Chapter 4
Systems of Fractional Kinetic Equations

In this chapter, we consider some general classes of reaction–diffusion systems that contain some fractional kinetics occurring in applications (cf. Appendix C below), and then investigate their local and global existence of solutions in detail. In a preliminary step, we derive results that allow for the existence of sufficiently smooth solutions which are needed in order to rigorously justify other precise and explicit calculations (namely, maximum principles, energy estimates and comparison arguments) which will be performed on more specific models in the sequel. It turns out that the techniques employed for the scalar equation (3.1.1) in the previous chapter will prove quite useful in the analysis.

4.1 Nonlinear Fractional Reaction–Diffusion Systems

Let $m \in \mathbb{N}$ and $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ where each u_i $(i = 1, \ldots, m)$ is a measurable physical quantity. Let $d_i = 0$ for $i = 1, \ldots, r$ and $d_i > 0$ for $i = r+1, \ldots, m$. We allow the case $r = 0$ to occur so that all $d_i > 0$ for $i = 1, \ldots, m$ in some cases. Next, let $D = \operatorname{diag}(d_1, \ldots, d_m)$ be the diagonal matrix of diffusion coefficients and assume that $u_0 = (u_{01}, \ldots, u_{0m})(x) \in \mathbb{R}^m$, for $x \in \mathcal{X}$, models the initial data. Let $f = (f_1, \ldots, f_m)(x, t, u_1, \ldots, u_m)$ with

$$f : (x, t, u) \in \mathcal{X} \times [0, \infty) \times \mathbb{R}^m \to f(x, t, u) \in \mathbb{R}^m,$$

be a nonlinear function that models possible interactions between the various quantities u_i $(i = 1, \ldots, m)$. Finally, consider a family of closed operators $(A_i)_{i=1}^m$ that satisfies the assumptions of Propositions 2.2.1, 2.2.2. Namely, we assume that each A_i satisfies assumption (**HA**) with a possible different value $\beta_{A_i} > 0$ for $i = 1, \ldots, m$. In particular, let $(S_i(t))_{i=1}^m$ be the corresponding family of analytic semigroups associated with A_i, each S_i $(i = 1, \ldots, m)$ can be extended to a

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contraction semigroup $S_{i,p_i}(t)$ on $L^{p_i}(\mathcal{X})$, $p_i \in [1,\infty]$, whose generator is A_{i,p_i} such that $A_{i,2} = A_i$. For any $\alpha_i \in (0,1)$, the corresponding operators (2.1.9) associated with these semigroups S_i can also be defined analogously, to give the families of operators $\left(S_{\alpha_i}(t)\right)_{i=1}^m$ and $\left(P_{\alpha_i}(t)\right)_{i=1}^m$, satisfying the ultracontractivity estimates of (2.2.12), after setting $\beta_A := \beta_{A_i}$, $\alpha := \alpha_i$. Finally, we set the diagonal (matrix) operator $A = \operatorname{diag}(A_1, \ldots, A_m)$ and introduce the following notion of Caputo-fractional derivative

$$\partial_t^\alpha u = \left(\partial_t^{\alpha_1} u_1, \ldots, \partial_t^{\alpha_m} u_m\right) \in \mathbb{R}^m,$$

where each $\partial_t^{\alpha_i} u_i \in \mathbb{R}$ is understood in the sense of Definition 2.1.1 for $\alpha_i \in (0,1)$. As usual, when $\alpha_i = 1$, $\partial_t^1 = \partial_t = d/dt$.

Our problem is to look for solutions $u = (u_1, \ldots, u_m)(x,t) \in \mathbb{R}^m$ of the following system

$$\partial_t^\alpha u = DAu + f(x,t,u), \quad (x,t) \in \mathcal{X} \times (0,\infty), \tag{4.1.1}$$

subject to the initial condition

$$u_{|t=0} = u_0 \text{ in } \mathcal{X}. \tag{4.1.2}$$

Note that components which do not diffuse as well as different kinds of “diffusion” operators for the diffusing components may occur in (4.1.1)–(4.1.2). Our goal is to construct bounded mild solutions for this initial-value problem and then turn to strong solutions. To this end, our assumptions on the nonlinearity f, from Sect. 3, are adapted to our new case, as follows.

(SF1) $f(x,t,\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is a measurable function for all $(x,t) \in \mathcal{X} \times (0,\infty)$ such that, for every bounded set $B \subset \overline{\mathcal{X}} \times [0,\infty) \times \mathbb{R}^m$, there exists a constant $L = L(B) > 0$ such that

$$|f(x,t,\xi)| \leq L(B), \text{ for all } (x,t,\xi) \in B$$

and

$$|f(x,t,\xi) - f(x,t,\eta)| \leq L(B)|\xi - \eta|, \text{ for all } (x,t,\xi), (x,t,\eta) \in B.$$

(**SF2**) For every bounded $B \subset \overline{\mathcal{X}} \times [0,\infty) \times \mathbb{R}^m$, there exists a constant $L(B) > 0$ such that, for all $(x,t,\xi), (x,s,\eta) \in B$,

$$|f(x,t,\xi) - f(x,s,\eta)| \leq L(B)\left(|t-s|^\gamma + |\xi - \eta|\right),$$

for some $\gamma > 0$.

As in Chap. 3, the assumption (**SF2**) will only be needed when the nonlinear source f is also time dependent; when $f = f(x,\xi)$ this condition is no longer necessary for the existence of strong solutions, we refer the reader to the proof of Corollary 3.2.3 (and Theorem 4.1.3 below).

First, we consider the initial-value problem (4.1.1)–(4.1.2) with $f \equiv 0$ and let $u_0 \in L^\infty(\mathcal{X}, \mathbb{R}^m)$. The solution $u = u(x,t)$ of the linear system for bounded initial datum u_0 defines a formal solution operator in the space $L^\infty(\mathcal{X}, \mathbb{R}^m)$ by setting

$$u(x,t) = (\mathbb{S}_\alpha(t)\, u_0)(x) = \left(\mathbb{S}_{\alpha,1}(t)\, u_{01}(x), \ldots, \mathbb{S}_{\alpha,m}(t)\, u_{0m}(x)\right),$$

for all $x \in \mathcal{X}$, $t \in [0,\infty)$. The linear system is decoupled, the operator $\mathbb{S}_\alpha(t)$ acts componentwise; namely, for all $t \in [0,\infty)$, we set

$$\begin{cases} \mathbb{S}_{\alpha,i}(t) = S_{\alpha_i}(t) \equiv I, & \text{for } i = 1,\ldots,r \text{ (when } d_i = 0), \\ \mathbb{S}_{\alpha,i}(t) = S_{\alpha_i}(d_i t)_{|L^\infty(\mathcal{X})}, & \text{for } i = r+1,\ldots,m \text{ (when } d_i > 0). \end{cases}$$

Of course, $r = 0$ is still allowed. In view of Remark 3.1.2, $\mathbb{S}_\alpha(t)$ is not strongly continuous in the Banach space $L^\infty(\mathcal{X}, \mathbb{R}^m)$, which we equip with the canonical sup-norm

$$\|u\|_\infty = \max_{1 \le i \le m} \|u_i\|_{L^\infty(\mathcal{X})}.$$

However, owing to the statement of Proposition 2.2.1, we have $\|\mathbb{S}_\alpha(t)\, u_0\|_\infty \le \|u_0\|_\infty$, for all $u_0 \in L^\infty(\mathcal{X}, \mathbb{R}^m)$ and $t \in [0,\infty)$. In order to deal with the full nonlinear system, we also define the operator $(\mathbb{P}_\alpha(t)\, f)$, for $f = (f_1,\ldots,f_m) \in \mathbb{R}^m$, also acting componentwise:

$$\mathbb{P}_\alpha(t)\, f = \left(\mathbb{P}_{\alpha,1}(t)\, f_1, \ldots, \mathbb{P}_{\alpha,m}(t)\, f_m\right),$$

where, for $i = r+1,\ldots,m$,

$$\mathbb{P}_{\alpha,i}(t)\, f_i \equiv P_{\alpha_i}(d_i t)\, f_i = \alpha_i t^{\alpha_i - 1} \int_0^\infty \tau \Phi_{\alpha_i}(\tau) S_i(\tau (d_i t)^{\alpha_i}) f_i \, d\tau$$

and $\mathbb{P}_{\alpha,i}(t)\, f_i \equiv \left(g_{\alpha_i} * f_i\right)(t)$, for $i = 1,\ldots,r$, in the case of non-diffusing components. Of course, we keep the convention that when some $\alpha_i \equiv 1$ for $i = r+1,\ldots,m$, we let $S_{\alpha_i} \equiv S_i$ and $P_{\alpha_i} \equiv S_i$.

Our notion of mild solution for (4.1.1)–(4.1.2) is stronger than the notion of mild solution from Definition 3.1.3. For the sake of notational convenience, we again let $u(t) = u(\cdot,t)$ and $f(t, u(t)) = f(\cdot, t, u(\cdot,t))$.

Definition 4.1.1 Let $T > 0$ be given, but otherwise arbitrary (and, possibly, $T = \infty$) and let $u_0 \in L^\infty(\mathcal{X}, \mathbb{R}^m)$. We say $u \in E_{\infty,0,T}$ is a mild solution of problem (4.1.1)–(4.1.2) on the time interval $[0,T)$ if:

(a) $u : (x,t) \in \mathcal{X} \times (0,T) \mapsto u(x,t) \in \mathbb{R}^m$ is measurable and $u(\cdot,t) \in L^\infty(\mathcal{X}, \mathbb{R}^m)$ such that

$$\sup_{s\in(0,t)} \|u(\cdot,s)\|_\infty =: U < \infty, \text{ for all } t \in (0,T).$$

(b) $u(t) = \mathbb{S}_\alpha(t)\,u_0 + \int_0^t \mathbb{P}_\alpha(t-\tau)\, f(\tau, u(\tau))\, d\tau$, for all $t \in (0,T)$, where the integral is an absolutely converging Bochner integral in $L^\infty(\mathcal{X}, \mathbb{R}^m)$.

(c) u satisfies the initial condition in the following sense:

$$\lim_{t\to 0^+} \|u(t) - \mathbb{S}_\alpha(t)\,u_0\|_\infty = 0.$$

We have the first result of this section.

Theorem 4.1.2 (Existence of Maximal Bounded Mild Solutions) *Let assumption **(HA)** be verified for each operator A_i $(i = 1,\ldots,m)$ and the conditions of **(SF1)** for the nonlinearity f. For any given $u_0 \in L^\infty(\mathcal{X}, \mathbb{R}^m)$, there exists a time $T \in (0,\infty]$ such that the initial-value problem* (4.1.1)–(4.1.2) *possesses a unique mild solution in the sense of Definition 4.1.1 on the interval $[0,T)$. Furthermore, the existence time $T \in (0,\infty]$ can be chosen maximal (i.e., the previous statement does not hold for a larger time). In that case, $T = T_{\max}$ and*

$$\lim_{t\to T_{\max}} \|u(t)\|_\infty = \infty, \text{ if } T_{\max} < \infty.$$

Proof Consider the following

$$\begin{aligned} &\textbf{Case1}: \text{all } \alpha_i \equiv 1, i = 1,\ldots,m.\\ &\textbf{Case2}: \text{at least one } \alpha_i \in (0,1), i = 1,\ldots,m. \end{aligned}$$

Let $U_0 \in [0,\infty)$ be such that $\|u_0\|_\infty \le U_0$. Choose $U > U_0$, $T_0 \in (0,1]$ arbitrarily and define the bounded set $B := \overline{\mathcal{X}} \times [0,T_0] \times [-U,U]^m$. Let $L(B) > 0$ be the constant from assumption (**SF1**) and choose $T \in (0,T_0]$ such that

$$U_0 + \widetilde{\theta}(T) \le U, \tag{4.1.3}$$

where the function $\widetilde{\theta}$ is defined as follows:

$$\begin{cases} \widetilde{\theta}(t) := e^{L(B)t} - 1, & \text{in case 1,} \\ \widetilde{\theta}(t) := E_{\alpha_{m_0},1}\left(\frac{\Gamma(\alpha_{m_0})L(B)}{\Gamma(\alpha_{M_0})} t^{\alpha_{m_0}}\right) - 1, & \text{in case 2.} \end{cases} \tag{4.1.4}$$

Here, we have set $\alpha_{m_0} := \min_{1\le i\le m}(\alpha_i) \in (0,1)$ and $\alpha_{M_0} := \max_{1\le i\le m}(\alpha_i) \in (0,1]$, and we recall that $E_{\kappa,\beta}(z)$ $(\kappa > 0, \beta \in \mathbb{C})$ is the generalized Mittag-Leffler function

$$E_{\kappa,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\kappa n + \beta)}, \ k \in (0, 1).$$

For initial data $u_0 \in \mathcal{L}^\infty(\mathcal{X}, \mathbb{R}^m)$ with $\|u_0\|_\infty \le U_0$, we define the sequence $u^{(l)}$ in $L^\infty(\mathcal{X} \times [0, T], \mathbb{R}^m)$, by

$$\begin{cases} u^{(1)}(t) = \mathbb{S}_\alpha(t)\, u_0, \\ u^{(l+1)}(t) = \mathbb{S}_\alpha(t)\, u_0 + \int_0^t \mathbb{P}_\alpha(t-s)\, f\left(s, u^{(l)}(s)\right) ds, \end{cases}$$

for $l \in \mathbb{N}$ and $t \in [0, T]$. We claim by induction that the following (4.1.5)–(4.1.9) are satisfied for all $t \in [0, T]$, and all $l \in \mathbb{N}$. We have the estimates:

$$\begin{aligned} &\left\|\left(u^{(l+1)} - u^{(l)}\right)(t)\right\|_\infty \\ &\le L(B) \max_{1\le i\le m} \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i - 1} \left\|\left(u^{(l)} - u^{(l-1)}\right)(\tau)\right\|_\infty d\tau \end{aligned} \tag{4.1.5}$$

and

$$\left\|\left(u^{(l+1)} - u^{(l)}\right)(t)\right\|_\infty = \max_{1\le i\le m} q_l^i(t) \le \theta_l(t), \tag{4.1.6}$$

where

$$\begin{cases} \theta_l(t) := \frac{(L(B)t)^l}{l!}, & \text{in case 1,} \\ \theta_l(t) := \frac{1}{\Gamma\left(l\alpha_{m_0}+1\right)} \left(\frac{\Gamma\left(\alpha_{m_0}\right)L(B)}{\Gamma\left(\alpha_{M_0}\right)} t^{\alpha_{m_0}}\right)^l, & \text{in case 2.} \end{cases} \tag{4.1.7}$$

Also, we have

$$\left\|u^{(l)}(t)\right\|_\infty \le U \tag{4.1.8}$$

and

$$\sum_{1\le j\le l} \left\|\left(u^{(j+1)} - u^{(j)}\right)(t)\right\|_\infty \le \widetilde{\theta}(t). \tag{4.1.9}$$

We can easily check these claims for $l = 1$ on account of the definition for $u^{(l)}$, the properties of $\mathbb{S}_\alpha (t)$, $\mathbb{P}_\alpha (t)$ and the conditions of (**SF1**). For instance,

$$\left\| u^{(1)} (t) \right\|_\infty = \| \mathbb{S}_\alpha (t) u_0 \|_\infty \leq \| u_0 \|_\infty < U;$$

since $\| P_{\alpha_i} (t) \|_{L^\infty(\mathcal{X})} \leq t^{\alpha_i - 1} / \Gamma (\alpha_i)$, for $t \in (0, T]$, we also get

$$\begin{aligned} q_2^1 (t) &:= \left\| u_i^{(2)} (t) - u_i^{(1)} (t) \right\|_{L^\infty} \\ &\leq \int_0^t \| P_{\alpha_i} (t - \tau) \|_{\infty,\infty} \left\| f_i \left(\tau, u^{(1)} (\tau) \right) \right\|_{L^\infty(\mathcal{X})} d\tau \\ &\leq \frac{L (B)}{\Gamma (\alpha_i)} \int_0^t (t - \tau)^{\alpha_i - 1} d\tau \leq \theta_1 (t) . \end{aligned}$$

Indeed, the Gamma function $\Gamma (x) > 0$ is non-increasing on $(0, \mu]$ and non-decreasing on $[\mu, \infty)$, for some $\mu \in (1, 2)$ (in fact, $\mu = 1.46163..$). This yields that $\Gamma \left(\alpha_{M_0} \right) \leq \Gamma (\alpha_i) \leq \Gamma \left(\alpha_{m_0} \right)$ in the second case, as well as $\max_{1 \leq i \leq m} t^{\alpha_i} \leq t^{\alpha_{m_0}}$, for $t \in [0, T] \subseteq [0, 1]$; henceforth, we deduce

$$\left\| u^{(2)} (t) - u^{(1)} (t) \right\|_\infty = \max_{1 \leq i \leq m} q_2^i (t) \leq \theta_1 (t) .$$

Suppose now (4.1.5)–(4.1.9) are already known for some $l - 1 \in \mathbb{N}$. We have to prove these assertions for $l \in \mathbb{N}$. Inequality (4.1.5) is immediate owing to the second condition of (**SF1**), while the assertions (4.1.6) and (4.1.9) are also immediate in the first case by merely performing explicit integration and recalling the series for the exponential function. We now show the same assertions in the second case when at least one of $\alpha_i \in (0, 1)$. By the second condition of (**SF1**), and a change of variable $rt = s$ we have

$$q_l^i (t) \leq \int_0^t \| P_{\alpha_i} (t - s) \|_{\infty,\infty} \left\| f \left(s, u^{(l)} (s) \right) - f \left(s, u^{(l-1)} (s) \right) \right\|_{L^\infty(\mathcal{X})} ds \tag{4.1.10}$$

$$\begin{aligned} &\leq \frac{L (B)}{\Gamma \left(\alpha_{M_0} \right)} \int_0^t (t - s)^{\alpha_i - 1} \theta_{l-1} (s) ds \\ &= \left(\frac{L (B)}{\Gamma \left(\alpha_{M_0} \right)} \right)^l \frac{\Gamma \left(\alpha_{m_0} \right)^{l-1} t^{\alpha_i + (l-1)\alpha_{m_0}}}{\Gamma \left((l - 1) \alpha_{m_0} + 1 \right)} \mathbb{B} \Big(\alpha_i, (l - 1) \alpha_{m_0} + 1 \Big), \end{aligned}$$

where

$$\mathbb{B} (x, y) = \int_0^1 (1 - r)^{x-1} r^{y-1} dr$$

is the standard (symmetric) Beta function. Since

$$\mathbb{B}\Big(\alpha_i, (l-1)\,\alpha_{m_0}+1\Big) \le \mathbb{B}\Big(\alpha_{m_0}, (l-1)\,\alpha_{m_0}+1\Big)$$

and

$$\mathbb{B}\Big(\alpha_{m_0}, (l-1)\,\alpha_{m_0}+1\Big) = \Gamma\left(\alpha_{m_0}\right)\frac{\Gamma\left((l-1)\,\alpha_{m_0}+1\right)}{\Gamma\left(l\alpha_{m_0}+1\right)}$$

we deduce from (4.1.10) that

$$q_l^i\left(t\right) \le \theta_l\left(t\right), \text{ for all } i = 1, \ldots, m,$$

for all $t \in [0, T] \subseteq [0, 1]$. This proves (4.1.6) for any $l \in \mathbb{N}$; (4.1.9) follows from the series of the Mittag-Leffler function $E_{\alpha_{m_0},1}$. It remains to show (4.1.8). Using the definition of the sequence $u^{(l)}$, we get

$$\begin{aligned}\left\|u^{(l)}\left(t\right)\right\|_\infty &\le \left\|u^{(1)}\left(t\right)\right\|_\infty + \sum_{1\le j<l}\left\|\left(u^{(j+1)}-u^{(j)}\right)\left(t\right)\right\|_\infty\\ &\le U_0 + \widetilde{\theta}\left(t\right) \le U,\end{aligned}$$

owing to (4.1.3). Therefore, (4.1.5)–(4.1.9) holds for all $l \in \mathbb{N}$. It follows that there exists a function $u \in L^\infty\left(\mathcal{X}\times[0,T];\mathbb{R}^m\right)$ such that

$$\sup_{t\in[0,T]}\left\|\left(u^{(l)}-u\right)\left(t\right)\right\|_\infty \le \sum_{l\le j<\infty}\left\|\left(u^{(j+1)}-u^{(j)}\right)\left(T\right)\right\|_\infty \to 0,$$

as $l \to \infty$. It is now straightforward to show that the limit u is a solution of our initial-value problem (4.1.1)–(4.1.2) on the time interval $[0, T]$. Since this interval is determined uniformly for all $u_0 \in L^\infty\left(\mathcal{X};\mathbb{R}^m\right)$ such that $\|u_0\|_\infty \le U_0$, we also have

$$\inf\left\{T\left(u_0\right) : u_0 \in L^\infty\left(\mathcal{X};\mathbb{R}^m\right),\ \ \|u_0\|_\infty \le U_0\right\} > 0, \tag{4.1.11}$$

for all $U_0 \in [0, \infty)$. We prove the second part in the statement of the theorem. We argue by contradiction. Suppose now that there exists $U_0 \in (0, \infty)$ and a sequence $t_n > 0$ such that

$$\lim_{n\to\infty} t_n = T_{\max} < \infty \text{ and } \sup_{n\in\mathbb{N}}\left\|u\left(t_n\right)\right\|_\infty \le U_0.$$

Hence by (4.1.11), there exists a number $\tau \in (0, \infty)$ and mild solutions $v_n : (x, t) \in \mathcal{X}\times[t_n, t_n+\tau) \mapsto v_n\left(x, t\right) \in \mathbb{R}^m$ of problem (4.1.1) for an initial datum $u\left(t_n\right)$ on the interval $[t_n, t_n+\tau)$. Hence by uniqueness, we get a mild solution u for the initial datum u_0 on the larger interval $[0, T_{\max}+\tau)$, which is a contradiction. This completes the proof. □

If the initial datum is sufficiently regular, the mild solution becomes a strong one on any time interval $[T_0, T_{\max})$, for any $T_0 > 0$.

Theorem 4.1.3 (Existence of Maximal Strong Solutions) *Let assumption **(HA)** for each operator A_i $(i = 1, \dots, m)$ and the conditions of **(SF1)–(SF2)** for the nonlinearity f be satisfied. Assume $u_{0i} \in L^{\infty}(\mathcal{X})$ for $i = 1, \dots, r$ (of course, $r = 0$ is allowed) and $u_{0i} \in D\left(A_{i,p_i}\right) \subset L^{\infty}(\mathcal{X})$ with $p_i \in \left(\beta_{A_i}, \infty\right) \cap (1, \infty)$ for $i = r+1, \dots, m$. Then every bounded mild solution of Theorem 4.1.2 satisfies*

$$u \in C^{0,\kappa}\left([0, T_{\max}); L^{\infty}\left(\mathcal{X}, \mathbb{R}^m\right)\right) \tag{4.1.12}$$

and

$$g_{1-\alpha_i} * (u_i - u_i(0)) \in C^{1,\gamma}\left([0, T_{\max}); L^{\infty}(\mathcal{X})\right),\ i = 1, \dots, r, \tag{4.1.13}$$

$$g_{1-\alpha_i} * (u_i - u_i(0)) \in C^{1}\left((0, T_{\max}); L^{\infty}(\mathcal{X})\right),\ i = r+1, \dots, m, \tag{4.1.14}$$

$$u_i \in C((0, T_{\max}); D(A_{i,p_i})),\ i = r+1, \dots, m, \tag{4.1.15}$$

for some $\kappa, \gamma > 0$. The bounded mild solution also satisfies the initial-value problem (4.1.1)–(4.1.2) in a strong sense, namely, all equations are satisfied for $t \in (0, T_{\max})$ and almost all $x \in \mathcal{X}$, and

$$\lim_{t \to 0^+} \|u(t) - u_0\|_{L^{\infty}(\mathcal{X})} = 0. \tag{4.1.16}$$

Proof We define for $i = 1, \dots, m$, $v_i(x,t) = \mathbb{S}_{\alpha,i}(t) u_{0i}$, $G_i(x,t) = f_i(x,t,u(x,t)) \in \mathbb{R}$, for $(x,t) \in \mathcal{X} \times (0, T_{\max})$. The integral solution for the mild solution can be written, with the usual convention, for $i = 1, \dots, m$,

$$u_i(t) = v_i(t) + \int_0^t P_{\alpha_i}(t-s) G_i(s)\, ds,\ \text{for } t \in (0, T_{\max}).$$

We argue separately for the diffusing and nondiffusing components. Let $i = r + 1, \dots, m$, and recall that each A_i generates an analytic semigroup $S_i \left(= S_{i,p_i}\right)$ in the space $L^{p_i}(\mathcal{X})$. Let $T \in [T_0, T_{\max})$ be arbitrary for any $T_0 > 0$ and let $0 < T_0 \le t < t + h \le T$ in the estimates below. We first show in what sense the initial datum is satisfied. We first have

$$\begin{aligned} \|u(t) - \mathbb{S}_{\alpha}(t) u_0\|_{\infty} &= \max_{1 \le i \le m} \left\|u_i(t) - \mathbb{S}_{\alpha,i}(t) u_{0i}\right\|_{L^{\infty}(\mathcal{X})} \qquad (4.1.17) \\ &\le \max_{1 \le i \le m} \int_0^t \left\|P_{\alpha_i}(t-s)\right\|_{\infty,\infty} \|f_i(s, u(s))\|_{\infty}\, ds \end{aligned}$$

$$\leq L(U) \max_{1\leq i\leq m} \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1}\, ds$$
$$\leq L(U) \max_{1\leq i\leq m} \frac{t^{\alpha_i}}{\Gamma(\alpha_i)\,\alpha_i} \to 0,$$

as $t \to 0^+$. Next, choose $\theta_i > \eta_i$, $\theta_i, \eta_i \in \left(\beta_{A_i}/p_i, 1\right)$ such that $\theta_i = \eta_i + \mu_i$. Recall that we have (by Step 2 of the proof of Theorem 3.2.2),

$$t^{1-\alpha_i} \left\| (-A_{i,p_i})^{\eta_i} P_{\alpha_i}(t) \right\|_{p,\infty} \leq C_i t^{-\eta_i \alpha_i}$$

as well as

$$\left\| (-A_{i,p_i})^{-(1-\eta_i)} \left(S_{\alpha_i}(t) - I \right) \right\|_{p_i, p_i} \leq C_i t^{\alpha_i (1-\eta_i)}.$$

Thus, we can argue as in Step 2 of the proof of Theorem 3.2.2 (see, in particular (3.2.16)–(3.2.18) by letting $\overline{q} = 1$, $\chi + 1/\overline{q} = \alpha_i (1-\theta_i)$) to deduce

$$\begin{aligned} \| u_i(t) - u_{0i} \|_{L^\infty(\mathcal{X})} &\leq C \left\| (-A_{i,p_i})^{\theta_i} (u_i(t) - u_{0i}) \right\|_{L^{p_i}(\mathcal{X})} \\ &\leq C t^{\alpha_i (1-\theta_i)} \left(\| u_{0i} \|_{D(A_{i,p_i})} + L(U) \right), \end{aligned} \tag{4.1.18}$$

for some $C > 0$, independent of t, which clearly shows (4.1.16) for the diffusing components. We also need to prove that u_i is continuous with respect to the time variable. As in the proof of Theorem 3.2.2 ($p := p_i$, $A_p := A_{i,p_i}$, $q_1 = q_2 = \infty$, and so forth), see (3.2.19)–(3.2.31), we get

$$\| u_i(t+h) - u_i(t) \|_{L^\infty(\mathcal{X})} \leq C_T L(U) h^\kappa, \; i = r+1, \ldots, m,$$

for some $\kappa > 0$ which depends on $\alpha_i, \beta_{A_i}, \eta_i$. This yields (4.1.12) for the diffusing components, namely

$$u_i \in C^{0,\kappa}\left([0, T_{\max}); L^\infty(\mathcal{X}) \right), \; i = r+1, \ldots, m. \tag{4.1.19}$$

Now we consider the non-diffusing components. Let $w = (u_1, \ldots, u_r)$ be the vector of non-diffusing components and define

$$H : (x, t, w) \in X \times [0, T_{\max}) \times \mathbb{R}^r \mapsto H(x, t, w) = f(x, t, w, u_{r+1}, \ldots, u_m) \in \mathbb{R}^r$$

with $f = (f_1, \ldots, f_r)$. The components $i = 1, \ldots, r$, of the integral solution for the mild solution then yields

$$w_i(x,t) = w_{0i}(x) + \int_0^t g_{\alpha_i}(t-s) H_i(x, s, w(x,s))\, ds. \tag{4.1.20}$$

The assumption (**SF2**) together with (4.1.19) implies that

$$|H(x,t,\xi) - H(x,s,\eta)| \le L_T(U)\left(|t-s|^{\gamma} + |\xi - \eta|\right), \tag{4.1.21}$$

for all $(x,t,\xi), (x,s,\eta) \in \overline{\mathcal{X}} \times [0,T] \times [-U,U]^r$ (the existence of $U > 0$ follows by construction of the mild solution u). Then we deduce for $t \ge \tau \ge 0$ and $i = 1,\dots,r$,

$$\begin{aligned} &\|w_i(x,t) - w_i(x,\tau)\|_{L^\infty(\mathcal{X})} \\ &\le \frac{L_T(U)}{\Gamma(\alpha_i)} \int_\tau^t (t-s)^{\alpha_i - 1}\, ds = \frac{L_T(U)(t-\tau)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \end{aligned}$$

and so $w_i \in C^{0,\alpha_i}([0,T_{\max}); L^\infty(\mathcal{X}))$, which together with (4.1.19) yields (4.1.12). The foregoing inequality also implies that

$$\lim_{t\to 0^+} \|w_i(t) - w_{0i}\|_{L^\infty(\mathcal{X})} = 0,\ i = 1,\dots,r.$$

Recalling once again (4.1.19) and the Hölder-Lipschitz condition (4.1.21), we easily infer that $H_i \in C^{0,\gamma}([0,T_{\max}); L^\infty(\mathcal{X}))$, $i = 1,\dots,r$, for some $\gamma > 0$. Hence, for $i = 1,\dots,r$, by (4.1.20) and the fact that $g_{1-\alpha} * g_\alpha = 1$, it follows that

$$\partial_t^{\alpha_i} w_i(t) = H_i(\cdot,t,w) \in C^{0,\gamma}([0,T_{\max}); L^\infty(\mathcal{X})). \tag{4.1.22}$$

We finally get the first of (4.1.13). To prove (4.1.14)–(4.1.15) and the remaining part of the statement of the theorem, we argue in a similar fashion as in Step 3 of the proof of Theorem 3.2.2. Thus, the theorem is proved. □

4.2 The Fractional Volterra–Lotka Model

We assume Ω is a bounded domain with Lipschitz continuous boundary $\partial\Omega$ and consider a predator-prey model which assumes a fractional version of the mass action law for the interaction of the two species, predator and prey. As usual, denote the density of prey by $u = u(x,t)$ and of the predator by $v = v(x,t)$. The dynamics of the prey-predator interaction is governed by the following system of reaction–diffusion equations

$$\begin{cases} \partial_t^\alpha u + D_u(-\Delta)_\Omega^s u = u(f - bv), & (x,t) \in \Omega \times (0,\infty), \\ \partial_t v + D_v(-\Delta)_\Omega^l v = v(-g + au), & (x,t) \in \Omega \times (0,\infty), \end{cases} \tag{4.2.1}$$

subject to the following set of boundary conditions

$$\mathcal{N}^{2-2s}u = \mathcal{N}^{2-2l}v = 0 \text{ on } \partial\Omega \times (0, \infty), \tag{4.2.2}$$

and initial conditions

$$(u, v)_{|t=0} = (u_0, v_0) \text{ in } \Omega. \tag{4.2.3}$$

We refer to Appendix B below for a complete description of the boundary operator appearing in (4.2.2). In general a, b, f, g are assumed to be positive constants. Following a discussion in [10], a more general situation can be considered such as, an inhomogeneous environment, symbiosis and saturation can be included by letting the sources f, g depends on x and u, v. We shall consider this situation in a forthcoming article. We assume diffusion rates $D_u, D_v \in (0, \infty)$ and consider the case $s, l \in (1/2, 1)$ since the boundary conditions (4.2.2) makes sense only in this case. Indeed, following an accumulation of evidence from a variety of experimental, theoretical, and field studies [6, 9] we observe that both diffusion operators $(-\Delta)^s_\Omega$, $(-\Delta)^l_\Omega$ offer a better foraging mechanism, than the classical counterpart of Laplacian Δ, for the movement of animals around their natural habitat (cf. also Appendix C, part I). When $s, l \in (0, 1/2]$ and/or $s, l \in \{1\}$, the subsequent results also hold with some minor modifications and different boundary conditions than in (4.2.2) (see Sect. 2.3, for many other possible examples of diffusion operators). The boundary conditions (4.2.2) play a similar role as in the case of no-flux Neumann boundary conditions in that both populations of predator and prey cannot penetrate the boundary $\partial\Omega$. Indeed, we recall that each unforced equation of (4.2.1)–(4.2.2) corresponds to a reflected Lévy process forced to stay inside Ω (see, for instance, [1–3, 5, 7, 8, 12] for the probabilistic point of view). The possible occurrence of a nonlocal derivative ∂_t^α, $\alpha \in (0, 1]$ in the first equation of (4.2.1) accounts for possible effects due to processes with time delay (i.e., "trapping" due environmental and/or predatory effects) in the population of prey (see, for instance, Appendix C.1).

Let $\mathbb{A}_{s,2}$ and $\mathbb{A}_{l,2}$ be the operators on $L^2(\Omega)$ associated with the closed forms

$$\mathcal{E}_s(\varphi, \phi) = \frac{C_{N,s}}{2} D_u \int_\Omega \int_\Omega \frac{(\varphi(x) - \varphi(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} dxdy, \quad \varphi, \phi \in W^{s,2}(\Omega),$$

and

$$\mathcal{E}_l(\varphi, \phi) = \frac{C_{N,l}}{2} D_v \int_\Omega \int_\Omega \frac{(\varphi(x) - \varphi(y))(\phi(x) - \phi(y))}{|x - y|^{N+2l}} dxdy, \quad \varphi, \phi \in W^{l,2}(\Omega),$$

respectively. Using the Green formula (B.0.7) we can show that $\mathbb{A}_{s,2}$ and $\mathbb{A}_{l,2}$ are realizations in $L^2(\Omega)$ of $D_u(-\Delta)^s_\Omega$ and $D_v(-\Delta)^l_\Omega$ with the fractional Neumann boundary conditions $\mathcal{N}^{2-2s}\varphi = 0$ on $\partial\Omega$ and $\mathcal{N}^{2-2l}\varphi = 0$ on $\partial\Omega$, respectively.

More precisely we have that

$$\begin{cases} D(\mathbb{A}_{s,2}) = \Big\{\varphi \in W^{s,2}(\Omega) : \ D_u(-\Delta)^s_\Omega \varphi \in L^2(\Omega), \ \mathcal{N}^{2-2s} u = 0 \ \text{ on } \ \partial\Omega\}, \\ \mathbb{A}_{s,2}\varphi = D_u(-\Delta)^s_\Omega \varphi. \end{cases}$$

and

$$\begin{cases} D(\mathbb{A}_{l,2}) = \{\varphi \in W^{l,2}(\Omega), \ D_v(-\Delta)^l_\Omega \varphi \in L^2(\Omega), \ \mathcal{N}^{2-2l} u = 0 \ \text{ on } \ \partial\Omega\}, \\ \mathbb{A}_{l,2}\varphi = D_v(-\Delta)^l_\Omega \varphi. \end{cases}$$

It follows from [4] (cf. also Sect. 2.3) that the operators $A_s := A_{s,2} = -\mathbb{A}_{s,2}$ and $A_l := A_{l,2} := \mathbb{A}_{l,2}$ satisfy the assumption **(HA)**. Throughout the following for $p \in [1, \infty]$ we shall denote by $A_{s,p}$ and $A_{l,p}$ the generator of the associated semigroup on $L^p(\Omega)$. For $p \geq 2$, each such generator possesses in fact the explicit characterization (2.2.10). In addition we shall let

$$\mathcal{L}^\infty_s(\Omega) := \overline{D(A_{s,\infty})}^{L^\infty(\Omega)} \ \text{ and } \ \mathcal{L}^\infty_l(\Omega) := \overline{D(A_{l,\infty})}^{L^\infty(\Omega)}.$$

We have the following existence result of global strong solutions in the sense introduced in the previous section (see Theorem 4.1.3).

Theorem 4.2.1 *Let* $1/2 < s, l < 1$, $\beta_{A_s} := N/(2s)$ *and* $\beta_{A_l} := N/(2l)$. *Take initial data* $u_0 \in D(A_{s,p_s}) \subset L^\infty(\Omega)$, $v_0 \in D(A_{l,p_l}) \subset L^\infty(\Omega)$ *for some* $p_s \in (\beta_{A_s}, \infty) \cap (1, \infty)$, $p_l \in (\beta_{A_l}, \infty) \cap (1, \infty)$ *such that* $u_0 \geq 0$, $v_0 \geq 0$. *Then the system* (4.2.1)–(4.2.3) *has a unique global strong solution* $u \geq 0$, $v \geq 0$ *on the time interval* $(0, \infty)$ *satisfying*

$$\lim_{t\to 0} \|u(t) - u_0\|_{L^\infty(\Omega)} = 0, \tag{4.2.4}$$

$$\lim_{t\to 0} \|v(t) - v_0\|_{L^\infty(\Omega)} = 0. \tag{4.2.5}$$

In addition for every $T \in (0, \infty)$ *the following estimates hold:*

$$\sup_{0<t<T} \|u(t)\|_{L^\infty(\Omega)} < \infty, \tag{4.2.6}$$

$$\sup_{0<t<T} \|v(t)\|_{L^\infty(\Omega)} < \infty. \tag{4.2.7}$$

Proof Let u_0 and v_0 be as in the statement of the theorem. Recall that the operator $A_{s,2}$ and $A_{l,2}$ satisfy the assumption **(HA)**. Define the function $F : \Omega \times [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ by $F(x, t, (\xi, \eta)) := \Big(\xi(f - b\eta), \eta(-g + a\xi)\Big)$. Then F satisfies the assumptions **(SF1)** and **(SF2)** with $\gamma = 1$. It follows from Theorem 4.1.3 that the system (4.2.1)–(4.2.3) has a strong solution (u, v) on some maximal interval

$[0, T_{\max})$. The strong solution is given by the integral representation

$$u(t) = S_{u,\alpha}(t)u_0 + \int_0^t P_{u,\alpha}(t-\tau)\,(u(f-bv))\,(\tau)d\tau, \tag{4.2.8}$$

$$v(t) = S_v(t)v_0 + \int_0^t S_v(t-\tau)\,(v(-g+au))\,(\tau)d\tau, \tag{4.2.9}$$

where $S_{u,\alpha}(t)$, $S_v(t)$ denote the resolvent family and semigroup on $L^2(\Omega)$ generated by the operators $A_{s,2}$ and $A_{l,2}$, respectively. Here we have also defined

$$S_{u,\alpha}\,(t)\,\omega := \int_0^\infty \Phi_\alpha(\tau)S_u(\tau t^\alpha)\omega d\tau,\;\; P_{u,\alpha}(t)\omega := \alpha t^{\alpha-1}\int_0^\infty \tau\Phi_\alpha(\tau)S_u(\tau t^\alpha)\omega d\tau$$

and set $P_{u,1}(t) \equiv S_{u,1}\,(t) := S_u\,(t)$. Recall that $u_0 \geq 0$ and $v_0 \geq 0$ on Ω and define the functions $k, h : \Omega \times (0,\infty) \times \mathbb{R} \to \mathbb{R}$ by $k(x,t,\xi) = \xi(f - bv(x,t))$ and $h(x,t,\xi) = \xi(-g + au(x,t))$. Since $k(x,t,0) = h(x,t,0) = 0$, it follows from Theorem 3.6.1 that $u(x,t) \geq 0$ and $v(x,t) \geq 0$ for a.e. $(x,t) \in \Omega \times [0, T_{\max})$.

Note that each component of F satisfies the assumption (**F6**) for $u \geq 0$, $v \geq 0$. Applying Theorem 3.4.11, we get that there exist two constants $C, C_0 > 0$ such that for every $0 < T < \infty$,

$$\sup_{0<t<T} \|u(t)\|_{L^\infty(\Omega)} \leq C\left(\|u_0\|_{L^\infty(\Omega)} + E_{\alpha,1}\,(C_0T) + T^\alpha E_{\alpha,1}\,(C_0T)\,f\right) \tag{4.2.10}$$

and we have shown (4.2.6) for any $\alpha \in (0,1]$. Next we consider (4.2.9) and let $p(x,t,\xi) = (-g + au)$. Since g is a positive constant and $u \geq 0$, we have

$$h\,(x,t,\xi)\,\xi = p(x,t,\xi)\xi^2 = (-g + au)\xi^2 \leq au(x,t)\xi^2.$$

It follows from (4.2.10) that $c_0 := \sup_{t\in(0,T)} \|au(x,t)\|_{L^\infty(\Omega)} < \infty$. We have shown that h also satisfies the assumption **(F6)**. Then applying Theorem 3.4.11 once again and recalling Corollary 3.4.14, we get that there there exist two constants $C, C_0 > 0$ such that for every $0 < T < \infty$,

$$\sup_{t\in(0,T)} \|v(t)\|_{L^\infty(\Omega)} \leq C\left(\|v_0\|_{L^\infty(\Omega)} + c_0\,(T+1)\,e^{C_0T}\right). \tag{4.2.11}$$

We have shown (4.2.7). Together with (4.2.6) we can conclude that $T_{\max} = \infty$ (see Sect. 4.1). For (4.2.4)–(4.2.5), we refer once again to the proof of Theorem 4.1.3 (see (4.1.18), (4.1.17) and set $\alpha_1 = \alpha$, $\alpha_2 = 1$, in which case $P_{\alpha_1} \equiv P_{u,\alpha}$, $P_{\alpha_2} \equiv S_v$); they easily follow now on the account of (4.2.10)–(4.2.11). The proof is finished. □

We recall from (3.1.4) that $\mathfrak{n}_{\mathfrak{s}} := \beta_{A_s}\alpha$, $\alpha \in (0,1]$ and $\mathfrak{n}_{\mathfrak{l}} := \beta_{A_l}$.

Theorem 4.2.2 *Let $p_0, q_0 \in [1, \infty]$ such that $\beta_{A_s}/p_0 < 1$ and consider initial data $0 \le u_0 \in L^{p_0}(\Omega)$, $0 \le v_0 \in L^{q_0}(\Omega)$. Then the fractional Lotka-Volterra system (4.2.1)–(4.2.3) has a unique global mild solution $u \ge 0, v \ge 0$ on the time interval $[0, \infty)$, given by (4.2.8)–(4.2.9), which is also a strong solution on $(0, \infty)$. Moreover, the pair (u, v) satisfies*

$$\lim_{t \to 0^+} \|u(t) - u_0\|_{L^1(\Omega)} = 0, \quad \lim_{t \to 0^+} \|v(t) - v_0\|_{L^k(\Omega)} = 0, \text{ for } k \in [1, q_0), \tag{4.2.12}$$

and the following estimates, for any $T \in (0, \infty)$:

$$\sup_{t \in (0,T)} (t \wedge 1)^{\delta_s} \|u(t)\|_{L^p(\Omega)} < \infty, \; p \in [p_0, \infty], \tag{4.2.13}$$

$$\sup_{t \in (0,T)} (t \wedge 1)^{\delta_l} \|v(t)\|_{L^q(\Omega)} < \infty, \; q \in [q_0, \infty], \tag{4.2.14}$$

where

$$\delta_s := \frac{\mathfrak{n}_{\mathfrak{s}}}{p_0}\left(1 - \frac{p_0}{p}\right),$$

$$\delta_l := \frac{\mathfrak{n}_{\mathfrak{l}}}{q_0}\left(1 - \frac{q_0}{q}\right).$$

The proof of the theorem follows from a series of propositions and lemmas that we subsequently give. In what follows one can start with more regular initial data due to the statement of Theorem 4.2.1 and then deduce all the required estimates with less regular initial data by exploiting a standard approximation argument.

Proposition 4.2.3 *Every nonnegative solution (u, v) satisfies the following estimate*[1]

$$\|u(t)\|_{L^{p_0}(\Omega)} \le \|u_0\|_{L^{p_0}(\Omega)} \left(E_{\alpha,1}\left(f t^{\alpha}\right)\right)^{\frac{1}{p_0}}, \tag{4.2.15}$$

for all $t \in (0, \infty)$ and $\alpha \in (0, 1]$, $p_0 \in [1, \infty)$.

Proof We derive the estimate in case $p_0 \in (1, \infty)$, the cases $p_0 \in \{1, \infty\}$ follow directly from a limit argument in (4.2.15). Multiply the first equation of (4.2.1) by $p_0 u^{p_0 - 1}$, integrate the resulting identity over Ω, then exploit the first inequality of Proposition 3.4.9 if $\alpha \in (0, 1)$ and use the fact that $u, v \ge 0$. We find that

$$\partial_t^{\alpha}\left(\|u(t)\|_{L^{p_0}(\Omega)}^{p_0}\right) \le f \|u(t)\|_{L^{p_0}(\Omega)}^{p_0},$$

[1] The Mittag-Leffler function $E_{1,1}(x) = e^x$.

for all $t \geq 0$. The comparison principle of Lemma A.0.7 then immediately yields the result since the unique solution of $\partial_t^\alpha y = fy$ is $y = y(0)\, E_{\alpha,1}(ft^\alpha)$. □

Lemma 4.2.4 *Let $p_0 \in [1,\infty]$ such that $\beta_{A_s}/p_0 < 1$ and assume sufficiently smooth data (u_0, v_0). Then for every $T \in (0,\infty)$ there exists a constant $M = M(T,\Omega,a,b,f,g) \in (0,\infty)$ such that*

$$\sup_{t\in(0,T)} (t\wedge 1)^{\delta_s} \|u(t)\|_{L^p(\Omega)} \leq M,\ \ p \in [p_0,\infty]. \tag{4.2.16}$$

Proof *We apply Lemma 3.4.3 to the equation in u and use the one-sided version due to Remark 3.4.5 since $u, v \geq 0$. The weight function $c(x,t) = f$ is constant, we have $q_1 = q_2 = r_2 = \infty$, $r_1 = p_0$ and $\gamma = 1$ and we can find a number $\tilde{b} \in [0,1)$ satisfying $\left(1-\tilde{b}\right)\frac{\beta_{A_s}\alpha}{p_0} < \alpha - \varepsilon$, for some $\varepsilon \in (0,\alpha)$. Note that $\beta_{A_s}/p_0 < 1$ is equivalent to $\beta_{A_s}\alpha/p_0 = \mathfrak{n}_{\mathfrak{s}}/p_0 < \alpha$. The assertion (3.4.18) of Lemma 3.4.3 then implies the existence of a constant $C_* > 0$ independent of u_0, u, U, t and T such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C_*\,(t\wedge 1)^{-\frac{\mathfrak{n}_{\mathfrak{s}}}{p_0}} \left[\|u_0\|_{L^{p_0}(\Omega)} + \Upsilon(t)\left(U + U^{1/\left(1-\tilde{b}\right)}\right)\right], \tag{4.2.17}$$

for all $t \in (0,T]$. This yields (4.2.16) for $p = \infty$ since by the definition of U, c and (4.2.15), we have

$$U := f\,\|1+|u|\|_{p_0,\infty,T}^{\left(1-\tilde{b}\right)} < \infty.$$

In case $p = p_0$, estimate (4.2.16) is just the a priori estimate (4.2.15), namely, it follows that

$$\sup_{t\in(0,T)} \|u(t)\|_{L^{p_0}(\Omega)} \leq M_1(T,f) < \infty. \tag{4.2.18}$$

Since both $L^{p_0}(\Omega)$ and $L^\infty(\Omega)$-estimates are now readily available by (4.2.18) and (4.2.17), we can use the interpolation inequality

$$\|u\|_{L^p(\Omega)} \leq \|u\|_{L^{p_0}(\Omega)}^{p_0/p}\, \|u\|_{L^\infty(\Omega)}^{1-p_0/p},\ \ p \in [p_0,\infty], \tag{4.2.19}$$

to derive the desired estimate in (4.2.16) for arbitrary p. □

Lemma 4.2.5 *Under the assumptions of Lemma 4.2.4, every nonnegative solution v of* (4.2.1)–(4.2.3) *satisfies*

$$\sup_{t\in(0,T)} \|v(t)\|_{L^{q_0}(\Omega)} < \infty,\ \textit{for}\ q_0 \in [1,\infty]. \tag{4.2.20}$$

Proof Consider the weight function $c(x,t) = -g + au(x,t)$ and notice that $v \geq 0$ is a solution of

$$\partial_t v + \mathbb{A}_{l,2} v = cv,\ v(0) = v_0.$$

Multiply this equation by qv^{q-1} and integrate the resulting identity over Ω. Since

$$\left(\mathbb{A}_{l,2} v,\ v^{q-1}\right)_{L^2(\Omega)} \geq 0$$

by (2.2.11), we find

$$\partial_t \|v(t)\|^q_{L^q(\Omega)} \leq q \|v(t)\|^q_{L^q(\Omega)} \|c(t,\cdot)\|_{L^\infty(\Omega)},$$

for all $t \in [0,T]$. This inequality implies that

$$\partial_t \|v(t)\|_{L^q(\Omega)} \leq \|v(t)\|_{L^q(\Omega)} \|c(t,\cdot)\|_{L^\infty(\Omega)}$$

and the application of Gronwall's inequality yields

$$\|v(t)\|_{L^q(\Omega)} \leq \|v_0\|_{L^q(\Omega)} e^{\int_0^t \|c(s,\cdot)\|_{L^\infty(\Omega)} ds}, \tag{4.2.21}$$

for any $q \in [1,\infty)$. Notice that in view of (4.2.17), $\|c(t,\cdot)\|_{L^\infty(\Omega)} \sim t^{-\mathfrak{n}_\mathfrak{s}/p_0}$ for $t \in (0,1)$ and $\|c(t,\cdot)\|_{L^\infty(\Omega)} \leq C_T$ for $t \geq 1$; thus we have $\|c(t,\cdot)\|_{L^\infty(\Omega)} \in L^1(0,T)$ since $\mathfrak{n}_\mathfrak{s}/p_0 < \alpha \leq 1$ by assumption. In particular, we infer from (4.2.21) and (4.2.17) the existence of a constant $M = M(T, p_0, \mathfrak{n}_\mathfrak{s}, g, a, f, u_0) > 0$, independent of q, such that

$$\|v(t)\|_{L^q(\Omega)} \leq M \|v_0\|_{L^q(\Omega)},\ t \in [0,T],$$

which is exactly the primary estimate (4.2.20) for $q = q_0 \in [1,\infty)$. Passing to the limit as $q \to \infty$ in the previous inequality, we also get the estimate (4.2.20) for $q_0 = \infty$. We thus conclude the proof. □

We can now show that v satisfies the smoothing property (4.2.14).

Lemma 4.2.6 *Under the assumptions of Lemma 4.2.4, for every $T \in (0,\infty)$ there exists a constant $M = M(T, \Omega, a, b, f, g) \in (0,\infty)$ such that*

$$\sup_{t\in(0,T)} (t \wedge 1)^{\delta_l} \|v(t)\|_{L^q(\Omega)} \leq M,\ q \in [q_0,\infty]. \tag{4.2.22}$$

Proof We first notice that estimate (4.2.16) with $p = \infty$ implies that $\|u\|_{\infty,q_2} \leq M$, for some $q_2 \in (1, 1/\delta)$, $\delta := \mathfrak{n}_\mathfrak{s}/p_0 < \alpha \leq 1$. This time we apply Lemma 3.4.3 to the equation in v with the weight function $c(x,t) = -g + au(x,t)$ which now satisfies $\|c\|_{\infty,q_2} \leq M_1$, and set $q_1 = r_2 = \infty$, $q_2 := q_2 \in (1, 1/\delta)$, $r_1 = q_0$

and $\gamma = 1$. The constant $M_1 \in (0, \infty)$ depends on the final time $T > 0$ but is independent of t. Indeed, we can find a *new* number $\tilde{b} \in [0, 1)$, sufficiently close to 1, satisfying

$$\frac{1}{q_2} + \left(1 - \tilde{b}\right) \frac{\mathfrak{n}_{\mathfrak{l}}}{q_0} < 1 - \varepsilon,$$

for some $\varepsilon \in (0, 1)$. It follows from the assertion (3.4.18) of Lemma 3.4.3 that there exists a constant $C_* > 0$ independent of v_0, v, V and t such that

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C_* (t \wedge 1)^{-\frac{\mathfrak{n}_{\mathfrak{l}}}{q_0}} \left[\|u_0\|_{L^{q_0}(\Omega)} + \Upsilon(t) \left(V + V^{1/\left(1-\tilde{b}\right)} \right) \right], \tag{4.2.23}$$

for all $t \in (0, T]$. Here, $V < \infty$ is defined as

$$V := \|1 + |v|\|_{q_0, \infty, T}^{\left(1-\tilde{b}\right)} \|c\|_{\infty, q_2}.$$

This yields estimate (4.2.22) for $q = \infty$. Next, recall that v also satisfies (4.2.20); this allows us to exploit an interpolation similar to (4.2.19) in the spaces $L^\infty(\Omega) \subset L^q(\Omega) \subset L^{q_0}(\Omega)$. Thus we arrive at the desired estimate (4.2.14) for an arbitrary $q \in [q_0, \infty]$ and we conclude the proof. □

Proposition 4.2.7 *Assume* $p_0, q_0 \in [1, \infty]$ *are such that* $\beta_{A_s}/p_0 < 1$. *Then the following assertions hold.*

(a) There exists a constant $M > 0$ *such that for every* $t \in (0, 1)$, *we have*

$$\left\| u(t) - S_{u,\alpha}(t) u_0 \right\|_{L^{q_0}(\Omega)} \leq M t^{\varepsilon}, \quad \|v(t) - S_v(t) v_0\|_{L^{q_0}(\Omega)} \leq M t^{\varepsilon}, \tag{4.2.24}$$

for some $\varepsilon > 0$, *for* $p_0, q_0 \in [1, \infty)$.

(b) For $i = 1, 2$, *let* (u_i, v_i) *be a solution of* (4.2.1)–(4.2.3) *corresponding to an initial datum* (u_{0i}, v_{0i}). *Then for every* $T \in (0, \infty)$, *there exists a constant* $C = C(T) \in (0, \infty)$, *independent of* (u_i, v_i), *such that*

$$\begin{aligned} &|\|u_1 - u_2\||_{\infty, \delta, T} + |\|v_1 - v_2\||_{q_0, 0, T} \\ &\leq C \left(\|u_{01} - u_{02}\|_{L^{p_0}(\Omega)} + \|v_{01} - v_{02}\|_{L^{q_0}(\Omega)} \right). \end{aligned} \tag{4.2.25}$$

Proof By the integral formula (4.2.8) and estimate (4.2.13) with $p = \infty$ and $\delta = \mathfrak{n}_{\mathfrak{s}}/p_0$, for $t \in (0, 1)$ we have as in (3.1.16) (with $s_0 := q_0$, $p_0 := q_0$,

$q_2 := \infty$, $q_1 := q_0$),

$$\left\| u(t) - S_{u,\alpha}(t) u_0 \right\|_{L^{q_0}(\Omega)} \leq C \int_0^t (\tau \wedge 1)^{-\delta} d\tau \left(|||u|||_{\infty,\delta,1} \right) \left(1 + |||v|||_{q_0,0,1} \right)$$
$$\leq M t^{\varepsilon},$$

for some $\varepsilon > 0$. The same argument applied to the difference $v(t) - S_v(t) v_0$ in (4.2.9) gives the required estimate in (4.2.24).

In order to show (4.2.25), we take $\varepsilon := 1 - \delta > 0$, where $\delta = \mathfrak{n}_{\mathfrak{s}} / p_0$. Subtracting the integral equations (4.2.8) corresponding to each $i = 1, 2$ and u_i, we obtain

$$\|u_1(t) - u_2(t)\|_{L^\infty(\Omega)} \leq \left\| S_{u,\alpha}(t) \right\|_{\infty,p_0} \|u_{01} - u_{02}\|_{L^{p_0}(\Omega)} \tag{4.2.26}$$
$$+ \Lambda(u_i, v_i) \int_0^t C \left\| P_{u,\alpha}(t-s) \right\|_{\infty,q_0} (s \wedge 1)^{-\delta} ds,$$

where

$$\Lambda(u_i, v_i) := |||u_1 - u_2|||_{\infty,\delta,T} \left(1 + |||v_1|||_{q_0,0,T} \right) \tag{4.2.27}$$
$$+ \left(1 + |||u_2|||_{\infty,\delta,T} \right) |||v_1 - v_2|||_{q_0,0,T} .$$

We can apply Lemma A.0.1 to the second summand in (4.2.26) and exploit the global bounds (4.2.13)–(4.2.14) to estimate the corresponding norms for u_2 and v_1. We deduce

$$|||u_1 - u_2|||_{\infty,\delta,t} \tag{4.2.28}$$
$$\leq C \|u_{01} - u_{02}\|_{L^{p_0}(\Omega)}$$
$$+ C_T \Upsilon(t) \left(|||u_1 - u_2|||_{\infty,\delta,T} + |||v_1 - v_2|||_{q_0,0,T} \right),$$

for all $t \in (0, T]$, for some $C > 0$ independent of t. Arguing similarly for the v-component, we find

$$\|v_1(t) - v_2(t)\|_{L^{q_0}(\Omega)} \leq \|S_v(t)\|_{q_0,q_0} \|v_{01} - v_{02}\|_{L^{q_0}(\Omega)}$$
$$+ C_T \Lambda(u_i, v_i) \int_0^t (s \wedge 1)^{-\delta} ds$$

which yields

$$|||v_1 - v_2|||_{q_0,0,t} \leq C \|v_{01} - v_{02}\|_{L^{q_0}(\Omega)} \tag{4.2.29}$$
$$+ C_T \Upsilon(t) \left(|||u_1 - u_2|||_{\infty,\delta,T} + |||v_1 - v_2|||_{q_0,0,T} \right),$$

for all $t \in (0, T]$. Choose now a small enough $h > 0$ such that $C_T \Upsilon(h) \leq 1/2$ into (4.2.28)–(4.2.29). We obtain

$$|||u_1 - u_2|||_{\infty,\delta,h} + |||v_1 - v_2|||_{q_0,0,h} \leq M(T) \left(\|u_{01} - u_{02}\|_{L^{p_0}(\Omega)} + \|v_{01} - v_{02}\|_{L^{q_0}(\Omega)} \right). \tag{4.2.30}$$

With the same proof, we can also infer that

$$\begin{aligned} &|||(u_1 - u_2)(\cdot, t_0 + \cdot)|||_{\infty,\delta,h} + |||(v_1 - v_2)(\cdot, t_0 + \cdot)|||_{q_0,0,h} \\ &\leq M(T) \left(\|(u_1 - u_2)(t_0)\|_{L^{p_0}(\Omega)} + \|(v_1 - v_2)(t_0)\|_{L^{q_0}(\Omega)} \right), \end{aligned} \tag{4.2.31}$$

for all $t_0 \in [0, T)$. We can now apply the estimate (4.2.31) successively for $j = 0, 1, 2, \ldots,$ with initial data $(u, v)(t_0 + jh)$. Then the assertion (4.2.25) follows by induction on j and we finish the proof of the proposition. □

Proof (Proof of Theorem 4.2.2) The proof follows now by a simple procedure where we approximate any rough nonnegative initial data $(u_0, v_0) \in L^{p_0}(\Omega) \times L^{q_0}(\Omega)$ by a sequence of nonnegative functions $(u_{0n}, v_{0n}) \in D(A_{s,p_s}) \times D(A_{l,p_l})$ (for some sufficiently large $p_s \in \left(\beta_{A_s}, \infty\right)$, $p_l \in \left(\beta_{A_l}, \infty\right)$ and $p_s, p_l \geq 2$) such that

$$\|u_{0n} - u_0\|_{L^{p_0}(\Omega)} \to 0, \|v_{0n} - v_0\|_{L^{q_0}(\Omega)} \to 0, \text{ as } n \to \infty$$

with

$$\|u_{0n}\|_{L^{p_0}(\Omega)} \leq \|u_0\|_{L^{p_0}(\Omega)}, \quad \|v_{0n}\|_{L^{q_0}(\Omega)} \leq \|v_0\|_{L^{q_0}(\Omega)}.$$

The above lemmata and propositions then hold with the constants $M, M_1, C, C_* > 0$ independent of n for the sequence of strong solutions (u_n, v_n). Thus, assertion (4.2.25) of Proposition 4.2.7 implies that the sequence (u_n, v_n) converges to (u, v) in $E_{\infty,\delta,T} \times E_{q_0,0,T}$, and all the a priori estimates derived in this section also hold for the limit solution (u, v). It is then straightforward to show from (4.2.8)–(4.2.9) that (u, v) is also the mild solution of system (4.2.1)–(4.2.3) for an initial datum $(u_0, v_0) \in L^{p_0}(\Omega) \times L^{q_0}(\Omega)$ (see Chap. 3 and Sect. 4.1). In particular every such mild solution (u, v) is global and bounded on $[T_0, \infty)$ for every $T_0 > 0$, and one can use arguments as in the proofs of Theorems 4.1.3 and 3.2.6, respectively, to show that (u, v) is also a strong solution on $[2T_0, \infty)$. The continuity properties in (4.2.12) follow also immediately by virtue of (4.2.24) and Remark 3.1.2. □

4.3 A Fractional Nuclear Reactor Model

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$ and consider the following parabolic system as a prototype for a nuclear reactor model that we believe has a more realistic physical interpretation than the classical one (see Appendix C). Let $u = u(x,t)$ represent the fast neutron density and $v = v(x,t)$ be the fuel temperature at any point $x \in \Omega$ and for any time $t \geq 0$. The system for (u, v) reads

$$\begin{cases} \partial_t^\alpha u + (-\Delta)_\Omega^s u = u(\lambda - bv), & (x,t) \in \Omega \times (0,\infty), \\ \partial_t^\beta v = -cv + au, & (x,t) \in \Omega \times (0,\infty), \end{cases} \tag{4.3.1}$$

subject to the following set of boundary and initial conditions:

$$\mathcal{N}^{2-2s} u = 0 \text{ on } \partial\Omega \times (0,\infty), \quad (u,v)_{|t=0} = (u_0, v_0) \text{ in } \Omega. \tag{4.3.2}$$

Here $s \in (1/2, 1)$, $\alpha, \beta \in (0,1)$ and λ, a, b, c are positive constants in the model equations (4.3.1)–(4.3.2) and $(-\Delta)_\Omega^s$ is the regional fractional Laplace operator in Ω (see (2.3.19)) and $\mathcal{N}^{2-2s}$ denotes the corresponding fractional Neumann derivative (see Sect. 2.3). The first (unforced) equation of (4.3.1) may be derived from a continuous-time random walk with temporal memory (see Appendix C.3), while incorporating avalanche-like transport effects in the neutron density and the second equation can be analogously derived on similar principles as those considered in Appendix C, by ignoring any diffusion effects in the fluid temperature v. We note that the case $s = \alpha = \beta = 1$ has been treated by Rothe [10] in some detail as a simple reactor model proposed in [11].

Note that the first equation of (4.3.1) is structurally the same as the equation for prey in the fractional Lotka-Volterra model investigated in Sect. 4.2. Thus the arguments appear to be even more simple than in that case provided that we can derive suitable a priori estimates for the fluid temperature in (4.3.1). Let $S_u(t)$ denote the semigroup on $L^2(\Omega)$ generated by the operator $A_{s,2}$, as given previously. Consider a sufficiently smooth initial datum (u_0, v_0) and its corresponding solution.

Proposition 4.3.1 *The fluid temperature* v *satisfies the following estimate*

$$\sup_{t\in(0,T)} \|v(t)\|_{L^q(\Omega)} \leq C^{1/q} \max\left\{ \|v_0\|_{L^q(\Omega)}, \frac{\varepsilon^{1/q-1}}{(C_\varepsilon q)^{1/\alpha q}} \sup_{t\in(0,T)} \|u(t)\|_{L^\infty(\Omega)} \right\}, \tag{4.3.3}$$

for some $\varepsilon > 0$ *depending only on* a, c, *and some constants* $C, C_\varepsilon > 0$ *independent of* $q \in [1, \infty]$.

Proof By application of Proposition 3.4.9 into the second equation of (4.3.1), we get

$$\begin{aligned}
&\partial_t^\beta \left(\|v(t)\|^q_{L^q(\Omega)} \right) + cq \|v(t)\|^q_{L^q(\Omega)} \\
&\leq aq \|v(t)\|^{q-1}_{L^q(\Omega)} \|u(t)\|_{L^\infty(\Omega)} \\
&\leq a\varepsilon^{1-q} \left(\sup_{t\in(0,T)} \|u(t)\|_{L^\infty(\Omega)} \right)^q + a\varepsilon(q-1) \|v(t)\|^q_{L^q(\Omega)},
\end{aligned}$$

for all $t \geq 0$. This inequality implies for a sufficiently small $\varepsilon \in (0, ca/2]$ and $C_\varepsilon = c/2$, that

$$\begin{aligned}
&\partial_t^\beta \left(\|v(t)\|^q_{L^q(\Omega)} \right) + C_\varepsilon q \|v(t)\|^q_{L^q(\Omega)} \\
&\leq M := a \left(\varepsilon^{1/q-1} \sup_{t\in(0,T)} \|u(t)\|_{L^\infty(\Omega)} \right)^q.
\end{aligned}$$

We infer by Lemma A.0.8 the existence of a constant $C > 0$, independent of q, such that

$$\|v(t)\|^q_{L^q(\Omega)} \leq C \max \left\{ \|v_0\|^q_{L^q(\Omega)}, \frac{a}{(C_\varepsilon q)^{1/q}} \left(\varepsilon^{1/q-1} \sup_{t\in(0,T)} \|u(t)\|_{L^\infty(\Omega)} \right)^q \right\}.$$

Taking the $1/q$-root on both sides, this inequality gives the desired assertion in (4.3.3) for every $q \in [1, \infty)$. Since the constants C, C_ε involved in (4.3.3) are independent of q, we also recover the estimate in case $q = \infty$, by passing to the limit as $q \to \infty$ in (4.3.3). □

In view of the simple estimate of Proposition 4.3.1, we can derive the existence of unique global strong solution in the sense of Theorem 4.1.3.

Theorem 4.3.2 *Let* $1/2 < s < 1$ *and* $\beta_{A_s} := N/(2s)$. *Take initial data* $u_0 \in D(A_{s,p_s}) \subset L^\infty(\Omega)$, $v_0 \in L^\infty(\Omega)$ *for some* $p_s \in (\beta_{A_s}, \infty) \cap (1, \infty)$ *such that* $u_0 \geq 0$, $v_0 \geq 0$. *Then the system* (4.3.1)–(4.3.2) *has a unique global strong solution* $u \geq 0$, $v \geq 0$ *on the time interval* $(0, \infty)$ *satisfying*

$$\lim_{t\to 0} \|u(t) - u_0\|_{L^\infty(\Omega)} = 0, \quad \lim_{t\to 0} \|v(t) - v_0\|_{L^\infty(\Omega)} = 0. \tag{4.3.4}$$

In addition for every $T \in (0, \infty)$ *the following estimates hold:*

$$\sup_{0<t<T} \|u(t)\|_{L^\infty(\Omega)} < \infty, \quad \sup_{0<t<T} \|v(t)\|_{L^\infty(\Omega)} < \infty. \tag{4.3.5}$$

Proof Let u_0 and v_0 be as in the statement of the theorem. We can infer the existence of a maximally defined strong solution by Theorem 4.1.3, given as

$$u(t) = S_{u,\alpha}(t)u_0 + \int_0^t P_{u,\alpha}(t-\tau)\,(u(\lambda - bv))\,(\tau)d\tau, \tag{4.3.6}$$

$$v(t) = v_0 + \int_0^t g_\beta\,(t-\tau)\,(-cv + au)\,(\tau)d\tau, \tag{4.3.7}$$

for $t \in (0, T_{\max})$. A similar argument to the proof of Theorem 4.2.1 successively yields that $u(x,t) \geq 0$ and then $v(x,t) \geq 0$ for a.e. $(x,t) \in \Omega \times [0, T_{\max})$ since $g\,(u, 0) = au \geq 0$ (for $g\,(u, v) := -cv + au$). Moreover, the first bound of (4.3.5) is satisfied by the same arguments of Theorem 4.2.1. Consequently, so is the second bound of (4.3.5) on account of (4.3.3) in case $q = \infty$. The continuity properties in (4.3.4) follow also by similar arguments on account of (4.3.5) with the exception that for the integral solution v we have a more direct estimate from (4.3.7). The proof is finished. □

As a consequence of Proposition 4.2.3 and Lemma 4.2.4 we immediately have the following estimate since the equation for u is the same as for the fractional system (4.2.1)–(4.2.2).

Proposition 4.3.3 *Let $p_0 \in [1, \infty]$ such that $\beta_{A_s}/p_0 < 1$ and assume a sufficiently smooth datum u_0. Then for every $T \in (0, \infty)$ there exists a constant $M = M\,(T, \Omega, b, \lambda) \in (0, \infty)$ such that*

$$\sup_{t\in(0,T)} (t \wedge 1)^{\delta_s}\,\|u\,(t)\|_{L^p(\Omega)} \leq M, \;\; p \in [p_0, \infty]\,, \tag{4.3.8}$$

where $\delta_s \geq 0$ is as in the statement of Theorem 4.2.2.

We now derive some uniform a priori L^q-estimate for the temperature. Of course, there is no smoothing effect in the component v other than the one implied by u. In other words, v turns out to be as regular as u but no more. Since $u, v \geq 0$, we have by (4.3.7) that pointwise in time,

$$v\,(t) \leq \overline{v}\,(t) := v_0 + a\int_0^t g_\beta\,(t-\tau)\,u(\tau)d\tau \tag{4.3.9}$$

and so it suffices to derive the required estimate for $\overline{v}$.

Proposition 4.3.4 *Under the assumptions of Proposition 4.3.3, it holds for any $v_0 \in L^q\,(\Omega)$, $q \leq p$ with $p \in [p_0, \infty]$ and $T \in (0, \infty)$, the estimate*

$$\sup_{t\in(0,T)} \|v\,(t)\|_{L^q(\Omega)} < \infty \text{ if } \beta \geq \delta_s$$

and

$$\sup_{t\in(0,T)} (t \wedge 1)^{\delta_s - \beta} \|v(t)\|_{L^q(\Omega)} < \infty \text{ } \textit{if } \beta < \delta_s.$$

Proof We have

$$\begin{aligned} \|\overline{v}(t)\|_{L^q(\Omega)} &\le \|v_0\|_{L^q(\Omega)} + a\int_0^t g_\beta(t-\tau)\|u(\tau)\|_{L^q(\Omega)}\,d\tau \\ &\le \|v_0\|_{L^q(\Omega)} + C\left(\sup_{t\in(0,T)}(t\wedge 1)^{\delta_s}\|u(t)\|_{L^p(\Omega)}\right) \\ &\quad \times \int_0^t g_\beta(t-\tau)(\tau\wedge 1)^{-\delta_s}\,d\tau. \end{aligned} \tag{4.3.10}$$

A basic change of variable $s = \tau/t$ gives for $t < 1$,

$$\int_0^t g_\beta(t-\tau)(\tau\wedge 1)^{-\delta_s}\,d\tau = C_\beta t^{\beta-\delta_s}\int_0^1 s^{-\delta_s}(1-s)^{\beta-1}\,d\tau,$$

where the latter integral is convergent since $\beta > 0$ and $\delta_s = \frac{\mathfrak{n}_\mathfrak{s}}{p_0}\left(1-\frac{p_0}{p}\right) < \alpha\left(1-\frac{p_0}{p}\right) < \alpha < 1$. When $t > 1$ we argue as in the proof of Lemma A.0.1 to split the integral over intervals $k < t \le k+1$, such that

$$\int_0^t = \int_0^1 + \int_1^2 + \ldots + \int_{k-2}^{k-1} + \int_{k-1}^{t-1} + \int_{t-1}^t.$$

It follows that

$$\int_0^t g_\beta(t-\tau)(\tau\wedge 1)^{-\delta_s}\,d\tau \le C(k+1) \le 2Ct,$$

for some constant $C > 0$ independent of t, T. We then infer the existence of a positive constant $M_2 = M_2(M, \Omega, a, T, v_0) \in (0, \infty)$ such that

$$\sup_{t\in(0,T)}(t\wedge 1)^{\delta_s-\beta}\|\overline{v}(t)\|_{L^q(\Omega)} \le M_2, \text{ if } \beta \le \delta_s$$

and

$$\sup_{t\in(0,T)}\|\overline{v}(t)\|_{L^q(\Omega)} \le M_2, \text{ if } \beta > \delta_s.$$

We may now conclude using (4.3.9). □

Proposition 4.3.5 *Let the assumptions of Proposition 4.3.3 be satisfied and let* $q_0 \in [1, p_0]$ *such that* $v_0 \in L^{q_0}(\Omega)$. *Then the following estimate holds:*

$$\sup_{t \in (0,T)} \|v(t)\|_{L^{q_0}(\Omega)} \leq \|v_0\|_{L^{q_0}(\Omega)} + C_T, \tag{4.3.11}$$

for some constant $C_T \in (0, \infty)$ *that depends only on the* $L^{p_0}(\Omega)$*-norm of* u_0, T *and the other physical parameters of the problem.*

Proof Let $T \in (0, \infty)$ be arbitrary. The Hölder inequality on the bounded interval $[0, T]$ yields in (4.3.11), owing to the fact that $g_\beta \in L^1(0, T)$,

$$\sup_{t \in (0,T)} \|\overline{v}(t)\|_{L^{q_0}(\Omega)} \leq \|v_0\|_{L^{q_0}(\Omega)} + C_T \|u\|_{L^\infty(0,T;L^{p_0}(\Omega))},$$

for some $C_T = C(a, \Omega, T, \beta) > 0$ independent of t. Application of (4.3.8) with $p = p_0$ then gives the desired estimate in (4.3.11) since $\delta_s = 0$ and $v \leq \overline{v}$. □

Proposition 4.3.6 *Assume* $p_0 \in [1, \infty]$ *such that* $\beta_{A_s}/p_0 < 1$ ($\Leftrightarrow \mathfrak{n}_\mathfrak{s}/p_0 < \alpha$) *and* $q_0 \in [1, p_0] \cap (\beta_{A_s}, \infty]$. *Then the following assertions hold.*

(a) There exists a constant $M > 0$ *such that for every* $t \in (0, 1)$*, we have*

$$\left\|u(t) - S_{u,\alpha}(t) u_0\right\|_{L^{p_0}(\Omega)} \leq M t^{\varepsilon}, \quad \|v(t) - v_0\|_{L^{q_0}(\Omega)} \leq M t^{\beta}, \tag{4.3.12}$$

for some small $\varepsilon > 0$.

(b) For $i = 1, 2$, *let* (u_i, v_i) *be a solution of* (4.3.1)–(4.3.2) *corresponding to an initial datum* (u_{0i}, v_{0i}). *Then for every* $t \in (0, T)$, *there exists a constant* $C = C(T) \in (0, \infty)$, *independent of* (u_i, v_i), *such that*

$$\begin{aligned} &|||u_1 - u_2|||_{p_0,0,t} + |||v_1 - v_2|||_{q_0,0,t} \\ &\leq C \left(\|u_{01} - u_{02}\|_{L^{p_0}(\Omega)} + \|v_{01} - v_{02}\|_{L^{q_0}(\Omega)}\right). \end{aligned} \tag{4.3.13}$$

Proof We first prove (4.3.12) by following a similar argument that we employed in the proof of Lemma 3.1.5 (see (3.1.16)) by viewing $c(x, t) := \lambda - bv$, $f(x, t, u) = c(x, t) u$, with $q_1 := q_0$, $q_2 := \infty$. To this end, let $T \in (0, 1)$, $0 \leq t \leq T$ and recall the uniform estimates (4.3.8), (4.3.11), which imply that

$$|||c|||_{q_0,0,T} \leq C\,|||1 + v|||_{q_0,0,T} \leq N_1, \quad ||||u||||_{p,\delta_s,T} \leq N_2. \tag{4.3.14}$$

Then let $s_0 \in [1, \infty)$ be such that

$$\delta_s \leq \frac{1}{s_0} \text{ and } \frac{\mathfrak{n}_\mathfrak{s}}{s_0} + \delta_s + \varepsilon < \alpha + \frac{\mathfrak{n}}{p_0},$$

for a sufficiently small $\varepsilon \in (0, \alpha]$ such that $\varepsilon + \delta_s \leq \alpha$. We subsequently apply the statement of Lemma A.0.1 with the choices $p := p_0$, $s_1 := s_0$, $s_2 := \infty$, $\theta := \delta_s$,

$\delta := 0$ and $\varepsilon := \varepsilon$ (note again that $r(\tau) \equiv \|c(\cdot,\tau)\|_{L^{q_0}(\mathcal{X})}$ and $p_{s_2}(r) = \|c\|_{q,\infty}$). Once again if $s_0 \geq p_0$ is arbitrary we have that $\mathfrak{n}_{\mathfrak{s}}/s_0 - \mathfrak{n}_{\mathfrak{s}}/p_0 \in [0, 2\alpha)$ is trivially satisfied, while if $s_0 < p_0$ one may choose s_0 sufficiently close to $p_0 \in [1,\infty)$ such that $1/s_0 < 2/\beta_{A_s} + 1/p_0$. Note that the assumptions of Lemma A.0.1 are satisfied with the above choices of $\delta, s_1, s_2, p, \varepsilon, \theta$, since $0 \leq \delta_s < \alpha < 1$ and $\varepsilon + \delta_s \leq \alpha$, and

$$\frac{\mathfrak{n}_{\mathfrak{s}}}{s_0} < \alpha + \frac{\mathfrak{n}_{\mathfrak{s}}}{p_0}.$$

Indeed, by virtue of Hölder's inequality, for all $t \in (0,T] \subset (0,1)$ we have

$$\begin{aligned}
&\left\|u(\cdot,t) - S_{u,\alpha}u_0\right\|_{L^{p_0}(\Omega)} &(4.3.15)\\
&\leq \left(\int_0^t \left\|P_{u,\alpha}(t-\tau)\right\|_{p_0,s_0}(\tau\wedge 1)^{-\delta_s}\|\lambda - bv\|_{L^{q_0}(\mathcal{X})}\,d\tau\right)|||u|||_{p,\delta_s,T}\\
&\leq C\,|||1+v|||_{q_0,0,T}\,t^{\varepsilon}\,||||u||||_{p,\delta_s,T}\\
&\leq CN_1N_2t^{\varepsilon}
\end{aligned}$$

owing once again to (4.3.14). This gives the first of the assertion (4.3.12). For the second estimate, by (4.3.7) we have for every $1 \leq q \leq q_0$,

$$\begin{aligned}
\|v(t) - v_0\|_{L^q(\Omega)} &\leq \int_0^t g_\beta(t-\tau)\,\|(-cv+au)(\tau)\|_{L^q(\Omega)}\,d\tau &(4.3.16)\\
&\leq C\left(||||v||||_{q_0,0,T} + ||||u||||_{p_0,0,T}\right)\int_0^t g_\beta(t-\tau)\,d\tau\\
&\leq C(N_1+N_2)\,t^{\beta},
\end{aligned}$$

for all $0 \leq t \leq T < 1$.

Next, we prove the continuous dependence estimate (4.3.13). By virtue of (4.3.14), from (4.2.27) we have the uniform bound

$$\Lambda(u_i,v_i) \leq |\|u_1-u_2\||_{p_0,0,t}(1+N_1) + (1+N_2)\,|\|v_1-v_2\||_{q_0,0,t}$$

so that the same argument exploited in (4.3.15) in the integral formulation (4.3.6) for the difference u, yields for $t \in (0,1)$,

$$||||u_1-u_2||||_{p_0,0,t} \leq C\,\|u_{01}-u_{02}\|_{L^{p_0}(\Omega)} + Ct^{\varepsilon}\,||||v_1-v_2||||_{q_0,0,t}\,. \qquad (4.3.17)$$

By (4.3.7), we obtain as in (4.3.16), for $q \leq q_0 \leq p_0$, that

$$\|v_1(t) - v_2(t)\|_{L^q(\Omega)} \qquad (4.3.18)$$

$$\leq \|v_{01} - v_{02}\|_{L^q(\Omega)} + \int_0^t g_\beta (t-\tau) \|(-cv+au)(\tau)\|_{L^q(\Omega)} \, d\tau$$
$$\leq \|v_{01} - v_{02}\|_{L^q(\Omega)} + Ct^\beta \left(||||v_1 - v_2||||_{q_0,0,t} + ||||u_1 - u_2||||_{p_0,0,t} \right).$$

Define $\rho := \min\{\varepsilon, \beta\} > 0$ and the function

$$\psi(t) := ||||v_1 - v_2||||_{q_0,0,t} + ||||u_1 - u_2||||_{p_0,0,t}.$$

By the estimates (4.3.17)–(4.3.18), for a sufficiently small $t < 1$, it holds

$$\psi(t) \leq C \left(\|u_{01} - u_{02}\|_{L^{p_0}(\Omega)} + \|v_{01} - v_{02}\|_{L^{q_0}(\Omega)} \right) + Ct^\rho \psi(t), \tag{4.3.19}$$

for some constant $C > 0$ independent of t. Further choose $t_0 \ll 1$ such that $Ct_0^\rho \leq 1/2$ and observe that (4.3.19) also implies

$$\psi(t) \leq 2C \left(\|u_{01} - u_{02}\|_{L^{p_0}(\Omega)} + \|v_{01} - v_{02}\|_{L^{q_0}(\Omega)} \right), \tag{4.3.20}$$

for all $t \in (0, t_0]$. Finally, we can employ (4.3.20) successively with initial data $(u, v)(t + it_0)$, for $i = 0, 1, 2, \ldots$, since by (4.3.8) and (4.3.11),

$$\sup_{i \in \mathbb{N}} \left(\|u(t + it_0)\|_{L^{p_0}(\Omega)} + \|v(t + it_0)\|_{L^{q_0}(\Omega)} \right) \leq N_3.$$

Indeed, for the same step size t_0, the assertion (4.3.20) yields the estimate

$$\sup_{t \in [t_0 + it_0, t_0 + (i+1)t_0]} \left(||||v_1 - v_2||||_{q_0,0,t} + ||||u_1 - u_2||||_{p_0,0,t} \right) \tag{4.3.21}$$
$$\leq C \left(\|(u_1 - u_2)(t_0 + it_0)\|_{L^{p_0}(\Omega)} + \|(v_1 - v_2)(t_0 + it_0)\|_{L^{q_0}(\Omega)} \right),$$

for all $i \in \{0, 1, 2, \ldots.\}$. Then the assertion (4.3.13) on the whole interval $(0, T)$ follows by an induction procedure on i, applied successively in (4.3.21). Thus, the proposition is proved. □

We conclude the section with the second result concerning the well-posed problem of mild solutions.

Theorem 4.3.7 *Assume $p_0 \in [1, \infty]$ such that $\beta_{A_s}/p_0 < 1$ and $q_0 \in [1, p_0] \cap (\beta_{A_s}, \infty]$, and let $0 \leq u_0 \in L^{p_0}(\Omega)$, $0 \leq v \in L^{q_0}(\Omega)$ be such that $u_0 \geq 0$ and $v_0 \geq 0$ a.e. on Ω. If $p_0 = \infty$, in addition assume $u_0 \in \mathcal{L}_s^\infty(\Omega)$. Then the fractional system (4.3.1)–(4.3.2) has a unique global mild solution $u \geq 0, v \geq 0$ on the time interval $[0, \infty)$, given by (4.3.6)–(4.3.7), which hold as absolutely convergent Bochner integrals in $L^1(\Omega)$. Moreover, the pair (u, v) satisfies*

$$\lim_{t \to 0^+} \|u(t) - u_0\|_{L^{p_0}(\Omega)} = 0, \quad \lim_{t \to 0^+} \|v(t) - v_0\|_{L^{q_0}(\Omega)} = 0 \tag{4.3.22}$$

and the uniform estimates stated in Propositions 4.3.3 and 4.3.5.

Proof The proof follows by a standard approximation procedure. The initial datum $(u_0, v_0) \in L^{p_0}(\Omega) \times L^{q_0}(\Omega)$ can be approximated by a convenient sequence of regular initial data (u_{0n}, v_{0n}), according to the statement of Theorem 4.3.2. In particular, this sequence may be chosen such that

$$\lim_{n\to\infty} \left[\|u_{0n} - u_0\|_{L^{p_0}(\Omega)} + \|v_{0n} - v_0\|_{L^{q_0}(\Omega)} \right] = 0 \tag{4.3.23}$$

and

$$\|u_{0n}\|_{L^{p_0}(\Omega)} \leq \|u_0\|_{L^{p_0}(\Omega)}, \quad \|v_{0n}\|_{L^{q_0}(\Omega)} \leq \|v_0\|_{L^{q_0}(\Omega)}, \tag{4.3.24}$$

for all $n \in \mathbb{N}$. Let now (u_n, v_n) be the global strong solution for an initial datum (u_{0n}, v_{0n}). All the constants occurring in Propositions 4.3.3–4.3.5 can be chosen independent of $n \in \mathbb{N}$, owing to (4.3.24). Furthermore, the assertion (4.3.13) of Proposition 4.3.6, together with (4.3.23), implies that the sequence (u_n, v_n) converges to $(u, v) \in E_{p_0,\delta_s,T} \times E_{q_0,0,T}$, in the sense that

$$||||v_n - v||||_{q_0,0,t} + ||||u_n - u||||_{p_0,0,t} \to 0, \text{ as } n \to \infty,$$

for all $t \in (0, T)$. Besides, all the estimates of Propositions 4.3.3–4.3.5 hold for the limit solution (u, v) as well. By the same arguments as in the proof of Lemma 3.1.5, it is now straightforward to show that (u, v) is indeed the mild solution of the system (4.3.1)–(4.3.2), for any initial datum (u_0, v_0). The conclusion (4.3.22) is also a consequence of Proposition 4.3.6 and Remark 3.1.2. □

Corollary 4.3.8 *The mild solution (u, v) of (4.3.1)–(4.3.2) is also regularizing in the sense that its first component u becomes a global strong solution on $[T_0, \infty)$, for every $T_0 > 0$, as well as, the second component $v \in L^\infty([T_0, \infty); L^\infty(\Omega))$.*

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