Chapter 4
Regularity of Powers of Ideals and the Combinatorial Framework

Castelnuovo-Mumford regularity (or simply *regularity*) is an important invariant in commutative algebra and algebraic geometry. Computing or finding bounds for the regularity is a difficult problem. In the next three chapters, we shall address the regularity of ordinary and symbolic powers of squarefree monomial ideals.

Our interest in squarefree monomial ideals comes from their strong connections to topology and combinatorics via the construction of Stanley-Reisner ideals and edge ideals. In recent years advances in computer technology and speed of computation have drawn significant attention toward problems and questions involving this class of ideals.

The collection of problems and questions presented in these three chapters originates from a celebrated result proved independently by Cutkosky, Herzog and Trung [48] and Kodiyalam [127] (see also Trung and Wang [162] for the module case and Bagheri, Chardin, and Hà,[6] and Whieldon [170] for the multigraded case), which states that for a homogeneous ideal I in a standard graded algebra R over a Noetherian commutative ring, the regularity of I^q is asymptotically a linear function. The problem of determining this linear function and the smallest value of q starting from which $\operatorname{reg} I^q$ becomes linear remains wide open and has evolved into a highly active research area in the last few decades.

We shall discuss this problem primarily for the class of squarefree monomial ideals. Our focus will be on studies of the asymptotic linear function $\operatorname{reg} I^q$ for a squarefree monomial ideal I via combinatorial data and structures of the corresponding simplicial complex and/or hypergraph.

4.1 Regularity of Powers of Ideals: The General Question

The main object of our discussion in this part of the book is the Castelnuovo-Mumford regularity. This notion can be defined in various ways. We shall first give the definition for modules over polynomial rings as this situation is our focus. A

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more general definition in terms of local cohomology will also be given for the more advanced interested reader. The motivating theorem and general question are given at the end of the section.

Definition 4.1 Let R be a standard graded polynomial ring over a field and let $\mathfrak{m}$ be its maximal homogeneous ideal. Let M be a finitely generated graded R-module and let

$$0 \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{p,j}(M)} \to \cdots \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{0,j}(M)} \to M \to 0$$

be its minimal free resolution. Then the regularity of M is given by

$$\operatorname{reg} M = \max\{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

Remark 4.2 It is clear from the definition that the regularity of M gives an upper bound for the generating degrees of M.

Example 4.3 Consider

$$I = \langle x^2y - 2yz^2 + 3z^3, 2xw - 3yw, yw^4 - y^2z^3 - 2x^5 \rangle \subseteq R = \mathbb{Q}[x, y, z, w].$$

Then I has the following minimal free resolution:

$$0 \longrightarrow R(-10) \longrightarrow R(-5) \oplus R(-7) \oplus R(-8) \longrightarrow R(-2) \oplus R(-3) \oplus R(-5) \longrightarrow I \longrightarrow 0.$$

Thus, $\operatorname{reg} I = 8$.

If R is a general standard graded algebra over a ring, then the minimal free resolution of an R-module M may not be finite. In this case, the regularity can still be defined via local cohomology. See, for example, Chardin [35], and Eisenbud and Goto [64] for the equivalence between the two definitions when R is a polynomial ring over a field.

Definition 4.4 Let R be a standard graded algebra over a Noetherian commutative ring with identity and let $\mathfrak{m}$ be its maximal homogeneous ideal. Let M be a finitely generated graded R-module. For $i \geq 0$, let

$$a^i(M) = \begin{cases} \max\left\{l \in \mathbb{Z} \,\middle|\, \left[H^i_{\mathfrak{m}}(M)\right]_l \neq 0\right\} & \text{if } H^i_{\mathfrak{m}}(M) \neq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

The *regularity* of M is defined to be

$$\operatorname{reg} M = \max_{i \geq 0}\{a^i(M)\}.$$

Note that $a^i(M) = 0$ for $i > \dim M$, so the regularity of M is well-defined.

Example 4.5 Consider $R = \mathbb{K}[x_1, \dots, x_n]$, a polynomial ring over a field $\mathbb{K}$. Then $H^i_{\mathfrak{m}}(R) = 0$ for all $i < n$, and $a^n(R) = -n$. Thus, $\operatorname{reg} R = 0$.

This definition of regularity works especially well with short exact sequences. For instance, the following lemma is well-known (cf. [63, Corollary 20.19]).

Lemma 4.6 *Let* $0 \to M \to N \to P \to 0$ *be a short exact sequence of graded* R*-modules. Then*

1. $\operatorname{reg} N \leq \max\{\operatorname{reg} M, \operatorname{reg} P\}$,
2. $\operatorname{reg} M \leq \max\{\operatorname{reg} N, \operatorname{reg} P + 1\}$,
3. $\operatorname{reg} P \leq \max\{\operatorname{reg} M - 1, \operatorname{reg} N\}$,
4. $\operatorname{reg} M = \operatorname{reg} P + 1$ *if* $\operatorname{reg} N < \operatorname{reg} P$*,*
5. $\operatorname{reg} P = \operatorname{reg} M - 1$ *if* $\operatorname{reg} N < \operatorname{reg} M$*,*
6. $\operatorname{reg} P = \operatorname{reg} N$ *if* $\operatorname{reg} N > \operatorname{reg} M$*, and*
7. $\operatorname{reg} N = \operatorname{reg} M$ *if* $\operatorname{reg} P + 1 < \operatorname{reg} M$*.*

The motivation of our discussion is the following celebrated result, which was first independently proved by Cutkosky, Trung and Herzog [48] and Kodiyalam [127] (the constant a was determined in Trung and Wang [162]).

Theorem 4.7 ([48, 127, 162]) *Let* R *be a standard graded algebra over a Noetherian commutative ring with identity. Let* $I \subseteq R$ *be a homogeneous ideal. Then there exist constants* a *and* b *such that*

$$\operatorname{reg} I^q = aq + b \text{ for all } q \gg 0.$$

Moreover,

$$a = \min\{d(J) \mid J \text{ is a minimal homogeneous reduction of } I\}.$$

Here, $J \subseteq I$ is a *reduction* of I if $I^{s+1} = JI^s$ for some (and all) $s \geq 0$, and $d(J)$ denotes the maximal generating degree of J.

The following problem remains wide open despite much effort from researchers.

Problem 4.8 Determine b and $q_0 = \min\{t \in \mathbb{Z} \mid \operatorname{reg} I^q = aq + b \text{ for all } q \geq t\}$.

In general, when I is generated in the same degree, the constant b can be related to a *local* invariant, namely, the regularity of preimages of germs of schemes via certain projection maps from the blowup of $X = \operatorname{Proj} R$ along I.

For the interested reader, we expanded upon the above comment; we do not refer to this discussion in future sections. Let $I = \langle F_0, \dots, F_m \rangle$, where $F_0, \dots, F_m$ are homogeneous elements of degree d in R. Let $\pi : \tilde{X} \to X$ be the blow up of $X = \operatorname{Proj} R \subseteq \mathbb{P}^n$ along the subscheme defined by I. Let $\mathscr{R} = R[It] = \bigoplus_{q \geq 0} I^q t^q$ be the Rees algebra of I. By letting $\deg t = (0, 1)$ and $\deg F_i t = (d, 1)$, the Rees algebra $\mathscr{R}$ is naturally bi-graded with $\mathscr{R} = \bigoplus_{p,q \in \mathbb{Z}} \mathscr{R}_{(p,q)}$, where $\mathscr{R}_{(p,q)} = (I^q)_{p+qd} t^q$. Under this bi-gradation of $\mathscr{R}$, we can define the bi-projective scheme

$\operatorname{Proj}\mathscr{R}$ of $\mathscr{R}$ as follows (cf. Hà [92]):

$$\operatorname{Proj}\mathscr{R} = \{\mathfrak{p} \in \operatorname{Spec}\mathscr{R} \bigm| \mathfrak{p} \text{ is a bihomogeneous ideal and } \mathscr{R}_{++} \not\subseteq \mathfrak{p}\},$$

where $\mathscr{R}_{++} = \bigoplus_{p,q\geq 1} \mathscr{R}_{(p,q)}$. It can be seen that $\operatorname{Proj}\mathscr{R} \subseteq \mathbb{P}^n \times \mathbb{P}^m$ and $\tilde{X} \simeq \operatorname{Proj}\mathscr{R}$.

Let $\phi : \operatorname{Proj}\mathscr{R} \to \mathbb{P}^m$ denote the natural projection from $\operatorname{Proj}\mathscr{R}$ onto its second coordinate, and let $\overline{X} = \operatorname{im}(\phi)$. Note that ϕ is the morphism given by the divisor $D = dE_0 - E$, where E is the exceptional divisor of π and E_0 is the pullback of a general hyperplane in X. For a point $\wp \in \overline{X}$, let $\tilde{X}_\wp = \tilde{X} \times_{\overline{X}} \operatorname{Spec}\mathscr{O}_{\overline{X},\wp}$ be the preimage of ϕ over the affine scheme $\operatorname{Spec}\mathscr{O}_{\overline{X},\wp}$.

Let S denote the homogeneous coordinate ring of $\overline{X} \subseteq \mathbb{P}^m$. For a homogeneous prime $\wp \subseteq S$ (i.e., a point in $\overline{X}$), let $\mathscr{R}_\wp = \mathscr{R} \otimes_S S_\wp$ be the *localization* of $\mathscr{R}$ at $\wp$. The *homogeneous localization* of $\mathscr{R}$ at $\wp$, denoted by $\mathscr{R}_{(\wp)}$, is defined to be the collection of elements in $\mathscr{R}_\wp$ of degree 0 in terms of powers of t. Then $\tilde{X}_\wp = \operatorname{Proj}\mathscr{R}_{(\wp)}$. We define the *regularity* of $\tilde{X}_\wp$, denoted by $\operatorname{reg}\tilde{X}_\wp$, to be that of its homogeneous coordinate ring $\mathscr{R}_{(\wp)}$, and let $\operatorname{reg}\phi = \max\{\operatorname{reg}\tilde{X}_\wp \mid \wp \in \overline{X}\}$.

The following result follows from a series of work of Chardin [36], Eisendbud and Harris [65], and Hà [92]. Partial results on the stability index q_0 were obtained by Eisenbud and Ulrich [66], when I is $\mathfrak{m}$-primary, and by Chardin [35] and Bisui, Hà, and Thomas [18], when I is equi-generated.

Theorem 4.9 *Let R be a standard graded algebra over a Noetherian commutative ring with identity. Let $I \subseteq R$ be a homogeneous ideal generated in degree d. For $q \gg 0$, we have*

$$\operatorname{reg} I^q = qd + \operatorname{reg}\phi.$$

The invariant $\operatorname{reg}\phi$, in practice, is difficult to compute. Even when I is generated by “enough” (i.e., more than $\dim R$) general linear forms, it is still an open problem to compute $\operatorname{reg}\phi$.

In the next three chapters, we shall see a different approach to computing $\operatorname{reg}\phi$ (or equivalently, the free constant b) when I is a squarefree monomial ideal.

4.2 Squarefree Monomial Ideals and Combinatorial Framework

Our aim in this part of the book is to study a restricted version of Problem 4.8, which is applied to the class of squarefree monomial ideals. For this purpose, we shall now fix some notation. From now on, $\mathbb{K}$ will denote an infinite field, $R = \mathbb{K}[x_1, \ldots, x_n]$ will be a polynomial ring over $\mathbb{K}$, and $\mathfrak{m}$ will denote the maximal homogeneous ideal in R. For obvious reasons, we shall identify the variables $x_1, \ldots, x_n$ with the vertices of simplicial complexes and hypergraphs being discussed. By abusing notation, we also often identify a subset V of the vertices $X = \{x_1, \dots, x_n\}$ with the squarefree monomial $x^V = \prod_{x \in V} x$ in the polynomial ring R.

The combinatorial framework we shall use is the construction of Stanley-Reisner ideals and edge ideals corresponding to simplicial complexes and hypergraphs. The notion of edge ideals of hypergraphs is the generalization of that of edge ideals of graphs defined in Chap. 2.

4.2.1 Simplicial Complexes

A *simplicial complex* Δ over the vertex set $X = \{x_1, \dots, x_n\}$ is a collection of subsets of X such that if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$. Elements of Δ are called *faces*. Maximal faces (with respect to inclusion) are called *facets*. For $F \in \Delta$, the *dimension* of F is defined to be $\dim F = |F| - 1$. The *dimension* of Δ is $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$. The complex is called *pure* if all of its facets are of the same dimension. A graph can be viewed as a 1-dimensional simplicial complex.

Let Δ be a simplicial complex, and let $Y \subseteq X$ be a subset of its vertices. The *induced subcomplex* of Δ on Y, denoted by $\Delta[Y]$, is the simplicial complex with vertex set Y and faces $\{F \in \Delta \mid F \subseteq Y\}$.

Definition 4.10 Let Δ be a simplicial complex over the vertex set X, and let $\sigma \in \Delta$.

1. The *deletion* of σ in Δ, denoted by $\mathrm{del}_\Delta(\sigma)$, is the simplicial complex obtained by removing σ and all faces containing σ from Δ.
2. The *link* of σ in Δ, denoted by $\mathrm{link}_\Delta(\sigma)$, is the simplicial complex whose faces are

$$\{F \in \Delta \mid F \cap \sigma = \emptyset, \sigma \cup F \in \Delta\}.$$

Definition 4.11 A simplicial complex Δ is recursively defined to be *vertex decomposable* if either

1. Δ is a simplex (or the empty simplicial complex); or
2. there is a vertex v in Δ such that both $\mathrm{link}_\Delta(v)$ and $\mathrm{del}_\Delta(v)$ are vertex decomposable, and all facets of $\mathrm{del}_\Delta(v)$ are facets of Δ.

A vertex satisfying condition (2) is called a *shedding vertex*, and the recursive choice of shedding vertices are called a *shedding order* of Δ.

Definition 4.12 A simplicial complex Δ is said to be *shellable* if there exists a linear order of its facets $F_1, F_2, \dots, F_t$ such that for all $k = 2, \dots, t$, the subcomplex $\left(\bigcup_{i=1}^{k-1} \overline{F_i}\right) \bigcap \overline{F_k}$ is pure and of dimension $(\dim F_k - 1)$. Here $\overline{F}$ represents the simplex over the vertices of F.

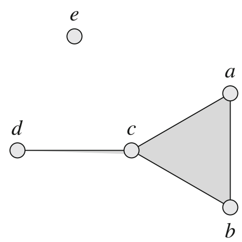

Fig. 4.1 A vertex decomposable simplicial complex

It is a celebrated fact that *pure* shellable complexes give rise to *Cohen-Macaulay Stanley-Reisner rings*. For more details on Cohen-Macaulay rings and modules, we refer the reader to Bruns and Herzog [25]. The notion of Stanley-Reisner rings will be discussed later in the section. Note also that a ring or module is *sequentially Cohen-Macaulay* if it has a filtration in which the factors are Cohen-Macaulay and their dimensions are increasing. This property corresponds to (*nonpure*) shellability in general.

Vertex decomposability can be thought of as a combinatorial criterion for shellability and sequentially Cohen-Macaulayness. In particular, for a simplicial complex Δ,

$$\Delta \text{ vertex decomposable } \Rightarrow \Delta \text{ shellable } \Rightarrow \Delta \text{ sequentially Cohen-Macaulay.}$$

Example 4.13 The simplicial complex Δ in Fig. 4.1 is a nonpure simplicial complex of dimension 2. It has 3 facets; the facet $\{a, b, c\}$ is of dimension 2, the facet $\{c, d\}$ is of dimension 1, and the facet $\{e\}$ is of dimension 0. The complex Δ is vertex decomposable with $\{e, d\}$ as a shedding order.

4.2.2 Hypergraphs

Hypergraphs are a generalization of graphs that where introduced in Chap. 2. We now introduce this combinatorial object; note that some of graph theoretic terms introduced in Chap. 2 have a hypergraph analog.

A hypergraph $H = (X, \mathscr{E})$ over the vertex set $X = \{x_1, \ldots, x_n\}$ consists of X and a collection $\mathscr{E}$ of nonempty subsets of X; these subsets are called the *edges* of H. A hypergraph H is *simple* if there is no nontrivial containment between any pair of its edges. Simple hypergraphs are also referred to as *clutters* or *Sperner systems*. All hypergraphs we consider will be simple.

When working with a hypergraph H, we shall use $X(H)$ and $\mathscr{E}(H)$ to denote its vertex and edge sets, respectively. We shall assume that hypergraphs under consideration have no *isolated vertices*, those are vertices that do not belong to any edge. An edge $\{v\}$ consisting of a single vertex is often referred to as an *isolated loop* (this is not to be confused with an isolated vertex).

Let $Y \subseteq X$ be a subset of the vertices in H. The *induced subhypergraph* of H on Y, denoted by $H[Y]$, is the hypergraph with vertex set Y and edge set $\{E \in \mathscr{E} \mid E \subseteq Y\}$. In Definition 2.23 we introduced a matching in a graph; we now extend this definition to the hypergraph context.

Definition 4.14 Let H be a simple hypergraph.

1. A collection C of edges in H is called a *matching* if the edges in C are pairwise disjoint. The maximum size of a matching in H is called its *matching number*.
2. A collection C of edges in H is called an *induced matching* if C is a matching, and C consists of all edges of the induced subhypergraph $H[\cup_{E \in C} E]$ of H. The maximum size of an induced matching in H is called its *induced matching number*.

Example 4.15 Figure 4.1 can be viewed as a hypergraph over the vertex set $V = \{a, b, c, d, e\}$ with edges $\{a, b, c\}, \{c, d\}$ and $\{e\}$. The collection $\{\{a, b, c\}, \{e\}\}$ forms an induced matching in this hypergraph.

Note that a *graph*, as introduced in Chap. 2, is a hypergraph in which all edges are of cardinality 2. We shall also need the following special family of graphs.

Definition 4.16 Let G be a simple graph on n vertices.

1. G is called *chordal* if it has no induced cycles of length ≥ 4.
2. G is called *very well-covered* if it has no isolated vertices and its minimal vertex covers all have cardinality $\frac{n}{2}$.

A hypergraph H is *d-uniform* if all its edges have cardinality d. For an edge E in H, let

$$N(E) = \{x \in X \mid \text{there exists } F \subseteq E \text{ such that } F \cup \{x\} \in \mathscr{E}\}$$

be the set of *neighbors* of E, and let $N[E] = N(E) \cup E$.

Definition 4.17 Let $H = (X, \mathscr{E})$ be a simple hypergraph and let E be an edge in H.

1. Define $H \setminus E$ to be the hypergraph obtained by deleting E from the edge set of H. This is often referred to as the *deletion* of E from H.
2. Define H_E to be the *contraction* of H to the vertex set $X \setminus N[E]$ (i.e., edges of H_E are minimal nonempty sets of the form $F \cap (X \setminus N[E])$, where $F \in \mathscr{E}$).

Definition 4.18 Let $H = (X, \mathscr{E})$ be a simple hypergraph.

1. A collection of vertices V in H is called an *independent set* if there is no edge $E \in \mathscr{E}$ such that $E \subseteq V$.
2. The *independence complex* of H, denoted by $\Delta(H)$, is the simplicial complex whose faces are independent sets in H.

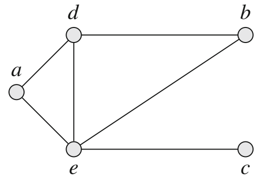

Fig. 4.2 A simple graph whose independence complex is in Fig. 4.1

Example 4.19 The simplicial complex Δ in Fig. 4.1 is the independence complex of the graph in Fig. 4.2.

Remark 4.20 We call a hypergraph H vertex decomposable (shellable, sequentially Cohen-Macaulay) if its independence complex $\Delta(H)$ is vertex decomposable (shellable, sequentially Cohen-Macaulay).

4.2.3 Stanley-Reisner Ideals and Edge Ideals

The Stanley-Reisner ideal and edge ideal constructions are well-studied correspondences between commutative algebra and combinatorics. Those constructions arise by identifying minimal generators of a squarefree monomial ideal with the minimal nonfaces of a simplicial complex or the edges of a simple hypergraph.

Stanley-Reisner ideals were developed in the 1970s and the early 1980s (cf. [155]) and have led to many important homological results (cf. books of Bruns and Herzog [25] and Peeva [146]).

Definition 4.21 Let Δ be a simplicial complex on X. The *Stanley-Reisner ideal* of Δ is defined to be

$$I_\Delta = \langle x^F \mid F \subseteq X \text{ is not a face of } \Delta \rangle.$$

Example 4.22 Let Δ be the simplicial complex in Fig. 4.1, and we set $R = \mathbb{K}[a, b, c, d, e]$. Then the minimal nonfaces of Δ are $\{a, d\}$, $\{a, e\}$, $\{b, d\}$, $\{b, e\}$, $\{c, e\}$ and $\{d, e\}$. Thus,

$$I_\Delta = \langle ad, ae, bd, be, ce, de \rangle.$$

Example 4.23 The simplicial complex Δ in Fig. 4.3 represents a minimal triangulation of the real projective plane. Its Stanley-Reisner ideal is

$$I_\Delta = \langle abc, abe, acf, ade, adf, bcd, bdf, bef, cde, cef \rangle.$$

The edge ideal construction for hypergraphs (first studied by Hà and Van Tuyl [94]) generalizes that of graphs (already presented in Definition 2.10). This construction is similar to that of facet ideals of Faridi [71].

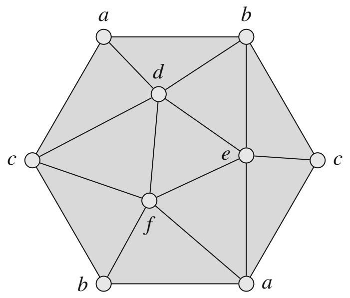

Fig. 4.3 A minimal triangulation of the real projective plane

Definition 4.24 Let H be a simple hypergraph on X. The *edge ideal* of H is defined to be

$$I(H) = \langle x^E \mid E \subseteq X \text{ is an edge in } H \rangle.$$

The notions of a Stanley-Reisner ideal and an edge ideal give the following one-to-one correspondences that allow us to pass back and forth from squarefree monomial ideals to simplicial complexes and simple hypergraphs.

$$\left\{ \begin{array}{c} \text{simplicial complexes} \\ \text{over } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{squarefree monomial} \\ \text{ideals in } R \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{simple hypergraphs} \\ \text{over } X \end{array} \right\}.$$

In fact, every edge ideal is a Stanley-Reisner ideal and vice-versa via the notion of the independence complex. The following lemma follows directly from the definition of independence complexes and the construction of Stanley-Reisner and edge ideals.

Lemma 4.25 *Let H be a simple hypergraph and let $\Delta = \Delta(H)$ be its independence complex. Then*

$$I_\Delta = I(H).$$

Example 4.26 The edge ideal of the graph G in Fig. 4.2 is the same as the Stanley-Reisner ideal of its independence complex, the simplicial complex in Fig. 4.1.

Remark 4.27 For simplicity, if $I = I_\Delta$, then we sometimes write reg Δ for reg I, and if $I = I(H)$, then we write reg H for reg I.

For a monomial ideal in general one can pass to a squarefree monomial ideal via the polarization and still keep many important properties and invariants. We shall briefly recall the notion of polarization; see Herzog and Hibi [106] for more details.

Definition 4.28 Let $I \subseteq R = \mathbb{K}[x_1, \ldots, x_n]$ be a monomial ideal. For each $i = 1, \ldots, n$ let a_i be the maximum power of x_i appearing in the monomial generators of I. The *polarization* of I, denoted by I^{pol}, is constructed as follows.

- Let $R^{\text{pol}} = \mathbb{K}[x_{11}, \ldots, x_{1a_1}, \ldots, x_{n1}, \ldots, x_{na_n}]$.
- The ideal I^{pol} is generated by monomials in R^{pol} that are obtained from generators of I under the following substitution, for each $(\gamma_1, \ldots, \gamma_n) \le (a_1, \ldots, a_n)$,

$$x_1^{\gamma_1} \ldots x_n^{\gamma_n} \longrightarrow x_{11} \ldots x_{1\gamma_1} \ldots x_{n1} \ldots x_{n\gamma_n}.$$

Note, for example, that $\operatorname{reg} R/I = \operatorname{reg} R^{\text{pol}}/I^{\text{pol}}$.

4.3 Hochster's and Takayama's Formulas

Hochster's and Takayama's formulas allow us to relate (multi)graded Betti numbers of a monomial ideal to the dimension of reduced homology groups of simplicial complexes. Hochster's formula deals specifically with squarefree monomial ideals, which are reflected in the next two lemmas, while Takayama's formula works for an arbitrary monomial ideal and is given later on.

The polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ has a natural $\mathbb{N}^n$-graded structure, and for any *monomial* ideal $I \subset R$, the quotient ring R/I inherits this $\mathbb{N}^n$-graded structure from that of R. Therefore, the torsion $\operatorname{Tor}_i^R(I, \mathbb{K})$ and the local cohomology module $H^i_{\mathfrak{m}}(R/I)$ has a $\mathbb{Z}^n$-graded structure. Let $[1, n]$ denote the set $\{1, \ldots, n\}$. For $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, set $x^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$. For a monomial $\mathbf{m}$ in R, by abusing notation, we view degree $\mathbf{m}$ component of a $\mathbb{Z}^n$-graded R-module as its degree $\operatorname{supp} \mathbf{m}$ component. We shall introduce Hochster's formula following, for example, [106, Theorem 8.1.1].

Lemma 4.29 (Hochster's Formula) *Let Δ be a simplicial complex on the vertex set $X = \{x_1, \ldots, x_n\}$ and let $\mathbf{m}$ be a monomial of R. Then,*

$$\dim_{\mathbb{K}} \operatorname{Tor}_i^R(I_\Delta, \mathbb{K})_{\mathbf{m}} = \begin{cases} \dim_{\mathbb{K}} \widetilde{H}^{\deg(\mathbf{m})-i-2}(\Delta[\operatorname{supp} \mathbf{m}]; \mathbb{K}) & \textit{if } \mathbf{m} \textit{ is squarefree} \\ 0 & \textit{otherwise.} \end{cases}$$

In particular,

$$\beta_{ij}(I_\Delta) = \sum_{\substack{\deg(\mathbf{m}) = j, \\ \mathbf{m} \text{ is squarefree}}} \dim_{\mathbb{K}} \widetilde{H}^{j-i-2}(\Delta[\operatorname{supp} \mathbf{m}]; \mathbb{K}) \ \textit{for all } i, j \ge 0.$$

Here, $\Delta[\operatorname{supp} \mathbf{m}]$ is the induced subcomplex of Δ on the support of $\mathbf{m}$.

Proof We shall outline the proof of Hochster's formula following that given by Herzog and Hibi [106, Theorem 8.1.1].

1. Let $\mathscr{K}$ be the Koszul complex of I_Δ with respect to the variables $x = \{x_1, \ldots, x_n\}$, let K_i be the i-th module, and let $H_i(\mathscr{K})$ be the i-th homology

group of $\mathscr{K}$. Since $\mathscr{K}$ is a complex of $\mathbb{Z}^n$-graded modules, $H_i(\mathscr{K})$ is also a $\mathbb{Z}^n$-graded $\mathbb{K}$-vector space. Thus, for a monomial $\mathbf{m}$ in R, we have

$$\operatorname{Tor}_i^R(I_\Delta, \mathbb{K})_{\mathbf{m}} = H_i(\mathscr{K})_{\mathbf{m}}.$$

2. For $F = \{j_0 < \cdots < j_i\} \subseteq [1, n]$, set $\mathbf{e}_F = e_{j_0} \wedge \cdots \wedge e_{j_i}$. The elements $\mathbf{e}_F$'s with $|F| = i$ form a basis for the i-th free module in the Koszul complex of R with respect to x. The $\mathbb{Z}^n$-degree of $\mathbf{e}_F$ is $\epsilon(F) \in \mathbb{Z}^n$, where $\epsilon(F)$ is the (0,1)-vector with support F.
3. A $\mathbb{K}$-basis for $(K_i)_{\mathbf{m}}$ is given by

$$x^{\mathbf{b}}\mathbf{e}_F, \quad \text{where } \mathbf{b} + \epsilon(F) = \mathbf{m} \text{ and } \operatorname{supp} \mathbf{b} \notin \Delta.$$

4. Define the simplicial complex

$$\Delta^{\mathbf{m}} = \left\{ F \subseteq [1, n] \;\middle|\; F \subseteq \operatorname{supp} \mathbf{m}, \operatorname{supp} \frac{\mathbf{m}}{x^{\epsilon(F)}} \notin \Delta \right\}.$$

Let $\widetilde{\mathscr{C}}(\Delta^{\mathbf{m}})[-1]$ be the oriented augmented chain complex of $\Delta^{\mathbf{m}}$ shifted by -1 in homological degree. Then, we have an isomorphism of complexes

$$\widetilde{\mathscr{C}}(\Delta^{\mathbf{m}})[-1] \longrightarrow \mathscr{K}_{\mathbf{m}}$$

obtained by $F = [j_0, \ldots, j_{i-2}] \mapsto \dfrac{\mathbf{m}}{x^{\epsilon(F)}}\mathbf{e}_F$. This, in turn, gives

$$H_i(\mathscr{K})_{\mathbf{m}} \simeq H_i(\widetilde{\mathscr{C}}(\Delta^{\mathbf{m}})[-1]).$$

5. If $\mathbf{m}$ is not squarefree, then there exists j such that x_j appears with power greater than 1 in $\mathbf{m}$. Define $\mathbf{m}(r) = \mathbf{m}x_j^r$ for $r \in \mathbb{N}$. It is easy to see that $\Delta^{\mathbf{m}} = \Delta^{\mathbf{m}(r)}$ for all $r \in \mathbb{N}$. Moreover, for $r \gg 0$, $H_i(\mathscr{K})_{\mathbf{m}(r)} = 0$. Thus,

$$H_i(\mathscr{K})_{\mathbf{m}} \simeq H_i(\widetilde{\mathscr{C}}(\Delta^{\mathbf{m}})[-1]) = H_i(\widetilde{\mathscr{C}}(\Delta^{\mathbf{m}(r)})[-1]) = H_i(\mathscr{K})_{\mathbf{m}(r)} = 0.$$

6. Suppose that $\mathbf{m}$ is squarefree. It can be seen that $F \subseteq \Delta^{\mathbf{m}}$ if and only if $F \subseteq \operatorname{supp} \mathbf{m}$ and $\operatorname{supp} \mathbf{m} \setminus F \notin \Delta[\operatorname{supp} \mathbf{m}]$. That is, $\Delta^{\mathbf{m}} = \Delta[\operatorname{supp} \mathbf{m}]^\vee$ where $(-)^\vee$ denotes the Alexander dual of a simplicial complex. Hence, we have

$$H_i(\widetilde{\mathscr{C}}(\Delta^{\mathbf{m}})[-1]) \simeq \widetilde{H}_{i-1}(\Delta[\operatorname{supp} \mathbf{m}]^\vee; \mathbb{K}) \simeq \widetilde{H}^{\deg \mathbf{m} - i - 2}(\Delta[\operatorname{supp} \mathbf{m}]; \mathbb{K})$$

where the second isomorphism is a standard fact about Alexander duality.

Lemma 4.30 *For a simplicial complex Δ, the following are equivalent:*

1. $\operatorname{reg} R/I_\Delta \geq d$.
2. $\tilde{H}_{d-1}(\Delta[S], \mathbb{K}) \neq 0$*, where* $\Delta[S]$ *denotes the induced subcomplex on some subset* S *of vertices.*
3. $\tilde{H}_{d-1}(\operatorname{link}_\Delta \sigma, \mathbb{K}) \neq 0$ *for some face* σ *of* Δ.

Proof The equivalence of (1) and (2) follows directly from Definition 4.1, together with Hochster's formula in Lemma 4.29. The equivalence of (1) and (3) follows directly from the local cohomology characterization of regularity, together with the fact that $H^i_{\mathfrak{m}}(R/I_\Delta, \mathbb{K})_{-\sigma} \simeq \tilde{H}^{i-|\sigma|-1}(\operatorname{link}_\Delta \sigma, \mathbb{K})$ (see Miller and Sturmfels book [137, Chapter 13.2]).

We will also make use of a variation of Hochster's formula following [137, Theorem 1.34]. This variation of Hochster's formula is given via upper-Koszul simplicial complexes associated to monomial ideals.

Definition 4.31 Let $I \subseteq R$ be a monomial ideal and let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ be a $\mathbb{N}^n$-graded degree. The *upper-Koszul simplicial complex* associated to I at degree $\boldsymbol{\alpha}$, denoted by $K^{\boldsymbol{\alpha}}(I)$, is the simplicial complex over $X = \{x_1, \ldots, x_n\}$ whose faces are:

$$\left\{ W \subseteq X \;\middle|\; \frac{x^{\boldsymbol{\alpha}}}{\prod\limits_{u \in W} u} \in I \right\}.$$

Theorem 4.32 ([137, Theorem 1.34]) *Let $I \subseteq R$ be a monomial ideal. Then its $\mathbb{N}^n$-graded Betti numbers are given as follows:*

$$\beta_{i,\boldsymbol{\alpha}}(I) = \dim_{\mathbb{K}} \widetilde{H}_{i-1}(K^{\boldsymbol{\alpha}}(I); \mathbb{K}) \text{ for } i \geq 0 \text{ and } \boldsymbol{\alpha} \in \mathbb{N}^n. \tag{4.1}$$

Takayama's formula [160, Theorem 1] describes the the dimension of the $\mathbb{Z}^n$-graded component $H^i_{\mathfrak{m}}(R/I)_{\mathbf{a}}$, for $\mathbf{a} \in \mathbb{Z}^n$, in terms of a simplicial complex $\Delta_{\mathbf{a}}(I)$. We shall recall the construction of $\Delta_{\mathbf{a}}(I)$, as given by Minh and Trung [138], which is simpler than the original construction of [160].

For $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, set $G_{\mathbf{a}} := \{j \in [1, n] \mid a_j < 0\}$. For every subset $F \subseteq [1, n]$, let $R_F = R[x_j^{-1} \mid j \in F]$. Define

$$\Delta_{\mathbf{a}}(I) = \{F \setminus G_{\mathbf{a}} \mid G_{\mathbf{a}} \subseteq F,\ x^{\mathbf{a}} \notin IR_F\}.$$

We call $\Delta_{\mathbf{a}}(I)$ a *degree complex* of I.

Lemma 4.33 (Takayama's Formula) *For any* $\mathbf{a} \in \mathbb{Z}^n$*, we have*

$$\dim_{\mathbb{K}} H^i_{\mathfrak{m}}(R/I)_{\mathbf{a}} = \dim_{\mathbb{K}} \widetilde{H}_{i-|G_{\mathbf{a}}|-1}(\Delta_{\mathbf{a}}(I), \mathbb{K}).$$

The original formula in [160, Theorem 1] is slightly different. It contains additional conditions on $\mathbf{a}$ for $H^i_{\mathfrak{m}}(R/I)_{\mathbf{a}} = 0$. However, the proof in [160] shows that we may drop these conditions, which is more convenient for our investigation.

From Takayama's formula we immediately obtain the following characterizations of depth and regularity of monomial ideals in terms of the degree complexes.

Lemma 4.34 *Let $I \subseteq R$ be a monomial ideal. Then*

$$\operatorname{reg} R/I = \max\{|\mathbf{a}| + |G_{\mathbf{a}}| + i \mid \mathbf{a} \in \mathbb{Z}^n, i \geq 0, \widetilde{H}_{i-1}(\Delta_{\mathbf{a}}(I), \mathbb{K}) \neq 0\}.$$