathbb{R}^d$ by P_H. For $p = 1, 2, \infty$, the closed unit ball of ℓ_p^d centered at the origin is denoted by $\mathbf{B}_p^d$. Let K and L be convex bodies in $\mathbb{R}^d$ with $L = -L$. We define their *distance* as

$$d(K, L) = \inf\{\lambda > 0 \colon \; L \subset T(K - a) \subset \lambda L \text{ for some } a \in \mathbb{R}^d \text{ and } T \in GL(\mathbb{R}^d)\}.$$

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By compactness, this infimum is attained, and when $K = -K$, it is attained with $a = 0$.

Alon and Milman [1] proved the following theorem in the case when K is centrally symmetric.

Theorem 11.1 *For every $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ with the property that in any dimension $d \in \mathbb{Z}^+$, and for any convex body K in $\mathbb{R}^d$, at least one of the following two statements hold:*

(i) there is an m-dimensional linear subspace H of $\mathbb{R}^d$ such that $d(P_H(K), \mathbf{B}_2^m) < 1 + \varepsilon$, for some m satisfying $\ln \ln m \geq \frac{1}{2} \ln \ln d$, or
(ii) there is an m-dimensional linear subspace H such that $d(P_H(K), \mathbf{B}_1^m) < 1 + \varepsilon$, for some m satisfying $\ln \ln m \geq \frac{1}{2} \ln \ln d - C(\varepsilon)$.

The main contribution of the present note is a way to deduce Theorem 11.1 from the original result of Alon and Milman, that is, the centrally symmetric case. By polarity, one immediately obtains

Corollary 11.1 *For every $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ with the property that in any dimension $d \in \mathbb{Z}^+$, and for any convex body K in $\mathbb{R}^d$ containing the origin in its interior, at least one of the following two statements hold:*

(i) there is an m-dimensional linear subspace H of $\mathbb{R}^d$ such that $d(H \cap K, \mathbf{B}_2^m) < 1 + \varepsilon$, for some m satisfying $\ln \ln m \geq \frac{1}{2} \ln \ln d$, or
(ii) there is an m-dimensional linear subspace H such that $d(H \cap K, \mathbf{B}_\infty^m) < 1 + \varepsilon$, for some m satisfying $\ln \ln m \geq \frac{1}{2} \ln \ln d - C(\varepsilon)$.

11.2 Proof of Theorem 11.1

For a convex body K in $\mathbb{R}^d$, we denote its polar by $K^* = \{x \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}$. The *support function* of K is $h_K(x) = \sup\{\langle x, y \rangle : y \in K\}$. For basic properties, see [2, 12].

First in Lemma 11.2, by a standard argument, we show that if the *difference body* $L - L$ of a convex body L is close to the Euclidean ball, then so is some linear dimensional section of L. For this, we need Milman's theorem whose proof (cf. [4, 7, 10]) does not use the symmetry of K even if it is stated with that assumption. We use $\mathbb{S}^{d-1}$ to denote the boundary of $\mathbf{B}_2^d$.

Lemma 11.1 (Milman's Theorem) *For every $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ with the property that in any dimension $d \in \mathbb{Z}^+$, and for any convex body K in $\mathbb{R}^d$ with $\mathbf{B}_2^d \subseteq K$, there is an m-dimensional linear subspace H of $\mathbb{R}^d$ such that*

$(1-\varepsilon)r(\mathbf{B}_2^d \cap H) \subseteq K \subseteq (1+\varepsilon)r(\mathbf{B}_2^d \cap H)$, for some m satisfying $m \geq C(\varepsilon)M^2 d$, where

$$M = M(K) = \int_{\mathbb{S}^{d-1}} ||x||_K d\sigma(x),$$

and $r = \frac{1}{M}$.

Lemma 11.2 *Let $\alpha, \varepsilon > 0$ be given. Then there is a constant $c = c(\alpha, \varepsilon)$ with the property that in any dimension $m \in \mathbb{Z}^+$, and for any convex body L in $\mathbb{R}^m$ with $d(L-L, \mathbf{B}_2^m) < 1+\alpha$, there is a k dimensional linear subspace F of $\mathbb{R}^m$ such that $d(P_F(L), \mathbf{B}_2^k) < 1+\varepsilon$ for some $k \geq cm$.*

Proof Let $\delta = d(L-L, \mathbf{B}_2^m)$. We may assume that $\frac{1}{\delta}\mathbf{B}_2^m \subseteq L-L \subseteq \mathbf{B}_2^m$. Thus, for the support function of $L-L$, we have $h_{L-L}(x) \geq \frac{1}{\delta}$ for any $x \in \mathbb{S}^{d-1}$. With the notations of Lemma 11.1, we have

$$\begin{aligned} M(L^*) &= \int_{\mathbb{S}^{d-1}} ||x||_{L^*} d\sigma(x) = \frac{1}{2}\int_{\mathbb{S}^{d-1}} h_L(x) + h_L(-x) d\sigma(x) \qquad (11.1)\\ &= \frac{1}{2}\int_{\mathbb{S}^{d-1}} h_{L-L}(x) d\sigma(x) \geq \frac{1}{2\delta} \geq \frac{1}{2(1+\alpha)}. \end{aligned}$$

Note that $L^* \supset (L-L)^* \supset \mathbf{B}_2^d$, thus, by Lemma 11.1 and polarity, we obtain that L has a k dimensional projection P_F with $d(P_F L, \mathbf{B}_2^d \cap F) \leq 1+\varepsilon$ and $k \geq C(\varepsilon)\frac{1}{4(1+\alpha)^2}m$. Here, $C(\varepsilon)$ is the same as in Lemma 11.1. □

The novel geometric idea of our proof is the following. We call a convex body $T = \operatorname{conv}(T_1 \cup \{\pm e\})$ in $\mathbb{R}^m$ a *double cone* if $T_1 = -T_1$ is convex set, span T_1 is an $(m-1)$-dimensional linear subspace, and $e \in \mathbb{R}^m \setminus \operatorname{span} T_1$. Double cones are *irreducible convex bodies*, that is, for any double cone T, if $T = L - L$ then $L = T/2$, see [11, 13]. We prove a stability version of this fact.

Lemma 11.3 (Stability of Irreducibility of Double Cones) *Let L be a convex body in $\mathbb{R}^m$ with $m \geq 2$, and T be a double cone of the form $T = \operatorname{conv}(T_1 \cup \{\pm e\})$. Assume that $T \subseteq L-L \subseteq \delta T$ for some $1 \leq \delta < \frac{3}{2}$. Then*

$$\left(\frac{3}{2} - \delta\right) T \subseteq L - a \subseteq \left(\delta - \frac{1}{2}\right) T.$$

for some $a \in \mathbb{R}^m$.

Proof By the assumptions, $e \in T \subseteq L - L$, thus, by translating L, we may assume that $o, e \in L$. Thus,

$$L \subseteq (L - L) \cap (L - L + e) \subseteq \delta T \cap (\delta T + e). \tag{11.2}$$

We claim that

$$\delta T \cap (\delta T + e) = \frac{e}{2} + \left(\delta - \frac{1}{2}\right) T. \tag{11.3}$$

Indeed, let H_λ denote the hyperplane $H_\lambda = \lambda e + \operatorname{span} T_1$. To prove (11.3), we describe the sections of the right hand side and the left hand side by the hyperplanes H_λ for all relevant values of λ. For any $\lambda \in [-\delta, \delta]$, we have

$$\delta T \cap H_\lambda = \delta(T \cap H_{\lambda/\delta}) = \lambda e + \delta \left(1 - \frac{|\lambda|}{\delta}\right) T_1.$$

For any $\lambda \in [-\delta + 1, \delta + 1]$, we have

$$(\delta T + e) \cap H_\lambda = e + (\delta T \cap H_{\lambda - 1}) = \lambda e + \delta \left(1 - \frac{|\lambda - 1|}{\delta}\right) T_1.$$

Thus, for any $\lambda \in [-\delta + 1, \delta]$, we have

$$\delta T \cap (\delta T + e) \cap H_\lambda = \lambda e + \delta \left(1 - \frac{1}{\delta} \max\{|\lambda|, |\lambda - 1|\}\right) T_1.$$

On the other hand, for any $\lambda \in [-\delta + 1, \delta]$, we have

$$(e/2 + (\delta - 1/2)T) \cap H_\lambda = \lambda e + (\delta - 1/2) \left(1 - \frac{|\lambda - 1/2|}{\delta - 1/2}\right) T_1.$$

Combining these two equations yields (11.3).

Thus,

$$T \subseteq L - L = \left(L - \frac{e}{2}\right) - \left(L - \frac{e}{2}\right) \subseteq \left(L - \frac{e}{2}\right) - \left(\delta - \frac{1}{2}\right) T.$$

Using the fact that $T = -T$, and $1 \leq \delta < 3/2$, we obtain

$$\left(\frac{3}{2} - \delta\right) T \subseteq L - \frac{e}{2},$$

finishing the proof of Lemma 11.3. □

Now, we are ready to prove Theorem 11.1. With the notations of the theorem, let $D = K - K$, and apply the symmetric version of the theorem for D in place of K. We may assume that $\varepsilon < 1/2$. In case (1), we use Lemma 11.2 and loose a linear factor in the dimension of the almost-Euclidean projection. In case (2), we use Lemma 11.3 with $T = \mathbf{B}_1^m$ and $\delta = 1 + \varepsilon$, and obtain the same dimension for the almost-ℓ_1^m projection.

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