Chapter 5
Final Remarks and Open Problems

In this monograph, we first consider a semilinear fractional kinetic equation that is characterized by the presence of a nonlinear time-dependent source $f = f(x, t, u)$, a generalized time derivative ∂_t^α in the sense of Caputo and the presence of a large class of diffusion operators A. Many examples of diffusion operators that satisfy our assumptions are given in Sect. 2.3. We give a unified analysis, using tools in semigroups theory and the theory of partial differential equations (Sects. 2.1 and 2.2), in order to obtain sharp results for the well-posedness problem of mild and strong solutions (Sects. 3.1 and 3.2), as well as for the global regularity problem in Sect. 3.4. Further properties, such as nonnegativity of the mild (and/or strong) solutions and their limiting behavior as $\alpha \to 1$, are also provided in Sects. 3.6 and 3.5, respectively. Finally, in Sect. 3.7 an application of these results is given.

The framework we develop for the scalar equation in Chap. 3 is then extended in the second part of the monograph (Chap. 4) to nonlinear systems of fractional kinetic equations. Here, we first develop a general scheme that allows to establish sharp results for the well-posedness problem of (locally-defined) mild and strong solutions associated with such general systems (Sect. 4.1). We then combine this analysis with that of the previous chapters to derive well-posedness results in terms of globally defined mild and strong solutions, for a fractional prey-predator model (Sect. 4.2) and a simple fractional nuclear reaction model (Sect. 4.3). In addition, we provide a number of important technical tools in Appendix A, in support of the analysis developed in this monograph; this appendix is followed by Appendix B, which contains several results concerning the regional fractional Laplace operator associated with fractional Neumann and/or Robin boundary conditions. Finally, in Appendix C, we recall the current scientific literature for different kinds of fractional kinetic equations that are suggested by concrete problems in mathematical physics, probability and finance, and which fully motivated the analysis in this monograph.

We give next a number of final comments and discuss possible open problems.

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Remark 5.0.1 Our main working hypothesis in this monograph is that the underlying physical space $\mathcal{X}$ is a (relatively) compact Hausdorff space. However, we note that this assumption has been placed just for the sake of technical convenience. Much of the results developed in Chap. 3 are true for instance when $\mathcal{X}$ is only locally compact (say when $\mathcal{X}$ is replaced by either $\mathbb{R}^N$, or half-space $\mathbb{R}^N_+$ or an unbounded open set $\Omega \subset \mathbb{R}^N$). Indeed, all the supporting technical results given in Appendix A, with the exception of Lemma A.0.2, are still valid when $\mathcal{X}$ is only locally compact. In particular, it means that the results on well-posedness of (locally-defined) mild and strong solutions are still true in that case, with the exception of case (c) of Theorem 3.1.4; we recall that this case uses Lemma A.0.2 in a crucial way. Moreover, one may obtain the same global bounds derived in Sect. 3.4 by making proper modifications in the proofs when $\mathcal{X}$ is only locally compact.

Problem 1 Prove the analogue of Lemma A.0.2 when $\mathcal{X}$ *is only locally compact.*

Remark 5.0.2 Let us consider the semilinear parabolic problem (3.1.1) with the nonlinearity $f(x,t,u) = c(x,t)|u|^{\gamma-1}u$, for some $c \in L_{q_1,q_2}$. We note that the critical exponent γ, as stated by Theorem 3.1.4,

$$\frac{\mathfrak{n}}{q_1} + \frac{1}{q_2} + (\gamma - 1)\frac{\mathfrak{n}}{p_0} \le \alpha, \ \mathfrak{n} := \beta_A \alpha, \ \alpha \in (0, 1], \tag{5.0.1}$$

is in fact optimal in the sense that there are always locally-defined mild solutions for some $u_0 \in L^{p_0}(\mathcal{X})$. When instead $\gamma \ge 1$ and $p_0 \ge 1$ satisfy the inequality

$$\frac{\mathfrak{n}}{q_1} + \frac{1}{q_2} + (\gamma - 1)\frac{\mathfrak{n}}{p_0} > \alpha, \tag{5.0.2}$$

we conjecture that problem (3.1.1) does not have any locally-defined mild solution for certain initial data $u_0 \in L^{p_0}(\mathcal{X})$. Indeed, this was already discovered by Weissler [12, 13] for the classical problem when $\alpha = 1$, $\beta_A = N/2$ and $q_1 = q_2 = \infty$; (5.0.2) recovers the super-critical range $(\gamma - 1)\frac{N}{2p_0} > 1$ in that case.

Problem 2 Prove the above conjecture in the super-critical case (5.0.2).

Problem 3 Consider the problem (3.1.1) in the subcritical and limiting cases as defined by (5.0.1). Several further open problems can be considered:

(a) Under the same assumptions of Chap. 3, investigate the long-term behavior of (3.1.1) in terms of global attractors and ω-limit sets.
(b) Under proper conditions on the nonlinearity and the diffusion operator A, show that each globally defined solution converges to a unique steady state u_* as time goes to infinity, where u_* is a proper solution of the corresponding stationary problem.
(c) Investigate the blow-up phenomenon for Problem (3.1.1) for various diffusion operators. We refer the reader to [11] when $A = \Delta$.

(d) Give a further refined regularity analysis to show the (Hölder) continuity of solutions for the abstract problem (3.1.1) for a large class of diffusion operators A. We recall that such result has already been proven in [1] for the corresponding problem with $f = f(x,t)$ and A is given by Example 2.3.6(a). When $A = \Delta$ or a second-order operator in divergence form, this has been proven in [14].

Problem 4 The current framework can be extended to accommodate more general transmission problems than the ones considered in [4, 5].

Problem 5 The framework in Sect. 3.3 can be further developed to show higher-order differentiability properties for the strong solution under additional assumptions of the nonlinear function f.

Remark 5.0.3 The framework developed in this monograph can be exploited to obtain global existence of solutions to other interesting reaction–diffusion systems that contain some fractional kinetics. Among them, one can consider more general systems based on ecological interactions and physical models based on chemical reactions with anomalous diffusion that may occur in spatially inhomogeneous media (cf. Appendix C). Among such interesting systems, one may mention the fractional Brusselator for reaction kinetics [9] which was considered in [6] as a physical model for activator-inhibitor dynamics that exhibits anomalous behavior.

Problem 6 Consider the fractional Brusselator discussed by Henry and Wearne [6] and prove the existence of globally-defined strong and mild solutions. This is an open problem in light of the difficulties that arise from the nature of the coupling in the system and the corresponding nonlinear terms. We refer the reader to the survey paper of Pierre [8] for more information regarding the classical reaction–diffusion problem when $\alpha_i \equiv 1, i \in \{1, \ldots, m\}$.

Problem 7 Investigate the long-term behavior of solutions, as time goes to infinity, to the fractional Volterra–Lotka and nuclear reactor systems introduced in Sects. 4.2 and 4.3, respectively.

Remark 5.0.4 The contribution [7] contains an analogue of the classical Aubin-Lion compactness lemma in order to obtain existence of weak solutions to some nonlinear systems that involve a fractional Caputo derivative. This approach can be also applied to the semilinear problem (1.0.1) in order to develop a well-defined L^2-theory. However, our approach doesn't require any compactness arguments and is of more general interest since it is developed in the L^p-setting. Moreover, our theory can be also extended for problems (1.0.1) with notions other than the Caputo fractional derivative for as long as one can provide a formula for the solution similar to (3.1.2). This is in particular very useful in those situations where the integral kernel in the Caputo derivative is slightly more general than $g_{1-\alpha}$ (see (2.1.1)). These issues shall be addressed in future contributions.

To conclude this section we list some open problems regarding the exterior value elliptic problems (Eqs. (2.3.21), (2.3.25) and (2.3.27)) for the fractional Laplace operator. We refer to [3] for more details.

Problem 8 Let $u \in W_0^{s,2}(\overline{\Omega})$ be a weak solution of the Dirichlet exterior value problem (2.3.21). Prove or disprove that u is a strong solution of (2.3.21).

Remark 5.0.5 Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain of class $C^{1,1}$. Let $(\varphi_n)_{n\geq 0}$ be the eigenfunctions of the operator $(-\Delta)_D^s$ (see Example 2.3.6(a)). It has been shown in [2, Section 5] (see also [10] for the case $N = 1$) that for every $n \geq 1$, $\varphi_n \in C^{0,s}(\overline{\Omega})$ and $\varphi_n \notin C^{0,\gamma}(\overline{\Omega})$ for any $\gamma > s$.

Problem 9 Let $u \in W_\Omega^{s,2}$ be a weak solution of the Neumann exterior value problem (2.3.25). Prove that $u \in C(\mathbb{R}^N)$ and $u|_\Omega \in W_{\text{loc}}^{2s,2}(\Omega)$. Prove or disprove that u is a strong solution of (2.3.25).

Problem 10 Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain of class $C^{1,1}$. Let $(\psi_n)_{n\geq 0}$ be the eigenfunctions of the operator $(-\Delta)_{\mathcal{N}}^s$ (see Example 2.3.6(b)). Prove that for every $n \geq 1$, $\psi_n \in C^{0,s}(\overline{\Omega})$ and $\psi_n \notin C^{0,\gamma}(\overline{\Omega})$ for any $\gamma > s$.

Problem 11 Let $u \in W_{\beta,\Omega}^{s,2}$ be a weak solution of the Robin exterior value problem (2.3.27). Prove that $u|_\Omega \in W_{\text{loc}}^{2s,2}(\Omega)$. Prove or disprove that u is a strong solution of (2.3.27). Assume that $\beta \in L^1(\mathbb{R}^N \setminus \Omega) \cap L^\infty(\mathbb{R}^N \setminus \Omega)$. Prove that $u \in C(\mathbb{R}^N)$.

Problem 12 Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain of class $C^{1,1}$ and that $\beta \in C_c^1(\mathbb{R}^N \setminus \Omega)$. Let $(\phi_n)_{n\geq 0}$ be the eigenfunctions of the operator $(-\Delta)_R^s$ (see Example 2.3.6(c)). Prove that for every $n \geq 1$, $\phi_n \in C^{0,s}(\overline{\Omega})$ and $\phi_n \notin C^{0,\gamma}(\overline{\Omega})$ for any $\gamma > s$.

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