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Operator Algebras, Toeplitz Operators and Related Topics

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Nikolai Vasilevski. *Credit*: Photo taken from the archive of Larisa Vasilevskaya

Preface

The aim of this volume is to present new results in the theory of Toeplitz operators, algebras of Toeplitz operators, and its applications. In particular, the book is devoted to the topic of boundedness and compactness of Toeplitz operators in p-Fock spaces and to the study of commutative C^*-algebras of Toeplitz operators in various spaces and over various multidimensional domains. Moreover, new results in the theory of Dirac operators, in the theory of algebras of singular integral operators on general Lebesgue spaces, and in the field of pseudodifferential arithmetic are presented. A number of related topics are discussed too. In total, 20 research papers are included.

The volume is dedicated to professor Nikolai Vasilevski on the occasion of his 70th birthday, and it begins with personal notes written by Wolfram Bauer, Raul Quiroga-Barranco, and Grigori Rozenblum covering the activities of Nikolai Vasilevski over the past 10–15 years. This introductory part ends with an article by Sergei Grudsky, Yuri Latushkin, and Michael Shapiro, which appeared earlier on the occasion of the 60th birthday of professor Vasilevski and is devoted to the mathematical achievements and the path of the life of our hero.

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Contents

Part I
Biographical Material

The Life and Work of Nikolai Vasilevski

Sergei Grudsky, Yuri Latushkin, and Michael Shapiro

Nikolai Leonidovich Vasilevski was born on January 21, 1948 in Odessa, Ukraine. His father, Leonid Semenovich Vasilevski, was a lecturer at Odessa Institute of Civil Engineering, his mother, Maria Nikolaevna Krivtsova, was a docent at the Department of Mathematics and Mechanics of Odessa State University.

In 1966 Nikolai graduated from Odessa High School Number 116, a school with special emphasis in mathematics and physics, that made a big impact at his creative and active attitude not only to mathematics, but to life in general. It was a very selective high school accepting talented children from all over the city, and famous for a high quality selection of teachers. A creative, nonstandard, and at the same time highly personal approach to teaching was combined at the school with a demanding attitude towards students. His mathematics instructor at the high school was Tatjana Aleksandrovna Shevchenko, a talented and dedicated teacher. The school was also famous because of its quite unusual by Soviet standards system of self-government by the students. Quite a few graduates of the school later became well-known scientists, and really creative researchers.

The article consists of the main text of the paper Sergei Grudsky, Yuri Latushkin, Michael Shapiro. The life and work of Nikolai Vasilevski. Operator Theory: Advances and Applications, 210 (2010).

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In 1966 Nikolai became a student at the Department of Mathematics and Mechanics of Odessa State University. Already at the third year of studies, he began his serious mathematical work under the supervision of the well known Soviet mathematician Georgiy Semenovich Litvinchuk. Litvinchuk was a gifted teacher and scientific adviser. He, as anyone else, was capable of fascinating his students by new problems which have been always interesting and up-to-date. The weekly Odessa seminar on boundary value problems, chaired by Prof. Litvinchuk for more than 25 years, very much influenced Nikolai Vasilevski as well as others students of G. S. Litvinchuk.

N. Vasilevski started to work on the problem of developing the Fredholm theory for a class of integral operators with nonintegrable integral kernels. In essence, the integral kernel was the Cauchy kernel multiplied by a logarithmic factor. The integral operators of this type lie between the singular integral operators and the integral operators whose kernels have weak (integrable) singularities. A famous Soviet mathematician F. D. Gakhov posted this problem in early 1950-th, and it remained open for more than 20 years. Nikolai managed to provide a complete solution in the setting which was much more general than the original. Working on this problem, Nikolai has demonstrated one of the main traits of his mathematical talent: his ability to achieve a deep penetration in the core of the problem, and to see rather unexpected connections between different theories. For instance, in order to solve Gakhov's Problem, Nikolai utilized the theory of singular integral operators with coefficients having discontinuities of first kind, and the theory of operators whose integral kernels have fixed singularities—both theories just appeared at that time. The success of the young mathematician was well recognized by a broad circle of experts working in the area of boundary value problems and operator theory. In 1971 Nikolai was awarded the prestigious M. Ostrovskii Prize, given to the young Ukrainian scientists for the best research work. Due to his solution of the famous problem, Nikolai quickly entered the mathematical community, and became known to many prominent mathematicians of that time. In particular, he was very much influenced by the his regular interactions with such outstanding mathematicians as M. G. Krein and S. G. Mikhlin.

In 1973 N. Vasilevski defended his PhD thesis entitled "To the Noether theory of a class of integral operators with polar-logarithmic kernels". In the same year he became an Assistant Professor at the Department of Mathematica and Mechanics of Odessa State University, where he was later promoted to the rank of Associate Professor, and, in 1989, to the rank of Full Professor.

Having received the degree, Nikolai continued his active mathematical work. Soon, he displayed yet another side of his talent in approaching mathematical problems: his vision and ability to use general algebraic structures in operator theory, which, on one side, simplify the problem, and, on another, can be used in many other problems. We will briefly describe two examples of this.

The first example is the method of orthogonal projections. In 1979, studying the algebra of operators generated by the Bergman projection, and by the operators of multiplication by piece-wise continuous functions, N. Vasilevski gave a description of the C^*-algebra generated by two selfadjoint elements s and n satisfying the properties $s^2 + n^2 = \mathrm{e}$ and $sn + ns = 0$. A simple substitution $p = (\mathrm{e} + s - n)/2$ and $q = (\mathrm{e} - s - n)/2$ shows that this algebra is also generated by two selfadjoint idempotents (orthogonal projections) p and q (and the identity element e). During the last quarter of the past century, the latter algebra has been rediscovered by many authors all over the world. Among all algebras generated by orthogonal projections, the algebra generated by two projections is the only tame algebra (excluding the trivial case of the algebra with identity generated by one orthogonal projection). All algebras generated by three or more orthogonal projections are known to be wild, even when the projections satisfy some additional constrains. Many model algebras arising in operator theory are generated by orthogonal projections, and thus any information of their structure essentially broadens the set of operator algebras admitting a reasonable description. In particular, two and more orthogonal projections naturally appear in the study of various algebras generated by the Bergman projection and by piece-wise continuous functions having two or more different limiting values at a point. Although these projections, say, P, $Q_1, \ldots, Q_n$, satisfy an extra condition $Q_1 + \ldots + Q_n = I$, they still generate, in general, a wild C^*-algebra. At the same time, it was shown that the structure of the algebra just mentioned is determined by the joint properties of certain positive injective contractions C_k, $k = 1, \ldots, n$, satisfying the identity $\sum_{k=1}^{n} C_k = I$, and, therefore, the structure is determined by the structure of the C^*-algebra generated by the contractions. The principal difference between the case of two projections and the general case of a finite set of projections is now completely clear: for $n = 2$ (and the projections P and $Q + (I - Q) = I$) we have only one contraction, and the spectral theorem directly leads to the desired description of the algebra. For $n \geq 2$ we have to deal with the C^*-algebra generated by a finite set of noncommuting positive injective contractions, which is a wild problem. Fortunately, for many important cases related to concrete operator algebras, these projections have yet another special property: the operators $PQ_1P, \ldots, PQ_nP$ mutually commute. This property makes the respective algebra tame, and thus it has a nice and simple description as the algebra of all $n \times n$ matrix valued functions that are continuous on the joint spectrum Δ of the operators $PQ_1P, \ldots, PQ_nP$, and have certain degeneration on the boundary of Δ.

Another notable example of the algebraic structures used and developed by N. Vasilevski is his version of the Local Principle. The notion of locally equivalent operators, and localization theory were introduced and developed by I. Simonenko in mid-sixtieth. According to the tradition of that time, the theory was focused on the study of individual operators, and on the reduction of the Fredholm properties of an operator to local invertibility. Later, different versions of the local principle have been elaborated by many authors, including, among others, G. R. Allan, R. Douglas, I. Ts. Gohberg and N. Ia. Krupnik, A. Kozak, B. Silbermann. In spite of the fact that many of these versions are formulated in terms of Banach- or C^*-algebras, the main result, as before, reduces invertibility (or the Fredholm property) to local invertibility. On the other hand, at about the same time, several papers on the description of algebras and rings in terms of continuous sections were published by J. Dauns and K. H. Hofmann, M. J. Dupré, J. M. G. Fell, M. Takesaki and J. Tomiyama. These two directions have been developed independently, with no known links between the two series of papers. N. Vasilevski was the one who proposed a local principle which gives the global description of the algebra under study in terms of continuous sections of a certain canonically defined C^*-bundle. This approach is based on general constructions of J. Dauns and K. H. Hofmann, and results of J. Varela. The main contribution consists of a deep re-comprehension of the traditional approach to the local principles unifying the ideas coming from both directions mentioned above, which results in a canonical procedure that provides the global description of the algebra under consideration in terms of continuous sections of a C^*-bundle constructed by means of local algebras.

In the eighties and even later, the main direction of the work of Nikolai Vasilevski has been the study of multidimensional singular integral operators with discontinuous coefficients. The main philosophy here was to study first algebras containing these operators, thus providing a solid foundation for the study of various properties (in particular, the Fredholm property) of concrete operators. The main tool has been the described above version of the local principle. This principle was not merely used to reduce the Fredholm property to local invertibility but also for a global description of the algebra as a whole based on the description of the local algebras. Using this methodology, Nikolai Vasilevski obtained deep results in the theory of operators with Bergman's kernel and piece-wise continuous coefficients, in the theory of multidimensional Toeplitz operators with pseudodifferential presymbols, in the theory of multidimensional Bitsadze operators, in the theory of multidimensional operators with shift, etc. In 1988 N. Vasilevski defended the Doctor of Sciences dissertation, based on these results, and entitled "Multidimensional singular integral operators with discontinuous classical symbols".

Besides being a very active mathematician, N. Vasilevski has been an excellent lecturer. His lectures are always clear, and sparkling, and full of humor, which so natural for someone who grew up in Odessa, a city with a longstanding tradition of humor and fun. He was the first at Odessa State University who designed and started to teach a class in general topology. Students happily attended his lectures in Calculus, Real Analysis, Complex Analysis, Functional Analysis. He has been one of the most popular professor at the Department of Mathematics and Mechanics of Odessa State University. Nikolai is a master of presentations, and his colleagues always enjoy his talks at conferences and seminars.

In 1992 Nikolai Vasilevski moved to Mexico. He started his career there as an Investigator (Full Professor) at the Mathematics Department of CINVESTAV (Centro de Investagacion y de Estudios Avansados). His appointment significantly strengthen the department which is one of the leading mathematical centers in Mexico. His relocation also visibly revitalized mathematical activity in the country in the field of operator theory. Actively pursuing his own research agenda, Nikolai also served as the organizer of several important conferences. For instance, let us mention the (regular since 1998) annual workshop "Análisis Norte-Sur", and the well-known international conference IWOTA-2009. He initiated the relocation to

Mexico a number of active experts in operator theory such as Yu. Karlovich and S. Grudsky, among others.

During his tenure in Mexico, Nikolai Vasilevski produced a sizable group of students and younger colleagues; five of young mathematicians received PhD under his supervision.

The contribution of N. Vasilevski in the theory of multidimensional singular integral operators found its rather unexpected development in his work on quaternionic and Clifford analysis, published mainly with M. Shapiro in 1985–1995, starting still in the Soviet Union, with the subsequent continuation during the Mexican period of his life. Among others, the following topics have been considered: The settings for the Riemann boundary value problem for quaternionic functions that are taking into account both the noncommutative nature of quaternionic multiplication and the presence of a family of classes of hyperholomorphic functions, which adequately generalize the notion of holomorphic functions of one complex variable; algebras, generated by the singular integral operators with quaternionic Cauchy kernel and piece-wise continuous coefficients; operators with quaternion and Clifford Bergman kernels. The Toeplitz operators in quaternion and Clifford setting have been introduced and studied in the first time. This work found the most favorable response and initiated dozens of citations.

During his life in Mexico, the scientific interests of Nikolai Vasilevski mainly concentrated around the theory of Toeplitz operators on Bergman and Fock spaces. In the end of 1990-th, N. Vasilevski discovered a quite surprising phenomenon in the theory of Toeplitz operators on the Bergman space. Unexpectedly, there exists a rich family of commutative C^*-algebras generated by Toeplitz operators with non-trivial defining symbols. In 1995 B. Korenblum and K. Zhu proved that the Toeplitz operators with radial defining symbols acting on the Bergman space over the unit disk can be diagonalized with respect to the standard monomial basis in the Bergman space. The C^*-algebra generated by such Toeplitz operators is therefore obviously commutative. Four years later N. Vasilevski also showed the commutativity of the C^*-algebra generated by the Toeplitz operators acting on the Bergman space over the upper half-plane and with defining symbols depending only on $\operatorname{Im} z$. Furthermore, he discovered the existence of a rich family of commutative C^*-algebras of Toeplitz operators. Moreover, it turned out that the smoothness properties of the symbols do not play any role in commutativity: the symbols can be merely measurable. Surprisingly, everything is governed by the geometry of the underlying manifold, the unit disk equipped with the hyperbolic metric. The precise description of this phenomenon is as follows. Each pencil of hyperbolic geodesics determines the set of symbols which are constant on the corresponding cycles, the orthogonal trajectories to geodesics forming the pencil. The C^*-algebra generated by the Toeplitz operators with such defining symbols is commutative. An important feature of such algebras is that they remain commutative for the Toeplitz operators acting on each of the commonly considered weighted Bergman spaces. Moreover, assuming some natural conditions on “richness” of the classes of symbols, the following complete characterization has been obtained: A C^*-algebra generated by the Toeplitz operators is commutative on each weighted Bergman space if and only if the corresponding defining symbols are constant on cycles of some pencil of hyperbolic geodesics. Apart from its own beauty, this result reveals an extremely deep influence of the geometry of the underlying manifold on the properties of the Toeplitz operators over the manifold. In each of the mentioned above cases, when the algebra is commutative, a certain unitary operator has been constructed. It reduces the corresponding Toeplitz operators to certain multiplication operators, which also allows one to describe their representations of spectral type. This gives a powerful research tool for the subject, in particular, yielding direct access to the majority of the important properties such as boundedness, compactness, spectral properties, invariant subspaces, of the Toeplitz operators under study.

The results of the research in this directions became a part of the monograph "Commutative Algebras of Toeplitz Operators on the Bergman Space" published by N. Vasilevski in Birkhauser in 2008.

Nikolai Leonidovich Vasilevski passed his sixties birthday on full speed, and being in excellent shape. We, his friends, students, and colleagues, wish him further success and, above all, many new interesting and successfully solved problems.

Principal Publications of Nikolai Vasilevski

Book

1. N. L. Vasilevski. *Commutative Algebras of Toeplitz Operators on the Bergman Space*, Operator Theory: Advances and Applications, Vol. 183, Birkhäuser Verlag, Basel-Boston-Berlin, 2008, XXIX, 417 p.

Articles

1. N. L. Vasilevski. *On the Noether conditions and a formula for the index of a class of integral operators*. Doklady Akad. Nauk SSSR, 1972, v. 202, N° 4, p. 747–750 (Russian). English translation: Soviet Math. Dokl., v. 13, no. 1, 1972, p. 175–179.
2. N. L. Vasilevski. *On properties of a class of integral operators in the space* L_p. Matemat. Zametki, 1974, v. 16, N° 4, p. 529–535 (Russian). English translation: Math. Notes, v. 16, no. 4, 1974, p. 905–909.
3. N. L. Vasilevski. *The Noether theory of a class of potential type integral operators*. Izvestija VUZov. Matematika, 1974, N° 7, p. 12–20 (Russian). English translation: Soviet Math. (Izv. VUZ), v. 18, no. 7, 1974, p. 8–15.
4. N. L. Vasilevski. *On the Noetherian theory of integral operators with a polar logarithmic kernel*. Doklady Akad. Nauk SSSR, 1974, v. 215, N° 3, p. 514–517 (Russian). English translation: Soviet Math. Dokl., v. 15, no. 2, 1974, p. 522–527.
5. N. L. Vasilevski, E. V. Gutnikov. *On the symbol of operators forming finite-dimensional algebras.* Doklady Akad. Nauk SSSR, 1975, v. 221, N° 1, p. 18–21 (Russian). English translation: Soviet Math. Dokl., v. 16, no. 2, 1975, p. 271–275.
6. N. L. Vasilevski, G. S. Litvinchuk. *Theory of solvability of a class of singular integral equations with involution*. Doklady Akad. Nauk SSSR, 1975, v. 221, N° 2, p. 269–271 (Russian). English translation: Soviet Math. Dokl., v. 16, no. 2, 1975, p. 318–321.
7. N. L. Vasilevski, M. V. Shapiro. *On an algebra generated by singular integral operators with the Carleman shift and in the case of piece-wise continuous coefficients*. Ukrainski Matematicheski Zurnal, 1975, v. 27, N° 2, p. 216–223 (Russian). English translation: Ukrainian Math. J., v. 27, no. 2, 1975, p. 171–176.

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https://doi.org/10.1007/978-3-030-44651-2_2

8. N. L. Vasilevski. *On a class of singular integral operators with kernels of polar-logarithmic type*. Izvestija Akad. Nauk SSSR, ser. matem., 1976, v. 40, N° 1, p. 131–151 (Russian). English translation: Math. USSR Izvestija, v. 10, no. 1, 1976, p. 127–143.
9. N. L. Vasilevski, E. V. Gutnikov. *On the structure of the symbol of operators forming finite-dimensional algebras*. Doklady Akad. Nauk SSSR, 1976, v. 230, N° 1, p. 11–14 (Russian). English translation: Soviet Math. Dokl., v. 17, no. 5, 1976, p. 1225–1229.
10. N. L. Vasilevski, A. A. Karelin, P. V. Kerekesha, G. S. Litvinchuk. *On a class of singular integral equations with shift and its applications in the theory of boundary value problems for partial differential equations.* I. Differentsialnye Uravnenija, 1977, v. 13, N° 9, p. 1692–1700 (Russian). English translation: Diff. Equations, v. 13, no. 9, 1977, p. 1180–1185.
11. N. L. Vasilevski. A. A. Karelin, P. V. Kerekesha, G. S. Litvinchuk. *On a class of singular integral equations with shift and its applications in the theory of boundary value problems for partial differential equations*. II. Differentsialnye Uravnenija, 1977, v. 13, N° 11, p. 2051–2062 (Russian). English translation: Diff. Equations, v. 13, no. 11, 1977, p. 1430–1438.
12. N. L. Vasilevski. *Symbols of operator algebras*. Doklady Akad. Nauk SSSR, 1977, v. 235, N° 1, p. 15–18 (Russian). English translation: Soviet Math. Dokl., v. 18, no. 4, 1977, p. 872–876.
13. N. L. Vasilevski, A. A. Karelin. *An investigation of a boundary value problem for the partial differential equation of the mixed type with the help of reduction to the singular integral equation with Carleman shift*. Izvestija VUZov. Matematika, 1978, N° 3, p. 15–19, (Russian). English translation: Soviet Math. (Izv. VUZ), v. 22, no. 3, 1978, p. 11–15.
14. N. L. Vasilevski, R. Trujillo. *On Φ_R-operators in matrix algebras of operators*. Doklady Akad. Nauk SSSR, 1979, v. 245, N° 6, p. 1289–1292 (Russian). English translation: Soviet Math. Dokl., v. 20, no. 2, 1979, p. 406–409.
15. N. L. Vasilevski, R. Trujillo. *On the theory of Φ_R-operators in matrix algebras of operators*. Linejnye Operatory, Kishinev, 1980, p. 3–15 (Russian).
16. N. L. Vasilevski. *On an algebra generated by some two-dimensional integral operators with continuous coefficients in a subdomain of the unit disc*. Journal of Integral Equations, 1980, v. 2, p. 111–116.
17. N. L. Vasilevski. *Banach algebras generated by some two-dimensional integral operators*. I. Math. Nachr., 1980, b. 96, p. 245–255 (Russian).
18. N. L. Vasilevski. *Banach algebras generated by some two-dimensional integral operators.* II. Math. Nachr., 1980, b. 99, p. 136–144 (Russian).
19. N. L. Vasilevski. *On the symbol theory for Banach operator algebras which generalizes algebras of singular integral operators*. Differentsialnye Uravnennija, 1981, v. 17, N° 4, p. 678–688 (Russian). English translation: Diff. Equations, v. 17, no. 4, 1981, p. 462–469.
20. N. L. Vasilevski, I. M. Spitkovsky. *On an algebra generated by two projectors*. Doklady Akad. Nauk UkSSR, Ser. "A", 1981, N° 8, p. 10–13 (Russian).
21. N. L. Vasilevski. *On the algebra generated by two-dimensional integral operators with Bergman kernel and piece-wise continuous coefficients*. Doklady Akad. Nauk SSSR, 1983, v. 271, N° 5, p. 1041–1044 (Russian). English translation: Soviet Math. Dokl., v. 28, no. 1, 1983, p. 191–194.
22. N. L. Vasilevski. *On certain algebras generated by a space analog of the singular operator with Cauchy kernel*. Doklady Akad. Nauk SSSR, 1983, v. 273, N° 3, p. 521–524 (Russian). English translation: Soviet Math. Dokl., v. 28, no. 3, 1983, p. 654–657.
23. N. L. Vasilevski. *On an algebra generated by abstract singular operators and Carleman shift.* Soobshchenija Akad. Nauk GSSR, 1984, v. 115, N° 3, p. 473–476 (Russian).
24. N. L. Vasilevski. *On an algebra generated by multivariable Wiener-Hopf operators*. Reports of Enlarged Session of Seminars of the I. N. Vekua Institute of Applied Mathematics. Tbilisi, 1985, v. 1, p. 59–62 (Russian).
25. N. L. Vasilevski, M. V. Shapiro. *On an analogy of monogenity in the sense of Moisil-Teodoresko and some applications in the theory of boundary value problems*. Reports of Enlarged Sessions of Seminars of the I. N. Vekua Institute of Applied Mathematics. Tbilisi, 1985, v. 1, p. 63–66 (Russian).

26. N. L. Vasilevski. *Algebras generated by multivariable Toeplitz operators with piece-wise continuous presymbols*. Scientific Proceedings of the Boundary Value Problems Seminar Dedicated to 75-birthday of Academician BSSR Academy of Sciences F. D. Gahov. Minsk, 1985, p. 149–150 (Russian).
27. N. L. Vasilevski. *Two-dimensional Mikhlin-Calderon-Zygmund operators and bisingular operators*. Sibirski Matematicheski Zurnal, 1986, v. 27, N° 2, p. 23–31 (Russian). English translation: Siberian Math. J., v. 27, no. 2, 1986, p. 161–168.
28. N. L. Vasilevski. *Banach algebras generated by two-dimensional integral operators with Bergman Kernel and piece-wise continuous coefficients*. I. Izvestija VUZov, Matematika, 1986, N° 2, p. 12–21 (Russian). English translation: Soviet Math. (Izv. VUZ), v. 30, no. 2, 1986, p. 14–24.
29. N. L. Vasilevski. *Banach algebras generated by two-dimensional integral operators with Bergman Kernel and piece-wise continuous coefficients*. II. Izvestija VUZov, Matematika, 1986, N° 3, p. 33–38 (Russian). English translation: Soviet Math. (Izv. VUZ), v. 30, no. 3, 1986, p. 44–50.
30. N. L. Vasilevski. *Algebras generated by multidimensional singular integral operators and by coefficients admitting discontinuities of homogeneous type*. Matematicheski Sbornik, 1986, v. 129, N° 1, p. 3–19 (Russian). English translation: Math. USSR Sbornik, v. 57, no. 1, 1987, p. 1–19.
31. N. L. Vasilevski. *On an algebra generated by Toeplitz operators with zero-order pseudo-differential presymbols*. Doklady Akad. Nauk SSSR, 1986, v. 289, N° 1, p. 14–18 (Russian). English translation: Soviet Math. Dokl., v. 34, no. 1, 1987, p. 4–7.
32. N. L. Vasilevski, M. V. Shapiro. *On quaternion Ψ-monogenic function*. “Methods of solving of the direct and inverse geoelectrical problems”. 1987, p. 54–65 (Russian).
33. N. L. Vasilevski. *On an algebra connected with Toeplitz operators on the tube domains.* Izvestija Akad. Nauk SSSR, ser. matem., 1987, v. 51, N° 1, p. 79–95 (Russian). English translation: Math. USSR Izvestija, v. 30, no.1, 1988, p. 71–87.
34. N. L. Vasilevski, R. Trujillo. *On C^*-algebra generated by almost-periodic two-dimensional singular integral operators with discontinuous presymbols*. Funkcionalny Analiz i ego Prilogenija, 1987, v. 21, N° 3, p. 75–76 (Russian). English translation: Func. Analysis and its Appl., v. 21, no. 3, 1987, p. 235–236.
35. N. L. Vasilevski. *Toeplitz operators associated with the Siegel domains*. Matematicki Vesnik, 1988, v. 40, p. 349–354.
36. N. L. Vasilevski. *Hardy spaces associated with the Siegel domains*. Reports of Enlarged Sessions of Seminars of the I. N. Vekua Institute of Applied Mathematics. Tbilisi, 1988, v. 3, N° 1, p. 48–51 (Russian).
37. N. L. Vasilevski, M. V. Shapiro. *Holomorphy, hyperholomorphy Toeplitz operators*. Uspehi Matematicheskih Nauk, 1989, v. 44, N° 4 (268), p. 226–227 (Russian). English translation: Russian Math. Surveys, v. 44, no. 4, 1989, p. 196–197.
38. N. L. Vasilevski, M. V. Shapiro. *Some questions of hypercomplex analysis* “Complex Analysis and Applications ’87”, Sofia , 1989, p. 523–531.
39. N. L. Vasilevski. *Non-classical singular integral operators and algebras generated by them*. Integral Equations and Boundary Value Problems. World Scientific. 1991, p. 210–215.
40. M. V. Shapiro, N. L. Vasilevski. *Singular integral operator in Clifford analysis*, Clifford Algebras and Their Applications in Mathematical Physics, Kluwer Academic Publishers, Netherlands, 1992, p. 271–277.
41. N. L. Vasilevski. *On an algebra generated by abstract singular operators and a shift operator*. Math. Nachr., v. 162, 1993, p. 89–108.
42. R. M. Porter, M. V. Shapiro, N. L. Vasilevski. *On the analogue of the $\overline{\partial}$-problem in quaternionic analysis.* Clifford Algebras and Their Applications in Mathematical Physics, F. Brackx et al., eds, Kluwer Academic Publishers, Netherlands, 1993, p. 167–173.
43. E. Ramírez de Arellano, M. V. Shapiro, N. L. Vasilevski. *Hurwitz pairs and Clifford algebra representations.* Clifford Algebras and Their Applications in Mathematical Physics, F. Brackx et al., eds, Kluwer Academic Publishers, Netherlands, 1993, p. 175–181.

44. M. V. Shapiro, N. L. Vasilevski. *On the Bergman kernel function in the Clifford analysis.* Clifford Algebras and Their Applications in Mathematical Physics, F. Brackx et al., eds, Kluwer Academic Publishers, Netherlands, 1993, p. 183–192.
45. N. L. Vasilevski. *On "discontinuous" boundary value problems for pseudodifferential operators.* International Conference on Differential Equations, Vol. 1,2, (Barcelona, 1991), World Sci. Publishing, River Edge, NJ, 1993, p. 953–958.
46. N. L. Vasilevski, R. Trujillo. *Convolution operators on standard CR - manifolds. I. Structural Properties.* Integral Equations and Operator Theory, v. 19, no. 1, 1994, p. 65–107.
47. N. L. Vasilevski. *Convolution operators on standard CR - manifolds. II. Algebras of convolution operators on the Heisenberg group.* Integral Equations and Operator Theory, v. 19, no. 3, 1994, p. 327–348.
48. N. L. Vasilevski. *On an algebra generated by two-dimensional singular integral operators in plane domains.* Complex Variables, v. 26, 1994, p 79–91.
49. R. M. Porter, M. Shapiro, N. Vasilevski. *Quaternionic differential and integral operators and the $\overline{\partial}$-problem.* Journal of Natural Geometry, v. 6, no. 2, 1994, p. 101–124.
50. E. Ramírez de Arellano, M. V. Shapiro, N. L. Vasilevski. *Two types of analysis associated to the notion of Hurwitz pairs.* Differential Geometric Methods in Theoretical Physics, Ed. J. Keller, Z. Oziewich, Advances in Applied Clifford Algebras, v. 4 (S1), 1994, p. 413–422.
51. M. V. Shapiro, N. L. Vasilevski. *Quaternionic Ψ-hyperholomorphic functions, singular integral operators and boundary value problems. I. Ψ-hyperholomorphic function theory.* Complex Variables, v. 27, 1995, p. 17–46.
52. M. V. Shapiro, N. L. Vasilevski. *Quaternionic Ψ-hyperholomorphic functions, singular integral operators and boundary value problems. II. Algebras of singular integral operators and Riemann type boundary value problems.* Complex Variables, v. 27, 1995, p. 67–96.
53. N. Vasilevski, V. Kisil, E. Ramirez de Arellano, R. Trujillo. *Toeplitz operators with discontinuous presymbols on the Fock space.* Russian Math. Doklady, v. 345, no. 2, 1995, p. 153–155 (Russian). English translation: Russian Math. Doklady
54. E. Ramírez de Arellano, N. L. Vasilevski. *Toeplitz operators on the Fock space with presymbols discontinuous on a thick set,* Mathematische Nachrichten, v. 180, 1996, p. 299–315.
55. E. Ramírez de Arellano, N. L. Vasilevski. *Algebras of singular integral operators generated by three orthogonal projections,* Integral Equations and Operator Theory, v. 25, no. 3, 1996, p. 277–288.
56. N. Vasilevski, E. Ramirez de Arellano, M. Shapiro. *Hurwitz classical problem and associated function theory.* Russian Math. Doklady, v. 349, no. 5, 1996, p. 588–591 (Russian). English translation: Russian Math. Doklady
57. E. Ramírez de Arellano, M. V. Shapiro, N. L. Vasilevski. *The hyperholomorphic Bergman projector and its properties.* In: *Clifford Algebras and Related Topics,* J. Ryan, Ed. CRC Press, Chapter 19, 1996, p. 333–344.
58. E. Ramírez de Arellano, M. V. Shapiro, N. L. Vasilevski. *Hurwitz analysis: basic concepts and connection with Clifford analysis.* In: *Generalizations of Complex Analysis and their Applications in Physics,* J. Lawrynowicz, Ed. Banach Center Publications, V. 37, Warszawa, 1996, p. 209–221.
59. M. V. Shapiro, N. L. Vasilevski. *On the Bergman kernel function in hyperholomorphic analysis,* Acta Applicandae Mathematicae, v. 46, 1997, p. 1–27.
60. E. Ramírez de Arellano, N. L. Vasilevski. *Bargmann projection, three-valued functions and corresponding Toeplitz operators,* Contemporary Mathematics, v. 212, 1998, p. 185–196.
61. N. L. Vasilevski. *C*-algebras generated by orthogonal projections and their applications.* Integral Equations and Operator Theory, v. 31, 1998, p. 113–132.
62. N. L. Vasilevski, M. V. Shapiro. *On the Bergman kern-function on quaternionic analysis.* Izvestiia VUZov, Matematika, no. 2, 1998, p. 84–88 (Russian). English translation: Russian Math. (Izvestiia VUZ), v. 42, no. 2, 1998, p. 81–85.
63. N. L. Vasilevski, *On the structure of Bergman and poly-Bergman spaces,* Integral Equations and Operator Theory, v. 33, 1999, p. 471–488.

64. N. L. Vasilevski, *On Bergman-Toeplitz operators with commutative symbol algebras,* Integral Equations and Operator Theory, v. 34, no. 1, 1999, p. 107–126.
65. E. Ramírez de Arellano, M. V. Shapiro, N. L. Vasilevski. *The hyperholomorphic Bergman projector and its properties.* In: *Clifford Algebras and Related Topics,* J. Ryan, Ed. CRC Press, Chapter 19, 1996, p. 333–344.
66. E. Ramírez de Arellano, M. V. Shapiro, N. L. Vasilevski. *Hurwitz analysis: basic concepts and connection with Clifford analysis.* In: *Generalizations of Complex Analysis and their Applications in Physics,* J. Lawrynowicz, Ed. Banach Center Publications, V. 37, Warszawa, 1996, p. 209–221.
67. N. L. Vasilevski, *On quaternionic Bergman and poly-Bergman spaces,* Complex Variables, v. 41, 2000, p. 111–132.
68. V. S. Rabinovish, N. L. Vasilevski, *Bergman-Toeplitz and pseudodifferential operators,* Operator Theory. Advances and Applications v. 114, 2000, p. 207–234.
69. N. L. Vasilevski, *Poly-Fock Spaces,* Operator Theory. Advances and Applications v. 117, 2000, p. 371–386.
70. V. V. Kucherenko, N. L. Vasilevski, *A shift operator generated by a trigonometric system,* Mat. Zametki, v. 67, no. 4, 2000, p. 539–548 (Russian). English translation: Mat. Notes.
71. N. L. Vasilevski, *The Bergman space in tube domains, and commuting Toeplitz operators,* Doklady RAN, v. 372, no. 1, 2000, p. 9–12 (Russian). Doklady Mathematics, v. 61, no. 3, 2000.
72. N. L. Vasilevski. *Bergman space on tube domains and commuting Toeplitz operators.* In: *Proceedings of the Second ISAAC Congress, Volume 2,* H. G. W. Begehr et al. (eds.), Kluwer Academic Publishers, The Netherlands, Chapter 163, 2000, p. 1523–1537.
73. S. Grudsky, N. Vasilevski, *Bergman-Toeplitz operators: Radial component influence,* Integral Equations and Operator Theory, v. 40, no. 1, 2001, p. 16–33.
74. A. N. Karapetyants, V. S. Rabinovich, N. L. Vasilevski, *On algebras of two dimensional singular integral operators with homogeneous discontinuities in symbols,* Integral Equations and Operator Theory, v. 40, no. 3, 2001, p. 278–308.
75. N. L. Vasilevski. *Toeplitz Operators on the Bergman Spaces: Inside-the-Domain Effects*, Contemporary Mathematics, v. 289, 2001, p. 79–146.
76. N. L. Vasilevski. *Bergman spaces on the unit disk.* In: *Clifford Analysis and Its Applications* F. Brackx et al. (eds.), Kluwer Academic Publishers, The Netherlands, 2001, p. 399–409.
77. S. Grudsky, N. Vasilevski. *Toeplitz operators on the Fock space: Radial component effects,* Integral Equations and Operator Theory, v. 44, no. 1, 2002, p. 10–37.
78. N. L. Vasilevski. *Commutative algebras of Toeplitz operators and hyperbolic geometry.* In: *Proceedings of the Ukranian Mathematical Congress - 2001, Functional Analysis, Section 11*, Institute of Mathematics of the National Academy of Sciences, Ukraine, 2002, p. 22–35.
79. N. L. Vasilevski. *Bergman Space Structure, Commutative Algebras of Toeplitz Operators and Hyperbolic Geometry,* Integral Equations and Operator Theory, v. 46, 2003, p. 235–251.
80. S. Grudsky, A. Karapetyants, N. Vasilevski. *Toeplitz Operators on the Unit Ball in* $\mathbb{C}^n$ *with Radial Symbols,* J. Operator Theory, v. 49, 2003, p. 325–346.
81. J. Ramírez Ortega, N. Vasilevski, E. Ramírez de Arellano *On the algebra generated by the Bergman projection and a shift operator. I.* Integral Equations and Operator Theory, v. 46, no. 4, 2003, p. 455–471.
82. N. L. Vasilevski. *Toeplitz operators on the Bergman space.* In: *Factorization, Singular Operators and Related Problems*, Edited by S. Samko, A. Lebre, A. F. dos Santos, Kluwer Academic Publishers, 2003, p. 315–333.
83. J. Ramírez Ortega, E. Ramírez de Arellano, N. Vasilevski *On the algebra generated by the Bergman projection and a shift operator. II.* Bol. Soc. Mat. Mexicana (3), v. 10, 2004, p. 105–117.
84. S. Grudsky, A. Karapetyants, N. Vasilevski. *Dynamics of properties of Toeplitz operators on the upper half-plane: Hyperbolic case,* Bol. Soc. Mat. Mexicana (3), v. 10, 2004, p. 119–138.
85. S. Grudsky, A. Karapetyants, N. Vasilevski. *Dynamics of properties of Toeplitz operators on the upper half-plane: Parabolic case,* J. Operator Theory, v. 52, no. 1, 2004, p. 185–204.

86. S. Grudsky, A. Karapetyants, N. Vasilevski. *Dynamics of properties of Toeplitz operators with radial Symbols,* Integral Equations and Operator Theory, v. 50, no. 2, 2004, p. 217–253.
87. N. L. Vasilevski. *On a general local principle for C^*-algebras,* Izv.VUZ North-Caucasian Region, Natural Sciences, Special Issue, " Pseudodifferential operators and some problems of mathematical physics", 2005, p. 34–42 (Russian).
88. S. Grudsky, N. Vasilevski. *Dynamics of Spectra of Toeplitz Operators,* Advances in Analysis. Proceedings of the 4th International ISAAC Congress. (York University, Toronto, Canada 11–16 August 2003), World Scientific, New Jersey London Singapore, 2005, p. 495–504.
89. S. Grudsky, R. Quiroga-Barranco, N. Vasilevski. *Commutative C^*-algebras of Toeplitz operators and quantization on the unit disk,* J. Functional Analysis, v. 234, 2006, p. 1–44.
90. N. L. Vasilevski, S. M. Grudsky, A. N. Karapetyants. *Dynamics of properties of Toeplitz operators on weighted Bergman spaces,* Siberian Electronic Math. Reports, v. 3, 2006, p. 362–383 (Russian).
91. N. Vasilevski. *On the Toeplitz operators with piecewise continuous symbols on the Bergman space,* In: "Modern Operator Theory and Applications", Operator Theory: Advances and Applications, v. 170, 2007, p. 229–248.
92. N. Vasilevski. *Poly-Bergman spaces and two-dimensional singular integral operators,* Operator Theory: Advances and Applications, v. 171, 2007, p. 349–359.
93. N. Tarkhanov, N. Vasilevski. *Microlocal analysis of the Bochner-Martinelli integral,* Integral Equations and Operator Theory, v. 57, 2007, p. 583–592.
94. R. Quiroga-Barranco, N. Vasilevski. *Commutative algebras of Toeplitz operators on the Reinhardt domains*, Integral Equations and Operator Theory, v. 59, no. 1, 2007, p. 67–98.
95. R. Quiroga-Barranco, N. Vasilevski. *Commutative C^*-algebras of Toeplitz operators on the unit ball, I. Bargmann-type transforms and spectral representations of Toeplitz operators*, Integral Equations and Operator Theory, v. 59, no. 3, 2007, p. 379–419.
96. R. Quiroga-Barranco, N. Vasilevski. *Commutative C^*-algebras of Toeplitz operators on the unit ball, II. Geometry of the level sets of symbols*, Integral Equations and Operator Theory, v. 60, no. 1, 2008, p. 89–132.
97. N. Vasilevski. *Commutative algebras of Toeplitz operators and Berezin quantization*, Contemporary Mathematics, v. 462, 2008, p. 125–143.
98. S. Grudsky, N. Vasilevski. *On the structure of the C^*-algebra generated by Toeplitz operators with piece-wise continuous symbols*, Complex Analysis and Operator Theory, v. 2, no. 4, 2008, p. 525–548.
99. S. Grudsky, N. Vasilevski. *Anatomy of the C^*-algebra generated by Toeplitz operators with piece-wise continuous symbols*, Operator Theory: Advances and Applications, v. 190, 2009, p. 243–265.
100. N. Vasilevski. *Quasi-Radial Quasi-Homogeneous Symbols and Commutative Banach Algebras of Toeplitz Operators*, Integral Equations and Operator Theory, v. 66, no. 1, 2010, p. 141–152.
101. N. Vasilevski. *Parabolic Quasi-radial Quasi-homogeneous Symbols and Commutative Algebras of Toeplitz Operators*, Operator Theory: Advances and Applications, v. 202, 2010, p. 553–568.
102. N. L. Vasilevki, S. M. Grudsky, E. A. Maximenko. *Toeplitz operators on the Bergman space generated by radial symbols and slowly oscillating sequences*, Proceedings of the Scientific School of I. B. Simonenko, Rostov-on-Don, Russia, 2010, p. 38–43 (En russo).
103. Yu. I. Karlovich, V. S. Rabinovich, and N. L. Vasilevski. *Algebras of pseudodifferential operators with discontinuous oscillating symbols*, Commun. Math. Anal. 2011, Conference 3, 2011, 108–130.
104. N. Vasilevski. *On compactness of commutators and semi-commutators of Toeplitz operators on the Bergman space*, Commun. Math. Anal. 2011, Conference 3, 2011, 225–234.
105. W. Bauer, N. Vasilevski. *Commutative Toeplitz Banach algebras on the ball and quasi-nilpotent group action*, Integral Equations Operator Theory v. 72, no. 2, 2012, 223–240.

106. W. Bauer, N. Vasilevski. *Banach Algebras of Commuting Toeplitz Operators on the Unit Ball via the Quasi-hyperbolic Group*, Operator Theory: Advances and Applications, v. 218, 2012, p. 155–175.
107. W. Bauer, N. Vasilevski. *On the structure of a commutative Banach algebra generated by Toeplitz operators with quasi-radial quasi-homogeneous symbols*, Integral Equations and Operator Theory, v. 74, no.2, 2012, 199–231.
108. N. Vasilevski. *On the algebras generated by Toeplitz operators with piecewise continuous symbols*, Indag. Math. (N.S.) v. 23, no. 3, 2012, 556–570.
109. S. M. Grudsky, E. A. Maximenko, and N. L. Vasilevski, *Radial Toeplitz Operators on the Unit Ball and Slowly Oscillating Sequences*, Commun. Math. Anal., v. 14, No. 2, 2013, 77–94.
110. A. Sánchez-Nungaray, N. Vasilevski. *Toeplitz Operators on the Bergman Spaces with Pseudodifferential Defining Symbols*, Operator Theory: Advances and Applications, v. 228, 2013, p. 355–374.
111. C. Herrera Yañez, E. A. Maximenko, and N. Vasilevski. *Vertical Toeplitz Operators on the Upper Half-Plane and Very Slowly Oscillating Functions*, Integral Equations and Operator Theory, v. 77, no. 2, 2013, 149–166.
112. W. Bauer, N. Vasilevski. *On the structure of commutative Banach algebras generated by Toeplitz operators on the unit ball. Quasi-elliptic case. I: Generating subalgebras*, J. Functional Analysis, v. 265, 2013, p. 2956–2990.
113. W. Bauer, C. Herrera Yañez, N. Vasilevski. *Eigenvalue characterization of radial operators on weighted Bergman spaces over the unit ball*, Integral Equations Operator Theory, v. 78, no. 2, 2014, p. 271–300.
114. G. Rozenblum, N. Vasilevski. *Toeplitz operators defined by sesquilinear forms: Fock space case*, Journal of Functional Analysis, v. 267, 2014, p. 4399–4430.
115. W. Bauer, C. Herrera Yañez, N. Vasilevski. (m, λ)*-Berezin transform and approximation of operators on weighted Bergman spaces over the unit ball*, Operator Theory: Advances and Applications, v. 240, 2014, p. 45–68.
116. Alma García, Nikolai Vasilevski. *Quasi-radial Operators on the Weighted Bergman Space over the Unit Ball*, Commun. Math. Anal., v. 17, No. 2, 2014, p. 178–188.
117. Erick Lee, Maribel Loaiza, Nikolai Vasilevski. *On Toeplitz operators with piecewise continuous and slowly oscillating radial symbols*, Commun. Math. Anal., v. 17, No. 2, 2014, p. 231–238.
118. W. Bauer, N. Vasilevski. *On the Structure of Commutative Banach Algebras Generated by Toeplitz Operators on the Unit Ball. Quasi-Elliptic Case. II: Gelfand Theory*, Complex Analysis and Operator Theory, v. 9, no. 3, 2015, p. 593–630. https://doi.org/10.1007/s11785-014-0385-z, 38 p.
119. Alma García, Nikolai Vasilevski. *Toeplitz Operators on the Weighted Bergman Space over the Two-Dimensional Unit Ball*, Journal of Function Spaces, vol. 2015, Article ID 306168, 10 pages, 2015. https://doi.org/10.1155/2015/306168.
120. Crispin Herrera Yañez, Nikolai Vasilevski, Egor A. Maximenko. *Radial Toeplitz Operators Revisited: Discretization of the Vertical Case*, Integral Equations Operator Theory, v. 83, no. 1, 2015, p. 49–60.
121. Kevin Esmeral, Egor A. Maximenko, Nikolai Vasilevski. C^**-Algebra Generated by Angular Toeplitz Operators on the Weighted Bergman Spaces Over the Upper Half-Plane*, Integral Equations Operator Theory, v. 83, no. 3, 2015, p. 413–428.
122. G. Rozenblum, N. Vasilevski. *Toeplitz operators defined by sesquilinear forms: Bergman space case*, Journal of Mathematical Sciences, v. 213, no. 4, 2016, p. 582–609.
123. G. Rozenblum, N. Vasilevski. *Toeplitz Operators in the Herglotz Space*, Integral Equations and Operator Theory, v. 86, no. 3, 2016, p. 409–438.
124. W. Bauer, N. Vasilevski. *On algebras generated by Toeplitz operators and their representations*, J. Functional Analysis, v. 272, 2017, p. 705–737.
125. A. Sánchez-Nungaray, N. Vasilevski. *Commutative Algebras of Toeplitz Operators on a Siegel Domain Associated with the Nilpotent Group of Its Biholomorphisms*, Operator Theory: Advances and Applications, v. 258, 2017, p. 275–300.

126. N. Vasilevski. *On Toeplitz Operators with Quasi-radial and Pseudo-homogeneous Symbols*, In: Harmonic Analysis, Partial Differential Equations, Banach Spaces, and Operator Theory, Volume 2, 2017, p. 401–417.
127. W. Bauer, R. Hagger, N. Vasilevski. *Uniform continuity and quantization on bounded symmetric domains*, J. London Math. Soc., v. 96, Issue 2, 2017, p. 345–366.
128. G. Rozenblum, N. Vasilevski. *Toeplitz Operators via Sesquilinear Forms*, Operator Theory: Advances and Applications, v. 262, 2018, p. 287–304.
129. A. Snchez-Nungaray, N. Vasilevski. *Algebras of Toeplitz Operators on the Three-Dimensional Siegel Domain*, Integral Equations and Operator Theory, v. 90, no. 4, 2018, p. 1–42.
130. Nikolai Vasilevski. *On commutative C^*-algebras generated by Toeplitz operators with $\mathbb{T}^m$-invariant symbols*, In: Operator Theory: Advances and Applications, v. 271, 2018, p. 443–464.
131. W. Bauer, R. Hagger, N. Vasilevski, *Algebras of Toeplitz operators on the n-dimensional unit ball*, Complex Anal. Oper. Theory, v. 13, no. 2, 2019, p. 493–524.
132. G. Rozenblum, N. Vasilevski, *Toeplitz operators in polyanalytic Bergman type spaces*, Functional analysis and geometry: Selim Grigorievich Krein centennial, p. 273–290, Contemp. Math., 733, Amer. Math. Soc., Providence, RI, 2019.
133. K. Esmeral, G. Rozenblum, N. Vasilevski, *L-invariant Fock-Carleson type measures for derivatives of order k and the corresponding Toeplitz operators*, J. Math. Sci. (N.Y.) v. 242 no. 2, 2019, Problems in mathematical analysis. No. 99, p. 337–358.

Ph. D. Dissertations Directed by Nikolai Vasilevski

References

1. Rafael Trujillo, *Fredholm Theory of Tensor Product of Operator Algebras*, Odessa State University, 1986.
2. Zhelko Radulovich, *Algebras of Multidimensional Singular Integral Operators with Discontinuous Symbols with Respect to Dual Variable*, Odessa State University, 1991.
3. Vladimir Kisil, *Algebras of Pseudodifferential Operators Associated with the Heisenberg Group*, Odessa State University, 1992.
4. Josué Ramírez Ortega, *Algebra generada por la proyección de Bergman y un operador de translación*, CINVESTAV del I.P.N., Mexico City, 1999.
5. Maribel Loaiza Leyva, *Algebra generada por la proyección de Bergman y por los operadores de multiplicación por funciones continuas a trozos*, CINVESTAV del I.P.N., Mexico City, 2000.
6. Ernesto Prieto Sanabría, *Operadores de Toeplitz en la 2-esfera en los espacios de Bergman con peso*, CINVESTAV del I.P.N., Mexico City, 2007.
7. Armando Sánchez Nungaray, *Super operadores de Toeplitz en la dos-esfera*, CINVESTAV del I.P.N., Mexico City, 2008.
8. Carlos Moreno Muñoz, *Operadores de Toeplitz en el espacio de Bergman con peso: Caso parabólico*, CINVESTAV del I.P.N., Mexico City, 2009.
9. Iliana Cristina Carrillo Ibarra, *C^*-algebra ganerada por operadores singulares integrales cuaterniónicos y funciones continuas a trozos*, INVESTAV del I.P.N., Mexico City, 2012
10. Kevin Michael Esmeral Garcia, *Characterization of some model commutative C^*-algebras generated by Toeplitz operators*, CINVESTAV del I.P.N., Mexico City, 2015
11. Breitner A. Ocampo Gomez, *C^*-algebras generadas por operadores de Toeplitz en el espacio de Bergman con ciertas classes de simbolos doscontinuos*, CINVESTAV del I.P.N., Mexico City, 2015
12. Crispin Herrera Yañez, *Operadores de Toeplitz radiales y verticales*, CINVESTAV del I.P.N., Mexico City, 2016
13. Alma Itzel García Salas, *Algebras generadas por operadores de Toeplitz con simbolos radiales psuedo-homogeneos*, CINVESTAV del I.P.N., Mexico City, 2017.

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The Mathematician Nikolai Vasilevski

Wolfram Bauer

Nikolai Vasilevski's scientific productivity and his influential ideas are reflected by his many fundamental contributions to mathematics. In particular, this concerns the field of Toeplitz and integral operators or operator algebras. Taking a look at his achievements in different aspects of a "mathematical life" is rather impressive. Let me just name a few: Nikolai has published more than 130 scientific papers with more than 25 coauthors. He is a member of the Editorial Board of 5 journals and has supervised over 20 Master and 10 Ph.D. students. He has been organizer of more than 15 workshops and conferences. During his time at CINVESTAV, Mexico, he has been—and still is—very influential on his students and the mathematical community. Many researches in the field have studied Nikolai's papers and his results had an impact on their works and approaches to mathematics. Due to the expertise and guidance of Nikolai and his colleagues a "Mexican school on Toeplitz operators" has emerged during the last years and is rather visible.

I met Nikolai for the first time in 2005 at a special session on *Toeplitz and Toeplitz like structures* at the Fifth ISAAC Congress in Catania, Italy. It was the beginning of a close collaboration which lasts until today. Since our first projects, I acknowledged and admired Nikolais deep insight into mathematics and his rigorous way of thinking. To my impression all his approaches are based on a clear mathematical philosophy. Even in case of a concrete and specific problem his way of thinking is lead by more general principles, which allow him to keep track of the relevant questions. In combination with his wide mathematical knowledge this makes it a joy and a challenge to work with Nikolai. Another aspect which I would like to mention is his ability to hold the mathematical community together and pass his enthusiasm for mathematics to the next generation. Typically, he has been organizer

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of a special session on Toeplitz operators or related topics at IWOTA. These events support the interaction between different worldwide groups working in the field of operator theory. In particular, they provide a valuable platform for the young researchers to share their ideas and to connect to the established experts.

I value Nikolai not only as a mathematician but also as a person. He has an open and friendly personality and a great sense of humor, which one can also sense during his well structured talks.

Dear Nikolai, congratulations to your 70th birthday. I wish you and your family all the best. Surely, the mathematical journey will go on. I am convinced that your endless curiosity combined with your intellectual capacity will lead to many new and fundamental discoveries.

The Mathematics of Nikolai Vasilevski

Raúl Quiroga-Barranco

Nikolai Vasilevski has been an inspiring source for many of us. I will try to resume the influence Nikolai has had on my own work.

I first met Nikolai Vasilevski as a fellow researcher in Cinvestav, Mexico City. Nikolai was already a well known and established mathematician. I had just arrived at Cinvestav to start my own career. I consider myself a Lie group theorist focusing on smooth actions on geometric manifolds. In particular, this involves everything related to the geometry of Riemannian symmetric spaces.

It was probably around 2004 that Nikolai approached me looking for a solution to a geometric problem that caught his attention. This problem can be formulated as follows. Suppose there is a pencil of hyperbolic geodesics on the unit disk $\mathbb{D}$, can you describe the structure of the pencil? A pencil is simply a family of curves that partitions (in this case) the unit disk $\mathbb{D}$ so that it is locally equivalent to the partition of $\mathbb{R}^2$ by horizontal lines. One allows to have finitely many singular points where the local description just described fails. There are three very well known examples of such pencils corresponding to the three types of Möbius transformations on the unit disk: elliptic, parabolic and hyperbolic. For a Möbius transformation of each one of these types one can consider a corresponding 1-parameter group $(g_t)_t$ of biholomorphisms. The orbits of one such $(g_t)_t$ is a pencil of curves, and by taking the orthogonal curves to these orbits we obtain a new pencil whose curves are hyperbolic geodesics in $\mathbb{D}$.

At that time, Nikolai and his collaborators Sergei Grudsky and Alexey Karapetyants had already discovered that the 1-parameter groups described above gave rise to commutative C^*-algebras generated by Toeplitz operators. More precisely, the family of symbols invariant by one of such subgroups provided

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Toeplitz operators generating a commutative C^*-algebra. After that, Nikolai and Sergei had conjectured that these ought to be the only possible commutative C^*-algebras generated by Toeplitz operators, under the additional assumption that the commutativity held for every weighted Bergman space on the unit disk $\mathbb{D}$. One also has to assume some condition ensuring that the C^*-algebras are large. Using Berezin quantization they had proved that, under such conditions, the symbols had to have the same level lines as well as the same gradient lines, and that the latter had to be hyperbolic geodesics. Hence, the classification of (non-trivial) commutative C^*-algebras generated by Toeplitz operators on the unit disk was reduced to the above mentioned problem on pencils of geodesics.

It turns out that one has to apply with further detail Berezin quantization to have more information and solve these problems. Once this was done, the three of us discovered that the common level sets of the symbols had constant curvature. Then, using differential geometry we proved that a geodesic pencil in $\mathbb{D}$, whose normal curves have constant curvature, is always given (as described above) by a 1-parameter subgroup of biholomorphisms. And this solved the original conjecture: a (non-trivial) family of symbols gives Toeplitz operators generating a commutative C^*-algebra if and only if the symbols are invariant under some 1-parameter subgroup of biholomorphisms.

The solution of this problem was a real breakthrough in the way Toeplitz operators are now studied. It pointed out the fact that the geometry of the domain and its biholomorphisms are the basis to understand Toeplitz operators on bounded symmetric domains.

Nikolai and I got down to work on similar results for higher dimensional irreducible bounded symmetric domains. More precisely, the problem was to construct commutative C^*-algebras generated by Toeplitz operators from symbols that have some invariance with respect to biholomorphisms for some domain other than the unit disk $\mathbb{D}$. Our first attempt was to consider the n-dimensional unit ball $\mathbb{B}^n$. The first problem we encountered was how to move from one-dimensional geometry to the n-dimensional case. At the moment, our best guess was to consider n-dimensional subgroups of biholomorphisms of $\mathbb{B}^n$, and we thought it was natural to ask them to be Abelian. Notice that this condition is trivial in the case $n = 1$ of the unit disk $\mathbb{D}$.

The group of biholomorphism of the unit ball $\mathbb{B}^n$ is realized by the special pseudo-unitary group $\mathrm{SU}(n, 1)$. Lie theorists had already classified all maximal Abelian subalgebras of the Lie algebra $\mathfrak{su}(n, 1)$, and this provides the classification of all connected maximal Abelian subgroups (MASG's) of $\mathrm{SU}(n, 1)$. There are exactly $n + 2$ conjugacy classes of MASG's in $\mathrm{SU}(n, 1)$. Nikolai and I proved that any subgroup H in one of such conjugacy classes gives rise to commutative C^*-algebras: the Toeplitz operators with H-invariant symbols commute with each other on every weighted Bergman space over $\mathbb{B}^n$.

In these results, we were able to provide very important and deep information, both geometric and analytic. On one hand, it turns out that the orbits of MASG's have constant curvature in the intrinsic (sectional) and extrinsic (second fundamental form) sense. The normal bundle to the orbits is integrable thus defining a foliation. This and the orbit foliation are both Lagrangian. On the other hand, the proof that the Toeplitz operators with H-invariant symbols commute with each other is obtained by simultaneously diagonalizing the operators. This was done through a Bargmann type transform, where some Fourier transforms (both discrete and continuous) are applied while restricting the values of the functions involved to the subgroup orbits.

The latter idea turns out be the key to understand the higher rank cases of bounded symmetric domains. It was this sort of computation that paved the way to the introduction of representation theory and its very powerful tools to find a plethora of commutative C^*-algebras generated by Toeplitz operators. We know now that multiplicity-free restrictions of the holomorphic discrete series yield, and in some cases characterize, commutative C^*-algebras generated by Toeplitz operators. Also, representation theory and the general Segal-Bargmann transform developed by Gestur Ólafsson and Bent Ørsted allow to describe the spectra of invariant Toeplitz operators.

This development, this surprising turn of events, has been possible thanks to Nikolai's mathematical curiosity, enthusiasm, and hard work. To my good friend Nikolai goes my appreciation and admiration.

Nikolai's Spaces

Grigori Rozenblum

Oh, it is a hard job to write an article for a jubilee of such a great person. Surely, it is not his first jubilee, and all the nice words have been already said, and all the proper jokes are already laughed at. Thus, about Nikolai, there are two excellent detailed reports [1, 2].

So, nothing can be added about early and earlier life and mathematics of Nikolai. And this means that almost nothing can be added at all—because his early life continues. Everyone knows him and he knows everyone. At least everyone of importance in Analysis. His mathematical creativity has developed even more since, with further expansions of the research field, deepening of understanding, and improvement of technical abilities. His students and descendants strive to be worthy of such a leader.

I will not write about mathematical details, since it is not interesting to write about the things I participated in, while I understand nothing of the numerous Nikolai's works I did not participate in.

So, just a few personal remarks. We could not decide when we met first. It might have happened in the Voronezh Winter School, when we were very young (WRONG! I was very young—Nikolai is still and forever young!), there was such a highly informal meeting of well advanced mathematicians with the youngest ones, with some dissident flavor, headed by the most brilliant Selim Krein. Probably, on the other hand, we met at a seminar in Lenigrad, in the same 70s, at an operator theory seminar headed by Birman and Solomyak. Hard to say now. But it happened somehow long-long ago. Having moved to the West, we continued meeting each other at some conferences, but, unexpectedly, it turned out that there exist some

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topics of common interest too. And thus we started working together. As usual, it all started with some great ideas that, surely, turned out to be wrong. I remember that on some of my visits to CINVESTAV we spent 2 or 3 weeks discussing hopelessly some *netlenka* (a non-translatable Russian expression meaning something like 'imperishable' or 'unforgettable' result) for Bergman-Toeplitz operators in the disk, and on the last day, during some of the final minutes of my visit, we found out that for the Fock space all obstacles disappear, and an important paper had arisen. (One must note that later the corresponding theory for the Bergman space has been established as well.) And so it went. I think that we published 7 or 8 joint papers, and our co-operation was almost conflict-free. Just two points where we keep quarreling. First: Does an operator act IN the space—or ON the space? Second. To denote the scalar product in a Hilbert space as $(.,.)$ or $\langle .,. \rangle$? A task for readers. Browse our joint papers and decide who of us had upper hand in these quarrels.

Among friends it is a bad idea to develop some feeling of envy. I, personally, do not envy Nikolai's mathematical achievements—you cannot envy the Sun that it shines that bright. But there is one thing that I envy him in so much. You cannot see it in his papers or books. But you can see it any time he talks to you, directly, or via Skype.

Guess!
What is it?
Well!
Yes!
His great moustache!
No one has anything comparable!!.

References

1. *The life and work of Nikolai Vasilevski* by S. Grudsky; Y. Latushkin; M. Shapiro, Recent trends in Toeplitz and pseudodifferential operators, 1–14, Oper. Theory Adv. Appl., **210**, Birkhauser Verlag, Basel, 2010.
2. *The Mathematical Life of Nikolai Vasilevski* by S. Grudsky, Morfismos, **16**, 1, 2012, 1–8 (https://www.morfismos.cinvestav.mx/Portals/morfismos/SiteDocs/Articulos/Vol16/No1/PARTE-4-Grudsky.pdf).

Part II
Papers

Non-geodesic Spherical Funk Transforms with One and Two Centers

M. Agranovsky and B. Rubin

Dedicated to Professor Nikolai Vasilevski on the occasion of his 70th anniversary

Abstract We study non-geodesic Funk-type transforms on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ associated with cross-sections of $\mathbb{S}^n$ by k-dimensional planes passing through an arbitrary fixed point inside the sphere. The main results include injectivity conditions for these transforms, inversion formulas, and connection with geodesic Funk transforms. We also show that, unlike the case of planes through a single common center, the integrals over spherical sections by planes through two distinct centers provide the corresponding reconstruction problem a unique solution.

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1 Introduction

Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$. Given a point a inside $\mathbb{S}^n$, we denote by $\mathrm{Gr}_a(n+1, k)$, $1 \le k \le n$, the Grassmann manifold of k-dimensional affine planes in $\mathbb{R}^{n+1}$ passing through a. The aim of the paper is to study injectivity of the generalized

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Funk transform

$$(F_a f)(\tau) = \int_{\mathbb{S}^n \cap \tau} f(x)\, d\sigma(x), \qquad \tau \in \mathrm{Gr}_a(n+1, k), \tag{1.1}$$

and obtain inversion formulas for F_a in suitable classes of functions.

The classical case $F_a = F_o$, when $a = o$ is the origin, goes back to the pioneering works by Funk [2, 3] ($n = 2$), which were inspired by Minkowski [10]. A generalization of the Funk transform F_o to arbitrary $1 \le k \le n$ is due to Helgason [8]; see also [9, 18, 20] and references therein. Operators of this kind play an important role in convex geometry, spherical tomography, and various branches of Analysis [4, 6, 7, 13, 14, 20].

The case when a differs from the origin is relatively new in modern literature, though Funk-type transforms on $\mathbb{S}^2$ for noncentral plane sections were considered by Gindikin et al. [7] in the framework of the kappa-operator theory. One should also mention non-geodesic Funk-type transforms studied by Palamodov [14, Section 5.2]. Inversion formulas for these transforms were obtained in terms of delta functions and differential forms. Operators (1.1) with $a \neq o$ are non-geodesic too, however, they differ from those in [14]. In particular, they are non-injective. Non-geodesic Funk-type transforms over subspheres of fixed radius were studied by the second co-author in [17], where the results fall into the scope of number theory.

The case $a \neq o$ in (1.1) with $k = n$ was considered by Salman; see [23] for $n = 2$ and [24] for any $n \ge 2$. To avoid non-uniqueness, he imposed restriction on the support of the functions that makes his operator different from ours. The stereographic projection method of [23, 24] makes it possible to reduce inversion of Salman's operator to the similar problem for a certain Radon-like transform over spheres in $\mathbb{R}^n$.

The next step was due to Quellmalz [15] for $n = 2$, who expressed F_a through the totally geodesic Funk transform F_o and thus explicitly inverted this operator on a certain subclass of continuous functions. If $a = o$ this subclass consists of even functions on $\mathbb{S}^n$. The results from [15] were generalized by Quellmalz [16] and Rubin [21] to any $n \ge 2$ with $k = n$. The paper [21] also contains an alternative inversion method for Salman's operators.

Our aim in the present article is twofold. First, we characterize the kernel (the null subspace) of F_a and the subclass of all continuous functions on which F_a is injective. We also obtain inversion formulas for F_a on that subclass for *any* $1 \le k \le n$ and thus generalize the corresponding results from [21].

Second, to achieve uniqueness in the reconstruction problem, we consider sections by planes through *two distinct centers*. To the best of our knowledge, this approach is new. We shall prove that for any pair of distinct points a and b inside the sphere, the kernels of the corresponding transforms F_a and F_b have trivial intersection. The latter means that, unlike the case of a single center, the collection of data from two distinct centers provides the reconstruction problem a unique solution. We also develop an analytic procedure of the reconstruction, that reduces to a certain dynamical system on $\mathbb{S}^n$.

The results of this paper extend to the case when the point a lies outside $\mathbb{S}^n$, and to arbitrary pairs of distinct centers a, b in $\mathbb{R}^{n+1}$. We plan to address these cases elsewhere.

Plan of the Paper Section 2 contains notation and necessary preliminaries related to Möbius-type automorphisms of the sphere. In Sect. 3 we describe the kernel of the operator F_a on continuous functions and characterize the subclass of functions on which F_a is injective. We also obtain an explicit inversion formula for F_a on that subclass. Section 4 deals with the system of two equations, $F_a f = g$, $F_b f = h$, corresponding to distinct centers a and b inside the sphere. Unlike the case of a single common center, such a system determines f uniquely and the function f can be reconstructed by a certain pointwise convergent series. Norm convergence of this series is studied in Sect. 5. It turns out that the series does not converge uniformly on the entire sphere $\mathbb{S}^n$ (only on some compact subsets of $\mathbb{S}^n$), however, it converges in the $L^p(\mathbb{S}^n)$-norm for all $1 \le p \le p_0$, $p_0 = n/(k-1)$, and this bound is sharp. In Sect. 6 we prove Theorem 3.1, which was formulated without proof in Sect. 3. This theorem plays a key role in the paper. It states that the shifted transform F_a is represented as $F_a = N_a F_o M_a$, where N_a and M_a are the suitable bijections and F_o is the classical Funk transform corresponding to $a = o$.

The main results are contained in Theorems 3.4, 4.2, 5.2, and 5.4.

2 Preliminaries

2.1 Notation

In the following, $\mathbb{B}^{n+1} = \{x \in \mathbb{R}^{n+1} : |x| < 1\}$ is the open unit ball in $\mathbb{R}^{n+1}$, $\mathbb{S}^n$ is its boundary, $x \cdot y$ is the usual dot product. The notation $C(\mathbb{S}^n)$ and $L^p(\mathbb{S}^n)$ for the corresponding spaces of continuous and L^p functions on $\mathbb{S}^n$ is standard. If x is the variable of integration over $\mathbb{S}^n$, then dx stands for the $O(n+1)$-invariant surface area measure on $\mathbb{S}^n$, so that $\int_{\mathbb{S}^n} dx = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$. We write $d\sigma(x)$ for the induced surface area measure on lower dimensional spherical sections. The letter x can be replaced by another one, depending on the context.

We denote by $\mathfrak{M}_{n,m}$ the space of real matrices having n rows and m columns; M' is the transpose of the matrix M, I_m is the identity $m \times m$ matrix. For $n \ge m$, $\mathrm{St}(n, m) = \{\mathrm{M} \in \mathfrak{M}_{n,m} : \mathrm{M}'\mathrm{M} = \mathrm{I}_m\}$ denotes the Stiefel manifold of orthonormal m-frames in $\mathbb{R}^n$; $\mathrm{Gr}_a(n, m)$ is the Grassmann manifold of m-dimensional affine planes in $\mathbb{R}^n$ passing through a fixed point a. We will be mainly dealing with the manifolds $\mathrm{St}(n+1, n+1-k)$, $\mathrm{Gr}_a(n+1, k)$, and $\mathrm{Gr}_o(n+1, k)$ (i.e. $a = o$), $1 \le k \le n$. Given a frame $\xi \in \mathrm{St}(n+1, n+1-k)$, the notation $\xi^\perp$ stands for the k-dimensional linear subspace orthogonal to ξ; $\{\xi\}$ denotes an $(n+1-k)$-dimensional linear subspace spanned by ξ. All points in $\mathbb{R}^{n+1}$ are identified with the corresponding column vectors.

2.2 Spherical Automorphisms

We recall some basic facts; see, e.g., Rudin [22, Section 2.2.1)], Stoll [25, Section 2.1]. Given a point $a \in \mathbb{B}^{n+1} \setminus \{o\}$, we denote by P_a and $\mathrm{Q}_a = \mathrm{I}_{n+1} - \mathrm{P}_a$ the orthogonal projections of $\mathbb{R}^{n+1}$ onto the direction of a and the subspace $a^\perp$, respectively. If $x \in \mathbb{R}^{n+1}$, then

$$\mathrm{P}_a x = \frac{a \cdot x}{|a|^2}\, a.$$

Let

$$\varphi_a x = \frac{a - \mathrm{P}_a x - s_a \mathrm{Q}_a x}{1 - x \cdot a}, \qquad s_a = \sqrt{1 - |a|^2}, \tag{2.1}$$

which is a one-to-one Möbius transformation satisfying

$$\varphi_a(o) = a, \qquad \varphi_a(a) = o, \qquad \varphi_a(\varphi_a x) = x, \tag{2.2}$$

$$1 - |\varphi_a x|^2 = \frac{(1 - |a|^2)(1 - |x|^2)}{(1 - x \cdot a)^2}, \qquad x \cdot a \neq 1. \tag{2.3}$$

If $x \in \mathbb{S}^n$, then

$$\frac{1 - a \cdot \varphi_a x}{1 + a \cdot \varphi_a x} = \frac{1 - |a|^2}{|a - x|^2}. \tag{2.4}$$

Properties (2.2)–(2.3) can be checked by straightforward computation. By (2.3), φ_a maps the ball $\mathbb{B}^{n+1}$ onto itself and preserves $\mathbb{S}^n$.

Remark 2.1 It is known that the ball $\mathbb{B}^{n+1}$ with the relevant metric can be considered as the Poincaré model of the real $(n + 1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$. There is an intimate connection between the Möbius transformations of $\mathbb{B}^{n+1}$ and the group $O(1, n + 1)$ in the hyperboloid model of $\mathbb{H}^{n+1}$. In the present article we do not exploit this connection. An interested reader may be referred, e.g., to Beardon [1, Section 3.7], Gehring et al. [5, Section 3.7], Mostow [12, Theorem 1.1].

Lemma 2.2 *For any $f \in L^1(\mathbb{S}^n)$,*

$$\int\limits_{\mathbb{S}^n} f(x)\, dx = s_a^n \int\limits_{\mathbb{S}^n} \frac{(f \circ \varphi_a)(y)}{(1 - a \cdot y)^n}\, dy, \qquad s_a = \sqrt{1 - |a|^2}. \tag{2.5}$$

Proof We write x in spherical coordinates

$$x = \sqrt{1-u^2}\,\theta + u\tilde{a}, \qquad \tilde{a} = \frac{a}{|a|}, \quad |u| \le 1, \quad \theta \in \mathbb{S}^n \cap a^\perp,$$

to obtain

$$\int\limits_{\mathbb{S}^n} f(x)\,dx = \int\limits_{-1}^{1} (1-u^2)^{(n-2)/2}\,du \int\limits_{\mathbb{S}^n \cap a^\perp} f\left(\sqrt{1-u^2}\,\theta + u\,\tilde{a}\right) d\theta. \tag{2.6}$$

By (2.1),

$$\varphi_a x = -\sqrt{1-v^2}\,\theta + v\tilde{a}, \qquad v = \frac{|a|-u}{1-|a|u}. \tag{2.7}$$

Note that the map $u \to v$ is an involution. Changing variable

$$u = \frac{|a|-v}{1-|a|v}$$

and taking into account that

$$\frac{du}{dv} = \frac{|a|^2-1}{(1-|a|v)^2}, \qquad 1-u^2 = \frac{(1-|a|^2)(1-v^2)}{(1-|a|v)^2},$$

we have

$$\begin{aligned}
\int\limits_{\mathbb{S}^n} f(x)\,dx &= (1-|a|^2)^{n/2} \int\limits_{-1}^{1} \frac{(1-v^2)^{(n-2)/2}}{(1-|a|v)^n}\,dv \\
&\quad \times \int\limits_{\mathbb{S}^n \cap a^\perp} f\left(\frac{\sqrt{1-|a|^2}\,\sqrt{1-v^2}}{1-|a|v}\,\theta + \frac{|a|-v}{1-|a|v}\,\tilde{a}\right) d\theta \\
&= s_a^n \int\limits_{\mathbb{S}^n} \frac{(f \circ \varphi_a)(y)}{(1-a\cdot y)^n}\,dy, \quad \text{as desired.}
\end{aligned}$$

□

We also define the reflection $\tau_a : \mathbb{S}^n \to \mathbb{S}^n$ about the point $a \in \mathbb{B}^{n+1}$:

$$\tau_a x = \frac{(|a|^2-1)\,x + 2(1-x\cdot a)\,a}{|x-a|^2}. \tag{2.8}$$

It assigns to $x \in \mathbb{S}^n$ the antipodal point $\tau_a x \in \mathbb{S}^n$ that lies on the line passing through x and a. A similar reflection map about the origin o is denoted by τ_o, so that $\tau_o x = -x$.

The map φ_a intertwines reflections τ_a and τ_o, that is,

$$\varphi_a \tau_a = \tau_o \varphi_a. \tag{2.9}$$

Indeed, φ_a maps chords of the ball onto chords. Hence, for any $x \in \mathbb{S}^n$, the segment $[x, \tau_a x]$ is mapped onto the segment $[\varphi_a x, \varphi_a \tau_a x]$. Since the first segment contains a, the second one contains $\varphi_a(a) = o$. The latter means that the points $\varphi_a x$ and $\varphi_a \tau_a x$ are symmetric with respect to the origin, that is, $\varphi_a \tau_a x = \tau_o \varphi_a x$.

Lemma 2.3 *If $f \in L^1(\mathbb{S}^n)$ and $a \in \mathbb{B}^{n+1}$, then*

$$\int\limits_{\mathbb{S}^n} f(\tau_a x)\, dx = \int\limits_{\mathbb{S}^n} f(x) \left(\frac{1-|a|^2}{|a-x|^2} \right)^n dx, \tag{2.10}$$

$$\int\limits_{\mathbb{S}^n} f(x)\, dx = \int\limits_{\mathbb{S}^n} f(\tau_a x) \left(\frac{1-|a|^2}{|a-x|^2} \right)^n dx. \tag{2.11}$$

Proof By (2.9) and (2.5),

$$\begin{aligned}
\int\limits_{\mathbb{S}^n} f(\tau_a x)\, dx &= \int\limits_{\mathbb{S}^n} f(\varphi_a \tau_o \varphi_a x)\, dx \qquad (\text{set } x = \varphi_a \tau_o y) \\
&= (1-|a|^2)^{n/2} \int\limits_{\mathbb{S}^n} \frac{(f \circ \varphi_a)(y)}{(1-a\cdot y)^n} \left(\frac{1-a\cdot y}{1+a\cdot y} \right)^n dy \\
&= \int\limits_{\mathbb{S}^n} f(x) \left(\frac{1-a\cdot \varphi_a x}{1+a\cdot \varphi_a x} \right)^n dx.
\end{aligned}$$

It remains to apply (2.4). The second equality follows from the first one: just replace $f(x)$ by $f(\tau_a x)$ and use $\tau_a \tau_a x = x$. □

3 The Shifted Funk Transform

3.1 Inversion Procedure

The following theorem establishes connection between the shifted Funk transform

$$(F_a f)(\tau) = \int\limits_{\mathbb{S}^n \cap \tau} f(x)\, d\sigma(x), \qquad \tau \in \mathrm{Gr}_a(n+1, k), \tag{3.1}$$

and the classical Funk transform $F_o = F_a|_{a=o}$ that takes functions on $\mathbb{S}^n$ to functions on $\mathrm{Gr}_o(n+1,k)$. Given a function f on $\mathbb{S}^n$ and a function Φ on $\mathrm{Gr}_o(n+1,k)$, we denote

$$(M_a f)(y)=\left(\frac{s_a}{1-a\cdot y}\right)^{k-1}(f\circ\varphi_a)(y), \quad (N_a\Phi)(\tau)=\Phi(\varphi_a\tau), \tag{3.2}$$

where $s_a=\sqrt{1-|a|^2}$ and φ_a is an automorphism (2.1).

Theorem 3.1 *Let $1\le k\le n$, $a\in\mathbb{B}^{n+1}$. If $f\in C(\mathbb{S}^n)$, then*

$$F_a f = N_a F_o M_a f. \tag{3.3}$$

The proof of this theorem is given in Sect. 6.

The Funk transform F_o is injective on the subspace $C^+(\mathbb{S}^n)$ of even functions, whilst the subspace $C^-(\mathbb{S}^n)$ of odd functions is the kernel of F_o in $C(\mathbb{S}^n)$; see, e.g., [9, 18–20]. We denote by $\tilde{F}_o$ the restriction of F_o onto $C^+(\mathbb{S}^n)$.

There exist several different approaches to inversion of $\tilde{F}_o$. We recall one of them. Given $\varphi=\tilde{F}_o f$, $f\in C^+(\mathbb{S}^n)$, consider the mean value operator

$$(F_x^*\varphi)(r)=\int\limits_{\{\zeta\in\mathrm{Gr}_o(n+1,k):\, d(x,\zeta)=r\}}\varphi(\zeta)\,dm(\zeta), \qquad 0<r<1, \tag{3.4}$$

where integration is performed with respect to the relevant probability measure over the set of all planes $\zeta\in\mathrm{Gr}_o(n+1,k)$ at geodesic distance $d(x,\zeta)=\cos^{-1} r$ from x.

Theorem 3.2 (cf. [19, Theorem 5.3]) *A function $f\in C^+(\mathbb{S}^n)$ can be reconstructed from $\varphi=\tilde{F}_o f$ by*

$$\begin{aligned} f(x) &\equiv (\tilde{F}_o^{-1}\varphi)(x) \\ &= \lim_{s\to 1}\left(\frac{1}{2s}\frac{\partial}{\partial s}\right)^k\left[\frac{\pi^{-k/2}}{\Gamma(k/2)}\int\limits_0^s (s^2-r^2)^{k/2-1}\,(F_x^*\varphi)(r)\,r^k\,dr\right]. \end{aligned} \tag{3.5}$$

In particular, for k even,

$$(\tilde{F}_o^{-1}\varphi)(x)=\lim_{s\to 1}\frac{1}{2\pi^{k/2}}\left(\frac{1}{2s}\frac{\partial}{\partial s}\right)^{k/2}[s^{k-1}(F_x^*\varphi)(s)]. \tag{3.6}$$

The limit in these formulas is understood in the sup*-norm.*

Now we proceed to inversion of F_a, which, by Theorem 3.1, is factorized as $F_a = N_a F_o M_a$. Here the operators M_a and N_a are injective, so that

$$(M_a^{-1} f)(x) = (1 - a \cdot \varphi_a x)^{k-1} (f \circ \varphi_a)(x), \qquad N_a^{-1}\Phi = \Phi \circ \varphi_a. \tag{3.7}$$

The following definition is motivated by the factorization $F_a = N_a F_o M_a$ and nicely agrees with the case $a = o$.

Definition 3.3 A function $f \in C(\mathbb{S}^n)$ is called a-even (or a-odd) if $M_a f$ is even (or odd, resp.) in the usual sense. The subspaces of all a-even and a-odd continuous functions on $\mathbb{S}^n$ will be denoted by $C_a^+(\mathbb{S}^n)$ and $C_a^-(\mathbb{S}^n)$, respectively. The restriction of F_a onto $C_a^+(\mathbb{S}^n)$ will be denoted by $\tilde{F}_a$.

Theorem 3.4 *Let* $1 < k \le n$. *Then* $\ker(F_a) = C_a^-(\mathbb{S}^n)$ *and the restricted operator* $\tilde{F}_a$ *is injective. A function* $f \in C_a^+(\mathbb{S}^n)$ *can be uniquely reconstructed from* $g = \tilde{F}_a f$ *by*

$$f \equiv \tilde{F}_a^{-1} g = M_a^{-1} \tilde{F}_o^{-1} N_a^{-1} g, \tag{3.8}$$

where M_a^{-1}, $\tilde{F}_o^{-1}$, *and* N_a^{-1} *are defined by (3.7) and Theorem 3.2.*

This statement is an immediate consequence of (3.3) and the corresponding results for F_o.

Remark 3.5 In the case $k = 1$, which is not included in Theorem 3.4, the plane τ is a line and the integral (1.1) is a sum of the values of f at the points where this line intersects the sphere. If x is one of such points and $L_{a,x}$ is the line through a and x, then

$$(F_a f)(L_{a,x}) = f(x) + f(\tau_a x). \tag{3.9}$$

The a-odd functions, for which $f(x) = -f(\tau_a x)$, form the kernel of the operator (3.9). An a-even function f, satisfying $f(x) = f(\tau_a x)$, can be reconstructed from $(F_a f)(L_{a,x})$ by the formula

$$f(x) = \frac{1}{2}(F_a f)(L_{a,x}). \tag{3.10}$$

3.2 Alternative Description of the Subspaces $C_a^{\pm}(\mathbb{S}^n)$

We set

$$\rho_a(x) = \left(\frac{1-|a|^2}{|a-x|^2}\right)^{k-1}, \qquad (W_a f)(x) = \rho_a(x) f(\tau_a x), \tag{3.11}$$

where τ_a is the reflection (2.8).

Lemma 3.6 *The operator W_a is an involution, i.e., $W_a W_a f = f$.*

Proof The statement is obvious for $a = o$, when $(W_0 f)(x) = f(-x)$. It is also obvious for any $a \in \mathbb{B}^n$ if $k = 1$. In the general case, taking into account that $\tau_a \tau_a x = x$, we have

$$(W_a W_a f)(x) = \left[\frac{1 - |a|^2}{|a - x|^2} \, \frac{1 - |a|^2}{|a - \tau_a x|^2} \right]^{k-1} f(x).$$

By (2.4) and (2.9), the expression in square brackets can be written as

$$\frac{(1 - a \cdot \varphi_a x)\,(1 - a \cdot \varphi_a \tau_a x)}{(1 + a \cdot \varphi_a x)\,(1 + a \cdot \varphi_a \tau_a x)} = \frac{(1 - a \cdot \varphi_a x)\,(1 + a \cdot \varphi_a x)}{(1 + a \cdot \varphi_a x)\,(1 - a \cdot \varphi_a x)} = 1.$$

This gives the result. □

Theorem 3.7 *A function $f \in C(\mathbb{S}^n)$ is a-even (or a-odd) if and only if $f = W_a f$ (or $f = -W_a f$, respectively).*

Proof By Definition 3.3, $f \in C(\mathbb{S}^n)$ is a-even if and only if $(M_a f)(y) = (M_a f)(-y)$ for all $y \in \mathbb{S}^n$. The latter is equivalent to

$$(f \circ \varphi_a)(y) = \left(\frac{1 - a \cdot y}{1 + a \cdot y} \right)^{k-1} (f \circ \varphi_a)(-y),$$

or (set $y = \varphi_a x$ and use (2.4) and (2.9))

$$f(x) = \left(\frac{1 - a \cdot \varphi_a x}{1 + a \cdot \varphi_a x} \right)^{k-1} f(\tau_a x) = \rho_a(x) f(\tau_a x) = (W_a f)(x).$$

The proof for the a-odd functions is similar. □

Corollary 3.8 *Every function $f \in C(\mathbb{S}^n)$ can be represented as a sum of its a-even and a-odd parts. Specifically,*

$$f = f_a^+ + f_a^-, \qquad f_a^\pm = \frac{f \pm W_a f}{2}. \tag{3.12}$$

Proof The first equality follows from the second one. Further, by Lemma 3.6,

$$W_a f_a^\pm = \frac{W_a f \pm W_a W_a f}{2} = \frac{W_a f \pm f}{2} = \pm f_a^\pm.$$

Hence, by Theorem 3.7, f_a^+ is a-even and f_a^- is a-odd. □

4 Reconstruction from Two Centers

As we have seen in Sect. 3, a generic function $f \in C(\mathbb{S}^n)$ cannot be reconstructed from $F_a f$. Because $F_a f = \tilde{F}_a f_a^+$, one can reconstruct only the a-even part f_a^+ of f, whilst the a-odd part is lost. Our aim is to show that complete reconstruction becomes possible if we consider two distinct centers instead of one. Specifically, let $a, b \in \mathbb{B}^{n+1}$, $a \neq b$. Consider the system of two equations

$$F_a f = g, \qquad F_b f = h, \tag{4.1}$$

and suppose that a function $f \in C(\mathbb{S}^n)$ satisfies this system. Then $\tilde{F}_a f_a^+ = g$, $\tilde{F}_b f_b^+ = h$, and therefore, by (3.12),

$$f_a^+ \equiv \frac{f + W_a f}{2} = \tilde{F}_a^{-1} g, \qquad f_b^+ \equiv \frac{f + W_b f}{2} = \tilde{F}_b^{-1} h,$$

where

$$(W_a f)(x) = \rho_a(x) f(\tau_a x), \qquad (W_b f)(x) = \rho_b(x) f(\tau_b x). \tag{4.2}$$

Setting

$$g_1 = 2\tilde{F}_a^{-1} g, \qquad h_1 = 2\tilde{F}_b^{-1} h,$$

we obtain a pair of functional equations

$$f = g_1 - W_a f, \qquad f = h_1 - W_b f. \tag{4.3}$$

Then we substitute f from the second equation into the right-hand side of the first one to get

$$f = Wf + q, \qquad W = W_a W_b, \qquad q = g_1 - W_a h_1. \tag{4.4}$$

Iterating (4.4), we obtain

$$f = W^m f + \sum_{j=0}^{m-1} W^j q; \qquad m = 1, 2, \dots . \tag{4.5}$$

This equation generates a dynamical system on $\mathbb{S}^n$.

Lemma 4.1 *Let a^* and b^* be the points on $\mathbb{S}^n$ that lie on the straight line through a and b. Suppose that a is closer to a^* than b. If $W = W_a W_b$, then $\lim_{m\to\infty} (W^m f)(x) = 0$ for all $x \in \mathbb{S}^n \setminus \{a^*\}$ and $1 < k \le n$. If $k = 1$ and $x \in \mathbb{S}^n \setminus \{a^*\}$, then $\lim_{m\to\infty} (W^m f)(x) = f(b^*)$.*

Proof We observe that

$$(Wf)(x) = (W_a W_b f)(x) = \rho_a(x)\rho_b(\tau_a x) f(\tau_b \tau_a x). \tag{4.6}$$

Denote

$$\rho(x)=\rho_a(x)\rho_b(\tau_a x)=\left[\frac{(1-|a|^2)\,(1-|b|^2)}{|a-x|^2\,|b-\tau_a x|^2}\right]^{k-1}, \quad \mathrm{T}=\tau_b\tau_a. \tag{4.7}$$

Then $(Wf)(x) = \rho(x) f(\mathrm{T}x)$ and, by iteration,

$$(W^m f)(x) = \omega_m(x)\, f(\mathrm{T}^{m+1}x), \qquad \omega_m(x) = \prod_{j=0}^{m} \rho(\mathrm{T}^j x). \tag{4.8}$$

For any $x \neq a^*$, the mapping T preserves the circle $C_{x,a,b}$ in the 2-plane spanned by x, a and b, and leaves the points a^* and b^* fixed. A simple geometric consideration in the 2-plane shows that the distance from the points $\mathrm{T}^j x \in C_{x,a,b}$ to b^* monotonically decreases, and therefore, the sequence $\mathrm{T}^j x$ has a limit. This limit must be a fixed point of the mapping T, and hence $\mathrm{T}^j x \to b^*$ as $j \to \infty$. Because ρ is continuous, it follows that

$$\lim_{j\to\infty} \rho(\mathrm{T}^j x) = \rho(b^*). \tag{4.9}$$

Using this fact, let us show that if $k > 1$, then

$$\lim_{m\to\infty} \omega_m(x) = 0. \tag{4.10}$$

Once (4.10) has been proved, the statement of the lemma for $k > 1$ will follow because the factor $f(\mathrm{T}^{m+1}x)$ has finite limit $f(b^*)$.

To prove (4.10), it suffices to show that

$$\rho(b^*) < 1, \tag{4.11}$$

where, by (4.7),

$$\rho(b^*) = \left[\frac{(1-|a|^2)\,(1-|b|^2)}{|a-b^*|^2\,|a^*-b|^2}\right]^{k-1}. \tag{4.12}$$

Let

$$a = a^* + t(b^* - a^*), \quad b = a^* + s(b^* - a^*), \qquad 0 < t < s < 1. \tag{4.13}$$

Taking into account that $|a^*| = |b^*| = 1$ and using (4.13), we obtain

$$1-|a|^2 = 2t(1-t)(1-a^*\cdot b^*), \qquad 1-|b|^2 = 2s(1-s)(1-a^*\cdot b^*), \tag{4.14}$$
$$|a-b^*|^2 = 2(1-t)^2(1-a^*\cdot b^*), \qquad |a^*-b|^2 = 2s^2(1-a^*\cdot b^*).$$

Hence

$$\rho(b^*) = \left[\frac{t(1-s)}{s(1-t)}\right]^{k-1} < 1. \tag{4.15}$$

The last inequality is an immediate consequence of the assumption $0 < t < s < 1$.

The case $k = 1$ is simpler. In this case $\rho(x) = 1$, and therefore, $(W^m f)(x) = f(\mathrm{T}^{m+1}x) \to f(b^*)$ as $m \to \infty$, $x \in \mathbb{S}^n \setminus \{a^*\}$. □

The above reasoning yields the following preliminary conclusion. If $a \neq b$, then, by Theorem 3.4, the kernel of the map $f \to (F_a f, F_b f)$, $f \in C(\mathbb{S}^n)$, is $C_a^-(\mathbb{S}^n) \cap C_b^-(\mathbb{S}^n)$. But if f is odd with respect to both a and b, then, by Theorem 3.7, $Wf = f$. By Lemma 4.1 it follows that $f(x) = 0$ for all $x \in \mathbb{S}^n \setminus \{a^*\}$. However, since f is continuous, we must have $f = 0$ *everywhere* on $\mathbb{S}^n$. In particular, it follows that f can be reconstructed from the knowledge of $F_a f$ and $F_b f$, or, what is the same, from f_a^+ and f_b^+.

More precisely, we have the following result.

Theorem 4.2 *Let W_a and W_b be involutions (4.2), $1 < k \leq n$. If the system of equations $F_a f = g$ and $F_b f = h$ has a solution $f \in C(\mathbb{S}^n)$, then this solution is unique and can defined by the pointwise convergent series*

$$f(x) = \sum_{j=0}^{\infty} W^j q(x), \qquad x \neq a^*, \tag{4.16}$$

where $W = W_a W_b$, $q = 2[\tilde{F}_a^{-1} g - W_a \tilde{F}_b^{-1} h]$, $\tilde{F}_a^{-1}$ and $\tilde{F}_b^{-1}$ being defined as in (3.8). Alternatively,

$$f(x) = \sum_{j=0}^{\infty} \tilde{W}^j r(x), \qquad x \neq b^*, \tag{4.17}$$

where $\tilde{W} = W_b W_a$ and $r = 2[\tilde{F}_b^{-1} h - W_b \tilde{F}_a^{-1} g]$.

Proof To prove (4.16), it suffices to pass to the limit in (4.5), taking into account that, by Lemma 4.1, the remainder $(W^m f)(x)$ of the series (4.16) converges to zero for every $x \neq a^*$. An alternative formula (4.17) then follows if we interchange a and b, g and h. □

Remark 4.3 In the case $k = 1$, a function $f \in C(\mathbb{S}^n)$ can be reconstructed from the system $F_a f = g$, $F_b f = h$ as follows. By Lemma 4.1, $(W^m f)(x) \to f(b^*)$ as $m \to \infty$. Hence

$$f(x) = \sum_{j=0}^{\infty} q(\mathrm{T}^{j+1}x) + f(b^*), \qquad x \neq a^*, \quad \mathrm{T} = \tau_b \tau_a, \tag{4.18}$$

where $q(x) = 2\,[(\tilde{F}_a^{-1} g)(x) - (\tilde{F}_b^{-1} h)(\tau_a x)]$. By (3.10),

$$(\tilde{F}_a^{-1} g)(x) = \frac{1}{2}\, g(L_{a,x}), \qquad (\tilde{F}_b^{-1} h)(\tau_a x) = \frac{1}{2}\, h(L_{b,\tau_a x}),$$

where the line $L_{a,x}$ passes through a and x and $L_{b,\tau_a x}$ passes through b and $\tau_a x$. It follows that

$$q(x) = g(L_{a,x}) - h(L_{b,\tau_a x}). \tag{4.19}$$

Similarly, $(\tilde{W}^m f)(x) \to f(a^*)$, and we have

$$f(x) = \sum_{j=0}^{\infty} r(\tilde{\mathrm{T}}^{j+1}x) + f(a^*), \qquad x \neq b^*, \quad \tilde{\mathrm{T}} = \tau_a \tau_b, \tag{4.20}$$

$$r(x) = h(L_{b,x}) - g(L_{a,\tau_b x}). \tag{4.21}$$

The series (4.18) and (4.20) reconstruct f up to unknown additive constants $f(a^*)$ or $f(b^*)$, where a^* and b^* are the endpoints of the chord through a and b. However, complete reconstruction is still possible, if we apply symmetrization, by summing (4.18) and (4.20). This gives the following result.

Theorem 4.4 *Let $k = 1$. Then*

$$2f(x) = \sum_{j=0}^{\infty} q(\mathrm{T}^{j+1}x) + \sum_{j=0}^{\infty} r(\tilde{\mathrm{T}}^{j+1}x) + F_a(L_{a,b}), \qquad x \neq a^*, b^*, \tag{4.22}$$

where q and r are defined by (4.19) and (4.21), respectively, $L_{a,b}$ is the line through a and b, and $F_a(L_{a,b}) = f(a^*) + f(b^*)$ $(= F_b(L_{a,b}))$ is known. The values of f at the points a^* and b^* can be reconstructed by continuity.

5 Norm Convergence of the Reconstructing Series

Reconstruction of f by the pointwise convergent series (4.16) and (4.17) gives a little possibility to control the accuracy of the result because the rate of the pointwise convergence depends on the point. Therefore, it is natural to look at the convergence in certain normed spaces. Below we explore such convergence in the spaces $C(\mathbb{S}^n)$ and $L^p(\mathbb{S}^n)$. As above, we keep the notation a^* and b^* for the endpoints of the chord through a and b.

Consider the most interesting case $k > 1$. By (4.5), the convergence of the series (4.16) to f is equivalent to convergence of its remainder $W^m f$ to 0 as $m \to \infty$. Thus, it suffices to confine to $W^m f$.

We first note that the series (4.16) may diverge at the point a^*. Indeed, because $(W^m f)(a^*) = f(\mathrm{T}^{m+1}a^*) \prod\limits_{j=0}^{m} \rho(\mathrm{T}^j a^*)$ and a^* is a fixed point of the mapping T, we have

$$(W^m f)(a^*) = \rho(a^*)^{m+1} f(a^*), \qquad \rho(a^*) = \left[\frac{(1-|a|^2)(1-|b|^2)}{|a-a^*|^2|b-b^*|^2}\right]^{k-1}.$$

Suppose that a and b are symmetric with respect to the origin and $|a| = |b| = 1/2$. Then

$$\rho(a^*) = \left[\frac{(1+|a|)(1+|b|)}{(1-|a|)(1-|b|)}\right]^{k-1} = 9^{k-1},$$

and therefore $(W^m f)(a^*) = 9^{(k-1)(m+1)} f(a^*) \to \infty$ as $m \to \infty$ whenever $f(a^*) \neq 0$. The latter means that if $f(a^*) \neq 0$, then the series (4.16) diverges at a^* and its uniform convergence on the entire sphere fails. Below it will be shown that the uniform convergence of this series fails for any $a, b \in \mathbb{B}^{n+1}$.

To understand the type of convergence, we need a deeper insight in the dynamics of involved reflections.

5.1 Dynamics of the Double Reflection Mapping $\mathrm{T} = \tau_b \tau_a$

We know that the trajectory $\{\mathrm{T}^m x : m = 0, 1, 2, \dots\}$ of any point $x \in \mathbb{S}^n \setminus \{a^*\}$ converges to the point b^*, which is the endpoint of the chord containing a and b. Let us specify the character of this convergence.

Lemma 5.1 *The mapping* $\mathrm{T} = \tau_b \tau_a$ *maps the punctured sphere* $\mathbb{S}^n_a = \mathbb{S}^n \setminus \{a^*\}$ *onto itself. The point* b^* *is the attracting point of the dynamical system* $T^m : \mathbb{S}^n_a \to \mathbb{S}^n_a$ *uniformly on compact subsets, that is, for any open neighborhood* $U \subset \mathbb{S}^n_a$ *of* b^* *and any compact set* $K \subset \mathbb{S}^n_a$ *there exists* $\overline{m}$ *such that* $T^m K \subset U$ *for all* $m \geq \overline{m}$.

Proof The first statement is obvious, because $\mathrm{T}a^* = a^*$ and $\mathrm{T}^{-1}a^* = \tau_a \tau_b a^* = a^*$. The second statement follows by a standard argument for monotone pointwise convergence on compacts. In fact, it suffices to prove this statement for the sets U

and K having the form

$$U = U_\varepsilon = B(b^*, \varepsilon), \qquad K = K_\delta = \mathbb{S}^n \setminus B(a^*, \delta),$$

where $B(a^*, \varepsilon)$ and $B(b^*, \delta)$ are geodesic balls in $\mathbb{S}^n$ of sufficiently small radii.

The pointwise convergence yields that for any fixed $x_0 \in K_\delta$ there exists a number m_0 such that $\mathrm{T}^{m_0} x_0 \in U_\varepsilon$. By continuity, the same is true for every x in some neighborhood V_{x_0} of x_0. Thus, the compact K_δ is covered by open sets V_x, $x \in K_\delta$, and therefore we can cover K_δ by a finite family $\{V_{x_1}, \dots V_{x_M}\}$. For each x_i, there is a number m_i such that $\mathrm{T}^{m_i} x_i \in U_\varepsilon$. Setting $\overline{m} = \max\{m_1, \dots, m_M\}$, we have

$$\mathrm{T}^{\overline{m}} K_\delta \subset U_\varepsilon.$$

A simple geometric consideration shows that the sequence $\mathrm{T}^{m+1} K_\delta$ monotonically decreases, i.e., $\mathrm{T}^{m+1} K_\delta \subset \mathrm{T}^m K_\delta$. Hence $\mathrm{T}^m K_\delta \subset U_\varepsilon$ for all $m \geq \overline{m}$. □

5.2 *Uniform Convergence on Compact Subsets of the Punctured Sphere*

Theorem 5.2 *If $f \in C(\mathbb{S}^n)$, then the series (4.16) converges to f uniformly on compact subsets of the punctured sphere $\mathbb{S}^n \setminus \{a^*\}$.*

Proof Consider the remainder $(W^m f)(x)$ of the series (4.16). By (4.8),

$$(W^m f)(x) = \omega_m(x)\, f(\mathrm{T}^{m+1} x), \qquad \omega_m(x) = \prod_{j=0}^{m} \rho(\mathrm{T}^j x).$$

Because $\rho(b^*) < 1$ (see (4.15)), for a fixed γ satisfying $\rho(b^*) < \gamma < 1$, there is an open neighborhood $U \subset \mathbb{S}^n \setminus \{a^*\}$ of the point b^* such that $0 < \rho(y) < \gamma$ for all $y \in U$. On the other hand, Lemma 5.1 says that there exists $\overline{m}$ such that $\mathrm{T}^m K_\delta \subset U$ for $m \geq \overline{m}$ and hence $0 < \rho(\mathrm{T}^m x) < \gamma$ for all $x \in K_\delta$ and $m \geq \overline{m}$. Thus

$$\omega_m(x) \leq \gamma^{m-\overline{m}} \max_{x \in K_\delta} \prod_{j=0}^{\overline{m}} \rho(\mathrm{T}^j x)$$

for all $m \geq \overline{m}$ and all $x \in K_\delta$. It follows that $\omega_m(x) \to 0$ as $m \to \infty$ uniformly on K_δ. Since $|f(T^m x)| \leq \|f\|_{C(S^n)}$, we conclude that $W^m f \to 0$ uniformly on K_δ. This gives the result. □

5.3 L^p-Convergence

Lemma 5.3 *The operators W_a, W_b, $W = W_a W_b$, and $\tilde{W} = W_b W_a$ are isometries of the space $L^{p_0}(\mathbb{S}^n)$ with $p_0 = n/(k-1)$.*

Proof The statement about W_a follows from (2.11), which reads

$$\int_{\mathbb{S}^n} \big(\rho_a(x)\big)^{p_0} f(\tau_a x) dx = \int_{\mathbb{S}^n} f(x)\, dx.$$

The equality holds for any $f \in L^1(\mathbb{S}^n)$ and therefore, if $f \in L^{p_0}(\mathbb{S}^n)$, then, using $|f(x)|^{p_0}$ instead of f, we obtain $\|W_a f\|_{p_0} = \|f\|_{p_0}$. The statement for W_b follows analogously. The operators W and $\tilde{W}$ are also isometries, as the products of two isometries. □

Theorem 5.4 *Let $f \in C(S^n)$, $p_0 = n/(k-1)$. The series (4.16) and (4.17) converge to f in the norm of $L^p(\mathbb{S}^n)$ for any $1 \le p < p_0$. The convergence to f fails in any space $L^p(\mathbb{S}^n)$ with $p_0 \le p \le \infty$.*

Proof It is clear that f belongs to $L^p(\mathbb{S}^n)$ for any $1 \le p \le \infty$. Fix $\delta > 0$ and consider the function $W^m f = (W_a W_b)^m f$. Suppose that $p < p_0$ and set $r = p_0/p > 1$. We write

$$\begin{aligned} \|W^m f\|_p^p &= \int_{B(a^*,\delta)} |(W^m f)(x)|^p\, dx + \int_{K_\delta} |(W^m f)(x)|^p dx \\ &= I_1(m,\delta) + I_2(m,\delta), \end{aligned} \tag{5.1}$$

where, as above, $K_\delta = \mathbb{S}^n \setminus B(a^*, \delta)$. By Hölder's inequality,

$$I_1(m,\delta) \le \Bigg(\int_{B(a^*,\delta)} \Big(|(W^m f)(x)|^p\Big)^r dx\Bigg)^{1/r} \Bigg(\int_{B(a^*,\delta)} dx\Bigg)^{r/(r-1)}.$$

Owing to Lemma 5.3, the operator W^m preserves the L^{p_0}-norm, and therefore

$$\Bigg(\int_{B(a^*,\delta)} \Big(|(W^m f)(x)|^p\Big)^r dx\Bigg)^{1/r} = \Bigg(\int_{B(a^*,\delta)} |(W^m f)(x)|^{p_0} dx\Bigg)^{1/r}$$

$$\le \|W^m f\|_{p_0}^{p_0/r} = \|f\|_{p_0}^p.$$

Hence

$$I_1(m,\delta) \le A(\delta)^{r/(r-1)} \|f\|_{p_0}^p, \tag{5.2}$$

where $A(\delta)$ is the n-dimensional surface area of the geodesic ball $B(a^*, \delta)$. For the second integral in (5.1) we have

$$I_2(m, \delta) < \sigma_n \sup_{x \in K_\delta} |(W^m f)(x)|^p, \tag{5.3}$$

where σ_n is the area of the unit sphere $\mathbb{S}^n$.

Now we fix sufficiently small $\varepsilon > 0$. Using (5.2), let us choose $\delta > 0$ so that $I_1(m, \delta) < \varepsilon/2$ for all $m \geq 0$. By Theorem 5.2, the inequality (5.3) implies that there exists $\tilde{m} = \tilde{m}(\delta)$ such that $I_2(m, \delta) < \varepsilon/2$ for all $m \geq \tilde{m}$. Hence, by (5.1), $\|W^m f\|_p^p < \varepsilon$ for $m \geq \tilde{m}$, and therefore $W^m f$ tends to 0 as $m \to \infty$ in the L^p-norm. The latter gives the desired convergence of the series (4.16).

On the other hand, if $p > p_0$, then, by Hölder's inequality, $\|f\|_{p_0} = \|W^m f\|_{p_0} \leq c\,\|W^m f\|_p$, $c = \text{const} > 0$. It follows that the L^p-norm of the remainder $W^m f$ of the series does not tend to 0 as $m \to \infty$, unless $f = 0$.

The proof for $\tilde{W} f$ is similar. □

Remark 5.5 As we can see, the iterative method in terms of the series (4.16) and (4.17) does not provide uniformly convergent reconstruction of continuous functions. The reconstruction is guaranteed only in the L^p-norm with $1 \leq p < p_0 = n/(k-1)$. For instance, in the case of the hyperplane sections, when $k = n$ and $p_0 = 1 + 1/(n-1)$, the L^2-convergence fails because p_0 does not exceed 2. The less the dimension k is, the greater exponent p can be chosen. The case $p = 1$ works for all $1 < k \leq n$.

6 Proof of Theorem 3.1

We recall that $a \in \mathbb{B}^{n+1}$, $s_a = \sqrt{1 - |a|^2}$, and φ_a is an automorphism (2.1). The following lemma allows us to exploit the language of Stiefel manifolds when dealing with affine planes.

Lemma 6.1 *Let $1 \leq k \leq n$. The map φ_a extends as a bijection from* $\mathrm{Gr}_a(n+1, k)$ *onto* $\mathrm{Gr}_o(n+1, k)$. *Specifically, if $\tau \in \mathrm{Gr}_a(n+1, k)$ is defined by*

$$\tau = \{x \in \mathbb{R}^{n+1} : \xi' x = \xi' a\}, \qquad \xi \in \mathrm{St}(n+1, n+1-k), \tag{6.1}$$

then $\zeta \equiv \varphi_a \tau \in \mathrm{Gr}_o(n+1, k)$ has the form

$$\zeta = \{y \in \mathbb{R}^{n+1} : \eta' y = 0\}, \qquad \eta \in \mathrm{St}(n+1, n+1-k), \tag{6.2}$$

where

$$\eta = -(\mathrm{A}\xi)\,\alpha^{-1/2}, \qquad \mathrm{A} = s_a \mathrm{P}_a + \mathrm{Q}_a, \qquad \alpha = (\mathrm{A}\xi)'(\mathrm{A}\xi). \tag{6.3}$$

Conversely, if $\zeta \in \mathrm{Gr}_o(n+1,k)$ is defined by (6.2), then $\tau \equiv \varphi_a \zeta$ has the form (6.1) with

$$\xi = (\mathrm{A}_1\eta)\,\beta^{-1/2}, \qquad \mathrm{A}_1 = \mathrm{P}_a + s_a\mathrm{Q}_a, \qquad \beta = (\mathrm{A}_1\eta)'(\mathrm{A}_1\eta). \tag{6.4}$$

Proof Let $\tau \in \mathrm{Gr}_a(n+1,k)$ be defined by (6.1). Then

$$\zeta \equiv \varphi_a \tau = \{y \in \mathbb{R}^{n+1} : \xi'(\varphi_a y - a) = 0\}.$$

By (2.1),

$$\varphi_a y - a = \frac{s_a \mathrm{A} y}{1 - y \cdot a}. \tag{6.5}$$

Hence $\zeta = \{y \in \mathbb{R}^{n+1} : (\mathrm{A}\xi)'y = 0\}$. Now (6.2) follows if we represent the $(n+1)\times(n+1-k)$ matrix $\mathrm{A}\xi$ in the polar form

$$\mathrm{A}\xi = \eta\,\alpha^{1/2}, \quad \alpha = (\mathrm{A}\xi)'(\mathrm{A}\xi), \quad \eta = (\mathrm{A}\xi)\,\alpha^{-1/2}; \tag{6.6}$$

see, e.g., [11, pp. 66, 591].

Conversely, let $\zeta \in \mathrm{Gr}_o(n+1,k)$ be defined by (6.2). Then

$$\tau \equiv \varphi_a \zeta = \{x \in \mathbb{R}^{n+1} : \eta'\varphi_a x = 0\}.$$

By (2.1), the equality $\eta'\varphi_a x = 0$ is equivalent to

$$(\mathrm{P}_a\eta + s_a\mathrm{Q}_a\eta)'x = \eta' a \quad \text{or} \quad (\mathrm{A}_1\eta)'x = \eta' a. \tag{6.7}$$

We write $\mathrm{A}_1\eta$ in the form $\mathrm{A}_1\eta = \xi\,\beta^{1/2}$ with $\beta = (\mathrm{A}_1\eta)'(\mathrm{A}_1\eta)$ and $\xi = (\mathrm{A}_1\eta)\,\beta^{-1/2} \in \mathrm{St}(n+1, n+1-k)$. Then (6.7) yields $\xi' x = \beta^{-1/2}\eta' a$. To complete the proof, it remains to note that $\beta^{-1/2}\eta' a = \xi' a$. Indeed,

$$\begin{aligned}\xi' a &= \beta^{-1/2}(\mathrm{A}_1\eta)'a = \beta^{-1/2}(\mathrm{P}_a\eta + s_a\mathrm{Q}_a\eta)'a \\ &= \beta^{-1/2}(\eta'(\mathrm{P}_a a + s_a\eta'\mathrm{Q}_a a) = \beta^{-1/2}\eta' a.\end{aligned}$$

□

Proof of the Theorem The case $k = 1$ is almost obvious; cf. Remark 3.5. Assuming $1 < k \le n$, let $\tau \in \mathrm{Gr}_a(n+1,k)$ have the form (6.1) and write

$$(F_a f)(\tau) \equiv (F_a f)(\xi) = \int\limits_{\{x \in \mathbb{S}^n : \xi'(x-a)=0\}} f(x)\,d\sigma(x), \quad \xi \in \mathrm{St}(n+1, n+1-k).$$

□

We make use of the standard approximation machinery. Given a sufficiently small $\varepsilon > 0$, let

$$(F_{a,\varepsilon} f)(\xi) = \int\limits_{\mathbb{S}^n} f(x)\, \omega_\varepsilon(\xi'(x-a))\, dx, \tag{6.8}$$

where ω_ε is a smooth bump function supported on the ball in $\mathbb{R}^{n+1-k}$ of radius ε with center at the origin, so that $\lim\limits_{\varepsilon\to 0} \int_{|t|<\varepsilon} \omega_\varepsilon(t)\, g(t)\, dx = g(o)$ for any function g which is continuous in a neighborhood of the origin.

Step I Let us show that

$$\lim_{\varepsilon\to 0} (F_{a,\varepsilon} f)(\xi) = (1-|\xi' a|^2)^{-1/2} (F_a f)(\xi). \tag{6.9}$$

We pass to bispherical coordinates (see, e.g., [20, p. 31])

$$x = \begin{bmatrix} \varphi \sin\theta \\ \psi \cos\theta \end{bmatrix}, \quad \varphi \in \mathbb{S}^n \cap \xi^\perp, \quad \psi \in \mathbb{S}^n \cap \{\xi\}, \quad 0 \le \theta \le \pi/2, \tag{6.10}$$

$$dx = \sin^{k-1}\theta\, \cos^{n-k}\theta\, d\theta d\varphi d\psi,$$

and set $s = \cos\theta$. This gives

$$\begin{aligned}(F_{a,\varepsilon} f)(\xi) &= \int_0^1 s^{n-k}(1-s^2)^{(k-2)/2} ds \int\limits_{\mathbb{S}^n\cap\xi^\perp} d\varphi \\ &\times \int\limits_{\mathbb{S}^n\cap\{\xi\}} f\left(\begin{bmatrix} \varphi\sqrt{1-s^2} \\ s\psi \end{bmatrix}\right) \omega_\varepsilon(s\psi - \xi' a)\, d\psi \\ &= \int\limits_{\mathbb{R}^{n-k+1}} H(y)\, \omega_\varepsilon(y - \xi' a)\, dy, \end{aligned} \tag{6.11}$$

where

$$H(y) = (1-|y|^2)^{(k-2)/2} \int\limits_{\mathbb{S}^n\cap\xi^\perp} f\left(\begin{bmatrix} \varphi\sqrt{1-|y|^2} \\ y \end{bmatrix}\right) d\varphi$$

if $|y| \le 1$ and $H(y) = 0$, otherwise. Passing to the limit, we obtain

$$\lim_{\varepsilon\to 0} (F_{a,\varepsilon} f)(\xi) = H(\xi' a),$$

where

$$H(\xi' a) = (1-|\xi' a|^2)^{(k-2)/2} \int\limits_{\mathbb{S}^n \cap \xi^\perp} f\left(\begin{bmatrix} \varphi\sqrt{1-|\xi' a|^2} \\ \xi' a \end{bmatrix}\right) d\varphi. \tag{6.12}$$

If the argument of f is denoted by x, then $x - a$ lies in the subspace perpendicular to ξ. Further, the integration in (6.12) is performed over the $(k-1)$-dimensional sphere of radius $\sqrt{1-|\xi' a|^2}$. Switching to the surface area measure, we can write (6.12) as

$$H(\xi' a) = (1-|\xi' a|^2)^{-1/2} \int\limits_{\{x\in\mathbb{S}^n :\, \xi'(x-a)=0\}} f(x)\, d\sigma(x),$$

as desired.

Step II Let us obtain an alternative expression for the limit (6.9), now in terms of the automorphism φ_a. By Lemma 2.2,

$$(F_{a,\varepsilon} f)(\xi) = s_a^n \int\limits_{\mathbb{S}^n} \frac{(f\circ\varphi_a)(y)}{(1-a\cdot y)^n}\, \omega_\varepsilon(\xi'[\varphi_a y - a])\, dy,$$

where

$$\xi'[\varphi_a y - a] = -\frac{s_a \xi' \mathrm{A} y}{1-a\cdot y} = -\frac{s_a (\mathrm{A}\xi)' y}{1-a\cdot y}, \qquad \mathrm{A} = s_a \mathrm{P}_a + \mathrm{Q}_a;$$

see (6.5). Denote

$$\tilde{f}(y) = s_a^n \frac{(f\circ\varphi_a)(y)}{(1-a\cdot y)^n}.$$

Then

$$(F_{a,\varepsilon} f)(\xi) = \int\limits_{\mathbb{S}^n} \tilde{f}(y)\, \omega_\varepsilon\left(\frac{s_a (\mathrm{A}\xi)' y}{1-a\cdot y}\right) dy.$$

As in (6.6), the polar decomposition yields

$$\mathrm{A}\xi = \eta\, \alpha^{1/2}, \quad \alpha = (\mathrm{A}\xi)'(\mathrm{A}\xi), \quad \eta = (\mathrm{A}\xi)\, \alpha^{-1/2}. \tag{6.13}$$

Then we pass to bispherical coordinates (cf. (6.10))

$$y = \begin{bmatrix} \varphi \sin\theta \\ \psi \cos\theta \end{bmatrix}, \quad \varphi \in \mathbb{S}^n \cap \eta^\perp, \quad \psi \in \mathbb{S}^n \cap \{\eta\}, \quad 0 \le \theta \le \pi/2,$$

$$dy = \sin^{k-1}\theta \, \cos^{n-k}\theta \, d\theta d\varphi d\psi,$$

and set $s = \cos\theta$. This gives

$$(F_{a,\varepsilon} f)(\xi) = \int_0^1 s^{n-k}(1-s^2)^{(k-2)/2} ds \int_{\mathbb{S}^n \cap \eta^\perp} d\varphi$$

$$\times \int_{\mathbb{S}^n \cap \{\eta\}} \tilde{f}\left(\begin{bmatrix} \sqrt{1-s^2}\varphi \\ s\psi \end{bmatrix}\right) \omega_\varepsilon\left(\frac{s_a\, \alpha^{1/2} s\psi}{1 - a\cdot(\sqrt{1-s^2}\,\varphi + s\psi)}\right) d\psi,$$

or (set $z = s\psi \in \{\eta\} \sim \mathbb{R}^{n+1-k}$, $|z| < 1$)

$$(F_{a,\varepsilon} f)(\xi) = \int_{|z|<1} (1-|z|^2)^{(k-2)/2} dz$$

$$\times \int_{\mathbb{S}^n \cap \eta^\perp} \tilde{f}\left(\begin{bmatrix} \sqrt{1-|z|^2}\varphi \\ z \end{bmatrix}\right) \omega_\varepsilon\left(\frac{s_a\, \alpha^{1/2} z}{1 - a\cdot(\sqrt{1-|z|^2}\,\varphi + z)}\right) d\varphi,$$

We set

$$t \equiv t(z) = \frac{s_a\, \alpha^{1/2} z}{1 - a\cdot(\sqrt{1-|z|^2}\,\varphi + z)} = \frac{\Lambda z}{1-h(z)}, \tag{6.14}$$

$$\Lambda = s_a\, \alpha^{1/2}, \qquad h(z) = a\cdot(\sqrt{1-|z|^2}\,\varphi + z),$$

so that $t = o$ if and only if $z = o$, where o is the origin in the corresponding space. Further, we write (6.14) as

$$\Phi(t, z) \equiv \Lambda z - t + th(z) = 0$$

and denote $m = n + 1 - k$. Because the $m \times m$ matrix $(\partial\Phi_i/\partial z_j)(o, o) = \Lambda$ is invertible, there exists an inverse function $z = z(t)$, which is well-defined and differentiable in a small neighborhood of $t = 0$. Hence, for sufficiently small $\varepsilon > 0$,

$$(F_{a,\varepsilon} f)(\xi) = \int\limits_{|t|<\varepsilon} (1 - |z(t)|^2)^{(k-2)/2} \omega_\varepsilon(t)\, |\det(z'(t))|\, dt$$
$$\times \int\limits_{\mathbb{S}^n \cap \eta^\perp} \tilde{f}\left(\begin{bmatrix} \sqrt{1-|z(t)|^2}\varphi \\ z(t) \end{bmatrix} \right) d\varphi,$$

where

$$z'(t) = -\left[\frac{\partial \Phi(t,z)}{\partial z}\right]^{-1} \frac{\partial \Phi(t,z)}{\partial t}, \qquad z = z(t).$$

Passing to the limit, we obtain

$$\lim_{\varepsilon\to 0} (F_{a,\varepsilon} f)(\xi) = |\det(z'(o))| \int\limits_{\mathbb{S}^n \cap \eta^\perp} \tilde{f}(\varphi)\, d\varphi,$$

where

$$z'(o) = (1 - a\cdot\varphi)\Lambda^{-1} = s_a^{-1}(1 - a\cdot\varphi)\,\alpha^{-1/2}, \qquad \alpha = (\mathrm{A}\xi)'(\mathrm{A}\xi).$$

This gives

$$\lim_{\varepsilon\to 0} (F_{a,\varepsilon} f)(\xi) = \frac{s_a^{k-1}}{\det(\alpha)^{1/2}} \int\limits_{\mathbb{S}^n \cap \eta^\perp} \frac{(f\circ\varphi_a)(\varphi)}{(1 - a\cdot\varphi)^{k-1}}\, d\varphi.$$

Note that

$$\begin{aligned} \alpha &= (\mathrm{A}\xi)'(\mathrm{A}\xi) = (s_a \mathrm{P}_a \xi + \mathrm{Q}_a \xi)'(s_a \mathrm{P}_a \xi + \mathrm{Q}_a \xi) \\ &= I_{n+1-k} - \xi' a a' \xi, \end{aligned}$$

and therefore $\det(\alpha) = \det(I_{n+1-k} - \xi' a a' \xi)$. The last expression can be transformed by making use of the known fact from Algebra (see, e.g., [11, Theorem A3.5]). Specifically, if U and V are $m \times n$ and $n \times m$ matrices, respectively, then

$$\det(\mathrm{I}_m + \mathrm{UV}) = \det(\mathrm{I}_n + \mathrm{VU}). \tag{6.15}$$

By this formula, $\det(\alpha) = 1 - (a'\xi)(\xi'a) = 1 - |\xi'a|^2$. Thus, changing notation, as in (3.2), we have

$$\lim_{\varepsilon \to 0} (F_{a,\varepsilon} f)(\xi) = (1 - |\xi'a|^2)^{-1/2} \int\limits_{\mathbb{S}^n \cap \eta^\perp} (M_a f)(y)\, d\sigma(y), \tag{6.16}$$

where $\eta = (\mathrm{A}\xi)\, \alpha^{-1/2}$; cf. (6.13).

Step III Comparing (6.9) with (6.16) and switching backward to the Grassmannian language (use Lemma 6.1), we obtain the statement of the theorem.

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References

1. A. Beardon *The Geometry of Discrete Groups*, Graduate Texts in Mathematics, Springer-Verlag, New York Inc., 1983.
2. P.G. Funk, Über Flächen mit lauter geschlossenen geodätischen Linien, Thesis, Georg-August-Universität Göttingen, 1911.
3. P.G. Funk, Über Flächen mit lauter geschlossenen geodätschen Linen. *Math. Ann.,* **74** (1913), 278–300.
4. R.J. Gardner, *Geometric Tomography* (second edition). Cambridge University Press, New York, 2006.
5. F.W. Gehring, G.J. Martin, B.P. Palka, *An Introduction to the Theory of Higher-Dimensional Quasiconformal Mappings*, AMS, Providence, Rhode Island, 2017.
6. I.M. Gelfand, S.G. Gindikin, M.I. Graev. *Selected Topics in Integral Geometry*, Translations of Mathematical Monographs, AMS, Providence, Rhode Island, 2003.
7. S. Gindikin, J. Reeds, L. Shepp, Spherical tomography and spherical integral geometry. In *Tomography, impedance imaging, and integral geometry (South Hadley, MA, 1993)*, 83–92, Lectures in Appl. Math., **30**, Amer. Math. Soc., Providence, RI (1994).
8. S. Helgason, The totally geodesic Radon transform on constant curvature spaces. *Contemp. Math.*, **113** (1990), 141–149.
9. S. Helgason, *Integral geometry and Radon transform*. Springer, New York-Dordrecht-Heidelberg-London, 2011.
10. H. Minkowski, Über die Körper konstanter Breite [in Russian]. *Mat. Sbornik*. **25** (1904), 505–508; German translation in *Gesammelte Abhandlungen* 2, Bd. (Teubner, Leipzig, (1911), 277–279.
11. R.J. Muirhead, *Aspects of multivariate statistical theory*, John Wiley & Sons. Inc., New York, 1982.
12. G.D. Mostow, Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms. Inst. Hautes Études Sci. Publ. Math. **34** (1968), 53–104.
13. V. Palamodov. *Reconstructive Integral Geometry*. Monographs in Mathematics, 98. Birkhäuser Verlag, Basel, 2004.
14. V.P. Palamodov, *Reconstruction from Integral Data*. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2016.
15. M. Quellmalz, A generalization of the Funk-Radon transform. Inverse Problems **33**, no. 3, 035016, 26 pp. (2017).

16. M. Quellmalz, The Funk-Radon transform for hyperplane sections through a common point, Preprint, arXiv:1810.08105 (2018).
17. B. Rubin, Generalized Minkowski-Funk transforms and small denominators on the sphere. *Fractional Calculus and Applied Analysis* (2) **3** (2000), 177–203.
18. B. Rubin, Inversion formulas for the spherical Radon transform and the generalized cosine transform. *Advances in Appl. Math.*, **29** (2002), 471–497.
19. B. Rubin, On the Funk-Radon-Helgason inversion method in integral geometry. *Contemp. Math.*, **599** (2013), 175–198.
20. B. Rubin, *Introduction to Radon transforms: With elements of fractional calculus and harmonic analysis* (Encyclopedia of Mathematics and its Applications), Cambridge University Press, 2015.
21. B. Rubin, Reconstruction of functions on the sphere from their integrals over hyperplane sections, Analysis and Mathematical Physics, https://doi.org/10.1007/s13324-019-00290-1, 2019.
22. W. Rudin, *Function Theory in the Unit Ball of* C^n, Springer, 1980.
23. Y. Salman, An inversion formula for the spherical transform in S^2 for a special family of circles of integration. Anal. Math. Phys., **6**, no. 1 (2016), 43–58.
24. Y. Salman, Recovering functions defined on the unit sphere by integration on a special family of sub-spheres. Anal. Math. Phys. **7**, no. 2 (2017), 165–185.
25. M. Stoll, *Harmonic and subharmonic function theory on the hyperbolic ball*. LMS Lecture Notes in Mathematics, vol. **155**, Cambridge University Press, 2016.

Berger-Coburn Theorem, Localized Operators, and the Toeplitz Algebra

Wolfram Bauer and Robert Fulsche

Dedicated to Nikolai Vasilevski on the occasion of his 70th birthday

Abstract We give a simplified proof of the Berger-Coburn theorem on the boundedness of Toeplitz operators and extend one part of this theorem to the setting of p-Fock spaces ($1 \leq p \leq \infty$). We present an overview of recent results by various authors on the compactness characterization via the Berezin transform for certain operators acting on the Fock space. Based on these results we present three new characterizations of the Toeplitz C^* algebra generated by Toeplitz operators with bounded symbols.

Keywords Boundedness of Toeplitz operators · Toeplitz algebra · Localized operators

Mathematics Subject Classification (2010) Primary 47B35; Secondary 47L80

1 Introduction

The present paper combines a survey part with some new results in the area of Toeplitz operators on Fock and Bergman spaces. They are among the most intensively studied concrete operators on function Banach or Hilbert spaces. Basic questions concern boundedness and compactness criteria, membership in operator ideals or their index and spectral theory. For a list of classical and more recent results we refer to [1, 7, 8, 14, 15, 27, 28] and the literature cited therein. Instead of considering single operators, the study of C^* or Banach algebras generated

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by Toeplitz operators with specific types of symbols has attracted attention and essential progress has been made during the last years [6, 21, 23–25].

On the one hand the purpose of this paper is to present an outline of some known classical and recent results on boundedness and compactness of Toeplitz operators on p-Fock spaces. Moreover, we highlight some relations between them that became apparent by applying more recent observations. On the other hand we add some new aspects to the theory. We present a simplified proof of one part of the Berger-Coburn theorem from [8]. Our proof uses little "machinery" and even extends the boundedness criterion of the theorem to the setting of p-Fock spaces or weighted L^p-spaces for any $1 \leq p \leq \infty$. In the second part we discuss compactness characterizations of bounded operators on Bergman and Fock spaces via the Berezin transform starting from the classical result by S. Axler and D. Zheng [1] in the case of the Bergman space over the unit disc and its Fock space version in [11].

Theorem 1.1 (S. Axler, D. Zheng [1]) *Let A be an element of the non-closed algebra generated by Toeplitz operators with bounded symbols acting on the Bergman space over the unit disc. Then A is compact if and only if its Berezin symbol $\tilde{A}$ vanishes at the boundary.*

Subsequently Theorem 1.1 has been extended to larger operator algebras, such as operators acting on a scale of Banach spaces in the Fock space when $p = 2$ [4], C^* algebras generated by sufficiently localized operators [26], or the full Toeplitz algebra. The latter is the C^* algebra generated by all Toeplitz operators with essentially bounded symbols [5, 20, 23]. Surprising identities between these algebras or their completions in [25] show that these generalizations in fact are manifestations of the same result. More precisely, they just provide different characterizations of the Toeplitz algebra.

In the Fock space setting and $p = 2$ we prove three new characterizations of the Toeplitz algebra. One of the results (Corollary 4.13) involves bounded Toeplitz operators with (in general unbounded) symbols in the space BMO of functions having bounded mean oscillation. This observation links the discussion to the first part of the paper. In particular, Corollary 4.13 gives an extension of a compactness characterization in [10] from single Toeplitz operators to elements in the generated algebra.

Recently there have been great advances by combining ideas from operator theory on function spaces with techniques that originally have been developed for the spectral theory of band and band-dominated operators (see [5, 13–15, 17, 23]). This has provided efficient tools in the analysis. We will discuss how the space of band-dominated operators gives rise to a new characterization of the Toeplitz algebra. Throughout the text we have collected various open questions which we believe are interesting and may be subject of a future work.

The paper is organized as follows. In Sect. 2 we introduce definitions and notation. As for a comprehensive source on the analysis of Fock spaces we refer to the textbook [28].

A simple proof of an upper bound for the norm $\|T_f^t\|$ of a Toeplitz operator T_f^t with (possibly unbounded) symbol f acting on p-Fock space or weighted L^p-space is given in Sect. 3. This result generalizes a theorem in [8]. We recall examples of Toeplitz operators with highly oscillating unbounded symbols which are well-known in the literature (e.g. [8]) and illuminate the result. Finally, we comment on some progress in [3] on a conjecture by C. Berger and L. Coburn in [8] concerning a boundedness characterization of Toeplitz operators.

Section 4 starts with a (certainly non-complete) survey of the literature on compactness characterizations via the Berezin transform. In particular, we relate the results by applying surprising characterizations of the Toeplitz algebra $\mathcal{T}$ in [25]. By combining some of these results and using an inequality of Sect. 2 we can provide three new characterizations of $\mathcal{T}$. In particular, our observation generalizes a theorem in [10] on a compactness characterization of Toeplitz operators with BMO-symbols.

2 Preliminaries

On $\mathbb{C}^n$ consider the one-parameter family of probability measures

$$d\mu_t(z) = \frac{1}{(\pi t)^n} e^{-\frac{1}{t}|z|^2} dV(z), \qquad t > 0$$

where V denotes the usual Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Here we write $|z| = (|z_1|^2 + \ldots + |z_n|^2)^{\frac{1}{2}}$ for the Euclidean norm of $z \in \mathbb{C}^n$. Throughout the paper we write $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ for the non-negative integers. Let $1 \leq p < \infty$ and $t > 0$ and define

$$L_t^p := L^p(\mathbb{C}^n, \mu_{2t/p}) \qquad \textit{and} \qquad F_t^p := L_t^p \cap \operatorname{Hol}(\mathbb{C}^n).$$

Here $\operatorname{Hol}(\mathbb{C}^n)$ denotes the space of holomorphic functions on $\mathbb{C}^n$. Further, for measurable $g : \mathbb{C}^n \to \mathbb{C}$ we use the notation

$$\|g\|_{L_t^\infty} = \operatorname*{ess\,sup}_{z \in \mathbb{C}^n} |g(z) e^{-\frac{1}{2t}|z|^2}|$$

and set

$$L_t^\infty := \big\{ g : \mathbb{C}^n \to \mathbb{C} \ : \ g \textit{ measurable}, \|g\|_{L_t^\infty} < \infty \big\},$$
$$F_t^\infty := L_t^\infty \cap \operatorname{Hol}(\mathbb{C}^n).$$

Recall that F_t^2 is a reproducing kernel Hilbert space equipped with the standard L_t^2-inner product $\langle f, g \rangle_t := \int_{\mathbb{C}^n} f(z)\overline{g(z)} d\mu_t(z)$ for $f, g \in F_t^2$, the derived L_t^2-norm

$\|f\|_t = (\langle f, f\rangle_t)^{1/2}$ and reproducing kernel

$$K_z^t(w) = K^t(w, z) = e^{\frac{w\cdot\overline{z}}{t}} \in F_t^2.$$

In particular, the orthogonal projection $P^t : L_t^2 \to F_t^2$ is given by

$$P^t f(z) = \langle f, K_z^t\rangle_t = \int_{\mathbb{C}^n} f(w) e^{\frac{z\cdot\overline{w}}{t}} d\mu_t(w).$$

For $p \neq 2$, the integral operator

$$\begin{aligned} P^t f(z) &= \int_{\mathbb{C}^n} f(w) e^{\frac{z\cdot\overline{w}}{t}} d\mu_t(w) \\ &= \left(\frac{p}{2}\right)^n \int_{\mathbb{C}^n} f(w) e^{\frac{z\cdot\overline{w}}{t}} e^{\frac{1}{t}(\frac{p}{2}-1)|w|^2} d\mu_{2t/p}(w) \end{aligned}$$

still defines a bounded projection $P^t : L_t^p \to F_t^p$ [19, Theorem 7.1]. Given a suitable measurable function $f : \mathbb{C}^n \to \mathbb{C}$ and any $t > 0$ we define the Toeplitz operator $T_f^t : F_t^p \to F_t^p$ by

$$T_f^t = P^t M_f.$$

If not further specified we will consider T_f^t on the domain

$$D(T_f^t) = \left\{h \in F_t^p \ : \ fh \in L_t^p\right\}.$$

In the case $t = 1$ we shortly write $P = P^1$ and $T_f = T_f^1$. Especially for $p = 2$ Toeplitz operators on the Fock space are well-studied under different aspects (see [18, 28] and the literature therein). However, due to a number of open problems, in particular concerning their algebraic and analytic properties, they remain interesting objects of current research.

Denote by

$$k_z^t(w) = \frac{K^t(w, z)}{\sqrt{K^t(z, z)}}, \qquad \textit{where} \quad z, w \in \mathbb{C}^n \tag{2.1}$$

the normalized reproducing kernel. Again, for $t = 1$ we write $k_z = k_z^1$ and $K_z = K_z^1$. We define the *Berezin transform* of an operator $A \in \mathcal{L}(F_t^2)$ by

$$\widetilde{A}^{(t)}(z) := \langle Ak_z^t, k_z^t\rangle_t \tag{2.2}$$

and we use the short notation $\widetilde{A} = \widetilde{A}^{(1)}$. Note that (2.2) defines a complex valued function on $\mathbb{C}^n$ which is bounded whenever A is a bounded operator. The Berezin transform of a measurable function f with the property that $fK_z^t \in L_t^2$ for all $z \in \mathbb{C}^n$ is denoted by

$$\widetilde{f}^{(t)}(z) := \langle fk_z^t, k_z^t\rangle_t.$$

$\widetilde{f}^{(4t)}$ coincides with the *heat transform* of f on $\mathbb{C}^n$ at time t. Observe that $\widetilde{T_f^t}^{(t)} = \widetilde{f}^{(t)}$.

3 On the Berger-Coburn Theorem

One of the simple properties of Toeplitz operators is the fact that boundedness of the symbol implies boundedness of the operator. The converse in general is false. Indeed, one of the important questions in the theory of Toeplitz operators is a characterization of the boundedness of the operator in terms of its (unbounded) symbol. In the case $p = 2$ we recall the classical Berger-Coburn theorem on the boundedness of Toeplitz operators on the Fock space:

Theorem 3.1 (Berger-Coburn [8]) *Assume that $f \in L_t^2$. Then the following norm estimates hold true for $T_f^t : F_t^2 \to F_t^2$:*

$$C(s)\|T_f^t\| \geq \|\widetilde{f}^{(s)}\|_\infty, \quad 2t > s > t/2$$

$$c(s)\|\widetilde{f}^{(s)}\|_\infty \geq \|T_f^t\|, \quad t/2 > s > 0.$$

Here, $C(s), c(s) > 0$ are universal constants depending only on s, t and n.

Berger and Coburn proved this result for $t = 2$. However, the proof directly generalizes to the case $t > 0$. We start by a short outline of the original proof of Theorem 3.1 in [8]:

The first estimate is obtained by a trace-estimate of an operator product. More precisely, in [8] a trace-class operator $S_a^{(s)}$ is constructed depending on $a \in \mathbb{C}^n$ and s in the above range such that

$$\operatorname{trace}\left(T_f^t S_a^{(s)}\right) = \widetilde{f}^{(s)}(a).$$

Then the standard trace estimate

$$|\operatorname{trace}(AB)| \leq \|A\|\|B\|_{\mathrm{tr}}$$

can be applied, where A is a bounded operator, B a trace class operator and $\|\cdot\|_{\mathrm{tr}}$ denotes the *trace norm*. The second inequality provides a boundedness criterion for Toeplitz operators and its proof consists of the following steps:

1. Transform T_f^t into a Weyl-pseudodifferential operator on $L^2(\mathbb{R}^n)$ via the Bargmann transform $\mathcal{B}_t : F_t^2 \to L^2(\mathbb{R}^n)$, which defines a bijective Hilbert space isometry.
2. Estimate the symbol of the pseudodifferential operator $W_{\sigma(f)} = \mathcal{B}_t T_f^t \mathcal{B}_t^{-1}$.
3. Apply the Calderón-Vaillancourt Theorem.

Here we will present a proof of the second inequality based on simple estimates for integral operators. In particular, we avoid the theory of pseudodifferential operators and the application of the Calderón-Vaillancourt Theorem. Instead of working on $L^2(\mathbb{R}^n)$ all calculations will be done in the Fock space setting. Our proof has also the advantage that it generalized to the case of the p-Fock space F_t^p for $1 \leq p \leq \infty$. Moreover, it applies to Toeplitz operators interpreted as integral operators on the enveloping space L_t^p.

In the following we assume that $f : \mathbb{C}^n \to \mathbb{C}$ is a measurable function such that $fK_z^t \in L^2(\mathbb{C}^n, \mu_t)$ for all $z \in \mathbb{C}^n$. In particular, the Berezin transform and its off-diagonal extension

$$\widetilde{f}^{(t)}(z, w) := \left\langle f k_z^t, k_w^t \right\rangle_t$$

exist for all $z, w \in \mathbb{C}^n$.

Without any further assumptions T_f^t is an unbounded operator in general. Let $p = 2$ and note that T_f^t is densely defined since $\{K^t(\cdot, z) : z \in \mathbb{C}^n\}$ is a total set in F_t^2. Hence we can consider its adjoint $(T_f^t)^*$. Recall that

$$h \in D\big((T_f^t)^*\big) \Leftrightarrow \exists C > 0 : |\langle T_f^t g, h\rangle_t| \leq C\|g\|_t \ \forall g \in D(T_f^t).$$

For $z \in \mathbb{C}^n$ and $g \in D(T_f^t)$ it holds

$$|\langle T_f^t g, K_z^t\rangle_t| = |\langle g, \overline{f} K_z^t\rangle_t| \leq \|g\|_t \|f K_z^t\|_t,$$

and therefore $\operatorname{span}\{K_z^t : z \in \mathbb{C}^n\} \subseteq D((T_f^t)^*)$. In particular, the adjoint operator $(T_f^t)^*$ is densely defined as well.

We compute a useful representation for the integral kernel of T_f^t on F_t^2:

$$\begin{aligned} T_f^t g(z) &= \langle T_f^t g, K_z^t\rangle_t \\ &= \langle g, (T_f^t)^* K_z^t\rangle_t = \int_{\mathbb{C}^n} \overline{\big((T_f^t)^* K_z^t\big)(w)} g(w) d\mu_t(w). \end{aligned} \tag{3.1}$$

This integral kernel can further be computed as

$$\begin{aligned}\overline{\big((T_f^t)^* K_z^t\big)(w)} &= \overline{\langle (T_f^t)^* K_z^t, K_w^t\rangle_t} = \langle K_w^t, (T_f^t)^* K_z^t\rangle_t \\ &= \langle T_f^t K_w^t, K_z^t\rangle_t = \langle f K_w^t, K_z^t\rangle_t \\ &= \sqrt{K^t(w,w)K^t(z,z)}\langle f k_w^t, k_z^t\rangle_t \\ &= e^{\frac{1}{2t}(|w|^2+|z|^2)}\widetilde{f}^{(t)}(w,z) \\ &= e^{\frac{1}{2t}|z-w|^2+\frac{1}{t}\operatorname{Re}(z\cdot\overline{w})}\widetilde{f}^{(t)}(w,z).\end{aligned}$$

Inserting this expression above gives:

$$T_f^t g(z) = \int_{\mathbb{C}^n} e^{\frac{1}{2t}|z-w|^2+\frac{1}{t}\operatorname{Re}(z\cdot\overline{w})}\widetilde{f}^{(t)}(w,z)g(w)d\mu_t(w), \tag{3.2}$$

which can be considered as an integral operator either on F_t^2 or L_t^2.

Recall that P^t is defined on L_t^p for each $1 \leq p \leq \infty$ by the same integral expression, hence $T_f^t : F_t^p \to F_t^p$ and $T_f^t : F_t^2 \to F_t^2$ act in the same way on the space span$\{K_z^t : z \in \mathbb{C}^n\}$. Therefore, the integral kernel gives the same operator for the F_t^p-version of the Toeplitz operator on $F_t^p \cap F_t^2$, which is a dense subset of F_t^p (for $p < \infty$).

We are now going to derive estimates for $\widetilde{f}^{(t)}(w,z)$. Easy computations show that $fK_z^s \in L^2(\mathbb{C}^n, \mu_s)$ for $0 < s < t$. Hence $\widetilde{f}^{(s)}(w,z)$ exists for all s in the range $(0,t]$. Moreover, from the *semigroup property* of the heat transform it follows that:

$$\big\langle f k_z^t, k_z^t\big\rangle_t = \widetilde{f}^{(t)}(z) = \Big(\widetilde{f}^{(s)}\Big)^{\sim(t-s)}(z) = \big\langle \widetilde{f}^{(s)} k_z^{t-s}, k_z^{t-s}\big\rangle_{t-s}$$

for all $0 \leq s < t$. In particular,

$$\big\langle f K_z^t, K_z^t\big\rangle_t = e^{-\frac{s}{t(t-s)}|z|^2}\big\langle \widetilde{f}^{(s)} K_z^{t-s}, K_z^{t-s}\big\rangle_{t-s}. \tag{3.3}$$

We can extend this relation to off-diagonal values:

Lemma 3.2 *For $z, w \in \mathbb{C}^n$ and $0 \leq s < t$ it holds:*

$$\widetilde{f}^{(t)}(w,z) = e^{\frac{s}{2t(t-s)}|w-z|^2-\frac{is}{t(t-s)}\operatorname{Im}(z\cdot\overline{w})}\big\langle \widetilde{f}^{(s)} k_w^{t-s}, k_z^{t-s}\big\rangle_{t-s}. \tag{3.4}$$

Proof Recall that $\langle f K_w^t, K_z^t\rangle_t$ is anti-holomorphic in w and holomorphic in z. The same holds for

$$e^{-\frac{s}{t(t-s)}z\cdot\overline{w}}\big\langle \widetilde{f}^{(s)} K_w^{t-s}, K_z^{t-s}\big\rangle_{t-s}.$$

Since both functions agree on the diagonal $w = z$ by Eq. (3.3), they agree for all choices of $w, z \in \mathbb{C}^n$ by a well-known identity theorem [12, Proposition 1.69]. Hence,

$$\langle f K_w^t, K_z^t \rangle_t = e^{-\frac{s}{t(t-s)} z\cdot\overline{w}} \langle \widetilde{f}^{(s)} K_w^{t-s}, K_z^{t-s} \rangle_{t-s}.$$

Division by the normalizing factors implies (3.4). □

Lemma 3.3 *Let $g \in L^\infty(\mathbb{C}^n)$ and $t > 0$. Then, it holds*

$$\left| \langle g k_w^t, k_z^t \rangle_t \right| \leq \|g\|_\infty e^{-\frac{1}{4t}|w-z|^2}.$$

Proof Let $z, w \in \mathbb{C}^n$, then:

$$\begin{aligned}
\left| \langle g k_w^t, k_z^t \rangle_t \right| &= \frac{1}{\sqrt{K^t(w,w)K^t(z,z)}} \left| \int_{\mathbb{C}^n} g(u) e^{\frac{1}{t}(u\cdot\overline{w} + z\cdot\overline{u})} d\mu_t(u) \right| \\
&\leq e^{-\frac{1}{2t}(|z|^2+|w|^2)} \|g\|_\infty \int_{\mathbb{C}^n} e^{\frac{1}{t}\operatorname{Re}(u\cdot\overline{w}+z\cdot\overline{u})} d\mu_t(u) \\
&= e^{-\frac{1}{2t}(|z|^2+|w|^2)} \|g\|_\infty \int_{\mathbb{C}^n} e^{\frac{1}{2t} u\cdot\overline{(w+z)} + \frac{1}{2t}\overline{u}\cdot(w+z)} d\mu_t(u) \\
&= e^{-\frac{1}{2t}(|z|^2+|w|^2)} \|g\|_\infty \langle K_{(w+z)/2}^t, K_{(w+z)/2}^t \rangle_t \\
&= e^{-\frac{1}{2t}(|z|^2+|w|^2)} \|g\|_\infty K^t((w+z)/2, (w+z)/2) \\
&= e^{-\frac{1}{2t}(|z|^2+|w|^2) + \frac{1}{4t}|w+z|^2} \|g\|_\infty \\
&= \|g\|_\infty e^{-\frac{1}{4t}|w-z|^2}.
\end{aligned}$$

□

Theorem 3.4 *Assume that f is such that $\widetilde{f}^{(s)}$ is bounded for some $s \in (0, t/2)$.*

(i) *For $1 \leq p \leq \infty$, the integral operator*

$$I_f^t : L_t^p \to F_t^p$$

defined by

$$I_f^t g(z) := \int_{\mathbb{C}^n} e^{\frac{1}{2t}|z-w|^2 + \frac{1}{t}\operatorname{Re}(z\cdot\overline{w})} \widetilde{f}^{(t)}(w,z) g(w) d\mu_t(w)$$

is bounded. Moreover,

$$\|I_f^t\|_{L_t^p \to F_t^p} \leq C \|\widetilde{f}^{(s)}\|_\infty$$

for some constant C which only depends on t, s and n.

(ii) *In particular, for* $1 \le p \le \infty$ *the Toeplitz operator* $T_f^t : F_t^p \to F_t^p$ *is bounded with the same norm bound*

$$\|T_f^t\|_{F_t^p \to F_t^p} \le C \|\widetilde{f}^{(s)}\|_\infty.$$

Proof We prove (i) first. By Lemmas 3.2 and 3.3 we obtain

$$|\widetilde{f}^{(t)}(w,z)| \le \|\widetilde{f}^{(s)}\|_\infty e^{\left(\frac{s}{2t(t-s)} - \frac{1}{4(t-s)}\right)|w-z|^2}.$$

Set $\frac{s}{2t(t-s)} - \frac{1}{4(t-s)} = -\gamma_{s,t}$ and observe that

$$\gamma_{s,t} > 0 \Leftrightarrow s < \frac{t}{2}.$$

For $p = \infty$, it holds

$$\begin{aligned}
\|I_f^t g\|_{F_t^\infty} &\le \left(\frac{1}{\pi t}\right)^n \|\widetilde{f}^{(s)}\|_\infty \times \\
&\times \int_{\mathbb{C}^n} e^{\left(\frac{1}{2t} - \gamma_{s,t}\right)|z-w|^2 + \frac{1}{t}\operatorname{Re}(z\cdot\overline{w}) - \frac{1}{2t}|z|^2 - \frac{1}{t}|w|^2} |g(w)| dV(w) \\
&= \left(\frac{1}{\pi t}\right)^n \|\widetilde{f}^{(s)}\|_\infty \int_{\mathbb{C}^n} |g(w)| e^{-\frac{1}{2t}|w|^2} e^{-\gamma_{s,t}|w-z|^2} dV(w) \\
&\le \left(\frac{1}{\pi t}\right)^n \|\widetilde{f}^{(s)}\|_\infty \|g\|_{L_t^\infty} \int_{\mathbb{C}^n} e^{-\gamma_{s,t}|w-z|^2} dV(w) \\
&= \|\widetilde{f}^{(s)}\|_\infty \|g\|_{L_t^\infty} \left(\frac{1}{\gamma_{s,t} t}\right)^n.
\end{aligned}$$

This proves

$$\|I_f^t\|_{L_t^\infty \to F_t^\infty} \le \|\widetilde{f}^{(s)}\|_\infty \left(\frac{1}{\gamma_{s,t} t}\right)^n.$$

For $p = 1$, we obtain the following:

$$\begin{aligned}
&\|I_f^t g\|_{F_t^1} = \int_{\mathbb{C}^n} |I_f^t g(z)| d\mu_{2t}(z) \\
&= \left(\frac{1}{2t^2\pi^2}\right)^n \times \\
&\times \int_{\mathbb{C}^n} \left| \int_{\mathbb{C}^n} e^{\frac{1}{2t}|z-w|^2 + \frac{1}{t}\operatorname{Re}(z\cdot\overline{w}) - \frac{1}{t}|w|^2 - \frac{1}{2t}|z|^2} \widetilde{f}^{(t)}(w,z) g(w) dV(w) \right| dV(z)
\end{aligned}$$

$$\leq \left(\frac{1}{2t^2\pi^2}\right)^n \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} |g(w)||\widetilde{f}^{(t)}(w,z)|e^{-\frac{1}{2t}|w|^2} dV(w)dV(z)$$
$$\leq \left(\frac{1}{2t^2\pi^2}\right)^n \|\widetilde{f}^{(s)}\|_\infty \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} |g(w)|e^{-\frac{1}{2t}|w|^2} e^{-\gamma_{s,t}|w-z|^2} dV(w)dV(z)$$
$$= \left(\frac{1}{2t^2\pi^2}\right)^n \|\widetilde{f}^{(s)}\|_\infty \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} e^{-\gamma_{s,t}|w-z|^2} dV(z)|g(w)|e^{-\frac{1}{2t}|w|^2} dV(w)$$
$$= \left(\frac{1}{\gamma_{s,t}t}\right)^n \|\widetilde{f}^{(s)}\|_\infty \|g\|_{L_t^1}.$$

Therefore,

$$\|I_f^t\|_{L_t^1 \to F_t^1} \leq \|\widetilde{f}^{(s)}\|_\infty \left(\frac{1}{\gamma_{s,t}t}\right)^n.$$

Using complex interpolation (i.e. the fact that $[L_t^1, L_t^\infty]_\theta = L_t^{p_\theta}$ and $[F_t^1, F_t^\infty]_\theta = F_t^{p_\theta}$, where $p_\theta = 1/(1-\theta)$, c.f. [19, 28]), one obtains

$$\|I_f^t\|_{L_t^p \to F_t^p} \leq \|\widetilde{f}^{(s)}\|_\infty \left(\frac{1}{\gamma_{s,t}t}\right)^n.$$

For (ii) observe that for $1 \leq p \leq \infty$, T_f^t acts by the same integral expression as I_f^t, hence it holds $T_f^t = I_f^t|_{F_t^p}$. Therefore, the Toeplitz operator inherits the norm estimate from I_f^t. □

Remark 3.5

1. From the proof we see that the constant in the estimate essentially behaves as $\left(\frac{t}{t/2-s}\right)^n$ when $s \to t/2$.
2. For $p = 2$ one can obtain a direct proof of the statement without interpolation, using Lemmas 3.2 and 3.3 and Schur's test (cf. the proof of Lemma 4.9 below, which uses a similar argument).

There are various problems left open concerning the characterization of bounded operators. Here, we mention some of them:

Question 1 Does an L_t^p-version of the first estimate in Theorem 3.1 hold?

Question 2 Can one give a reasonable characterization of boundedness for products of Toeplitz operators with unbounded symbols.

Recall that for $p = 2$ and under certain growth conditions of the symbols at infinity finite products of unbounded Toeplitz operators have been well-defined in [2]. They can be interpreted as elements in an algebra of operators acting on a scale of Banach spaces in the Fock space. We mention that boundedness of products $T_f T_{\bar{g}}$ on the 2-Fock space with holomorphic symbols f, g has recently been characterized in [9].

The following conjecture in the case $t = \frac{1}{2}$ was made by C. Berger and L. Coburn in [8].

Conjecture 1 T_f^t is bounded if and only if $\widetilde{f}^{(\frac{t}{2})}$ is bounded.

There are indications that this conjecture may actually hold true. The authors of [8] accompanied their conjecture with the following example:

Example 1 ([8]) Let $\lambda \in \mathbb{C}$ be a parameter. Consider the functions

$$g_\lambda(z) := e^{\lambda|z|^2}, \quad z \in \mathbb{C}^n$$

for $\operatorname{Re}\lambda < \frac{1}{2t}$. The latter condition guarantees that the heat transforms $\widetilde{g_\lambda}^{(s)}$ exist for all $0 < s < 2t$. Moreover, since g_λ is radial, a simple calculation shows that the Toeplitz operator $T_{g_\lambda}^t$ acts diagonally on the standard orthonormal basis $\{e_m^t : m \in \mathbb{N}_0^n\}$ of F_t^2, where

$$e_m^t(z) := \frac{z^m}{\sqrt{t^{|m|}m!}}. \tag{3.5}$$

Here, we used standard multiindex notation. One has

$$T_{g_\lambda}^t e_m^t = (1 - t\lambda)^{-(|m|+n)}\, e_m^t.$$

This implies that $T_{g_\lambda}^t$ is bounded on F_t^2 if and only if $|1 - t\lambda| \geq 1$.

Using the well-known formula

$$\int_{\mathbb{R}} e^{bx - \frac{a}{2}x^2} dx = \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}}, \quad a, b \in \mathbb{C}, \ \operatorname{Re}(a) > 0,$$

one can show that

$$\widetilde{g_\lambda}^{(s)}(w) = \frac{1}{(1 - s\lambda)^n} e^{\frac{\lambda}{1 - s\lambda}|w|^2}, \quad 0 < s < 2t.$$

This function is obviously bounded if and only if

$$\operatorname{Re}\left(\frac{\lambda}{1 - s\lambda}\right) \leq 0 \iff |1 - 2s\lambda| \geq 1.$$

In particular, $T_{g_\lambda}^t$ is bounded if and only if $\widetilde{g_\lambda}^{(\frac{t}{2})}$ is a bounded function.

There are two other known cases for which the conjecture has been verified. The first concerns operators with non-negative symbols.

Proposition 3.6 (Berger-Coburn [7]) *Let $f \geq 0$ be such that $f \in L_t^2$. Then, $T_f^t : F_t^2 \to F_t^2$ is bounded if and only if $\widetilde{f}^{(\frac{t}{2})}$ is bounded.*

Since this result is not directly stated in [7] (even though [8] refers this fact to that article), we give a short proof based on a lemma from that paper:

Proof In the case of $t = 2$, [7, Lemma 14] states and proves the following estimate, which holds with the same proof for general $t > 0$:

$$\| \widetilde{|g|^2}^{(t)} \|_\infty \leq \| T_{|g|^2}^t \| \leq 4^n \| \widetilde{|g|^2}^{(t)} \|_\infty.$$

If $f \geq 0$, letting $g = \sqrt{f}$ and applying this inequality proves that T_f^t is bounded if and only if $\widetilde{f}^{(t)}$ is bounded. A simple estimate yields $\widetilde{f}^{(t)}(z) \geq \frac{1}{2^n} \widetilde{f}^{(\frac{t}{2})}(z)$. Further, $\widetilde{f}^{(t)}(z) \leq \| \widetilde{f}^{(\frac{t}{2})} \|_\infty$ can be seen to hold by the semigroup property of the heat transform. Therefore, $\widetilde{f}^{(t)}$ is bounded if and only if $\widetilde{f}^{(\frac{t}{2})}$ is bounded. □

The second result is about symbols with certain oscillatory behaviour at infinity. For $z \in \mathbb{C}^n$ let $\tau_z(w) = z - w, \ w \in \mathbb{C}^n$. For $f \in L^1_{\text{loc}}(\mathbb{C}^n)$, we say that f is of *bounded mean oscillation* and write $f \in \text{BMO}$ if

$$\sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} |f \circ \tau_z - \widetilde{f}^{(t)}(z)| d\mu_t < \infty.$$

This notion can be seen to be independent of $t > 0$, cf. [3] for details. As is known BMO contains unbounded function. One has:

Theorem 3.7 (Bauer-Coburn-Isralowitz [3, Theorem 6]) *Let $f \in \text{BMO}$. Then, $\widetilde{f}^{(t)}$ is bounded for one $t > 0$ if and only if it is bounded for all $t > 0$. In particular, $T_f^t : F_t^2 \to F_t^2$ is bounded if and only if $\widetilde{f}^{(\frac{t}{2})}$ is bounded.*

We will come back to the last result when discussing different characterizations of the Toeplitz algebra.

4 Characterizations of the Toeplitz Algebra

From now on we only consider the case $p = 2$ and $t = 1$. It is tautological to say that operator-theoretic properties of Toeplitz operators are tightly related to properties of their symbols. One of the most basic results of this kind is the following: Let $f \in C(\mathbb{C}^n)$ be such that

$$f(z) \to 0 \quad as \quad |z| \to \infty.$$

Then, T_f is compact. As is well-known the Berezin transform is one-to-one on the algebra $\mathcal{L}(F_1^2)$ of all bounded operators. This indicates that operator theoretic properties are also tightly connected to properties of the Berezin transform, e.g. if T_f is compact, then it holds

$$\widetilde{T_f}(z) = \widetilde{f}(z) \to 0 \quad as \quad |z| \to \infty.$$

Conversely, it is an obvious question how compactness of an operator can be characterized in terms of the symbol (in case of a Toeplitz operator) or its Berezin transform. The first significant progress in this context was made for certain operators acting on the Bergman space over the unit disc $A^2(\mathbb{D})$. We will not introduce all objects involved and refer to [27] for an introduction to Toeplitz operators on $A^2(\mathbb{D})$.

Theorem 4.1 (Axler-Zheng [1, Theorem 2.2]) *Let $A \in \mathcal{L}(A^2(\mathbb{D}))$ be a finite sum of finite products of Toeplitz operators with essentially bounded symbols. Then*

$$A \text{ } is \text{ } compact \iff \mathcal{B}(A)(z) \to 0 \ \ as \ z \to \partial\mathbb{D}.$$

Here, $\mathcal{B}(A)$ denotes the Berezin transform of A.

The corresponding statement for the weighted Bergman spaces over the unit ball in $\mathbb{C}^n$ or even general bounded symmetric domains (with some extra conditions on the weights) was derived shortly afterwards [11, 22]. In [11] also the Fock space is treated:

Theorem 4.2 (Engliš [11, Theorem B]) *Let $A \in \mathcal{L}(F_1^2)$ be a finite sum of finite products of Toeplitz operators with essentially bounded symbols. Then, the following are equivalent:*

$$A \text{ is compact} \iff \widetilde{A}(z) \to 0 \quad as \ |z| \to \infty.$$

The next progress made towards a characterization of compactness for an even larger set of operators acting on F_1^2 followed several years later. Although we put $t = 1$ in our discussion most of the remaining statements hold true in the more general case $t > 0$ with almost identical proofs.

Set

$$c_0 := 0, \quad c_{j+1} := \frac{1}{4(1 - c_j)} \qquad where \qquad j \in \mathbb{N}_0$$

and note that the sequence $(c_j)_j$ is monotonely increasing with $c_j < \frac{1}{2}$. Let $f : \mathbb{C}^n \to \mathbb{C}$ be measurable and define:

$$\|f\|_{\mathcal{D}_{c_j}} := \operatorname*{ess\,sup}_{z \in \mathbb{C}^n} |f(z)e^{-c_j|z|^2}|.$$

Then, letting

$$\mathcal{D}_{c_j} = \big\{ f : \mathbb{C}^n \to \mathbb{C}\, measurable \, : \, \|f\|_{\mathcal{D}_{c_j}} < \infty \big\},$$

the norm $\|\cdot\|_{\mathcal{D}_{c_j}}$ induces the structure of a Banach space on $\mathcal{D}_{c_j}$. By varying $j \in \mathbb{N}_0$ we obtain an increasing scale of Banach spaces:

$$L^\infty(\mathbb{C}^n) = \mathcal{D}_{c_0} \subset \mathcal{D}_{c_1} \cdots \subset \mathcal{D}_{c_j} \subset \cdots \subset \mathcal{D} := \bigcup_{j \in \mathbb{N}_0} \mathcal{D}_{c_j} \subset L_1^2.$$

Setting now

$$\mathcal{H}_{c_j} = \mathcal{D}_{c_j} \cap \mathrm{Hol}(\mathbb{C}^n),$$

equipped with the norm of $\mathcal{D}_{c_j}$ we obtain a second scale of Banach spaces in the Fock space F_1^2:

$$\mathbb{C} \cong \mathcal{H}_0 \subset \mathcal{H}_{c_1} \subset \cdots \subset \mathcal{H}_{c_j} \subset \cdots \subset \mathcal{H} := \bigcup_{j \in \mathbb{N}_0} \mathcal{H}_{c_j} \subset F_1^2. \tag{4.1}$$

For $z \in \mathbb{C}^n$ we define the *Weyl operators* $W_z : F_1^2 \to F_1^2$ through

$$W_z(f)(w) = k_z(w) f(w - z).$$

The operators W_z are well known to be unitary. Further, it holds for all $z, w \in \mathbb{C}^n$:

1. $W_z^* = W_z^{-1} = W_{-z}$
2. $W_z W_w = e^{-i\,\mathrm{Im}(z \cdot \overline{w})} W_{z+w}$.

For a linear operator A on F_1^2 and $z \in \mathbb{C}$ we set

$$A_z := W_z^* A W_z.$$

Definition 4.3 A bounded $\mathbb{R}$-linear operator $A : F_1^2 \to F_1^2$ with $A(\mathcal{H}) \subset \mathcal{H}$ is said to *act uniformly continuously* on the scale (4.1) if for each $j_1 \in \mathbb{N}_0$ there exists $j_2 \geq j_1$ and $d > 0$ independent of $z \in \mathbb{C}^n$ such that for all $f \in \mathcal{H}_{j_1}$:

$$\|A_z f\|_{\mathcal{D}_{c_{j_2}}} \leq d \|f\|_{\mathcal{D}_{c_{j_1}}}.$$

We denote the spaces of $\mathbb{C}$-linear operators and the $\mathbb{C}$-antilinear operators acting uniformly continuously on the scale (4.1) by $\mathcal{F}^l$ and $\mathcal{F}^{al}$, respectively.

Several properties of $\mathcal{F}^l$ and $\mathcal{F}^{al}$ are known:

Proposition 4.4 (Bauer-Furutani [4])

1. $\mathcal{F}^l$ *is a* $*$*-algebra,*
2. $\mathcal{F}^l$ *contains all Toeplitz operators with essentially bounded symbol,*
3. $\mathcal{F}^{al}$ *contains the operators* w_f*.*

Here, for $f \in L^\infty(\mathbb{C}^n)$, we define the antilinear operator w_f on F_1^2 by

$$w_f(g) := P(fCg),$$

where $Cg(z) = \overline{g(z)}$ denotes complex conjugation. Those operators are closely related to the little Hankel operators. In [4] they played a role in compactness characterizations of Toeplitz operators on the pluriharmonic Fock space. Since w_f is not a finite sum of finite products of Toeplitz operators, the following result was an improvement (in the case $t = 1$) of Theorem 4.2:

Theorem 4.5 (Bauer-Furutani [4, Theorem 3.11]) *Let* $A \in \mathcal{F}^l \cup \mathcal{F}^{al}$*. Then,* A *is compact if and only if* $\widetilde{A}(z) \to 0$ *as* $|z| \to \infty$*.*

Recall that the (full) Toeplitz algebra is the C^* algebra generated by Toeplitz operators with bounded symbols, i.e.

$$\mathcal{T} = C^*\big(\{T_f \ : \ f \in L^\infty(\mathbb{C}^n)\}\big),$$

where $C^*(M)$ denotes the C^* algebra generated by a given set $M \subset \mathcal{L}(F_1^2)$.

The next step forward was a complete characterization of compact operators in terms of the Toeplitz algebra and the Berezin transform. This result was first obtained in [23, Theorem 9.5] and [20, Theorem 1.1] in the setting of the Bergman space and standard weighted Bergman space over the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$, respectively. There also is a version for the p-Fock space which we state next in the special case $p = 2$.

Theorem 4.6 (Bauer-Isralowitz [5, Theorem 1.1]) *Let* $A \in \mathcal{L}(F_1^2)$*. Then, it holds*

$$A \text{ is compact } \iff A \in \mathcal{T} \text{ and } \widetilde{A}(z) \to 0 \text{ as } |z| \to \infty.$$

Remark 4.7 Theorem 4.6 and its versions on Bergman spaces over $\mathbb{B}_n$ have been proven in the general setting of standard weighted p-Bergman spaces $A_\alpha^p(\mathbb{B}_n)$ over the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$ (here the parameter α refers to the weight) and p-Fock spaces F_t^p for $1 < p < \infty$ in the original papers. For simplicity, we have only quoted the Hilbert space result. Moreover, a version of this theorem on p-Bergman spaces over bounded symmetric domains has recently been studied in [15].

One could think that this is the end of the story for characterizing compact operators on F_1^2. However, there is a link between Theorems 4.5 and 4.6 which we will discuss next. If a priori an operator A is not given as a finite sum of finite products of Toeplitz operators it may be difficult to either check whether $A \in \mathcal{F}^l \cup \mathcal{F}^{al}$ or $A \in \mathcal{T}$ as is assumed in the above theorems. On the other hand, there are well-known examples in the literature of operators $A \in \mathcal{L}(F_1^2)$ which are not compact but with $\widetilde{A}(z) \to 0$ as $|z| \to \infty$. For completeness we will present such an example which even is a bounded Toeplitz operator (with unbounded symbol) below. Hence some additional assumptions are required to ensure that vanishing of the Berezin transform at infinity implies compactness of the operator. It is therefore desirable to extend Theorem 4.6 to a class $\mathcal{A} \subset \mathcal{L}(F_1^2)$ containing the Toeplitz algebra and for which membership $A \in \mathcal{A}$ is easier to check. On the Fock space a new approach appeared in [17, 26]:

Definition 4.8 An operator $A \in \mathcal{L}(F_1^2)$ is said to be *sufficiently localized* if there exist constants β, C with $2n < \beta < \infty$ and $0 < C$ such that

$$\left|\langle Ak_z, k_w\rangle_1\right| \leq \frac{C}{(1+|z-w|)^\beta}. \tag{4.2}$$

We denote the set of all sufficiently localized operators by $\mathcal{A}_{sl}$.

We note at this point that boundedness of an operator A with $k_z \in D(A)$ for all $z \in \mathbb{C}^n$ and with (4.2) already follows under certain natural assumptions on the domain of A^*:

Lemma 4.9 *Let A be a densely defined operator on F_1^2 such that*

$$\operatorname{span}\{K_z \;:\; z \in \mathbb{C}^n\} \subset D(A) \cap D(A^*).$$

If there is a positive function $H \in L^1(\mathbb{C}^n, V)$ with

$$|\langle Ak_z, k_w\rangle_1| \leq H(z-w)$$

for all $z, w \in \mathbb{C}^n$, then A is bounded. In particular, (4.2) implies boundedness of A since

$$\frac{1}{(1+|z|)^\beta} \in L^1(\mathbb{C}^n, V) \quad \text{if} \quad \beta > 2n.$$

Proof Similarly to the computation in (3.1) one obtains:

$$\begin{aligned}(Af)(w) &= \int_{\mathbb{C}^n} f(z)\overline{(A^*K_w)(z)}d\mu_1(z)\\ &= \int_{\mathbb{C}^n} \|K_z\|_1\|K_w\|_1\langle Ak_z, k_w\rangle_1 f(z)d\mu_1(z).\end{aligned}$$

Letting

$$\widetilde{A}(z,w) := \langle Ak_z, k_w\rangle_1,$$

this implies

$$(Af)(w) = \int_{\mathbb{C}^n} e^{\frac{|z|^2+|w|^2}{2}} \widetilde{A}(z,w) f(z) d\mu_1(z).$$

Observe that it suffices to prove boundedness of the integral operator with kernel $|\widetilde{A}(z,w)|e^{\frac{|z|^2+|w|^2}{2}}$. It holds

$$\begin{aligned}\frac{1}{\pi^n}\int_{\mathbb{C}^n} |\widetilde{A}(z,w)|e^{\frac{|z|^2+|w|^2}{2}} e^{\frac{|w|^2}{2}} e^{-|w|^2} dV(w) &= \\ &= e^{\frac{|z|^2}{2}} \frac{1}{\pi^n}\int_{\mathbb{C}^n} |\widetilde{A}(z,w)| dV(w) \\ &\leq e^{\frac{|z|^2}{2}} \frac{1}{\pi^n}\int_{\mathbb{C}^n} H(z-w) dV(w) \\ &= \frac{1}{\pi^n}\|H\|_{L^1(\mathbb{C}^n,V)} e^{\frac{|z|^2}{2}}.\end{aligned}$$

Analogously:

$$\frac{1}{\pi^n}\int_{\mathbb{C}^n} |\widetilde{A}(z,w)|e^{\frac{|z|^2+|w|^2}{2}} e^{\frac{|z|^2}{2}} e^{-|z|^2} dV(z) \leq \frac{1}{\pi^n}\|H\|_{L^1(\mathbb{C}^n,V)} e^{\frac{|w|^2}{2}}.$$

Using Schur's test [16, Theorem 5.2] with the function $h(w) = e^{\frac{|w|^2}{2}}$, one therefore obtains

$$\|A\| \leq \frac{1}{\pi^n}\|H\|_{L^1(\mathbb{C}^n,V)}.$$

For the particular case

$$H(w) := \frac{1}{(1+|w|)^\beta}$$

with $\beta > 2n$, one easily sees that $H \in L^1(\mathbb{C}^n, V)$ using polar coordinates. □

Checking localizedness of an operator in the above sense is indeed simpler than checking membership in the Toeplitz algebra. In fact, (4.2) reduces (in principle) to some integral estimate. Further, any Toeplitz operator with bounded symbol is indeed sufficiently localized according to the simple estimate in Lemma 3.3. Hence we obtain:

$$\mathcal{T} \subseteq C^*(\mathcal{A}_{sl}).$$

We now relate the notion of localized operators to the result in Theorem 4.5.

Lemma 4.10 *Let $A \in \mathcal{F}^l$. Then, A is sufficiently localized.*

Proof With $z, w \in \mathbb{C}^n$ we obtain:

$$\begin{aligned}\langle Ak_z, k_w\rangle_1 &= \langle AW_z 1, W_z W_z^* W_w 1\rangle_1\\ &= \langle A_z 1, W_{-z} W_w 1\rangle_1\\ &= \langle A_z 1, W_{w-z} 1\rangle_1 e^{-i\,\mathrm{Im}(z\cdot\overline{w})}\\ &= \langle A_z 1, k_{w-z}\rangle_1 e^{-i\,\mathrm{Im}(z\cdot\overline{w})}.\end{aligned}$$

Using this, we obtain for a suitable index $k \in \mathbb{N}_0$ and from $\|1\|_{\mathcal{D}_{c_k}} = 1$:

$$\begin{aligned}|\langle Ak_z, k_w\rangle_1| &\le \frac{1}{\pi^n}\int_{\mathbb{C}^n} \big|[A_z 1](u) k_{w-z}(u)\big| e^{-|u|^2} dV(u)\\ &\le \frac{1}{\pi^n}\int_{\mathbb{C}^n} \|A_z 1\|_{\mathcal{D}_{c_k}} e^{\mathrm{Re}((w-z)\cdot\overline{u}) - \frac{|w-z|^2}{2} - (1-c_k)|u|^2} dV(u)\\ &\le \frac{d}{\pi^n}\int_{\mathbb{C}^n} e^{\mathrm{Re}((w-z)\cdot\overline{u}) - (1-c_k)|u|^2} dV(u)\, e^{-\frac{|w-z|^2}{2}},\end{aligned}$$

where $d > 0$ is the constant from the uniformly continuous action of A on the scale (4.1). Recall that for $a \in \mathbb{C}^n$, $\gamma > 0$ and by applying the properties of the reproducing kernel:

$$\begin{aligned}e^{\gamma|a|^2} &= \int_{\mathbb{C}^n} |e^{\gamma a\cdot\overline{u}}|^2 d\mu_{1/\gamma}(u)\\ &= \int_{\mathbb{C}^n} e^{2\gamma\,\mathrm{Re}(a\cdot\overline{u})} d\mu_{1/\gamma}(u).\end{aligned}$$

Letting $\gamma = 1 - c_k$ and $a = \frac{w-z}{2\gamma}$ gives

$$\int_{\mathbb{C}^n} e^{\mathrm{Re}((w-z)\cdot\overline{u}) - (1-c_k)|u|^2} dV(u) = \frac{\pi^n}{(1-c_k)^n} e^{\frac{|z-w|^2}{4(1-c_k)}}.$$

We hence obtain

$$\begin{aligned}\big|\big\langle Ak_z, k_w\big\rangle_1\big| &\le \frac{d}{(1-c_k)^n} e^{\frac{|z-w|^2}{4(1-c_k)} - \frac{|z-w|^2}{2}}\\ &= \frac{d}{(1-c_k)^n} e^{-(\frac{1}{2} - \frac{1}{4(1-c_k)})|z-w|^2}\\ &= \frac{d}{(1-c_k)^n} e^{-(\frac{1}{2} - c_{k+1})|z-w|^2}.\end{aligned}$$

Since $c_{k+1} < \frac{1}{2}$ the statement follows. □

We can slightly relax the boundedness assumption of the operator symbol and still obtain sufficiently localized Toeplitz operators. More precisely:

Lemma 4.11 *Let $f \in$ BMO be such that the Toeplitz operator T_f is bounded. Then, T_f is sufficiently localized.*

Proof According to Lemma 3.2 and for $0 < s < 1$ we have the identity

$$\langle fk_z, k_w \rangle_1 = e^{\frac{s}{2(1-s)}|z-w|^2 - \frac{is}{(1-s)}\operatorname{Im}(w\cdot\overline{z})} \langle \widetilde{f}^{(s)} k_z^{1-s}, k_w^{1-s} \rangle_{1-s}.$$

Applying Lemma 3.3 with $g \in L^\infty(\mathbb{C}^n)$ shows

$$\left| \langle g k_z^{1-s}, k_w^{1-s} \rangle_{1-s} \right| \leq \|g\|_\infty e^{-\frac{1}{4(1-s)}|z-w|^2}.$$

By Theorem 3.7 the heat transform $\widetilde{f}^{(s)}$ is bounded for each $0 < s < 1$. Choosing now $s < 1/2$, we obtain from the above estimates

$$\left| \langle fk_z, k_w \rangle_1 \right| \leq \|\widetilde{f}^{(s)}\|_\infty e^{\frac{2s-1}{4(1-s)}|z-w|^2},$$

where $\frac{2s-1}{4(1-s)} < 0$. □

It was proven by J. Xia and D. Zheng that $C^*(\mathcal{A}_{sl})$ is not too large in the sense that it still allows the desired compactness characterization:

Theorem 4.12 (Xia-Zheng [26, Theorem 1.2]) *Let $A \in C^*(\mathcal{A}_{sl})$. Then, A is compact if and only if $\widetilde{A}(z) \to 0$ as $z \to \infty$.*

At this point we want to mention that the compactness characterization was already known to hold for bounded single Toeplitz operators with (possibly unbounded) symbol in BMO. This has been achieved by N. Zorboska in [29] for the Bergman space $A^2(\mathbb{D})$ and later by L. Coburn, J. Isralowitz and B. Li in the setting of the Fock space [10]. By combining Lemma 4.11 and Theorem 4.12, we obtain a generalization of the latter result:

Corollary 4.13 *Let $A \in \mathcal{L}(F_1^2)$ be a finite sum of finite products of bounded Toeplitz operators with (possibly unbounded) symbols in* BMO. *Then, A is compact if and only if $\widetilde{A}(z) \to 0$ as $|z| \to \infty$.*

A next step was taken in [17]:

Definition 4.14 An operator $A \in \mathcal{L}(F_1^2)$ is called *weakly localized* if:

$$\sup_{z\in\mathbb{C}^n} \int_{\mathbb{C}^n} |\langle Ak_z, k_w\rangle_1| dV(w) < \infty,$$

$$\sup_{z\in\mathbb{C}^n} \int_{\mathbb{C}^n} |\langle A^*k_z, k_w\rangle_1| dV(w) < \infty,$$

$$\lim_{r\to\infty} \sup_{z\in\mathbb{C}^n} \int_{|z-w|\geq r} |\langle Ak_z, k_w\rangle_1| dV(w) = 0,$$

$$\lim_{r\to\infty} \sup_{z\in\mathbb{C}^n} \int_{|z-w|\geq r} |\langle A^*k_z, k_w\rangle_1| dV(w) = 0.$$

The set of all weakly localized operators is denoted by $\mathcal{A}_{wl}$.

It is easy to see that $\mathcal{A}_{sl} \subseteq \mathcal{A}_{wl}$. The analogue of Theorem 4.6 for the setting $C^*(\mathcal{A}_{wl})$ was proven in [17] in case of the Fock space as well as for the Bergman space over the unit ball.

Finally, the following surprising result was obtained by J. Xia [25] in the setting of the Bergman space $A^2(\mathbb{B}_n)$ as well as for the Fock space F_1^2:

Theorem 4.15 (Xia [25, Section 4]) *It holds*

$$\mathcal{T}^{(1)} = \mathcal{T} = C^*(\mathcal{A}_{sl}) = C^*(\mathcal{A}_{wl}).$$

Here, $\mathcal{T}^{(1)}$ denotes the norm closure of $\{T_f \ : \ f \in L^\infty\}$. Theorem 4.15 indicates that indeed all the generalizations of Theorem 4.6 were just new characterizations of the Toeplitz algebra. Based on the observations in Lemmas 4.10 and 4.11 we can add two other characterizations:

Theorem 4.16 *Denote by $\overline{\mathcal{F}^l}$ the operator norm closure of $\mathcal{F}^l$, which is a C^* algebra. It then holds:*

$$\mathcal{T} = \overline{\mathcal{F}^l} = C^*\big(\{T_f;\ f \in \text{BMO} \ \textit{ and } T_f \textit{ bounded}\}\big).$$

Proof Obviously, $\overline{\mathcal{F}^l}$ defines a C^* algebra in $\mathcal{L}(F_1^2)$ and since all Toeplitz operators with essentially bounded symbols are uniformly continuously acting on the scale (4.1) (cf. [4] for a calculation) it follows that $\mathcal{T} \subset \overline{\mathcal{F}^l}$. Since elements in $\mathcal{F}^l$ are sufficiently localized according to Lemma 4.10 we have:

$$\overline{\mathcal{F}^l} \subset C^*(\mathcal{A}_{sl}) = \mathcal{T}.$$

The last identity follows from J. Xia's result in Theorem 4.15. The assertion $\mathcal{T} = \overline{\mathcal{F}^l}$ is obtained by combining both inclusions.

As for the second characterization of the Toeplitz algebra recall that $L^\infty \subset$ BMO. It follows, using Lemma 4.11

$$\mathcal{T} \subset C^*\big(\{T_f;\ f \in \text{BMO} \ \textit{ and } T_f \ \textit{ bounded}\}\big) \subset C^*(\mathcal{A}_{sl}).$$

Equality again follows from Theorem 4.15. □

It is worth mentioning, that there are bounded Toeplitz operators (necessarily with symbols not in BMO) which do not satisfy the desired compactness characterization. In particular, they cannot define elements in the Toeplitz algebra. We give one such example (which has been previously known in the literature):

Example 2 Consider again the function $g_\lambda(z) = e^{\lambda|z|^2}$ from Example 1. Let $\lambda \in \mathbb{C}$ be such that

$$\mathrm{Re}(\lambda) < \frac{1}{2} \tag{4.3}$$

$$|1-\lambda| = 1 \tag{4.4}$$

$$|1-2\lambda| > 1. \tag{4.5}$$

It is easy to see that such λ exist. Assumption (4.3) guarantees that $\widetilde{T_{g_\lambda}}$ is well-defined. Recall that it is given by

$$\widetilde{T_{g_\lambda}}(z) = \frac{1}{(1-\lambda)^n} e^{\frac{\lambda}{1-\lambda}|z|^2}.$$

Assumption (4.5) is equivalent to $\mathrm{Re}(\lambda/(1-\lambda)) < 0$. Hence, it implies that

$$\widetilde{T_{g_\lambda}}(z) \to 0 \qquad as \qquad |z| \to \infty.$$

Further, recall that T_{g_λ} acts on the standard orthonormal basis in (3.5) as:

$$T_{g_\lambda} e_m^1 = \frac{1}{(1-\lambda)^{|m|+n}} e_m^1, \qquad m \in \mathbb{N}_0^n.$$

Put $\nu = \frac{1}{1-\lambda}$. Since e_m^1 is a homogeneous polynomial of degree $|m|$ we have:

$$T_{g_\lambda} e_m^1(z) = \nu^n \cdot e_m^1(\nu z).$$

As assumption (4.4) implies that T_{g_λ} is bounded, this equation extends to all of F_1^2:

$$T_{g_\lambda} f(z) = \nu^n \cdot f(\nu z), \qquad f \in F_1^2.$$

Because of $|\nu| = 1$ the Toeplitz operator T_{g_λ} is unitary, hence cannot be compact.

Let us return to the initial proof of Theorem 4.6 in [5] (which is similar to the proof in the case of the Bergman space $A^2(\mathbb{B}_n)$ [20, 23]). Among other ideas the arguments used results on limit operators. Based on such limit operator techniques, a characterization of the Fredholm property of operators in $\mathcal{T}$ has been proven in [13] (subsequently to [14] which contains the analysis in the case of the standard weighted Bergman spaces $A^p_\alpha(\mathbb{B}_n)$). There, the so-called band-dominated operators were introduced:

Definition 4.17

1. An operator $A \in \mathcal{L}(L^2_1)$ is called a *band operator* if there is a number $\omega > 0$ such that $M_f A M_g = 0$ for all $f, g \in L^\infty(\mathbb{C}^n)$ with
$$d\big(\operatorname{supp} f, \operatorname{supp} g\big) > \omega.$$
Here d denotes the Euclidean distance and we write M_f for the multiplication operator by f. The infimum of all such ω will be called the *band-width* of A.
2. The set BDO of *band-dominated operators* is defined as the norm-closure of all band operators in $\mathcal{L}(L^2_1)$.

The following properties of BDO have been derived, which relate compressions to F^2_1 of band-dominated operators to the Toeplitz algebra.

Proposition 4.18 (Fulsche-Hagger [13])

1. BDO *is a* C^* *algebra of operators on* L^2_1 *containing* P *and multiplication operators* M_f *for* $f \in L^\infty$.
2. P BDO $P \subset \mathcal{L}(F^2_1)$ *contains* $\mathcal{T}$.

In principle, a similar limit operator approach as in [5] can be used to derive a compactness characterization for operators from P BDO P. This has not been worked out for the Fock space, but was done for the Bergman space over bounded symmetric domains in [15, Theorem A]. After the preceding discussions, this naturally leads to the question whether P BDO $P = \mathcal{T}$. That this is indeed true will be the content of the last part of this work. Lemma 4.19 and the main result in Theorem 4.20 were communicated to us by Raffael Hagger.

Lemma 4.19 *For all* $z \in \mathbb{C}^n$ *and* $r > 0$ *it holds*

$$\|M_{1-\chi_{B(z,r)}} k_z\|_1 \leq C_n e^{-\frac{r^2}{2n}},$$

where $C_n > 0$ *is some constant depending only on the dimension* n*. Here,* $B(z,r) \subset \mathbb{C}^n$ *is the Euclidean ball of radius* r *around* z *and* $\chi_{B(z,r)}$ *is the indicator function of that ball.*

Proof For $z \in \mathbb{C}^n$ and $r > 0$ denote by $Q(z,r)$ the set

$$\begin{aligned} Q(z,r) &:= \{w \in \mathbb{C}^n \ : \ |w_j - z_j| < r \text{ for all } \ j = 1, \dots, n\} \\ &= \prod_{j=1}^n D(z_j, r), \end{aligned}$$

where $D(z_j, r) \subset \mathbb{C}$ is the disc around $z_j \in \mathbb{C}$ of radius r. It is immediate that $Q(0, \frac{r}{\sqrt{n}}) \subset B(0, r)$.

We estimate the norm for $z = 0$ first:

$$\begin{aligned}\|M_{1-\chi_{B(0,r)}} k_0\|_1^2 &= 1 - \frac{1}{\pi^n} \int_{B(0,r)} e^{-|z|^2} dV(z) \\ &\leq 1 - \frac{1}{\pi^n} \int_{Q(0,\frac{r}{\sqrt{n}})} e^{-|z|^2} dV(z) \\ &= 1 - \left(\frac{1}{\pi} \int_{D(0,\frac{r}{\sqrt{n}})} e^{-|z_1|^2} dV_1(z_1) \right)^n,\end{aligned}$$

where in the last integral V_1 denotes the Lebesgue measure on $\mathbb{C}$. Using polar coordinates, one easily sees that

$$\frac{1}{\pi} \int_{D(0,\frac{r}{\sqrt{n}})} e^{-|z_1|^2} dV_1(z_1) = 1 - e^{-\frac{r^2}{n}}.$$

This gives

$$\begin{aligned}\|M_{1-\chi_{B(0,r)}} k_0\|_1 &\leq 1 - \left(1 - e^{-\frac{r^2}{n}}\right)^n = 1 - \sum_{k=0}^n \binom{n}{k} (-1)^k e^{-\frac{kr^2}{n}} \\ &= \sum_{k=1}^n \binom{n}{k} (-1)^{k+1} e^{-\frac{kr^2}{n}} \leq \sum_{k=1}^n \binom{n}{k} e^{-\frac{r^2}{n}}.\end{aligned}$$

For general $z \in \mathbb{C}^n$, we obtain

$$\begin{aligned}\|M_{1-\chi_{B(z,r)}} k_z\|_1 &= \|W_{-z} M_{1-\chi_{B(z,r)}} W_z k_0\|_1 \\ &= \|M_{1-\chi_{B(0,r)}} k_0\|_1 \leq C_n e^{-\frac{r^2}{2n}},\end{aligned}$$

where we used the facts that W_{-z} is an isometry and $k_z = W_z k_0$ in the first equality and $W_{-z} M_f W_z = M_{f \circ \tau_{-z}}$ for arbitrary functions $f \in L^\infty(\mathbb{C}^n)$ in the second equality. ◻

Theorem 4.20 *It holds $\mathcal{T} = P$ BDO P.*

Proof It suffices to prove that P BDO $P \subset \mathcal{T}$. We will do this by proving that P BO $P \subset \mathcal{A}_{sl}$, the result then follows from Xia's Theorem 4.15.

Let $A \in$ BO have band-width ω. Let $z, w \in \mathbb{C}^n$ be such that $|z - w| \leq 3\omega$. Then,

$$|\langle PAPk_z, k_w \rangle_1| \leq \|A\| = \|A\| e^{\frac{\omega^2}{2n}} e^{-\frac{\omega^2}{2n}} \leq \|A\| e^{\frac{\omega^2}{2n}} e^{-\frac{|z-w|^2}{18n}}.$$

For $|z-w|>3\omega$ we put $r=\frac{|z-w|}{3}$. Observe that this implies

$$d(B(z,r),B(w,r))>\omega. \tag{4.6}$$

We obtain

$$\begin{aligned}|\langle PAPk_z,k_w\rangle_1| &= |\langle Ak_z,k_w\rangle_1|\\ &= \big|\langle AM_{\chi_{B(z,r)}}k_z, M_{\chi_{B(w,r)}}k_w\rangle_1\\ &\quad + \langle AM_{1-\chi_{B(z,r)}}k_z, M_{\chi_{B(w,r)}}k_w\rangle_1 + \langle Ak_z, M_{1-\chi_{B(w,r)}}k_w\rangle_1\big|.\end{aligned}$$

The first term vanishes by Eq. (4.6). For the other two terms, we apply Lemma 4.19 and obtain the estimate

$$|\langle PAPk_z,k_w\rangle_1|\le 2C_n\|A\|e^{-\frac{|z-w|^2}{18n}}.$$

Adjusting the constants, we obtain a uniform estimate for $|\langle PAPk_z,k_w\rangle_1|$ for all $z,w\in\mathbb{C}^n$ which proves that A is sufficiently localized. □

Since many of the introduced objects also exist in the setting of p-Fock spaces and several of the mentioned results carry over to this setting one may also ask:

Question 3 Is there an F_t^p-analogue of Theorem 4.15?

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References

1. S. Axler and D. Zheng, *Compact operators via the Berezin transform*, Indiana Univ. Math. J. **47** (1998), 387–400.
2. W. Bauer, *Berezin-Toeplitz quantization and composition formulas*, J. Funct. Anal. **256** (2009), 3107–3142.
3. W. Bauer, L. A. Coburn, and J. Isralowitz, *Heat flow, BMO and the compactness of Toeplitz operators*, J. Funct. Anal. **259** (2010), 57–78.
4. W. Bauer and K. Furutani, *Compact operators and the pluriharmonic Berezin transform*, Internat. J. Math **19** (2008), 645–669.
5. W. Bauer and J. Isralowitz, *Compactness characterization of operators in the Toeplitz algebra of the Fock space F_α^p*, J. Funct. Anal. **263** (2012), 1323–1355.
6. W. Bauer and N. Vasilevski, *On the structure of commutative Banach algebras generated by Toeplitz operators on the unit ball. Quasi-elliptic case. I: Generating subalgebras*, J. Funct. Anal. **265** (2013), 2956–2990.
7. C. L. Berger and L. A. Coburn, *Toeplitz operators on the Segal-Bargmann space*, Trans. Amer. Math. Soc. **301** (1987), 813–829.
8. ———, *Heat flow and Berezin-Toeplitz estimates*, Amer. J. Math. **116** (1994), 563–590.

9. H. R. Cho, J.-D. Park, and K. Zhu, *Products of Toeplitz operators on the Fock space*, Proc. Amer. Math. Soc. **142** (2014), 2483–2489.
10. L. A. Coburn, J. Isralowitz, and B. Li, *Toeplitz operators with BMO symbols on the Segal-Bargmann space*, Trans. Amer. Math. Soc. **363** (2011), 3015–3030.
11. M. Engliš, *Compact Toeplitz operators via the Berezin transform on bounded symmetric domains*, Integr. Equ. Oper. Theory **33** (1999), 426–455.
12. G. B. Folland, *Harmonic analysis in phase space*, Princeton University Press, 1989.
13. R. Fulsche and R. Hagger, *Fredholmness of Toeplitz operators on the Fock space*, Complex Anal. Oper. Theory **13** (2019), 375–403.
14. R. Hagger, *The essential spectrum of Toeplitz operators on the unit ball*, Integr. Equ. Oper. Theory **89** (2017), 519–556.
15. ———, *Limit operators, compactness and essential spectra on bounded symmetric domains*, J. Math. Anal. Appl. **470** (2019), 470–499.
16. P. R. Halmos and V. S. Sunder, *Bounded integral operators on L^2*, Springer Verlag, 1978.
17. J. Isralowitz, M. Mitkovski, and B. D. Wick, *Localization and compactness in Bergman and Fock spaces*, Indiana Univ. Math. J. **64** (2015), 1553–1573.
18. J. Isralowitz and K. Zhu, *Toeplitz operators on the Fock space*, Integr. Equ. Oper. Theory **66** (2010), 593–611.
19. S. Janson, J. Peetre, and R. Rochberg, *Hankel forms and the Fock space*, Rev. Mat. Iberoamericana **3** (1987), no. 1, 61–138.
20. M. Mitkovski, D. Suárez, and B. D. Wick, *The essential norm of operators on $A^p_\alpha(\mathbb{B}_n)$*, Integr. Equ. Oper. Theory **75** (2013), 197–233.
21. R. Quiroga-Barranco, N. Vasilevski, *Commutative C^*-algebras of Toeplitz operators on the unit ball, I. Bargmann-type transforms and spectral representations of Toeplitz operators*, Integr. Equ. Oper. Theory, **59** (2007), 379–419.
22. R. Raimondo, *Toeplitz operators on the Bergman space of the unit ball*, Bull. Austral. Math. Soc. **62** (2000), 273–285.
23. D. Suárez, *The essential norm of operators in the Toeplitz algebra on $A^p(\mathbb{B}_n)$*, Indiana Univ. Math. J. **56** (2007), 2185–2232.
24. N. Vasilevski, *Commutative algebras of Toeplitz operators on the Bergman space*, Birkhäuser Verlag, Basel, Boston, Berlin, 2008.
25. J. Xia, *Localization and the Toeplitz algebra on the Bergman space*, J. Funct. Anal. **269** (2015), 781–814.
26. J. Xia and D. Zheng, *Localization and Berezin transform on the Fock space*, J. Funct. Anal. **264** (2013), 97–117.
27. K. Zhu, *Operator theory in function spaces*, 2nd ed., Mathematical Surveys and Monographs, vol. 138, American Mathematical Society, 2007.
28. ———, *Analysis on Fock spaces*, Graduate Texts in Mathematics, vol. 263, Springer US, New York, 2012.
29. N. Zorboska, *Toeplitz operators with BMO symbols and the Berezin transform*, Int. J. Math and Math. Sci. **46** (2003), 2929–2945.

Toeplitz Operators on the Domain $\{Z \in M_{2\times 2}(\mathbb{C}) \mid ZZ^* < I\}$ with $\mathrm{U}(2) \times \mathbb{T}^2$-Invariant Symbols

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Dedicated to Nikolai Vasilevski on the occasion of his 70th birthday

Abstract Let D be the irreducible bounded symmetric domain of 2×2 complex matrices that satisfy $ZZ^* < I_2$. The biholomorphism group of D is realized by $\mathrm{U}(2,2)$ with isotropy at the origin given by $\mathrm{U}(2) \times \mathrm{U}(2)$. Denote by $\mathbb{T}^2$ the subgroup of diagonal matrices in $\mathrm{U}(2)$. We prove that the set of $\mathrm{U}(2) \times \mathbb{T}^2$-invariant essentially bounded symbols yield Toeplitz operators that generate commutative C^*-algebras on all weighted Bergman spaces over D. Using tools from representation theory, we also provide an integral formula for the spectra of these Toeplitz operators.

1 Introduction

In this work we consider the problem of the existence of commutative C^*-algebras that are generated by families of Toeplitz operators on weighted Bergman spaces over irreducible bounded symmetric domains. More precisely, we are interested in the case where the Toeplitz operators are those given by symbols invariant by some closed subgroup of the group of biholomorphisms. See [1, 6, 11–13] for related

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previous works. This problem has turned out to be a quite interesting one thanks in part to the application of representation theory.

An important particular case is given when one considers the subgroup fixing some point in the domain, in other words, a maximal compact subgroup of the group of biholomorphisms. In [1], we proved that for such maximal compact subgroups, the corresponding C^*-algebra is commutative. On the other hand, there is another interesting family of subgroups to consider: the maximal tori in the group of biholomorphisms. By the results from [1] (see also [2]) it is straightforward to check that the C^*-algebra generated by the Toeplitz operators whose symbols are invariant under a fixed maximal torus is commutative if and only if the irreducible bounded symmetric domain is biholomorphically equivalent to some unit ball.

These results have inspired Nikolai Vasilevski to pose the following question. Let D be an irreducible bounded symmetric domain that is not biholomorphically equivalent to a unit ball (that is, it is not of rank one), K a maximal compact subgroup and T a maximal torus in the group of biholomorphisms of D. Does there exist a closed subgroup H such that $T \subsetneq H \subsetneq K$ for which the C^*-algebras (for all weights) generated by Toeplitz operators with H-invariant symbols are commutative? The goal of this work is to give a positive answer to this question for the classical Cartan domain of type I of 2×2 matrices. In the rest of this work we will denote simply by D this domain.

The group of biholomorphisms of D is realized by the Lie group $\mathrm{U}(2, 2)$ acting by fractional linear transformations. A maximal compact subgroup is given by $\mathrm{U}(2) \times \mathrm{U}(2)$, which contains the maximal torus $\mathbb{T}^2 \times \mathbb{T}^2$, where $\mathbb{T}^2$ denotes the group of 2×2 diagonal matrices with diagonal entries in $\mathbb{T}$. We prove that there are exactly two subgroups properly between $\mathrm{U}(2) \times \mathrm{U}(2)$ and $\mathbb{T}^2 \times \mathbb{T}^2$, and these are $\mathrm{U}(2) \times \mathbb{T}^2$ and $\mathbb{T}^2 \times \mathrm{U}(2)$ (see Proposition 3.3), for which it is also proved that the corresponding C^*-algebras generated by Toeplitz operators are unitarily equivalent (see Proposition 3.4). In Sect. 4 we study the properties of $\mathrm{U}(2) \times \mathbb{T}^2$-invariant symbols. The main result here is Theorem 5.3, where we prove the commutativity of the C^*-algebras generated by Toeplitz operators whose symbols are $\mathrm{U}(2) \times \mathbb{T}^2$-invariant. As a first step to understand the structure of these C^*-algebras we provide in Sect. 6 a computation of the spectra of the Toeplitz operators. The main result here is Theorem 6.4.

We would like to use this opportunity to thank Nikolai Vasilevski, to whom this work is dedicated. Nikolai has been a very good friend and an excellent collaborator. He has provided us all with many ideas to work with.

2 Preliminaries

Let us consider the classical Cartan domain given by

$$D = \{Z \in M_{2\times 2}(\mathbb{C}) : ZZ^* < I_2\},$$

where $A < B$ means that $B - A$ is positive definite. This domain is sometimes denoted by either $D^I_{2,2}$ or $D_{2,2}$.

We consider the Lie group

$$\mathrm{U}(2,2) = \{M \in \mathrm{GL}(4,\mathbb{C}) : M^* I_{2,2} M = I_{2,2}\},$$

where

$$I_{2,2} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

and the Lie group

$$\mathrm{SU}(2,2) = \{M \in \mathrm{U}(2,2) : \det M = 1\}.$$

Then $\mathrm{SU}(2,2)$, and hence also $\mathrm{U}(2,2)$, act transitively on D by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1},$$

where we have a block decomposition by matrices with size 2×2. And $\mathrm{SU}(2,2)$ is, up to covering, the group of biholomorphic isometries of D and the action of $\mathrm{SU}(2,2)$ is locally faithful. We observe that the action of $\mathrm{U}(2,2)$ on D is not faithful. More precisely, the kernel of its action is the subgroup of matrices of the form tI_4, where $t \in \mathbb{T}$.

The maximal compact subgroup of $\mathrm{U}(2,2)$ that fixes the origin 0 in D is given by

$$\mathrm{U}(2)\times\mathrm{U}(2) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in \mathrm{U}(2), B \in \mathrm{U}(2) \right\}.$$

For simplicity, we write the elements of $\mathrm{U}(2)\times\mathrm{U}(2)$ as (A, B) instead of using their block diagonal representation. A maximal torus of $\mathrm{U}(2)\times\mathrm{U}(2)$ is given by

$$\mathbb{T}^4 = \{(D_1, D_2) \in \mathrm{U}(2)\times\mathrm{U}(2) : D_1, D_2 \text{ diagonal}\}.$$

The corresponding maximal compact subgroup and maximal torus in $\mathrm{SU}(2,2)$ are given by

$$\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2)) = \{(A, B) \in \mathrm{U}(2)\times\mathrm{U}(2) : \det(A)\det(B) = 1\},$$
$$\mathbb{T}^3 = \{(D_1, D_2) \in \mathbb{T}^4 : \det(D_1)\det(D_2) = 1\}.$$

For every $\lambda > 3$ we will consider the weighted measure v_λ on D given by

$$dv_\lambda(Z) = c_\lambda \det(I_2 - ZZ^*)^{\lambda-4}\, dZ$$

where the constant c_λ is chosen so that v_λ is a probability measure. In particular, we have, see [4, Thm. 2.2.1]:

$$c_\lambda = \frac{(\lambda-3)(\lambda-2)^2(\lambda-1)}{\pi^4}, \quad \lambda > 3.$$

The Hilbert space inner product defined by v_λ will be denoted by $\langle \cdot, \cdot \rangle_\lambda$. We will from now on always assume that $\lambda > 3$. The weighted Bergman space $\mathcal{H}^2_\lambda(D)$ is the Hilbert space of holomorphic functions that belong to $L^2(D, v_\lambda)$. This is a reproducing kernel Hilbert space with Bergman kernel given by

$$k_\lambda(Z, W) = \det(I_2 - ZW^*)^{-\lambda},$$

which yields the Bergman projection $B_\lambda : L^2(D, v_\lambda) \to \mathcal{H}^2_\lambda(D)$ given by

$$B_\lambda f(Z) = \int_D f(W) k_\lambda(Z, W) dv_\lambda(W).$$

We recall that the space of holomorphic polynomials $\mathcal{P}(M_{2\times 2}(\mathbb{C}))$ is dense on every weighted Bergman space. Furthermore, it is well known that one has, for every $\lambda > 3$, the decomposition

$$\mathcal{H}^2_\lambda(D) = \bigoplus_{d=0}^{\infty} \mathcal{P}^d(M_{2\times 2}(\mathbb{C}))$$

into a direct sum of Hilbert spaces, where $\mathcal{P}^d(M_{2\times 2}(\mathbb{C}))$ denotes the subspace of homogeneous holomorphic polynomials of degree d.

For every essentially bounded symbol $\varphi \in L^\infty(D)$ and for every $\lambda > 3$ we define the corresponding Toeplitz operator by

$$T^{(\lambda)}_\varphi(f) = B_\lambda(\varphi f), \quad f \in \mathcal{H}^2_\lambda(D).$$

In particular, these Toeplitz operators are given by the following expression

$$T^{(\lambda)}_\varphi(f)(Z) = c_\lambda \int_D \frac{\varphi(W) f(W) \det(I_2 - WW^*)^{\lambda-4}}{\det(I_2 - ZW^*)^\lambda}\, dW.$$

On the other hand, for every $\lambda > 3$ there is an irreducible unitary representation of U(2, 2) acting on $\mathcal{H}^2_\lambda(D)$ given by

$$\pi_\lambda : \widetilde{\mathrm{U}}(2,2) \times \mathcal{H}^2_\lambda(D) \to \mathcal{H}^2_\lambda(D)$$
$$(\pi_\lambda(g)f)(Z) = j(g^{-1}, Z)^{\frac{\lambda}{4}} f(g^{-1}Z),$$

where $j(g, Z)$ denotes the complex Jacobian of the transformation g at the point Z.

We note that every $g \in \mathrm{U}(2) \times \mathrm{U}(2)$ defines a linear unitary transformation of D that preserves all the measures $\mathrm{d}v_\lambda$.

If $\lambda/4$ is not an integer, then $j(g, Z)^{\lambda/4}$ is not always well defined which makes it necessary to consider a covering of U(2, 2). We therefore consider the universal covering group $\widetilde{\mathrm{U}}(2,2)$ of U(2, 2) and its subgroup $\mathbb{R} \times \mathrm{SU}(2) \times \mathbb{R} \times \mathrm{SU}(2)$, the universal covering group of $\mathrm{U}(2) \times \mathrm{U}(2)$. Here the covering map is given by

$$(x, A, y, B) \mapsto (e^{ix}A, e^{iy}B).$$

Hence, the action of $\mathbb{R} \times \mathrm{SU}(2) \times \mathbb{R} \times \mathrm{SU}(2)$ on D is given by the expression

$$(x, A, y, B)Z = e^{i(x-y)}AZB^{-1}.$$

It follows that the restriction of π_λ to the subgroup $\mathbb{R} \times \mathrm{SU}(2) \times \mathbb{R} \times \mathrm{SU}(2)$ is given by the expression

$$(\pi_\lambda(x, A, y, B)f)(Z) = e^{i\lambda(y-x)} f(e^{i(y-x)}A^{-1}ZB).$$

It is well known that this restriction is multiplicity-free for every $\lambda > 3$ (see [1, 8–10]).

It is useful to consider as well the representation

$$\pi'_\lambda : (U(2) \times \mathrm{U}(2)) \times \mathcal{H}^2_\lambda(D) \to \mathcal{H}^2_\lambda(D)$$
$$(\pi'_\lambda(g)f)(Z) = f(g^{-1}Z),$$

which is well-defined and unitary as a consequence of the previous remarks. Note that the representations π_λ and π'_λ are defined on groups that differ by a covering, but they also differ by the factor $e^{i\lambda(y-x)}$. It follows that π'_λ is multiplicity-free with the same isotypic decomposition as that of π_λ.

3 Toeplitz Operators Invariant Under Subgroups of U(2) × U(2)

For a closed subgroup $H \subset \mathrm{U}(2) \times \mathrm{U}(2)$ we will denote by $\mathcal{A}^H$ the complex vector space of essentially bounded symbols φ on D that are H-invariant, i.e. such that for every $h \in H$ we have

$$\varphi(hZ) = \varphi(Z)$$

for almost every $Z \in D$. Denote by $\mathcal{T}^{(\lambda)}(\mathcal{A}^H)$ the C^*-algebra generated by Toeplitz operators with symbols in $\mathcal{A}^H$ acting on the weighted Bergman space $\mathcal{H}^2_\lambda(D)$. We have $\mathrm{U}(2) \times \mathrm{U}(2) = \mathbb{T}(\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2)))$ and the center acts trivially on D. We also point to the special case that will be the main topic of this article.

Let us denote

$$\mathrm{U}(2) \times \mathbb{T} = \left\{ (A, t) = \left(A, \begin{pmatrix} t & 0 \\ 0 & \overline{t} \end{pmatrix} \right) : A \in \mathrm{U}(2), t \in \mathbb{T} \right\}.$$

We now prove that $\mathrm{U}(2) \times \mathbb{T}$-invariance is equivalent to $\mathrm{U}(2) \times \mathbb{T}^2$-invariance.

Lemma 3.1 *The groups* $\mathrm{U}(2) \times \mathbb{T}^2$ *and* $\mathrm{U}(2) \times \mathbb{T}$ *have the same orbits. In other words, for every* $Z \in D$*, we have*

$$(\mathrm{U}(2) \times \mathbb{T})Z = (\mathrm{U}(2) \times \mathbb{T}^2)Z.$$

In particular, an essentially bounded symbol φ *is* $\mathrm{U}(2) \times \mathbb{T}^2$*-invariant if and only if it is* $\mathrm{U}(2) \times \mathbb{T}$*-invariant.*

Proof We observe that $\mathrm{U}(2) \times \mathbb{T}^2$ is generated as a group by $\mathrm{U}(2) \times \mathbb{T}$ and the subgroup

$$\{I_2\} \times \mathbb{T} I_2.$$

But for every $t \in \mathbb{T}$ and $Z \in D$ we have

$$(I_2, t I_2)Z = \overline{t} Z = (\overline{t} I_2, I_2)Z$$

which is a biholomorphism of D already realized by elements of $\mathrm{U}(2) \times \mathbb{T}$. Hence, $\mathrm{U}(2) \times \mathbb{T}^2$ and $\mathrm{U}(2) \times \mathbb{T}$ yield the same transformations on their actions on D, and so the result follows. □

The following is now a particular case of [1, Thm. 6.4] and can be proved directly in exactly the same way.

Theorem 3.2 *For a closed subgroup H of $\mathrm{U}(2) \times \mathrm{U}(2)$ the following conditions are equivalent for every $\lambda > 3$:*

(1) *The C^*-algebra $\mathcal{T}^{(\lambda)}(\mathcal{A}^H)$ is commutative.*
(2) *The restriction $\pi_\lambda|_H$ is multiplicity-free.*

As noted in Sect. 2, the unitary representation π_λ is multiplicity-free on $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2))$ and thus the C^*-algebra generated by Toeplitz operators by $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2))$-invariant symbols is commutative for every weight $\lambda > 3$. Such operators are also known as radial Toeplitz operators.

On the other hand, it follows from Example 6.5 of [1] that the restriction $\pi_\lambda|_{\mathbb{T}^3}$ is not multiplicity-free, where $\mathbb{T}^3$ is the maximal torus of $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2))$ described in Sect. 2. Hence, we conclude that $\mathcal{T}^{(\lambda)}(\mathcal{A}^{\mathbb{T}^3})$ is not commutative for any $\lambda > 3$.

We now consider subgroups H such that $\mathbb{T}^3 \subset H \subset \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2))$ or, equivalently, subgroups H such that $\mathbb{T}^4 \subset H \subset \mathrm{U}(2) \times \mathrm{U}(2)$. For simplicity, we will assume that H is connected.

Proposition 3.3 *Let $\mathbb{T}^4$ denote the subgroup of diagonal matrices in $\mathrm{U}(2) \times \mathrm{U}(2)$. Then the only connected subgroups strictly between $\mathrm{U}(2) \times \mathrm{U}(2)$ and $\mathbb{T}^4$ are $\mathrm{U}(2) \times \mathbb{T}^2$ and $\mathbb{T}^2 \times \mathrm{U}(2)$. In particular, the only connected subgroups strictly between $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2))$ and $\mathbb{T}^3$ are $\mathrm{S}(\mathrm{U}(2) \times \mathbb{T}^2)$ and $\mathrm{S}(\mathbb{T}^2 \times \mathrm{U}(2))$.*

Proof It is enough to prove the first claim for the corresponding Lie algebras.

First note that ($x_1, x_2 \in \mathbb{R}, z \in \mathbb{C}$)

$$\left[\begin{pmatrix} ix_1 & 0 \\ 0 & ix_2 \end{pmatrix}, \begin{pmatrix} 0 & z \\ -\overline{z} & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & i(x_1 - x_2)z \\ -\overline{i(x_1 - x_2)z} & 0 \end{pmatrix},$$

which proves that the space

$$V = \left\{ \begin{pmatrix} 0 & z \\ -\overline{z} & 0 \end{pmatrix} : z \in \mathbb{C} \right\}$$

is an irreducible $i\mathbb{R}^2$-submodule of $\mathfrak{u}(2)$. Hence, the decomposition of $\mathfrak{u}(2) \times \mathfrak{u}(2)$ into irreducible $i\mathbb{R}^4$-submodules is given by

$$\mathfrak{u}(2) \times \mathfrak{u}(2) = i\mathbb{R}^4 \oplus (V \times \{0\}) \oplus (\{0\} \times V).$$

We conclude that $\mathfrak{u}(2) \times i\mathbb{R}^2$ and $i\mathbb{R}^2 \times \mathfrak{u}(2)$ are the only $i\mathbb{R}^4$-submodules strictly between $\mathfrak{u}(2) \times \mathfrak{u}(2)$ and $i\mathbb{R}^4$, and both are Lie algebras. □

There is natural biholomorphism

$$\begin{aligned} F : D &\to D \\ Z &\mapsto Z^\top \end{aligned}$$

that clearly preserves all the weighted measures dv_λ. Hence, F induces a unitary map

$$F^* : L^2(D, v_\lambda) \to L^2(D, v_\lambda)$$
$$F^*(f) = f \circ F^{-1}$$

that preserves $\mathcal{H}^2_\lambda(D)$. And the same expression

$$\varphi \mapsto F^*(\varphi) = \varphi \circ F^{-1}$$

defines an isometric isomorphism on the space $L^\infty(D)$ of essentially bounded symbols.

Furthermore, we consider the automorphism $\rho \in \mathrm{Aut}(\mathrm{U}(2) \times \mathrm{U}(2))$ given by $\rho(A, B) = (\overline{B}, \overline{A})$. Thus, we clearly have

$$F((A, B)Z) = F(AZB^{-1}) = \overline{B}Z^\top \overline{A}^{-1} = \rho(A, B)F(Z),$$

for all $(A, B) \in \mathrm{U}(2) \times \mathrm{U}(2)$ and $Z \in D$. In other words, the map F intertwines the $\mathrm{U}(2) \times \mathrm{U}(2)$-action with that of the image of ρ.

We observe that $\rho(\mathrm{U}(2) \times \mathbb{T}^2) = \mathbb{T}^2 \times \mathrm{U}(2)$. Hence, the previous constructions can be used to prove that both groups define equivalent C^*-algebras from invariant Toeplitz operators.

Proposition 3.4 *The isomorphism of $L^\infty(D)$ given by F^* maps $\mathcal{A}^{\mathrm{U}(2)\times\mathbb{T}^2}$ onto $\mathcal{A}^{\mathbb{T}^2\times\mathrm{U}(2)}$. Furthermore, for every weight $\lambda > 3$ and for every $\varphi \in \mathcal{A}^{\mathrm{U}(2)\times\mathbb{T}^2}$ we have*

$$T^{(\lambda)}_{F^*(\varphi)} = F^* \circ T^{(\lambda)}_\varphi \circ (F^*)^{-1}.$$

In particular, the C^*-algebras $\mathcal{T}^{(\lambda)}(\mathcal{A}^{\mathrm{U}(2)\times\mathbb{T}^2})$ and $\mathcal{T}^{(\lambda)}(\mathcal{A}^{\mathbb{T}^2\times\mathrm{U}(2)})$ are unitarily equivalent for every $\lambda > 3$.

Proof From the above computations, for a given $\varphi \in L^\infty(D)$ we have

$$\varphi \circ (A, B) \circ F^{-1} = \varphi \circ F^{-1} \circ \rho(A, B)$$

for every $(A, B) \in \mathrm{U}(2) \times \mathrm{U}(2)$. Hence, φ is $U(2) \times \mathbb{T}^2$-invariant if and only if $F^*(\varphi)$ is $\mathbb{T}^2 \times \mathrm{U}(2)$-invariant. This proves the first part.

On the other hand, we use that the map F^* is unitary on $L^2(D, v_\lambda)$ to conclude that for every $f, g \in \mathcal{H}^2_\lambda(D)$ we have

$$\begin{aligned}\left\langle T^{(\lambda)}_{F^*(\varphi)}(f), g\right\rangle_\lambda &= \langle F^*(\varphi) f, g\rangle_\lambda \\ &= \left\langle (\varphi \circ F^{-1}) f, g\right\rangle_\lambda \\ &= \langle \varphi(f \circ F), g \circ F\rangle_\lambda \\ &= \left\langle T^{(\lambda)}_\varphi \circ (F^*)^{-1}(f), (F^*)^{-1} g\right\rangle_\lambda \\ &= \left\langle F^* \circ T^{(\lambda)}_\varphi \circ (F^*)^{-1}(f), g\right\rangle_\lambda,\end{aligned}$$

and this completes the proof. □

4 U(2) × $\mathbb{T}^2$-Invariant Symbols

As noted in Sect. 2, the subgroup $\mathrm{U}(2) \times \mathrm{U}(2)$ does not act faithfully. Hence, it is convenient to consider suitable subgroups for which the action is at least locally faithful. This is particularly important when describing the orbits of the subgroups considered. We also noted before that the most natural choice is to consider subgroups of $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2))$, however for our setup it will be useful to consider other subgroups.

For the case of the subgroup $\mathrm{U}(2) \times \mathbb{T}^2$ it turns out that $\mathrm{U}(2) \times \mathbb{T}^2$-invariance is equivalent to $\mathrm{S}(\mathrm{U}(2) \times \mathbb{T}^2)$-invariance. This holds for the action through biholomorphisms on D and so for every induced action on function spaces over D.

To understand the structure of the $\mathrm{U}(2) \times \mathbb{T}$-orbits the next result provides a choice of a canonical element on each orbit.

Proposition 4.1 *For every $Z \in M_{2\times 2}(\mathbb{C})$ there exists $r \in [0, \infty)^3$ and $(A, t) \in \mathrm{U}(2) \times \mathbb{T}$ such that*

$$(A, t)Z = \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix}.$$

Furthermore, if $Z = (Z_1, Z_2)$ satisfies $\det(Z), \langle Z_1, Z_2\rangle \neq 0$, then r is unique and (A, t) is unique up to a sign.

Proof First assume that $\det(Z) = 0$, so that we can write $Z = (au, bu)$ for some unitary vector $u \in \mathbb{C}^2$ and for $a, b \in \mathbb{C}$. For $Z = 0$ the claim is trivial. If either a or b is zero, but not both, then we can choose $A \in \mathrm{U}(2)$ that maps the only nonzero column into a positive multiple of e_1 and the result follows. Finally, we assume that

a and b are both non-zero. In this case, choose $A \in \mathrm{U}(2)$ such that $A(au) = |a|e_1$ and $t \in \mathbb{T}$ such that

$$t^2 = \frac{a|b|}{b|a|}.$$

Then, one can easily check that

$$(tA, t)Z = \begin{pmatrix} |a| & |b| \\ 0 & 0 \end{pmatrix}.$$

Let us now assume that $\det(Z) \neq 0$. From the unit vector

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{Z_1}{|Z_1|},$$

we define

$$A = \begin{pmatrix} \overline{a} & \overline{b} \\ -b & a \end{pmatrix} \in \mathrm{SU}(2).$$

Then, it follows easily that we have

$$AZ = \begin{pmatrix} |Z_1| & \frac{1}{|Z_1|}\langle Z_2, Z_1 \rangle \\ 0 & \frac{1}{|Z_1|}\det(Z) \end{pmatrix}.$$

If $s, t \in \mathbb{T}$ are given, then we have

$$\left(\begin{pmatrix} t & 0 \\ 0 & s \end{pmatrix} A, t \right) Z = \begin{pmatrix} |Z_1| & \frac{t^2}{|Z_1|}\langle Z_2, Z_1 \rangle \\ 0 & \frac{st}{|Z_1|}\det(Z) \end{pmatrix}.$$

Hence, it is enough to choose $s, t \in \mathbb{T}$ so that $r_2 = t^2 \langle Z_2, Z_1 \rangle$ and $r_3 = st \det(Z)$ are both non-negative to complete the existence part with $r_1 = |Z_1|$.

For the uniqueness, let us assume that $\det(Z), \langle Z_1, Z_2 \rangle \neq 0$ and besides the identity in the statement assume that we also have

$$(A', t')Z = \begin{pmatrix} r_1' & r_2' \\ 0 & r_3' \end{pmatrix},$$

with the same restrictions. Then, we obtain the identity

$$(A'A^{-1}, t'\overline{t}) \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} = \begin{pmatrix} r_1' & r_2' \\ 0 & r_3' \end{pmatrix}. \tag{4.1}$$

This implies that $A'A^{-1}$ is a diagonal matrix of the form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

with $a, b \in \mathbb{T}$. Then, taking the determinant of (4.1) we obtain $abr_1r_3 = r_1'r_3'$, which implies that $ab = 1$. If we now use the identities from the entries in (4.1), then one can easily conclude that $r = r'$ and $(A', t') = \pm(A, t)$. □

The following result is an immediate consequence.

Corollary 4.2 *Let $\varphi \in L^\infty(D)$ be given. Then, φ is* $\mathrm{U}(2)\times\mathbb{T}^2$*-invariant if and only if for a.e. $Z \in D$ we have*

$$\varphi(Z) = \varphi \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix}$$

where $r = (r_1, r_2, r_3)$ are the (essentially) unique values obtained from Z in Proposition 4.1.

5 Toeplitz Operators with $\mathrm{U}(2) \times \mathbb{T}^2$-Invariant Symbols

As noted in Sect. 2, for every $\lambda > 3$ the restriction of π_λ to $\mathbb{R}\times\mathrm{SU}(2)\times\mathbb{R}\times\mathrm{SU}(2)$ is multiplicity-free. We start this section by providing an explicit description of the corresponding isotypic decomposition.

Let us consider the following set of indices

$$\vec{\mathbb{N}}^2 = \{\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2 : \nu_1 \geq \nu_2 \geq 0\}.$$

Then, for every $\nu \in \vec{\mathbb{N}}^2$, we let F_ν denote the complex irreducible $\mathrm{SU}(2)$-module with dimension $\nu_1 - \nu_2 + 1$. For example, F_ν can be realized as the $\mathrm{SU}(2)$-module given by $\mathrm{Sym}^{\nu_1-\nu_2}(\mathbb{C}^2)$ or by the space of homogeneous polynomials in two complex variables and degree $\nu_1 - \nu_2$. Next, we let the center $\mathbb{T}I_2$ of $\mathrm{U}(2)$ act on the space F_ν by the character $t \mapsto t^{\nu_1+\nu_2}$. It is easy to check that the actions on F_ν of $\mathrm{SU}(2)$ and $\mathbb{T}I_2$ are the same on their intersection $\{\pm I_2\}$. This turns F_ν into a complex irreducible $\mathrm{U}(2)$-module. We note (and will use without further remarks) that the $\mathrm{U}(2)$-module structure of F_ν can be canonically extended to a module structure over $\mathrm{GL}(2,\mathbb{C})$.

We observe that the dual F_ν^* as $\mathrm{U}(2)$-module is realized by the same space with the same $\mathrm{SU}(2)$-action but with the action of the center $\mathbb{T}I_2$ now given by the character $t \mapsto t^{-\nu_1-\nu_2}$.

If V is any $\mathbb{R} \times \mathrm{SU}(2) \times \mathbb{R} \times \mathrm{SU}(2)$-module, then for every λ we consider a new $\mathbb{R} \times \mathrm{SU}(2) \times \mathbb{R} \times \mathrm{SU}(2)$-module given by the action

$$(x, A, y, B) \cdot v = e^{i\lambda(y-x)}(x, A, y, B)v \tag{5.1}$$

where $(x, A, y, B) \in \mathbb{R} \times \mathrm{SU}(2) \times \mathbb{R} \times \mathrm{SU}(2)$, $v \in V$ and the action of (x, A, y, B) on v on the left-hand side is given by the original structure of V. We will denote by V_λ this new $\mathbb{R} \times \mathrm{SU}(2) \times \mathbb{R} \times \mathrm{SU}(2)$-module structure.

In particular, for every $\nu \in \overrightarrow{\mathbb{N}}^2$ the space $F_\nu^* \otimes F_\nu$ is an irreducible module over $\mathrm{U}(2) \times \mathrm{U}(2)$ and, for every $\lambda > 3$, the space $(F_\nu^* \otimes F_\nu)_\lambda$ is an irreducible module over $\mathbb{R} \times \mathrm{SU}(2) \times \mathbb{R} \times \mathrm{SU}(2)$. Note that two such modules defined for $\nu, \nu' \in \overrightarrow{\mathbb{N}}^2$ are isomorphic (over the corresponding group) if and only if $\nu = \nu'$.

Proposition 5.1 *For every $\lambda > 3$, the isotypic decomposition of the restriction of π_λ to $\mathbb{R} \times \mathrm{SU}(2) \times \mathbb{R} \times \mathrm{SU}(2)$ is given by*

$$\mathcal{H}_\lambda^2(D) \cong \bigoplus_{\nu \in \overrightarrow{\mathbb{N}}^2} (F_\nu^* \otimes F_\nu)_\lambda,$$

and this decomposition is multiplicity-free. With respect to this isomorphism and for every $d \in \mathbb{N}$, the subspace $\mathcal{P}^d(M_{2\times 2}(\mathbb{C}))$ corresponds to the sum of the terms for ν such that $|\nu| = d$. Furthermore, for the Cartan subalgebra given by the diagonal matrices of $\mathfrak{u}(2) \times \mathfrak{u}(2)$ and a suitable choice of positive roots, the irreducible $\mathbb{R} \times \mathrm{SU}(2) \times \mathbb{R} \times \mathrm{SU}(2)$-submodule of $\mathcal{H}_\lambda^2(D)$ corresponding to $(F_\nu^* \otimes F_\nu)_\lambda$ has a highest weight vector given by

$$p_\nu(Z) = z_{11}^{\nu_1 - \nu_2} \det(Z)^{\nu_2},$$

for every $\nu \in \overrightarrow{\mathbb{N}}^2$.

Proof By the remarks in Sect. 2 we can consider the representation π'_λ. Furthermore, it was already mentioned in that section that $\mathcal{P}^d(M_{2\times 2}(\mathbb{C}))$ is $\mathbb{R} \times \mathrm{SU}(2) \times \mathbb{R} \times \mathrm{SU}(2)$-invariant and so we compute its decomposition into irreducible submodules. In what follows we consider both π_λ and π'_λ always restricted to $\mathbb{R} \times \mathrm{SU}(2) \times \mathbb{R} \times \mathrm{SU}(2)$. We also recall that for π'_λ we already have an action for $\mathrm{U}(2) \times \mathrm{U}(2)$ without the need of passing to the universal covering group.

Note that the representation π'_λ on each $\mathcal{P}^d(M_{2\times 2}(\mathbb{C}))$ naturally extends with the same expression from $\mathrm{U}(2) \times \mathrm{U}(2)$ to $\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C})$. This action is regular in the sense of representations of algebraic groups. By the Zariski density of $\mathrm{U}(2)$ in $\mathrm{GL}(2, \mathbb{C})$ it follows that invariance and irreducibility of subspaces as well as isotypic decompositions with respect to either $\mathrm{U}(2)$ or $\mathrm{GL}(2, \mathbb{C})$ are the same for

π'_λ in $\mathcal{P}^d(M_{2\times 2}(\mathbb{C}))$. Hence, we can apply Theorem 5.6.7 from [3] (see also [5]) to conclude that

$$\mathcal{P}^d(M_{2\times 2}(\mathbb{C})) \cong \bigoplus_{\substack{\nu\in\overrightarrow{\mathbb{N}}^2 \\ |\nu|=d}} F_\nu^* \otimes F_\nu$$

as $\mathrm{U}(2)\times\mathrm{U}(2)$-modules for the representation π'_λ. Since the representations π_λ and π'_λ differ by the factor $e^{i\lambda(y-x)}$ for elements of the form (x, A, y, B), taking the sum over $d \in \mathbb{N}$ we obtain the isotypic decomposition of $\mathcal{H}^2_\lambda(D)$ as stated. This is multiplicity-free as a consequence of the remarks in this section.

Finally, the claim on highest weight vectors is contained in the proof of Theorem 5.6.7 from [3], and it can also be found in [5]. □

We now consider the subgroup $\mathrm{U}(2)\times\mathbb{T}^2$. Note that the subgroup of $\mathbb{R}\times\mathrm{SU}(2)\times\mathbb{R}\times\mathrm{SU}(2)$ corresponding to $\mathrm{U}(2)\times\mathbb{T}^2$ is realized by $\mathbb{R}\times\mathrm{SU}(2)\times\mathbb{R}\times\mathbb{T}$ with covering map given by the expression

$$(x, A, y, t) \mapsto \left(e^{ix}A, e^{iy}\begin{pmatrix} t & 0 \\ 0 & \overline{t}\end{pmatrix}\right).$$

In particular, the action of $\mathbb{R}\times\mathrm{SU}(2)\times\mathbb{R}\times\mathbb{T}$ on D is given by

$$(x, A, y, t)Z = e^{i(x-y)}AZ\begin{pmatrix} t & 0 \\ 0 & \overline{t}\end{pmatrix},$$

and the representation π_λ restricted to $\mathbb{R}\times\mathrm{SU}(2)\times\mathbb{R}\times\mathbb{T}$ is given by

$$(\pi_\lambda(x, A, y, t)f)(Z) = e^{i\lambda(y-x)}f\left(e^{i(y-x)}A^{-1}Z\begin{pmatrix} t & 0 \\ 0 & \overline{t}\end{pmatrix}\right).$$

We recall that for any Cartan subgroup of $\mathrm{U}(2)$ we have a weight space decomposition

$$F_\nu = \bigoplus_{j=0}^{\nu_1-\nu_2} F_\nu(\nu_1-\nu_2-2j),$$

where $F_\nu(k)$ denotes the one-dimensional weight space corresponding to the weight $k = -\nu_1+\nu_2, -\nu_1+\nu_2+2, \ldots, \nu_1-\nu_2-2, \nu_1-\nu_2$. For simplicity, we will always consider the Cartan subgroup $\mathbb{T}^2$ of $\mathrm{U}(2)$ given by its subset of diagonal matrices. We conclude that $F_\nu(k)$ is isomorphic, as a $\mathbb{T}^2$-module, to the one-dimensional representation corresponding to the character $(t_1, t_2) \mapsto t_1^{\nu_2}t_2^k$. We will denote by $\mathbb{C}_{(m_1,m_2)}$ the one-dimensional $\mathbb{T}^2$-module defined by the character $(t_1, t_2) \mapsto t_1^{m_1} t_2^{m_2}$, where $(m_1, m_2) \in \mathbb{Z}^2$. In particular, we have $F_\nu(k) \cong \mathbb{C}_{(\nu_2,k)}$ for every $k = -\nu_1 + \nu_2, -\nu_1 + \nu_2 + 2, \ldots, \nu_1 - \nu_2 - 2, \nu_1 - \nu_2$.

Using the previous notations and remarks we can now describe the isotypic decomposition for the restriction of π_λ to $\mathbb{R} \times \mathrm{SU}(2) \times \mathbb{R} \times \mathbb{T}$. As before, for a module V over the group $\mathbb{R} \times \mathrm{SU}(2) \times \mathbb{R} \times \mathbb{T}$ we will denote by V_λ the module over the same group obtained by the expression (5.1).

Proposition 5.2 *For every $\lambda > 3$, the isotypic decomposition of the restriction of π_λ to $\mathbb{R} \times \mathrm{SU}(2) \times \mathbb{R} \times \mathbb{T}$ is given by*

$$\mathcal{H}^2_\lambda(D) \cong \bigoplus_{\nu \in \overrightarrow{\mathbb{N}}^2} \bigoplus_{j=0}^{\nu_1-\nu_2} (F^*_\nu \otimes \mathbb{C}_{(\nu_2,\nu_1-\nu_2-2j)})_\lambda,$$

and this decomposition is multiplicity-free. Furthermore, for the Cartan subalgebra given by the diagonal matrices of $\mathfrak{u}(2) \times i\mathbb{R}^2$ and a suitable choice of positive roots, the irreducible $\mathbb{R} \times \mathrm{SU}(2) \times \mathbb{R} \times \mathbb{T}$-submodule of $\mathcal{H}^2_\lambda(D)$ corresponding to $(F^*_\nu \otimes \mathbb{C}_{(\nu_2,\nu_1-\nu_2-2j)})_\lambda$ has a highest weight vector given by

$$p_{\nu,j}(Z) = z_{11}^{\nu_1-\nu_2-j} z_{12}^{j} \det(Z)^{\nu_2},$$

for every $\nu \in \overrightarrow{\mathbb{N}}^2$ and $j = 0, \ldots, \nu_1 - \nu_2$.

Proof We build from Proposition 5.1 and its proof so we follow its notation.

As noted above in this section we have a weight space decomposition

$$F_\nu = \bigoplus_{j=0}^{\nu_1-\nu_2} F_\nu(\nu_1 - \nu_2 - 2j) \cong \bigoplus_{j=0}^{\nu_1-\nu_2} \mathbb{C}_{(\nu_2,\nu_1-\nu_2-2j)},$$

where the isomorphism holds term by term as modules over the Cartan subgroup $\mathbb{T}^2$ of diagonal matrices of $\mathrm{U}(2)$. It follows from this and Proposition 5.1 that we have an isomorphism

$$\mathcal{H}^2_\lambda(D) \cong \bigoplus_{\nu \in \overrightarrow{\mathbb{N}}^2} \bigoplus_{j=0}^{\nu_1-\nu_2} F^*_\nu \otimes \mathbb{C}_{(\nu_2,\nu_1-\nu_2-2j)},$$

of modules over $\mathrm{U}(2) \times \mathbb{T}^2$ for the restriction of π'_λ to this subgroup. Hence, with the introduction of the factor $e^{i\lambda(y-x)}$ from (5.1) we obtain the isomorphism of modules over $\mathbb{R} \times \mathrm{SU}(2) \times \mathbb{R} \times \mathbb{T}$ for the restriction of π_λ to this subgroup. This proves the first part of the statement.

We also note that the modules $(F_\nu^* \otimes \mathbb{C}_{(\nu_2,\nu_1-\nu_2-2j)})_\lambda$ are clearly irreducible over $\mathbb{R}\times \mathrm{SU}(2)\times\mathbb{R}\times\mathbb{T}$ and non-isomorphic for different values of ν and j. Hence, the restriction of π_λ to $\mathbb{R}\times \mathrm{SU}(2)\times\mathbb{R}\times\mathbb{T}$ is multiplicity-free.

On the other hand, the proof of Theorem 5.6.7 from [3], on which that of Proposition 5.1 is based, considers the Cartan subalgebra defined by diagonal matrices in $\mathfrak{u}(2)\times\mathfrak{u}(2)$ and the order on roots for which the positive roots correspond to matrices of the form (X, Y) with X lower triangular and Y upper triangular. With these choices, for every $\nu \in \overrightarrow{\mathbb{N}}^2$, the highest weight vector $p_\nu(Z)$ from Proposition 5.1 lies in the subspace corresponding to the tensor product of two highest weight spaces. Hence, $p_\nu(Z)$ lies in the subspace corresponding to $(F_\nu^* \otimes \mathbb{C}_{(\nu_2,\nu_1-\nu_2)})_\lambda$. In particular, $p_\nu(Z)$ is a highest weight vector for $(F_\nu^* \otimes \mathbb{C}_{(\nu_2,\nu_1-\nu_2)})_\lambda$.

It is well known from the description of the representations of $\mathfrak{sl}(2,\mathbb{C})$ that the element

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C})$$

acts on F_ν so that it maps

$$F_\nu(\nu_1-\nu_2-2j) \to F_\nu(\nu_1-\nu_2-2j-2)$$

isomorphically for every $j = 0, \ldots, \nu_1-\nu_2-1$. This holds for the order where the upper triangular matrices in $\mathfrak{sl}(2,\mathbb{C})$ define positive roots. Since the action of $\mathrm{U}(2)\times\{I_2\}$ commutes with that of Y it follows that the element $(0, Y) \in \mathfrak{sl}(2,\mathbb{C})\times\mathfrak{sl}(2,\mathbb{C})$ maps a highest weight vector of $F_\nu^* \otimes \mathbb{C}_{(\nu_2,\nu_1-\nu_2-2j)}$ onto a highest weight vector of $F_\nu^* \otimes \mathbb{C}_{(\nu_2,\nu_1-\nu_2-2j-2)}$. Hence, a straightforward computation that applies j-times the element $(0, Y)$ starting from $p_\nu(Z)$ shows that the vector

$$p_{\nu,j}(Z) = z_{11}^{\nu_1-\nu_2-j} z_{12}^{j} \det(Z)^{\nu_2}$$

defines a highest weight vector for the submodule corresponding to space $F_\nu^* \otimes \mathbb{C}_{(\nu_2,\nu_1-\nu_2-2j)}$ for the representation π'_λ restricted to $\mathbb{R}\times\mathrm{SU}(2)\times\mathbb{R}\times\mathbb{T}$. Again, it is enough to consider the factor from (5.1) to conclude the claim on the highest weight vectors for π_λ restricted to $\mathbb{R}\times\mathrm{SU}(2)\times\mathbb{R}\times\mathbb{T}$. □

As a consequence we obtain the following result.

Theorem 5.3 *For every $\lambda > 3$, the C^*-algebra $\mathcal{T}^{(\lambda)}(\mathcal{A}^{\mathrm{U}(2)\times\mathbb{T}^2})$ generated by Toeplitz operators with essentially bounded* $\mathrm{U}(2)\times\mathbb{T}^2$*-invariant symbols is commutative. Furthermore, if H is a connected subgroup between* $\mathbb{T}^4$ *and* $\mathrm{U}(2)\times\mathrm{U}(2)$ *such that $\mathcal{T}^{(\lambda)}(\mathcal{A}^H)$ is commutative, then H is either of* $\mathrm{U}(2)\times\mathrm{U}(2)$, $\mathrm{U}(2)\times\mathbb{T}^2$ *or* $\mathbb{T}^2\times\mathrm{U}(2)$. *Also, for the last two choices of H, the corresponding C^*-algebras $\mathcal{T}^{(\lambda)}(\mathcal{A}^H)$ are unitarily equivalent.*

Proof The commutativity of $\mathcal{T}^{(\lambda)}(\mathcal{A}^{\mathrm{U}(2)\times\mathbb{T}^2})$ follows from Proposition 5.2 and Theorem 3.2. The possibilities on the choices of H follows from Proposition 3.3 and the remarks from Sect. 2. The last claim is the content of Proposition 3.4. □

We also obtain the following orthogonality relations for the polynomials $p_{\nu,j}$.

Proposition 5.4 *Let* $\nu \in \overrightarrow{\mathbb{N}}^2$ *be fixed. Then, we have*

$$\int_{\mathrm{U}(2)} p_{\nu,j}(A)\overline{p_{\nu,k}(A)}\,\mathrm{d}A = \frac{\delta_{jk}}{\nu_1 - \nu_2 + 1}\binom{\nu_1 - \nu_2}{j}$$

for every $j, k = 0, \ldots, \nu_1 - \nu_2$.

Proof We remember that the irreducible U(2)-module F_ν can be realized as the space of homogeneous polynomials of degree $\nu_1 - \nu_2$ in two complex variables. For this realization, the U(2)-action is given by

$$(\pi_\nu(A)p)(z) = \det(A)^{\nu_1} p(A^{-1}z)$$

for $A \in U(2)$ and $z \in \mathbb{C}^2$.

Also, the computation of orthonormal bases on Bergman spaces on the unit ball (see for example [14]) implies that there is a U(2)-invariant inner product $\langle \cdot, \cdot \rangle$ on F_ν for which the basis

$$\left\{ v_j(z_1, z_2) = \binom{\nu_1 - \nu_2}{j}^{\frac{1}{2}} z_1^{\nu_1 - \nu_2 - j} z_2^j : j = 0, 1, \ldots, \nu_1 - \nu_2 \right\},$$

is orthonormal. We fix the inner product and this orthonormal basis for the rest of the proof.

With these choices it is easy to see that the map given by

$$Z \mapsto \left\langle \pi_\nu(Z)v_j, v_0 \right\rangle,$$

for $Z \in \mathrm{GL}(2, \mathbb{C})$, is polynomial and is a highest weight vector for the $\mathrm{U}(2) \times \mathbb{T}^2$-module corresponding to $F_\nu^* \otimes \mathbb{C}_{(\nu_2, \nu_1 - \nu_2 - 2j)}$ in the isomorphism given by Proposition 5.2. Hence there is a complex number $\alpha_{\nu,j}$ such that

$$p_{\nu,j}(Z) = \alpha_{\nu,j} \left\langle \pi_\nu(Z)v_j, v_\nu \right\rangle$$

for all $Z \in \mathrm{GL}(2, \mathbb{C})$ and $j = 0, \ldots, \nu_1 - \nu_2$.

By Schur's orthogonality relations we conclude that

$$\int_{\mathrm{U}(2)} p_{\nu,j}(Z)\overline{p_{\nu,k}(Z)}\,\mathrm{d}Z = \frac{\delta_{jk}|\alpha_{\nu,j}|^2}{\nu_1 - \nu_2 + 1}$$

for every $j, k = 0, \dots, \nu_1 - \nu_2$.

Next we choose

$$A_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \in \mathrm{SU}(2).$$

and evaluate at this matrix to compute the constant $\alpha_{\nu,j}$.

First, we compute

$$\begin{aligned}
(\pi_\nu(A_0^{-1})v_0)(z_1, z_2) &= v_0\left(\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) \\
&= v_0\left(\frac{1}{\sqrt{2}}(z_1 - z_2), \frac{1}{\sqrt{2}}(z_1 + z_2)\right) \\
&= \frac{1}{\sqrt{2^{\nu_1-\nu_2}}}(z_1 - z_2)^{\nu_1-\nu_2} \\
&= \frac{1}{\sqrt{2^{\nu_1-\nu_2}}}\sum_{j=0}^{\nu_1-\nu_2}(-1)^j\binom{\nu_1-\nu_2}{j}z_1^{\nu_1-\nu_2-j}z_2^j,
\end{aligned}$$

which implies that

$$\left\langle \pi_\nu(A_0)v_j, v_0\right\rangle = \left\langle v_j, \pi_\nu(A_0^{-1})v_0\right\rangle = \frac{(-1)^j}{\sqrt{2^{\nu_1-\nu_2}}}\binom{\nu_1-\nu_2}{j}^{\frac{1}{2}}.$$

Meanwhile,

$$p_{\nu,j}(A_0) = \left(\frac{1}{\sqrt{2}}\right)^{\nu_1-\nu_2-j}\left(-\frac{1}{\sqrt{2}}\right)^j \det(A_0)^{\nu_2} = \frac{(-1)^j}{\sqrt{2^{\nu_1-\nu_2}}},$$

thus implying that

$$\alpha_{\nu,j} = \binom{\nu_1-\nu_2}{j}^{\frac{1}{2}}.$$

This completes our proof. □

6 The Spectra of Toeplitz Operators with $\mathrm{U}(2) \times \mathbb{T}^2$-Invariant Symbols

We recall that the Haar measure μ on $\mathrm{GL}(2, \mathbb{C})$ is given by

$$\mathrm{d}\mu(Z) = |\det(Z)|^{-4}\, \mathrm{d}Z = \det(ZZ^*)^{-2}\, \mathrm{d}Z.$$

where $\mathrm{d}Z$ denotes the Lebesgue measure on the Euclidean space $M_{2\times 2}(\mathbb{C})$. Furthermore, we have the following expression for the integration with respect to the Haar measure:

Lemma 6.1 *For every function* $f \in C_c(\mathrm{GL}(2, \mathbb{C}))$ *we have*

$$\int_{\mathrm{GL}(2,\mathbb{C})} f(Z)\, \mathrm{d}\mu(Z) = \int_{\mathbb{C}} \int_{(0,\infty)^2} \int_{\mathrm{U}(2)} f\left(A \begin{pmatrix} a_1 & z \\ 0 & a_2 \end{pmatrix}\right) a_2^{-2}\, \mathrm{d}A\, \mathrm{d}a\, \mathrm{d}z.$$

Proof For the moment let

$$n_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}.$$

We start with the Iwasawa decomposition of $GL(2, \mathbb{C})$ that allows us to decompose any $Z \in \mathrm{GL}(2, \mathbb{C})$ as

$$Z = A\, \mathrm{diag}(a_1, b_1) n_z$$

where $A \in \mathrm{U}(2)$, $a_1, a_2 > 0$ and $z \in \mathbb{C}$. Then, by [7, Prop. 8.43] and some changes of coordinates we obtain the result as follows.

$$\begin{aligned}
&\int_{\mathrm{GL}(2,\mathbb{C})} f(Z)\, \mathrm{d}\mu(Z) \\
&\qquad = \int_{\mathbb{C}} \int_0^\infty \int_0^\infty \int_{\mathrm{U}(2)} f\left(A \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} n_z\right) a_1^2 a_2^{-2}\, \mathrm{d}A\, \mathrm{d}a_1\, \mathrm{d}a_2\, \mathrm{d}z \\
&\qquad = \int_{\mathbb{C}} \int_0^\infty \int_0^\infty \int_{\mathrm{U}(2)} f\left(A \begin{pmatrix} a_1 & a_1 z \\ 0 & a_2 \end{pmatrix}\right) a_1^2 a_2^{-2}\, \mathrm{d}A\, \mathrm{d}a_1\, \mathrm{d}a_2\, \mathrm{d}z \\
&\qquad = \int_{\mathbb{C}} \int_0^\infty \int_0^\infty \int_{\mathrm{U}(2)} f\left(A \begin{pmatrix} a_1 & z \\ 0 & a_2 \end{pmatrix}\right) a_2^{-2}\, \mathrm{d}A\, \mathrm{d}a_1\, \mathrm{d}a_2\, \mathrm{d}z.
\end{aligned}$$

□

By the remarks above, the weighted measure v_λ on D can be written in terms of the Haar measure on $\mathrm{GL}(2,\mathbb{C})$ as follows

$$\begin{aligned} \mathrm{d}v_\lambda(Z) &= c_\lambda|\det(Z)|^4 \det(I_2 - ZZ^*)^{\lambda-4}\,\mathrm{d}\mu(Z) \\ &= c_\lambda \det(ZZ^*)^2 \det(I_2 - ZZ^*)^{\lambda-4}\,\mathrm{d}\mu(Z). \end{aligned} \tag{6.1}$$

We use this and Lemma 6.1 to write down the measure v_λ in terms of measures associated to the foliation on $M_{2\times2}(\mathbb{C})$ given by the action of $\mathrm{U}(2)\times\mathbb{T}^2$ (see Proposition 4.1). The next result applies only to suitably invariant functions, but this is enough for our purposes.

Proposition 6.2 *Let $\lambda > 3$ be fixed. If $f \in C_c(M_{2\times2}(\mathbb{C}))$ is a function that satisfies $f(t_\theta Z t_\theta^{-1}) = f(Z)$ for every $Z \in M_{2\times2}(\mathbb{C})$ where*

$$t_\theta = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix}, \quad \theta \in \mathbb{R},$$

then we have

$$\int_{M_{2\times2}(\mathbb{C})} f(Z)\,\mathrm{d}v_\lambda(Z) = 2\pi c_\lambda \int_{\mathbb{R}_+^3}\int_{\mathrm{U}(2)} f\left(A\begin{pmatrix} r_1 & r_2 \\ 0 & r_3\end{pmatrix}\right) r_1^4 r_2 r_3^2 b(r)^{\lambda-4}\,\mathrm{d}A\,\mathrm{d}r,$$

where $b(r) = 1 - r_1^2 - r_2^2 - r_3^2 + r_1^2 r_3^2$ for $r \in (0,\infty)^3$.

Proof First we observe that for every $A \in U(n)$, $a_1, a_2 > 0$ and $z \in \mathbb{C}$ we have

$$\begin{aligned} \det\left(I_2 - A\begin{pmatrix} a_1 & z \\ 0 & a_2\end{pmatrix}\begin{pmatrix} a_1 & 0 \\ \overline{z} & a_2\end{pmatrix}A^*\right) &= \det\left(I_2 - \begin{pmatrix} a_1^2 + |z|^2 & a_2 z \\ a_2\overline{z} & a_2^2\end{pmatrix}\right) \\ &= \det\left(\begin{pmatrix} 1 - a_1^2 - |z|^2 & -a_2 z \\ -a_2\overline{z} & 1 - a_2^2\end{pmatrix}\right) \\ &= 1 - a_1^2 - a_2^2 - |z|^2 + a_1^2 a_2^2 \\ &= b(a_1, |z|, a_2), \end{aligned}$$

where b is defined as in the statement. Using this last identity, (6.1) and Lemma 6.1 we compute the following for f as in the statement. We apply some coordinates changes and use the bi-invariance of the Haar measure of $\mathrm{U}(n)$.

$$\begin{aligned}
&\int_{M_{2\times 2}(\mathbb{C})} f(Z)\,\mathrm{d}v_\lambda(Z)\\
&\quad = c_\lambda \int_{\mathbb{C}}\int_{(0,\infty)^2}\int_{\mathrm{U}(2)} f\left(A\begin{pmatrix} a_1 & z\\ 0 & a_2\end{pmatrix}\right)\\
&\qquad \times a_1^4 a_2^2 b(a_1,|z|,a_2)^{\lambda-4}\,\mathrm{d}A\,\mathrm{d}a\,\mathrm{d}z\\
&\quad = 2\pi c_\lambda \int_0^1\int_{(0,\infty)^3}\int_{\mathrm{U}(2)} f\left(A\begin{pmatrix} a_1 & re^{2\pi i\theta}\\ 0 & a_2\end{pmatrix}\right)\\
&\qquad \times a_1^4 a_2^2 r b(a_1,r,a_2)^{\lambda-4}\,\mathrm{d}A\,\mathrm{d}a\,\mathrm{d}r\,\mathrm{d}\theta\\
&\quad = 2\pi c_\lambda \int_0^1\int_{(0,\infty)^3}\int_{\mathrm{U}(2)} f\left(At_{\theta/2}\begin{pmatrix} a_1 & r\\ 0 & a_2\end{pmatrix}t_{\theta/2}^{-1}\right)\\
&\qquad \times a_1^4 a_2^2 r b(a_1,r,a_2)^{\lambda-4}\,\mathrm{d}A\,\mathrm{d}a\,\mathrm{d}r\,\mathrm{d}\theta\\
&\quad = 2\pi c_\lambda \int_0^1\int_{(0,\infty)^3}\int_{\mathrm{U}(2)} f\left(t_{\theta/2}^{-1}At_{\theta/2}\begin{pmatrix} a_1 & r\\ 0 & a_2\end{pmatrix}\right)\\
&\qquad \times a_1^4 a_2^2 r b(a_1,r,a_2)^{\lambda-4}\,\mathrm{d}A\,\mathrm{d}a\,\mathrm{d}r\,\mathrm{d}\theta\\
&\quad = 2\pi c_\lambda \int_0^1\int_{(0,\infty)^3}\int_{\mathrm{U}(2)} f\left(A\begin{pmatrix} a_1 & r\\ 0 & a_2\end{pmatrix}\right)\\
&\qquad \times a_1^4 a_2^2 r b(a_1,r,a_2)^{\lambda-4}\,\mathrm{d}A\,\mathrm{d}a\,\mathrm{d}r\,\mathrm{d}\theta.
\end{aligned}$$

□

In view of Proposition 6.2 the following formula will be useful.

Lemma 6.3 *For every* $\nu \in \overrightarrow{\mathbb{N}}^2$ *and* $j = 0,\dots,\nu_1-\nu_2$ *we have*

$$p_{\nu,j}\left(A\begin{pmatrix} r_1 & r_2\\ 0 & r_3\end{pmatrix}\right) = \sum_{k=0}^{j}\binom{j}{k} p_{\nu,k}(A) r_1^{\nu_1-j} r_2^{j-k} r_3^{\nu_2+k}$$

for every $A \in \mathrm{U}(2)$ *and* $r \in (0,\infty)^3$.

Proof Let $A \in \mathrm{U}(2)$ be given and write

$$A = \begin{pmatrix} \alpha & \beta \\ -\gamma\overline{\beta} & \gamma\overline{\alpha} \end{pmatrix},$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$ and $|\gamma| = 1$. Hence, we have

$$A \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} = \begin{pmatrix} \alpha r_1 & \alpha r_2 + \beta r_3 \\ * & * \end{pmatrix},$$

and so we conclude that

$$\begin{aligned} p_{\nu,j}&\left(A\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix}\right) \\ &= (\alpha r_1)^{\nu_1-\nu_2-j}(\alpha r_2 + \beta r_3)^j \det\left(A\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix}\right)^{\nu_2} \\ &= (\alpha r_1)^{\nu_1-\nu_2-j}\sum_{k=0}^{j}\binom{j}{k}(\alpha r_2)^{j-k}(\beta r_3)^k \det\left(A\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix}\right)^{\nu_2} \\ &= \sum_{k=0}^{j}\binom{j}{k}\alpha^{\nu_1-\nu_2-k}\beta^k \det(A)^{\nu_2} r_1^{\nu_1-j} r_2^{j-k} r_3^{\nu_2+k} \\ &= \sum_{k=0}^{j}\binom{j}{k} p_{\nu,k}(A) r_1^{\nu_1-j} r_2^{j-k} r_3^{\nu_2+k}. \end{aligned}$$

Note that in the last line we have used the expression obtained in the first line. □

We now apply the previous results to compute the spectra of the Toeplitz operators with $\mathrm{U}(2)\times\mathbb{T}^2$-invariant symbols.

Theorem 6.4 *Let $\lambda > 3$ and $\varphi \in \mathcal{A}^{\mathrm{U}(2)\times\mathbb{T}^2}$ be given. With the notation of Proposition 5.2, the Toeplitz operator T_φ acts on the subspace of $\mathcal{H}^2_\lambda(D)$ corresponding to $(F^*_\nu \otimes \mathbb{C}_{(\nu_2,\nu_1-\nu_2-2j)})_\lambda$ as a multiple of the identity by the constant*

$$\gamma(\varphi,\nu,j) = \frac{\langle \varphi p_{\nu,j}, p_{\nu,j}\rangle_\lambda}{\langle p_{\nu,j}, p_{\nu,j}\rangle_\lambda} = \frac{\displaystyle\sum_{k=0}^{j}\binom{j}{k}^2\binom{\nu_1-\nu_2}{k}\int_\Omega \varphi\begin{pmatrix} r_1 & r_2 \\ 0 & r_3\end{pmatrix} a(r,\nu,j,k)b(r)^{\lambda-4}\,\mathrm{d}r}{\displaystyle\sum_{k=0}^{j}\binom{j}{k}^2\binom{\nu_1-\nu_2}{k}\int_\Omega a(r,\nu,j,k)b(r)^{\lambda-4}\,\mathrm{d}r}$$

for every $\nu \in \overrightarrow{\mathbb{N}}^2$ *and* $j = 0, \ldots, \nu_1 - \nu_2$*, where*

$$\Omega = \left\{ r \in (0,\infty)^3 : \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} \in D \right\}.$$

with the functions $a(r,\nu,j,k) = r_1^{2(\nu_1-j)+4} r_2^{2(j-k)+1} r_3^{2(\nu_2+k)+2}$*, for* $0 \le k \le j$*, and* $b(r) = 1 - r_1^2 - r_2^2 - r_3^2 + r_1^2 r_3^2$ *for* $r \in (0,\infty)^3$.

Proof Let $\varphi \in \mathcal{A}^{\mathrm{U}(2)\times\mathbb{T}^2}$ be given and fix $\nu \in \overrightarrow{\mathbb{N}}^2$ and $j = 0, \ldots, \nu_1 - \nu_2$. First, we observe that we have

$$|p_{\nu,j}(tZt^{-1})|^2 = |p_{\nu,j}(Z)|^2$$

for all $Z \in M_{2\times 2}(\mathbb{C})$ and $t \in \mathbb{T}^2$. The symbol φ is bi-$\mathbb{T}^2$-invariant as well. Hence, we can apply Proposition 6.2 to $\varphi|p_{\nu,j}|^2$ to compute as follows

$$\begin{aligned}
\langle \varphi p_{\nu,j}, p_{\nu,j}\rangle_\lambda &= \int_D \varphi(Z)|p_{\nu,j}(Z)|^2 \,\mathrm{d}v_\lambda(Z) \\
&= 2\pi c_\lambda \int_\Omega \int_{U(2)} \varphi\left(A\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix}\right) \left| p_{\nu,j}\left(A\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix}\right)\right|^2 \\
&\quad \times r_1^4 r_2 r_3^2 b(r)^{\lambda-4} \,\mathrm{d}A\,\mathrm{d}r \\
&= 2\pi c_\lambda \sum_{k=0}^{j} \binom{j}{k}^2 \int_\Omega \varphi\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} \int_{U(2)} |p_{\nu,k}(A)|^2 \,\mathrm{d}A \\
&\quad \times a(r,\nu,j,k) b(r)^{\lambda-4} \,\mathrm{d}r \\
&= \frac{2\pi c_\lambda}{\nu_1 - \nu_2 + 1} \sum_{k=0}^{j} \binom{j}{k}^2 \binom{\nu_1-\nu_2}{k} \int_\Omega \varphi\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} \\
&\quad \times a(r,\nu,j,k) b(r)^{\lambda-4} \,\mathrm{d}r.
\end{aligned}$$

The second identity applies Proposition 6.2. For the third identity we apply Proposition 5.4 and the invariance of φ. In the last identity we apply again the orthogonality relations from Proposition 5.4.

The proof is completed by taking $\varphi \equiv 1$ in the above computation. □

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References

1. M. Dawson, G. Ólafsson, R. Quiroga-Barranco, *Commuting Toeplitz Operators on Bounded Symmetric Domains and Multiplicity-Free Restrictions of Holomorphic Discrete Series*. Journal of Functional Analysis **268** (2015), No. 7, 1711–1732.
2. M. Englis, *Density of algebras generated by Toeplitz operators on Bergman spaces*, Arkiv för Matematik **30** (1992), No. 2, 227–243.
3. R. Goodman and N. Wallach, *Symmetry, representations, and invariants*. Graduate Texts in Mathematics, 255. Springer, Dordrecht, 2009.
4. L. K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains*. Translated from the Russian by Leo Ebner and Adam Korányi American Mathematical Society, Providence, R.I. 1963.
5. K. Johnson, *On a ring of invariant polynomials on a Hermitian symmetric space*. J. Algebra 67 (1980), no. 1, 72–81.
6. S. Grudsky, R. Quiroga-Barranco and N. Vasilevski: *Commutative C^*-algebras of Toeplitz operators and quantization on the unit disk*. J. Funct. Anal. 234 (2006), no. 1, 1–44.
7. Anthony W. Knapp, *Lie groups beyond an introduction*. Second edition. Progress in Mathematics, 140. Birkhäuser Boston, Inc., Boston, MA, 2002.
8. T. Kobayashi, *Multiplicity-free theorems of the restrictions of unitary highest-weight modules with respect to reductive symmetric pairs*, “Representation Theory and Automorphic Forms”, Progress in Mathematics **255** (2008), 45–109.
9. G. Ólafsson, *Analytic continuation in representation theory and harmonic analysis*. Global analysis and harmonic analysis (Marseille-Luminy, 1999), 201–233, Sémin. Congr. **4**, Soc. Math. France, Paris, 2000.
10. G. Ólafsson and B. Ørsted, *Generalizations of the Bargmann transform*, Lie theory and its applications in physics (Clausthal, 1995), 3–14, World Sci. Publ., River Edge, NJ, 1996.
11. R. Quiroga-Barranco, *Separately radial and radial Toeplitz operators on the unit ball and representation theory*, Boletin de la Sociedad Matematica Mexicana 22 (2016), no. 2, 605–623.
12. R. Quiroga-Barranco and N. Vasilevski, *Commutative C^*-algebras of Toeplitz operators on the unit ball. I. Bargmann-type transforms and spectral representations of Toeplitz operators*, Integral Equations Operator Theory **59** (2007), No. 3, 379–419.
13. R. Quiroga-Barranco and N. Vasilevski, *Commutative C^*-algebras of Toeplitz operators on the unit ball. II. Geometry of the level sets of symbols*, Integral Equations Operator Theory **60** (2008), No. 1, 89–132.
14. K. Zhu. *Spaces of holomorphic functions in the unit ball*, volume 226 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

Separately Radial Fock-Carleson Type Measures for Derivatives of Order *k*

Kevin Esmeral

Dedicated to Nikolai L. Vasilevski, professor, mentor, coworker and friend, on the occasion of his 70th birthday

Abstract The separately radial Fock-Carleson type measures for derivatives of order k are introduced and characterized on the Fock space. Also, we study the separately radial Toeplitz operators generated by derivatives of k-FC type measure and give a criterion for Toeplitz operators to be separately radial. Finally, we show that the C*-algebra generated by these Toeplitz operators is isometrically isomorphic to a C*-subalgebra of the bounded sequences.

Keywords Fock space · Toeplitz operators · Separately radial

Mathematics Subject Classification (2010) 47A75 (primary), 58J50 (secondary)

1 Introduction

In linear algebra an infinite Toeplitz matrix T is defined by the rule:

$$T = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \ddots \\ a_2 & a_1 & a_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

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where $a_n \in \mathbb{C},\ n \in \mathbb{Z}$. In 1911 *Otto Toeplitz* proved that the matrix T defines a bounded operator on $\ell_2(\mathbb{Z}_+)$, where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, if and only if the numbers a_n are the Fourier coefficients of a function $a \in L_\infty(S^1)$, where S^1 is the unit circle.

The classical Hardy space $\mathcal{H}^2$ can be viewed as the closed linear span in $L_2(S^1)$ of $\{z^n : n \geq 0\}$. For $g \in L_\infty(S^1)$, the Toeplitz operator $\mathbf{T}_g$ defined by $\mathbf{T}_g h = \mathrm{P}(gh)$, where P denotes the orthogonal projection from $L_2(S^1)$ onto $\mathbf{H}^2$, is bounded and satisfies $\|\mathbf{T}_g\| \leq \|g\|_\infty$. The matrix of $\mathbf{T}_g$ with respect to the orthonormal basis $\{z^n : n \geq 0\}$ is the Toeplitz matrix T with a_n being the Fourier coefficients of g. Thus, the Toeplitz operators are a generalization of the Toeplitz matrices T.

Let $\mathcal{X}$ be a function space and let P be a projection of $\mathcal{X}$ onto some closed subspace $\mathcal{Y}$ of $\mathcal{X}$. Then the Toeplitz operator $\mathbf{T}_g : \mathcal{Y} \longrightarrow \mathcal{Y}$ with defining symbol g is given by $\mathbf{T}_g f = \mathrm{P}(gf)$. The most studied cases are when $\mathcal{Y}$ is either the Bergman space, the Hardy space, or the Fock space. More recently Toeplitz operators have been also studied on many other spaces, for example on the harmonic Bergman space [29].

The Toeplitz operators have been extensively studied in several branches of mathematics: complex analysis, theory of normed algebras, operator theory [4, 23, 27, 36], harmonic analysis [1, 11], and mathematical physics, particularly in connection with quantum mechanics [7, 10], etc. Recently, G. Rozenblum and N. L. Vasilevski considered a new approach of Toeplitz operators that permits to enrich the class of Toeplitz operators and turn into Toeplitz operators that failed to be Toeplitz in the classical sense [30, 31].

For a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and a reproducing kernel Hilbert subspace $\mathcal{A}$ of $\mathcal{H}$ with reproducing kernel K_z at the point z, Rozenblum and Vasilevski [30] introduced Toeplitz operators $\mathbf{T}_\mathrm{F}$ acting on $\mathcal{A}$ defined by bounded sesquilinear forms F on $\mathcal{A}$. Here, the Riesz theorem for bounded sesquilinear form is used to justify the existence of an operator $\mathbf{A} \in \mathcal{B}(\mathcal{F}^2(\mathbb{C}^n))$ with $\mathrm{F}(\cdot,\cdot) = \langle \mathbf{A}\cdot, \cdot \rangle$, and thus to consider a wider class of Toeplitz operators given by

$$(\mathbf{T}_\mathrm{F} f)(z) = \mathrm{F}(f, K_z) = \langle \mathbf{A} f, K_z \rangle. \tag{1.1}$$

Unlike, classical Toeplitz operators which multiply and project back into the subspace, the set of all Toeplitz operators generated by sesquilinear forms is a *-algebra, noncommutative in general, but it is a very important property that in the classical sense we could not have, see [30, Theorem 4.1].

In the classical sense, one of the common strategies in the study of commutative C*-algebras generated by Toeplitz operators is to select a specific class of defining symbols: radial [3, 5, 13, 20, 22, 23, 25, 35], vertical, angular [12, 14, 21, 24, 26, 34] and horizontal [15, 16], depending if they are acting on Bergman spaces or Fock spaces.

Many mathematicians work in search for new families of symbols to get wider the class of Toeplitz operators, see [30] and the references given there. We mention one case among others in Fock spaces, Isralowitz and Zhu [27] introduced Toeplitz operators $\mathbf{T}_\mu$ with Borel regular measures μ as symbols. The Rozenblum and

Vasilevski's approach extend this point of view and even permit to take coderivatives of order k of Fock-Carleson measures, see for example [30].

In the study of new commutative C*-algebras generated by Toeplitz operators, Esmeral et al. [15] extended to the n-dimensional case the definition of the *coderivative* of a measure μ, denoted by $\boldsymbol{\partial}^\alpha\overline{\boldsymbol{\partial}}^\beta\mu$. In particular, they characterized the C*-algebra generated by all horizontal Toeplitz operators $\mathbf{T}_{\mathrm{F}_{\mu,\alpha,\beta}}$ where

$$\mathrm{F}_{\mu,\alpha,\beta}(f,g) = \pi^{-n}\int_{\mathbb{C}^n} \partial^\alpha f(z)\overline{\partial^\beta g(z)}e^{-|z|^2}d\mu(z), \tag{1.2}$$

and k, β, α are multi-index such that $2k = \alpha + \beta$.

In present paper, we introduce the separately radial Fock-Carleson type measures μ for derivatives of order k and we characterize the C*-algebra generated by all Toeplitz operators $\mathbf{T}_{\mathrm{F}_{\mu,\alpha,\beta}}$, where the sesquilinear forms $\mathrm{F}_{\mu,\alpha,\beta}$ are given by these measures.

The paper is organized as follows. Sections 2 and 3 present some preliminaries: here we fix notation and establish some basic properties of separately radial operators, Fock-Carleson measures, such type measures for derivatives of order k and Toeplitz operators generated by sesquilinear forms. In Sect. 4 we introduce separately radial Fock-Carleson type measures μ for derivatives of order k and we show that the Toeplitz operators $\mathbf{T}_{\boldsymbol{\partial}^\alpha\overline{\boldsymbol{\partial}}^\beta\mu}$ generated by such Fock-Carleson type measures μ are "almost diagonal". Then we give an explicit formula for the sequences of the eigenvalues (Proposition 4.3). As a by-product of such proposition, we show that $\mathbf{T}_{\boldsymbol{\partial}^\alpha\overline{\boldsymbol{\partial}}^\beta\mu}$ is separately radial if and only $\alpha = \beta = k \in \mathbb{Z}_+^n$. Finally, in Sect. 5 we proceed with the study of separately radial Toeplitz operators $\mathbf{T}_{\boldsymbol{\partial}^\alpha\overline{\boldsymbol{\partial}}^\alpha\mu}$ generated by separately radial k-FC measures μ for $\mathcal{F}^2(\mathbb{C}^n)$ and we establish a criterion for such Toeplitz operators to be separately radial (Theorem 5.2). We prove further that the C*-algebra generated by Toeplitz operators given by coderivatives of separately radial k-FC type measures is commutative an isometrically isomorphic to a C*-subalgebra of $\ell_\infty(\mathbb{Z}_+^n)$ (Theorem 5.3).

2 Separately Radial Operators

In this section we fix notation and we compile some basic facts on separately radial operators on the Fock space $\mathcal{F}^2(\mathbb{C}^n)$. The results presented here are a natural extension to higher dimensions of the radial case, see for example [13, 22, 36], and their proofs can be found in [28]. We use the following standard notation: $z = x + iy \in \mathbb{C}^n$, where $x = (\operatorname{Re} z_1, \ldots, \operatorname{Re} z_n)$ and $y = (\operatorname{Im} z_1, \ldots, \operatorname{Im} z_n)$. For $z, w \in \mathbb{C}^n$ we write

$$z\cdot w = \sum_{k=1}^n z_k w_k,\ z^2 = z\cdot z = \sum_{k=1}^n z_k^2, \quad |z|^2 = z\cdot\overline{z} = \sum_{k=1}^n |z_k|^2,\ \mathbf{1} = (1,1,\ldots,1) \in \mathbb{Z}_+^n.$$

$$\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!,\ |\alpha| = \sum_{k=1} \alpha_k,\ \alpha \le \beta \Leftrightarrow \alpha_j \le \beta_j, \text{ for } j = 1, 2, \ldots, n, \alpha, \beta \in \mathbb{Z}_+^n.$$

$$z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n},\ \overline{z}^\alpha = \overline{z_1}^{\alpha_1}\cdots\overline{z_n}^{\alpha_n},\ \partial^\alpha f = \frac{\partial^{\alpha_1}}{z_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_1}}{z_n^{\alpha_n}} f,\ \overline{\partial}^\alpha = \frac{\partial^{\alpha_1}}{\overline{z_1}^{\alpha_1}}\cdots\frac{\partial^{\alpha_1}}{\overline{z_n}^{\alpha_n}} f.$$

The Fock space [17], denoted by $\mathcal{F}^2(\mathbb{C}^n)$ (also known as the Segal–Bargmann space, see [2, 33]), of all entire functions that are square integrable on $\mathbb{C}^n$ with respect to the Gaussian measure $dg_n(z) = (\pi)^{-n} e^{-|z|^2} d\nu_n(z)$, where ν_n is the usual Lebesgue measure on $\mathbb{C}^n$. It is well-known that $\mathcal{F}^2(\mathbb{C}^n)$ is a closed subspace of $L_2(\mathbb{C}^n, dg_n)$, thus there exists a unique orthogonal projection P from $L_2(\mathbb{C}, dg_n)$ onto $\mathcal{F}^2(\mathbb{C}^n)$. This projection has the integral form

$$(\mathrm{P}f)(z) = \int_{\mathbb{C}^n} f(w)\overline{K_z(w)}dg_n(w), \tag{2.1}$$

where the function $K_z\colon \mathbb{C}^n \to \mathbb{C}$ is the *reproducing kernel* at a point z, and it is given by the formula

$$K_z(w) = e^{\overline{z}\cdot w} \quad w \in \mathbb{C}^n. \tag{2.2}$$

As it was mentioned in Introduction, given $\varphi \in L_\infty(\mathbb{C}^n)$, the *Toeplitz operator* $\mathbf{T}_\varphi$ with defining symbol φ acts on the Fock space $\mathcal{F}^2(\mathbb{C}^n)$ by the rule $T_\varphi f = \mathrm{P}(f\varphi)$, where P is (2.1), for details see for example [8, 36].

Let $\mathrm{H_U}$ be the Haar measure of the compact group $\mathbf{U}(n, \mathbb{C})$. Denote by $\mathbf{U}_d(n, \mathbb{C})$ the compact subgroup of $\mathbf{U}(n, \mathbb{C})$ consisting of all unitary matrices that are diagonal. For $\mathbf{X} \in \mathbf{U}(n, \mathbb{C})$, we denote by $V_\mathbf{X}$ the linear operator $V_\mathbf{X}\colon L_2(\mathbb{C}^n, dg_n) \to L_2(\mathbb{C}^n, dg_n)$ given by

$$(V_\mathbf{X} f)(z) = f(\mathbf{X}^* z), \quad z \in \mathbb{C}^n. \tag{2.3}$$

Since $\mathbf{X}^* = \mathbf{X}^{-1} \in \mathbf{U}(n, \mathbb{C})$, $V_\mathbf{X}$ is a unitary operator, with $V_\mathbf{X}^* = V_{\mathbf{X}^{-1}}$.

Definition 2.1 (Separately Radial Operator) Let $S \in \mathcal{B}(\mathcal{F}^2(\mathbb{C}^n))$. The operator S is said to be *separately radial* if it commutes with $V_\mathbf{X}$ for all $\mathbf{X} \in \mathbf{U}_d(n, \mathbb{C})$. i.e.,

$$SV_\mathbf{X} = V_\mathbf{X} S. \tag{2.4}$$

The separately radialization of a bounded operator S is defined by

$$\mathbf{SRad}(S) = \int_{\mathbf{U}_d(n,\mathbb{C})} V_\mathbf{X}^* S V_\mathbf{X}\, d\,\mathrm{H_U}(\mathbf{X}), \tag{2.5}$$

where the integral is taken in the weak sense. Note that S is separately radial if and only if $\mathbf{SRad}(S) = S$.

Next, we consider separately radial functions and some of its properties, see details in [28].

Definition 2.2 (Separately Radial Function) A function $\varphi \in L_\infty(\mathbb{C}^n)$ is called *separately radial* if there exists $a \in L_\infty(\mathbb{R}^n_+)$ such that $\varphi(z_1, z_2, \ldots, z_n) = a(|z_1|, |z_2|, \ldots, |z_n|)$ a.e. $z \in \mathbb{C}^n$.

Definition 2.3 (The Radialization of a Function) Let $\varphi \in L_\infty(\mathbb{C})$. The function $\mathbf{srad}(\varphi)$ given by

$$\mathbf{srad}(\varphi)(z) = \frac{1}{(2\pi)^n} \int_{\mathbf{U}_d(n,\mathbb{C})} \varphi(\mathbf{X}z)\, d\,\mathrm{H}_{\mathrm{U}}(\mathbf{X}) \tag{2.6}$$

is called the *separately radialization* of φ.

By the periodicity of the mapping $t \mapsto e^{it}$, the formula (2.6) can rewritten as

$$\mathbf{srad}(\varphi)(z) = \frac{1}{(2\pi)^n} \int_{\mathbf{U}_d(n,\mathbb{C})} \varphi\left(\mathbf{X}\begin{bmatrix} |z_1| \\ |z_2| \\ \vdots \\ |z_n| \end{bmatrix}\right) d\,\mathrm{H}_{\mathrm{U}}(\mathbf{X}). \tag{2.7}$$

Lemma 2.4 (Criterion for a Function to be Radial) *A function $\varphi \in L_\infty(\mathbb{C}^n)$ is radial if and only if $\varphi(z) = \mathbf{srad}(\varphi)(z)$ a.e. $z \in \mathbb{C}^n$.*

It is well known [36] that the set consisting of all normalized monomials $e_\alpha(z) = z^\alpha/\sqrt{\alpha!}$, $\alpha \in \mathbb{Z}^n_+$, form an orthonormal basis of $\mathcal{F}^2(\mathbb{C}^n)$. The following result states a criterion for a bounded operator on $\mathcal{F}^2(\mathbb{C}^n)$ be separately radial.

Proposition 2.5 (Criterion of Separately Radial Operators) *Let $\boldsymbol{T} \in \mathcal{B}(\mathcal{F}^2(\mathbb{C}^n))$. The following conditions are equivalent.*

1. *$\boldsymbol{T}$ is separately radial.*
2. *$\boldsymbol{T}$ is a diagonal operator with respect to the monomial basis.*
3. *The Berezin transform* [6]

$$\widetilde{\boldsymbol{T}}(z) = \frac{\langle \mathbf{T}K_z, K_z\rangle}{\langle K_z, K_z\rangle}, \quad z \in \mathbb{C}^n.$$

is a separately radial function.

An easy computation shows that $\mathbf{srad}(\widetilde{\varphi}) = \widetilde{\mathbf{srad}(\varphi)} = \widetilde{\mathbf{T}_{\mathbf{srad}(\varphi)}}$, for each $\varphi \in L_\infty(\mathbb{C}^n)$. Thus, by Proposition 2.5 and by injectivity of Berezin transform the following criterion holds.

Proposition 2.6 *Let $\varphi \in L_\infty(\mathbb{C}^n)$. The Toeplitz operator $\mathbf{T}_\varphi$ is separately radial if and only if φ is a separately radial function.*

3 *k*-FC Type Measures and Toeplitz Operators

In this section we summarize some results on Fock-Carleson type measures, such measures for derivatives of order k and Toeplitz operators defined by sesquilinear forms, for details see [15, 30, 36]. From now on, we denote by Borel($\mathbb{C}^n$) the Borel σ-algebra of $\mathbb{C}^n$ and by $\mathfrak{B}_{reg}(\mathbb{C}^n)$ the set of all *complex regular* Borel measures $\mu\colon$ Borel$(\mathbb{C}^n) \to \mathbb{C}$ with total variation

$$|\mu|(B) = \sup \sum_{n\in\mathbb{N}} |\,\mu(B_n)|,$$

where the supremum is taken over all partitions B_n of B, that satisfy the conditions:

- $|\mu|$ is locally finite: $|\mu|(\mathcal{K}) < \infty$ for each compact $\mathcal{K} \subset \Omega$;
- μ is regular, i.e.,

$$|\mu(A)| = \sup\left\{|\mu(\mathcal{K})| : \mathcal{K} \text{is compact } X \subset A\right\} = \inf\left\{|\mu(U)| \colon A \subset U \text{ and } U \text{ is open}\right\}.$$

3.1 Fock-Carleson Type Measures

As it was mentioned in Introduction, Isralowitz and Zhu introduced Toeplitz operators $\mathbf{T}_\mu$ acting on the Fock space $\mathcal{F}^2(\mathbb{C}^n)$ with $\mu \in \mathfrak{B}_{reg}(\mathbb{C}^n)$ as symbols [27]:

$$\mathbf{T}_\mu f(z) = \pi^{-n} \int_{\mathbb{C}^n} e^{z\cdot\overline{w}} f(w) e^{-|w|^2} d\mu(w), \quad z \in \mathbb{C}^n, \tag{3.1}$$

thus, for any $f, g \in \mathcal{F}^2(\mathbb{C}^n)$,

$$\langle \mathbf{T}_\mu f, g\rangle = \pi^{-n} \int_{\mathbb{C}^n} \langle K_w, g\rangle f(w) e^{-|w|^2} d\mu(w) = \pi^{-n} \int_{\mathbb{C}^n} f(w)\overline{g(w)} e^{-|w|^2} d\mu(w).$$

If μ is a complex Borel regular measure satisfying the *Condition* (M), namely

$$M = \sup_{z\in\mathbb{C}^n} \int_{\mathbb{C}^n} |K_z(w)|^2 e^{-|w|^2} d|\mu|(w) < \infty, \tag{3.2}$$

then the operator $\mathbf{T}_\mu$ given in (3.1) is well-defined on the dense subset of all finite linear combinations of kernel function. It is important to mention that if μ is absolutely continuous with respect to the usual Lebesgue measure, all the results can be reformulated in terms of the density function and $\mathbf{T}_\mu$ is a Toeplitz operator in the classic sense. From now on we will assume that μ satisfies (3.2).

It is well-known that the Berezin transform of a function $\varphi \in L_\infty(\mathbb{C}^n)$ coincides with the Berezin transform of the Toeplitz operator $\mathbf{T}_\varphi$, and we will denote it by $\widetilde{\varphi}$. By the integral representation of the classical Toeplitz operator $\mathbf{T}_\varphi$, Isralowitz and Zhu in [27] extended to positive Borel measures the classical Berezin transformation as follows:

$$\widetilde{\mu}(z) = \pi^{-n} \int_{\mathbb{C}^n} |k_z(w)|^2 e^{-|w|^2} d\mu(w) = \pi^{-n} \int_{\mathbb{C}^n} e^{-|z-w|^2} d\mu(w), \quad z \in \mathbb{C}^n, \tag{3.3}$$

where

$$k_z(w) = K_z(w)(K_z(z))^{-1/2} = e^{w\cdot\overline{z} - \frac{|z|^2}{2}} \tag{3.4}$$

is the *normalized reproducing kernel* of $\mathcal{F}^2(\mathbb{C}^n)$. In particular, if $\mathbf{T}_\mu$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$, then $\widetilde{\mu}$ is the Berezin transform of $\mathbf{T}_\mu$.

Definition 3.1 (Fock-Carleson Type Measures) A positive measure μ is said to be a *Fock-Carleson* type measure for $\mathcal{F}^2(\mathbb{C}^n)$ (FC measure, in short), if there exists a constant $\omega(\mu) > 0$ such that for every $f \in \mathcal{F}^2(\mathbb{C}^n)$

$$\int_{\mathbb{C}^n} |f(w)|^2 e^{-|w|^2} d\mu(w) \leq \omega(\mu) \, \|f\|^2_{\mathcal{F}^2(\mathbb{C}^n)}$$

The next result provides a criterion for a Toeplitz operator $\mathbf{T}_\mu$ with a positive measure μ as defining symbol to be bounded. For more details we refer the reader to [27, Theorems 2.3 and 3.1].

Proposition 3.2 *Let μ be a positive Borel regular measure on $\mathbb{C}^n$. Then the following conditions are equivalent:*

1. *The Toeplitz operator $\mathbf{T}_\mu$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$.*
2. *The sesquilinear form*

$$\mathrm{F}(f, g) = \pi^{-n} \int_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-|z|^2} d\mu(z)$$

   is bounded in $\mathcal{F}^2(\mathbb{C}^n)$.
3. *$\widetilde{\mu}$ is bounded on $\mathbb{C}^n$.*
4. *For any fixed $\mathbf{r} = (r_j)_{j=1}^n$ with $r_j > 0$,*

$$\mu(B_{\mathbf{r}}(z)) < C, \quad \text{for all } z \in \mathbb{C},$$

   for some constant $C > 0$, where $B_{\mathbf{r}}(z)$ denotes the polydisk centered at z with radius $\mathbf{r}$.
5. *μ is a Fock-Carleson type measure.*

A natural generalization is to admit a complex valued Borel measure μ such that its variation $|\mu|$ is a FC measure. In such a case, as a by-product of Proposition 3.2, the results of [27] imply that the following norms for μ are equivalent:

1. $\|\mu\|_1 = \|\mathbf{T}_\mu\|$.
2. $\|\mu\|_2 = \sup\limits_{z\in\mathbb{C}^n} \widetilde{|\mu|}(z)$.
3. $\|\mu\|_3 = \sup\limits_{z\in\mathbb{C}^n} |\mu|(B_r(z))$, where r is any fixed positive number.
4. $\|\mu\|_4 = \sup\limits_{\substack{f\in\mathcal{F}^2(\mathbb{C}^n)\\ \|f\|=1}} \left\{\int_{\mathbb{C}^n} |f(w)|^2 e^{-|w|^2} d|\mu|(w)\right\}$.

3.2 *Fock-Carleson Type Measures for Derivatives of Order k*

Next, we make a slight change to the approach of Esmeral, Rozenblum and Vasilevski [15] to define the measures μ_p and thus studying the separately radial Fock-Carleson type measures for derivatives of order k.

Proposition 3.3 *Let $f \in \mathcal{F}^2(\mathbb{C}^n)$ and $k \in \mathbb{Z}_+^n$. Then for every $z \in \mathbb{C}^n$*

$$|\partial^k f(z)| \le C\, k! \|f\|_{\mathcal{F}^2(\mathbb{C}^n)} \prod_{j=1}^n (1+|z_j|^2)^{k_j/2} e^{\frac{|z_j|^2}{2}}$$

with the constant $C > 0$ not depending on $k \in \mathbb{Z}_+^n$.

Proof The proof is almost literal as that of Proposition 3.2 in [15]. We only take $r_j = (1+|z_j|^2)^{-1/2}$ instead of $r_j = (1+x_j^2)^{-1/2}(1+y_j^2)^{-1/2}$. □

Definition 3.4 (*k*-FC Measures) Let $k \in \mathbb{Z}_+^n$. A positive Borel regular measure μ is called a *Fock-Carleson* type measure for derivatives of order k (k-FC, in short) for $\mathcal{F}^2(\mathbb{C}^n)$ if there exists $\omega_k(\mu) > 0$ such that for every $f \in \mathcal{F}^2(\mathbb{C}^n)$

$$\int_{\mathbb{C}^n} \left|\partial^k f(w)\right|^2 e^{-|w|^2} d\mu(w) \le \omega_k(\mu) \int_{\mathbb{C}^n} |f(w)|^2 e^{-|w|^2} d\nu_{2n}(w). \tag{3.5}$$

If $|k| = 0$, then any 0-FC type measure is just a FC-measure for $\mathcal{F}^2(\mathbb{C}^n)$. A complex regular Borel measure is called k-FC if (3.5) is satisfied for μ replaced by $|\mu|$.

Denote by μ_p the Borel measure on $\mathbb{C}^n$ given by

$$\mu_p(B) = \int_B \prod_{j=1}^n (1+|z_j^2|)^{p_j} d\mu(z), \quad B \subset \mathbb{C}^n. \tag{3.6}$$

The following results relate the k-FC and Fock-Carleson type measures for $\mathcal{F}^2(\mathbb{C}^n)$. The proofs are almost literally the same as [30, Theorem 5.4 and Corollary 5.5]. It is enough to replace Proposition 5.1 of [30] by the above Proposition 3.3 and take the product of the lattices used in the proof of [30, Theorem 5.4].

Proposition 3.5 *Let $k \in \mathbb{Z}_+^n$. A positive measure μ is a k-FC type measure if and only if, for some (and, therefore for any) $r > 0$, the following quantity is finite:*

$$C_k(\mu, r) = (k!)^2 \sup_{z \in \mathbb{C}^n} \mu_k(B_r(z)).$$

For a fixed r, the constant $\omega_k(\mu)$ in (3.5) can be taken as $\omega_k(\mu) = C(r)C_k(\mu, r)$ where $C(r)$ depends only on r. For a complex measure μ, the 'if' part holds true.

Proposition 3.6 *For any $p, k \in \mathbb{Z}_+^n$, a positive Borel measure μ is a k-FC type measure if and only if the measure μ_{k-p} is a p-FC type measure. Furthermore, $C_{k-p}(\mu_p, r) = C_k(\mu, r)$.*

Following [15], by means of (3.6) and Proposition 3.6 we consider k-FC type measure for half positive integer multi-indices k.

Definition 3.7 (k-FC Measures: Extended Version) If $k \in (\mathbb{Z}_+/2)^n$, then we say that μ is a k-FC type measure if the quantity

$$C_k(\mu) = (k!)^2 \sup_{z \in \mathbb{C}^n} |\mu_k|(B_{\sqrt{n}}(z))$$

is finite. Here $B_{\sqrt{n}}(z)$ denotes the polydisk in $\mathbb{C}^n$ centered at $z = (z_1, \ldots, z_n)$ and radius $|\mathbf{1}| = \sqrt{n}$ (the value of $\mathbf{1} = (1, 1, \ldots, 1)$ is taken here just for convenience.)

Remark 1 Note that any measure with compact support is a k-FC type measure for each k. On the other hand, given $k \in (\mathbb{Z}_+/2)^n$, the Borel measure

$$d\mu(z) = \prod_{j=1}^{n} \frac{dr_j d\theta_j}{(1 + r_j^2)^{k_j}}, \quad z = (z_1, z_2, \ldots, z_n), \text{ and } z_j = r_j e^{i\theta_j},$$

is, by Proposition 3.6, a k-FC type measure for $\mathcal{F}^2(\mathbb{C}^n)$. In fact,

$$\mu_k(B) = \int_B \prod_{j=1}^{n} (1 + |z_j|^2)^{k_j} d\mu(z) = \nu_n(B).$$

for every Borel set $B \subset \mathbb{C}^n$. Here $\mu_k = \nu_n$ is the usual Lebesgue measure of $\mathbb{C}^n$. Therefore μ_k is a FC for $\mathcal{F}^2(\mathbb{C}^n)$.

3.3 Toeplitz Operators Generated by k-FC Type Measures

Next, Subsection 3.3 of [15] is summarized in order to have the necessary tools to study the separately radial Toeplitz operators generated by k-FC types measures.

Let $\mu \in \mathfrak{B}_{reg,M}(\mathbb{C}^n)$ be a k-FC type measure for $\mathcal{F}^2(\mathbb{C}^n)$, where $k \in (\mathbb{Z}_+/2)^n$. For $\alpha,\ \beta \in \mathbb{Z}_+^n$, with $2k = \alpha + \beta$, the *coderivative* $\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu$ is given by Esmeral et al. [15] and Rozenblum et al. [30]: for a function $h = f\overline{g} \in L_1(\mathbb{C}^n, e^{-|w|^2} d\nu_n(w))$ with $f, g \in \mathcal{F}^2(\mathbb{C}^n)$

$$(\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu, h) = (-1)^{\alpha+\beta}(\mu\, G, \partial^\alpha \overline{\partial}^\beta h) = (-1)^{\alpha+\beta}(\mu\, G, \partial^\alpha f \overline{\partial^\beta g}), \quad G(z) = e^{-|z|^2},$$

(where $(\cdot, \cdot)$ is the intrinsic pairing between measures and functions and $(-1)^{\alpha+\beta} = \prod_{j=1}^n (-1)^{\alpha_j+\beta_j}$), provided that the right-hand side makes sense. The sesquilinear form $\mathrm{F}_{\mu,\alpha,\beta}$ on $\mathcal{F}^2(\mathbb{C}^n)$ associated with the *coderivative* $\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu$ is given by

$$\mathrm{F}_{\mu,\alpha,\beta}(f, g) = (\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu, f\overline{g}) = (-1)^{\alpha+\beta}\pi^{-n} \int_{\mathbb{C}^n} \partial^\alpha f(z) \overline{\partial^\beta g(z)} e^{-|z|^2} d\mu(z).$$

For $\alpha, \beta \in \mathbb{Z}_+^n$, $k \in (\mathbb{Z}_+/2)^n$, $2k = \alpha + \beta$, and the coderivative $\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu$ of a k-FC type measure μ, the Toeplitz operator $\mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu}$ generated by the sesquilinear form (1.2) and the Berezin transform of the coderivatives $\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu$ of a k-FC type measure have the following integral representation

$$(\mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu} f)(z) = \mathrm{F}_{\mu,\alpha,\beta}(f, K_z) = \pi^{-n} z^\beta \int_{\mathbb{C}^n} \partial^\alpha f(w) e^{z\cdot\overline{w}} e^{-|w|^2} d\mu(w), \quad f \in \mathcal{F}^2(\mathbb{C}^n). \tag{3.7}$$

$$\widetilde{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu}(z) = z^\beta \overline{z}^\alpha \int_{\mathbb{C}^n} e^{-|z-w|^2} d\mu(w), \quad z \in \mathbb{C}^n. \tag{3.8}$$

In particular, if the Toeplitz operator $\mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu}$ is bounded, then $\widetilde{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu} = \widetilde{\mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu}}$, i.e.,

$$\widetilde{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu}(z) = \frac{\left\langle \mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu} K_z, K_z \right\rangle}{\left\langle K_z, K_z \right\rangle} = e^{-|z|^2}\, \mathrm{F}_{\mu,\alpha,\beta}(K_z, K_z), \quad z \in \mathbb{C}^n. \tag{3.9}$$

Lemma 3.8 *Let μ be a positive regular measure on $\mathbb{C}^n$ and $k \in \left(\mathbb{Z}_+/2\right)^n$. Then for any $\mathbf{r} = (r_j)_{j=1}^n$ with $r_j > 0$, there exists a positive constant $C = C(\mathbf{r}, k) > 0$ such that*

$$\mu_k\left(B_{\mathbf{r}}(z)\right) \leq C \left| \widetilde{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu}(z) \right|, \tag{3.10}$$

for each $z \in \mathbb{C}^n$ with $|z_j| \geq 1$, $j = 1, 2, \ldots, n$.

Proof Let $r_j > 0$ and $z_j \in \mathbb{C}$ with $|z_j| \geq 1$. Then $(1+|w_j|^2)^{k_j} \leq (1+r_j+|z_j|)^{2k_j}$ for each $|w_j - z_j| < r_j$ and hence

$$\begin{aligned}
\mu_k(B_{\mathbf{r}}(z)) &= \int_{B_{\mathbf{r}}(z)} \prod_{j=1}^{n} (1+|w_j|^2)^{k_j} d\mu(w) \\
&\leq e^{|r|^2} \int_{B_{\mathbf{r}}(z)} \prod_{j=1}^{n} e^{-|z_j - w_j|^2} (1+|w_j|^2)^{k_j} d\mu(w) \\
&\leq e^{|r|^2} \prod_{j=1}^{n} (1+r_j+|z_j|)^{2k_j} \int_{B_{\mathbf{r}}(z)} \prod_{j=1}^{n} e^{-|z_j-w_j|^2} d\mu(w) \\
&= e^{|r|^2} \prod_{j=1}^{n} \left(\sum_{l=0}^{2k_j} \binom{2k_j}{l} (1+r_j)^l |z_j|^{2k_j - l} \right) \int_{B_{\mathbf{r}}(z)} \prod_{j=1}^{n} e^{-|z_j-w_j|^2} d\mu(w) \\
&\leq e^{|r|^2} \prod_{j=1}^{n} \left(\sum_{l=0}^{2k_j} \binom{2k_j}{l} (1+r_j)^l \right) \int_{B_{\mathbf{r}}(z)} \prod_{j=1}^{n} |z_j|^{2k_j} e^{-|z_j-w_j|^2} d\mu(w) \\
&\leq \left| \widetilde{\boldsymbol{\partial}^{\alpha} \overline{\boldsymbol{\partial}}^{\beta} \mu}(z) \right| \left(\prod_{j=1}^{n} \pi e^{r_j^2} (2+r_j)^{2k_j} \right).
\end{aligned}$$

□

The next result provides a criterion for a Toeplitz operator $\mathbf{T}_{\boldsymbol{\partial}^{\alpha}\overline{\boldsymbol{\partial}}^{\beta}\mu}$ defined by the derivatives of the k-FC type measure $\mu \in \mathfrak{B}_{reg}(\mathbb{C}^n)$ to be bounded and it is analogous to Proposition 3.2.

Theorem 3.9 *Let μ be a positive regular measure on $\mathbb{C}^n$ and $k \in (\mathbb{Z}_+/2)^n$. Then for every $\alpha, \beta \in \mathbb{Z}_+$ with $\alpha + \beta = 2k$ the following conditions are equivalent:*

1. *The sesquilinear form*

$$\mathrm{F}_{\mu,\alpha,\beta}(f,g) = \pi^{-n} \int_{\mathbb{C}^n} \partial^{\alpha} f(z) \overline{\partial^{\beta} g(z)} e^{-|z|^2} d\mu(z),$$

   is bounded in $\mathcal{F}^2(\mathbb{C}^n)$.
2. *The Toeplitz operator $\mathbf{T}_{\boldsymbol{\partial}^{\alpha}\overline{\boldsymbol{\partial}}^{\beta}\mu}$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$.*
3. *$\widetilde{\boldsymbol{\partial}^{\alpha}\overline{\boldsymbol{\partial}}^{\beta}\mu}$ is bounded on $\mathbb{C}^n$.*
4. *For every $\mathbf{r} = (r_j)_{j=1}^n$ with $r_j > 0$, there exists $C = C(\mathbf{r}, k) > 0$ such that*

$$\mu_k(B_{\mathbf{r}}(z)) < C, \quad \textit{for all } z \in \mathbb{C}^n. \tag{3.11}$$

5. *μ is a k-FC type measure.*

Proof The proof of (1) ⇒ (2), (2) ⇒ (3), (4) ⇒ (5) and (5) ⇒ (1) follow easily by (3.7), (3.9), [30, Theorem 5.4] and [30, Proposition 6.1] respectively. It remains to prove that (3) implies (4).

If $|z_j| \geq 1$ for every $j = 1, 2, \ldots, n$ then the boundedness holds by Lemma 3.8. Now, if there exists $j_0 \in \{1, 2, \ldots, n\}$ such that $|z_{j_0}| < 1$. Then $|w_{j_0}| < r_{j_0} + |z_{j_0}| < r_{j_0} + 1$ for every $w \in B_{\mathbf{r}}(z)$ and hence

$$\mu_k(B_{\mathbf{r}}(z)) \leq \int_{B_{\mathbf{r}}(z)} \left(\prod_{\substack{j=1 \\ j \neq j_0}}^{n} (1 + |w_j|^2)^{k_j} \right) \left(1 + |w_{j_0}|^2\right)^{k_{j_0}} e^{-\left|\left(\sqrt{2}+ir_{j_0}^{1/2}\right)-w_{j_0}\right|^2}$$

$$e^{\left|\left(\sqrt{2}+ir_{j_0}^{1/2}\right)-w_{j_0}\right|^2} d\mu(w)$$

$$\leq \pi^n e^{|r|^2 - r_{j_0}^2 + 4(r_{j_0}+2)^2} \prod_{j=1}^{n} (2 + r_j)^{2k_j} \int_{B_{\mathbf{r}}(z)} \left(\prod_{\substack{j=1 \\ j \neq j_0}}^{n} |z_j|^{2k_j} e^{-|z_j - w_j|^2} \right)$$

$$e^{-\left|\left(\sqrt{2}+ir^{1/2}\right)-w_{j_0}\right|^2} d\mu(w)$$

$$\leq C(\mathbf{r}, k) \int_{\mathbb{C}^n} \left(\prod_{\substack{j=1 \\ j \neq j_0}}^{n} |z_j|^{2k_j} e^{-|z_j - w_j|^2} \right) \left(\left|\sqrt{2} + ir_{j_0}^{1/2}\right|^{2k_{j_0}} e^{-\left|\left(\sqrt{2}+ir_{j_0}^{1/2}\right)-w_{j_0}\right|^2} \right) d\mu(w)$$

$$= C(\mathbf{r}, k) \left| \widetilde{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu} \left(z_1, z_2, \ldots, z_{j_0-1}, \sqrt{2} + ir_{j_0}^{1/2}, z_{j_0+1}, \ldots, z_n\right) \right| \leq C(\mathbf{r}, k) \, \|\widetilde{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu}\|_\infty. \square$$

As in FC type measure, we may to admit complex valued Borel measures μ such that their variation $|\mu|$ are k-FC measures. In such a case, as a by-product of Proposition 3.9, the results imply that the following norms for μ are equivalent:

1. $\|\mu\|_1 = \|\mathbf{T}_{\widetilde{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu}}\|$.
2. $\|\mu\|_2 = \|\widetilde{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta |\mu|}\|_\infty$.
3. $\|\mu\|_3 = \sup_{z \in \mathbb{C}^n} |\mu_k|(B_{\mathbf{r}}(z))$, where $\mathbf{r} = (r_j)_{j=1}^n$ is any fixed positive radius.
4. $\|\mu\|_4 = \sup_{\substack{f \in \mathcal{F}^2(\mathbb{C}^n) \\ \|f\|=1}} \left\{ \int_{\mathbb{C}^n} |\partial^k f(w)|^2 e^{-|w|^2} d|\mu|(w) \right\}$.

4 Separately Radial k-FC Type Measures

Let $\mathbb{T}^n$ be the n-dimensional torus. Given complex Borel regular measures $\varrho \in \mathfrak{B}_{reg}(\mathbb{R}^n_+)$ and $\eta \in \mathfrak{B}_{reg}(\mathbb{T}^n)$ we denote by $\mu = \varrho \otimes \eta$ the tensor product of the measures ϱ and η. i.e., for any $A \in \text{Borel}(\mathbb{R}^n_+)$ and any $B \in \text{Borel}(\mathbb{T}^n)$, $\mu(A \times B) = \varrho(A)\eta(B)$, with the usual extension to all Borel sets in $\mathbb{C}^n$.

Definition 4.1 (Separately Radial Measures) We say that $\mu \in \mathfrak{B}_{reg}(\mathbb{C}^n)$ is *separately radial* if there exists $\varrho \in \mathfrak{B}_{reg}(\mathbb{R}^n_+)$ such that $\mu = \varrho \otimes \mathfrak{m}$ where $\mathfrak{m}$ is the Haar measure of $\mathbb{T}^n$. Furthermore, if $\mu = \varrho \otimes \mathfrak{m}$ is a k-FC type measure for $\mathcal{F}^2(\mathbb{C}^n)$ we say that μ is *k-srFC*, in particular, for 0-srFC we say *srFC* for short.

Proposition 4.2 *Let $k \in (\mathbb{Z}_+/2)^n$. A complex Borel measure μ is k-srFC type measure for $\mathcal{F}^2(\mathbb{C}^n)$ if and only if $\mu_{k-\alpha}$ is α-srFC for any $\alpha \in \mathbb{Z}^n_+$*

Proof Suppose that μ is k-srFC. i.e., $\mu = \varrho \otimes \mathfrak{m}$ for some $\varrho \in \mathfrak{B}_{reg}(\mathbb{R}^n_+)$ and $|\mu|$ is k-srFC. Then, $|\mu|_{k-\alpha} = |\mu_{k-\alpha}|$ is α-FC for any $\alpha \in \mathbb{Z}^n_+$ by Proposition 3.6. For every $A \in \text{Borel}(\mathbb{C}^n)$,

$$
\begin{aligned}
\mu_{k-\alpha}(A) &= \int_A \prod_{j=1}^{n} (1+|w_j|^2)^{k_j-\alpha_j} d\mu(w) \\
&= \int_{\left\{(r_1,r_2,\ldots,r_n,e^{i\theta_1},\ldots,e^{i\theta_n}) :\, \left(r_j e^{i\theta_j}\right)_{j=1}^n \in A\right\}} \left(\prod_{j=1}^{n}(1+r_j^2)^{k_j-\alpha_j}\right) \\
&\qquad d\varrho(r)d\mathfrak{m}((e^{i\theta_j})_{j=1}^n) \\
&= (\varrho_{k-\alpha} \otimes \mathfrak{m})(A).
\end{aligned}
$$

Here $d\varrho_{k-\alpha}(r) = \left(\prod_{j=1}^{n}(1+r_j^2)^{k_j-\alpha_j}\right) d\varrho(r)$. Conversely, for any $\alpha \in \mathbb{Z}^n_+$ suppose that $\mu_{k-\alpha}$ is separately radial. i.e., $\mu_{k-\alpha} = \lambda \otimes \mathfrak{m}$ for some $\lambda \in \mathfrak{B}_{reg}(\mathbb{R}^n_+)$. Then for every $A \in \text{Borel}(\mathbb{C}^n)$

$$
\begin{aligned}
\mu(A) &= \int_A \prod_{j=1}^{n} (1+|w_j|^2)^{\alpha_j-k_j} d\mu_{k-\alpha}(w) \\
&= \int_{\left\{(r_1,r_2,\ldots,r_n,e^{i\theta_1},e^{i\theta_2},\ldots,e^{i\theta_n}) :\, \left(r_j e^{i\theta_j}\right)_{j=1}^n \in A\right\}} \left(\prod_{j=1}^{n}(1+r_j^2)^{\alpha_j-k_j}\right) \\
&\qquad d\lambda(r)d\mathfrak{m}((e^{i\theta_j})_{j=1}^n) \\
&= (\lambda_{\alpha-k} \otimes \mathfrak{m})(A).
\end{aligned}
$$

Here $d\lambda_{\alpha-k}(r) = \left(\prod_{j=1}^{n}(1+r_j^2)^{\alpha_j-k_j}\right) d\varrho(r)$. □

Next, we show that every Toeplitz operator with coderivatives of a separately radial measure as symbol is unitarily equivalent to the composition of the multiplication operator by some ℓ_∞-sequence and the shift operator acting on $\ell_2(\mathbb{Z}_+^n)$.

Proposition 4.3 *Let $\alpha, \beta \in \mathbb{Z}_+^n$ and $k \in (\mathbb{Z}_+/2)^n$ be such that $2k = \alpha + \beta$. If $\mu = \varrho \otimes \mathfrak{m}$ is a k-srFC type measure on $\mathcal{F}^2(\mathbb{C}^n)$, then $\mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu} e_m = \gamma_{\varrho,\alpha,\beta}(m) e_{m+\beta-\alpha}$ for any $m \in \mathbb{Z}_+^n$, where*

$$\gamma_{\varrho,\alpha,\beta}(m) = \begin{cases} \dfrac{2^n\sqrt{m!(m-\alpha+\beta)!}}{[(m-\alpha)!]^2} \displaystyle\int_{\mathbb{R}_+^n} \left(\prod_{j=1}^{n} r_j^{2(m_j-\alpha_j)} e^{-r_j^2}\right) d\varrho(r), & \text{if } m \geq \alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Proof Let $\alpha, \beta \in \mathbb{Z}_+^n$, $k \in (\mathbb{Z}_+/2)^n$ be such that $2k = \alpha + \beta$ and $\mu = \varrho \otimes \mathfrak{m}$. Then by (3.7) for any $m, v \in \mathbb{Z}_+^n$,

$$\left\langle \mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu} e_m, e_v \right\rangle = \begin{cases} 0, & \text{if } \alpha > m \text{ or } \beta > v \\ \dfrac{\sqrt{v!m!}}{\pi^n\,(m-\alpha)!(v-\beta)!} \displaystyle\int_{\mathbb{C}^n} z^{m-\alpha}\overline{z^{v-\beta}} e^{-|z|^2} d\mu(z), & \text{otherwise.} \end{cases}$$

Now, since $\int_{\mathbb{T}^n} f(w) d\mathfrak{m}(w) = \int_0^{2\pi} \cdots \int_0^{2\pi} f(e^{i\theta_1}, \ldots, e^{i\theta_n}) \prod_{j=1}^{n} \frac{d\theta_j}{2\pi}$, we have that

$$= \begin{cases} 0, & \text{if } \alpha > m \text{ or } \beta > v \\ \gamma_{\varrho,\alpha,\beta}(m) \displaystyle\prod_{j=1}^{n} \delta_{m_j-\alpha_j, v_j-\beta_j}, & \text{otherwise.} \end{cases}$$

Therefore, since $\overline{\operatorname{span}\{e_m : m \in \mathbb{Z}_+^n\}} = \mathcal{F}^2(\mathbb{C}^n)$ we have for every $m \in \mathbb{Z}_+^n$ that

$$\mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu} e_m = \sum_{v \in \mathbb{Z}_+^n} \left\langle \mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu} e_m, e_v \right\rangle e_v = \gamma_{\varrho,\alpha,\beta}(m) e_{m+\beta-\alpha}.$$

□

Corollary 4.4 *Let $\alpha, \beta \in \mathbb{Z}_+^n$, $k \in (\mathbb{Z}_+/2)^n$ be such that $2k = \alpha + \beta$ and $\mu = \varrho \otimes \mathfrak{m}$ be a k-srFC type measure for $\mathcal{F}^2(\mathbb{C}^n)$. Then the operator $\mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu}$ is separately radial if and only if $\alpha = \beta = k \in \mathbb{Z}_+^n$.*

Corollary 4.5 *Let $\alpha, \beta \in \mathbb{Z}_+^n$, $k \in (\mathbb{Z}_+/2)^n$ be such that $2k = \alpha + \beta$ and $\varrho \in \mathfrak{B}_{reg}(\mathbb{R}_+^n)$. Then $\mu = \varrho \otimes \mathfrak{m}$ is a k-FC type measure for $\mathcal{F}^2(\mathbb{C}^n)$ if and only if $\gamma_{\varrho,\alpha,\beta} \in \ell_\infty(\mathbb{Z}_+^n)$ where $\gamma_{\varrho,\alpha,\beta} = (\gamma_{\varrho,\alpha,\beta}(n))_{n\in\mathbb{Z}_+^n}$ is given in (4.1).*

Corollary 4.6 *Let $k \in (\mathbb{Z}_+/2)^n$ and $\varrho \in \mathfrak{B}_{reg}(\mathbb{R}_+^n)$. Then $\gamma_{\varrho,k,k} = \gamma_{\varrho,k} \in \ell_\infty(\mathbb{Z}_+^n)$ if and only if $\gamma_{\varrho_{k-\alpha},\alpha} \in \ell_\infty(\mathbb{Z}_+^n)$ for every $\alpha \in \mathbb{Z}_+^n$. Here $d\varrho_{k-\alpha}(r) = \left(\prod_{j=1}^{n}(1+r_j^2)^{k_j-\alpha_j}\right) d\varrho(r)$.*

Proof Let $\varrho \in \mathfrak{B}_{reg}(\mathbb{R}_+^n)$. If $\gamma_{\varrho,k} \in \ell_\infty(\mathbb{Z}_+^n)$ then $\mu = \varrho \otimes \mathfrak{m}$ is a k-FC type measure for $\mathcal{F}^2(\mathbb{C}^n)$ by Corollary 4.5 and hence $\mu_{k-\alpha} = \varrho_{k-\alpha} \otimes \mathfrak{m}$, where $d\varrho_{k-\alpha}(r) = \left(\prod_{j=1}^{n}(1+r_j^2)^{k_j-\alpha_j}\right) d\varrho(r)$, is a α-srFC type measure for $\mathcal{F}^2(\mathbb{C}^n)$ by Proposition 4.2. Therefore, $\gamma_{\varrho_{k-\alpha},\alpha} \in \ell_\infty(\mathbb{Z}_+^n)$ by Corollary 4.5. The rest of the proof runs as before. □

Example 2 (Unidimensional Case[30]) Given $k, \alpha, \beta \in \mathbb{Z}_+$, consider the k-FC type measure

$$d\mu(z) = \frac{d\nu(z)}{(1+|z|^2)^k}, \tag{4.2}$$

and the corresponding sesquilinear form $\mathrm{F}_{\mu,\alpha,\beta}$:

$$\mathrm{F}_{\mu,\alpha,\beta}(f,g) = (-1)^{\alpha+\beta}\int_{\mathbb{C}} \partial^\alpha f(z)\overline{\partial^\beta g(z)}\frac{e^{-|z|^2}}{\pi(1+|z|^2)^k}d\nu(z).$$

The exact formula for $\gamma_{\varrho,\alpha,\beta}$ is rather complicated, but its asymptotic behaviour for large values of n is quite simple. For $n \geq \alpha + k$,

$$\mathrm{F}_{\mu,\alpha,\beta}(e_n, e_{n+\beta-\alpha}) = (-1)^{\alpha+\beta}\frac{(n-\alpha)^{\frac{\alpha+\beta}{2}}}{(n-\alpha-k)^k}\left(1+O\left(\frac{1}{n}\right)\right).$$

Thus, $\gamma_{\varrho,\alpha,\beta} = (\gamma_{\varrho,\alpha,\beta}(n))_{n\in\mathbb{Z}_+}$ is bounded if and only if $\alpha + \beta \leq 2k$, and if $\alpha = \beta = k$ then $\mathbf{T}_{\partial^k\overline{\partial}^k\mu}$ is a compact perturbation of the identity, for details see [30, Example 6.7].

Remark 1 Let $\alpha, \beta \in \mathbb{Z}_+^n$, $k \in (\mathbb{Z}_+/2)^n$ be such that $2k = \alpha + \beta$ and $\mu = \varrho \otimes \mathfrak{m}$ be a k-srFC type measure for $\mathcal{F}^2(\mathbb{C}^n)$. By Proposition 4.3, for any $m \in \mathbb{Z}_+^n$ with $\alpha \leq m$ and any $v \in \mathbb{Z}_+^n$

$$\left\langle \mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu} e_m, e_v \right\rangle = \gamma_{\varrho,\alpha,\beta}(m) \left\langle e_{m+\beta-\alpha}, e_v \right\rangle = \gamma_{\varrho,\alpha,\beta}(v + \alpha - \beta)\delta_{m,v+\alpha-\beta}$$
$$= \gamma_{\varrho,\alpha,\beta}(v + \alpha - \beta)\delta_{m,v+\alpha-\beta} = \left\langle e_m, \tilde{\gamma}_{\varrho,\alpha,\beta}(v) e_{v+\alpha-\beta} \right\rangle,$$

where

$$\tilde{\gamma}_{\varrho,\alpha,\beta}(m) = \overline{\gamma_{\varrho,\alpha,\beta}(m + \alpha - \beta)}. \tag{4.3}$$

Therefore,

$$\mathbf{T}^*_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu} e_m = \begin{cases} \tilde{\gamma}_{\varrho,\alpha,\beta}(m) e_{m+\alpha-\beta}, & \text{if } m \geq \beta, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

Let $\mathcal{S}\colon \mathcal{F}^2(\mathbb{C}^n) \to \mathcal{F}^2(\mathbb{C}^n)$ be the *shift* operator given by

$$\mathcal{S} e_\alpha = \begin{cases} e_{\alpha-\mathbf{1}}, & \text{if } \alpha \geq \mathbf{1} \\ 0, & \text{otherwise} \end{cases}, \tag{4.5}$$

where $e_\alpha(z) = \dfrac{z^\alpha}{\sqrt{\alpha!}}$. This operator is bounded with $\|\mathcal{S}\| = 1$, its adjoint operator $\mathcal{S}^*$ is given by

$$\mathcal{S}^* e_\alpha = e_{\alpha+\mathbf{1}}, \quad \alpha \in \mathbb{Z}_+^n \tag{4.6}$$

and it is a partial isometry with $\mathcal{S}\mathcal{S}^* = \mathrm{Id}$. Furthermore, for any $\beta \in \mathbb{Z}_+^n$

$$\mathcal{S}^\beta e_\alpha = \begin{cases} e_{\alpha-\beta}, & \text{if } \alpha \geq \beta, \\ 0, & \text{otherwise.} \end{cases} \qquad (\mathcal{S}^*)^\beta e_\alpha = e_{\alpha+\beta}, \quad \alpha \in \mathbb{Z}_+^n. \tag{4.7}$$

Corollary 4.7 *Let $\alpha, \beta \in \mathbb{Z}_+^n$, $k \in (\mathbb{Z}_+/2)^n$ be such that $2k = \alpha + \beta$ and $\mu = \varrho \otimes \mathfrak{m}$ be a k-srFC type measure for $\mathcal{F}^2(\mathbb{C}^n)$. Then the operator $\mathcal{S}^\beta \mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu} (\mathcal{S}^*)^\alpha$ is separately radial.*

Proof Let $\{e_v : v \in \mathbb{Z}_+^n\}$ be the monomial basis of $\mathcal{F}^2(\mathbb{C}^n)$. Then by Proposition 4.3 and (4.7) we have that $\mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu} (\mathcal{S}^*)^\alpha e_v = \gamma_{\varrho,\alpha,\beta}(v + \alpha) e_{v+\beta} = \gamma_{\varrho,\alpha,\beta}(v + \alpha)(\mathcal{S}^*)^\beta e_v$ and hence $\mathcal{S}^\beta \mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu} (\mathcal{S}^*)^\alpha e_v = \gamma_{\varrho,\alpha,\beta}(v + \alpha) e_v$ for each $v \in \mathbb{Z}_+^n$. i.e., $\mathcal{S}^\beta \mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu} (\mathcal{S}^*)^\alpha$ is separately radial by Proposition 2.5. $\square$

Proposition 4.8 *Let $\alpha, \beta \in \mathbb{Z}_+^n$, $k \in (\mathbb{Z}_+/2)^n$ be such that $2k = \alpha + \beta$ and $\mu = \varrho \otimes \mathfrak{m}$ be a k-srFC type measure for $\mathcal{F}^2(\mathbb{C}^n)$. If the Toeplitz operator $\mathbf{T}_{\partial^\beta \overline{\partial}^\beta \mu}$ is bounded then $\mathbf{T}_{\partial^\alpha \overline{\partial}^\beta \mu}$ is bounded for $\alpha \leq \beta$. Analogously, If the Toeplitz operator $\mathbf{T}_{\partial^\alpha \overline{\partial}^\alpha \mu}$ is bounded then $\mathbf{T}_{\partial^\alpha \overline{\partial}^\beta \mu}$ is bounded for $\beta \leq \alpha$.*

Proof By (4.1) and Proposition 4.3 it follows that for any $\alpha, \beta \in \mathbb{Z}_+^n$, $k \in (\mathbb{Z}_+/2)^n$ with $2k = \alpha + \beta$ and any $v \in \mathbb{Z}_+^n$ with $v \geq \alpha$,

$$\mathbf{T}_{\partial^\alpha \overline{\partial}^\beta \mu} e_v = \sqrt{\frac{(v+\beta-\alpha)!}{v!}} (\mathcal{S}^*)^\beta \mathcal{S}^\alpha \mathbf{T}_{\partial^\alpha \overline{\partial}^\alpha \mu} e_v.$$

$$\mathbf{T}^*_{\partial^\alpha \overline{\partial}^\beta \mu} e_v = \sqrt{\frac{(v+\alpha-\beta)!}{v!}} (\mathcal{S}^*)^\alpha \mathcal{S}^\beta \mathbf{T}_{\partial^\beta \overline{\partial}^\beta \mu} e_v.$$

Suppose that $\alpha \leq \beta$ and $\mathbf{T}_{\partial^\beta \overline{\partial}^\beta \mu}$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$. Then for every $f \in \mathcal{F}^2(\mathbb{C}^n)$,

$$\begin{aligned}
\left\| \mathbf{T}_{\partial^\alpha \overline{\partial}^\beta \mu} f \right\|^2 &= \sum_{v \in \mathbb{Z}_+^n} \left| \left\langle \mathbf{T}_{\partial^\alpha \overline{\partial}^\beta \mu} f, e_v \right\rangle \right|^2 = \sum_{v \in \mathbb{Z}_+^n} \left| \left\langle f, \mathbf{T}^*_{\partial^\alpha \overline{\partial}^\beta \mu} e_v \right\rangle \right|^2 \\
&= \sum_{v \geq \beta} \frac{(v+\alpha-\beta)!}{v!} \left| \left\langle f, (\mathcal{S}^*)^\alpha \mathcal{S}^\beta \mathbf{T}_{\partial^\beta \overline{\partial}^\beta \mu} e_v \right\rangle \right|^2 \\
&\leq \left(\prod_{j=1}^n \sup_{v_j \geq \beta_j} \frac{(v_j+\alpha_j-\beta_j)!}{v_j!} \right) \sum_{v \in \mathbb{Z}_+^n} \left| \left\langle \mathbf{T}^*_{\partial^\beta \overline{\partial}^\beta \mu} (\mathcal{S}^*)^\beta \mathcal{S}^\alpha f, e_v \right\rangle \right|^2 \\
&= \left(\prod_{j=1}^n \sup_{v_j \geq \beta_j} \frac{(v_j+\alpha_j-\beta_j)!}{v_j!} \right) \left\| \mathbf{T}^*_{\partial^\beta \overline{\partial}^\beta \mu} (\mathcal{S}^*)^\beta \mathcal{S}^\alpha f \right\|^2.
\end{aligned}$$

Now, by Stirling approximation, for any $j = 1, 2, \ldots, n$

$$\sqrt{\frac{(v_j+\alpha_j-\beta_j)!}{v_j!}} \sim (v_j+1)^{\frac{\alpha_j-\beta_j}{2}}$$

and hence

$$\prod_{j=1}^n \sup_{v_j \geq \beta_j} \frac{(v_j+\alpha_j-\beta_j)!}{v_j!} < +\infty$$

for $\alpha \leq \beta$. Thus, the statement holds since any linear operator $\mathbf{T}$ is bounded if and only if $\mathbf{T}^*$ is bounded. The rest of the proof runs as before since for $\alpha \geq \beta$ and $\mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\alpha \mu}$ bounded on $\mathcal{F}^2(\mathbb{C}^n)$ we have

$$\left\| \mathbf{T}^*_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu} f \right\|^2 \leq \left(\prod_{j=1}^n \sup_{v_j \geq \beta_j} \frac{(v_j + \beta_j - \alpha_j)!}{v_j!} \right) \left\| \mathbf{T}^*_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\alpha \mu} (\mathcal{S}^*)^\alpha \mathcal{S}^\beta f \right\|^2. \qquad \square$$

The following result follows from Corollary 4.5 and Proposition 4.8.

Corollary 4.9 *Let $\alpha \in \mathbb{Z}_+^n$, $k \in (\mathbb{Z}_+/2)^n$ be such that $2k - \alpha \in \mathbb{Z}_+^n$ and ϱ be a Borel measure on $\mathbb{R}_+^n$ such that $\mu = \varrho \otimes \mathfrak{m}$ is a complex regular Borel measure on $\mathbb{C}^n$, where $\mathfrak{m}$ is the Haar measure on $\mathbb{T}^n$. If the sequence $\hat{\varrho}_\alpha = \big(\hat{\varrho}_\alpha(v)\big)_{v \in \mathbb{Z}_+^n}$ belongs to $\ell_\infty(\mathbb{Z}_+^n)$, where*

$$\hat{\varrho}_\alpha(v) = \begin{cases} \dfrac{2^n v!}{[(v-\alpha)!]^2} \displaystyle\int_{\mathbb{R}_+^n} r^{2(v-\alpha)} e^{-r^2} d\varrho(r), & \text{if } v \geq \alpha, \\[2ex] 0 & \text{otherwise}. \end{cases} \tag{4.8}$$

then $\mu = \varrho \otimes \mathfrak{m}$, is a k-srFC radial type measure on $\mathcal{F}^2(\mathbb{C}^n)$.

5 Separately Radial Toeplitz Operators

Let $\alpha, \beta \in \mathbb{Z}_+^n$, $k \in (\mathbb{Z}_+/2)^n$ be such that $2k = \alpha + \beta$. If μ is a k-srFC then by Corollary 4.4 the Toeplitz operator $\mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\beta \mu}$ is separately radial if and only if $\alpha = \beta = k \in \mathbb{Z}_+^n$. In this section we explore this situation and give a criterion for a Toeplitz operator $\mathbf{T}_{\boldsymbol{\partial}^\alpha \overline{\boldsymbol{\partial}}^\alpha \mu}$ to be separately radial.

The following lemma is analogous to the injectivity property of the Berezin transform for measures satisfying the (M)-condition.

Lemma 5.1 *Let $k \in (\mathbb{Z}_+/2)^n$ and $\mu \in \mathfrak{B}_{reg}(\mathbb{C}^n)$ be a complex measure satisfying the (M)-condition (3.2). If*

$$\left(\prod_{j=1}^n |z_j|^{2k_j} \right) \int_{\mathbb{C}^n} e^{-|z-w|^2} d\mu(w) = 0$$

for any z in $\mathbb{C}^n$ then μ is the zero measure.

Proof If $\left(\prod_{j=1}^{n} |z_j|^{2k_j}\right)\int_{\mathbb{C}^n} e^{-|z-w|^2} d\mu(w) = 0$ for any $z \in \mathbb{C}^n$ then $\int_{\mathbb{C}^n} e^{-|z-w|^2} d\mu(w) = 0$ for any z with $\prod_{j=1}^{n} z_j \neq 0$. Let $\Phi\colon \mathbb{C}^n \times \overline{\mathbb{C}^n} \to \mathbb{C}$ be the mapping given by

$$\Phi(z, w) = \int_{\mathbb{C}^n} e^{z\cdot\overline{\zeta}}\, e^{w\cdot\zeta} e^{-|\zeta|^2} d\mu(\zeta).$$

If μ satisfies the (M)-condition (3.2) for any $z \in \mathbb{C}^n$ by Cauchy–Schwarz inequality we have,

$$\begin{aligned}\left|\int_{\mathbb{C}^n} e^{z\cdot\overline{\zeta}}\, e^{w\cdot\zeta} e^{-|\zeta|^2} d\mu(\zeta)\right| &\leq \left(\int_{\mathbb{C}^n} |K_z(\zeta)|^2 e^{-|\zeta|^2} d|\mu|(\zeta)\right)^{1/2} \\ &\quad \left(\int_{\mathbb{C}^n} |K_{\overline{w}}(\zeta)|^2 e^{-|\zeta|^2} d|\mu|(\zeta)\right)^{1/2} \\ &\leq M.\end{aligned}$$

Therefore, the Toeplitz operator $\mathbf{T}_\mu$ given in (3.1) is well-defined and bounded on the dense subset of all finite linear combinations of kernel function and hence Φ is well-defined and continuous on $\mathbb{C}^n \times \overline{\mathbb{C}^n}$ since $\Phi(z, w) = \mathbf{T}_\mu K_{\overline{w}}(z)$. On the other hand, note that for any triangle Δ_j in $\mathbb{C}$ by the Fubini's Theorem one gets that

$$\int_{\partial\Delta_j} \Phi(z, w) dz = \int_{\mathbb{C}^n} e^{w\cdot\zeta} \left(\left(\prod_{\substack{l=1\\ l\neq j}}^{n} e^{z_l\cdot\overline{\zeta_l}}\right) \int_{\partial\Delta_j} e^{z_j\cdot\overline{\zeta_j}} dz_j\right) d\mu(\zeta) = 0$$

$$\int_{\partial\Delta_j} \Phi(z, w) dw = \int_{\mathbb{C}^n} e^{z\cdot\overline{\zeta}} \left(\left(\prod_{\substack{l=1\\ l\neq j}}^{n} e^{w_l\cdot\zeta_l}\right) \int_{\partial\Delta_j} e^{w_j\cdot\zeta_j} dw_j\right) d\mu(\zeta) = 0$$

Thus, by Morera's Theorem Φ is a separately analytic on $\mathbb{C}^n \times \overline{\mathbb{C}^n}$ and hence by the Hartogs's Theorem Φ is analytic on $\mathbb{C}^n \times \overline{\mathbb{C}^n}$. Now, observe that the mapping $\Psi\colon \mathbb{C}^n \times \overline{\mathbb{C}^n} \to \mathbb{C}$ given by $\Psi(z, w) = z^k w^k \Phi(z, w)$ is analytic on $\mathbb{C}^n \times \overline{\mathbb{C}^n}$ and

$$\Psi(z, \overline{z}) = e^{|z|^2} \left(\prod_{j=1}^{n} |z_j|^{2k_j}\right) \int_{\mathbb{C}^n} e^{-|\zeta-z|^2} d\mu(\zeta) = 0.$$

Then $\Psi \equiv 0$ by [19, Proposition 1.69] and hence $\Phi(z, w) = 0$ for all (z, w) belonging to any ball $B_r((0,0))$ centered at $(0, 0)$ and radius $r > 0$. Therefore, by identity Theorem for several complex variables, $\Phi \equiv 0$ in $\mathbb{C}^n \times \overline{\mathbb{C}^n}$. Now, letting $d\varsigma(u, v) = e^{-|u+iv|^2} d\mu(u + iv)$ we have for any $x, y \in \mathbb{R}^n$,

$$\begin{aligned}\int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-i(x,y)\cdot(u,v)} d\varsigma(u, v) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{(-y+ix)\cdot(u+iv)+(y+ix)\cdot(u-iv)} d\varsigma(u, v) \\ &= \int_{\mathbb{C}^n} e^{(-y+ix)\cdot\zeta+(y+ix)\cdot\overline{\zeta}} e^{-|\zeta|^2} d\mu(\zeta) \\ &= \Phi(y + ix, -y + ix) = 0.\end{aligned}$$

i.e., the Fourier-Stieltjes transformation of the bounded complex measure ς in $\mathbb{R}^n \times \mathbb{R}^n$ is 0. Thus by the injectivity of Fourier-Stieltjes transform, $\varsigma \equiv 0$, see [9, Proposition 3.8.6], and hence $\mu \equiv 0$. □

Theorem 5.2 *Let $k \in \mathbb{Z}^n_+$ and μ be a positive k-FC type measure for $\mathcal{F}^2(\mathbb{C}^n)$. then the following conditions are equivalent:*

1. $\mathbf{T}_{\boldsymbol{\partial}^k \overline{\boldsymbol{\partial}}^k \mu}$ *is separately radial.*
2. *The Berezin transform $\widetilde{\boldsymbol{\partial}^k \overline{\boldsymbol{\partial}}^k \mu}$ of $\boldsymbol{\partial}^k \overline{\boldsymbol{\partial}}^k \mu$ is a separately radial function.*
3. *μ is invariant under the action of $\mathbf{U}_d(n, \mathbb{C})$, i.e., for every $A \in \mathrm{Borel}(\mathbb{C}^n)$ and every $\mathbf{X} \in \mathbf{U}_d(n, \mathbb{C})$,*

$$\mu\left(\mathbf{X}A\right) = \mu(A).$$

4. *For any Borel sets $Y \in \mathrm{Borel}(\mathbb{R}^n_+)$ and $Z \in \mathrm{Borel}(\mathbb{T}^n)$, and every $\mathbf{X} \in \mathbf{U}_d(n, \mathbb{C})$,*

$$\mu\left(\mathbf{X}[Y \times Z]\right) = \mu(Y \times Z)$$

5. *μ is separately radial. i.e., there exists a positive regular Borel measure ϱ on $\mathbb{R}^n_+$ such that $\mu = \varrho \otimes \mathfrak{m}$.*

Proof (1) ⇒ (2) Let $\mathbf{X} \in \mathbf{U}_d(n, \mathbb{C})$. Then for every $z \in \mathbb{C}^n$ that

$$V^*_{\mathbf{X}} \widetilde{\mathbf{T}_{\boldsymbol{\partial}^k \overline{\boldsymbol{\partial}}^k \mu}} V_{\mathbf{X}}(z) = e^{-|\mathbf{X}z|^2} \left\langle \mathbf{T}_{\boldsymbol{\partial}^k \overline{\boldsymbol{\partial}}^k \mu} K_{\mathbf{X}z}, K_{\mathbf{X}z} \right\rangle = \widetilde{\mathbf{T}_{\boldsymbol{\partial}^k \overline{\boldsymbol{\partial}}^k \mu}} \left(\mathbf{X}z\right). \tag{5.1}$$

Thus, if $\mathbf{T}_{\boldsymbol{\partial}^k \overline{\boldsymbol{\partial}}^k \mu}$ is a separately radial operator, then, by (3.9), $\widetilde{\boldsymbol{\partial}^k \overline{\boldsymbol{\partial}}^k \mu}(\mathbf{X}z) = \widetilde{\boldsymbol{\partial}^k \overline{\boldsymbol{\partial}}^k \mu}(z)$ for every $z \in \mathbb{C}^n$. Now, since $\mathbf{X} \in \mathbf{U}_d(n, \mathbb{C})$ is arbitrary, we have that the function $\widetilde{\boldsymbol{\partial}^k \overline{\boldsymbol{\partial}}^k \mu}$ is separately radial by Lemma 2.4, i.e., it depends only on $(|z_1|, |z_2|, \ldots, |z_n|)$. (2) ⇒ (3) Let $\mathbf{X} \in \mathbf{U}_d(n, \mathbb{C})$, $A \in \mathrm{Borel}(\mathbb{C}^n)$ and

$\mu_{\mathbf{X}}(A) = \mu\left(\mathbf{X}A\right)$. By the chain rule for every $f \in \mathcal{F}^2(\mathbb{C}^n)$ and every $\mathbf{X} \in \mathbf{U}_d(n, \mathbb{C})$,

$$\begin{aligned}\left\langle V_{\mathbf{X}}^* \mathbf{T}_{\partial^k \overline{\partial}^k \mu} V_{\mathbf{X}} f, g\right\rangle &= \left\langle \mathbf{T}_{\partial^k \overline{\partial}^k \mu} V_{\mathbf{X}} f, V_{\mathbf{X}} g\right\rangle = \int_{\mathbb{C}^n} \partial^k (V_{\mathbf{X}} f)(z) \overline{\partial^k (V_{\mathbf{X}} g)(z)} d\mu(z) \\ &= \int_{\mathbb{C}^n} (\partial^k f)\left(\mathbf{X}^* w\right) \overline{(\partial^k g)\left(\mathbf{X}^* w\right)} d\mu(w) = \int_{\mathbb{C}^n} \partial^k f\,(w)\, \overline{\partial^k g\,(w)} d\mu_{\mathbf{X}}(w) \\ &= \left\langle \mathbf{T}_{\partial^k \overline{\partial}^k \mu_{\mathbf{X}}} f, g\right\rangle, \quad \text{for all } g \in \mathcal{F}^2(\mathbb{C}^n).\end{aligned}$$

Therefore $\widetilde{\partial^k \overline{\partial}^k \mu_{\mathbf{X}}}(z) = \widetilde{\partial^k \overline{\partial}^k \mu}\left(\mathbf{X}z\right)$. However, since $\widetilde{\partial^k \overline{\partial}^k \mu}$ depends only on $(|z_1|, |z_2|, \ldots, |z_n|) \in \mathbb{R}_+^n$, we have for almost all $z \in \mathbb{C}^n$,

$$0 = \widetilde{\partial^k \overline{\partial}^k \lambda_{\mathbf{X}}}(z) = \left(\prod_{j=1}^n \frac{|z_j|^{2k_j}}{\pi}\right) \int_{\mathbb{C}^n} e^{-|z-w|^2} d\lambda_{\mathbf{X}}(w), \quad \text{where } \lambda_{\mathbf{X}} = \mu_{\mathbf{X}} - \mu.$$

Since $|\lambda_{\mathbf{X}}|(E) \leq |\mu_{\mathbf{X}}|(E) + |\mu|(E)$ for any $E \in \text{Borel}(\mathbb{C}^n)$ by (3.5) it is easy to see that for each $f \in \mathcal{F}^2(\mathbb{C}^n)$

$$\begin{aligned}\int_{\mathbb{C}^n} |\partial^k f(w)|^2 e^{-|w|^2} d|\lambda_{\mathbf{X}}|(w) &\leq \int_{\mathbb{C}^n} |\partial^k f(w)|^2 e^{-|w|^2} d|\mu_{\mathbf{X}}|(w) \\ &\quad + \int_{\mathbb{C}^n} |\partial^k f(w)|^2 e^{-|w|^2} d|\mu|(w) \\ &= \int_{\mathbb{C}^n} |\partial^k (V_{\mathbf{X}} f)(w)|^2 e^{-|w|^2} d|\mu|(\zeta) \\ &\quad + \int_{\mathbb{C}^n} |\partial^k f(w)|^2 e^{-|w|^2} d|\mu|(w) \\ &\leq \omega_k(\mu) \left(\| V_{\mathbf{X}} f\|^2 + \|f\|^2\right) = 2\omega_k(\mu)\|f\|^2.\end{aligned}$$

Now, it follows that $\lambda_{\mathbf{X}} \equiv 0$ by Lemma 5.1. i.e., $\mu(\mathbf{X}A) = \mu(A)$ for every Borel set $A \subset \mathbb{C}^n$ and every $\mathbf{X} \in \mathbf{U}_d(n, \mathbb{C})$.

(3) $\Rightarrow$ (4) It is immediately.

(4) $\Rightarrow$ (5) For every Borel set $Y \in \text{Borel}(\mathbb{R}_+^n)$ define the mapping $\Psi_Y\colon \text{Borel}(\mathbb{T}^n) \to [0, +\infty]$ by $\Psi_Y(Z) = \mu(Y \times Z)$. Then Ψ_Y is a locally finite regular Borel measure on $\mathbb{T}^n$ by Rudin [32, Theorem 2.18]. By hypothesis we have that for every $Z \in \text{Borel}(\mathbb{T}^n)$ and $\mathbf{X} \in \mathbf{U}_d(n, \mathbb{C})$

$$\Psi_Y(\mathbf{X}Z) = \mu\left(\mathbf{X}(Y \times Z)\right) = \mu(Y \times Z) = \Psi_Y(Z).$$

Thus Ψ_Y is invariant under the matrix product and hence by uniqueness of the Haar measure, [18, Theorem 2.20], there exists a number $\varrho(Y)$ such that $\mu(Y \times Z) = \varrho(Y)\mathfrak{m}(Z)$ for every $Z \in \text{Borel}(\mathbb{T}^n)$. Now, since $\mathbb{T}^n$ is a compact group we have that $\mathfrak{m}(\mathbb{T}^n) = 1$ and hence $\mu(Y \times \mathbb{T}^n) = \varrho(Y)$ for each $Y \in \text{Borel}(\mathbb{R}^n_+)$. Thus by Rudin [32, Theorem 2.18], ϱ is a positive regular Borel measure on $\mathbb{R}^n_+$ and hence μ is separately radial.

(5) $\Rightarrow$ (1) If $\mu = \varrho \otimes \mathfrak{m}$ then by Proposition 4.4 the Toeplitz operator $\mathbf{T}_{\boldsymbol{\partial}^k \overline{\boldsymbol{\partial}}^k \mu}$ is separately radial. $\square$

Since a complex regular Borel measure μ is a k-FC type measure if and only if its variation $|\mu|$ is a k-FC measure, it follows that Theorem 5.2 remains valid for such type of measures.

Let $k \in (\mathbb{Z}_+/2)^n$. If $k \in \mathbb{Z}^n_+$ then we denote by $\mathcal{T}_{\mathbb{Z}_+}(ksrFC)$ and by $\mathfrak{G}_{k,\mathbb{Z}_+}$ the C*-algebra generated by the set $\left\{\mathbf{T}_{\boldsymbol{\partial}^k \overline{\boldsymbol{\partial}}^k \mu} : \mu \text{ is } k\text{-srFC}\right\}$ and $\{\gamma_{\varrho,k} : \varrho \otimes \mathfrak{m} \text{ is } k\text{-FC}\}$ respectively, we write $\mathcal{T}(srFC)$ for $|k| = 0$. In addition, if $k \in \left((\mathbb{Z}_+/2) \setminus \mathbb{Z}_+\right)^n$ we denote by $\mathcal{T}_{\mathbb{Z}_+/2}(ksrFC)$ and by $\mathfrak{G}_{k,\mathbb{Z}_+/2}$ the C*-algebra generated by the set $\left\{\mathbf{T}_{\mu_k} : \mu \text{ is } k\text{-srFC}\right\}$ and $\{\gamma_{\varrho_k} : \varrho_k \otimes \mathfrak{m} \text{ is FC}\}$ respectively. Observe that $\gamma_{\varrho,0} = \gamma_\varrho = \gamma_{\varrho_0}$.

Theorem 5.3 *For any $k \in (\mathbb{Z}_+/2)^n$ the C*-algebras $\mathcal{T}_{ksrFC}$ and $\mathcal{T}_{\mathbb{Z}_+/2}(ksrFC)$ are commutative. In particular, if $k \in \mathbb{Z}^n_+$ then $\mathcal{T}_{ksrFC}$ is isometrically isomorphic to $\mathfrak{G}_{k,\mathbb{Z}_+}$ and if $k \in \left((\mathbb{Z}_+/2) \setminus \mathbb{Z}_+\right)^n$ then $\mathcal{T}_{ksrFC}$ is isometrically isomorphic to $\mathfrak{G}_{k,\mathbb{Z}_+/2}$.*

For any $k \in (\mathbb{Z}_+/2)^n$, by Proposition 4.2 a Borel regular measure μ is k-srFC if and only if μ_k is a FC type measure for $\mathcal{F}^2(\mathbb{C}^n)$, where μ_k is given in (3.6). Therefore, it follows that the C*-subalgebras $\mathfrak{G}_{k,\mathbb{Z}_+/2}$ and $\mathfrak{G}_{k,\mathbb{Z}_+}$ are isomorphic.

Corollary 5.4 *For any $k \in (\mathbb{Z}_+/2)^n$ the C*-algebras $\mathfrak{G}_{k,\mathbb{Z}_+/2}$ and $\mathfrak{G}_{k,\mathbb{Z}_+}$ are isomorphic to the C*-algebra $\mathfrak{G}$ generated by the set $\left\{\gamma_\varrho : \varrho \otimes \mathfrak{m} \text{ is FC}\right\}$.*

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References

1. Abreu, L.D., Faustino, N.: On Toeplitz operators and localization operators. Proc. Amer. Math. Soc. **143**, 4317–4323 (2015) http://dx.doi.org/10.1090/proc/12211
2. Bargmann, V.: On a Hilbert space of analytic functions and an associated integral transform. Comm. Pure Appl. Math. **3**, 187–214 (1961) http://dx.doi.org/10.1002/cpa.3160140303
3. Bauer, W., Herrera Yañez, C., Vasilevski, N.: Eigenvalue characterization of radial operators on weighted Bergman spaces over the unit ball. Integr. Equ. Oper. Theory **78**, no. 2, 271–300 (2014) http://dx.doi.org/10.1007/s00020-013-2101-1
4. Bauer, W., Issa, H.: Commuting Toeplitz operators with quasi-homogeneous symbols on the Segal–Bargmann space. J. Math. Anal. Appl. **386**, 213–235 (2012) http://dx.doi.org/10.1016/j.jmaa.2011.07.058

5. Bauer, W., Le, T.: Algebraic properties and the finite rank problem for Toeplitz operators on the Segal–Bargmann space. J. Funct. Anal. **261**, 2617–2640 (2011) http://dx.doi.org/10.1016/j.jfa.2011.07.006
6. Berezin, F.A.: General concept of quantization. Comm. Math. Phys. **40**, 153–174 (1975) http://dx.doi.org/10.1007/BF01609397
7. Berger, C.A., Coburn, L.A.: Toeplitz operators and quantum mechanics. J. Funct. Anal. **68**, 273–299 (1986) http://dx.doi.org/10.1016/0022-1236(86)90099-6
8. Berger, C.A., Coburn, L.A.: Toeplitz operators on the Segal-Bargmann space. Trans. Amer. Math. Soc. **2**, 813–829 (1987) http://dx.doi.org/10.2307/2000671
9. Bogachev, V. I. : Measure Theory, Springer-Verlag, Berlin (2007)
10. Coburn, L.A.: Deformation estimates for the Berezin-Toeplitz quantization. Commun. Math. Phys. **149**, 415–424 (1992) http://projecteuclid.org/euclid.cmp/1104251229
11. Engliš, M.: Toeplitz operators and group representations. J. Fourier Anal. Appl. **13**, 243–265 (2007) http://dx.doi.org/10.1007/s00041-006-6009-x
12. Esmeral, K., Maximenko, E.A.: C^*-algebra of angular Toeplitz operators on Bergman spaces over the upper half-plane. Commun. Math. Anal. **17**, no. 2, 151–162 (2014) http://projecteuclid.org/euclid.cma/1418919761
13. Esmeral, K., Maximenko, E.A.: Radial Toeplitz Operators on the Fock Space and Square-Root-Slowly Oscillating Sequences. Complex Anal. Oper. Theory **10**, 1655–1677 (2016) http://dx.doi.org/10.1007/s11785-016-0557-0
14. Esmeral, K., Maximenko, E. A., Vasilevski, N.: C^*-algebra generated by angular Toeplitz operators on the weighted Bergman spaces over the upper half-plane. Integr. Equ. Oper. Theory **83**, 413–428 (2015) https://doi.org/10.1007/s00020-015-2243-4
15. Esmeral, K., Rozenblum, G., Vasilevski, N.: $\mathcal{L}$-invariant Fock-Carleson type measures for derivatives of order k and the corresponding Toeplitz operators. To appear in Journal of Mathematics, 2019.
16. Esmeral, K., Vasilevski, N.: C*-algebra generated by horizontal Toeplitz operators on the Fock space. Bol. Soc. Mat. Mex. **22**, 567–582 (2016) http://dx.doi.org/10.1007/s40590-016-0110-1
17. Fock, V.A.: Konfigurationsraum und zweite Quantelung. Z. Phys. **75**, 622–647 (1932) http://dx.doi.org/10.1007/BF01344458
18. Folland, G.B.: A Course in Abstract Harmonic Analysis. Studies in Advanced Mathematics, Boca raton (1995)
19. Folland, G.B.: Harmonic Analysis in Phase Space. Annals of Mathematics Studies, Princeton University Press (1989)
20. Grudsky, S., Karapetyants, A., Vasilevski, N.: Dynamics of properties of Toeplitz operators with radial symbols. Integr. Equ. Oper. Theory **20**, 217–253 (2004) http://dx.doi.org/10.1007/s00020-003-1295-z
21. S. Grudsky, S., Karapetyants, A., Vasilevski, N.: Dynamics of properties of Toeplitz operators on the upper half-plane: Hyperbolic case. Bol. Soc. Mat. Mexicana **10**, no. 3, 119–138 (2004) https://doi.org/10.1007/978-3-7643-8726-6_14
22. Grudsky, S.M., Maximenko, E.A., Vasilevski, N.L.: Radial Toeplitz operators on the unit ball and slowly oscillating sequences. Commun. Math. Anal. **14**, no. 2, 77–94 (2013) http://projecteuclid.org/euclid.cma/1356039033
23. Grudsky, S., Vasilevski, N.: Toeplitz operators on the Fock space: Radial component effects. Integr. Equ. Oper. Theory **44**, 10–37 (2002) http://dx.doi.org/10.1007/BF01197858
24. Herrera Yañez, C., Hutník, O., Maximenko, E.A.: Vertical symbols, Toeplitz operators on weighted Bergman spaces over the upper half-plane and very slowly oscillating functions. C. R. Acad. Sci. Paris, Ser. I **352**, 129–132 (2014) http://dx.doi.org/10.1016/j.crma.2013.12.004
25. Herrera Yañez, C., Maximenko, E.A., Vasilevski, N.L.: Radial Toeplitz operators revisited: Discretization of the vertical case. Integr. Equ. Oper. Theory **81**, 49–60 (2015) http://dx.doi.org/10.1007/s00020-014-2213-2
26. Herrera Yañez, C., Maximenko, E.A., Vasilevski, N.L.: Vertical Toeplitz operators on the upper half-plane and very slowly oscillating functions. Integr. Equ. Oper. Theory **77**, 149–166 (2013) http://dx.doi.org/10.1007/s00020-013-2081-1

27. Isralowitz, J., Zhu, K.: Toeplitz operators on Fock spaces. Integr. Equ. Oper. Theory **66**, 593–611 (2010) http://dx.doi.org/10.1007/s00020-010-1768-9
28. Appuhamy, A., Le, T.: Commutants of Toeplitz Operators with Separately Radial Polynomial Symbols. Complex Anal. Oper. Theory **10**, 1–12 (2016) http://dx.doi.org/10.1007/s11785-014-0439-2
29. Loaiza, M., Lozano, C.: On C^*-Algebras of Toeplitz Operators on the Harmonic Bergman Space. Integr. Equ. and Oper. Theory **76**, 105–130 (2013) http://dx.doi.org/10.1007/s00020-013-2046-4
30. Rozenblum, G., Vasilevski, N.: Toeplitz operators defined by sesquilinear forms: Fock space case. J. Funct. Anal. **267**, 4399–4430 (2014) https://doi.org/10.1016/j.jfa.2014.10.001
31. Rozenblum, G., Vasilevski, N.: Toeplitz operators defined by sesquilinear forms: Bergman space case. J. Math. Sci. **213**, no. 2, 582–609 (2016) https://doi.org/10.1007/s10958-016-2726-0
32. Rudin, W.: Real and Complex Analysis. 2nd Edition, McGraw-Hill, New York (1974)
33. Segal, I.E.: Lectures at the summer seminar on Applied Math. Boulder, Colorado (1960)
34. Vasilevski, N.L.: Commutative Algebras of Toeplitz Operators on the Bergman Space. Operator Theory: Advances and Applications, vol. 185, Birkhäuser, Boston (2008)
35. Zorboska, N.: The Berezin transform and radial operators. Proc. Amer. Math. Soc. **131**, 793–800 (2003) http://www.jstor.org/stable/1194482
36. Zhu, K.: Analysis on Fock Spaces. In series: Graduate Texts in Mathematics, vol. 263, Springer, New York (2012)

Toeplitz Operators with $\mathbb{T}_q^m$-Invariant Symbols on Some Weakly Pseudoconvex Domains

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Dedicated to Nikolai Vasilevski on the occasion of his 70 *birthday.*

Abstract In this paper, we study the Banach algebra $\mathcal{T}(\mathbb{T}_m^q)$ which is generated by Toeplitz operators whose symbols are invariant under the action of the $\mathbb{T}_m^q$ subgroup of the maximal torus $\mathbb{T}^n$, which are acting on the Bergman space on weakly pseudo-convex domains Ω_p^n. Moreover, we proved that the commutator of the C^*-algebra $\mathcal{T}(\mathcal{R}_k(\Omega_p^n))$ is equal to the Toeplitz algebra $\mathcal{T}(\mathbb{T}_m^q)$, where $\mathcal{T}(\mathcal{R}_k(\Omega_p^n))$ is the C^*-algebra generated by Toeplitz operators where the symbols are k-quasi-radial. Finally, using this relationship we found some commutative Banach algebras generated by Toeplitz operators which generalize the Banach algebra generated by Toeplitz operators with quasi-homogeneous quasi-radial symbols.

1 Introduction

In recent years, a subject of study has been the connection between commutative C^*-algebras generated by Toeplitz operators and the action of subgroups of biholomorphisms on the underlying manifold. In particular, there is a classification of commutative C^*-algebras generated by Toeplitz operators acting on the weighted Bergman space which were described for the case of the unit ball in $\mathbb{C}^n$ and the complex projective space. The important point to note here is following statement: Given a maximal Abelian group G of biholomorphisms of the unit ball or the complex projective space, the C^*-algebra generated by Toeplitz operators whose symbols are invariant under the action of G is commutative on each weighted Bergman space. There are five different models of maximal Abelian subgroups of

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biholomorphisms of the unit ball: quasi-elliptic, quasi-parabolic, quasi-hyperbolic, nilpotent, and quasi-nilpotent, giving in total $n + 2$ subgroups, see [9, 12–14] for further results and details.

On the other hand, several authors showed that there exist many Banach (not C^*) algebras generated by Toeplitz operators which are commutative on each weighted Bergman space for all balls or Siegel Domains of dimension $n \geq 2$. The main idea is to provide a family of functions, which, in a sense, were subordinated to one of the previously mentioned models associated to maximal Abelian groups G of biholomorphisms and we obtain commutative Banach algebras generated by Toeplitz operators with symbols in this family of functions. In all of so far described commutative cases the symbols were invariant under a certain action of a subgroup of the corresponding group G, see [3–7, 15–18], for details.

Another step to finding Banach (not C^*) algebras generated by Toeplitz operators was given in [19], here N. Vasileski introduced $\mathbb{T}^m$-invariant symbols that are invariant under the action of the group $\mathbb{T}^m$, with $m \leq n$, being a subgroup of the quasi-elliptic group $\mathbb{T}^n$ of biholomorphisms of the unit ball $\mathbb{B}^n$ or complex space $\mathbb{C}^n$. Note that this set of all $\mathbb{T}^m$-invariant symbols produces a non-commutative C^*-algebra. Moreover, they characterized the action of the Toeplitz operators with $\mathbb{T}^m$-invariant symbols, which leave invariant the space of homogeneous polynomials. Using this result the author obtained a new class of commutative Banach algebras generated by Toeplitz operators with $\mathbb{T}^m$-invariant symbols.

In [11], they introduced quasi-homogeneous quasi-radial symbols on a family of weakly pseudoconvex domains Ω^n_p. Such family of domains contains the unit ball $\mathbb{B}^n$ as a particular case. This was a generalization of the symbols considered in [16] for the unit ball $\mathbb{B}^n$ as well as those considered in [10] for the complex projective space.

In [1, 8], T. Le considered φ a bounded separately radial polynomial on the unit ball or the complex space and he characterized bounded functions ϕ such that the Toeplitz operator T_ϕ commutes with T_φ on the Bergman or Fock space respectively. In particular, the functions ϕ are invariant under the action of a subgroup of the maximal torus. However, this result is valid only for polynomials, in this work we extend the latter results to some quasi radial functions.

In this paper, we study the Banach algebras $\mathcal{T}(\mathbb{T}^q_m)$ which are generated by Toeplitz operators whose symbols are invariant under the action of the $\mathbb{T}^q_m$ subgroup of the maximal torus $\mathbb{T}^n$, which are acting on the Bergman space over weakly pseudoconvex domains Ω^n_p. We consider the C^*-algebra $\mathcal{T}(\mathcal{R}_k(\Omega^n_p))$, which is generated by Toeplitz operators with quasi-radial symbols and we prove that the commutator restricted to Toeplitz operators of this C^*-algebra is equal to the Banach algebra $\mathcal{T}(\mathbb{T}^q_m)$. Finally, using this relationship we found some commutative Banach algebras generated by Toeplitz operators which generalize the Banach algebra generated by Toeplitz operators with quasi-homogeneous quasi-radial symbols.

The paper is organized as follows. Section 2 contains preliminary material. In Sect. 3 we studied Toeplitz operators with $\mathbb{T}^q_m$-invariant symbols and showed that this kind of operators leave invariant each subspace of the weighted homogeneous polynomials. Moreover, we studied the C^*-algebra $\mathcal{T}_k$, which is generated by

Toeplitz operators with quasi-radial symbols and showed that every of these operators are multiple of identity on each subspace of weighted homogeneous polynomials. Finally, in Sect. 4 we found some new Banach algebras generated by Toeplitz operators.

2 Preliminaries

Given a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ we will use the standard notation

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$
$$\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!,$$
$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}.$$

For $p \in \mathbb{Z}_+^n$ a fixed multi-index, we define the following sets

$$\Omega_p^n(r) = \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{j=1}^n |z_j|^{2p_j} < r^2\right\}, \tag{2.1}$$

$$\mathbb{S}_p^n(r) = \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{j=1}^n |z_j|^{2p_j} = r^2\right\}. \tag{2.2}$$

For the case $r = 1$, we simply write Ω_p^n and $\mathbb{S}_p^n$, respectively. Note that for $p = (1, \ldots, 1)$, we have that $\Omega_p^n = \mathbb{B}^n$ is the unit ball and $\mathbb{S}_p^n = \mathbb{S}^n$ is the unit sphere in $\mathbb{C}^n$, both centered at the origin.

For every $z \in \mathbb{C}^n$, we also denote

$$r = \|z\|_p = \sqrt{|z_1|^{2p_1} + \cdots + |z_n|^{2p_n}},$$
$$\xi_j = \frac{z_j}{\|z\|_p^{\frac{1}{p_j}}} = \frac{z_j}{r^{\frac{1}{p_j}}},$$

for all $j = 1, \ldots, n$. In particular we have

$$\sum_{j=1}^n |\xi_j|^{2p_j} = 1,$$

which implies that $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{S}_p^n$. Note that these expressions define a set of coordinates (r, ξ) for every $z \in \mathbb{C}^n$, this coordinates are called p-polar coordinates.

We let dv denote the Lebesgue measure on Ω_p^n normalized so that $v(\Omega_p^n) = 1$. Also we let σ denote the hypersurface measure on $\mathbb{S}_p^n$, also normalized so that $\sigma(\mathbb{S}_p^n) = 1$.

The next lemma relates the measures v and σ, for the proof we refer to [19].

Lemma 2.1 *The measures on v and σ satisfy*

$$\int_{\Omega_p^n} f(z)dv(z) = \left(2\sum_{j=1}^{n}\frac{1}{p_j}\right)\int_0^1 r^{2(\sum_{j=1}^{n}\frac{1}{p_j})-1}\int_{\mathbb{S}_p^n} f(r,\xi)d\sigma(\xi)dr,$$

for every non-negative measurable function f on Ω_p^n.

Remark 2.2 For the case $p = (1, \dots, 1)$ we have the well known result

$$\int_{\mathbb{B}^n} f(z)dv(z) = 2n\int_0^1 r^{2n-1}\int_{\mathbb{S}^n} f(r,\xi)d\sigma(\xi)dr,$$

for every non-negative measurable function f on $\mathbb{B}^n$.

The Hilbert spaces $L^2(\Omega_p^n)$ and $L^2(S_p^n)$ are those associated to the usual Lebesgue measure dV on Ω_p^n and the hypersurface measure dS on S_p^n. We denote by $\mathcal{A}^2(\Omega_p^n)$ the closed subspace of $L^2(\Omega_p^n)$ consisting of those functions which are holomorphic in Ω_p^n, and we let $P_p : L^2(\Omega_p^n) \to \mathcal{A}^2(\Omega_p^n)$ be the orthogonal projection. If $a \in L_\infty(\Omega_p^n)$ then the Toeplitz operator T_a with symbol a is the bounded operator on $\mathcal{A}^2(\Omega_p^n)$ defined by $T_a(f) = P_p(af)$.

The following identity is proved in [2],

$$\langle z^\alpha, z^\beta\rangle = \delta_{\alpha,\beta}\frac{\pi^n\prod_{j=1}^{n}\Gamma\left(\frac{\alpha_j+1}{p_j}\right)}{\left[\prod_{j=1}^{n}p_j\right]\Gamma\left(\sum_{j=1}^{n}\frac{\alpha_j+1}{p_j}+1\right)}. \tag{2.3}$$

As a consequence of 2.3 a monomial orthonormal basis in $\mathcal{A}^2(\Omega_p^n)$ is given by

$$e_\alpha = \left(\frac{\left[\prod_{j=1}^{n}p_j\right]\Gamma\left(\sum_{j=1}^{n}\frac{\alpha_j+1}{p_j}+1\right)}{\pi^n\prod_{j=1}^{n}\Gamma\left(\frac{\alpha_j+1}{p_j}\right)}\right)^{\frac{1}{2}} z^\alpha, \quad \alpha\in\mathbb{Z}_+^n. \tag{2.4}$$

If $\alpha, \beta \in \mathbb{N}^n$ for $\mathbb{S}^n_p$ we have the formula

$$\int_{\mathbb{S}^n_p} \xi^\alpha \overline{\xi^\beta} d\sigma(\xi) = \delta_{\alpha,\beta} \frac{\Gamma\left(\sum_{j=1}^n \frac{1}{p_j}\right) \prod_{j=1}^n \Gamma\left(\frac{\alpha_j+1}{p_j}\right)}{\left[\prod_{j=1}^n \Gamma\left(\frac{1}{p_j}\right)\right] \Gamma\left(\sum_{j=1}^n \frac{\alpha_j+1}{p_j}\right)}, \tag{2.5}$$

or equivalently

$$\int_{\mathbb{S}^n_p} \xi^\alpha \overline{\xi^\beta} dS(\xi) = \delta_{\alpha,\beta} \frac{2\pi^n \prod_{j=1}^n \Gamma\left(\frac{\alpha_j+1}{p_j}\right)}{\left[\prod_{j=1}^n p_j\right] \Gamma\left(\sum_{j=1}^n \frac{\alpha_j+1}{p_j}\right)}. \tag{2.6}$$

For a proof of formulas 2.5 and 2.6 we refer the reader to [19]. Note that when $p = (1, \ldots, 1)$ we have

$$\int_{\mathbb{S}^n} \xi^\alpha \overline{\xi^\beta} dS(\xi) = \delta_{\alpha,\beta} \frac{2\pi^n \alpha!}{(n-1+|\alpha|)!}.$$

3 $\mathbb{T}^m_q$−Invariant Symbols

Let $k = (k_1, \ldots, k_m) \in \mathbb{Z}^m_+$ such that $|k| = n$. Rearrange the n coordinates of $z \in \mathbb{B}^n$ in m groups $z_{(j)}$ of lenght k_j, $j = 1, \ldots, m$. We use the notation $z_{(j)} = (z_{j,1}, \ldots, z_{j,k_j})$, $j = 1, \ldots, m$, with

$$z_{1,1} = z_1, \quad z_{1,2} = z_2, \quad \ldots, z_{1,k_1} = z_{k_1};$$
$$z_{2,1} = z_{k_1+1}, \quad \ldots, z_{2,k_2} = z_{k_1+k_2}; \quad \ldots;$$
$$z_{m,1} = z_{n-k_m+1}, \quad \ldots, z_{m,k_m} = z_n.$$

In general for any n−tuple u we will use alternative representations

$$u = (u_1, \ldots, u_n) = (u_{(1)}, \ldots, u_{(m)}).$$

We denote

$$r_j = \|z_{(j)}\|_p = \sqrt{\sum_{s=1}^{k_j} |z_{j,s}|^{2p_{j,s}}}, \tag{3.1}$$

for every $j = 1, \dots, m$. Further more for every $z \in \mathbb{C}^n$ we denote

$$\xi_{(j)} = \left(\frac{z_{j,1}}{r_j^{\frac{1}{p_{j,1}}}}, \dots, \frac{z_{j,k_j}}{r_j^{\frac{1}{p_{j,k_j}}}} \right), \tag{3.2}$$

for every $j = 1, \dots, m$. Note that $\xi_{(j)} \in \mathbb{S}_{p_{(j)}}^{k_j}$ for every $j = 1, \dots, m$.

For a fixed $q = (q_1, \dots, q_n) \in \mathbb{Z}_+^n$ and each $\kappa = (\kappa_1, \dots, \kappa_m) \in \mathbb{Z}_+^m$ let H_κ^q be the finite dimensional subspace of the Bergman space $\mathcal{A}^2(\Omega_p^n)$ defined by

$$H_\kappa^q := \operatorname{span}\{e_\alpha : \langle \alpha_{(j)}, q_{(j)} \rangle = \kappa_j, \quad j = 1, \cdots, m\};$$

where $\langle \alpha_{(j)}, q_{(j)} \rangle = \sum_{s=1}^{k_j} q_{j,s}\alpha_{j,s}$, $j = 1, \dots, m$. Then we have

$$\mathcal{A}^2(\Omega_p^n) = \bigoplus_{|\kappa|=0}^{\infty} H_\kappa^q.$$

We consider the subgroup $\mathbb{T}_q^m$ of $\mathbb{T}^n$ which consist of elements of the form

$$\eta = (\eta_1^{q_{1,1}}, \dots, \eta_1^{q_{1,k_1}}, \dots, \eta_m^{q_{m,1}}, \dots, \eta_m^{q_{m,k_m}}), \quad \eta_1, \dots \eta_m \in \mathbb{T}.$$

The action of $\mathbb{T}_q^m$ on $\mathcal{A}^2(\Omega_p^n)$ is given by

$$\begin{aligned} z = (z_1, \dots, z_n) \longmapsto \\ \eta z = (\eta_1^{q_{1,1}} z_{1,1}, \dots, \eta_1^{q_{1,k_1}} z_{1,k_1}, \dots, \eta_m^{q_{m,1}} z_{m,1}, \dots, \eta_m^{q_{m,k_m}} z_{m,k_m}) \end{aligned} \tag{3.3}$$

where $\eta \in \mathbb{T}_q^m$.

Definition 3.1 Let $k = (k_1, \dots, k_m) \in \mathbb{Z}_+^m$ be a partition of n. A $\mathbb{T}_q^m$-invariant symbol is a function $b : \Omega_p^n \mapsto \mathbb{C}$ such that $b(\eta z) = b(z)$ for every $\eta \in \mathbb{T}_q^m$ and every $z \in \Omega_p^n$, where the action ηz is given by 3.3.

Lemma 3.2 *Let $b \in L_\infty(\Omega_p^n)$ be invariant under the action of the group $\mathbb{T}_q^m$ on $\mathcal{A}^2(\Omega_p^n)$. Then the Toeplitz operator T_b leaves invariant all spaces H_κ^q.*

Proof Making the change of variable $w = \eta z$, we calculate

$$\begin{aligned}
\langle T_b e_\alpha, e_\beta\rangle = \langle b e_\alpha, e_\beta\rangle &= \int_{\Omega_p^n} b(z) e_\alpha(z)\overline{e_\beta(z)} dv(z) \\
&= \int_{\Omega_p^n} b(\eta z) e_\alpha(z)\overline{e_\beta(z)} dv(z) \\
&= \prod_{j=1}^m \eta_j^{\langle p_{(j)}, \beta_{(j)} - \alpha_{(j)}\rangle} \int_{\Omega_p^n} b(w) e_\alpha(w)\overline{e_\beta(w)} dv(w) \\
&= \prod_{j=1}^m \eta_j^{\langle p_{(j)}, \beta_{(j)} - \alpha_{(j)}\rangle} \langle T_b e_\alpha, e_\beta\rangle
\end{aligned}$$

It is clear that $\langle T_b e_\alpha, e_\beta\rangle = 0$ if $\langle p_{(j)}, \alpha_{(j)}\rangle \neq \langle p_{(j)}, \beta_{(j)}\rangle$ for some $j = 1, \ldots, m$, which proves the lemma. □

When $p = q = (1, \ldots, 1)$, this lemma is the same result for the unit ball $\mathbb{B}^n$ that we can find in [11]. A direct implication of Lemma 3.2 is the following corollary which describes the action of the Toeplitz operator as a direct sum of Toeplitz operator restricted to each finite dimensional subspace H_κ^q.

Corollary 3.3 *Let b a $\mathbb{T}_q^m$-invariant function, then the Toeplitz operator T_b acts in the Bergman space $\mathcal{A}^2(\Omega_p^n)$ as follows*

$$T_b = \bigoplus_{|\kappa|=0}^{\infty} T_b|_{H_\kappa^q}.$$

Definition 3.4 Let $k = (k_1, \ldots, k_m) \in \mathbb{Z}_+^m$ be a partition of n. A k-quasi-radial symbol is a function $a : \Omega_p^n \to \mathbb{C}$ that can be written as $a(z) = \widetilde{a}(r_1, \ldots, r_m)$ where r_j, $j = 1, \ldots, m$, is given by 3.1. We will denote by $\mathcal{R}_k(\Omega_p^n)$ (or simply $\mathcal{R}_k$) the set of k-quasi-radial symbols on Ω_p^n.

The C^*-algebra generated by Toeplitz operators with symbols in $\mathcal{R}_k$ will be denoted by $\tau(\mathcal{R}_k)$.

From now on, we consider $q = (q_1, \ldots, q_n) = (q_{(1)}, \ldots, q_{(m)}) \in \mathbb{Z}_+^n$ as follows

$$q_{j,s} = \prod_{\substack{t=1 \\ t\neq s}}^{k_j} p_{j,t}, \tag{3.4}$$

for every $s = 1, \ldots, k_j$ and $j = 1, \ldots, m$, where $p \in \mathbb{Z}_+^n$ is the multi-index which defines the pseudo convex domain Ω_p^n.

The following lemma is a result from [19], it shows that the Toeplitz operators with k-quasi-radial symbols are diagonal operators.

Lemma 3.5 *Let $k \in \mathbb{Z}_+^m$ be a partition of n. Then, for any k-quasi-radial bounded measurable symbol $a \in \mathcal{R}_k(\Omega_p^n)$, we have*

$$T_a z^\alpha = \gamma_{a,k}(\alpha) z^\alpha,$$

for every $\alpha \in \mathbb{N}^n$, where

$$\gamma_{a,k}(\alpha) = \frac{4^m \Gamma\left(\sum_{j=1}^{n} \frac{\alpha_j + 1}{p_j} + 1\right) \prod_{j=1}^{m} \left(\sum_{s=1}^{k_j} \frac{1}{p_{j,s}}\right)}{\prod_{j=1}^{m} \Gamma\left(\sum_{s=1}^{k_j} \frac{\alpha_{j,s} + 1}{p_{j,s}}\right)}$$

$$\times \int_{\Delta_m^n} a(r_1, \ldots, r_m) \prod_{j=1}^{m} r_j^{\left(2 \sum_{s=1}^{k_j} \left(\frac{\alpha_{j,s}+1}{p_{j,s}}\right) - 1\right)} dr_j, \tag{3.5}$$

and $\Delta_m^n = \{(r_1, \ldots, r_m) \in \mathbb{R}_+ : r_1^2 + \cdots + r_m^2 < 1\}$.

Example 3.6 Consider the symbol of the form $a_j(r_1, \ldots, r_m) = r_j^2$ for $j = 1, \ldots, m$, then we have that the spectral function of the Toeplitz operator T_{a_j} has the following form

$$\gamma_{a_j,k}(\alpha) = \frac{2^m \left(\sum_{s=1}^{k_j} \frac{\alpha_{j,s} + 1}{p_{j,s}}\right) \prod_{l=1}^{m} \left(\sum_{s=1}^{k_l} \frac{1}{p_{j,s}}\right)}{\sum_{l=1}^{n} \frac{\alpha_l + 1}{p_l} + 1}. \tag{3.6}$$

Remark 3.7 We observe that the function $\gamma_{a,k}(\alpha)$ only depends in the values

$$\sum_{s=1}^{k_j} \frac{\alpha_{j,s} + 1}{p_{j,s}}$$

for every $j = 1, \ldots, m$. Then

$$\begin{aligned}\gamma_{a,k}(\alpha) &= \gamma_{a,k}(\sum_{s=1}^{k_1} \frac{\alpha_{1,s}+1}{p_{1,s}}, \ldots, \sum_{s=1}^{k_m} \frac{\alpha_{m,s}+1}{p_{m,s}}) \\ &= \gamma_{a,k}(\sum_{s=1}^{k_1} \frac{\beta_{1,s}+1}{p_{1,s}}, \ldots, \sum_{s=1}^{k_m} \frac{\beta_{m,s}+1}{p_{m,s}}) \\ &= \gamma_{a,k}(\beta)\end{aligned}$$

if

$$\sum_{s=1}^{k_j} \frac{\alpha_{j,s}+1}{p_{j,s}} = \sum_{s=1}^{k_j} \frac{\beta_{j,s}+1}{p_{j,s}}$$

for every $j = 1, \ldots, m$.

Remark 3.8 Let $q \in \mathbb{Z}_+^n$ defined by 3.7, if e_α and e_β belongs to the subspace H_κ^q then

$$\langle \alpha_{(j)}, q_{(j)} \rangle = \langle \beta_{(j)}, q_{(j)} \rangle, \quad j = 1, \ldots, m,$$

it follows that

$$\sum_{s=1}^{k_j} \frac{\alpha_{j,s}+1}{p_{j,s}} = \sum_{s=1}^{k_j} \frac{\beta_{j,s}+1}{p_{j,s}}, \quad j = 1, \ldots, m,$$

finally, by the preceding remark $\gamma_{a,k}(\alpha) = \gamma_{a,k}(\beta)$.

In view of the previous remarks, we have that the Toeplitz operator with k-quasi-radial symbol acts as a constant operator in each subespace H_κ^q where q is given by 3.7, thus we have the following corollary.

Corollary 3.9 *Let $a \in \mathcal{R}_k(\Omega_p^n)$, then the Toeplitz operator acts in the Bergman space $\mathcal{A}^2(\Omega_p^n)$ as follows*

$$T_a = \bigoplus_{|\kappa|=0}^{\infty} T_a|_{H_\kappa^q} = \bigoplus_{|\kappa|=0}^{\infty} \gamma_{a,k}(\kappa) I_\kappa$$

where $I_\kappa = I|_{H^q_\kappa}$ *is the identity operator restricted to the subspace* H^q_κ *and* $\gamma_{a,k}(\kappa) = \gamma_{a,k}(\alpha)$ *for some* α *which satisfies* $\langle \alpha_{(j)}, q_{(j)} \rangle = \kappa_j$ *for every* $j = 1, \dots, m$. *Here* q *is defined by*

$$q_{j,s} = \prod_{\substack{t=1 \\ t \neq s}}^{k_j} p_{j,t}, \tag{3.7}$$

for every $s = 1, \dots, k_j$ *and* $j = 1, \dots, m$. *The function* $\gamma_{a,k}(\alpha)$ *is given by*

$$\gamma_{a,k}(\alpha) = \frac{4^m \Gamma\left(\sum_{j=1}^{n} \frac{\alpha_j + 1}{p_j} + 1\right) \prod_{j=1}^{m} \left(\sum_{s=1}^{k_j} \frac{1}{p_{j,s}}\right)}{\prod_{j=1}^{m} \Gamma\left(\sum_{s=1}^{k_j} \frac{\alpha_{j,s} + 1}{p_{j,s}}\right)}$$
$$\times \int_{\Delta^n_m} a(r_1, \dots, r_m) \prod_{j=1}^{m} r_j^{\left(2 \sum_{s=1}^{k_j} \left(\frac{\alpha_{j,s}+1}{p_{j,s}}\right) - 1\right)} dr_j,$$

with $\Delta^n_m = \{(r_1, \dots, r_m) \in \mathbb{R}_+ : r_1^2 + \cdots + r_m^2 < 1\}$.

Now we present the following theorem which is a direct consequence of Corollaries 3.3 and 3.9.

Theorem 3.10 *Let* $q \in \mathbb{Z}^n$ *defined by 3.7. If* $a \in \mathcal{R}_k(\Omega^n_p)$ *is a* k*-quasi-radial symbol and* b *is a symbol invariant under the action of* $\mathbb{T}^m_q$. *Then the Toeplitz operators* T_a *and* T_b *commute in the Bergman space* $\mathcal{A}^2(\Omega^n_p)$.

Proof Let $a \in \mathcal{R}_k(\Omega^n_p)$ and b a k-quasi-homogeneous symbol. Then we have

$$\begin{aligned} T_a T_b &= \left(\bigoplus_{|\kappa|=0}^{\infty} \gamma(a,k)(\kappa) I_\kappa\right) \left(\bigoplus_{|\kappa|=0}^{\infty} T_b|_{H^q_\kappa}\right) \\ &= \bigoplus_{|\kappa|=0}^{\infty} \gamma(a,k)(\kappa) T_b|_{H^q_\kappa} \\ &= \left(\bigoplus_{|\kappa|=0}^{\infty} T_b|_{H^q_\kappa}\right) \left(\bigoplus_{|\kappa|=0}^{\infty} \gamma(a,k)(\kappa) I_\kappa\right) \\ &= T_b T_a. \end{aligned}$$

□

We denote by $\mathcal{T}(\mathcal{R}_k(\Omega_p^n))$ the commutator of the set of k-quasi-radial symbols, that is

$$\mathcal{T}(\mathcal{R}_k(\Omega_p^n)) = \{T_b : [T_a, T_b] = T_a T_b - T_b T_a = 0, \quad \forall a \in \mathcal{R}_k(\Omega_p^n)\}.$$

It follows by the previous theorem that $T_b \in \mathcal{T}(\mathcal{R}_k(\Omega_p^n))$ if b is a $\mathbb{T}_q^m$-invariant symbol.

Lemma 3.11 *Let $b : \Omega_p^n \to \mathbb{C}$ be a symbol. Then the Topelitz operator T_b belongs to the commutator $\mathcal{T}(\mathcal{R}_k(\Omega_p^n))$ if and only if T_b leaves invariant each subspace H_κ.*

Proof Let $a : \Omega_p^n \to \mathbb{C}$ be a symbol such that $\overline{a} = a$. Then we have

$$\begin{aligned}\langle [T_b, T_a] e_\alpha, e_\beta \rangle &= \langle (T_b T_a - T_a T_b) e_\alpha, e_\beta \rangle \\ &= \langle T_b T_a e_\alpha, e_\beta \rangle - \langle T_a T_b z^\alpha, e_\beta \rangle \\ &= \langle T_b \gamma_{a,k}(\alpha) e_\alpha, e_\beta \rangle - \langle T_b e_\alpha, \gamma_{a,k}(\beta) e_\beta \rangle \\ &= (\gamma_{a,k}(\alpha) - \gamma_{a,k}(\beta)) \langle T_b e_\alpha, e_\beta \rangle. \end{aligned} \tag{3.8}$$

Note that, from the above equation we have that $[T_b, T_a] = 0$ is equivalent to the following statement

$$(\gamma_{a,k}(\alpha) - \gamma_{a,k}(\beta)) \langle T_b z^\alpha, z^\beta \rangle = 0, \tag{3.9}$$

In particular, if we considered the symbols of the Example 3.6 and e_α and e_β do not belong to the same subspace H_κ^q, then there exists a symbol a_{j_0} such that

$$(\gamma_{a_{j_0},k}(\alpha) - \gamma_{a_{j_0},k}(\beta)) \neq 0.$$

Therefore, from the above equation and Remark (3.12) we can conclude that $\langle T_b e_\alpha, e_\beta \rangle = 0$ if e_α and e_β do not belong to the same subspace H_κ^q, which is equivalent to T_b leaves invariant each subspace H_κ.

Conversely, suppose that T_b leaves invariant each subspace H_κ^q. Consider e_α, e_β in Bergman space thus we have two cases: First case, we have that $e_\alpha, e_\beta \in H_\kappa^q$ for some κ, which implies that $\gamma_{a,k}(\alpha) - \gamma_{a,k}(\beta) = 0$. Second case, we have that e_α and e_β do not belong to the same subspace H_κ^q, which implies that $\langle T_b e_\alpha, e_\beta \rangle = 0$. Therefore, from Remark (3.12) and above mentioned we obtain that $[T_b, T_a] = 0$.

□

Remark 3.12 For $\eta \in \mathbb{T}_q^n$ we define $b^\eta(z) = b(\eta z)$. Making the change of variable $w = \eta z$ we have

$$\begin{aligned}\langle T_{b^\eta} e_\alpha, e_\beta\rangle = \langle b^\eta e_\alpha, e_\beta\rangle &= \int_{\Omega_p^n} b(\eta z) e_\alpha(z)\overline{e_\beta(z)} dv(z) \\ &= \prod_{j=1}^{m} \eta_j^{\langle p_{(j)}, \beta_{(j)} - \alpha_{(j)}\rangle} \int_{\Omega_p^n} b(w) e_\alpha(w)\overline{e_\beta(w)} dv(w) \\ &= \prod_{j=1}^{m} \eta_j^{\langle p_{(j)}, \beta_{(j)} - \alpha_{(j)}\rangle} \langle T_b e_\alpha, e_\beta\rangle,\end{aligned}$$

Lemma 3.13 *Let $b : \Omega_p^n \to \mathbb{C}$ a symbol. Then the Toplitz operator T_b leaves invariant each subspace H_κ^q if and only if b is a $\mathbb{T}_q^m$-invariant symbol.*

Proof Suppose that T_b leaves invariant each subspace H_κ^q. Consider multi-indices α and β such that e_α and e_β do not belong to the same subspace H_κ^q, by Remark 3.12 we have

$$\langle T_{b^\eta} e_\alpha, e_\beta\rangle = 0,$$

then T_{b^η} leaves invariant each subspace H_κ^q, and we can write

$$T_{b^\eta} = \bigoplus_{|\kappa|=0}^{\infty} T_{b^\eta}|_{H_\kappa^q}.$$

Let e_α and e_β be elements of H_κ^q, by Remark 3.12 we have $\langle T_{b^\eta} e_\alpha, e_\beta\rangle = \langle T_b e_\alpha, e_\beta\rangle$. Then

$$T_{b^\eta}|_{H_\kappa^q} = T_b|_{H_\kappa^q}$$

for all $\kappa \in \mathbb{Z}_+^m$. It follows that $T_{b^\eta} = T_b$ which implies $b^\eta = b$ i.e. b is a $\mathbb{T}_q^m$-invariant symbol.

Conversely, suppose that b is a $\mathbb{T}_q^m$-invariant symbol, by Theorem 3.10 the Toeplitz operator T_b is in the commutator $C(\mathcal{R}_k(\Omega_p^n))$ and by Lemma 3.11 the Toeplitz operator T_b leaves invariant each subspace H_κ^q. □

As a direct consequence of Lemmas 3.11 and 3.13, we have the following theorem which is the fundamental result of this text.

Theorem 3.14 *Let $b : \Omega_p^n \to \mathbb{C}$ a symbol. Then the Toplitz operator T_b belongs to the commutator $\mathcal{T}(\mathcal{R}_k(\Omega_p^n))$ if and only if b is a $\mathbb{T}_q^m$-invarian symbol.*

4 Commutative Results for Quasi-Homogeneous Symbols

We fix a multi-index $\kappa \in \mathbb{Z}_+^m$. Since a Toeplitz operator with a $\mathbb{T}_q^m$-invariant symbol g leaves invariant each subspace H_κ^q as Lemma 3.2 asserts, the action of T_g on basis elements e_α in H_κ^q is as follows

$$T_g e_\alpha = \sum_{e_\beta \in H_\kappa^q} \langle T_g e_\alpha, e_\beta \rangle e_\beta$$

We give an alternative representation for points z in the pseudoconvex domain Ω_p^n. First, for each coordinate of z, we denote

$$z_j = |z_j| t_j \quad \text{or} \quad z_{j,l} = |z_{j,l}| t_{j,l},$$

where t_i and $t_{j,l}$ belong to $\mathbb{T}$. As before we introduce the radius

$$r_j = \|z_{(j)}\|_p = \sqrt{\sum_{s=1}^{k_j} |z_{j,s}|^{2p_{j,s}}} \tag{4.1}$$

for every $j = 1, \dots, m$. Now we represent the coordinates of $z_{(j)}$ in the form

$$z_{j,l} = r_j^{1/p_{j,l}} \xi_{j,l} = r_j^{1/p_{j,l}} s_{j,l} t_{j,l},$$

where

$$s_{j,l} = \frac{|z_{j,l}|}{r_j^{1/p_{j,l}}},$$

for $l = 1, \dots, k_j$, so that $s_{(j)} \in \mathbb{S}_{p_{(j)},+}^{k_j} := \mathbb{S}_{p_{(j)}}^{k_j} \cap \mathbb{R}_+^{k_j}$. Note that $\xi_{(j)} \in \mathbb{S}_p^{k_j}$, $j = 1, \dots, m$.

For a fixed j, we consider symbols of the form $g(z) = d_j(\xi_{(j)})$, where $d_j = d_j(\xi_{(j)}) \in L_\infty(\mathbb{S}_p^{k_j})$ and $d_j(\eta\xi_{(j)}) = d_j(\xi_{(j)})$ for all $\eta \in \mathbb{T}_q^m$. In this case $\langle T_g e_\alpha, e_\beta \rangle$ does not equal to 0 if and only if e_α and e_β are from the same H_κ and have the form

$$\alpha = (\alpha_{(1)}, \dots, \alpha_{(j-1)}, \alpha_{(j)}, \alpha_{(j+1)}, \dots, \alpha_{(m)}),$$
$$\beta = (\alpha(1), \dots, \alpha_{(j-1)}, \beta_{(j)}, \alpha_{(j+1)}, \dots, \alpha_{(m)}).$$

We describe the action of the Toeplitz operator T_g in H_κ^q. First we compute

$$\begin{aligned}\langle T_{d_j} z^\alpha, z^\beta\rangle &= \langle d_j z^\alpha, z^\beta\rangle \\ &= \int_{\Omega_p^n} d_j(\xi_{(j)}) z^\alpha \overline{z}^\beta dv(z).\end{aligned}$$

Consider the change of variables $z_{t,l} = r_t^{\frac{1}{p_{t,l}}} \xi_{t,l}$ for $l = 1, \dots, k_t$ and $t = 1, \dots, m$. Then we obtain

$$\begin{aligned}\langle T_{d_j} z^\alpha, z^\beta\rangle &= \int_{\Delta_m^n} \prod_{t=1}^m 2\left(\sum_{l=1}^{k_t} \frac{1}{p_{t,l}}\right) r_t^{\left(\sum_{l=1}^{k_t} \frac{\alpha_{t,l}+\beta_{t,l}+2}{p_{t,l}}-1\right)} dr_t \\ &\times \left(\prod_{\substack{t=1\\t\neq j}}^m \int_{S_{p_{(t)}}^{k_t}} \xi^{\alpha_{(t)}} \overline{\xi}^{\beta_{(t)}} dS_t\right) \int_{S_{p_{(j)}}^{k_j}} d_j(\xi_{(j)}) \xi^{\alpha_{(j)}} \overline{\xi}^{\beta_{(j)}} dS_j \\ &= \int_{\Delta_m^n} \prod_{t=1}^m 2\left(\sum_{l=1}^{k_t} \frac{1}{p_{t,l}}\right) r_t^{\left(\sum_{l=1}^{k_t} \frac{\alpha_{t,l}+\beta_{t,l}+2}{p_{t,l}}-1\right)} dr_t \\ &\times \left(\prod_{\substack{t=1\\t\neq j}}^m \frac{2\pi^{k_t} \prod_{l=1}^{k_t} \Gamma\left(\frac{\alpha_{t,l}+1}{p_{t,l}}\right)}{\left[\prod_{l=1}^{k_t} p_{t,l}\right] \Gamma\left(\sum_{l=1}^{k_t} \frac{\alpha_{t,l}+1}{p_{t,l}}\right)}\right) \int_{S_{p_{(j)}}^{k_j}} d_j(\xi_{(j)}) \xi^{\alpha_{(j)}} \overline{\xi}^{\beta_{(j)}} dS_j \\ &= \int_{\Delta_m^n} \prod_{t=1}^m r_t^{\left(\sum_{l=1}^{k_t} \frac{\alpha_{t,l}+\beta_{t,l}+2}{p_{t,l}}-1\right)} dr_t \\ &\times \left(\frac{2^{2m-k_j} \pi^{m-k_j} \prod_{l=1}^n \Gamma\left(\frac{\alpha_l+1}{p_l}\right) \left[\prod_{l=1}^{k_j} p_{j,l}\right] \prod_{t=1}^m \left(\sum_{l=1}^{k_t} \frac{1}{p_{t,l}}\right)}{\left[\prod_{l=1}^n p_l\right] \prod_{\substack{t=1\\t\neq j}}^m \Gamma\left(\sum_{l=1}^{k_t} \frac{\alpha_{t,l}+1}{p_{t,l}}\right) \prod_{l=1}^{k_j} \Gamma\left(\frac{\alpha_{j,l}+1}{p_{j,l}}\right)}\right) \\ &\times \int_{S_{p_{(j)}}^{k_j}} d_j(\xi_{(j)}) \xi^{\alpha_{(j)}} \overline{\xi}^{\beta_{(j)}} dS_j,\end{aligned}$$

where $\Delta_m^n = \{(r_1, \ldots, r_m) \in \mathbb{R}_+^m : r_1^2 + \cdots + r_m^2 < 1\}$. Recall that the Beta function of $m+1$ variables is defined by

$$B(x_1, \ldots, x_{m+1}) = \int_{\Delta_m} \left(\prod_{j=1}^{m} y_j^{x_j - 1} \right) \left(1 - \sum_{j=1}^{m} y_j \right)^{x_{m+1}-1} dy_1 \cdots dy_m,$$

where $\Delta_m = \{(y_1, \ldots, y_m) \in \mathbb{R}_+^m : y_1 + \cdots + y_m < 1\}$ is the standard m-dimensional simplex; recall as well that

$$B(x_1, \ldots, x_{m+1}) = \frac{\Gamma(x_1) \cdots \Gamma(x_{m+1})}{\Gamma(x_1 + \cdots + x_{m+1})}.$$

Since z^α and z^β are in H_κ^q by Remark 3.8 and changing the variables r_t^2 by r_t we have

$$\begin{aligned}
\int_{\Delta_m^n} \prod_{t=1}^{m} r_t^{\left(\sum_{l=1}^{k_t} \frac{\alpha_{t,l} + \beta_{t,l} + 2}{p_{t,l}} - 1 \right)} dr_t &= 2^{-m} \int_{\Delta_m} \prod_{t=1}^{m} r_t^{\sum_{l=1}^{k_t} \frac{\alpha_{t,l}+1}{p_{t,l}} - 1} dr_t \\
&= 2^{-m} B\left(\sum_{l=1}^{t_1} \frac{\alpha_{1,l}+1}{p_{1,l}}, \ldots, \sum_{l=1}^{t_m} \frac{\alpha_{m,l}+1}{p_{m,l}}, 1 \right) \\
&= \frac{2^{-m} \prod_{t=1}^{m} \Gamma\left(\sum_{l=1}^{k_t} \frac{\alpha_{t,l}+1}{p_{t,l}} \right)}{\Gamma\left(\sum_{l=1}^{n} \frac{\alpha_l + 1}{p_l} + 1 \right)}.
\end{aligned}$$

Finally

$$\begin{aligned}
&\langle T_{d_j} z^\alpha, z^\beta \rangle \\
&= \frac{(2\pi)^{m-k_j} \Gamma\left(\sum_{l=1}^{k_j} \frac{\alpha_{j,l}+1}{p_{j,l}} \right) \prod_{l=1}^{n} \Gamma\left(\frac{\alpha_l + 1}{p_l} \right) \left[\prod_{l=1}^{k_j} p_{j,l} \right] \prod_{t=1}^{m} \left(\sum_{l=1}^{k_t} \frac{1}{p_{t,l}} \right)}{\Gamma\left(\sum_{l=1}^{n} \frac{\alpha_l+1}{p_l} + 1 \right) \left[\prod_{l=1}^{n} p_l \right] \prod_{l=1}^{k_j} \Gamma\left(\frac{\alpha_{j,l}+1}{p_{j,l}} \right)} \\
&\times \int_{S_{p_{(j)}}^{k_j}} d_j(\xi_{(j)}) \xi^{\alpha_{(j)}} \overline{\xi}^{\beta_{(j)}} dS_j. \qquad (4.2)
\end{aligned}$$

In order to describe the action of Toeplitz operator T_{d_j} on the space H^q_κ we use the same notation introduced in [11]:

$$\alpha_{\widehat{(j)}} := (\alpha_1, \ldots, \alpha_{(j-1)}, \alpha_{(j+1)}, \ldots, \alpha_{(m)}),$$

being the tuple α with the part $\alpha_{(j)}$ omitted, and $\alpha_{\widehat{(j)}} \bowtie \alpha_{(j)} := \alpha$, being the tuple α restored by it parts $\alpha_{\widehat{(j)}}$ and $\alpha_{(j)}$.

Given α such that $e_\alpha \in H^q_\kappa$, let

$$H^q_\kappa(\alpha_{\widehat{(j)}}) := \{e_\beta = \alpha_{\widehat{(j)}} \bowtie \beta_{(j)} : \langle \beta_{(j)}, q_{(j)} \rangle = \kappa_j\}$$

be the $\alpha_{\widehat{(j)}}$-level of H^q_κ. The equality 4.2 implies that

$$T_{d_j} e_\alpha = \sum_{e_\beta \in H^q_\kappa(\alpha_{\widehat{(j)}})} \langle T_{d_j} e_\alpha, e_\beta \rangle e_\beta,$$

i.e. the Toeplitz operator T_{d_j} leaves invariant each $\alpha_{\widehat{(j)}}$-level $H^q_\kappa(\alpha_{\widehat{(j)}}) \subset H^q_\kappa$.

Note that symbols of the form $b_j(s_{(j)})c_j(t_{(j)})$ with $s_{(j)} \in \mathbb{S}^{k_j}_{p_{(j)},+}$ and $t_{(j)} \in \mathbb{T}^{k_j}$ are a special case of symbols $d_j(\xi_{(j)})$, $\xi_{(j)} \in \mathbb{S}^{k_j}_{p_{(j)}}$ that we have consider in the latter calculations. Then the preceding discussion and Theorem 3.14 leads to the following result which coincides with Theorem 4.3 of [11] when $p = (1, \ldots, 1)$.

Theorem 4.1 *Let*

$$g(z) = a(r_1, \ldots, r_m) \prod_{j=1}^{m} b_j(s_{(j)})c_j(t_{(j)}),$$

where $a \in R_k(\Omega^n_p)$ is a k-quasi-radial symbol, $b_j = b_j(s_{(j)}) \in L_\infty(\mathbb{S}^{k_j}_{p_{(j)},+})$, $c_j = c_j(t_{(j)}) \in L_\infty(\mathbb{T}^{k_j})$, and $c_j(\eta t_{(j)}) = c_j(t_{(j)})$ for all $\eta \in \mathbb{T}^m_q$, $j = 1, \ldots, m$.

The operators T_a and $T_{b_j c_j}$, for $j = 1, \ldots, m$, mutually commute and

$$T_g = T_a T_{b_1 c_1} \cdots T_{b_j c_j}.$$

The restriction of T_a on H^q_κ is a multiplication operator $\gamma_{a,k}(\kappa) I_\kappa$ which is described in Corollary 3.9, while the action of $T_{b_j c_j}$ on basis elements of H^q_κ is given by 4.2.

References

1. A. Appuhamy, T. Le. *Commutants of Toeplitz Operators with Separately Radial Polynomial Symbols.* Complex Analysis and Operator Theory (2016) 10, 1–12.
2. D. CROCKER AND I. RAEBURN, Toeplitz Operator on Certain weakly Pseudoconvex Domains, *J. Austral. Math. Soc. Ser. A*, **31** (1981), 1–14.
3. W. Bauer, N. Vasilevski. *Banach Algebras of Commuting Toeplitz Operators on the Unit Ball via the Quasi-hyperbolic Group*. Operator Theory: Advances and Applications, v. 218 (2012), 155–175.
4. W. Bauer, N. Vasilevski. *Commutative Toeplitz Banach algebras on the ball and quasi-nilpotent group action*. Integral Equations and Operator Theory, v. 72, no.2 (2012), 223–240.
5. W. Bauer, N. Vasilevski. *On the structure of a commutative Banach algebra generated by Toeplitz operators with quasi-radial quasi-homogeneous symbols*. Integral Equations and Operator Theory, v. 74, no.2 (2012), 199–231.
6. W. Bauer, N. Vasilevki. *On the structure of commutative Banach algebras generated by Toeplitz operators on the unit ball. Quasi-elliptic case. I: Generating subalgebras.* Journal of Functional Analysis, 265 (2013), 2956–2990.
7. W. Bauer, N. Vasilevki. *On the structure of commutative Banach algebras generated by Toeplitz operators on the unit ball. Quasi-elliptic case. II: Gelfand theory.* Complex Analysis and Operator Theory, 9 (2015), 593–630.
8. T. Le. *Commutants of separately radial Toeplitz operators in several variables.* Journal of Mathematical Analysis and Applications, 453 (2017), 48–63.
9. R. Quiroga-Barranco, A. Sánchez-Nungaray. *Commutative C*-Algebras of Toeplitz Operators on Complex Projective Spaces*. Integral Equations Operator Theory 71 (2011), 225–243.
10. R. Quiroga-Barranco, A. Sánchez-Nungaray. *Toeplitz Operators with Quasi-radial Quasi-homogeneous Symbols and Bundles of Lagrangian Frames*. Journal Operator Theory 71:1 (2014), 101–124.
11. R. Quiroga-Barranco, A. Sánchez-Nungaray. *Toeplitz Operators with Quasi-homogeneous Quasi-radial Symbols on some Weakly Pseudoconvex Domains*. Complex Analysis and Operator Theory (2015) 9, 1111–1134.
12. S. Grudsky, R. Quiroga-Barranco, N. Vasilevski. *Commutative C*-algebras of Toeplitz operators and quantization on the unit disk*. J. Functional Analysis, v. 234 (2006), 1–44.
13. R. Quiroga-Barranco, N. Vasilevski. *Commutative C*-algebra of Toeplitz operators on the unit ball, I. Bargmann type transform and spectral representations of Toeplitz operators*, Integral Equations and Operator Theory, 59, no. 3 (2007), 379–419.
14. R. Quiroga-Barranco, N. Vasilevski. *Commutative C*-algebra of Toeplitz operators on the unit ball, II. Geometriy of the level sets of symbols*, Integral Equations and Operator Theory, 59, no. 1 (2008), 89–132.
15. A. Sánchez-Nungaray, N. Vasilevski. *Commutative Algebras of Toeplitz Operators on a Siegel Domain Associated with the Nilpotent Group of Its Biholomorphisms*, Operator Theory: Advances and Applications, v. 258 (2017), 275–300.
16. N. Vasilevski. *Quasi-radial quasi-homogeneous symbols and commutative Banach algebras of Toeplitz operators*, Integral Equations Operator Theory, 66, no. 1 (2010), 141–152.
17. N. Vasilevski. *Parabolic Quasi-radial Quasi-homogeneous Symbols and Commutative Algebras of Toeplitz Operators*. Operator Theory: Advances and Applications, v. 202 (2010), 553–568.
18. N. Vasilevski. *On Toeplitz operators with quasi-radial and pseudo-homogeneous symbols*, In: Pereyra M., Marcantognini S., Stokolos A., Urbina W. (eds) Harmonic Analysis, Partial Differential Equations, Banach Spaces, and Operator Theory (Volume 2). Association for Women in Mathematics Series, vol 5. Springer, Cham.
19. N. Vasilevski. *On commutative C^*-algebras generated by Toeplitz operators with $\mathbb{T}^m$-invariant symbols*. Operator Theory: Advances and Applications, v. 271 (2018), 443–464.

The Twofold Ellis-Gohberg Inverse Problem for Rational Matrix Functions on the Real Line

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Dedicated to Nikolai Vasilevski on the occasion of his seventieth birthday.

Abstract A twofold Ellis-Gohberg inverse problem for rational matrix functions on the real line is considered in this paper. It is assumed that the data functions of the inverse problem are given by finite dimensional state space realizations. Necessary and sufficient conditions for the existence of a solution are given in terms of the matrices appearing in the state space realizations and solutions to associated Lyapunov equations. In case a solution exists, it is unique. We also provide explicit descriptions of this solution in terms of the matrices and Lyapunov equation solutions associated with the data functions.

Keywords Inverse problem · Rational matrix functions · State space realizations

Mathematics Subject Classification (2010) Primary: 15A29; Secondary: 47A56, 47A48, 93B15

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1 Introduction

From the mid-1980s R.L. Ellis, I. Gohberg and D.C. Lay wrote several papers on systems of orthogonal matrix polynomials and matrix functions, culminating in the monograph [3] by Ellis and Gohberg, where additional background and further references can be found. Inverse problems related to these orthogonal systems were first considered in [2] for scalar-valued Wiener functions on the circle, both for unilateral systems (onefold problem) and bilateral systems (twofold problem). In later work extensions of the onefold inverse problem on the circle were considered for square matrix-valued polynomials in [5] and for square matrix-valued Wiener functions in [4]. Nonsquare versions were only recently dealt with in [12] and [13] for the onefold problems on the circle and real line, respectively, while nonsquare twofold problems on the circle and real line were solved in [9] and [10], respectively. In this paper we further develop the solution to the twofold inverse problem on the real line from [10] to the case of rational matrix functions. In particular, we present necessary and sufficient conditions for the existence and uniqueness of a solution, which are computationally more attractive then the results for Wiener class matrix functions in [10]. For this purpose, we represent the rational data functions in realized form, which allows us to reduce the inverse problem to a linear algebra problem and to give the solutions in state space form.

In order to introduce the inverse problem some notation and terminology has to be introduced. This is done in the next section where also the main result (Theorem 2.1) is presented.

The paper consists of five sections (the present introduction included) and an appendix. The second section presents the main theorem and a simple scalar example illustrating the main theorem. Section 3 consists of three subsections which together yield the proof of the main theorem. In Sect. 4 alternative formulas for the unique solution to the twofold rational Ellis-Gohberg inverse problem are derived in two special cases. In Sect. 5 an example is presented showing that the conditions in the main theorem cannot be weakened. In the Appendix we review Theorem 1.2 in [10]. The latter theorem plays an important role in the proofs given in Sect. 3.

2 Main Theorem

The data of the inverse problem we are dealing with are proper rational matrix functions $\alpha, \beta, \gamma, \delta$ with

$$\alpha(\lambda) \in \mathbb{C}^{p\times p}, \quad \beta(\lambda) \in \mathbb{C}^{p\times q}, \quad \gamma(\lambda) \in \mathbb{C}^{q\times p}, \quad \delta(\lambda) \in \mathbb{C}^{q\times q}, \tag{2.1}$$

α and β having only poles in the open lower half plane $\mathbb{C}_-$, γ and δ having poles only in the open upper half plane $\mathbb{C}_+$, and with values at ∞ given by

$$\alpha(\infty) = I_p, \quad \beta(\infty) = 0, \quad \gamma(\infty) = 0, \quad \delta(\infty) = I_q.$$

The word "proper" refers to the fact that the four functions are analytic at infinity (see, e.g., [1, page 9] or [6, page 377]). Moreover, the functions β and γ are *strictly proper*, which means that these two functions are proper and the value at infinity is zero (see, e.g., [1, page 26]). When the above properties are fulfilled we call $\{\alpha, \beta, \gamma, \delta\}$ an *admissible rational data set.*

Given an admissible rational data set $\{\alpha, \beta, \gamma, \delta\}$ the rational version of the twofold Ellis-Gohberg inverse problem is to find a strictly proper $p \times q$ rational matrix function g which has all its poles in $\mathbb{C}_-$ such that

$$\begin{aligned} &\alpha(\lambda) + g(\lambda)\gamma(\lambda) - I_p \text{ has poles only in } \mathbb{C}_+; \\ &g(\bar{\lambda})^*\alpha(\lambda) + \gamma(\lambda) \text{ has poles only in } \mathbb{C}_-; \\ &\delta(\lambda) + g(\bar{\lambda})^*\beta(\lambda) - I_q \text{ has poles only in } \mathbb{C}_-; \\ &g(\lambda)\delta(\lambda) + \beta(\lambda) \text{ has poles only in } \mathbb{C}_+. \end{aligned} \tag{2.2}$$

We shall refer to the above problem as the *twofold Rat-EG inverse problem*.

In what follows we will write $g^*(\lambda)$ instead of $g(\bar{\lambda})^*$. More generally, for any rational matrix function $\varphi(\lambda)$ the function $\varphi(\bar{\lambda})^*$ will be denoted by $\varphi^*(\lambda)$ and will be called the *adjoint* of the function $\varphi(\lambda)$.

In this paper, using Theorem A.1, we exploit the fact that our data functions are rational matrix (Wiener class) functions to derive computationally effective solution criteria and a more explicit description of the solution. To achieve this, we assume our data functions are given in the form of finite dimensional state space realizations:

$$\begin{aligned} \alpha(\lambda) &= I_p + iC_1(\lambda I_{n_1} - iA_1)^{-1}B_1, \quad \beta(\lambda) = iC_2(\lambda I_{n_2} - iA_2)^{-1}B_2, \\ \gamma(\lambda) &= -iC_3(\lambda I_{n_3} + iA_3)^{-1}B_3, \quad \delta(\lambda) = I_q - iC_4(\lambda I_{n_4} + iA_4)^{-1}B_4. \end{aligned} \tag{2.3}$$

Here A_j, $1 \leq j \leq 4$, is a square matrix which is assumed to be stable, e.g., all eigenvalues of A_j are in the open left half plane $\mathbb{C}_{\text{left}}$. These stability conditions are automatically fulfilled if the realizations are minimal. In the latter case the McMillan degrees of $\alpha, \beta, \gamma, \delta$ are equal to n_1, n_2, n_3, n_4, respectively. Although the functions $\alpha, \beta, \gamma, \delta$ in an admissible rational data set can always be represented in this way, we shall not require the realizations in (2.3) to be minimal.

To state our solution to the twofold Rat-EG inverse problem we shall use the solution P_{ij}, for $i, j \in \{1, 2\}$ or $i, j \in \{3, 4\}$, to the following Lyapunov equation associated with the pairs (A_i, C_i) and (A_j, C_j):

$$A_i^* P_{ij} + P_{ij} A_j + C_i^* C_j = 0, \quad i, j \in \{1, 2\} \text{ or } i, j \in \{3, 4\}. \tag{2.4}$$

For $i = j$ we abbreviate P_{jj} to P_j. We also need the solution Q_j, for $1 \leq j \leq 4$, to the Lyapunov equation

$$A_j Q_j + Q_j A_j^* + B_j B_j^* = 0, \quad 1 \leq j \leq 4. \tag{2.5}$$

Since the matrices A_j, $1 \leq j \leq 4$, are all stable, the solutions P_{ij} and Q_j to the Lyapunov equations (2.4) and (2.5) are unique, and given explicitly by

$$P_{ij} = \int_0^\infty e^{sA_i^*} C_i^* C_j e^{sA_j}\, ds \quad \text{and} \quad Q_j = \int_0^\infty e^{sA_j} B_j B_j^* e^{sA_j^*}\, ds.$$

From the latter identities it follows that the matrices $P_j = P_{jj}$ and Q_j, $1 \leq j \leq 4$, are nonnegative. Furthermore, we have $P_{12}^* = P_{21}$ and $P_{34}^* = P_{43}$. See Section 3.8 in [15] for the basic theory of Lyapunov equations; see also Theorem I.5.5 in [7].

Since P_2 and Q_2 are nonnegative, the matrix $I_{n_2} + Q_2 P_2$ is invertible. Indeed, $I_{n_2} + P_2^{1/2} Q_2 P_2^{1/2} \geq I_{n_2}$, and therefore is invertible. But then

$$I_{n_2} + Q_2 P_2 = I_{n_2} + (Q_2 P_2^{1/2}) P_2^{1/2} \text{ is also invertible.}$$

Similarly, we see that $I_{n_3} + Q_3 P_3$ is invertible because P_3 and Q_3 are nonnegative. Using the matrices defined above, we set

$$\begin{aligned} N_1 &:= P_1 - P_{12}(I_{n_2} + Q_2 P_2)^{-1} Q_2 P_{21}, \\ N_4 &:= P_4 - P_{43}(I_{n_3} + Q_3 P_3)^{-1} Q_3 P_{34}. \end{aligned} \tag{2.6}$$

Now we are ready to formulate our main result.

Theorem 2.1 *The twofold Rat-EG inverse problem associated with the rational data set $\{\alpha, \beta, \gamma, \delta\}$ given by state space realizations (2.3) has a solution if and only if the following conditions are satisfied:*

(R1) $\big(C_1 + B_1^* P_1\big)(\lambda I_{n_1} - iA_1)^{-1} B_1 = B_3^*(\lambda I_{n_3} - iA_3^*)^{-1} P_3 B_3$;

(R2) $(C_4 + B_4^* P_4)(\lambda I_{n_4} + iA_4)^{-1} B_4 = B_2^*(\lambda I_{n_2} + iA_2^*)^{-1} P_2 B_2$;

(R3) *the following two identities hold:*

(a) $B_1^*(\lambda I_{n_1} + iA_1^*)^{-1} P_{12} B_2 = B_3^* P_{34}(\lambda I_{n_4} + iA_4)^{-1} B_4$,

(b) $(C_2 + B_1^* P_{12})(\lambda I_{n_2} - iA_2)^{-1} B_2 =$

$$= B_3^*(\lambda I_{n_3} - iA_3^*)^{-1}(C_3^* + P_{34} B_4);$$

(R4) *the matrices $I_{n_1} - Q_1 N_1$ and $I_{n_4} - Q_4 N_4$ are invertible.*

Moreover, in that case the solution is unique, and the unique solution g is given by

$$g(\lambda) = -iC_1(\lambda I_{n_1} - iA_1)^{-1}Y_1 - iC_2(\lambda I_{n_2} - iA_2)^{-1}\left(Y_2 - \widetilde{Y_2}\right). \tag{2.7}$$

Here Y_2 and $\widetilde{Y_2}$ are matrices of size $n_2 \times q$, and Y_1 is a matrix of size $n_1 \times q$, and these three matrices are defined by

$$Y_1 = (I_{n_1} - Q_1N_1)^{-1}Q_1P_{12}(I_{n_2} + Q_2P_2)^{-1}B_2, \tag{2.8}$$

$$Y_2 = (I_{n_2} + Q_2P_2)^{-1}B_2, \tag{2.9}$$

$$\widetilde{Y_2} = (I_{n_2} + Q_2P_2)^{-1}Q_2P_{21}Y_1. \tag{2.10}$$

This unique solution g is also given by

$$g(\lambda) = -iX_1(\lambda I_{n_4} - iA_4^*)^{-1}C_4^* - i\left(X_2 - \widetilde{X_2}\right)(\lambda I_{n_3} - iA_3^*)^{-1}C_3^*. \tag{2.11}$$

In this case X_2 and $\widetilde{X_2}$ are matrices of size $p \times n_3$, and X_1 is a matrix of size $p \times n_4$, and these three matrices are defined by

$$X_1 = B_3^*(I_{n_3} + P_3Q_3)^{-1}P_{34}Q_4(I_{n_4} - N_4Q_4)^{-1}, \tag{2.12}$$

$$X_2 = B_3^*(I_{n_3} + P_3Q_3)^{-1}, \tag{2.13}$$

$$\widetilde{X_2} = X_1P_{43}Q_3(I_{n_3} + P_3Q_3)^{-1}. \tag{2.14}$$

We conclude this section with a simple scalar example to illustrate the above theorem. Let the data α, β, γ, δ be the scalar rational functions given by

$$\alpha(\lambda) = \frac{\lambda - 5i}{\lambda + 3i}, \quad \beta(\lambda) = \frac{4}{\lambda + 3i}, \quad \gamma(\lambda) = \frac{4}{\lambda - 3i}, \quad \delta(\lambda) = \frac{\lambda + 5i}{\lambda - 3i}. \tag{2.15}$$

Note that $\alpha^* = \delta$ and $\beta^* = \gamma$. Therefore, and since the functions are also scalar, the first two conditions in (2.2) are equivalent to the last two. Hence, for (2.2) to hold, it suffices to verify the first two, or last two, of the conditions in (2.2). That $\alpha^* = \delta$ and $\beta^* = \gamma$ is not by accident, in the scalar case it is necessary for a solution to exist, as explained at the end of this example. In the matrix case this need not occur, which makes the scalar problem relatively simple compared to the matrix problem. We shall now use Theorem 2.1 to show that the twofold Rat-EG problem is solvable and to obtain the solution g.

The realizations of the functions $\alpha, \beta, \gamma, \delta$ in (2.15) are obtained by taking

$$A_1 = A_2 = A_3 = A_4 = -3, \quad B_1 = B_2 = B_3 = B_4 = 1,$$
$$C_1 = C_4 = -8, \quad C_2 = -4i, \quad C_3 = 4i.$$

Obviously, A_1, A_2, A_3, A_4 are stable 1×1 matrices. With this choice of the realizations the solutions to the Lyapunov equations (2.4) and (2.5) are given by

$$P_1 = P_4 = \tfrac{32}{3}, \quad P_2 = P_3 = \tfrac{8}{3}, \quad P_{12} = P_{21}^* = P_{34} = P_{43}^* = \tfrac{16}{3}i,$$
$$Q_1 = Q_2 = Q_3 = Q_4 = \tfrac{1}{6}.$$

Moreover the conditions (R1)–(R3) reduce to:

(R1) $\big(-8 + \tfrac{32}{3}\big)(\lambda + 3i)^{-1} = (\lambda + 3i)^{-1}\tfrac{8}{3}$;

(R2) $\big(-8 + \tfrac{32}{3}\big)(\lambda - 3i)^{-1} = (\lambda - 3i)^{-1}\tfrac{8}{3}$;

(R3)(a) $(\lambda - 3i)^{-1}\big(\tfrac{16}{3}i\big) = \big(\tfrac{16}{3}i\big)(\lambda - 3i)^{-1}$;

(R3)(b) $\big(-4i + \tfrac{16}{3}i\big)(\lambda + 3i)^{-1} = (\lambda + 3i)^{-1}\big(-4i + \tfrac{16}{3}i\big)$.

Clearly (R1)–(R3) are satisfied. Next we determine N_1 and N_4 in order to check (R4). A straightforward calculation yields

$$N_1 = \tfrac{32}{3} - (\tfrac{16}{3}i)(1 + \tfrac{1}{6}\tfrac{8}{3})^{-1}\tfrac{1}{6}(-\tfrac{16}{3}i) = \tfrac{96}{13} \quad \text{and} \quad N_4 = N_1.$$

Then

$$1 - Q_1N_1 = 1 - Q_4N_4 = 1 - \tfrac{1}{6}\tfrac{96}{13} = -\tfrac{3}{13}.$$

The latter shows that condition (R4) is satisfied too. The next step is to determine the function g by using (2.7). In this case we have

$$Y_2 = (1 + \tfrac{8}{3}\tfrac{1}{6})^{-1} = \tfrac{9}{13}, \quad Y_1 = -(\tfrac{3}{13})^{-1}\tfrac{1}{6}(\tfrac{16}{3}i)\tfrac{9}{13} = -\tfrac{8}{3}i,$$
$$\widetilde{Y}_2 = \tfrac{9}{13}\tfrac{1}{6}(-\tfrac{16}{3}i)(-\tfrac{8}{3}i) = -\tfrac{64}{39}.$$

and hence

$$g(\lambda) = -i(-8)(\lambda + 3i)^{-1}(-\tfrac{8}{3}i) + i(-4i)(\lambda + 3i)^{-1}(-\tfrac{64}{39} - \tfrac{9}{13}) =$$
$$= 12(\lambda + 3i)^{-1}.$$

Thus, by Theorem 2.1, the rational function g is a solution of the twofold Rat-EG inverse problem corresponding to the data $\{\alpha, \beta, \gamma, \delta\}$ given by (2.15).

An Additional Remark As we remarked above, the data set $\{\alpha, \beta, \gamma, \delta\}$ in this example is such that $\alpha^* = \delta$ and $\gamma^* = \beta$. These identities are in fact necessary for a solution of the scalar twofold EG inverse problem to exist. Indeed, if a solution to the twofold EG inverse problem associated with the data set $\{\alpha, \beta, \gamma, \delta\}$ of scalar functions exists then, according to [14, Theorem 2.1], the functions α and δ^* have the same zeros in $\mathbb{C}_+$. Lemma 2.2 in [14] implies that in that case α and δ^* have the same zeros in $\mathbb{C}_+$ if and only if $\alpha^* = \delta$ and hence also $\beta = \gamma^*$. Here we use that the existence of a solution to the twofold EG inverse problem implies that $\alpha^*\alpha - \gamma^*\gamma = 1$, $\delta^*\delta - \beta^*\beta = 1$ and $\alpha^*\beta = \gamma^*\delta$. (See also Theorem A.1 below.)

3 Proof of the Main Theorem

We split the section into three subsections. Throughout $\{\alpha, \beta, \gamma, \delta\}$ is an admissible rational data set. We assume that $\alpha, \beta, \gamma, \delta$ are given by the finite dimensional state space realizations (2.3) with A_1, A_2, A_3, A_4 being stable matrices.

3.1 The Conditions (R1), (R2), (R3)

This first subsection concerns the conditions (R1), (R2), and (R3) appearing in Theorem 2.1. We shall prove the following propositions.

Proposition 3.1 *Condition* (R1) *holds if and only if*

$$\alpha^*(\lambda)\alpha(\lambda) - \gamma^*(\lambda)\gamma(\lambda) = I_p. \tag{3.1}$$

Proposition 3.2 *Condition* (R2) *holds if and only if*

$$\delta^*(\lambda)\delta(\lambda) - \beta^*(\lambda)\beta(\lambda) = I_q. \tag{3.2}$$

Proposition 3.3 *Condition* (R3) *holds if and only if*

$$\alpha^*(\lambda)\beta(\lambda) = \gamma^*(\lambda)\delta(\lambda). \tag{3.3}$$

The following two elementary lemmas will be used to prove the above propositions.

Lemma 3.4 *Let F, G, H, K be matrices, $F \in \mathbb{C}^{n\times n}$, $G \in \mathbb{C}^{n\times p}$, $H \in \mathbb{C}^{p\times m}$, and $K \in \mathbb{C}^{m\times m}$, and assume that there exists a matrix $X \in \mathbb{C}^{n\times m}$ such that $FX - XK = GH$. Then*

$$(\lambda I_n - F)^{-1}GH(\lambda I_m - K)^{-1} = (\lambda I_n - F)^{-1}X - X(\lambda I_m - K)^{-1}. \tag{3.4}$$

Proof The fact that $FX - XK = GH$ implies that

$$GH = (F - \lambda I_n)X - X(K - \lambda I_m) = X(\lambda I_m - K) - (\lambda I_n - F)X.$$

Multiplying the latter identity from the left by $(\lambda I_n - F)^{-1}$ and from the right by $(\lambda I_m - K)^{-1}$ yields the identity (3.4). □

Lemma 3.5 *Let φ_1, φ_2, η_1 and η_2 be strictly proper rational $n \times m$ matrix functions. Assume that φ_1 and η_1 have all poles in $\mathbb{C}_-$, and φ_2 and η_2 have all poles in $\mathbb{C}_+$. Then $\varphi_1 = \eta_1$ and $\varphi_2 = \eta_2$ if and only if $\varphi_1 + \varphi_2 = \eta_1 + \eta_2$.*

Proof If $\varphi_1 = \eta_1$ and $\varphi_2 = \eta_2$, then clearly $\varphi_1 + \varphi_2 = \eta_1 + \eta_2$. Conversely, if $\varphi_1 + \varphi_2 = \eta_1 + \eta_2$, then $\varphi_1 - \eta_1 = \psi = -\varphi_2 + \eta_2$. Since all poles of φ_1 and η_1 are in $\mathbb{C}_-$, the same is true for ψ. Similarly, using that all poles of φ_2 and η_2 are in $\mathbb{C}_+$, all poles of ψ are in $\mathbb{C}_+$. Hence the rational function ψ has no poles in $\mathbb{C}$. But ψ is also strictly proper. Thus $\psi = 0$, and therefore $\varphi_1 = \eta_1$ and $\varphi_2 = \eta_2$. □

Proof of Proposition 3.1 We split the proof into three parts. First we compute the product $\alpha^*(\lambda)\alpha(\lambda)$ and next the product $\gamma^*(\lambda)\gamma(\lambda)$, using Lemma 3.4 in both cases.

Part 1 Since α is given by (2.3), we have

$$\alpha^*(\lambda) = I_p - iB_1^*(\lambda I_{n_1} + iA_1^*)^{-1}C_1^*. \tag{3.5}$$

It follows that

$$\begin{aligned}\alpha^*(\lambda)\alpha(\lambda) - I_p = -iB_1^*(\lambda I_{n_1} + iA_1^*)^{-1}C_1^* + iC_1(\lambda I_{n_1} - iA_1)^{-1}B_1 + \\ + B_1^*(\lambda I_{n_1} + iA_1^*)^{-1}C_1^*C_1(\lambda I_{n_1} - iA_1)^{-1}B_1.\end{aligned} \tag{3.6}$$

To compute the product $(\lambda I_{n_1} + iA_1^*)^{-1}C_1^*C_1(\lambda I_{n_1} - iA_1)^{-1}$ appearing in (3.6) we apply Lemma 3.4 with

$$F = -iA_1^*, \quad G = -C_1^*, \quad H = C_1, \quad K = iA_1, \quad X = iP_1.$$

In this case, using (2.4) with $j = 1$ and $P_{11} = P_1$, we see that $FX - XK = GH$, and hence Lemma 3.4 shows that

$$\begin{aligned}(\lambda I_{n_1} + iA_1^*)^{-1}(C_1^*C_1)(\lambda I_{n_1} - iA_1)^{-1} = \\ = (\lambda I_{n_1} + iA_1^*)^{-1}(-iP_1) + (iP_1)(\lambda I_{n_1} - iA_1)^{-1}.\end{aligned}$$

Using the later identity in (3.6) we obtain

$$\alpha^*(\lambda)\alpha(\lambda) - I = \varphi_1(\lambda) + \varphi_2(\lambda)$$

with

$$\varphi_1(\lambda) = i\left(C_1 + B_1^* P_1\right)(\lambda I_{n_1} - iA_1)^{-1} B_1 \quad \text{and} \quad \varphi_2 = \varphi_1^*.$$

Note that φ_1 is a rational $p \times p$ matrix function that has all its poles in $\mathbb{C}_-$.

Part 2 In this part we compute the product $\gamma^*(\lambda)\gamma(\lambda)$ in the same way as $a^*(\lambda)\alpha(\lambda)$ has been computed. Note that

$$\gamma^*(\lambda) = iB_3^*(\lambda I_{n_3} - iA_3^*)^{-1} C_3^*,$$

and thus $\gamma^*(\lambda)\gamma(\lambda) = B_3^*(\lambda I_{n_3} - iA_3^*)^{-1} C_3^* C_3 (\lambda I_{n_3} + iA_3)^{-1} B_3$. To compute this product we apply Lemma 3.4 with

$$F = iA_3^*, \quad G = C_3^*, \quad H = C_3, \quad K = -iA_3, \quad X = iP_3.$$

One checks that $FX - XK = GH$, and thus we can apply Lemma 3.4 to obtain

$$(\lambda I_{n_3} - iA_3^*)^{-1} C_3^* C_3 (\lambda I_{n_3} + iA_3)^{-1} =$$
$$= (\lambda I_{n_3} - iA_3^*)^{-1}(iP_3) - (iP_3)(\lambda I_{n_3} + iA_3)^{-1}.$$

Hence

$$\gamma^*(\lambda)\gamma(\lambda) = \eta_1(\lambda) + \eta_2(\lambda)$$

with

$$\eta_1(\lambda) = iB_3^*(\lambda I_{n_3} - iA_3^*)^{-1} P_3 B_3 \quad \text{and} \quad \eta_2 = \eta_1^*.$$

Part 3 Note that $\varphi_1, \varphi_2, \eta_1$ and η_2 as defined in the Parts 1 and 2 of this proof satisfy the conditions in Lemma 3.5. Hence we see that $\alpha^*(\lambda)\alpha(\lambda) - I = \gamma^*(\lambda)\gamma(\lambda)$ if and only if $\varphi_1 = \eta_1$ and $\varphi_2 = \eta_2$. Since $\eta_2 = \eta_1^*$ and $\varphi_2 = \varphi_1^*$ we have that $\alpha^*(\lambda)\alpha(\lambda) - I = \gamma^*(\lambda)\gamma(\lambda)$ if and only if $\varphi_1 = \eta_1$, i.e., condition (R1) is satisfied. □

Proof of Proposition 3.2 This proposition can be proved using arguments similar to those used to prove Proposition 3.1. Actually, one can obtain Proposition 3.2 as a corollary of Proposition 3.1 by applying the latter proposition using the data set $\{\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}\}$ given by

$$\widetilde{\alpha}(\lambda) = \delta(-\lambda), \; \widetilde{\beta}(\lambda) = \gamma(-\lambda), \; \widetilde{\gamma}(\lambda) = \beta(-\lambda), \; \widetilde{\delta}(\lambda) = \alpha(-\lambda).$$

That this is an admissible rational data set and how its solution relates to the original data set $\{\alpha, \beta, \gamma, \delta\}$ is explained in detail in Lemma 3.9 below. Given these additional facts, the proof proceeds analogously to the Proof of Proposition 3.1. We omit further details. □

Proof of Proposition 3.3 As for Proposition 3.1 we split the proof into three parts. But now we first compute the product $\alpha^*(\lambda)\beta(\lambda)$ and then continue with $\gamma^*(\lambda)\delta(\lambda)$. *Part 1*. Recall that $\alpha^*(\lambda)$ is given by (3.5) and $\beta(\lambda)$ by (2.3). It follows that

$$\begin{aligned}\alpha^*(\lambda)\beta(\lambda) =& iC_2(\lambda I_{n_2} - iA_2)^{-1}B_2+\\ &+ B_1^*(\lambda I_{n_1} + iA_1^*)^{-1}C_1^*C_2(\lambda I_{n_2} - iA_2)^{-1}B_2.\end{aligned}$$

To compute the product $(\lambda I_{n_1} + iA_1^*)^{-1}C_1^*C_2(\lambda I_{n_2} - iA_2)^{-1}$ we will apply Lemma 3.4 with

$$F = -iA_1^*, \quad G = -C_1^*, \quad H = C_2, \quad K = iA_2, \quad X = iP_{12}.$$

In this case, using (2.4), we see that $FX - XK = GH$, and hence we can apply Lemma 3.4 to show that

$$\begin{aligned}&(\lambda I_{n_1} + iA_1^*)^{-1}C_1^*C_2(\lambda I_{n_2} - iA_2)^{-1} =\\ &\qquad = iP_{12}(\lambda I_{n_2} - iA_2)^{-1} - i(\lambda I_{n_1} + iA_1^*)^{-1}P_{12}.\end{aligned}$$

Hence

$$\begin{aligned}\alpha^*(\lambda)\beta(\lambda) &= iC_2(\lambda I_{n_2} - iA_2)^{-1}B_2+\\ &\quad + iB_1^*P_{12}(\lambda I_{n_2} - iA_2)^{-1}B_2 - iB_1^*(\lambda I_{n_1} + iA_1^*)^{-1}P_{12}B_2\\ &= \varphi_1(\lambda) + \varphi_2(\lambda),\end{aligned}$$

where

$$\begin{aligned}\varphi_1(\lambda) &= i(C_2 + B_1^*P_{12})(\lambda I_{n_2} - iA_2)^{-1}B_2,\\ \varphi_2(\lambda) &= -iB_1^*(\lambda I_{n_1} + iA_1^*)^{-1}P_{12}B_2.\end{aligned} \tag{3.7}$$

Part 2. Since γ and δ are given by (2.3), we have

$$\gamma^*(\lambda) = iB_3^*(\lambda I_{n_3} - iA_3^*)^{-1}C_3^*,$$

and

$$\begin{aligned}\gamma^*(\lambda)\delta(\lambda) - \gamma^*(\lambda) &= -i\gamma^*(\lambda)C_4(\lambda I_{n_4} + iA_4)^{-1}B_4\\ &= B_3^*(\lambda I_{n_3} - iA_3^*)^{-1}C_3^*C_4(\lambda I_{n_4} + iA_4)^{-1}B_4.\end{aligned}$$

To compute the product $(\lambda I_{n_3} - iA_3^*)^{-1}C_3^*C_4(\lambda I_{n_4} + iA_4)^{-1}$ we will apply Lemma 3.4 with

$$F = iA_3^*, \quad G = C_3^*, \quad H = C_4, \quad K = -iA_4, \quad X = iP_{34}.$$

In this case, using (2.4), we see that $FX - XK = GH$, and hence we can apply Lemma 3.4 to show that

$$(\lambda I_{n_3} - iA_3^*)^{-1}C_3^*C_4(\lambda I_{n_4} + iA_4)^{-1} = \\ = i(\lambda I_{n_3} - iA_3^*)^{-1}P_{34} - iP_{34}(\lambda I_{n_4} + iA_4)^{-1}.$$

We conclude that

$$\begin{aligned} \gamma^*(\lambda)\delta(\lambda) &= iB_3^*(\lambda I_{n_3} - iA_3^*)^{-1}C_3^* + \\ &\quad + B_3^*\Big(i(\lambda I_{n_3} - iA_3^*)^{-1}P_{34} - iP_{34}(\lambda I_{n_4} + iA_4)^{-1}\Big)B_4 \\ &= \eta_1(\lambda) + \eta_2(\lambda) \end{aligned}$$

with

$$\begin{aligned} \eta_1(\lambda) &= iB_3^*(\lambda I_{n_4} - iA_3^*)^{-1}(C_3^* + P_{34}B_4) \\ \eta_2(\lambda) &= -iB_3^*P_{34}(\lambda I_{n_4} + iA_4)^{-1}B_4. \end{aligned} \tag{3.8}$$

Part 3. Note that $\varphi_1, \varphi_2, \eta_1$ and η_2 as defined in (3.7) and (3.8) satisfy the conditions in Lemma 3.5. Therefore $\alpha^*\beta = \gamma^*\delta$ if and only if $\varphi_1 = \eta_1$ and $\varphi_2 = \eta_2$, i.e., $\alpha^*\beta = \gamma^*\delta$ if and only if the conditions (b) and (a) in (R3) are satisfied. □

The three identities (3.1), (3.2), (3.3) play an important role in solving Ellis-Gohberg inverse problems. In fact (see, e.g., Proposition 2.1 in [11]) these identities are necessary conditions for the twofold inverse problems to be solvable. Thus the three propositions proved above tell us that conditions (R1), (R2), (R3) are necessary for the twofold Rat-EG problem to be solvable. Moreover, these three conditions are equivalent with condition (W1) appearing in Theorem A.1 provided the data set $\{\alpha, \beta, \gamma, \delta\}$ is an admissible rational data set, i.e., the matrix functions $\alpha, \beta, \gamma, \delta$ are of the type described in the first paragraph of Sect. 2.

3.2 The Condition (R4)

In this subsection we shall show that for our given data set the condition (R4) is equivalent to condition (W2) appearing in Theorem A.1. This requires some preliminaries.

First we introduce some additional notation and auxiliary results. We define operators

$$\begin{aligned}
\Gamma_1 &: \mathbb{C}^{n_1} \to L^2(\mathbb{R}_+)^p, & \Lambda_1 &: L^2(\mathbb{R}_-)^p \to \mathbb{C}^{n_1},\\
\Gamma_2 &: \mathbb{C}^{n_2} \to L^2(\mathbb{R}_+)^p, & \Lambda_2 &: L^2(\mathbb{R}_-)^q \to \mathbb{C}^{n_2},\\
\Gamma_3 &: \mathbb{C}^{n_3} \to L^2(\mathbb{R}_-)^q, & \Lambda_3 &: L^2(\mathbb{R}_+)^p \to \mathbb{C}^{n_3},\\
\Gamma_4 &: \mathbb{C}^{n_4} \to L^2(\mathbb{R}_-)^q, & \Lambda_4 &: L^2(\mathbb{R}_+)^q \to \mathbb{C}^{n_4},
\end{aligned}$$

by setting

$$(\Gamma_j x)(t) = C_j e^{tA_j} x \ (t \geq 0), \quad \Lambda_j f = \int_{-\infty}^{0} e^{-sA_j} B_j f(s)\,\mathrm{d}s \quad (j = 1, 2);$$

$$(\Gamma_j x)(t) = C_j e^{-tA_j} x \ (t \leq 0), \quad \Lambda_j f = \int_{0}^{\infty} e^{sA_j} B_j f(s)\,\mathrm{d}s \quad (j = 3, 4).$$

The adjoints of these operators are given by

$$\Gamma_j^* f = \int_0^\infty e^{sA_j^*} C_j^* f(s)\,ds, \quad (\Lambda_j^* x)(t) = B_j^* e^{-tA_j^*} x \quad (t \leq 0) \quad (j = 1, 2);$$

$$\Gamma_j^* f = \int_{-\infty}^0 e^{-sA_j^*} C_j^* f(s)\,ds, \quad (\Lambda_j^* x)(t) = B_j^* e^{tA_j^*} x \quad (t \geq 0) \quad (j = 3, 4).$$

Note that the operators P_j, $j = 1, 2, 3, 4$, P_{12}, P_{21}, P_{34}, P_{43} and the operators Q_j, $j = 1, \ldots, 4$, which have been defined by (2.4) and (2.5) as solutions of Lyapunov equations, are also defined by the identities:

$$P_j = \Gamma_j^* \Gamma_j \quad \text{and} \quad Q_j = \Lambda_j \Lambda_j^* \quad \text{for } j = 1, 2, 3, 4,$$

$$P_{ij} = \Gamma_i^* \Gamma_j \quad \text{for } (ij) = (12), (21), (34), (43).$$

Next we consider the operators Ξ_{11} and Ξ_{22} given by

$$\Xi_{11} = I + \Gamma_2 \Lambda_2 \Lambda_2^* \Gamma_2^* - \Gamma_1 \Lambda_1 \Lambda_1^* \Gamma_1^* = I + \Gamma_2 Q_2 \Gamma_2^* - \Gamma_1 Q_1 \Gamma_1^*, \tag{3.9}$$

$$\Xi_{22} = I + \Gamma_3 \Lambda_3 \Lambda_3^* \Gamma_3^* - \Gamma_4 \Lambda_4 \Lambda_4^* \Gamma_4^* = I + \Gamma_3 Q_3 \Gamma_3^* - \Gamma_4 Q_4 \Gamma_4^*. \tag{3.10}$$

These operators Ξ_{11} and Ξ_{22} act on $L^2(\mathbb{R}_+)^p$ and $L^2(\mathbb{R}_-)^q$, respectively. For reasons that will become clear further on, we are interested in invertibility of these operators. For that purpose we need the operators:

$$R_b = I + \Gamma_2 Q_2 \Gamma_2^* \quad \text{and} \quad R_c = I + \Gamma_3 Q_3 \Gamma_3^*.$$

Note that R_b acts on $L^2(\mathbb{R}_+)^p$ and R_c acts on $L^2(\mathbb{R}_-)^q$, and both operators are positive definite and thus invertible.

Lemma 3.6 *The inverses of the operators R_b and R_c are given by*

$$\begin{aligned} R_b^{-1} &= I - \Gamma_2(I_{n_2} + Q_2P_2)^{-1}Q_2\Gamma_2^*, \\ R_c^{-1} &= I - \Gamma_3(I_{n_3} + Q_3P_3)^{-1}Q_3\Gamma_3^*. \end{aligned} \tag{3.11}$$

Furthermore,

$$\Gamma_1^*R_b^{-1}\Gamma_1 = N_1 \quad \textit{and} \quad \Gamma_4^*R_c^{-1}\Gamma_4 = N_4, \tag{3.12}$$

where N_1 and N_4 are the operators on $\mathbb{C}^p$ and $\mathbb{C}^q$, respectively, defined by (2.6).

Proof To prove the first identity in (3.11) we use (see, e.g., Section 2.2 in [1]) the classical identity

$$(D + CB)^{-1} = D^{-1} - D^{-1}C(I + BD^{-1}C)^{-1}BD^{-1}, \tag{3.13}$$

where D is assumed to be invertible. We apply the above identity with $D = I$, $C = \Gamma_2Q_2$, and $B = \Gamma_2^*$. This yields

$$\begin{aligned} R_b^{-1} &= (I + \Gamma_2Q_2\Gamma_2^*)^{-1} = I - \Gamma_2Q_2(I + \Gamma_2^*\Gamma_2Q_2)^{-1}\Gamma_2^* \\ &= I - \Gamma_2(I_{n_2} + Q_2P_2)^{-1}Q_2\Gamma_2^*. \end{aligned} \tag{3.14}$$

The second identity in (3.11) is proved in a similar way.

To prove the first identity in (3.12), note that

$$\begin{aligned} \Gamma_1^*R_b^{-1}\Gamma_1 &= \Gamma_1^*\Gamma_1 - \Gamma_1^*\Gamma_2\left(I_{n_2} + Q_2P_2)^{-1}\right)Q_2\Gamma_2^*\Gamma_1 \\ &= P_1 - P_{12}\left(I_{n_2} + Q_2P_2)^{-1}\right)Q_2P_{21} = N_1. \end{aligned}$$

In a similar way one proves the second identity in (3.12). □

The next two lemma's relate the invertibility of Ξ_{11} and Ξ_{22} to the invertibility of $I - Q_1N_1$ and $I - Q_4N_4$, respectively.

Lemma 3.7 *Let N_1 be the operator on the finite dimensional space $\mathbb{C}^{n_1}$ given by (2.6). Then Ξ_{11} given by (3.9) is invertible if and only if the finite dimensional operator $I_{n_1} - Q_1N_1$ is invertible, and in that case*

$$\Xi_{11}^{-1} = R_b^{-1} + R_b^{-1}\Gamma_1(I_{n_1} - Q_1N_1)^{-1}Q_1\Gamma_1^*R_b^{-1}. \tag{3.15}$$

Proof According to (3.9) the operator $\Xi_{11} = D + CB$, where $D = R_b$ is invertible, $C = -\Gamma_1 Q_1$, and $B = \Gamma_1^*$. Again using (3.13) we see that Ξ_{11} is invertible if and only if $I + BD^{-1}C$ is invertible. Note that

$$I + BD^{-1}C = I - \Gamma_1^* R_b^{-1} \Gamma_1 Q_1 = I - N_1 Q_1.$$

Thus $I + BD^{-1}C$ is invertible if and only if $I - N_1 Q_1$ is invertible. Moreover, in that case

$$\begin{aligned}\Xi_{11}^{-1} &= D^{-1} - D^{-1}C(I + BD^{-1}C)^{-1}BD^{-1}\\ &= R_b^{-1} + R_b^{-1}\Gamma_1 Q_1 (I_{n_1} - N_1 Q_1)^{-1} \Gamma_1^* R_b^{-1}\\ &= R_b^{-1} + R_b^{-1}\Gamma_1 (I_{n_1} - Q_1 N_1)^{-1} Q_1 \Gamma_1^* R_b^{-1}.\end{aligned}$$

Hence (3.15) is proved too. □

The following lemma is proved in a similar way.

Lemma 3.8 *Let N_4 be the operator on the finite dimensional space $\mathbb{C}^{n_4}$ given by* (2.6)*. Then Ξ_{22} given by* (3.10) *is invertible if and only if the finite dimensional operator $I - Q_4 N_4$ is invertible, and in that case*

$$\Xi_{22}^{-1} = R_c^{-1} + R_c^{-1}\Gamma_4 (I_{n_4} - Q_4 N_4)^{-1} Q_4 \Gamma_4^* R_c^{-1}. \tag{3.16}$$

The inversion results presented by Lemmas 3.7 and 3.8 can be viewed as generalizations of the inversion result presented in Section 2 of [8].

Related Hankel Operators The operators $\Gamma_j \Lambda_j$, $j = 1, 2, 3, 4$, are the finite rank Hankel integral operators associated with the rational matrix functions $\alpha, \beta, \gamma, \delta$. More precisely, using the notation introduced in the second paragraph of the Appendix (see formulas (A.2)) we have

$$\Gamma_1\Lambda_1 = H_{+,\alpha}, \quad \Gamma_2\Lambda_2 = H_{+,\beta}, \quad \Lambda_3\Gamma_3 = H_{-.\gamma}, \quad \Gamma_4\Lambda_4 = H_{-,\delta},$$

Furthermore, taking adjoints, we have

$$\Lambda_1^*\Gamma_1^* = H_{-,\alpha^*}, \quad H_{-,\beta^*} = \Lambda_2^*\Gamma_2^*, \quad H_{+,\gamma^*} = \Lambda_3^*\Gamma_3^*, \quad H_{+,\delta^*} = \Lambda_4^*\Gamma_4^*.$$

Since the defining functions are rational matrix functions, the associate Hankel operators can be considered as operators on L^1-spaces as well as operators on L^2-spaces.

Using the above notation we see that

$$\begin{aligned}\Xi_{11} &= I + H_{+,\beta} H_{-,\beta^*} - H_{+,\alpha} H_{-,a^*},\\ \Xi_{22} &= I + H_{-,\gamma} H_{+,\gamma^*} - H_{-,\delta} H_{+\delta^*}.\end{aligned} \tag{3.17}$$

It follows that in the present context, where α, β, γ, δ are rational matrix functions, the operator Ξ_{11} defined by (3.9) is equal to the operator M_{11} defined by (A.4), and the operator Ξ_{22} defined by (3.10) is equal to the operator M_{22} defined by (A.4). But then Lemmas 3.7 and 3.8 show that for our data set the condition (R4) is equivalent to the condition (W2) in Theorem A.1 appearing in the Appendix. We proved the first part of Theorem 2.1.

To finish the proof of Theorem 2.1 it remains to prove formulas (2.7) and (2.11) for the solution of the twofold rational EG inverse problem. This will be done in the next subsection.

3.3 The Formulas for the Solution g

In this section we assume that the conditions (R1)–(R4) are satisfied. The aim is to prove formulas (2.7) and (2.11) for the solution g. This will be done by applying Theorem A.1. Recall (see the text after (3.17)) that in the present setting $\Xi_{11} = M_{11}$ and $\Xi_{22} = M_{22}$.

First we derive (2.7). From the identities (A.5) and (3.15) we know that the solution g is given by

$$g(\lambda) = \int_0^\infty e^{i\lambda t} h(t)\, dt, \text{ with } h \text{ being given by } h := -M_{11}^{-1} b = -\Xi_{11}^{-1} b.$$

Here b and β are related through (A.7). Since β is given by (2.3), we have

$$b(t) = C_2 e^{tA_2} B_2,\ t \geq 0, \text{ and hence } b = \Gamma_2 B_2.$$

Using (3.11) we obtain

$$\begin{aligned} R_b^{-1} b = R_b^{-1} \Gamma_2 B_2 &= \Gamma_2 \big(I_{n_2} - (I_{n_2} + Q_2 P_2)^{-1} Q_2 P_2\big) B_2 \\ &= \Gamma_2 (I_{n_2} + Q_2 P_2)^{-1} B_2 = \Gamma_2 Y_2, \end{aligned}$$

with Y_2 given by (2.9). Employing this formula together with (3.15) gives

$$\begin{aligned} -h = M_{11}^{-1} \Gamma_2 B_2 &= \Xi_{11}^{-1} \Gamma_2 B_2 \\ &= \big(R_b^{-1} + R_b^{-1} \Gamma_1 (I_{n_1} - Q_1 N_1)^{-1} Q_1 \Gamma_1^* R_b^{-1}\big) b \\ &= R_b^{-1} b + R_b^{-1} \Gamma_1 (I_{n_1} - Q_1 N_1)^{-1} Q_1 \Gamma_1^* R_b^{-1} b \\ &= \Gamma_2 Y_2 + R_b^{-1} \Gamma_1 (I_{n_1} - Q_1 N_1)^{-1} Q_1 P_{12} Y_2 \\ &= \Gamma_2 Y_2 + R_b^{-1} \Gamma_1 Y_1, \end{aligned}$$

with Y_1 as in (2.8). Again using the formula for R_b^{-1} in (3.14) we obtain

$$\begin{aligned} R_b^{-1}\Gamma_1 &= \big(I - \Gamma_2(I_{n_2} + Q_2P_2)^{-1}Q_2\Gamma_2^*\big)\Gamma_1 \\ &= \Gamma_1 - \Gamma_2(I_{n_2} + Q_2P_2)^{-1}Q_2\Gamma_2^*\Gamma_1 \\ &= \Gamma_1 - \Gamma_2(I_{n_2} + Q_2P_2)^{-1}Q_2P_{21}. \end{aligned}$$

Hence $R_b^{-1}\Gamma_1 Y_1 = \Gamma_1 Y_1 - \Gamma_2\widetilde{Y}_2$, where $\widetilde{Y}_2$ is given by (2.10). Note that $\widetilde{Y}_2$ is a matrix of size $n_2 \times q$. We find that

$$h = -\Gamma_2 Y_2 - (\Gamma_1 Y_1 - \Gamma_2\widetilde{Y}_2) = -\Gamma_2(Y_2 - \widetilde{Y}_2) - \Gamma_1 Y_1.$$

Thus $h = h_1 + h_2$ with h_1 and h_2 being given by

$$h_1(t) = -C_1 e^{tA_1} Y_1 \quad \text{and} \quad h_2(t) = -C_2 e^{tA_2}(Y_2 - \widetilde{Y}_2) \quad (t \geq 0).$$

It follows that the functions

$$\begin{aligned} g_1(\lambda) &= \int_0^\infty e^{i\lambda t} h_1(t)\,dt = -iC_1(\lambda I_{n_1} - iA_1)^{-1}Y_1, \\ g_2(\lambda) &= \int_0^\infty e^{i\lambda t} h_2(t)\,dt = -iC_2(\lambda I_{n_2} - iA_2)^{-1}(Y_2 - \widetilde{Y}_2). \end{aligned}$$

are rational matrix functions, and $g = g_1 + g_2$ is the unique solution. Hence we obtain (2.7).

Next we will derive formula (2.11) from (A.6) and (3.16). The computations are close to those in the previous paragraph, and therefore we leave out some details. The function c in (A.6) is determined by γ via (A.8). Since γ is as in (2.3), we have $c(t) = C_3 e^{tA_3} B_3 = \Gamma_3 B_3$. From the identity (3.16), the identity $\Xi_{22} = M_{22}$, and $c(t) = C_3 e^{tA_3} B_3 = \Gamma_3 B_3$ it follows that

$$M_{22}^{-1}c = R_c^{-1}\Gamma_3 B_3 + R_c^{-1}\Gamma_4 X_1^* \ \text{ with } \ X_1 \text{ as in (2.12)}.$$

Furthermore, using $R_c^{-1}\Gamma_3 B_3 = \Gamma_3(I_{n_3} + Q_3P_3)^{-1}B_3 = \Gamma_3 X_2^*$ with X_2 as in (2.13) one obtains that

$$M_{22}^{-1}c = \Gamma_4 X_1^* + \Gamma_3 X_2^* - \Gamma_3(\widetilde{X}_2)^* \ \text{ with } \widetilde{X}_2 \text{ as in (2.14)}.$$

Then the identity (A.6) yields

$$g^*(\lambda) = iC_4(\lambda I_{n_4} + iA_4)^{-1}X_1^* + iC_3(\lambda I_{n_3} + iA_3)^{-1}(X_2^* - (\widetilde{X}_2)^*).$$

By taking adjoints we obtain (2.11).

All together this completes the Proof of Theorem 2.1. □

We conclude this subsection with a remark about formulas (2.7) and (2.11) for the solution g. In fact, we shall show that formula (2.11) can be derived from formula (2.7) by a direct computation not using Theorem A.1. To do this we need the following lemma.

Lemma 3.9 *Let $\{\alpha, \beta, \gamma, \delta\}$ be an admissible rational data set, and put*

$$\widetilde{\alpha}(\lambda) = \delta(-\lambda), \quad \widetilde{\beta}(\lambda) = \gamma(-\lambda), \quad \widetilde{\gamma}(\lambda) = \beta(-\lambda), \quad \widetilde{\delta}(\lambda) = \alpha(-\lambda). \tag{3.18}$$

Then the quadruple $\{\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}\}$ is also an admissible rational data set. Moreover, if the twofold Rat-EG inverse problem for the data set $\{\alpha, \beta, \gamma, \delta\}$ is solvable, then the twofold Rat-EG inverse problem for the data set $\{\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}\}$ is also solvable. Furthermore, if g is the (unique) solution of the twofold Rat-EG inverse problem for the data set $\{\alpha, \beta, \gamma, \delta\}$, then the solution h of the twofold Rat-EG inverse problem for the data set $\{\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}\}$ is given by

$$h(\lambda) = g^*(-\lambda) \quad \textit{and} \quad h^*(\lambda) = g(-\lambda). \tag{3.19}$$

Applying the construction of the above lemma to the admissible rational data set $\{\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}\}$ we recover the original data set $\{\alpha, \beta, \gamma, \delta\}$. Hence the statements of the above lemma are in fact "if and only if" statements between these data sets.

Proof Since $\{\alpha, \beta, \gamma, \delta\}$ is an admissible rational data set, (2.1) tells us that

$$\widetilde{\alpha}(\lambda) \in \mathbb{C}^{q\times q}, \quad \widetilde{\beta}(\lambda) \in \mathbb{C}^{q\times p}, \quad \widetilde{\gamma}(\lambda) \in \mathbb{C}^{p\times q}, \quad \widetilde{\delta}(\lambda) \in \mathbb{C}^{p\times p}.$$

Moreover, the fact that γ and δ have poles only in the open upper half plane $\mathbb{C}_+$ implies that the functions $\widetilde{\alpha}$ and $\widetilde{\beta}$ have poles only in the open lower half plane $\mathbb{C}_-$. Similarly, one shows that $\widetilde{\gamma}$ and $\widetilde{\delta}$ have poles only in the upper half plane $\mathbb{C}_+$. Finally

$$\widetilde{\alpha}(\infty) = I_q, \quad \widetilde{\beta}(\infty) = 0, \quad \widetilde{\gamma}(\infty) = 0, \quad \widetilde{\delta}(\infty) = I_p.$$

Thus $\{\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}\}$ is an admissible rational data set.

Next assume that the twofold Rat-EG inverse problem for the data set $\{\alpha, \beta, \gamma, \delta\}$ is solvable, and let g be the (unique) solution. Let $h(\lambda)$ be the rational function defined by the first identity in (3.19). Then the second identity also holds true, because

$$h^*(\lambda) = h(\bar{\lambda})^* = [g^*(-\bar{\lambda})]^* = [g(-\lambda)]^{**} = g(-\lambda).$$

We will show that with appropriate modifications the four statements in (2.2) hold true. More precisely, we shall prove that

$$\widetilde{\alpha}(\lambda) + h(\lambda)\widetilde{\gamma}(\lambda) - I_q \text{ has poles only in } \mathbb{C}_+; \tag{3.20}$$

$$h(\bar{\lambda})^*\widetilde{\alpha}(\lambda) + \widetilde{\gamma}(\lambda) \text{ has poles only in } \mathbb{C}_-; \tag{3.21}$$

$$\widetilde{\delta}(\lambda) + h(\bar{\lambda})^*\widetilde{\beta}(\lambda) - I_p \text{ has poles only in } \mathbb{C}_-; \tag{3.22}$$

$$h(\lambda)\widetilde{\delta}(\lambda) + \widetilde{\beta}(\lambda) \text{ has poles only in } \mathbb{C}_+. \tag{3.23}$$

Let us prove statement (3.22). From the first line in (2.2) we know that

$$\alpha(\lambda) + g(\lambda)\gamma(\lambda) - I_p \ \text{ is analytic on } \overline{\mathbb{C}_-}. \tag{3.24}$$

Using the identities in (3.18) we see that $\widetilde{\delta}(-\lambda) + g(\lambda)\widetilde{\beta}(-\lambda) - I_p$ is analytic on $\overline{\mathbb{C}_-}$. Hence

$$\begin{aligned}
(3.24) \implies & \widetilde{\delta}(\lambda) + g(-\lambda)\widetilde{\beta}(\lambda) - I_p \ \text{ is analytic on } \overline{\mathbb{C}_+} \\
\implies & \widetilde{\delta}(\lambda) + h^*(\lambda)\widetilde{\beta}(\lambda) - I_p \ \text{ is analytic on } \overline{\mathbb{C}_+} \\
\implies & \widetilde{\delta}(\lambda) + h(\bar{\lambda})^*\widetilde{\beta}(\lambda) - I_p \ \text{ is analytic on } \overline{\mathbb{C}_+} \implies (3.22).
\end{aligned}$$

Next let us show that the second line in (2.2) implies (3.23). From the second line in (2.2) we know that

$$g(\bar{\lambda})^*\alpha(\lambda) + \gamma(\lambda) \ \text{ is analytic on } \overline{\mathbb{C}_+}. \tag{3.25}$$

Using the identities in (3.18) we see that $g(\bar{\lambda})^*\widetilde{\delta}(-\lambda) + \widetilde{\beta}(-\lambda)$ is analytic on $\overline{\mathbb{C}_+}$. Hence

$$\begin{aligned}
(3.25) \implies & g(-\bar{\lambda})^*\widetilde{\delta}(\lambda) + \widetilde{\beta}(\lambda) \ \text{ is analytic on } \overline{\mathbb{C}_-} \\
\implies & h(\lambda)\widetilde{\delta}(\lambda) + \widetilde{\beta}(\lambda) \ \text{ is analytic on } \overline{\mathbb{C}_-} \implies (3.23).
\end{aligned}$$

In a similar way one shows that the third line in (2.2) implies (3.20), and that the fourth line in (2.2) implies (3.21), as desired. □

The Identity (2.11) as a Corollary of the Identity (2.7) In what follows we assume that the functions α, β, γ, d are given by the finite dimensional state space realizations (2.3). Then the functions $\{\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}\}$ are given by

$$\widetilde{\alpha}(\lambda) = I_q + iC_4(\lambda I_{n_4} - iA_4)^{-1}B_4, \quad \widetilde{\beta}(\lambda) = iC_3(\lambda I_{n_3} - iA_3)^{-1}B_3,$$

$$\widetilde{\gamma}(\lambda) = -iC_2(\lambda I_{n_2} - iA_2)^{-1}B_2, \quad \widetilde{\delta}(\lambda) = I_p - iC_1(\lambda I_{n_1} + iA_1)^{-1}B_1.$$

Now assume that the twofold Rat-EG inverse problem for the data set $\{\alpha, \beta, \gamma, \delta\}$ is solvable, and let g be the (unique) solution. Then we know from Lemma 3.9 that the twofold Rat-EG inverse problem for the data set $\{\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}\}$ is also solvable, and that the (unique) solution h is given by the first identity in (3.19). Applying Theorem 2.1 with $\{\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}\}$ in place of the data set $\{\alpha, \beta, \gamma, d\}$, we can use formula (2.7) to obtain a formula for the function h. In fact

$$h(\lambda) = -iC_4(\lambda I_{n_4} - iA_4)^{-1}U_1 - iC_3(\lambda I_{n_3} - iA_3)^{-1}(U_1 - U_2), \qquad (3.26)$$

where

$$U_1 = (I_{n_4} - Q_4N_4)^{-1}Q_4P_{43}(I_{n_3} + Q_3P_3)^{-1}B_3,$$

$$U_2 = (I_{n_3} + Q_3P_3)^{-1}B_3, \quad \text{and} \quad U_2 = (I_{n_3} + Q_3P_3)^{-1}B_3P_{34}U_1.$$

According to the second identity in (3.19), we have $g(\lambda) = h(-\bar{\lambda})^*$. From (3.26) we know that

$$h(-\bar{\lambda}) = iC_4(\bar{\lambda}I_{n_4} + iA_4)^{-1}U_1 + iC_3(\bar{\lambda}I_{n_2} + iA_3)^{-1}\left(U_1 - \widetilde{U_2}\right).$$

Taking adjoints and using $g(\lambda) = h(-\bar{\lambda})^*$ we see that

$$\begin{aligned} g(\lambda) &= -iU_1^*(\lambda I_{n_4} - iA_4^*)^{-1}C_4^* - i\left(U_1^* - U_2{}^*\right)(\lambda I_{n_2} - iA_3^*)^{-1}C_3^* \\ &= -iX_1(\lambda I_{n_4} - iA_4^*)^{-1}C_4^* - i\,(X_1 - X_2)\,(\lambda I_{n_2} - iA_3^*)^{-1}C_3^*, \end{aligned}$$

where

$$X_1 = U_1^* = B_3^*(I_{n_3} + P_3Q_3)^{-1}P_{34}Q_4(I_{n_4} - N_4Q_4)^{-1},$$

$$X_2 = U_2^* = B_3^*(I_{n_3} + P_3Q_3)^{-1},$$

$$\widetilde{X_2} = U_2{}^* = X_1P_{43}Q_3(I_{n_3} + P_3Q_3)^{-1}.$$

Thus g is given by (2.11) as desired. □

4 Two Special Classes of Admissible Rational Data Sets

In this section we present alternative formulas for the unique solution to the twofold Rat-EG inverse problem in two special cases. Throughout $\{\alpha, \beta, \gamma, \delta\}$ is an admissible rational data set. In the first subsection we assume that 1 is not a singular value of the Hankel operators $H_{+,\alpha}$ and $H_{-,\delta}$. In other words, the operators

$I - H^*_{+,\alpha} H_{+,\alpha}$ and $I - H^*_{-,\delta} H_{-,\delta}$ are assumed to be invertible. In the second subsection we assume that the rational functions $\alpha(\lambda)^{-1}$ and $\delta(\lambda)^{-1}$ have poles only in $\mathbb{C}_-$ and $\mathbb{C}_+$, respectively. These additional properties of α and δ allow us to simplify considerably the formulas for the solution g given by (2.7) and (2.11).

4.1 *The Case When* 1 *is Not a Singular Value of* $H_{+,\alpha}$ *and* $H_{-,\delta}$

We first show that 1 is not a singular value of $H_{+,\alpha}$ (resp. of $H_{-,\delta}$) is equivalent to the matrix $I_{n_1} - Q_1 P_1$ (resp. the matrix $I_{n_4} - Q_4 P_4$) being invertible. Let us prove this for α. Recall that $I_{n_1} - Q_1 P_1 = I - \Lambda_1 \Lambda_1^* \Gamma_1^* \Gamma_1$. Thus $I_{n_1} - Q_1 P_1$ is invertible if and only if

$$I - \Lambda_1^* \Gamma_1^* \Gamma_1 \Lambda_1 = I - H^*_{+,\alpha} H_{+,\alpha} \text{ is invertible.}$$

Hence, invertibility of $I_{n_1} - Q_1 P_1$ is equivalent to 1 not being a singular value of $H_{+,\alpha}$.

Proposition 4.1 *Let* $\{\alpha, \beta, \gamma, \delta\}$ *be an admissible rational data set for the twofold Rat-EG inverse problem given by state space realizations* (2.3)*. Assume* $I_{n_1} - Q_1 P_1$ *and* $I_{n_4} - Q_4 P_4$ *are invertible. Define* N_1 *and* N_4 *as in* (2.6)*, and set*

$$N_2 = P_2 + P_{21}(I_{n_1} - Q_1 P_1)^{-1} Q_1 P_{12} \text{ and } N_3 = P_3 + P_{34}(I_{n_4} - Q_4 P_4)^{-1} Q_4 P_{43}.$$

Then $I_{n_1} - Q_1 N_1$ *is invertible if and only if* $I_{n_2} + Q_2 N_2$ *is invertible, and* $I_{n_4} - Q_4 N_4$ *is invertible if and only if* $I_{n_3} + Q_3 N_3$ *is invertible. In particular, condition* (R4) *of Theorem 2.1 is equivalent to:*

(R4)′ *the matrices* $I_{n_2} + Q_2 N_2$ *and* $I_{n_3} + Q_2 N_3$ *are invertible.*

Furthermore, if conditions (R1)–(R4) *of Theorem 2.1 are satisfied, then the unique solution* g *to the twofold Rat-EG inverse problem is given by*

$$\begin{aligned} g(\lambda) &= -iC_1(\lambda I_{n_1} - iA_1)^{-1} Y_1 - iC_2(\lambda I_{n_2} - iA_2)^{-1}(I_{n_2} + Q_2 N_2)^{-1} B_2 \\ &= -iX_1(\lambda I_{n_4} - iA_4^*)^{-1} C_4^* - iB_3^*(I_{n_3} + N_3 Q_3)^{-1}(\lambda I_{n_3} - A_3^*)^{-1} C_3^*. \end{aligned}$$

Here Y_1 *and* X_1 *are given by* (2.8) *and* (2.12)*, respectively.*

Proof We start with the claim related to the invertibility of $I_{n_2} + Q_2 N_2$. Define

$$M_1 = P_{12}(I_{n_2} + Q_2 P_2)^{-1} \quad \text{and} \quad M_2 = P_{21}(I_{n_1} - Q_1 P_1)^{-1},$$

so that

$$N_1 = P_1 - M_1 Q_2 P_{21} \quad \text{and} \quad N_2 = P_1 - M_2 Q_1 P_{12}.$$

Note that for matrices $T_1 \in \mathbb{C}^{m\times m}$, $T_2 \in \mathbb{C}^{m\times n}$ and $T_3 \in \mathbb{C}^{n\times m}$ with T_1 invertible we have that $T_1 + T_2T_3$ is invertible if and only if $I_n + T_3T_1^{-1}T_2$ is invertible, and in this case

$$T_3(T_1 + T_2T_3)^{-1}T_2 = I_n - (I_n + T_3T_1^{-1}T_2)^{-1}. \tag{4.1}$$

Apply this identity with

$$T_1 = I_{n_1} - Q_1P_1, \quad T_2 = Q_1M_1, \quad T_3 = Q_2P_{21}.$$

By assumption T_1 is invertible. Note that

$$T_1 + T_2T_3 = I_{n_1} - Q_1P_1 + Q_1M_1Q_2P_{21} = I_{n_1} - Q_1N_1.$$

On the other hand

$$\begin{aligned} I_{n_2} + T_3T_1^{-1}T_2 &= I_{n_2} + Q_2P_{21}(I_{n_1} - Q_1P_1)^{-1}Q_1M_1 \\ &= I_{n_2} + Q_2P_{21}(I_{n_1} - Q_1P_1)^{-1}Q_1P_{12}(I_{n_2} + Q_2P_2)^{-1} \\ &= (I_{n_2} + Q_2(P_2 + P_{21}(I_{n_1} - Q_1P_1)^{-1}Q_1P_{12}))(I_{n_2} + Q_2P_2)^{-1} \\ &= (I_{n_2} + Q_2N_2)(I_{n_2} + Q_2P_2)^{-1}. \end{aligned}$$

Hence we obtain that $I_{n_1} - Q_1N_1$ is invertible if and only if $I_{n_2} + Q_2N_2$ is invertible. Similarly it follows that $I_{n_4} - Q_4N_4$ is invertible if and only if $I_{n_3} + Q_3N_3$ is invertible.

Now assume (R1)–(R4) are satisfied, and hence (R4)$'$ is satisfied. By Theorem 2.1 a unique solution g for the twofold Rat-EG inverse problem exists and is given by (2.7) as well as by (2.11). It remains to show that

$$\widetilde{Y}_2 - Y_2 = -(I_{n_2} + Q_2N_2)^{-1}B_2, \quad \widetilde{X}_2 - X_2 = -B_3^*(I_{n_3} + N_3Q_3)^{-1}, \tag{4.2}$$

where $\widetilde{Y}_2$, Y_2, $\widetilde{X}_2$ and X_2 are given by (2.10), (2.9), (2.14) and (2.13), respectively. The above computations along with (4.1) yield

$$\begin{aligned} Q_2P_{21}(I_{n_1} - Q_1N_1)^{-1}Q_1M_1 &= I_{n_2} - ((I_{n_2} + Q_2N_2)(I_{n_2} + Q_2P_2)^{-1})^{-1} \\ &= I_{n_2} - (I_{n_2} + Q_2P_2)(I_{n_2} + Q_2N_2)^{-1}. \end{aligned}$$

Using this identity we obtain

$$\begin{aligned}\widetilde{Y}_2 &= (I_{n_2} + Q_2P_2)^{-1}Q_2P_{21}(I_{n_1} - Q_1N_1)^{-1}Q_1P_{12}(I_{n_2} + Q_2P_2)^{-1}B_2\\ &= (I_{n_2} + Q_2P_2)^{-1}Q_2P_{21}(I_{n_1} - Q_1N_1)^{-1}Q_1M_1B_2\\ &= (I_{n_2} + Q_2P_2)^{-1}B_2 - (I_{n_2} + Q_2N_2)^{-1}B_2 = Y_2 - (I_{n_2} + Q_2N_2)^{-1}B_2.\end{aligned}$$

This proves the first identity in (4.2). The second identity follows by similar computations. □

4.2 The Special Case When $\alpha(\lambda)^{-1}$ has Poles Only in $\mathbb{C}_-$ and $\delta(\lambda)^{-1}$ has Poles Only in $\mathbb{C}_+$

Let $\{\alpha, \beta, \gamma, \delta\}$ be an admissible rational data set given by the state space realizations (2.3). As before, we assume that A_j, $1 \le j \le 4$, is a stable matrix. It is well known that in that case

$$\begin{aligned}\alpha(\lambda)^{-1} &= I_p - iC_1\big(\lambda I_{n_1} - i(A_1 - B_1C_1)\big)^{-1}B_1,\\ \delta(\lambda)^{-1} &= I_q + iC_4\big(\lambda I_{n_4} + i(A_4 - B_4C_4)\big)^{-1}B_4.\end{aligned}$$

In this subsection we consider the case when $\alpha(\lambda)^{-1}$ has poles only in $\mathbb{C}_-$ and $\delta(\lambda)^{-1}$ has poles only in $\mathbb{C}_+$. In other words $\alpha(\lambda)^{-1}$ and $\delta(\lambda)^{-1}$ are analytic on $\overline{\mathbb{C}_+}$ and $\overline{\mathbb{C}_-}$, respectively. This implies that $A_1 - B_1C_1$ and $A_4 - B_4C_4$ are also stable. Let us prove the latter statement for $A_1 - B_1C_1$; the proof for $A_4 - B_4C_4$ is similar. Note (cf., the third formula in [7, Lemma XIII.5.3]) that

$$\begin{aligned}&\big(\lambda I_{n_1} - i(A_1 - B_1C_1)\big)^{-1} =\\ &\quad = (\lambda I_{n_1} - iA_1)^{-1} - i(\lambda I_{n_1} - iA_1)^{-1}B_1\alpha(\lambda)^{-1}C_1(\lambda I_{n_1} - iA_1)^{-1}.\end{aligned} \tag{4.3}$$

Since A_1 is stable, $(\lambda I_{n_1} - iA_1)^{-1}$ is analytic on $\overline{\mathbb{C}_+}$. By assumption the same is true for $\alpha(\lambda)^{-1}$. Then (4.3) tells us that $\big(\lambda I_{n_1} - i(A_1 - B_1C_1)\big)^{-1}$ is analytic on $\overline{\mathbb{C}_+}$ too, which implies that all eigenvalues of $A_1 - B_1C_1$ are in $\mathbb{C}_{\text{left}}$, and hence $A_1 - B_1C_1$ is stable.

Under the above assumptions, Theorem 9.1 of [14] presents alternative formulas for the solution to the twofold EG inverse problem. In the following theorem we derive the state space realizations analogues of these formulas for the case when the data functions are rational.

Theorem 4.2 *Let $\{\alpha, \beta, \gamma, \delta\}$ be an admissible rational data set, and let α, β, γ and δ be given by the state space realizations (2.3). Furthermore, assume that $\alpha(\lambda)^{-1}$ has poles only in $\mathbb{C}_-$ and $\delta(\lambda)^{-1}$ has poles only in $\mathbb{C}_+$. Then the twofold*

Rat-EG inverse problem has a solution if and only if the conditions (R1), (R2), (R3) *appearing in Theorem 2.1 are satisfied, and the solution, if it exists, is unique. Moreover, the unique solution g, if it exists, is given by the formulas:*

$$g(\lambda) = -i(B_3^* - B_1^* P_{13}^\times)(\lambda I_{n_3} - iA_3^*)^{-1} C_3^*, \tag{4.4}$$

$$= -iC_2(\lambda I_{n_2} - iA_2)^{-1}(B_2 - P_{24}^\times B_4). \tag{4.5}$$

Here $P_{13}^\times$ and $P_{24}^\times$ are the unique solutions of the Lyapunov equations

$$(A_1 - B_1C_1)^* P_{13}^\times + P_{13}^\times A_3^* = -C_1^* B_3^*, \tag{4.6}$$

$$P_{24}^\times(A_4 - B_4C_4) + A_2 P_{24}^\times = -B_2C_4, \tag{4.7}$$

which are well-defined because $A_1 - B_1C_1$ and $A_4 - B_4C_4$ are stable.

Proof Note that the rational function $-\big(\alpha^*(\lambda)\big)^{-1}\gamma^*(\lambda)$ has no poles on the real line and is zero at infinity. This allows us to decompose this function as follows:

$$-\big(\alpha^*(\lambda)\big)^{-1}\gamma^*(\lambda) = g(\lambda) + h(\lambda) \tag{4.8}$$

where g has all its poles in $\mathbb{C}_-$, the function h has all its poles in $\mathbb{C}_+$, and both are zero at infinity. According to the Propositions 3.1–3.3 the conditions (C1)–(C3) in [10, Theorem 9.1] are satisfied. Thus we conclude from [10, Theorem 9.1] that the rational function g appearing in the right hand side of (4.8) is the unique solution of the twofold EG inverse problem associated with the data set $\{\alpha, \beta, \gamma, \delta\}$.

Next we will show that g given by (4.8) is also given by (4.4). Note that

$$\big(\alpha^*(\lambda)\big)^{-1} = I_p + iB_1^*\left(\lambda I_{n_1} + i(A_1 - B_1C_1)^*\right)^{-1} C_1^*,$$

$$\gamma^*(\lambda) = iB_3^*\left(\lambda I_{n_3} - iA_3^*\right)^{-1} C_3^*.$$

This yields

$$\begin{aligned} -\big(\alpha^*(\lambda)\big)^{-1}\gamma^*(\lambda) &= -iB_3^*\left(\lambda I_{n_3} - iA_3^*\right)^{-1} C_3^* + \\ &\quad - iB_1^*\left(\lambda I_{n_1} + i(A_1 - B_1C_1)^*\right)^{-1} C_1^* iB_3^*\left(\lambda I_{n_3} - iA_3^*\right)^{-1} C_3^*. \end{aligned} \tag{4.9}$$

It follows from (4.6) that

$$P_{13}^\times(\lambda I_{n_3} - iA_3^\times) - (\lambda I_{n_1} + i\big(A_1 - B_1C_1)^*\big)P_{13}^\times = iC_1^* B_3^*.$$

Using this in the second term in the right hand side of Eq. (4.9) we get

$$\begin{aligned} -\big(\alpha^*(\lambda)\big)^{-1}\gamma^*(\lambda) &= -(iB_3^* - iB_1^* P_{13}^\times)(\lambda I_{n_3} - iA_3^*)^{-1} C_3^* + \\ &\quad - iB_1^*\big(\lambda I_{n_1} + i(A_1 - B_1C_1)^*\big)^{-1} P_{13}^\times C_3^* \end{aligned} \tag{4.10}$$

The first term in the right hand side of (4.10) has all poles in the open lower half plane and the second term has all poles in the open upper half plane. Therefore we see from (4.8) that $g(\lambda)$ is given by (4.4).

From Theorem 9.1 in [10] it also follows that g is given by

$$-\beta(\lambda)\delta(\lambda)^{-1} = g(\lambda) + k(\lambda),$$

where g is the solution of the Rat-EG inverse problem and k has all its poles in the open lower half plane. Similar computations as above show that this formula for g yields (4.5). □

Remark 4.3 Note that explicit formulas for the unique solution g of the Rat-EG inverse problem are given in (2.7) and (2.11), and under additional conditions also in the present Sects. 4.1 and 4.2; see Proposition 4.1 and Theorem 4.2. Given these explicit formulas, it is natural to try to prove the main parts of the theorems by direct computation, not using the roundabout by the way of Theorem A.1. So far this has not been done. We leave it as an open problem.

5 An Example

Assuming conditions (R1), (R2), (R3) in Theorem 2.1 are satisfied, we show in this section that it can happen that the matrix $I_{n_1} - Q_1 N_1$ appearing in (R4) is invertible while the matrix $I_{n_4} - Q_4 N_4$ is not invertible. In other words, Theorem 2.1 is not true if in (R4) the word "and" is replaced by"or", i.e., the two invertibility conditions in (R4) are independent. The example we present to prove the above statement is closely related to [14, Example 3].

The rational data functions we shall use are the scalar functions

$$\alpha(\lambda) = \frac{\lambda - 5i}{\lambda + 3i}, \qquad \beta(\lambda) = \frac{4}{\lambda + 3i},$$
$$\gamma(\lambda) = \frac{4(\lambda + i)}{(\lambda - 3i)(\lambda - i)}, \quad \delta(\lambda) = \frac{(\lambda + 5i)(\lambda + i)}{(\lambda - 3i)(\lambda - i)}.$$

One easily verifies that these functions satisfy the conditions (3.1)–(3.3) and therefore, according to Propositions 3.1–3.3, the conditions (R1)–(R3) in Theorem 2.1 are satisfied for any choice of the state space realizations of $\alpha, \beta, \gamma, \delta$ in (2.3). In this example we use the realizations (2.3) with

$$A_1 = A_2 = -3, \quad B_1 = B_2 = 1, \quad C_1 = -8, \quad C_2 = -4i,$$
$$A_3 = A_4 = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}, \ B_3 = B_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ C_3 = \begin{bmatrix} 8i & -4i \end{bmatrix}, \ C_4 = \begin{bmatrix} -16 & 6 \end{bmatrix}.$$

The solutions of the Lyapunov equations (2.4) and (2.5) are then given by

$$P_1 = \tfrac{32}{3}, \quad Q_1 = \tfrac{1}{6}, \quad P_2 = \tfrac{8}{3}, \quad Q_2 = \tfrac{1}{6}, \quad P_{12} = P_{21}^* = \tfrac{16}{3}i,$$
$$P_3 = \begin{bmatrix} \frac{32}{3} & -8 \\ -8 & 8 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} \frac{1}{6} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}, \quad P_4 = \begin{bmatrix} \frac{128}{3} & -24 \\ -24 & 18 \end{bmatrix}, \quad Q_4 = \begin{bmatrix} \frac{1}{6} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix},$$
$$P_{34}^* = P_{43} = 4i \begin{bmatrix} -\frac{16}{3} & 4 \\ 3 & -3 \end{bmatrix}$$

Recall that the operators N_1 and N_4 appearing in condition (R4) in Theorem 2.1 are given by (2.6). Thus, in the present setting we have

$$\begin{aligned} N_1 &= P_1 - P_{12}(1 + Q_2 P_2)^{-1} Q_2 P_{21} \\ &= \frac{32}{3} - \frac{16}{3}\left(1 + \frac{4}{9}\right)^{-1} \frac{1}{6}\frac{16}{3} = \frac{96}{13} \quad \text{and} \end{aligned}$$

$$\begin{aligned} N_4 &= P_4 - P_{43}(I_2 + Q_3 P_3)^{-1} Q_3 P_{34} \\ &= \begin{bmatrix} \frac{128}{3} & -24 \\ -24 & 18 \end{bmatrix} - 16 \begin{bmatrix} -\frac{16}{3} & 4 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} \frac{7}{9} & \frac{2}{3} \\ -\frac{4}{3} & 3 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{6} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{16}{3} & 3 \\ 4 & -3 \end{bmatrix} \\ &= \tfrac{4}{29} \begin{bmatrix} \frac{576}{3} & -90 \\ -90 & \frac{117}{2} \end{bmatrix}. \end{aligned}$$

It follows that

$$1 - Q_1 N_1 = -\frac{3}{13} \quad \text{and} \quad I_2 - Q_4 N_4 = \begin{bmatrix} -\frac{9}{29} & \frac{3}{58} \\ -\frac{12}{29} & \frac{2}{29} \end{bmatrix}.$$

Hence $1 - Q_1 N_1$ is invertible while $I_2 - Q_4 N_4$ is not invertible.

Although the twofold Rat-EG inverse problem associated with the above data is not solvable (because $I_2 - Q_4 N_4$ is not invertible), formula (2.7) can be used to define a candidate function g, namely,

$$g(\lambda) = \frac{12}{\lambda + 3i}. \tag{5.1}$$

However, for formula (2.11) this is not the case because $I_2 - Q_4 N_4$ is not invertible and hence formula (2.11) does not make sense. Note that in the formula (2.7) only the realizations of the functions α and β appear. Since these two functions in this example are identically equal to their namesakes in the example in Sect. 2, it is clear that the function g determined here should be identical to the function g in the example in Sect. 2.

We conclude with another remark. In the present setting the functions α and γ^* have no common zero in $\mathbb{C}_+$. Therefore, we know from [13, Theorem 4.1] that there exits a unique function k with poles only in $\mathbb{C}_-$ such that $\alpha + k\gamma - 1$ has poles only in $\mathbb{C}_+$ and $k^*\alpha + \gamma$ has poles in $\mathbb{C}_-$ only. Similarly, since β and δ^* have no common zero in $\mathbb{C}_+$, Theorem 4.1 in [13] shows that there exits a unique function h with poles only in $\mathbb{C}_+$ such that $\delta + h\beta - 1$ has poles only in $\mathbb{C}_-$ and $h^*\delta + \beta$ has poles in $\mathbb{C}_+$ only. Since condition (R4) is not satisfied it follows that $h \neq k^*$. Actually, we have

$$k(\lambda) = \frac{24}{\lambda + 3i} - \frac{4}{\lambda + i}, \quad h(\lambda) = \frac{24}{\lambda - 3i} - \frac{6}{\lambda - i}.$$

From this it follows that the function g in (5.1) determined by Eq. (2.7) cannot satisfy both of the first two equations in (2.2) nor both of the last two equations in (2.2). In terms of the terminology used in [13], the function k is a left onefold solution and h a right onefold solution but neither k nor h^* is a twofold solution.

Appendix A Wiener Space Twofold EG Inverse Theorem on $\mathbb{R}$

In this appendix we recall Theorem 1.2 in [10] (see also the first paragraph of the introduction), which is one of our main sources. We shall present the theorem in the language of Wiener class functions rather than in terms of L_1 functions as is done in [10]. The transition from L_1 functions to Wiener functions fits better with the fact that our rational matrix functions are also Wiener class functions.

We first introduce the required notations and terminology. Let s, r be positive integers and write $L^1(\mathbb{R})^{s\times r}$ for the space of $s \times r$ matrix functions with entries from $L^1(\mathbb{R})$. We define the subspaces $L^1(\mathbb{R}_+)^{s\times r}$ and $L^1(\mathbb{R}_-)^{s\times r}$ consisting of functions in $L^1(\mathbb{R})^{s\times r}$ with support on $\mathbb{R}_+ = [0, \infty)$ or $\mathbb{R}_- = (-\infty, 0]$, respectively.

The Wiener class $\mathcal{W}(\mathbb{R})^{s\times r}$ is defined as the space of functions φ of the form

$$\varphi(\lambda) = f_0 + \int_{-\infty}^{\infty} e^{i\lambda t} f(t)\, \mathrm{d}t\,, \quad \lambda \in \mathbb{R}, \tag{A.1}$$

with $f_0 \in \mathbb{C}^{s\times r}$ and $f \in L^1(\mathbb{R})^{s\times r}$. The subspaces $\mathcal{W}(\mathbb{R})^{s\times r}_{\pm,0}$ consist of the functions φ in $\mathcal{W}(\mathbb{R})^{s\times r}$ for which in the representation (A.1) the constant $f_0 = 0$

and $f \in L^1(\mathbb{R}_\pm)^{s\times r}$. With φ given by (A.1) we associate the Hankel operators $H_{-,\varphi} : L^1(\mathbb{R}_+)^r \to L^1(\mathbb{R}_-)^s$ and $H_{+,\varphi} : L^1(\mathbb{R}_-)^r \to L^1(\mathbb{R}_+)^s$ given by

$$\begin{aligned} \left(H_{-,\varphi}h\right)(t) &= \int_0^\infty f(t-\tau)h(\tau)\,d\tau, \quad t \le 0, \quad h \in L^1(\mathbb{R}_+)^r, \\ \left(H_{+,\varphi}h\right)(t) &= \int_{-\infty}^0 f(t-\tau)h(\tau)\,d\tau, \quad t \ge 0, \quad h \in L^1(\mathbb{R}_-)^r. \end{aligned} \tag{A.2}$$

For the twofold EG inverse problem for Wiener functions on the real line, as defined in Subsection 3.3.1 of [10], our data functions $\alpha, \beta, \gamma, \delta$ are

$$\begin{aligned} &\alpha \in e_p + \mathcal{W}(\mathbb{R})_{+,0}^{p\times p}, \quad &\gamma \in \mathcal{W}(\mathbb{R})_{-,0}^{q\times p}, \\ &\beta \in \mathcal{W}(\mathbb{R})_{+,0}^{p\times q}, \quad &\delta \in e_q + \mathcal{W}(\mathbb{R})_{-,0}^{q\times q}. \end{aligned} \tag{A.3}$$

Here e_p and e_q denote the functions identically equal to the unit matrix I_p and I_q, respectively. The twofold EG inverse problem is to find $g \in \mathcal{W}(\mathbb{R})_{+,0}^{p\times q}$ such that the following four inclusions are satisfied:

$$\begin{aligned} &\alpha + g\gamma - e_p \in \mathcal{W}(\mathbb{R})_{-,0}^{p\times p} \quad \text{and} \quad g^*\alpha + \gamma \in \mathcal{W}(\mathbb{R})_{+,0}^{q\times p}; \\ &\delta + g^*\beta - e_q \in \mathcal{W}(\mathbb{R})_{+,0}^{q\times q} \quad \text{and} \quad g\delta + \beta \in \mathcal{W}(\mathbb{R})_{-,0}^{p\times q}. \end{aligned}$$

If g has these properties, we refer to g as a *solution to the twofold EG inverse problem associated with the data set* $\{\alpha, \beta, \gamma, \delta\}$.

With the given data set $\{\alpha, \beta, \gamma, \delta\}$ we associate the following operators:

$$\begin{aligned} M_{11} &= I + H_{+,\beta}H_{-,\beta^*} - H_{+,\alpha}H_{-,\alpha^*}, \\ M_{22} &= I + H_{-,\gamma}H_{+,\gamma^*} - H_{-,\delta}H_{+,\delta^*}. \end{aligned} \tag{A.4}$$

Notice that these operators are uniquely determined by the data. We are now ready to state Theorem 1.2 in [10] in terms of Wiener class functions.

Theorem A.1 *Let* $\{\alpha, \beta, \gamma, \delta\}$ *be the Wiener matrix functions given by* (A.3). *Then the twofold EG inverse problem associated with the data set* $\{\alpha, \beta, \gamma, \delta\}$ *has a solution if and only the following two conditions are satisfied:*

(W1) $\alpha^*\alpha - \gamma^*\gamma = e_p,\ \delta^*\delta - \beta^*\beta = e_q,\ \alpha^*\beta = \gamma^*\delta;$

(W2) *the operators* M_{11} *and* M_{22} *defined by* (A.4) *are one-to-one.*

Furthermore, in that case M_{11} *and* M_{22} *are invertible, the solution is unique, and the unique solution* g *and its adjoint* g^* *are given by*

$$g(\lambda) = -\int_0^\infty e^{i\lambda t}\left(M_{11}^{-1}b\right)(t)\,dt, \quad Im\,\lambda \ge 0; \tag{A.5}$$

$$g^*(\lambda) = -\int_{-\infty}^{0} e^{i\lambda t}\left(M_{22}^{-1}c\right)(t)\,dt, \quad Im\,\lambda \leq 0. \tag{A.6}$$

Here b and c are the matrix functions determined by

$$\beta(\lambda) = \int_{0}^{\infty} e^{i\lambda t} b(t)\,dt, \quad \textit{where } b \in L^1(\mathbb{R}_+)^{p\times q}; \tag{A.7}$$

$$\gamma(\lambda) = \int_{-\infty}^{0} e^{i\lambda t} c(t)\,dt, \quad \textit{where } c \in L^1(\mathbb{R}_-)^{q\times p}. \tag{A.8}$$

The above theorem is used in Sect. 3 where we prove our main result.

Erratum Regarding Formula (10.5) in [10] We take the opportunity to correct the three identities in formula (10.5) in [10]. The correct identities are as follows:

$$\alpha a_0^{-1}\alpha^* - \beta d_0^{-1}\beta^* = e_p, \quad \delta d_0^{-1}\delta^* - \gamma a_0^{-1}\gamma^* = e_q, \quad \alpha a_0^{-1}\gamma^* = \beta d_0^{-1}\delta^*,$$

as can be seen from Remark 2.3 in [10].

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References

1. H. Bart, I. Gohberg, M.A. Kaashoek, and A.C.M. Ran, *Factorization of matrix and operator functions: the state space method*, Oper. Theory Adv. Appl. **178**, Birkhäuser Verlag, Basel, 2008.
2. R.L. Ellis and I. Gohberg, Orthogonal systems related to infinite Hankel matrices, *J. Funct. Analysis* **109** (1992), 155–198.
3. R.L. Ellis and I. Gohberg, *Orthogonal systems and convolution operators*, Oper. Theory Adv. Appl. **140**, Birkhäuser Verlag, Basel, 2003.
4. R.L. Ellis, I. Gohberg, and D.C. Lay, Infinite analogues of block Toeplitz matrices and related orthogonal functions. *Integral Equ. Oper. Theory* **22** (1995), 375–419.
5. R. L. Ellis, I. Gohberg, and D.C. Lay, On a Class of Block Toeplitz Matrices, *Linear Algebra Appl.* **241/243** (1996), 225–245.
6. A.E. Frazho and W. Bhosri, *An operator perspective on signals and systems*, Oper. Theory Adv. Appl. **204**, Birkhäuser Verlag, Basel, 2010.
7. I. Gohberg, S. Goldberg, and M.A. Kaashoek, *Classes of Linear Operators I*, Oper. Theory Adv. Appl. **49**, Birkhäuser Verlag, Basel, 1990.
8. G.J. Groenewald and M.A. Kaashoek, A Gohberg-Heinig type inversion formula involving Hankel operators, in: *Interpolation, Schur functions and moment problems*, Oper. Theory Adv. Appl. **165**, Birkhäuser Verlag, Basel, 2005, pp. 291–302.
9. S. ter Horst, M.A. Kaashoek, and F. van Schagen, The discrete twofold Ellis-Gohberg inverse problem. *J. Math. Anal. Appl.* **452** (2017), 846–870.

10. S. ter Horst, M.A. Kaashoek, and F. van Schagen, The twofold Ellis-Gohberg inverse problem in an abstract setting and applications, in: *Interpolation and realization theory with applications to control theory*, Oper. Theory Adv. Appl. **272** (2019), 155–212.
11. M.A. Kaashoek and F. van Schagen, Ellis-Gohberg identities for certain orthogonal functions II: Algebraic setting and asymmetric versions, *Math. Proc. R. Ir. Acad.*, **113A** (2) (2013), 107–130.
12. M.A. Kaashoek and F. van Schagen, The inverse problem for Ellis-Gohberg orthogonal matrix functions, *Integral Equ. Oper. Theory* **80** (2014), 527–555.
13. M.A. Kaashoek and F. van Schagen, The Ellis-Gohberg inverse problem for matrix-valued Wiener functions on the line, *Oper. Matrices* **10** (2016), 1009–1042.
14. M.A. Kaashoek and F. van Schagen, Onefold and twofold Ellis-Gohberg inverse problems for scalar Wiener class functions, in: *Positivity and Noncommutative Analysis: Festschrift in Honour of Ben de Pagter on the Occasion of his 65th Birthday*, Birkhäuser Verlag, Basel, 257–267 (2019).
15. K. Zhou with J. C. Doyle and K. Glover, *Robust and optimal control*, Prentice Hall, New Jersey, 1996.

Fractional Integrodifferentiation and Toeplitz Operators with Vertical Symbols

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Dedicated to the occasion of 70th birthday of Professor N. Vasilevski.

Abstract We consider the so-called vertical Toeplitz operators on the weighted Bergman space over the half plane. The terminology "vertical" is motivated by the fact that if a is a symbol of such Toeplitz operator, then $a(z)$ depends only on $y = \Im z$, where $z = x + iy$. The main question raised in this paper can be formulated as follows: given two bounded vertical Toeplitz operators T_a^λ and T_b^λ, under which conditions is there a symbol h such that $T_a^\lambda T_b^\lambda = T_h^\lambda$? It turns out that this problem has a very nice connection with fractional calculus! We shall formulate our main results using the well-known theory of Riemann–Liouville fractional integrodifferentiation.

Keywords Toepliz operators · Riemann–Liouville fractional integrodifferentiation · Spaces of holomorphic functions

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1 Introduction

Let $\Pi = \{z = x + iy : x \in \mathbb{R},\ y > 0\}$ be the upper half plane in the complex plane $\mathbb{C}$, $dA(z) = dxdy$ is the Lebesgue area measure, and $dA_\lambda(z) = (\lambda + 1)\,(2\Im z)^\lambda\, dA(z)$, $\lambda > -1$. We denote by $L^2_\lambda(\Pi) = L^2(\Pi; dA_\lambda)$ the space of square integrable functions on Π with respect to the measure dA_λ. Then the Bergman space $\mathcal{A}^2_\lambda(\Pi)$, known also as Bergman-Jerbashian space, (see [6, 7, 20, 21]), is the subspace of holomorphic functions in $L^2_\lambda(\Pi)$. Note that $\lambda = 0$ corresponds to the unweighted case. The corresponding orthogonal projection B^λ_Π, from $L^p_\lambda(\Pi)$ onto $\mathcal{A}^p_\lambda(\Pi)$, is given by the formula

$$B^\lambda_\Pi f(z) = \int_\Pi K_\lambda(z, w) f(w)\, dA_\lambda(w) = -\frac{1}{\pi}\int_\Pi \frac{f(w)}{(z - \overline{w})^{2+\lambda}}\, dA_\lambda(w),\ z \in \Pi,$$

and is bounded for $1 < p < \infty$.

In this study, we consider Toeplitz operators with symbols $g = g(2y)$, where $y = \Im z$, acting on $\mathcal{A}^2_\lambda(\Pi)$. These operators may be unbounded, but anyway, at least for $g \in L^1_\lambda(\Pi)$ they are densely defined by the rule

$$T^\lambda_g f = B^\lambda_\Pi g f. \tag{1}$$

We refer to the books [10, 15, 24–26] for a general modern theory of Bergman type spaces and operators on Bergman type spaces. More specifically, a very comprehensive study of special classes of Toeplitz operators, including the class in which we are interested in this paper, is presented in N.Vasilevski's monograph [24].

Using the well-known structural properties of $\mathcal{A}^2_\lambda(\Pi)$ in [11] (see also [12] and [24]), the Toeplitz operator T^λ_g with vertical symbol g is reduced to the operator of multiplication $\mathcal{M}^\lambda_g = \gamma^\lambda_g(x)I$ which acts on the space $L^2(\mathbb{R}_+)$. More details will be given by Theorem 2 below.

We shall exploit this idea to study the product (in a sense of composition) of two Toeplitz operators with vertical symbols. We recall our main problem which is: given two Toeplitz operators T^λ_a and T^λ_b is there a symbol h such that $T^\lambda_a T^\lambda_b = T^\lambda_h$? It appears that the solution to this problem is closely related to Laplace transform techniques and the theory of fractional integrodifferentiation in the general weighted cases.

It is worth mentioning here that products of Toeplitz operators in different holomorphic spaces have been studied extensively since the last two decades. Nevertheless and despite all the partial results obtained by different authors, we are still far from a complete answer to the question when the product of two Toeplitz operators is another Toeplitz operators. For more thorough treatments of this subject, the reader might refer to the following references [1–5, 8, 9, 14, 16–18].

Recently, a particular attention was paid to the so-called quasihomogeneous Toeplitz operators. A Toeplitz operator T_g is said to be quasihomogeneous if its symbol g can be written under the form $g(re^{i\theta}) = e^{ip\theta}\phi(r)$, where p is an integer and ϕ is a radial function. Quasihomogeneous symbols are considered to be a generalization of the class of radial symbols (i.e. when $p = 0$). Such operators act on the orthogonal basis of the Bergman space of the unit disk as a shift operator with holomorphic weight. Various promising results were obtained for this class of operators. We refer the reader to [4, 14, 16–19]. Furthermore, in [13] a partial answer to whether there exists or not a symbol h such that $T_a^\lambda T_b^\lambda = T_h^\lambda$ was given in the case of the unit ball of $\mathbb{C}$, was obtained by Fourier analysis and the Wiener ring theory. Vertical Toeplitz operators certainly admit realization in the framework of the unit disk (see Sect. 2), but the corresponding symbols will not be radial (or quasihomogeneous) functions. At our best knowledge, we are not aware of any manuscript in the current literature dealing with the product of two vertical Toeplitz operators in the specific context of the mentioned above main problem.

The paper is organized as follows. In Sect. 2 we collect auxiliary facts. Section 3 is devoted to the specified above main problem. For the sake of clarity, we consider the unweighted and weighted cases separately, the former being a simplification of the latter. Finally, we state some open questions.

2 Auxiliary Statements and Definitions

2.1 The Class $AC^k(\alpha, \beta)$

By $AC^1(\alpha, \beta) \equiv AC(\alpha, \beta)$ we denote the class of absolutely continuous functions on the interval (α, β). It is known that $f \in AC^1(\alpha, \beta)$ if and only if it is a primitive of a Lebesgues integrable function on (α, β). By $AC^k(\alpha, \beta)$, $k = 2, 3, \ldots,$ we denote the class of continuously differentiable up to the order $k - 1$ functions f on (α, β) with $f^{(k-1)} \in AC^1(\alpha, \beta)$. The following known fact sheds a light on the functions from $AC^k(\alpha, \beta)$.

Lemma 1 ([23], Lemma 2.4) *The class $AC^k(\alpha, \beta)$ consists only of functions f for which the following representation holds*

$$f(t) = \frac{1}{(k-1)!} \int_0^t (t-\tau)^{k-1}\varphi(\tau)d\tau + \sum_{j=0}^{k-1} C_j (t-\alpha)^j,$$

for some $\varphi \in L^1(\alpha, \beta)$, and some constants C_j.

2.2 On Spectral Representation of Toeplitz Operators with Vertical Symbols

In order to simplify formulas here and in what follows, we take $g = g(2y)$ for the (so-called vertical) symbol of Toeplitz operator T_g^λ acting on $\mathcal{A}^2_\lambda(\Pi)$. In [11] (see also [12] and [24]), it was shown that any such Toeplitz operator T_g^λ with vertical symbol g can be reduced to the operator of multiplication by the function

$$\gamma_g^\lambda(x) = \frac{x^{1+\lambda}}{\Gamma(1+\lambda)} \int_0^\infty g(t)\, t^\lambda\, e^{-xt} dt, \quad x \in \mathbb{R}_+, \tag{2}$$

which acts on the space $L^2(\mathbb{R}_+)$. The closure of the range of this function provides the spectrum for the corresponding Toeplitz operator. Precisely, the following theorem holds.

Theorem 2 ([11]) *Toeplitz operator T_g^λ with the symbol $g = g(2y)$ acting on $\mathcal{A}^2_\lambda(\Pi)$, is unitary equivalent to the operator of multiplication $\mathcal{M}_g^\lambda = \gamma_g^\lambda I$, acting on $L^2(\mathbb{R}_+)$. Moreover,* $\mathrm{sp} T_g^\lambda = \overline{\{\gamma \in \mathbb{C} : \gamma = \gamma_g^\lambda(x),\ \text{all } x \in \mathbb{R}_+^1\}}$.

Using Theorem 2, we can understand the action of the Toeplitz operator (1) on $L^2_\lambda(\Pi)$ considering it as being unitary equivalent to the operator of multiplication by $\gamma_g^\lambda I$, acting on $L^2(\mathbb{R}_+)$. Thus we may consider bounded and unbounded Toeplitz operators. However, we prefer to deal with operators which are initially bounded on $L^2_\lambda(\Pi)$. We shall specify the corresponding assumptions in the beginning of Sect. 3.

Recall that vertical Toeplitz operators have the following realization in the framework of the unit disk (see [24]). Let $\mathbb{D} = \{z = x + iy : |z| < 1\}$ stand for the unit disk, and $\partial\mathbb{D}$ stand for its boundary (unit circle). Consider the space

$$L^2_\lambda(\mathbb{D}) = L^2(\mathbb{D}, (\lambda+1)(1-|z|^2)^\lambda \frac{1}{\pi} dxdy), \quad \text{where} \quad z = x + iy,$$

and let $\mathcal{A}^2_\lambda(\mathbb{D})$ be the corresponding Bergman space. For a given point $z_0 \in \partial\mathbb{D}$, consider all Euclidean circles tangent to $\partial\mathbb{D}$ at z_0. Consider the class of all Toeplitz operators $T_{\widehat{g}}^\lambda$ acting on $L^2_\lambda(\mathbb{D})$ with the symbols $\widehat{g}$ which are constant on the above mentioned circles. All such classes are reduced (up to a rotation) to the class of operators corresponding to $z_0 = i$. The conformal map $z = \Phi(w) = \frac{w-i}{1-iw}$, from Π onto $\mathbb{D}$, maps the lines $\{z \in \Pi : z = x + iy_0,\ x \in \mathbb{R},\ y_0 > 0 \text{ is fixed}\}$ into the circles tangent to $\partial\mathbb{D}$ at the point $z_0 = i$. The unitary operator $U : L^2_\lambda(\mathbb{D}) \to L^2_\lambda(\Pi)$

$$Uf(z) = \left(\frac{\sqrt{2}}{1-iw}\right)^{2+\lambda} f \circ \Phi(w), \tag{3}$$

provides the following relation

$$T^{\lambda}_{g\circ\Phi} = U^{-1} T^{\lambda}_{g} U \tag{4}$$

between the Toeplitz operator T^{λ}_{g} with vertical symbol acting on $L^2_{\lambda}(\Pi)$ and the Toeplitz operators $T^{\lambda}_{\widehat{g}}$ acting on $L^2_{\lambda}(\mathbb{D})$ with the symbol $\widehat{g} = g \circ \Phi$ which is constant on the mentioned above circles. We note that the inverse map is given by the relation:

$$U^{-1}\varphi(w) = \left(\frac{\sqrt{2}}{1+iw}\right)^{2+\lambda} \varphi \circ \Psi(w), \quad \Psi(w) = \frac{w+1}{1+iw}.$$

2.3 On Fractional Integrodifferentiation and Laplace Transform

For fractional integrodifferentiation we refer to the books [22, 23]. We will use fractional integrals on the whole real axis which are defined by

$$I^{\alpha}_{+}\varphi(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{\varphi(t)}{(x-t)^{1-\alpha}} dt, \quad x \in \mathbb{R}, \tag{5}$$

$$I^{\alpha}_{-}\varphi(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{\varphi(t)}{(t-x)^{1-\alpha}} dt, \quad x \in \mathbb{R}, \tag{6}$$

or as for the convolution

$$I^{\alpha}_{\pm}\varphi(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha-1} \varphi(x \mp t) dt, \quad x \in \mathbb{R}. \tag{7}$$

Fractional integrals $I^{\alpha}_{\pm}$ are defined for functions $\varphi \in L^p(\mathbb{R})$ if $0 < \Re\alpha < 1$ and $1 < p < \frac{1}{\Re\alpha}$. It is known that

$$I^{\alpha}_{\pm} e^{\pm\theta x} = \theta^{-\alpha} e^{\pm\theta x}, \quad \text{for } \Re\theta > 0 \text{ and } \Re\alpha > 0. \tag{8}$$

We will also need the following fractional integration on the half-axis

$$I^{\alpha}_{0+}\varphi(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} \frac{\varphi(t)}{(x-t)^{1-\alpha}} dt, \quad x \in \mathbb{R}_{+}. \tag{9}$$

The Laplace transform of a function φ, defined for all real numbers $t \geqslant 0$, is the function $\mathcal{L}\varphi$, given by:

$$\mathcal{L}\varphi(z) = \int_{0}^{\infty} \varphi(t)\, e^{-zt}\, dt, \quad z = x + iy. \tag{10}$$

The meaning of $\mathcal{L}\varphi$ depends on the class of functions of interest. A necessary condition for existence of the integral is that φ must be locally integrable on $\mathbb{R}^1_+$. For locally integrable φ that is of exponential type, the integral can be understood as a (proper) Lebesgue integral. For instance, if $|\varphi(t)| \leqslant \alpha e^{\beta t}$ for some nonnegative constants α, β and all $t > s_0$, then the Laplace transform $\mathcal{L}\varphi$ is correctly defined as a function in $\{z = x + iy \in \mathbb{C} : x > \beta\}$. The Laplace convolution product of two functions φ and ψ is defined by the integral

$$\varphi \circ \psi(x) = \int_0^x \varphi(t)\psi(x-t)dt, \quad x \in \mathbb{R}_+,$$

so that the Laplace transform of the convolution is given by

$$\mathcal{L}\left(\varphi \circ \psi\right)(z) = \left(\mathcal{L}\varphi\right)(z)\left(\mathcal{L}\psi\right)(z),$$

as long as the objects in the above formula exist.

3 Product of Vertical Toeplitz Operators

3.1 Statement of the Main Problem

We formulate the main problem as follows. Given two vertical Toeplitz operators T_a^λ and T_b^λ bounded on $\mathcal{A}^2_\lambda(\Pi)$ with the symbols a and b, find a function (symbol) h such that

$$T_a^\lambda T_b^\lambda = T_h^\lambda. \tag{11}$$

Here and everywhere below we consider operator relations on a dense set of polynomials in $\mathcal{A}^2_\lambda(\Pi)$. In view of the Theorem 2, the above problem is equivalent to the problem of finding a function h such that

$$\gamma_a^\lambda(x)\gamma_b^\lambda(x) = \gamma_h^\lambda(x), \quad x \in \mathbb{R}^1_+. \tag{12}$$

Here and in what follows, we impose the following admissibility conditions on a symbol g of vertical Toeplitz operator T_g^λ :

(i) $g(t)t^\lambda \in L^1(0, B)$, for all positive constants B;
(ii) for any $\varepsilon > 0$ there exist $A_\varepsilon \geqslant 0$, $t_\varepsilon > 0$, such that $|g(t)| \leqslant A_\varepsilon e^{\varepsilon t}$, all $t \geqslant t_\varepsilon$.
(iii) Toeplitz operator T_g^λ is bounded on $\mathcal{A}^2_\lambda(\Pi)$.

We say that a symbol g of a Toeplitz operator T_g^λ is admissible provided g satisfies the above stated conditions (i)–(iii).

3.2 Unweighted Case ($\lambda = 0$)

We start with the unweighted case $\lambda = 0$. For the unweighted case we will avoid using the index "0" in the notation and simply write T_g, γ_g, $\mathcal{A}^2(\Pi)$, instead of T_g^0, γ_g^0, $\mathcal{A}_0^2(\Pi)$.

Theorem 3 *Let T_a and T_b be vertical Toeplitz operators on $\mathcal{A}^2(\Pi)$ with admissible symbols a and b. If there exists a function h on $\mathbb{R}_+^1$ satisfying (i)–(ii) and such that*

$$\int_0^t \left[h(\tau) - a(\tau)b(t-\tau)\right] d\tau = 0, \quad \text{all} \;\; t \in \mathbb{R}_+^1, \tag{13}$$

then

$$T_a T_b = T_h. \tag{14}$$

Proof In view of Theorem 2, for a Toeplitz operator T_g, we consider the corresponding function γ_g given by

$$\begin{aligned}\gamma_g(x) &= x \int_0^\infty g\,(t)\, e^{-xt} dt = x^2 \int_0^\infty g(t) \int_t^\infty e^{-x\tau} d\tau \\ &= x^2 \int_0^\infty e^{-x\tau} d\tau \int_0^\tau g(t) dt = x^2 \int_0^\infty \widetilde{g}(\tau)\, e^{-x\tau} d\tau,\end{aligned}$$

where we denoted

$$\widetilde{g}(t) = \int_0^t g(\tau) d\tau.$$

Hence, the relation

$$\gamma_a(x)\gamma_b(x) = \gamma_h(x), \quad x \in \mathbb{R}_+^1,$$

becomes

$$\left(\int_0^\infty a(t)e^{-xt}dt\right)\left(\int_0^\infty b(t)e^{-xt}dt\right) = \int_0^\infty \widetilde{h}(t)e^{-xt}dt, \quad x \in \mathbb{R}_+^1,$$

which, in terms of Laplace transform, reads as

$$\left(\mathcal{L}a\right)(x)\left(\mathcal{L}b\right)(x) = \left(\mathcal{L}\widetilde{h}\right)(x), \quad x \in \mathbb{R}_+^1.$$

It suffices to check the above equality for $\widetilde{h}(t) = \int_0^t h(\tau)d\tau$. In virtue of (13) we have for such $\widetilde{h}$ and for $x \in \mathbb{R}^1_+$,

$$\left(\mathcal{L}\widetilde{h}\right)(x) = \int_0^\infty \widetilde{h}(t)e^{-xt}dt = \int_0^\infty e^{-xt}dt \int_0^t h(\tau)d\tau$$
$$= \int_0^\infty e^{-xt}dt \int_0^t a(\tau)b(t-\tau)d\tau = \int_0^\infty a(\tau)d\tau \int_\tau^\infty b(t-\tau)e^{-xt}dt$$
$$= \int_0^\infty a(\tau)e^{-x\tau}d\tau \int_0^\infty b(t)e^{-xt}dt = \left(\mathcal{L}a\right)(x)\left(\mathcal{L}b\right)(x), \quad x \in \mathbb{R}^1_+,$$

where the change of order of integration is justified by Fubini's theorem. This formula, in view of Theorem 2, proves the statement of the theorem. □

Theorem 4 *Let T_a and T_b be vertical Toeplitz operators on $\mathcal{A}^2(\Pi)$ with admissible symbols a and b. Assume either a or b is differentiable and has finite limit value at the origin (for instance, let it be the function b). Then the function h defined by*

$$h(t) = a(t)b(0) + \int_0^t a(\tau)\, b'(t-\tau)\, d\tau$$

is such that

$$T_a T_b = T_h.$$

Proof By differentiating equation (13) we obtain

$$h(t) = \widetilde{h}'(t) = a(t)b(0) + \int_0^t a(\tau)\, b'(t-\tau)\, d\tau.$$

This formula, in view of Theorem 2, proves the statement of the theorem. □

3.3 Weighted Case $\lambda > -1$

Lemma 5 *If a function g satisfies (i)–(ii), then the following relation holds*

$$\gamma_g^\lambda(x) = \left(\frac{x^{1+\lambda}}{\Gamma(1+\lambda)}\right)^2 \int_0^\infty \widetilde{g}_\lambda(\xi)e^{-x\xi}d\xi, \quad x \in \mathbb{R}^1_+,$$

where

$$\widetilde{g}_\lambda(t) \equiv \int_0^t \frac{g(\xi)\xi^\lambda}{(t-\xi)^{-\lambda}}d\xi = \Gamma(1+\lambda)\left(I_{0+}^{1+\lambda} g(\xi)\xi^\lambda\right)(t). \quad (15)$$

Proof We start with the integral on the right side of (2). Taking into account equation (8), we have

$$\begin{aligned}\int_0^\infty g(t)t^\lambda e^{-xt}dt &= x^{1+\lambda}\int_0^\infty g(t)t^\lambda \frac{1}{x^{1+\lambda}}e^{-xt}dt\\ &= x^{1+\lambda}\int_0^\infty g(t)t^\lambda \left(I_-^{1+\lambda}e^{-x\xi}\right)(t)dt\\ &= x^{1+\lambda}\int_0^\infty g(t)t^\lambda \left(\frac{1}{\Gamma(1+\lambda)}\int_t^\infty \frac{e^{-x\xi}}{(\xi-t)^{-\lambda}}d\xi\right)dt\\ &= \frac{x^{1+\lambda}}{\Gamma(1+\lambda)}\int_0^\infty e^{-x\xi}d\xi\int_0^\xi \frac{g(t)t^\lambda}{(\xi-t)^{-\lambda}}dt\\ &= \frac{x^{1+\lambda}}{\Gamma(1+\lambda)}\int_0^\infty \widetilde{g}_\lambda(\xi)e^{-x\xi}d\xi.\end{aligned}$$

The change of order of integration is justified by Fubini's theorem. Comparing the obtained formula with (2) completes the proof.

□

Theorem 6 *Let* $\lambda > -1$ *and let* T_a^λ *and* T_b^λ *be vertical Toeplitz operators on* $\mathcal{A}_\lambda^2(\Pi)$ *with admissible symbols* a *and* b. *If there exists a function* h *on* $\mathbb{R}_+^1$ *satisfying* (i)–(ii) *and such that*

$$\int_0^t \left[h(\tau) - a(\tau)b(t-\tau)\right]\tau^\lambda(t-\tau)^\lambda d\tau = 0,\quad t\in\mathbb{R}_+^1, \tag{16}$$

then

$$T_a^\lambda T_b^\lambda = T_h^\lambda. \tag{17}$$

Proof In view of Lemma 5, (12), i.e.,

$$\gamma_a^\lambda(x)\gamma_b^\lambda(x) = \gamma_h^\lambda(x),\quad x\in\mathbb{R}_+^1,$$

becomes

$$\left(\mathcal{L}a(t)t^\lambda\right)(x)\left(\mathcal{L}b(t)t^\lambda\right)(x) = \Gamma(1+\lambda)\left(\mathcal{L}\left(I_{0+}^{1+\lambda}h(\xi)\xi^\lambda\right)(t)\right)(x).$$

It suffices to check the above equality for h satisfying (16). We have for $x \in \mathbb{R}^1_+$,

$$\begin{aligned}\Gamma(1+\lambda)\left(\mathcal{L}\left(I_{0+}^{1+\lambda}h(\xi)\xi^{\lambda}\right)(t)\right)(x) &= \int_0^{\infty}\left(I_{0+}^{1+\lambda}h(\xi)\xi^{\lambda}\right)(t)e^{-xt}dt\\ &= \int_0^{\infty}\left(I_{0+}^{1+\lambda}a(\xi)b(t-\xi)\xi^{\lambda}\right)(t)e^{-xt}dt\\ &= \int_0^{\infty}e^{-xt}dt\int_0^t a(\tau)\tau^{\lambda}\, b(t-\tau)(t-\tau)^{\lambda}\, d\tau\\ &= \left(\mathcal{L}a\right)(x)\left(\mathcal{L}b\right)(x), \quad x \in \mathbb{R}^1_+,\end{aligned}$$

where the change of order of integration is justified by Fubini's theorem.This formula, in view of Theorem 2, proves the statement of the theorem. □

In what follows, we shall give necessary and sufficient conditions for the product of two vertical Toeplitz operators to be again a vertical Toeplitz operator using Riemann–Liouville operators. For simplicity, we introduce the following notations:

$$\begin{aligned}\varphi_{\lambda}(t) &= h(t)t^{\lambda},\\ f_{\lambda}(t) &= \frac{1}{\Gamma(1+\lambda)}\int_0^t a(\tau)\tau^{\lambda}\, b(t-\tau)(t-\tau)^{\lambda}\, d\tau.\end{aligned}$$

First, we consider the case $-1 < \lambda < 0$.

Theorem 7 *Let $-1 < \lambda < 0$ and let T_a^{λ} and T_b^{λ} be vertical Toeplitz operators on $\mathcal{A}^2_{\lambda}(\Pi)$ with admissible symbols a and b. There exists admissible symbol h such that*

$$T_a^{\lambda}T_b^{\lambda} = T_h^{\lambda}$$

if and only if

$$I_{0+}^{-\lambda}f_{\lambda}(t) = \frac{1}{\Gamma(-\lambda)}\int_0^t \frac{f_{\lambda}(\tau)}{(t-\tau)^{1+\lambda}}d\tau \in AC(0,B), \text{ for any } B>0, \text{ and} \tag{18}$$

$$I_{0+}^{-\lambda}f_{\lambda}(0) = \frac{1}{\Gamma(-\lambda)}\int_0^t \frac{f_{\lambda}(\tau)}{(t-\tau)^{1+\lambda}}d\tau\bigg|_{t=0} = 0. \tag{19}$$

Proof As in Theorem 6, we see that our problem reduces to the well-known Abel equation

$$I_{0+}^{1+\lambda}\varphi(t) = f(t),$$

considered on an arbitrary interval $(0, B)$, $B > 0$, and where we should replace φ with φ_λ and f with f_λ :

$$I_{0+}^{1+\lambda}\varphi_\lambda(t) = f_\lambda(t), \quad t \in (0, B).$$

Certainly, if we solve the Abel equation with $\varphi = \varphi_\lambda$ and $f = f_\lambda$ for an arbitrary $B > 0$, then we recover the function h. It is known that if the solution φ of the Abel equation exists, then it must be of the form

$$\varphi(t) = \frac{1}{\Gamma(-\lambda)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{1+\lambda}} d\tau.$$

Thus the solution φ is unique for any arbitrary B, and hence it is uniquely defined on $\mathbb{R}_+^1$. Moreover, it is well-known that the Abel equation has a solution $\varphi \in L^1(0, B)$ (for any arbitrary $B > 0$) if and only if the conditions (18) and (19) are satisfied. Therefore, $\varphi_\lambda(t) = h(t)t^\lambda \in L^1(0, B)$ for arbitrary $B > 0$, and also satisfies (ii), and the operator identity $T_a^\lambda T_b^\lambda = T_h^\lambda$, considered on a dense set of polynomials in $\mathcal{A}_\lambda^2(\Pi)$, is valid by construction. Hence, h generates bounded operator which means that the symbol h is admissible. This finishes the proof. □

Remark 1 If $f \in AC(\alpha, \beta)$, then $I_{0+}^{\theta} f \in AC(\alpha, \beta)$, for any $\theta \in (0, 1)$. Therefore, the condition $f_\lambda \in AC(0, B)$ is sufficient for the validity of the condition (18) (i.e. for $I_{0+}^{-\lambda} f_\lambda \in AC(0, B)$).

Since the unweighted case $\lambda = 0$ was considered separately at the beginning of this section, we therefore turn our attention to the case $\lambda > 0$. We shall denote by $[\lambda]$ the entire part of λ.

Theorem 8 *Let $\lambda > 0$ and let T_a^λ and T_b^λ be vertical Toeplitz operators on $\mathcal{A}_\lambda^2(\Pi)$ with admissible symbols a and b. Then there exists an admissible symbol h such that*

$$T_a^\lambda T_b^\lambda = T_h^\lambda$$

if and only if

$$I_{0+}^{1+[\lambda]-\lambda} f_\lambda(t) = \frac{1}{\Gamma(1+[\lambda]-\lambda)} \int_0^t \frac{f_\lambda(\tau)}{(t-\tau)^{-[\lambda]+\lambda}} d\tau \in AC^{1+[\lambda]}(0, B), \tag{20}$$

for any $B > 0$, and

$$\left(\frac{d}{dt}\right)^k I_{0+}^{1+[\lambda]-\lambda} f_\lambda(0) = \frac{1}{\Gamma(1+[\lambda]-\lambda)} \left(\left(\frac{d}{dt}\right)^k \int_0^t \frac{f_\lambda(\tau)}{(t-\tau)^{-[\lambda]+\lambda}} d\tau \right)\Bigg|_{t=0} = 0,$$

for $k = 0, 1, \ldots, [\lambda]$.

Proof The proof follows the lines of the proof of Theorem 7 with the use of Theorem 2.3 from [23]. We leave it to the reader. □

Remark 2 The condition $f_\lambda \in AC^{1+[\lambda]}(0, B)$ is sufficient for the validity of the condition (20) (i.e. for $I_{0+}^{1+[\lambda]-\lambda} f_\lambda \in AC^{1+[\lambda]}(0, B)$).

4 Conclusion

We strongly believe that the technique developed here will help in studying particular problems and questions related to the product of vertical Toeplitz operators, such as the existence of Brown-Halmos type theorem i.e., a description of symbols a and b such that $T_a^\lambda T_b^\lambda = T_{ab}^\lambda$ or the nonzero zero divisor i.e., are there nonzero symbols a and b such that $T_a^\lambda T_b^\lambda = 0$? Partial results to these two problems were obtained for some specific classes of Toeplitz operators (e.g. quasihomogeneous Toeplitz operators) in the unit disk, the ball of $\mathbb{C}^n$ and other domains (see the references provided below). However, we believe that for the case of vertical Toeplitz operators more challenging computations may occur.

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References

1. P. Ahern, On the range of the Berezin transform, J. Funct. Anal.215, No. 1, 206–216 (2004).
2. P. Ahern and Ž. Čučković, Products of Toeplitz operators on the Bergman space, Illinois J. Math.45 (1) (2001), 113–121.
3. P. Ahern and Ž. Čučković, A theorem of Brown-Halmos type for Bergman spaceToeplitz operators, J. Funct. Anal.187 (2001), 200–210.
4. P. Ahern and Ž. Čučković, Some examples related to the Brown-Halmos theorem for the Bergman space, Acta Sci. Math. (Szeged) 70 (2004), 373–378.
5. S. Axler, Toeplitz operators, http://www.axler.net/ToeplitzOperators.pdf
6. Bergman S. Uber die kernfunctionen eines bereiches und verhalten am rande, I. J. Reine Agnew. Math. 1933;169:1–42.
7. Bergman S. The kernel function and conformal mapping. 2nd ed. Vol.5 AMS, Providence (RI): Mathematical Surveys and Monographs; 1970.
8. A. Brown and P. Halmos, Algebraic properties of Toeplitz operators, J. reine angew. Math., 213 (1964), 89–102.
9. Ž. Čučković and N. V. Rao, Mellin transform, monomial symbols and commuting Toeplitz operators. J. Funct. Anal.154 (1), 195–214 (1998).
10. Duren P., Schuster A. Bergman spaces. Vol. 100. Providence (RI): Mathematical Surveys and Monographs; 2004.
11. Grudsky S.M., Karapetyants A.N., Vasilevski N.L. Dynamics of properties of Toeplitz operators on the upper half plane: parabolic case. J. Operator Theory. 2004. Vol. 52, no 1. pp. 185–214.
12. Grudsky S.M., Karapetyants A.N., Vasilevski N.L. Dynamics of properties of Toeplitz operators in the weighted Bergman spaces. Siberian Electronic Mathematical Reports. 2006;3:362–383.

13. Grudsky S.M., Karapetyants A.N., Vasilevski N.L. Toeplitz operators on the unit ball in $\mathbb{C}^n$ with redial symbols. Journal of Operator Theory, Vol. 49, No. 2 (Spring 2003), pp. 325–346
14. S. Grudsky and N. Vasilevski, Bergman-Toeplitz operators: Radial component influence, Integral Equation Operator Theory 40 (2001), 16–33.
15. Hedenmalm H., Korenblum B., Zhu K. Theory of Bergman spaces. New York: Springer Verlag, Inc.; 2000.
16. I. Louhichi, Powers and roots of Toeplitz operators, Proc. Amer. Math. Soc.135 (2007), no. 5, 1465–1475.
17. I. Louhichi and N. V. Rao, Roots of Toeplitz operators on the Bergman space, Pacific J. Math.252 (2011), no. 1, 127–144.
18. I. Louhichi, E. Strouse, L. Zakariasy, Products of Toeplitz operators on the Bergman space, Integral Equations Operator Theory 54 (2006), no. 4, 525–539.
19. I. Louhichi, N. V. Rao, and A. Yousef, Two questions on products of Toeplitz operators on the Bergman space, Complex Anal. Oper. Theory 3, No. 4, 881–889 (2009).
20. Jerbashian M.M. On canonical representation of functions meronorphic in the unit disc. DAN of Armenia. 1945;3:3–9.
21. Jerbashian M.M. On the representability problem for analytic functions. Soobsh. Inst. Math. and Mech. AN of Armenia. 1948;2:3–40.
22. V.S. Kiryakova, Generalized Fractional Calculus and Applications. Longman Sci. Techn. and John Wiley and Sons, Harlow-N. York (1994).
23. S. Samko, A. Kilbas, O. Marichev, Fractional Integrals and Derivatives, Gordon and Breach Science Publishers, 1993.
24. Vasilevski N.L. Commutative Algebras of Toeplitz Operators on the Bergman Space. Vol.185: Operator Theory: Advances and Applications; 2008.
25. Zhu K. Operator theory in function spaces. Marcel Dekker, New York: Monographs and textbooks in pure and applied mathematics; 1990.
26. Zhu K. Spaces of holomorphic functions in the unit ball. Graduate texts in Mathematics, Springer 2004.

Toeplitz Operators with Radial Symbols on Weighted Holomorphic Orlicz Space

Alexey Karapetyants and Jari Taskinen

Dedicated to the occasion of 70th birthday of Professor N. Vasilevski.

Abstract We consider a class of Toeplitz operators with special radial symbols on weighted holomorphic Orlicz space. For an operator from such class we prove necessary and sufficient conditions of boundedness on holomorphic Orlicz space in terms of behaviour of averages of the radial symbol of the operator. In the case when either symbol or its certain average is nonnegative, we obtain characterization for boundedness of the corresponding Toeplitz operator.

1 Introduction

The weighted holomorphic Orlicz space $\mathcal{A}_\lambda^\Phi(\mathbb{D})$ on the unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$ is defined to consists of the functions from the weighted Orlicz space $L_\lambda^\Phi(\mathbb{D})$ which are also holomorphics in $\mathbb{D}$ (for details see Sect. 2). This is a direct generalization of the weighted classical Bergman space, sometimes called Bergman-Jerbashian space. We refer to the books [4, 8, 24, 27, 28] for a general modern theory of Bergman type spaces and operators on Bergman type spaces. For the definition of Orlicz space and some properties we refer to the corresponding section in this paper and also to the books [9, 13, 14, 18, 19].

Recently, the study of classical operators of complex analysis, such as Toeplitz, Hankel and composition operators, has attracted considerable attention in the

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framework of the holomorphic Orlicz spaces and their modification, see e.g. [1, 2, 10–12, 23].

Toeplitz operators form an important and much studied subclass of the classical operators. Given a symbol $a \in L^1_\lambda(\mathbb{D})$ on the unit disc and $\lambda > -1$, the Toeplitz operator T^λ_a is defined by the formula

$$T^{(\lambda)}_a f = P_\lambda(af), \tag{1.1}$$

where

$$P_\lambda f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^{2+\lambda}} dA_\lambda(w) \ , \quad z \in \mathbb{D}, \tag{1.2}$$

is the classical Bergman projection; here, $dA_\lambda = (1 - |z|^2)^\lambda dA$ and dA is the area measure on the complex plane normalized so that the measure of the disc equals 1. For an arbitrary $a \in L^1_\lambda(\mathbb{D})$ the operator $T^{(\lambda)}_a$ may fail to map the standard weighted Bergman space $\mathcal{A}^p_\lambda(\mathbb{D})$ into itself but anyway it is densely defined, namely on the subspace $H^\infty(\mathbb{D})$ of bounded analytic functions.

The question of characterizing symbols $a \in L^1_\lambda(\mathbb{D})$ such that the corresponding Toeplitz operator is bounded in $\mathcal{A}^p_\lambda(\mathbb{D})$ is still open even in the simplest case $p = 2$, $\lambda = 0$. There are however special cases, when a satisfactory or even complete answer is known. Bounded Toeplitz operators with positive symbols on $\mathcal{A}^2(\mathbb{D})$ and more general spaces were characterized in [17] and [26]. An extensive study of Toelpitz operators with (unbounded) radial symbols can be found in the [5–7], see also the monograph by Vasilevski, [24]. Zorboska [29] considered $L^1(\mathbb{D})$-symbols that satisfy the condition of bounded mean oscillation and determined the bounded Toeplitz operators in terms of the boundary behavior of the Berezin transform of their symbols.

The purpose of this work is to extend to weighted Orlicz space $\mathcal{A}^\Phi_\lambda(\mathbb{D})$ setting certain results concerning the boundedness of $T^{(\lambda)}_a$ for radial symbols. It is well known (see Theorem 7.5 in [28]), that the boundedness can be characterized in the case of standard weighted Bergman spaces and positive symbols. In [15], generalizing the results in [5, 24], the positivity requirement was considerably relaxed by a much weaker assumption on the positivity of certain repeated integrals of the (radial) symbol. The proof used certain estimates of the kernel of the Berezin transform. Here, we will give a further generalization of this argument to the case of Orlicz spaces, see Theorems 3.3, 3.4 and 3.5, where the result is formulated as a characterization of the boundedness of the Toeplitz operator and its Berezin transform.

The paper is organized as follows. In Sect. 2 we give some necessary definitions and collect auxiliary results. Section 3 is devoted to our main results. In Theorem 3.3 we give sufficient conditions for boundedness of Toeplitz operator with radial symbol in holomorphic Orlicz space in terms of behaviour of averages of the symbol under some additional conditions on Young function, which, roughly speaking, allow power-like Young functions. We also notice that in that case the Berezin transform of the Toeplitz operator is a bounded function. Without these power-like additional conditions for Young function, but under condition of positivity either of a symbol or a certain average of the symbol we prove the necessity: i.e., boundedness of Toeplitz operator and finiteness of Berezin transform imply certain behavior of average, see Theorem 3.4. In Theorem 3.5 we collect the above information in one statement, so presenting a characterization for boundedness of Toeplitz operators with radial symbols and with a condition that either symbol or certain average of this operator is a nonnegative function. We also present a reformulation for the weighted Lebesgue case, i.e. when the Young function is given by $\Phi(t) = t^p$.

2 Preliminaries

2.1 Notations

Given $1 \le p \le \infty$ and $-1 < \lambda < \infty$ we denote by $L^p_\lambda(\mathbb{D})$ the standard Lebesgue space of complex valued functions on the disc $\mathbb{D}$ that are p—integrable with respect to the measure dA_λ. By $\mathcal{A}^p_\lambda(\mathbb{D})$ we denote the standard weighted Bergman space, which is the closed subspace of $L^p_\lambda(\mathbb{D})$ consisting of analytic functions in $\mathbb{D}$. The index λ is suppressed from the notation, if it equals 0. By $C_0^\infty(\mathbb{D})$ we denote the space of compactly supported C^∞-functions on the disc. Given an analytic function f and $m \in \mathbb{N} = \{1, 2, 3, \ldots\}$ we write $f^{(m)}$ for the m-th complex derivative of f. By C, C_1 etc. we denote generic positive constant(s), the value of which may change from place to place but not in the same group of inequalities. We use the Landau notation so that for example "$f(r) = O(g(r)), \quad r \to 1$" means that the quantity $f(r)/g(r)$ remains bounded in the limit $r \to 1$.

2.2 Orlicz Spaces

Let us recall the definition of the weighted Orlicz space $L^\Phi_\lambda(\mathbb{D})$ and some properties of Young functions Φ. For more details we refer the reader to the books [9, 13, 14, 18, 19]. Let $\Phi : [0, \infty] \to [0, \infty]$ be a Young function, i.e., a convex function such that $\Phi(0) = 0$, $\lim_{x\to\infty} \Phi(x) = \Phi(\infty) = \infty$. From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. To each Young function Φ one identifies the complementary function Ψ, which possesses the same properties, by the rule $\Psi(y) = \sup_{x\geqslant 0}\{xy - \Phi(x)\}$. Note that

$$t \leqslant \Phi^{-1}(t)\Psi^{-1}(t) \leqslant 2t \quad \text{for all } t \geqslant 0. \tag{2.1}$$

Let $L^\Phi_\lambda(\mathbb{D})$ be the weighted Orlicz space consisting of all measurable functions f on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} \Phi(k|f(z)|)dA_\lambda(z) < \infty, \quad \text{for some } k > 0.$$

The functional

$$N_\Phi(f) = \|f\|_{L^\Phi_\lambda(\mathbb{D})} = \inf\left\{\lambda > 0 : \int_{\mathbb{D}} \Phi\left(\frac{|f(z)|}{\lambda}\right) dA_\lambda(z) \leqslant 1\right\}$$

defines a norm in $L^\Phi_\lambda(\mathbb{D})$. By definition, $L^\Phi_\lambda(\mathbb{D})$ is a lattice, i.e.

$$\begin{aligned} & f \in L^\Phi_\lambda(\mathbb{D}) \text{ and } |g(z)| \le |f(z)| \text{ for a.e. } z \in \mathbb{D} \;\; \Rightarrow \\ & g \in L^\Phi_\lambda(\mathbb{D}) \text{ and } \|g\|_{L^\Phi_\lambda(\mathbb{D})} \le \|f\|_{L^\Phi_\lambda(\mathbb{D})}. \end{aligned} \tag{2.2}$$

We will need the following indices

$$\begin{aligned} p_\Phi &= \sup\{s > 0 : t^{-s}\Phi(t) \text{ is non} - \text{decreasing for t} > 0\}; \\ q_\Phi &= \inf\{s > 0 : t^{-s}\Phi(t) \text{ is non} - \text{increasing for t} > 0\}. \end{aligned}$$

These indices were used first by Yamamuro [25] (see also [16]).

The following density result is a reformulation of Theorem 3.7.15 of [9], where the unweighted version is presented, and it can be proved by the same arguments as in the reference.

Lemma 2.1 *Let Φ be a Young function and there exists $q < \infty$ such that $\frac{\Phi(t)}{t^q}$ is almost decreasing. Then the space $C_0^\infty(\mathbb{D})$ is a dense subspace of $L^\Phi_\lambda(\mathbb{D})$.*

Proof Our space $L^\Phi_\lambda(\mathbb{D})$ equals the space $L^\varphi(A, d\mu)$ in Proposition 3.5.1 of [9], when $\mathbb{D}$, Φ and dA_λ are taken as A, φ and $d\mu$, respectively. Thus, according to the citation, simple functions (see e.g. Section 3.5. of [9]) on the unit disc form a dense subspace $L^\Phi_\lambda(\mathbb{D})$. Furthermore, by classical, elementary arguments, simple functions can be approximated by functions of $C_0^\infty(\mathbb{D})$ simultaneously with respect to the norms of $L^p_\lambda(\mathbb{D})$ and $L^q_\lambda(\mathbb{D})$. Now, according to Lemma 3.7.7 of [9], the space $L^p_\lambda(\mathbb{D}) \cap L^q_\lambda(\mathbb{D})$ embeds continuously into $L^\Phi_\lambda(\mathbb{D})$. This yields the result. □

2.3 Classical Operators

Along with the weighted Bergman projection P_λ, we will use the following operators, which are modifications of the classical Bergman projections:

$$P_\lambda^+ f(z) = \int_{\mathbb{D}} \frac{f(w)}{|1 - z\overline{w}|^{2+\lambda}} dA_\lambda(w), \tag{2.3}$$

$$T_{a,b} f(z) = (1 - |z|^2)^a \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^{2+a+b}} dA_b(w), \tag{2.4}$$

$$S_{a,b} f(z) = (1 - |z|^2)^a \int_{\mathbb{D}} \frac{f(w)}{|1 - z\overline{w}|^{2+a+b}} dA_b(w). \tag{2.5}$$

Theorem 2.2 ([3, 20]) *Let $\lambda, \beta > -1$. Let Φ be a Young function and assume that $1 < p_\Phi \leqslant q_\Phi < \infty$ and $\lambda + 1 < p_\Phi(\beta + 1)$. Then*

1. P_β is bounded in $L^\Phi_\lambda(\mathbb{D})$;
2. P_β^+ is bounded in $L^\Phi_\lambda(\mathbb{D})$.

The Theorem 2.2 was proved in [3] (and generalized to the weighted setting in [20]) for the case of the Bergman projection and under some slightly different assumption for Young function. In the citations the authors use the so-called lower and upper indices, introduced and used first by Simonenko [21] in the context of interpolation and extrapolation of Orlicz spaces. Here we use different Yamamuro type indices p_Φ, q_Φ, and we do not assume continuity of Young function. But, the proof of this modified version is the same as the proof in [20], moreover it clearly holds for the maximal Bergman operator.

We also need some slight generalization of the above result.

Theorem 2.3 *Let Φ be a Young function and assume that $1 < p_\Phi \leqslant q_\Phi < \infty$. Let $a \geqslant 0$, $\lambda > -1$, $b > -1$, and $\lambda + 1 < p_\Phi(\beta + 1)$. Then*

1. $T_{a,b}$ is bounded in $L^\Phi_\lambda(\mathbb{D})$;
2. $S_{a,b}$ is bounded in $L^\Phi_\lambda(\mathbb{D})$.

Proof There exists p_0, $1 < p_0 < p_\Phi$, such that $-p_0 a < \lambda + 1 < p_0(b + 1)$. Certainly, there exists $p_1 > q_\Phi$ such that $-p_1 a < \lambda + 1 < p_1(b + 1)$. Therefore by Theorem 2.10 from [27] the operators $T_{a,b}$ and $S_{a,b}$ are bounded on $L^{p_0}_\lambda(\mathbb{D})$ and on $L^{p_1}_\lambda(\mathbb{D})$. The rest of the proof follows by interpolation using Theorem 6.5, part (c) from [16]. □

Theorem 2.4 *Let Φ be a Young function and assume that $1 < p_\Phi \leqslant q_\Phi < \infty$. Let $\lambda > -1$, $m \in \mathbb{N}$ and let f be an analytic function on $\mathbb{D}$ and denote*

$$g_m(z) = \sum_{k=0}^{m} (1 - |z|^2)^k f^{(k)}(z).$$

If $f \in \mathcal{A}^{\Phi}_{\lambda}(\mathbb{D})$*, then* g_m *belongs to* $\mathcal{A}^{\Phi}_{\lambda}(\mathbb{D})$ *and there holds the norm bound*

$$\|g_m\|_{L^{\Phi}_{\lambda}(\mathbb{D})} \leqslant C_{\Phi,\lambda,m}\|f\|_{L^{\Phi}_{\lambda}(\mathbb{D})}, \tag{2.6}$$

for some positive constant $C_{\Phi,\lambda,m}$ *depending on* Φ, λ*, and* m.

Proof Assume first $f \in \mathcal{A}^{\Phi}_{\lambda}(\mathbb{D})$. We choose $b > 0$ so large that $\lambda + 1 < p_{\Phi}(b+1)$ and differentiate the reproducing formula

$$f(z) = C_b \int_{\mathbb{D}} \frac{(1-|w|^2)^b}{(1-z\overline{w})^{2+b}} f(w)\,dA(w)$$

with respect to z under the integral sign several times to write for all $k = 1, \ldots, m$

$$(1-|z|^2)^k f^{(k)}(z) = C_{k,b}(1-|z|^2)^k \int_{\mathbb{D}} \frac{(1-|w|^2)^b}{(1-z\overline{w})^{2+k+b}} \overline{w}^k f(w)\,dA(w),$$

Here, the function $\overline{z}^k f$ belongs $L^{\Phi}_{\lambda}(\mathbb{D})$, by (2.2), hence the right hand side also does by Theorem 2.3, since it is equal to $T_{k,b}(\overline{z}^k f)$. The norm bound (2.6) is clear due to (2.2). □

2.4 Berezin Transform

Recall that, on the unit disc, the weighted Berezin transform of a function f is defined by

$$\mathbb{B}_{\lambda} f(z) \equiv \widetilde{f}_{\lambda}(z) = \int_{\mathbb{D}} f(\varphi_z(w))\,dA_{\lambda}(w) = \int_{\mathbb{D}} f(w)|k^{\lambda}_z(w)|^2\,dA_{\lambda}(w). \tag{2.7}$$

Here

$$k^{\lambda}_z(w) = \frac{(1-|z|^2)^{1+\lambda/2}}{(1-z\overline{w})^{2+\lambda}} \tag{2.8}$$

are the normalized weighted reproducing kernels of the classical weighted Bergman space $\mathcal{A}^2_{\lambda}(\mathbb{D})$ and $\varphi_z(w) = (z-w)/(1-\overline{z}w)$ is the Moebius transform of the unit disc. The (weighted) Berezin transform of a bounded operator $T : \mathcal{A}^2_{\lambda}(\mathbb{D}) \to \mathcal{A}^2_{\lambda}(\mathbb{D})$ is defined as

$$\widetilde{T}(z) = \langle T k^{\lambda}_z, k^{\lambda}_z\rangle_{\lambda} = \int_{\mathbb{D}} T k^{\lambda}_z(w)\overline{k^{\lambda}_z(w)}\,dA_{\lambda}(w). \tag{2.9}$$

The Berezin transform $\widetilde{T}_a^{(\lambda)}$ of the bounded Toeplitz operator $T_a^{(\lambda)}$ in $\mathcal{A}_\lambda^2(\mathbb{D})$ with symbol a coincides with the Berezin transform of the symbol $\widetilde{a}_\lambda$. See [8, 27, 28] for details.

If $-1 < \lambda < \infty$ and X is any Banach space of analytic functions on the disc such that all bounded analytic functions are contained both to X and its dual X^* (determined by the integral duality with respect to the measure dA_λ), and $T : X \to X$ is a bounded operator, we define the Berezin transform $\widetilde{T}$ of T by the same formula (2.9).

Remark 2.5 It is known in the case of standard Bergman spaces that if the Toeplitz operator $T_a^{(\lambda)}$ maps $\mathcal{A}_\lambda^p(\mathbb{D})$ boundedly into itself, then its Berezin transform is a bounded function on the disc $\mathbb{D}$. In fact, this follows directly from the boundedness of $T_a^{(\lambda)}$ and the definition of $\widetilde{T}_a^{(\lambda)}$, since the norm of k_z^λ in $\mathcal{A}_\lambda^p(\mathbb{D})$, respectively in $\mathcal{A}_\lambda^q(\mathbb{D})$ with $1/p + 1/q = 1$, is proportional to $(1-|z|^2)^{-1-\frac{\lambda}{2}+\frac{\lambda+2}{p}}$, respectively $(1-|z|^2)^{-1-\frac{\lambda}{2}+\frac{\lambda+2}{q}}$; see also Theorem 7.5 of [28] and paper [22]. We do not known in what generality this result extends to Orlicz spaces.

3 Toeplitz Operators with Radial Symbols on $\mathcal{A}_\lambda^\Phi(\mathbb{D})$

We start by an elementary technical statement, but we prefer to prove it explicitly in order to show the dependence of the estimates on the constants.

Lemma 3.1 *For $w \in \mathbb{D}$, denote $w = \rho\sigma$, where $\rho = |w|$ and $\sigma = \frac{w}{|w|}$. There exist a constant C such that*

$$\frac{1}{C(\gamma-1)}\frac{1}{(1-\rho|z|)^{\gamma-1}} \leqslant \int_{\mathbb{T}} \frac{|d\sigma|}{|1-z\overline{w}|^\gamma} \leqslant \frac{C}{\gamma-1}\frac{1}{(1-\rho|z|)^{\gamma-1}} \tag{3.1}$$

Proof If $r = |z| \leqslant \frac{1}{2}$ the two sided estimate (3.1) is obvious even with only constants in the left and right side.

Let $r \geqslant \frac{1}{2}$. We have with the notation $\delta = 1 - \rho|z|$:

$$I_\gamma(z,\rho) = \int_{\mathbb{T}} \frac{|d\sigma|}{|1-z\overline{w}|^\gamma} = \int_{\mathbb{T}} \frac{|d\sigma|}{|\sigma - z\rho|^\gamma} = 4\int_0^{\frac{\pi}{2}} \frac{d\alpha}{(\delta^2 + 4r\sin^2\alpha)^{\frac{\gamma}{2}}}.$$

For the estimate from below we note that $\sin\alpha \leqslant \alpha$ on $[0, \frac{\pi}{2}]$, hence

$$\delta^2 + 4r\sin^2\alpha \leqslant \delta^2 + 4r\alpha^2 \leqslant (\delta + 2\sqrt{r}\alpha)^2,$$

and

$$I_\gamma(z,\rho) \geqslant 4\int_0^{\frac{\pi}{2}} \frac{d\alpha}{(\delta+2\sqrt{r}\alpha)^\gamma} = \frac{1}{\delta^{\gamma-1}} \frac{2}{\sqrt{r}(\gamma-1)} \left(1-\left(\frac{\delta}{\delta+\pi\sqrt{r}}\right)^{\gamma-1}\right)$$

$$\geqslant \frac{1}{\delta^{\gamma-1}} \frac{2}{(\gamma-1)} \left(1-\left(\frac{\delta}{\delta+\frac{\pi}{\sqrt{2}}}\right)^{\gamma-1}\right) \geqslant \frac{C_1}{\gamma-1} \frac{1}{(1-\rho|z|)^{\gamma-1}}.$$

For the estimate from above we note that since $\frac{\sin\alpha}{\alpha}$ is decreasing on $[0,\frac{\pi}{2}]$ one has $\sin\alpha \geqslant \frac{2}{\pi}\alpha$ on $[0,\frac{\pi}{2}]$, and then we obtain

$$\delta^2 + 4r\sin^2\alpha \geqslant \delta^2 + \frac{16}{\pi^2} r\alpha^2 \geqslant \frac{(\delta+\frac{4}{\pi}\sqrt{r}\alpha)^2}{2}.$$

Hence,

$$I_\gamma(z,\rho) \leqslant 2^{2+\frac{\gamma}{2}} \int_0^{\frac{\pi}{2}} \frac{d\alpha}{(\delta+\frac{4}{\pi}\sqrt{r}\alpha)^{\frac{\gamma}{2}}} \leqslant \frac{2^{\frac{\gamma+1}{2}}\pi}{(\gamma-1)} \frac{1}{\delta^{\gamma-1}} \left(1-\left(\frac{\delta}{\delta+2}\right)^{\gamma-1}\right)$$

$$\leqslant \frac{C_2}{\gamma-1} \frac{1}{(1-\rho|z|)^{\gamma-1}}.$$

□

Remark 3.2 It is a matter of calculus to show that

$$\frac{C_1}{(\gamma-1)} \frac{1}{(1-\rho|z|)^{\gamma+m-1}} \leqslant \frac{\partial^m}{\partial\rho^m} \int_{\mathbb{T}} \frac{1}{|1-z\rho\overline{\sigma}|^\gamma} |d\sigma| \leqslant \frac{C_2}{(\gamma-1)} \frac{1}{(1-\rho|z|)^{\gamma+m-1}},$$

for $m = 0, 1, 2, \ldots$.

We also make a remark that will be needed later: for all $n \in \mathbb{N}$, and $\lambda > -1$ there exists a constant $C_{\Phi,n,\lambda} > 0$ such that

$$|f^{(n)}(0)| \leq C_{\Phi,n,\lambda} \|f\|_{L^\Phi_\lambda(\mathbb{D})}, \tag{3.2}$$

for all $f \in \mathcal{A}^\Phi_\lambda(\mathbb{D})$. Indeed, for every $0 < r < 1$ one has by the Cauchy integral formula

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(re^{it})}{r^{n+1}e^{int}} re^{it} i\, dt$$

which yields for example for all $1/4 < r < 3/4$

$$|f^{(n)}(0)| \leqslant \frac{n!}{2\pi}\int_0^{2\pi}\frac{|f(re^{it})|}{r^n}\,dt \leqslant C\int_0^{2\pi}|f(re^{it})|(1-r^2)^{\lambda}r\,dt$$

for some constant $C > 0$ depending on n and λ. Integrating both sides over $(\frac{1}{4}, \frac{3}{4})$ with respect to r, we obtain

$$\begin{aligned}|f^{(n)}(0)| &\leqslant 2C\int_{\frac{1}{4}}^{\frac{3}{4}}dr\int_0^{2\pi}|f(re^{it})|(1-r^2)^{\lambda}r\,dt\\ &\leqslant C_1\int_{\mathbb{D}}|f(z)|\,dA_{\lambda}(z) \leqslant C_1\|f\|_{L_{\lambda}^{\Phi}(\mathbb{D})}.\end{aligned}$$

Here we used the Hölder inequality and the fact that the weight $(1-r^2)^{\lambda}$ is bounded from below and above by positive constants for the given r-interval.

Now we are in position to formulate and prove our main results. Let us introduce the averages:

$$\begin{aligned}B_{a,\lambda}^{(0)}(r) &= a(r),\\ B_{a,\lambda}^{(1)}(r) &= \int_r^1 a(t)(1-t^2)^{\lambda}tdt,\\ B_{a,\lambda}^{(j)}(r) &= \int_r^1 B_{a,\lambda}^{(j-1)}(t)dt, \quad j = 2, 3, \ldots.\end{aligned}$$

Theorem 3.3 *Let Φ be a Young function with $1 < p_{\Phi} \leqslant q_{\Phi} < \infty$. If there exists $m \in \mathbb{N} \cup \{0\}$ such that*

(i) $B_{a,\lambda}^{(m)}(r) = O\left((1-r)^{m+\lambda}\right), \quad r \to 1,$
then

(ii) the Toeplitz operator $T_a^{(\lambda)} : \mathcal{A}_{\lambda}^{\Phi}(\mathbb{D}) \to \mathcal{A}_{\lambda}^{\Phi}(\mathbb{D})$ is well-defined and bounded, and its Berezin transform $\widetilde{T}_{\lambda}$ is a bounded function.

Proof We first prove the boundedness of the Toeplitz operator. By Lemma 2.1 we know that polynomials form a dense set in $\mathcal{A}_{\lambda}^{\Phi}(\mathbb{D})$, hence at least for f being a polynomial we have:

$$\begin{aligned}T_a^{(\lambda)}f(z) &= \int_{\mathbb{D}}\frac{a(w)f(w)}{(1-z\overline{w})^{2+\lambda}}dA_{\lambda}(w)\\ &= \frac{\lambda+1}{\pi}\int_{\mathbb{T}}|d\sigma|\int_0^1\frac{a(\rho)f(\rho\sigma)}{(1-z\rho\overline{\sigma})^{2+\lambda}}(1-\rho^2)^{\lambda}\rho d\rho\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda+1}{\pi}\int_{\mathbb{T}}\Bigg[\sum_{j=1}^{m} B_{a,\lambda}^{(j)}(0)\left(\frac{\partial^{j-1}}{\partial\rho^{j-1}}\frac{f(\rho\sigma)}{(1-z\rho\overline{\sigma})^{2+\lambda}}\right)\Bigg|_{\rho=0} \\
&\quad + \int_0^1 B_{a,\lambda}^{(m)}(\rho)\left(\frac{\partial^m}{\partial\rho^m}\frac{f(\rho\sigma)}{(1-z\rho\overline{\sigma})^{2+\lambda}}\right)d\rho\Bigg]|d\sigma|. \qquad (3.3)
\end{aligned}
$$

We note that

$$
\begin{aligned}
\frac{\partial^j}{\partial\rho^j}\frac{f(\rho\sigma)}{(1-z\rho\overline{\sigma})^{2+\lambda}} &= \sum_{k=0}^{j} C_k^j\left(\frac{\partial^k}{\partial\rho^k}f(\rho\sigma)\right)\left(\frac{\partial^{j-k}}{\partial\rho^{j-k}}\frac{1}{(1-z\rho\overline{\sigma})^{2+\lambda}}\right) \\
= \sum_{k=0}^{j} A_{k,j,\lambda}(z,\sigma) f^{(k)}(\rho\sigma)\frac{1}{(1-z\rho\overline{\sigma})^{2+\lambda+j-k}},
\end{aligned}
$$

where $A_{k,j,\lambda}(z,\sigma) = C_{k,j,\lambda} z^{j-k}\sigma^{2k-j}$ with some constants $C_{k,j,\lambda}$. Therefore,

$$
\begin{aligned}
T_a^{(\lambda)} f(z) &= \frac{\lambda+1}{\pi}\sum_{j=1}^{m} B_{a,\lambda}^{(j)}(0)\sum_{k=0}^{j-1} f^{(k)}(0)\int_{\mathbb{T}} A_{k,j-1,\lambda}(z,\sigma)|d\sigma| \\
&\quad + \frac{\lambda+1}{\pi}\int_{\mathbb{T}}|d\sigma|\int_0^1 B_{a,\lambda}^{(m)}(\rho)\left(\frac{\partial^m}{\partial\rho^m}\frac{f(\rho\sigma)}{(1-z\rho\overline{\sigma})^{2+\lambda}}\right)d\rho
\end{aligned}
$$

and the first double sum is nonzero only if j is even and $k = \frac{j}{2}$.

Regarding the second term, taking into account that

$$
\frac{(1-\rho^2)^{m-k}}{|1-z\rho\overline{\sigma}|^{2+\lambda+m-k}} \leqslant C\frac{1}{|1-z\rho\overline{\sigma}|^{2+\lambda}}
$$

uniformly in $z\in\mathbb{D}$, $\sigma\in\mathbb{T}$, we calculate

$$
\begin{aligned}
\left|I_{m,a,\lambda} f(z)\right| &\equiv \left|\int_{\mathbb{T}}|d\sigma|\int_0^1 B_{a,\lambda}^{(m)}(\rho)\left(\frac{\partial^m}{\partial\rho^m}\frac{f(\rho\sigma)}{(1-z\rho\overline{\sigma})^{2+\lambda}}\right)d\rho\right| \\
&\leqslant C\int_{\mathbb{T}}|d\sigma|\int_0^1\left(\sum_{k=0}^{m}(1-\rho^2)^k|f^{(k)}(\rho\sigma)|\right)\frac{(1-\rho^2)^\lambda}{|1-z\rho\overline{\sigma}|^{2+\lambda}}d\rho \\
&= C_1\int_{\mathbb{D}}\frac{\sum_{k=0}^{m}(1-|w|^2)^k|f^{(k)}(w)|}{|1-z\overline{w}|^{2+\lambda}}dA_\lambda(w) \\
&= C_1 P_\lambda^+ g_m(z), \qquad (3.4)
\end{aligned}
$$

where P_λ^+ is as in (2.3) and we denoted

$$g_m(z) = \sum_{k=0}^{m} (1 - |z|^2)^k f^{(k)}(z).$$

The function g_m belongs to $L_\lambda^\Phi(\mathbb{D})$ due to $f \in \mathcal{A}_\lambda^\Phi(\mathbb{D})$ and Theorem 2.4, and we have the bound

$$\|g_m\|_{\mathcal{A}_\lambda^\Phi(\mathbb{D})} \leqslant C\|f\|_{\mathcal{A}_\lambda^\Phi(\mathbb{D})}, \quad 1 < p < \infty, \ \lambda > -1.$$

This, together with Theorem 2.2 imply $P_\lambda^+ g_m \in \mathcal{A}_\lambda^\Phi(\mathbb{D})$ and

$$\|P_\lambda^+ g_m\|_{\mathcal{A}_\lambda^\Phi(\mathbb{D})} \leqslant C\|g_m\|_{\mathcal{A}_\lambda^\Phi(\mathbb{D})}, \quad 1 < p < \infty, \ \lambda > -1. \tag{3.5}$$

Gathering all estimates (3.3)–(3.5), we arrive at the conclusion that the Toeplitz operator $T_a^{(\lambda)}$ is bounded,

$$\|T_a^{(\lambda)} f\|_{\mathcal{A}_\lambda^\Phi(\mathbb{D})} \leqslant C\|f\|_{\mathcal{A}_\lambda^\Phi(\mathbb{D})}$$

since the constant $C > 0$ can be chosen independently of $f \in \mathcal{A}_\lambda^\Phi(\mathbb{D})$.

As for the boundedness of the Berezin transform, we have for all $z \in \mathbb{D}$, by the Fubini theorem and the reproducing kernel property,

$$\begin{aligned}
&\widetilde{T}_a^{(\lambda)}(z) \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{a(w)(1-|w|^2)^\lambda}{(1-\zeta\bar{w})^{2+\lambda}} \frac{(1-|z|^2)^{1+\lambda/2}}{(1-\bar{z}w)^{2+\lambda}} \frac{(1-|z|^2)^{1+\lambda/2}}{(1-z\bar{\zeta})^{2+\lambda}} (1-|\zeta|^2)^\lambda dA(w) dA(\zeta) \\
&= \int_{\mathbb{D}} \frac{a(w)(1-|w|^2)^\lambda}{(1-z\bar{w})^{2+\lambda}} \frac{(1-|z|^2)^{2+\lambda}}{(1-\bar{z}w)^{2+\lambda}} dA(w) = (1-|z|^2)^{1+\lambda/2} T_a^{(\lambda)} k_z^\lambda(z)
\end{aligned} \tag{3.6}$$

The estimate

$$\left|\frac{\partial^k}{\partial w^k} k_z^\lambda(w)\right| \le C_k \frac{(1-|z|^2)^{1+\lambda/2}}{|1-z\bar{w}|^{2+\lambda+k}} \tag{3.7}$$

follows just by differentiating the formula (2.8). For a fixed $z \in \mathbb{D}$, we take $f = k_z^\lambda$ in (3.3)–(3.4) and apply this to (3.6) so that together with (3.7) we obtain the bound

$$\begin{aligned}
|\widetilde{T}_a^{(\lambda)}(z)| &= (1-|z|^2)^{1+\lambda/2} \left| T_a^{(\lambda)} k_z^\lambda(\zeta) \right| \\
&\le (1-|z|^2)^{1+\lambda/2} \int_{\mathbb{D}} \sum_{k=0}^{m} \frac{(1-|w|^2)^k}{|1-z\overline{w}|^{2+\lambda}} \left| \frac{\partial^k}{\partial w^k} k_z^\lambda(w) \right| dA_\lambda(w)
\end{aligned}$$

$$\leq (1-|z|^2)^{1+\lambda/2} \int_{\mathbb{D}} \sum_{k=0}^{m} \frac{(1-|w|^2)^k}{|1-z\overline{w}|^{2+\lambda}} \frac{(1-|z|^2)^{1+\lambda/2}}{|1-z\bar{w}|^{2+\lambda+k}} dA_\lambda(w)$$

$$= (1-|z|^2)^{2+\lambda} \int_{\mathbb{D}} \sum_{k=0}^{m} \frac{(1-|w|^2)^{k+\lambda}}{|1-z\bar{w}|^{4+2\lambda+k}} dA(w). \tag{3.8}$$

By the Forelli-Rudin-estimates

$$\int_{\mathbb{D}} \sum_{k=0}^{m} \frac{(1-|w|^2)^{\lambda+k}}{|1-z\overline{w}|^{4+2\lambda+k}} dA(w) \leq C_m (1-|z|^2)^{-2-\lambda}, \quad z \in \mathbb{D},$$

so that $|\widetilde{T}_a^{(\lambda)}(z)|$ has an upper bound independent of z, as claimed. □

Theorem 3.4 *Let Φ be a Young function, and assume that there exists $m \in \mathbb{N} \cup \{0\}$ such that the average $B_{a,\lambda}^{(m)}$ is nonnegative a.e. in* $(0, 1)$.
If

(ii) the Toeplitz operator $T_a^{(\lambda)} : \mathcal{A}_\lambda^\Phi(\mathbb{D}) \to \mathcal{A}_\lambda^\Phi(\mathbb{D})$ is well-defined and bounded, and its Berezin transform $\widetilde{T}_a$ is a bounded function,
then we have
(i) $B_{a,\lambda}^{(m+1)}(r) = O\left((1-r)^{m+1+\lambda}\right), \quad r \to 1.$

Proof We assume that $T_a^{(\lambda)}$ maps $\mathcal{A}_\lambda^\Phi(\mathbb{D})$ boundedly into itself and the Berezin transform $\widetilde{T}_a^{(\lambda)}$ is a bounded function of $z \in \mathbb{D}$. We have by the definition of Berezin transform

$$\begin{aligned}
\widetilde{T}_a^{(\lambda)}(z) = \widetilde{a}_\lambda(z) &= \int_{\mathbb{D}} a(w) |k_z^\lambda(w)|^2 dA_\lambda(w) \\
&= \frac{\lambda+1}{\pi} \int_{\mathbb{T}} |d\sigma| \int_0^1 a(\rho) |k_z^\lambda(\rho\sigma)|^2 (1-\rho^2)^\lambda \rho d\rho \\
&= \frac{\lambda+1}{\pi} (1-|z|^2)^{2+\lambda} \int_{\mathbb{T}} |d\sigma| \int_0^1 a(\rho) \frac{(1-\rho^2)^\lambda}{|1-z\rho\overline{\sigma}|^{4+2\lambda}} \rho d\rho.
\end{aligned}$$

Here we denoted $w = \rho\sigma$, where $\rho = |w|$ and $\sigma = \frac{w}{|w|}$. If $m = 0$, then using Lemma 3.1 we estimate

$$\begin{aligned}
\left|\widetilde{T}_a^{(\lambda)}(z)\right| &= \frac{\lambda+1}{\pi} (1-|z|^2)^{2+\lambda} \int_{\mathbb{T}} |d\sigma| \int_0^1 a(\rho) \frac{(1-\rho^2)^\lambda}{|1-z\rho\overline{\sigma}|^{4+2\lambda}} \rho d\rho \\
&= \frac{\lambda+1}{\pi} (1-|z|^2)^{2+\lambda} \int_0^1 a(\rho)(1-\rho^2)^\lambda \rho d\rho \int_{\mathbb{T}} \frac{|d\sigma|}{|1-z\rho\overline{\sigma}|^{4+2\lambda}} \\
&\geqslant C(1-|z|^2)^{-1-\lambda} \int_0^1 a(\rho)(1-\rho^2)^\lambda \rho d\rho
\end{aligned}$$

$$\geqslant C(1-|z|^2)^{-1-\lambda}\int_{|z|}^1 a(\rho)(1-\rho^2)^\lambda \rho d\rho$$
$$= C(1-|z|^2)^{-1-\lambda} B_{a,\lambda}^{(1)}(|z|).$$

This along with the boundedness of the function $\widetilde{T}_a^{(\lambda)}$ on $\mathbb{D}$ implies the validity of (i) for $m=0$.

Let now $m \in \mathbb{N}$. Integrating by parts and neglecting the outer terms when $\rho = 1$, we have

$$\begin{aligned}
J_{a,\lambda}(z,\sigma) &\equiv \int_0^1 a(\rho)|k_z^\lambda(\rho\sigma)|^2(1-\rho^2)^\lambda \rho d\rho \\
&= (1-|z|^2)^{2+\lambda} B_{a,\lambda}^{(1)}(0) + \int_0^1 B_{a,\lambda}^{(1)}(\rho)\frac{\partial}{\partial\rho}|k_z^\lambda(\rho\sigma)|^2 d\rho \\
&= \sum_{j=1}^m B_{a,\lambda}^{(j)}(0)\left(\frac{\partial^{j-1}}{\partial\rho^{j-1}}\left|k_z^\lambda(\rho\sigma)\right|^2\right)\Bigg|_{\rho=0} + \int_0^1 B_{a,\lambda}^{(m)}(\rho)\frac{\partial^m}{\partial\rho^m}\left|k_z^\lambda(\rho\sigma)\right|^2 d\rho \\
&\equiv J_{a,\lambda}^{(1)}(z) + J_{a,\lambda}^{(2)}(z,\sigma)
\end{aligned}$$

The term

$$J_{a,\lambda}^{(1)}(z) = (1-|z|^2)^{2+\lambda}\sum_{j=1}^m B_{a,\lambda}^{(j)}(0)\left(\frac{\partial^{j-1}}{\partial\rho^{j-1}}\frac{1}{|1-z\rho\overline{\sigma}|^{4+2\lambda}}\right)\Bigg|_{\rho=0}$$

is bounded for each $z \in \mathbb{D}$, and when $|z| \to 1$ it behaves as

$$J_{a,\lambda}^{(1)}(z) = O\left((1-|z|^2)^{2+\lambda}\right), \quad |z| \to 1.$$

Further,

$$\begin{aligned}
\int_{\mathbb{T}} J_{a,\lambda}^{(2)}(z,\sigma)|d\sigma| &= \int_{\mathbb{T}}|d\sigma|\int_0^1 B_{a,\lambda}^{(m)}(\rho)\frac{\partial^m}{\partial\rho^m}\left|k_z^\lambda(\rho\sigma)\right|^2 d\rho \\
&= \int_0^1 B_{a,\lambda}^{(m)}(\rho)d\rho\frac{\partial^m}{\partial\rho^m}\int_{\mathbb{T}}\left|k_z^\lambda(\rho\sigma)\right|^2|d\sigma| \\
&= (1-|z|^2)^{2+\lambda}\int_0^1 B_{a,\lambda}^{(m)}(\rho)d\rho\frac{\partial^m}{\partial\rho^m}\int_{\mathbb{T}}\frac{1}{|1-z\rho\overline{\sigma}|^{4+2\lambda}}|d\sigma|.
\end{aligned}$$

Hence,

$$\left|\int_{\mathbb{T}} J^{(2)}_{a,\lambda}(z,\sigma)|d\sigma|\right| \geqslant C(1-|z|^2)^{2+\lambda}\int_0^1 B^{(m)}_{a,\lambda}(\rho)\frac{1}{(1-\rho|z|)^{3+2\lambda+m}}d\rho$$

$$\geqslant C(1-|z|^2)^{2+\lambda}\int_{|z|}^1 B^{(m)}_{a,\lambda}(\rho)\frac{1}{(1-\rho|z|)^{3+2\lambda+m}}d\rho$$

$$\geqslant C(1-|z|^2)^{-m-1-\lambda}\int_{|z|}^1 B^{(m)}_{a,\lambda}(\rho)d\rho$$

$$= C(1-|z|^2)^{-m-1-\lambda}B^{(m+1)}_{a,\lambda}(|z|).$$

Therefore, as above, we see that the condition (i) must be satisfied. This concludes the proof of the necessity of the condition (i). □

We reformulate the previous results as the following necessary and sufficient condition concerning the boundedness of the Toeplitz operator. Of course, condition (ii) can be simplified in the (many) cases, where one can deduce the boundedness of the Berezin transform from the boundedness of the Toeplitz operator; see also Remark 2.5.

Theorem 3.5 *Let Φ be a Young function with $1 < p_\Phi \leqslant q_\Phi < \infty$ and assume that there exists $m \in \mathbb{N}\cup\{0\}$ such that the average $B^{(m)}_{a,\lambda}$ is nonnegative a.e. in $(0,1)$. Then, the following are equivalent:*

(i) $B^{(m+1)}_{a,\lambda}(r) = O\left((1-r)^{m+1+\lambda}\right), \quad r \to 1,$

(ii) the Toeplitz operator $T^{(\lambda)}_a : \mathcal{A}^\Phi_\lambda(\mathbb{D}) \to \mathcal{A}^\Phi_\lambda(\mathbb{D})$ is well-defined and bounded, and its Berezin transform $\widetilde{T}^{(\lambda)}_a$ is a bounded function.

As a corollary we reformulate the Theorem 3.5 for the case of the weighted Lebesgue space (i.e., $\Phi(t) = t^p$).

Theorem 3.6 *Let $1 < p < \infty$, and there exists $m \in \mathbb{N}\cup\{0\}$ such that the average $B^{(m)}_{a,\lambda}$ is nonnegative a.e. in $(0,1)$. Then, the following are equivalent:*

(i) $B^{(m+1)}_{a,\lambda}(r) = O\left((1-r)^{m+1+\lambda}\right), \quad r \to 1,$

(ii) the Toeplitz operator $T^{(\lambda)}_a : \mathcal{A}^p_\lambda(\mathbb{D}) \to \mathcal{A}^p_\lambda(\mathbb{D})$ is well-defined and bounded.

The reader will have no difficulty to reformulate Theorems 3.3 and 3.4 for the case of the weighted Lebesgue space with $\Phi(t) = t^p$.

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References

1. D. Bekolle, A. Bonami, E. Tchoundja, Atomic decomposition and Weak Factorization for Bergman-Orlicz spaces. arXiv:1805.03754
2. S.Charpentier, Small Bergman-Orlicz and Hardy-Orlicz spaces, and their composition operators. Math. Z. (2019). https://doi.org/10.1007/s00209-019-02240-w
3. Y. Deng, L. Huang, T. Zhao, D. Zheng, Bergman projection and Bergman spaces, Journal of Operator Theory. 46 (2001), 3–24.
4. P. Duren, A. Schuster, Bergman spaces. Vol. 100. Providence (RI): Mathematical Surveys and Monographs; 2004.
5. S.M. Grudsky, A.N. Karapetyants, N.L. Vasilevski, Toeplitz operators on the unit ball in $\mathbb{C}^n$ with radial symbols. J. Operator Theory. 2003;49:325–346.
6. S.M. Grudsky, A.N. Karapetyants, N.L. Vasilevski, Dynamics of properties of Toeplitz operators with radial symbols. Integral Equations Operator Theory. 2004;50:217–253.
7. S.M. Grudsky, A.N. Karapetyants, N.L. Vasilevski, Dynamics of properties of Toeplitz operators in the weighted Bergman spaces. Siberian Electronic Mathematical Reports. 2006;3:362–383.
8. H. Hedenmalm, B. Korenblum, and K. Zhu, Theory of Bergman Spaces, Springer Verlag, New York, 2000.
9. P. Harjulehto and P.Hästö, Orlicz spaces and Generalized Orlicz spaces. Lecture Notes in Mathematics, vol. 2236, Springer, Cham, 2019.
10. A.Karapetyants, H. Rafeiro, S.Samko, Boundedness of the Bergman projection and some properties of Bergman type spaces. Complex Analysis and Operator Theory. 1–15 (2018).
11. A.Karapetyants, S.Samko, On mixed norm Bergman–Orlicz–Morrey spaces. Georgian Mathematical Journal, Volume 25, Issue 2, 271–282.
12. A. Karapetyants,S.Samko, On Boundedness of Bergman Projection Operators in Banach Spaces of Holomorphic Functions in Half-Plane and Harmonic Functions in Half-Space. Journal of Mathematical Sciences. J Math Sci. (2017) Volume 226, Issue 4, pp 344–354
13. M.A. Krasnoselskii, Ya.B. Rutitski, Convex functions and Orlicz spaces. Translated from the first Russian edition. Noordhoff, Groningen (1961)
14. A. Kufner, O. John, S. Fucik, Function spaces, Leyden, Noordhoff International Publishing. Monographs and Textbooks on Mechanics of Solids and Fluids. Mechanics: Analysis, 1977
15. W. Lusky, and J. Taskinen, Toeplitz operators on Bergman spaces and Hardy multipliers, 2011. Studia Mathematica. 204, 2, 137–154
16. L.Maligranda, Indices and interpolation. Dissertationes Mathematicae 234 (1985).
17. G. McDonald and C. Sundberg, Toeplitz operators on the disc, Indiana Univ. Math. J. 28 (1979), 595–611.
18. L. Pick, A. Kufner, O. John, S.Fucik, Function Spaces, 1. De Gruyter Series in Nonlinear Analysis and Applications 14 (2013).
19. M.M.Rao, Z.D.Ren, Theory of Orlicz spaces, Monographs and Textbooks in Pure and Applied Mathematics, 146, Marcel Dekker, 1991.
20. B.Sehba, Derivatives characterization of Bergman-Orlicz spaces and applications. arXiv:1610.01954
21. I. B. Simonenko, Interpolation and extrapolation of linear operators in Orlicz spaces. Mat. Sb. (N.S.), 1964, Volume 63(105), Number 4, Pages 536–553.
22. J. Taskinen and J. Virtanen, Toeplitz operators on Bergman spaces with locally integrable symbols. Rev. Mat. Iberoam. 2010; 26 (2): 693–706.
23. E. Tchoundja, and R. Zhao, Weak Factorization and Hankel Forms for Bergman–Orlicz Spaces on the Unit Ball. Integr. Equ. Oper. Theory (2019) 91: 16.
24. N.L.Vasilevski, Commutative Algebras of Toeplitz Operators on the Bergman Space. Vol.185: Operator Theory: Advances and Applications; 2008.
25. S.Yamamuro, Exponents of modulared semi-ordered linear spaces. J. Fac. Sci. Hokkaido Univ. Ser. I Math., Volume 12, Number 4 (1953), 211–253.

26. K. Zhu, Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domains, *J. Operator Theory* **20** (1988), no. 2, 329–357.
27. K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer Verlag, New York, 2004.
28. K. Zhu, *Operator Theory in Function Spaces*, AMS Mathematical Surveys and Monographs **138**, 2007.
29. N. Zorboska, Toeplitz operators with BMO symbols and the Berezin transform, *Int. J. Math. Sci.*, no. 46, (2003) 2929–2945.

Algebras of Singular Integral Operators with PQC Coefficients on Weighted Lebesgue Spaces

Yu. I. Karlovich

To my friend Nikolai Vasilevski on the occasion of his 70th birthday

Abstract Let $\mathcal{B}_{p,w}$ be the Banach algebra of all bounded linear operators on the weighted Lebesgue space $L^p(\mathbb{T}, w)$ with $p \in (1, \infty)$ and a Muckenhoupt weight $w \in A_p(\mathbb{T})$ which is locally equivalent at open neighborhoods u_t of points $t \in \mathbb{T}$ to weights W_t for which the functions $\tau \mapsto (\tau - t)(\ln W_t)'(\tau)$ are quasicontinuous on u_t, and let PQC be the C^*-algebra of all piecewise quasicontinuous functions on $\mathbb{T}$. The Banach algebra

$$\mathfrak{A}_{p,w} = \text{alg}\{aI, S_{\mathbb{T}} : a \in PQC\} \subset \mathcal{B}_{p,w}$$

generated by all multiplication operators aI by functions $a \in PQC$ and by the Cauchy singular integral operator $S_{\mathbb{T}}$ is studied. A Fredholm symbol calculus for the algebra $\mathfrak{A}_{p,w}$ is constructed and a Fredholm criterion for the operators $A \in \mathfrak{A}_{p,w}$ in terms of their Fredholm symbols is established by applying the Allan-Douglas local principle, the two idempotents theorem and a localization of Muckenhoupt weights W_t to power weights by using quasicontinuous functions and Mellin pseudodifferential operators with non-regular symbols.

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1 Introduction

Let $\mathcal{B}(X)$ denote the Banach algebra of all bounded linear operators acting on a Banach space X, let $\mathcal{K}(X)$ be the closed two-sided ideal of all compact operators in $\mathcal{B}(X)$, and let $\mathcal{B}^\pi(X) = \mathcal{B}(X)/\mathcal{K}(X)$ be the Calkin algebra of the cosets $A^\pi := A + \mathcal{K}(X)$, where $A \in \mathcal{B}(X)$. An operator $A \in \mathcal{B}(X)$ is said to be *Fredholm*, if its image is closed and the spaces $\ker A$ and $\ker A^*$ are finite-dimensional (see, e.g., [6] and [12]). Equivalently, $A \in \mathcal{B}(X)$ is Fredholm if and only if the coset A^π is invertible in the algebra $\mathcal{B}^\pi(X)$.

A measurable function $w : \mathbb{T} \to [0, \infty]$ defined on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ is called a weight if the preimage $w^{-1}(\{0, \infty\})$ of the set $\{0, \infty\}$ has measure zero. For $p \in (1, \infty)$, a weight w belongs to the *Muckenhoupt class* $A_p(\mathbb{T})$ if

$$c_{p,w} := \sup_I \left(\frac{1}{|I|} \int_I w^p(\tau)|d\tau| \right)^{1/p} \left(\frac{1}{|I|} \int_I w^{-q}(\tau)|d\tau| \right)^{1/q} < \infty,$$

where $1/p + 1/q = 1$, and supremum is taken over all intervals $I \subset \mathbb{T}$ of finite length $|I|$. In what follows we assume that $p \in (1, \infty)$, $w \in A_p(\mathbb{T})$, and consider the weighted Lebesgue space $L^p(\mathbb{T}, w)$ equipped with the norm

$$\|f\|_{L^p(\mathbb{T},w)} := \left(\int_{\mathbb{T}} |f(\tau)|^p w^p(\tau)|d\tau| \right)^{1/p}.$$

As is known (see, e.g., [3, 11]), the Cauchy singular integral operator $S_{\mathbb{T}}$ given by

$$(S_{\mathbb{T}} f)(t) = \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{\mathbb{T} \setminus (te^{-i\varepsilon}, te^{i\varepsilon})} \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \mathbb{T},$$

is bounded on every space $L^p(\mathbb{T}, w)$ if and only if $p \in (1, \infty)$ and $w \in A_p(\mathbb{T})$.

Let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the *Fourier transform*,

$$(\mathcal{F} f)(x) := \int_{\mathbb{R}} f(t)e^{-itx} dt, \quad x \in \mathbb{R}.$$

A function $a \in L^\infty(\mathbb{R})$ is called a *Fourier multiplier* on $L^p(\mathbb{R}, w)$ if the convolution operator $W^0(a) := \mathcal{F}^{-1}a\mathcal{F}$ maps the dense subset $L^2(\mathbb{R}) \cap L^p(\mathbb{R}, w)$ of $L^p(\mathbb{R}, w)$ into itself and extends to a bounded linear operator on $L^p(\mathbb{R}, w)$. For a weight $w \in A_p(\mathbb{R})$, let $M_{p,w}$ stand for the unital Banach algebra of all Fourier multipliers on $L^p(\mathbb{R}, w)$ equipped with pointwise operations and the norm $\|a\|_{M_{p,w}} := \|W^0(a)\|_{\mathcal{B}(L^p(\mathbb{R},w))}$ (see [2, Corollary 2.9]).

Letting $\mathcal{B}_{p,w} := \mathcal{B}(L^p(\mathbb{T}, w))$ and $\mathcal{K}_{p,w} := \mathcal{K}(L^p(\mathbb{T}, w))$ for $p \in (1, \infty)$ and $w \in A_p(\mathbb{T})$, we consider the Banach algebra

$$\mathfrak{A}_{p,w} := \operatorname{alg}\left\{aI, S_{\mathbb{T}} : a \in PQC\right\} \subset \mathcal{B}_{p,w} \tag{1.1}$$

generated by all multiplication operators aI ($a \in PQC$) and by the Cauchy singular integral operator $S_{\mathbb{T}}$, where the C^*-algebra $PQC \subset L^\infty(\mathbb{T})$ of piecewise quasicontinuous functions is defined in Sect. 2. As is well known (see, e.g., the proof of [21, Theorem 4.1.5]), the ideal $\mathcal{K}_{p,w}$ is contained in the Banach algebra $\mathfrak{A}_{p,w}$ for all $p \in (1, \infty)$ and all $w \in A_p(\mathbb{T})$.

A Fredholm criterion and an index formula for Toeplitz operators with piecewise quasicontinuous symbols on the Hardy space H^2 on the unit circle $\mathbb{T}$ were established by D. Sarason in [26]. A Fredholm criterion for Toeplitz operators with piecewise quasicontinuous symbols on weighted Hardy spaces $H^p(\varrho)$ with $p \in (1, \infty)$ and weights of the form $\varrho(t) = \prod_{j=1}^n |t - t_j|^{\mu_j}$, where $t \in \mathbb{T}$, $t_1, \dots, t_n$ are pairwise distinct point on $\mathbb{T}$ and $\mu_1, \dots, \mu_n$ are real numbers subject to the condition $\mu_j \in (-1/p, 1 - 1/p)$ for all j, which means that $\varrho \in A_p(\mathbb{T})$, was obtained by A. Böttcher and I.M. Spitkovsky in [7]. Banach algebras of singular integral operators with piecewise quasicontinuous coefficients on weighted Lebesgue spaces $L^p(\mathbb{T}, \varrho)$ were studied in [5].

Fredholm criteria and index formulas are also established for singular integral operators with coefficients having semi-almost periodic discontinuities [23, 24] and for Wiener-Hopf operators with semi-almost periodic presymbols [9] (see also [4] and the references therein). Note that Fredholm results for semi-almost periodic data and piecewise quasicontinuous data essentially differ.

The present paper deals with studying the Fredholmness of singular integral operators with piecewise quasicontinuous coefficients on weighted Lebesgue spaces $L^p(\mathbb{T}, w)$ for a much larger class of Muckenhoupt weights w that are locally equivalent at open neighborhoods u_t of points $t \in \mathbb{T}$ to weights W_t for which the functions $\tau \mapsto (\tau - t)(\ln W_t)'(\tau)$ are in $QC|_{u_t}$. A Fredholm symbol calculus for the Banach algebra $\mathfrak{A}_{p,w}$ given by (1.1) is constructed and a Fredholm criterion of the operators $A \in \mathfrak{A}_{p,w}$ in terms of their Fredholm symbols is established by applying the Allan-Douglas local principle, the two idempotents theorem and a localization of Muckenhoupt weights W_t to power weights by using quasicontinuous functions and Mellin pseudodifferential operators with non-regular symbols.

The paper is organized as follows. In Sect. 2 the C^*-algebras $SO^\diamond$, QC and PQC of slowly oscillating, quasicontinuous and piecewise quasicontinuous functions are considered and their maximal ideal spaces are described. In Sect. 3, modifying

[18, Section 6.2], we introduce a slightly different class of Muckenhoupt weights locally equivalent to *slowly oscillating weights* that are obtained by a procedure of smoothness improvement. In Sect. 4 we recall results on the boundedness and compactness of Mellin pseudodifferential operators with non-regular symbols and give an application of such operators to weighted Cauchy singular integral operators. Section 5 deals with an application of the Allan-Douglas local principle (see, e.g., [8, Theorem 7.47] and [6, Theorem 1.35]) to local studying the quotient Banach algebra $\mathfrak{A}^\pi_{p,w} := \mathfrak{A}_{p,w}/\mathcal{K}_{p,w}$. By [26], the maximal ideal space $M(QC)$ of QC is the union of three pairwise disjoint sets $\widetilde{M}^-(QC)$, $M^0(QC)$ and $\widetilde{M}^+(QC)$ given by (2.3). In particular, in Sect. 5.3 we describe the structure of the local algebras for all $\xi \in M(QC)$ and present the invertibility criteria in local algebras for all $\xi \in \widetilde{M}^\pm(QC)$.

In Sect. 6, applying quasicontinuous functions and Mellin pseudodifferential operators with non-regular symbols and modifying [17, Section 6], we localize Muckenhoupt weights satisfying assumption **(A)** (see Sect. 3) and describe the spectra in local algebras of elements related to the operator $wS_\mathbb{T}w^{-1}I \in \mathcal{B}(L^p(\mathbb{T}))$. As a result, the localization reduces considered weights $w \in A_p(\mathbb{T})$ to power weights $w_\xi \in A_p(\mathbb{T})$ parameterized by points $\xi \in M^0(QC)$. Section 7 deals with the two idempotents theorem (see, e.g., [10, 13] and [3, Theorem 8.7]) and its application to studying the invertibility in local algebras for $\xi \in M^0(QC)$. In particular, we identify here the spectra of cosets $[X_t]^\pi_{p,w,\xi}$ being crucial in the two idempotents theorem. Section 8 contains the main results of the paper: a Fredholm symbol calculus for the Banach algebra $\mathfrak{A}_{p,w}$ with $p \in (1,\infty)$ and Muckenhoupt weights w satisfying condition **(A)**, the inverse closedness of the quotient algebra $\mathfrak{A}^\pi_{p,w}$ in the Calkin algebra $\mathcal{B}^\pi_{p,w}$, which means that for every coset $A^\pi \in \mathfrak{A}^\pi_{p,w}$ its spectra in the algebras $\mathfrak{A}^\pi_{p,w}$ and $\mathcal{B}^\pi_{p,w}$ coincide (see, e.g., [4, p. 3]), and a Fredholm criterion for the operators $A \in \mathfrak{A}_{p,w}$ in terms of their Fredholm symbols.

2 The C^*-Algebras $SO^\diamond$, QC and PQC

2.1 The C^*-Algebra $SO^\diamond$ of Slowly Oscillating Functions

Let $L^\infty(\mathbb{T})$ be the C^*-algebra of all bounded measurable functions on the unit circle $\mathbb{T}$. Let $C := C(\mathbb{T})$ and $PC := PC(\mathbb{T})$ denote the C^*-subalgebras of $L^\infty(\mathbb{T})$ consisting, respectively, of all continuous functions on $\mathbb{T}$ and all piecewise continuous functions on $\mathbb{T}$, that is, the functions having finite one-sided limits at each point $t \in \mathbb{T}$.

Following [1, Section 4], we say that a function $f \in L^\infty(\mathbb{T})$ is *slowly oscillating* at a point $t \in \mathbb{T}$ if

$$\lim_{\varepsilon\to 0} \text{ess sup}\left\{|f(\tau) - f(s)| : \tau, s \in \mathbb{T}_{r\varepsilon,\,\varepsilon}(t)\right\} = 0$$

for every $r \in (0, 1)$ (equivalently, for some $r \in (0, 1)$), where

$$\mathbb{T}_{r\varepsilon,\,\varepsilon}(t) := \big\{z \in \mathbb{T} :\ r\varepsilon \le |z - t| \le \varepsilon\big\} \ \text{ for } \ t \in \mathbb{T}.$$

For each $t \in \mathbb{T}$, let $SO_t(\mathbb{T})$ denote the C^*-subalgebra of $L^\infty(\mathbb{T})$ given by

$$SO_t(\mathbb{T}) := \Big\{f \in C_b(\mathbb{T} \setminus \{t\}) :\ f \text{ slowly oscillates at } t\Big\},$$

where $C_b(\mathbb{T} \setminus \{t\}) := C(\mathbb{T} \setminus \{t\}) \cap L^\infty(\mathbb{T})$. Let $SO^\diamond$ be the minimal C^*-subalgebra of $L^\infty(\mathbb{T})$ that contains all C^*-algebras $SO_t(\mathbb{T})$ for $t \in \mathbb{T}$. In particular, $C \subset SO^\diamond$.

Given a commutative unital C^*-algebra $\mathcal{A}$, we denote by $M(\mathcal{A})$ the maximal ideal space of $\mathcal{A}$. Since $M(C)$ can be identified with $\mathbb{T}$, we conclude that

$$M(SO^\diamond) = \bigcup_{t\in\mathbb{T}} M_t(SO^\diamond), \quad M_t(SO^\diamond) := \big\{\xi \in M(SO^\diamond) :\ \xi|_C = t\big\}, \tag{2.1}$$

where $M_t(SO^\diamond)$ are called the fibers of $M(SO^\diamond)$ over points $t \in \mathbb{T}$.

2.2 The C*-Algebra QC of Quasicontinuous Functions

For each arc $I \subset \mathbb{T}$ and each $f \in L^1(\mathbb{T})$, the *average* of f over I is given by $I(f) := |I|^{-1} \int_I f(\tau)|d\tau|$, where $|I| := \int_I |d\tau|$ is the Lebesgue measure of I. A function $f \in L^1(\mathbb{T})$ is said to have *vanishing mean oscillation* on $\mathbb{T}$ if

$$\lim_{\delta\to 0}\left(\sup_{I\subset\mathbb{T},\ |I|\le\delta} \frac{1}{|I|}\int_I |f(\tau) - I(f)|\,|d\tau|\right) = 0.$$

The set of functions of vanishing mean oscillation on $\mathbb{T}$ is denoted by VMO.

Let H^∞ be the closed subalgebra of $L^\infty(\mathbb{T})$ that consists of all functions being non-tangential limits on $\mathbb{T}$ of bounded analytic functions on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. According to [25] and [26], the C^*-algebra QC of *quasicontinuous* functions on $\mathbb{T}$ is defined by

$$QC := (H^\infty + C) \cap (\overline{H^\infty} + C) = VMO \cap L^\infty(\mathbb{T}).$$

By the proof of [19, Theorem 4.2], we immediately obtain the following.

Theorem 2.1 *The C^*-algebra $SO^\diamond$ is contained in the C^*-algebra QC.*

Since $C \subset QC$, it follows similarly to (2.1) that

$$M(QC) = \bigcup_{t\in\mathbb{T}} M_t(QC), \quad M_t(QC) := \big\{\xi \in M(QC) :\ \xi|_C = t\big\},$$

where $M_t(QC)$ are fibers of $M(QC)$ over points $t \in \mathbb{T}$. For each $(\lambda, t) \in (1, \infty) \times \mathbb{T}$ with $t = e^{i\theta}$, the map

$$\delta_{\lambda,t} : QC \to \mathbb{C}, \quad f \mapsto \delta_{\lambda,t}(f) := \frac{\lambda}{2\pi} \int_{\theta-\frac{\pi}{\lambda}}^{\theta+\frac{\pi}{\lambda}} f(e^{ix})dx,$$

defines a linear functional in QC^*, which is identified with the point (λ, t).

Let $M_t^0(QC)$ denote the set of functionals in $M_t(QC)$ that lie in the weak-star closure in QC^* of the set $(1, \infty) \times \{t\}$. For $t \in \mathbb{T}$, we also consider the sets

$$M_t^+(QC) := \Big\{\xi \in M_t(QC) : \ \xi(f) = 0 \ \text{if} \ f \in QC \ \text{and} \ \limsup_{z\to t^+} |f(z)| = 0\Big\},$$

$$M_t^-(QC) := \Big\{\xi \in M_t(QC) : \ \xi(f) = 0 \ \text{if} \ f \in QC \ \text{and} \ \limsup_{z\to t^-} |f(z)| = 0\Big\}.$$

For each $t \in \mathbb{T}$, it follows from [26, Lemma 8] that

$$M_t^+(QC) \cap M_t^-(QC) = M_t^0(QC), \quad M_t^+(QC) \cup M_t^-(QC) = M_t(QC). \tag{2.2}$$

Hence, the fiber $M_t(QC)$ splits into the three disjoint sets: $M_t^0(QC)$ and

$$\widetilde{M}_t^+(QC) := M_t^+(QC) \setminus M_t^0(QC), \quad \widetilde{M}_t^-(QC) := M_t^-(QC) \setminus M_t^0(QC).$$

We also define the sets

$$M^\pm(QC) = \bigcup_{t\in\mathbb{T}} M_t^\pm(QC), \quad M^0(QC) = \bigcup_{t\in\mathbb{T}} M_t^0(QC), \quad \widetilde{M}^\pm(QC) = \bigcup_{t\in\mathbb{T}} \widetilde{M}_t^\pm(QC). \tag{2.3}$$

Regarding functions in $L^\infty(\mathbb{T})$ as extended harmonically into the open unit disc $\mathbb{D}$, we deduce from [26, p. 821] that the restriction g_t of a function $f \in QC$ to the radius $\gamma_t = [0, t)$, where $t \in \mathbb{T}$, is a bounded continuous function on γ_t that slowly oscillates at the point t. Then the function f defined on $\mathbb{T}$ by $f(te^{ix}) = g_t[t(1-|x|/\pi)]$ $(0 < |x| \le \pi)$ belongs to $SO_t \subset SO^\diamond \subset QC$, and $\lim_{x\to 1}[g_t(tx) - f(tx)] = 0$ in view of [26, Lemma 5]. By [26, p. 823], this allows one for every $a \in SO^\diamond$ and every $t \in \mathbb{T}$ to relate the values $a(\zeta)$ for $\zeta \in M_t(SO^\diamond)$ with values $a(\xi)$ for $\xi \in M_t^0(QC)$ by setting $a(\xi) = a(\zeta)$ whenever $\xi|_{SO^\diamond} = \zeta$.

2.3 The C*-Algebra PQC of Piecewise Quasicontinuous Functions

Let $PQC := \text{alg}(QC, PC)$ be the C^*-subalgebra of $L^\infty(\mathbb{T})$ generated by the C^*-algebras QC and PC. The functions in PQC are referred to as the *piecewise* *quasicontinuous* functions. Since $QC \subset PQC$, we have

$$M(PQC) = \bigcup_{\xi \in M(QC)} M_\xi(PQC), \quad M_\xi(PQC) := \{y \in M(PQC) : \; y|_{QC} = \xi\}.$$

As is known, $M(PC) = \mathbb{T} \times \{0, 1\}$. There is a natural map w of $M(PQC)$ into $M(QC) \times \{0, 1\}$, which is given as follows: defining $\xi = y|_{QC}$, $t = y|_C$ and $v = y|_{PC}$ for every $y \in M(PQC)$, we conclude that $w(y) = (\xi, 0)$ if $v = (t, 0)$ and $w(y) = (\xi, 1)$ if $v = (t, 1)$. We have the following characterization of fibers $M_\xi(PQC)$ for $\xi \in M(QC)$ (see [26] and also [6, Theorem 3.36]).

Lemma 2.2 *Let $t \in \mathbb{T}$ and $\xi \in M_t(QC)$. Then*

(i) $M_\xi(PQC) = \{(\xi, 1)\}$ *whenever* $\xi \in \widetilde{M}_t^+(QC)$*;*
(ii) $M_\xi(PQC) = \{(\xi, 0)\}$ *whenever* $\xi \in \widetilde{M}_t^-(QC)$*;*
(iii) $M_\xi(PQC) = \{(\xi, 0), (\xi, 1)\}$ *whenever* $\xi \in M_t^0(QC)$*. In this case, if* $t = e^{i\theta}$ *and* $\{\lambda_n\} \subset (1, \infty)$ *is such that* $(\lambda_n, t) \to \xi$ *in the weak-star topology on* QC^**, then for every* $f \in PQC$*,*

$$(\xi, 1)f = \lim_{n\to\infty} \frac{\lambda_n}{\pi} \int_\theta^{\theta + \frac{\pi}{\lambda_n}} f(e^{ix})dx, \quad (\xi, 0)f = \lim_{n\to\infty} \frac{\lambda_n}{\pi} \int_{\theta - \frac{\pi}{\lambda_n}}^{\theta} f(e^{ix})dx.$$

For $a \in PQC$ and $\xi \in M(QC)$, we put

$$a(\xi^-) := a(\xi, 0) \;\text{ if }\; \xi \in M^-(QC), \quad a(\xi^+) := a(\xi, 1) \;\text{ if }\; \xi \in M^+(QC).$$

3 Muckenhoupt Weights Equivalent to Slowly Oscillating Weights

Let $1 < p < \infty$ and $w = e^v \in A_p(\mathbb{T})$. Then $v = \ln w \in BMO(\mathbb{T})$ (see, e.g., [11, p. 258] or [3, Theorem 2.5]). For every $t \in \mathbb{T}$, we define the real-valued functions v_t and V_t on the interval $U := (-\pi/2, \pi/2)$ by

$$v_t(x) := v(te^{ix}) \quad \text{for almost all} \quad x \in (-\pi/2, \pi/2), \tag{3.1}$$

$$V_t(x) := \frac{1}{x} \int_0^x v_t(s)ds \quad \text{for all} \quad |x| \in (0, \pi/2). \tag{3.2}$$

Then $v_t \in BMO(U)$ and $V_t \in C(\overline{U} \setminus \{0\})$, where $\overline{U}$ denotes the closure of U in $\mathbb{T}$.

In what follows we assume that

(A) *for every $t \in \mathbb{T}$ there is a symmetric neighborhood $U_t \subset U$ of zero such that the function $\Sigma_t : x \mapsto xV_t'(x)$ belongs to the C^*-algebra $QC(U_t) := VMO(U_t) \cap L^\infty(U_t)$.*

Since $QC(U_t) \subset BMO(U_t)$ and since

$$xV_t'(x) = v_t(x) - V_t(x) \text{ for almost all } x \in U, \tag{3.3}$$

we infer from **(A)** that $V_t \in BMO(U_t)$ along with $v_t \in BMO(U_t)$ for every $t \in \mathbb{T}$.

By **(A)** and the definition of $VMO(U_t)$ in [11, Chapter VI], the functions $\Sigma_t : x \mapsto xV_t'(x)$ belong to $VMO_0(U_t) \cap L^\infty(U_t)$ for all $t \in \mathbb{T}$, where $VMO_0(U_t)$ consists of all functions $v \in L^1(U_t)$ such that

$$\lim_{x\to 0} \frac{1}{x}\int_0^x |v(s) - I_x(v)|ds = 0, \quad I_x(v) := \frac{1}{x}\int_0^x v(s)ds \;\; (x \in U_t \setminus \{0\})$$

and $\lim_{x\to 0}\big(I_x(v) - I_{-x}(v)\big) = 0$.

Consequently, by [18, Lemma 6.1], for every $t \in \mathbb{T}$ the function

$$x \mapsto \frac{1}{x}\int_0^x \tau V_t'(\tau)d\tau \;\; (x \in U_t \setminus \{0\})$$

belongs to the set $SO_0(U_t) = \widetilde{SO}_0(U_t) \cap L^\infty(U_t)$, where $\widetilde{SO}_0(U_t)$ consists of all functions $f \in C(\overline{U}_t \setminus \{0\})$ such that

$$\lim_{x\to +0} \max\big\{|f(y) - f(z)| : \; y, z \in [-2x, x] \cup [x, 2x]\big\} = 0. \tag{3.4}$$

Following [18], we say that a weight w is locally equivalent to a weight W at a neighborhood $u_t \subset \mathbb{T}$ of a point $t \in \mathbb{T}$ if w/W, $W/w \in L^\infty(u_t)$. Setting $W_t(te^{ix}) = e^{V_t(x)}$ for $x \in U_t$ and $t \in \mathbb{T}$, we deduce from **(A)** and (3.3) that the weight $w = e^v$ is locally equivalent to the weights W_t at the neighborhoods $u_t = \{te^{ix} : x \in U_t\}$ of points $t \in \mathbb{T}$, and the functions $\tau \mapsto (\tau - t)(\ln W_t)'(\tau)$ are in $QC(u_t)$.

For every $t \in \mathbb{T}$, we also define the function

$$\widetilde{V}_t(x) := \frac{1}{x}\int_0^x V_t(\tau)d\tau \quad (x \in U_t \setminus \{0\}). \tag{3.5}$$

In view of (3.2), (3.3) and (3.5), we infer that for each $t \in \mathbb{T}$ and all $x \in U_t \setminus \{0\}$,

$$x\widetilde{V}_t'(x) = V_t(x) - \widetilde{V}_t(x) = \frac{1}{x}\int_0^x \tau V_t'(\tau)d\tau. \tag{3.6}$$

Hence the function $\widetilde{\Sigma}_t : x \mapsto x\widetilde{V}_t'(x)$ belongs to $SO_0(U_t)$ along with the function $x \mapsto \frac{1}{x}\int_0^x \Sigma_t(\tau)d\tau$ by [17, Lemma 4] and [18, Lemma 6.1]. Then the weights w and W_t are locally equivalent on u_t to the weight $\widetilde{W}_t$ given by $\widetilde{W}_t(te^{ix}) = e^{\widetilde{V}_t(x)}$ for $x \in U_t$. The weights $\widetilde{W}_t$ defined for each $t \in \mathbb{T}$ are called *slowly oscillating weights*. Since $w \in A_p(\mathbb{T})$, it follows that $W_t, \widetilde{W}_t \in A_p(u_t)$ and $e^{\widetilde{V}_t} \in A_p(U_t)$.

Since the function $\widetilde{V}_t$ is continuously differentiable on $\overline{U_t} \setminus \{0\}$ and since the function $\widetilde{\Sigma}_t : x \to x\widetilde{V}_t'(x)$ is in $SO_0(U_t)$, we deduce from [17, Theorem 4] (cf. [3, Theorem 2.36]) that the weight $e^{\widetilde{V}_t}$ belongs to $A_p(U_t)$ if and only if

$$-1/p < \liminf_{x \to 0} \left(x\widetilde{V}_t'(x)\right) \le \limsup_{x \to 0} \left(x\widetilde{V}_t'(x)\right) < 1/q. \tag{3.7}$$

Given $p \in (1, \infty)$ and a weight $w \in A_p(\mathbb{T})$ satisfying condition **(A)**, we associate with w and every point $t \in \mathbb{T}$ the locally equivalent weight $\widetilde{W}_t = e^{\mathcal{V}_t} \in A_p(u_t)$, where $\mathcal{V}_t(\tau) = \widetilde{V}_t(-i \ln(\tau/t))$ for all $\tau \in u_t$. Then the function σ_t given by

$$\sigma_t(\tau) = (\tau - t)\mathcal{V}_t'(\tau) \quad \text{for all } \tau \in u_t \setminus \{t\}, \tag{3.8}$$

belongs to $SO_t(u_t) = SO_t(\mathbb{T})|_{u_t}$. Since $x\widetilde{V}_t'(x) = x\mathcal{V}_t'(te^{ix})tie^{ix}$ for $x \in U_t$ and

$$\lim_{x \to 0} [(te^{ix} - t)/(xtie^{ix})] = 1,$$

we conclude that the function $\widetilde{\Sigma}_t : x \mapsto x\widetilde{V}_t'(x)$ is equivalent at the point 0 to the function $\sigma_t(te^{ix}) = (te^{ix} - t)\mathcal{V}_t'(te^{ix})$ for every $t \in \mathbb{T}$. Hence $\sigma_t \in SO_t(u_t)$ if and only if $\widetilde{\Sigma}_t \in SO_0(U_t)$. This allows us to identify the points $\zeta \in M_t(SO^\diamond)$ and $\eta \in M_0(SO_0(U_t))$ by the rule $\zeta(a) = \eta(a \circ te^{i(\cdot)})$ for all $a \in SO^\diamond$. Identifying the points $\zeta \in M_t(SO^\diamond)$ and $\xi \in M_t^0(QC)$ as in Sect. 2.2, and the points $\xi \in M_t^0(QC)$ and $\widetilde{\xi} \in M_0^0(QC(U_t))$ by the rule $\xi(a) = \widetilde{\xi}(a \circ te^{i(\cdot)})$ for all $a \in QC$, we can define the numbers

$$\delta_\xi := \xi(\sigma_t) = \zeta(\sigma_t) = \eta(\widetilde{\Sigma}_t) = \widetilde{\xi}(\widetilde{\Sigma}_t) = \widetilde{\xi}\widetilde{V}_t'(\widetilde{\xi}) \quad \text{for every } \xi \in M_t^0(QC). \tag{3.9}$$

4 Mellin Pseudodifferential Operators

If a is an absolutely continuous function of finite total variation on $\mathbb{R}$, then $a' \in L^1(\mathbb{R})$ and $V(a) = \int_{\mathbb{R}} |a'(x)|dx$ (see, e.g., [20, Chapter VIII, § 3; Chapter IX, § 4]). The set $V(\mathbb{R})$ of all absolutely continuous functions a of finite total variation on $\mathbb{R}$ forms a Banach algebra when equipped with the norm $\|a\|_V := \|a\|_{L^\infty(\mathbb{R})} + V(a)$. Following [14, 15], let $C_b(\mathbb{R}_+, V(\mathbb{R}))$ denote the Banach algebra of all bounded continuous $V(\mathbb{R})$-valued functions b on $\mathbb{R}_+ = (0, \infty)$ with the norm

$$\|b(\cdot, \cdot)\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))} = \sup_{r \in \mathbb{R}_+} \|b(r, \cdot)\|_V.$$

As usual, let $C_0^\infty(\mathbb{R}_+)$ be the set of all infinitely differentiable functions of compact support on $\mathbb{R}_+$. Let $d\mu(t) = dt/t$ for $t \in \mathbb{R}_+$.

Mellin pseudodifferential operators are generalizations of Mellin convolution operators. The following boundedness result for Mellin pseudodifferential operators was obtained in [15, Theorem 6.1] (see also [14, Theorem 3.1]).

Theorem 4.1 *If* $b \in C_b(\mathbb{R}_+, V(\mathbb{R}))$*, then the Mellin pseudodifferential operator* $\mathrm{Op}(b)$*, defined for functions* $f \in C_0^\infty(\mathbb{R}_+)$ *by the iterated integral*

$$\big[\mathrm{Op}(b)f\big](r) = \frac{1}{2\pi}\int_{\mathbb{R}} d\lambda \int_{\mathbb{R}_+} b(r,\lambda)\left(\frac{r}{\varrho}\right)^{i\lambda} f(\varrho)\,\frac{d\varrho}{\varrho} \quad \textit{for } r \in \mathbb{R}_+,$$

extends to a bounded linear operator on every space $L^p(\mathbb{R}_+, d\mu)$ *with* $p \in (1,\infty)$*, and there is a number* $C_p \in (0,\infty)$ *depending only on* p *such that*

$$\|\,\mathrm{Op}(b)\|_{\mathcal{B}(L^p(\mathbb{R}_+,d\mu))} \le C_p \|b\|_{C_b(\mathbb{R}_+,V(\mathbb{R}))}.$$

Following [26], a function $f \in C_b(\mathbb{R}_+)$ is called *slowly oscillating* (at 0 and ∞) if for each (equivalently, for some) $\lambda \in (0,1)$,

$$\lim_{x\to s}\max\big\{|f(r)-f(\varrho)| : r,\varrho \in [\lambda x, x]\big\} = 0 \quad (s \in \{0,\infty\}).$$

Obviously, the set $SO(\mathbb{R}_+)$ of all slowly oscillating (at 0 and ∞) functions in $C_b(\mathbb{R}_+)$ is a unital commutative C^*-algebra. This algebra properly contains $C(\overline{\mathbb{R}}_+)$, the C^*-algebra of all continuous functions on $\overline{\mathbb{R}}_+ := [0,+\infty]$.

Let $SO(\mathbb{R}_+, V(\mathbb{R}))$ denote the Banach subalgebra of $C_b(\mathbb{R}_+, V(\mathbb{R}))$ consisting of all $V(\mathbb{R})$-valued functions b on $\mathbb{R}_+$ that slowly oscillate at 0 and ∞, that is,

$$\lim_{x\to 0}\mathrm{cm}_x^C(b) = \lim_{x\to\infty}\mathrm{cm}_x^C(b) = 0,$$

where

$$\mathrm{cm}_x^C(b) = \max\big\{\big\|b(r,\cdot)-b(\varrho,\cdot)\big\|_{L^\infty(\mathbb{R})} : r,\varrho \in [x,2x]\big\}.$$

Let $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ be the Banach algebra of all $V(\mathbb{R})$-valued functions b belonging to $SO(\mathbb{R}_+, V(\mathbb{R}))$ and such that

$$\lim_{|h|\to 0}\sup_{r\in\mathbb{R}_+}\big\|b(r,\cdot)-b^h(r,\cdot)\big\|_V = 0$$

where $b^h(r,\lambda) := b(r,\lambda+h)$ for all $(r,\lambda) \in \mathbb{R}_+\times\mathbb{R}$.

The following result on compactness of commutators of Mellin pseudodifferential operators was obtained in [16, Theorem 3.5] (see also [14, Corollary 8.4]).

Theorem 4.2 *If* $a, b \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$*, then the commutator* $[\mathrm{Op}(a), \mathrm{Op}(b)]$ *is a compact operator on every space* $L^p(\mathbb{R}_+, d\mu)$ *with* $p \in (1,\infty)$.

Consider the isometric isomorphism

$$\mathcal{T} : \mathcal{B}(L^p(\mathbb{R}_+)) \to \mathcal{B}(L^p(\mathbb{R}_+, d\mu)), \quad A \mapsto \mathcal{G}A\mathcal{G}^{-1}, \tag{4.1}$$

where $\mathcal{G}$ is the isometric isomorphism of $L^p(\mathbb{R}_+)$ onto $L^p(\mathbb{R}_+, d\mu)$ given by

$$(\mathcal{G}f)(r) = r^{1/p} f(r) \ \text{ for } \ r \in \mathbb{R}_+. \tag{4.2}$$

Theorem 4.3 ([17, Theorem 7]) *Let $p \in (1,\infty)$ and let $w = e^V \in A_p(\mathbb{R}_+)$, where the function $x \mapsto xV'(x)$ belongs to $SO(\mathbb{R}_+)$ and*

$$-1/p < \inf_{x\in\mathbb{R}_+} (xV'(x)) \le \sup_{x\in\mathbb{R}_+} (xV'(x)) < 1 - 1/p.$$

Then $\mathcal{T}\big(wS_{\mathbb{R}_+}w^{-1}I\big) = \mathrm{Op}(b) + K$, where $K \in \mathcal{K}(L^p(\mathbb{R}_+, d\mu))$ and the function $b \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ is given by

$$b(r,\lambda) := \coth\big(\pi\lambda + \pi i(1/p + rV'(r))\big) \ \text{for all} \ (r,\lambda) \in \mathbb{R}_+ \times \mathbb{R}.$$

5 Local Study of the Banach Algebra $\mathfrak{A}^\pi_{p,w}$

5.1 An Application of the Allan-Douglas Local Principle

Given $p \in (1,\infty)$ and $w \in A_p(\mathbb{T})$, we consider the unital Banach algebra

$$\mathcal{Z}_{p,w} := \big\{aI : \ a \in QC\big\} \subset \mathcal{B}_{p,w} \tag{5.1}$$

and its quotient Banach algebra $\mathcal{Z}^\pi_{p,w} := (\mathcal{Z}_{p,w} + \mathcal{K}_{p,w})/\mathcal{K}_{p,w}$ consisting of the cosets $[aI]^\pi := aI + \mathcal{K}_{p,w}$ for all $a \in QC$. By [17, Theorem 2], $\mathcal{Z}^\pi_{p,w}$ is a central subalgebra of the quotient Banach algebra $\mathfrak{A}^\pi_{p,w} = \mathfrak{A}_{p,w}/\mathcal{K}_{p,w}$, where $\mathfrak{A}_{p,w}$ is defined by (1.1) (recall that $\mathcal{K}_{p,w} \subset \mathfrak{A}_{p,w}$).

Let $\Lambda_{p,w}$ denote the Banach subalgebra of $\mathcal{B}_{p,w}$ consisting of all operators in $\mathcal{B}_{p,w}$ that commute modulo compact operators with every operator $A \in \mathcal{Z}_{p,w}$. Clearly, $\Lambda_{p,w}$ contains the Banach algebra $\mathfrak{A}_{p,w}$ and $\mathcal{Z}^\pi_{p,w}$ is a central subalgebra of the quotient Banach algebra $\Lambda^\pi_{p,w} := \Lambda_{p,w}/\mathcal{K}_{p,w}$.

By [17, Lemma 1], $M(\mathcal{Z}^\pi_{p,w}) = M(QC)$. For every $\xi \in M(QC)$, let $\mathcal{J}^\pi_{p,w,\xi}$ be the smallest closed two-sided ideal of the Banach algebra $\Lambda^\pi_{p,w}$ that contains the maximal ideal

$$\mathcal{I}^\pi_{p,w,\xi} := \{[aI]^\pi : \ a \in QC, \ a(\xi) = 0\}$$

of the central algebra $\mathcal{Z}^\pi_{p,w}$ of $\Lambda^\pi_{p,w}$. Consider the quotient Banach algebra $\Lambda^\pi_{p,w,\xi} := \Lambda^\pi_{p,w}/\mathcal{J}^\pi_{p,w,\xi}$. Let $\mathcal{A}^\pi_{p,w,\xi}$ be the smallest closed subalgebra of $\Lambda^\pi_{p,w,\xi}$ that contains the cosets $A^\pi_{p,w,\xi} := A^\pi + \mathcal{J}^\pi_{p,w,\xi}$ for all $A \in \mathfrak{A}_{p,w}$.

By the Allan-Douglas local principle (see, e.g., [6, Theorem 1.35]), we immediately obtain the following result.

Lemma 5.1 *Given $p \in (1,\infty)$ and $w \in A_p(\mathbb{T})$, an operator $A \in \mathfrak{A}_{p,w}$ is Fredholm on the weighted Lebesgue space $L^p(\mathbb{T},w)$ (equivalently, the coset $A^\pi \in \mathfrak{A}^\pi_{p,w}$ is invertible in the Banach algebra $\mathfrak{A}^\pi_{p,w}$) if and only if for every $\xi \in M(QC)$ the coset $A^\pi_{p,w,\xi} \in \mathcal{A}^\pi_{p,w,\xi}$ is invertible in the Banach algebra $\Lambda^\pi_{p,w,\xi}$.*

5.2 Local Representatives

Let us identify the cosets $A^\pi_{p,w,\xi}$ for all $A \in \mathfrak{A}_{p,w}$ and all $\xi \in M(QC)$, where $p \in (1,\infty)$ and $w \in A_p(\mathbb{T})$. For $t \in \mathbb{T}$, let χ_t^- and χ_t^+ denote the characteristic functions of the intervals $(-t,t)$ and $(t,-t)$, respectively.

Lemma 5.2 *If $a \in PQC$, $t \in \mathbb{T}$ and one of the following conditions holds:*

$$\begin{gathered} a(\xi^-) = 0 \ \text{ if } \ \xi \in \widetilde{M}_t^-(QC), \quad a(\xi^+) = 0 \ \text{ if } \ \xi \in \widetilde{M}_t^+(QC), \\ a(\xi^\pm) = 0 \ \text{ if } \ \xi \in M_t^0(QC), \end{gathered} \tag{5.2}$$

then $[aI]^\pi_{p,w,\xi} = [0]^\pi_{p,w,\xi} = \mathcal{J}^\pi_{p,w,\xi}$ for given $\xi \in M_t(QC)$.

Proof Obviously, it suffices to prove the lemma only for functions $a \in PQC$ that have finite sets of piecewise quasicontinuous discontinuities on $\mathbb{T}$. If $t \in \mathbb{T}$ is a point of such discontinuity for a function $a \in PQC$ of that class, then there exist unique functions $a_t^\pm \in QC$ such that the function

$$\widetilde{a} := a - a_t^-\chi_t^- - a_t^+\chi_t^+ \in PQC \tag{5.3}$$

vanishes at an open neighborhood $u_t \subset \mathbb{T}$ of t. Consider a smaller open neighborhood $\widetilde{u}_t$ of t such that the closure of $\widetilde{u}_t$ is contained in u_t. For every $\xi \in M_t(QC)$, we now take a function $c_\xi \in QC$ such that $c_\xi(\xi) = 0$ and $c_\xi(\eta) = 1$ for all $\eta \in \bigcup_{\tau \in \mathbb{T}\setminus\widetilde{u}_t} M_\tau(QC)$. Then $\widetilde{a} = \widetilde{a}c_\xi$, which implies that $[\widetilde{a}I]^\pi \in \mathcal{J}^\pi_{p,w,\xi}$, and hence, by (5.3),

$$[aI]^\pi_{p,w,\xi} = [(a_t^-\chi_t^- + a_t^+\chi_t^+)I]^\pi_{p,w,\xi} \quad \text{for every} \quad \xi \in M_t(QC). \tag{5.4}$$

If $\xi \in M_t^0(QC)$, then we infer from (5.4) and (5.2) that

$$a_t^+(\xi) = a(\xi^+) = 0 \quad \text{and} \quad a_t^-(\xi) = a(\xi^-) = 0, \tag{5.5}$$

since $\chi_t^+(\xi^+) = \chi_t^-(\xi^-) = 1$ and $\chi_t^+(\xi^-) = \chi_t^-(\xi^+) = 0$ by [26, Lemma 13]. Hence, by (5.4) and (5.5), $[(a_t^-\chi_t^- + a_t^+\chi_t^+)I]^\pi \in \mathcal{J}_{p,w,\xi}^\pi$, and then $[aI]_{p,w,\xi}^\pi = \mathcal{J}_{p,w,\xi}^\pi$.

Further, if $\xi \in \widetilde{M}_t^+(QC)$, then the coset $[\chi_t^- I]^\pi \in \mathcal{J}_{p,w,\xi}^\pi$. Indeed, take a function $g \in QC$ such that $g(\xi) = 1$ and $g = 0$ on $M_t^0(QC)$. Then, by the proof of [26, Lemma 13], $\chi_t^- g \in QC$ and $(\chi_t^- g)(\xi) = \chi_t^-(t^+)g(\xi) = 0$. Hence,

$$[\chi_t^- I]^\pi = [\chi_t^- gI]^\pi - [\chi_t^-(g - g(\xi))I]^\pi,$$

where $[\chi_t^- gI]^\pi \in \mathcal{I}_{p,w,\xi}^\pi$ and $[\chi_t^-(g - g(\xi))I]^\pi \in \mathcal{J}_{p,w,\xi}^\pi$, which means that $[\chi_t^- I]^\pi \in \mathcal{J}_{p,w,\xi}^\pi$. Similarly, $[\chi_t^+ I]^\pi \in \mathcal{J}_{p,w,\xi}^\pi$ if $\xi \in \widetilde{M}_t^-(QC)$. Thus, by (5.4),

$$\begin{aligned} [(a - a_t^-)I]^\pi &= [(a_t^+ - a_t^-)\chi_t^+ I]^\pi \in \mathcal{J}_{p,w,\xi}^\pi \quad \text{if} \quad \xi \in \widetilde{M}_t^-(QC), \\ [(a - a_t^+)I]^\pi &= [(a_t^- - a_t^+)\chi_t^- I]^\pi \in \mathcal{J}_{p,w,\xi}^\pi \quad \text{if} \quad \xi \in \widetilde{M}_t^+(QC), \end{aligned} \tag{5.6}$$

which implies in view of (5.2) that

$$a_t^-(\xi) = a(\xi^-) = 0 \ \text{ if } \ \xi \in \widetilde{M}_t^-(QC), \ \ a_t^+(\xi) = a(\xi^+) = 0 \ \text{ if } \ \xi \in \widetilde{M}_t^+(QC).$$

Hence, $[a_t^\pm I]^\pi \in \mathcal{I}_{p,w,\xi}^\pi$ if $\xi \in \widetilde{M}_t^\pm(QC)$, respectively, and therefore, by (5.6), $[aI]_{p,w,\xi}^\pi = \mathcal{J}_{p,w,\xi}^\pi$ for all $\xi \in \widetilde{M}_t^\pm(QC)$ as well. ◻

Theorem 5.3 *For every $t \in \mathbb{T}$ and every $\xi \in M_t(QC)$, the mapping $\beta_\xi : A \mapsto A_{p,w,\xi}^\pi$ given on the generators aI $(a \in PQC)$ and $S_\mathbb{T}$ of the algebra $\mathfrak{A}_{p,w}$ by*

$$\beta_\xi(aI) := \begin{cases} \big[a(\xi^\pm)I\big]_{p,w,\xi}^\pi & \text{if } \xi \in \widetilde{M}_t^\pm(QC), \\ \big[(a(\xi^+)\chi_t^+ + a(\xi^-)\chi_t^-)I\big]_{p,w,\xi}^\pi & \text{if } \xi \in M_t^0(QC), \end{cases} \tag{5.7}$$

$$\beta_\xi(S_\mathbb{T}) := \big[S_\mathbb{T}\big]_{p,w,\xi}^\pi \ \text{ if } \ \xi \in M_t(QC), \tag{5.8}$$

extends to a Banach algebra homomorphism $\beta_\xi : \mathfrak{A}_{p,w} \to \mathcal{A}_{p,w,\xi}^\pi$. Moreover,

$$\sup_{\xi \in M(QC)} \|\beta_\xi(A)\|_{\mathcal{A}_{p,w,\xi}^\pi} \le \|A^\pi\| := \inf_{K \in \mathcal{K}_{p,w}} \|A + K\| \quad \text{for all} \quad A \in \mathfrak{A}_{p,w}.$$

Proof Let $\xi \in M(QC)$. Consider the Banach algebra homomorphisms

$$\beta_\xi : \mathfrak{A}_{p,w} \to \mathfrak{A}_{p,w}^\pi \to \mathcal{A}_{p,w,\xi}^\pi, \quad A \mapsto A^\pi \mapsto A_{p,w,\xi}^\pi.$$

Obviously, for every $A \in \mathfrak{A}_{p,w}$,

$$\sup\left\{\|\beta_\xi(A)\|_{\mathcal{A}_{p,w,\xi}^\pi} : \xi \in M(QC)\right\} \le \|A^\pi\|.$$

It remains to prove (5.7) for $a \in PQC$ because (5.8) for $S_\mathbb{T} \in \mathfrak{A}_{p,w}$ is evident.

Fix $t \in \mathbb{T}$ and $\xi \in M_t(QC)$. For every $a \in PQC$, the function

$$\widehat{a}_\xi := \begin{cases} a - a(\xi^\pm) & \text{if } \ \xi \in \widetilde{M}_t^\pm(QC), \\ a - a(\xi^-)\chi_t^- - a(\xi^+)\chi_t^+ & \text{if } \ \xi \in M_t^0(QC), \end{cases}$$

belongs to PQC. Moreover, by Lemma 5.2, $[\widehat{a}_\xi I]^\pi \in \mathcal{J}^\pi_{p,w,\xi}$. This gives (5.7). □

5.3 Structure of the Local Algebras $\mathcal{A}^\pi_{p,w,\xi}$

Clearly, $P_\pm := (I \pm S_\mathbb{T})/2$ are projections on the space $L^p(\mathbb{T}, w)$. Theorem 5.3 directly implies the following result on the structure of the local algebras $\mathcal{A}^\pi_{p,w,\xi}$.

Lemma 5.4 *Given $p \in (1, \infty)$, $w \in A_p(\mathbb{T})$ and $\xi \in M(QC)$, the local algebras $\mathcal{A}^\pi_{p,w,\xi}$ generated by the cosets $[S_\mathbb{T}]^\pi_{p,w,\xi}$ and $[aI]^\pi_{p,w,\xi}$ for all $a \in PQC$ have the following structure:*

(i) *if $t \in \mathbb{T}$ and $\xi \in M_t^0(QC)$, then $\mathcal{A}^\pi_{p,w,\xi}$ is generated by the unit $I^\pi_{p,w,\xi}$ and two idempotents*

$$P^\pi_{p,w,\xi} := [P_+]^\pi_{p,w,\xi}, \quad Q^\pi_{p,w,\xi} := [\chi_t^+ I]^\pi_{p,w,\xi}; \tag{5.9}$$

(ii) *if $t \in \mathbb{T}$ and $\xi \in \widetilde{M}_t^\pm(QC)$, then $\mathcal{A}^\pi_{p,w,\xi}$ is generated by the unit $I^\pi_{p,w,\xi}$ and one idempotent $P^\pi_{p,w,\xi} := [P_+]^\pi_{p,w,\xi}$.*

If $\xi \in \widetilde{M}^\pm(QC)$, then the Banach algebra $\mathcal{A}^\pi_{p,w,\xi}$ according to Lemma 5.4(ii) is commutative, any coset in this algebra has the form $[c_+P_+ + c_-P_-]^\pi_{p,w,\xi}$ $(c_\pm \in \mathbb{C})$, where $[P_+]^\pi_{p,w,\xi} = P^\pi_{p,w,\xi}$, $[P_-]^\pi_{p,w,\xi} = I^\pi_{p,w,\xi} - P^\pi_{p,w,\xi}$, and the map Θ_ξ defined by

$$\Theta_\xi(I^\pi_{p,w,\xi}) = \mathrm{diag}\{1, 1\}, \quad \Theta_\xi(P^\pi_{p,w,\xi}) = \mathrm{diag}\{1, 0\}, \tag{5.10}$$

extends to a Banach algebra isomorphism $\Theta_\xi : \mathcal{A}^\pi_{p,w,\xi} \to \mathrm{diag}\{\mathbb{C}, \mathbb{C}\}$ of the Banach algebra $\mathcal{A}^\pi_{p,w,\xi}$ onto the C^*-algebra of diagonal 2×2 complex-valued matrices. Since, for arbitrary functions $a_\pm \in PQC$,

$$[a_+P_+ + a_-P_-]^\pi_{p,w,\xi} = [a_+(\xi^\pm)P_+ + a_-(\xi^\pm)P_-]^\pi_{p,w,\xi} \quad \text{if } \ \xi \in \widetilde{M}^\pm(QC),$$

respectively, and therefore, by (5.10),

$$\Theta_\xi\big([a_+P_+ + a_-P_-]^\pi_{p,w,\xi}\big) = \mathrm{diag}\,\{a_+(\xi^\pm), a_-(\xi^\pm)\}, \tag{5.11}$$

we immediately obtain the following.

Theorem 5.5 *Given $p \in (1,\infty)$, $w \in A_p(\mathbb{T})$ and $\xi \in \widetilde{M}^{\pm}(QC)$, the Banach algebra $\mathcal{A}^{\pi}_{p,w,\xi}$ is inverse closed in the Banach algebra $\Lambda^{\pi}_{p,w,\xi}$, and a coset $A^{\pi}_{p,w,\xi}$ is invertible in the Banach algebra $\mathcal{A}^{\pi}_{p,w,\xi}$ if and only if* $\det \Theta_{\xi}\big(A^{\pi}_{p,w,\xi}\big) \neq 0$.

It remains to study the invertibility of the cosets $A^{\pi}_{p,w,\xi}$ in the Banach algebra $\mathcal{A}^{\pi}_{p,w,\xi}$ for every $\xi \in M^0(QC)$.

6 The Local Study of the Banach Algebra $\mathfrak{A}_{p,w}$ for $\xi \in M^0(QC)$

6.1 Required Quotient Banach Algebras and Their Ideals

Let $p \in (1,\infty)$ and let $w \in A_p(\mathbb{T})$ satisfy condition **(A)**. Taking $w \equiv 1$ in $\mathcal{B}_{p,w}$ and $\mathcal{K}_{p,w}$, we abbreviate $\mathcal{B}_p := \mathcal{B}_{p,1}$ and $\mathcal{K}_p := \mathcal{K}_{p,1}$. Let $\Lambda^{\pi}_p := \Lambda^{\pi}_{p,1}$ be the Banach algebra of all cosets in the quotient Banach algebra $\mathcal{B}^{\pi}_p := \mathcal{B}_p/\mathcal{K}_p$ that commute with each coset in the Banach algebra $\mathcal{Z}^{\pi}_p := \{aI + \mathcal{K}_p : a \in QC\} \subset \mathcal{B}^{\pi}_p$. Hence, $\mathcal{Z}^{\pi}_p$ is a central subalgebra of Λ^{π}_p.

For every $\xi \in M^0(QC)$, let $\mathcal{J}^{\pi}_{p,\xi}$ be the smallest closed two-sided ideal of the Banach algebra Λ^{π}_p that contains the maximal ideal

$$\mathcal{I}^{\pi}_{p,\xi} := \{[aI]^{\pi} : a \in QC,\ a(\xi) = 0\}$$

of $\mathcal{Z}^{\pi}_p$, and let $\Lambda^{\pi}_{p,\xi} := \Lambda^{\pi}_p/\mathcal{J}^{\pi}_{p,\xi}$.

Let E be one of the sets $U_t = [-\delta,\delta] \subset \mathbb{R}$, $\gamma = [0,\delta] \subset \mathbb{R}_+$ or $\mathbb{R}_+$, and let $QC(E) := VMO(E) \cap L^{\infty}(E)$. Given $p \in (1,\infty)$ and a set E, let $\Lambda_p(E)$ be the Banach algebra of all operators $A \in \mathcal{B}(L^p(E))$ for which $[aI, A] \in \mathcal{K}_p(E)$ for all $a \in QC(E)$, where $\mathcal{K}_p(E)$ is the ideal of compact operators on the space $L^p(E)$, and let $\Lambda^{\pi}_p(E) := \Lambda_p(E)/\mathcal{K}_p(E)$. Relating characters $\xi \in M^0_t(QC)$ and $\widetilde{\xi} \in M^0_0(QC(U_t))$ by $\widetilde{\xi}(a_t) = \xi(a)$, where $a_t(x) = a(te^{ix})$ for $a \in QC$ and $x \in U_t$, and identifying characters $\widetilde{\xi} \in M^0_0(QC(E))$ for all E, we consider the closed two-sided ideal $\mathcal{J}^{\pi}_{p,\widetilde{\xi},E}$ of the Banach algebra $\Lambda^{\pi}_p(E)$, which is generated by the maximal ideal $\mathcal{I}^{\pi}_{p,\widetilde{\xi},E} := \{aI + \mathcal{K}_p(E) : a \in QC(E),\ a(\widetilde{\xi}) = 0\}$ of the central subalgebra $\mathcal{Z}^{\pi}_p(E) = \{aI + \mathcal{K}_p(E) : a \in QC(E)\}$ of $\Lambda^{\pi}_p(E)$. Let $A^{\pi}_{p,\widetilde{\xi},E} := A^{\pi} + \mathcal{J}^{\pi}_{p,\widetilde{\xi},E}$ and let $\Lambda^{\pi}_{p,\widetilde{\xi}}(E) := \Lambda^{\pi}_p(E)/\mathcal{J}^{\pi}_{p,\widetilde{\xi},E}$.

6.2 Localization of Muckenhoupt Weights Satisfying Condition (A)

Let us simplify the cosets $[wS_{\mathbb{T}}w^{-1}I]^{\pi}_{p,\xi} := [wS_{\mathbb{T}}w^{-1}I]^{\pi} + \mathcal{J}^{\pi}_{p,\xi}$ for $\xi \in M^0(QC)$. Modifying [17, Theorem 8], we establish the following.

Theorem 6.1 *Let $p \in (1,\infty)$ and let $w = e^v \in A_p(\mathbb{T})$ be a weight satisfying condition* **(A)**. *If $t \in \mathbb{T}$ and $\xi \in M_t^0(QC)$, then*

$$[wS_{\mathbb{T}}w^{-1}I]^{\pi}_{p,\xi} = [w_\xi S_{\mathbb{T}}w_\xi^{-1}I]^{\pi}_{p,\xi}, \tag{6.1}$$

where $w_\xi(\tau) = |\tau - t|^{\delta_\xi}$ for all $\tau \in \mathbb{T}$, and the number $\delta_\xi \in (-1/p, 1-1/p)$ is given by (3.8) *and* (3.9).

Proof Fix $t \in \mathbb{T}$, and define the real-valued functions v_t, V_t and $\widetilde{V}_t$ on the interval $U := (-\pi/2, \pi/2)$ by formulas (3.1), (3.2), and (3.5), respectively. Then it follows that $v_t \in BMO(U)$, $V_t \in BMO(U_t)$, the function $\Sigma_t : x \mapsto xV_t'(x)$ is in $QC(U_t)$ by condition **(A)**, and the function $\widetilde{\Sigma}_t : x \mapsto x\widetilde{V}_t'(x)$ belongs to $SO_0(U_t)$, where $U_t \subset U$ is a symmetric closed neighborhood of zero. Let $\mathcal{V}_t(\tau) = \widetilde{V}_t(-i\log(\tau/t))$ for all $\tau \in u_t$, where $u_t = \{te^{ix} : x \in U_t\}$. Then we infer from (3.3) and (3.6) that $v - \mathcal{V}_t \in QC(u_t)$ because the function $x \mapsto v_t(x) - \widetilde{V}_t(x) = xV_t'(x) + x\widetilde{V}'(x)$ belongs to $QC(U_t)$. Hence, for $w = e^v$ and $\widetilde{W}_t = e^{\mathcal{V}_t}$, it follows that

$$[\chi_{u_t} wS_{\mathbb{T}}w^{-1}\chi_{u_t}I]^{\pi} = [\chi_{u_t}\widetilde{W}_t S_{\mathbb{T}}\widetilde{W}_t^{-1}\chi_{u_t}I]^{\pi} \tag{6.2}$$

because $e^{v-\mathcal{V}_t} \in QC(u_t)$ and therefore $e^{v-\mathcal{V}_t}S_{\mathbb{T}}e^{\mathcal{V}_t - v}I - S_{\mathbb{T}} \in \mathcal{K}_p$.

Let $\gamma = [0,\delta]$, where $U_t = [-\delta,\delta]$ and $0 < \delta < \pi/2$. Consider the isometric isomorphism

$$\Upsilon_t : L^p(u_t) \to L_2^p(\gamma), \quad (\Upsilon_t f)(x) = \begin{Bmatrix} f(te^{ix}) \\ f(te^{-ix}) \end{Bmatrix}, \quad x \in \gamma, \tag{6.3}$$

where the norm of vector-functions $\varphi = \{\varphi_k\}_{k=1}^2 \in L_2^p(\gamma)$ with entries in $L^p(\gamma)$ is given by $\|\varphi\| = \big(\|\varphi_1\|^p_{L^p(\gamma)} + \|\varphi_2\|^p_{L^p(\gamma)}\big)^{1/p}$. It is easily seen that

$$\big[\Upsilon_t(\chi_{u_t}\widetilde{W}_t S_{\mathbb{T}}\widetilde{W}_t^{-1}\chi_{u_t})\Upsilon_t^{-1}\big]^{\pi} = \begin{bmatrix} \big[e^{\widetilde{V}_t}S_\gamma e^{-\widetilde{V}_t}I\big]^{\pi} & -\big[e^{\widetilde{V}_t}R_\gamma e^{-\widetilde{V}_t^\circ}I\big]^{\pi} \\ \big[e^{\widetilde{V}_t^\circ}R_\gamma e^{-\widetilde{V}_t}I\big]^{\pi} & -\big[e^{\widetilde{V}_t^\circ}S_\gamma e^{-\widetilde{V}_t^\circ}I\big]^{\pi} \end{bmatrix}, \tag{6.4}$$

where $\widetilde{V}_t^\circ(x) = \widetilde{V}_t(-x)$ for $x \in \gamma$, the operators S_γ and R_γ are given by

$$(S_\gamma\psi)(x) = \frac{1}{\pi i}\int_\gamma \frac{\psi(y)dy}{y-x}, \quad (R_\gamma\psi)(x) = \frac{1}{\pi i}\int_\gamma \frac{\psi(y)dy}{y+x} \quad (x \in \gamma).$$

Since $\lim_{x\to 0}\big(\widetilde{V}_t(-x)-\widetilde{V}_t(x)\big)=0$ in view of (3.4) and therefore $e^{\widetilde{V}_t^\circ-\widetilde{V}_t}\in C(\overline{\gamma})$ with value 1 at zero, we infer from (6.4) that

$$\big[\Upsilon_t(\chi_{u_t}\widetilde{W}_t S_{\mathbb{T}}\widetilde{W}_t^{-1}\chi_{u_t})\Upsilon_t^{-1}\big]^\pi=\begin{bmatrix}\big[e^{\widetilde{V}_t}S_\gamma e^{-\widetilde{V}_t}I\big]^\pi & -\big[e^{\widetilde{V}_t}R_\gamma e^{-\widetilde{V}_t}I\big]^\pi\\ \big[e^{\widetilde{V}_t}R_\gamma e^{-\widetilde{V}_t}I\big]^\pi & -\big[e^{\widetilde{V}_t}S_\gamma e^{-\widetilde{V}_t}I\big]^\pi\end{bmatrix}. \tag{6.5}$$

Similarly, defining $\widetilde{w}_\xi(x)=x^{\delta_\xi}$ for all $x\in\gamma$, we obtain

$$\big[\Upsilon_t(\chi_{u_t}w_\xi S_{\mathbb{T}}w_\xi^{-1}\chi_{u_t})\Upsilon_t^{-1}\big]^\pi=\begin{bmatrix}\big[\widetilde{w}_\xi S_\gamma\widetilde{w}_\xi^{-1}I\big]^\pi & -\big[\widetilde{w}_\xi R_\gamma\widetilde{w}_\xi^{-1}I\big]^\pi\\ \big[\widetilde{w}_\xi R_\gamma\widetilde{w}_\xi^{-1}I\big]^\pi & -\big[\widetilde{w}_\xi S_\gamma\widetilde{w}_\xi^{-1}I\big]^\pi\end{bmatrix}. \tag{6.6}$$

Take the function $\sigma_t\in SO_t(u_t)$ defined by (3.8) with $\mathcal{V}_t(\tau)=\widetilde{V}_t(-i\log(\tau/t))$ for $\tau\in u_t$. Identifying the points $\xi\in M_t^0(QC)$ and $\widetilde{\xi}\in M_0^0(QC(\gamma))$ by the rule $\widetilde{\xi}\big(a\circ te^{i(\cdot)}\big)=\xi(a)$ for all $a\in QC$, we conclude from (3.9) that the values $\sigma_t(\xi)$ are given by $\delta_\xi=\xi(\sigma_t)=\widetilde{\xi}\,\widetilde{V}_t'(\widetilde{\xi})$ for all $\xi\in M_t^0(QC)$. Thus, to prove (6.1), it remains to show in view of (6.2), (6.5) and (6.6) that

$$\big[e^{\widetilde{V}_t}S_\gamma e^{-\widetilde{V}_t}I\big]^\pi_{p,\widetilde{\xi},\gamma}=\big[\widetilde{w}_\xi S_\gamma\widetilde{w}_\xi^{-1}I\big]^\pi_{p,\widetilde{\xi},\gamma}, \tag{6.7}$$

$$\big[e^{\widetilde{V}_t}R_\gamma e^{-\widetilde{V}_t}I\big]^\pi_{p,\widetilde{\xi},\gamma}=\big[\widetilde{w}_\xi R_\gamma\widetilde{w}_\xi^{-1}I\big]^\pi_{p,\widetilde{\xi},\gamma} \tag{6.8}$$

for every $t\in\mathbb{T}$ and every $\xi\in M_t^0(QC)$, where $A^\pi_{p,\widetilde{\xi},\gamma}=A^\pi+\mathcal{J}^\pi_{p,\widetilde{\xi},\gamma}$.

Extending the function $\widetilde{V}_t$ from $\gamma\setminus\{0\}$ to a continuous function on $\mathbb{R}_+$ that vanishes at a neighborhood of $+\infty$ and denoting this extension by $\widetilde{V}_t$ again, we conclude that $e^{\widetilde{V}_t}\in A_p(\mathbb{R}_+)$. Hence, by [17, Theorem 4],

$$0<1/p+\liminf_{x\to 0}\big(x\widetilde{V}_t'(x)\big)\le 1/p+\limsup_{x\to 0}\big(x\widetilde{V}_t'(x)\big)<1. \tag{6.9}$$

Moreover, setting $\overline{\mathbb{R}}_+=[0,+\infty]$ and replacing $e^{\widetilde{V}_t}$ by an equivalent weight $e^{\widehat{V}_t}$ such that $e^{\widetilde{V}_t-\widehat{V}_t}\in C(\overline{\mathbb{R}}_+)$, $\lim_{x\to s}(\widetilde{V}_t(x)-\widehat{V}_t(x))=0$ for all $s\in\{0,\infty\}$, and therefore $[e^{\widetilde{V}-\widehat{V}}I,S_{\mathbb{R}_+}]\in\mathcal{K}_p(\mathbb{R}_+)$, we may assume without loss of generality that

$$0<1/p+\inf_{x\in\mathbb{R}_+}\big(x\widetilde{V}_t'(x)\big)\le 1/p+\sup_{x\in\mathbb{R}_+}\big(x\widetilde{V}_t'(x)\big)<1$$

instead of (6.9). Then it follows from Theorem 4.3 that

$$\mathcal{T}(e^{\widetilde{V}_t}S_{\mathbb{R}_+}e^{-\widetilde{V}_t}I)=\mathrm{Op}(\widetilde{b})+K, \tag{6.10}$$

where $\mathcal{T}$ is given by (4.1)–(4.2), the function $\widetilde{b}\in\mathcal{E}(\mathbb{R}_+,V(\mathbb{R}))$ is given by

$$\widetilde{b}(r,\lambda):=\coth\big(\pi\lambda+\pi i(1/p+r\widetilde{V}_t'(r))\big),$$

and $K \in \mathcal{K}(L^p(\mathbb{R}_+, d\mu))$. Let $\mathcal{T}_\gamma : L^p(\gamma) \to \chi_\gamma L^p(\mathbb{R}_+, d\mu)$ be the restriction of $\mathcal{T}$ to γ. By (6.10), we obtain

$$\big[\mathcal{T}_\gamma\big(e^{\widetilde{V}_t} S_\gamma e^{-\widetilde{V}_t} I\big)\big]^\pi = \big[\chi_\gamma \mathcal{T}\big(e^{\widetilde{V}_t} S_{\mathbb{R}_+} e^{-\widetilde{V}_t} I\big)\chi_\gamma I\big]^\pi = \big[\chi_\gamma \operatorname{Op}(\widetilde{b})\chi_\gamma I\big]^\pi. \tag{6.11}$$

Since $[aI, e^{\widetilde{V}_t} S_\gamma e^{-\widetilde{V}_t} I] \in \mathcal{K}(L^p(\gamma))$ for all $a \in QC(\gamma)$, we infer that the commutators $\big[aI, \chi_\gamma \operatorname{Op}(\widetilde{b})\chi_\gamma I\big]$ belong to $\mathcal{K}(L^p(\gamma, d\mu))$ for all $a \in QC(\gamma)$. By [17, Lemma 4] and the proof of [17, Theorem 8], for any function $\theta \in SO(\mathbb{R}_+)$ being a sufficiently small perturbation in $SO(\mathbb{R}_+)$ of the function $\widetilde{\Sigma}_t : x \mapsto x\widetilde{V}_t'(x)$, the function b given by $b(r, \lambda) := \coth\big(\pi\lambda + \pi i(1/p + \theta(r))\big)$ belongs to $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$, and the commutators $[aI, \chi_\gamma \operatorname{Op}(b)\chi_\gamma I]$ belong to $\mathcal{K}(L^p(\gamma, d\mu))$ for all $a \in QC(\gamma)$.

Given $\widetilde{\xi} \in M_0^0(QC(\gamma))$ and taking $\widehat{\xi} = \widetilde{\xi}|_{SO(\mathbb{R}_+)} \in M_0(SO(\mathbb{R}_+))$, we choose a function $\theta \in SO(\mathbb{R}_+)$ such that $\theta(\eta) = \widetilde{\Sigma}_t(\eta) = \eta\widetilde{V}_t'(\eta) = \widetilde{\xi}\widetilde{V}_t'(\widetilde{\xi})$ for all η in an open neighborhood $U_{\widehat{\xi}} \subset M(SO(\mathbb{R}_+))$ of a point $\widehat{\xi} \in M_0(SO(\mathbb{R}_+))$, and the norm $\|\sigma - \theta\|_{L^\infty(\mathbb{R}_+)}$ is sufficiently small. Then $b(\eta, \lambda) = \widetilde{b}(\eta, \lambda) = \widetilde{b}(\widetilde{\xi}, \lambda)$ for all $\eta \in U_{\widehat{\xi}}$ and all $\lambda \in \mathbb{R}$. Hence there exists a function $d \in SO(\mathbb{R}_+)$ such that

$$d(\widetilde{\xi}) = 0 \quad \text{and} \quad d(\widetilde{b} - b) = \widetilde{b} - b. \tag{6.12}$$

Since $[aI, \chi_\gamma \operatorname{Op}(\widetilde{b} - b)\chi_\gamma I] \in \mathcal{K}(L^p(\gamma, d\mu))$ for all $a \in QC(\gamma)$, it follows that $\chi_\gamma \mathcal{T}^{-1}(\operatorname{Op}(\widetilde{b} - b))\chi_\gamma I \in \Lambda_p(\gamma)$. On the other hand, by (6.12) and Theorem 4.2,

$$\chi_\gamma \operatorname{Op}(\widetilde{b}-b)\chi_\gamma I = \chi_\gamma \operatorname{Op}(d(\widetilde{b}-b))\chi_\gamma I = d\chi_\gamma \operatorname{Op}(\widetilde{b}-b)\chi_\gamma I \simeq \chi_\gamma \operatorname{Op}(\widetilde{b}-b)\chi_\gamma dI,$$

where $d(\widetilde{\xi}) = 0$ and $A \simeq B$ means that $A - B \in \mathcal{K}(L^p(\gamma, d\mu))$. Hence the coset $[\chi_\gamma \mathcal{T}^{-1}(\operatorname{Op}(\widetilde{b} - b))\chi_\gamma I]^\pi$ belongs to the ideal $\mathcal{J}^\pi_{p,\widetilde{\xi},\gamma}$ for given $\widetilde{\xi} \in M_0^0(QC(\gamma))$.

Finally, since the function $\widetilde{b} - \widetilde{b}(\widetilde{\xi}, \cdot)$ can be approximated in the norm of $C_b(\mathbb{R}_+, V(\mathbb{R}))$ by functions of the form $\widetilde{b} - b$ in view of the estimate

$$\|\widetilde{b} - b\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))} \le C\|\widetilde{\Sigma}_t - \theta\|_{L^\infty(\mathbb{R}_+)}$$

(see [17, Lemma 5]), we conclude that

$$[\chi_\gamma \mathcal{T}^{-1}\big(\operatorname{Op}(\widetilde{b} - \widetilde{b}(\widetilde{\xi}, \cdot))\big)\chi_\gamma I]^\pi \in \mathcal{J}^\pi_{p,\widetilde{\xi},\gamma} \quad \text{for all } \widetilde{\xi} \in M_0^0(QC(\gamma)), \tag{6.13}$$

where the function $\widetilde{b}(\widetilde{\xi}, \cdot) \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ is given by

$$\widetilde{b}(\widetilde{\xi}, \lambda) := \coth\big(\pi\lambda + \pi i(1/p + \widetilde{\xi}\widetilde{V}_t'(\widetilde{\xi}))\big).$$

Hence, we infer from (6.11) and (6.13) that

$$\big[e^{\widetilde{V}_t} S_\gamma e^{-\widetilde{V}_t} I\big]^\pi_{p,\widetilde{\xi},\gamma} = \big[\chi_\gamma \mathcal{T}^{-1}(\operatorname{Op}(\widetilde{b}(\widetilde{\xi}, \cdot)))\chi_\gamma I\big]^\pi_{p,\widetilde{\xi},\gamma}. \tag{6.14}$$

Similarly to (6.11), we obtain

$$\left[\mathcal{T}_\gamma\left(\widetilde{w}_\xi S_\gamma \widetilde{w}_\xi^{-1} I\right)\right]^\pi = \left[\chi_\gamma \mathcal{T}\left(\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I\right)\chi_\gamma I\right]^\pi = \left[\chi_\gamma \operatorname{Op}(\widetilde{b}(\widetilde{\xi},\cdot))\chi_\gamma I\right]^\pi,$$

which implies that

$$\left[\widetilde{w}_\xi S_\gamma \widetilde{w}_\xi^{-1} I\right]^\pi_{p,\widetilde{\xi},\gamma} = \left[\chi_\gamma \mathcal{T}^{-1}(\operatorname{Op}(\widetilde{b}(\widetilde{\xi},\cdot)))\chi_\gamma I\right]^\pi_{p,\widetilde{\xi},\gamma}. \tag{6.15}$$

Applying (6.14) and (6.15), we obtain (6.7). Equality (6.8) is proved analogously, which completes the proof of (6.1). □

6.3 *Spectra of Necessary Cosets*

Consider the C^*-algebra of quasicontinuous on $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ functions given by

$$QC(\dot{\mathbb{R}}) := (H^\infty + C(\dot{\mathbb{R}})) \cap (\overline{H^\infty} + C(\dot{\mathbb{R}})),$$

where H^∞ is the closed subalgebra of $L^\infty(\mathbb{R})$ that consists of all functions being non-tangential limits on $\mathbb{R}$ of bounded analytic functions defined on the upper half-plane.

Given $p \in (1,\infty)$, $\xi \in M_t^0(QC)$, $\delta_\xi \in (-1/p, 1/q)$ and $\widetilde{w}_\xi(x) = x^{\delta_\xi}$ for $x \in \mathbb{R}_+$, we consider the Banach subalgebra of $\mathcal{B}(L^p(\mathbb{R}_+))$ of the form

$$\widetilde{\mathfrak{A}}_{p,\widetilde{w}_\xi}(\mathbb{R}_+) := \operatorname{alg}\{aI, \widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I : a \in QC(\mathbb{R}_+)\}, \tag{6.16}$$

which is generated by the multiplication operators aI for all $a \in QC(\mathbb{R}_+)$ and by the operator $\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I$, where $QC(\mathbb{R}_+) = QC(\dot{\mathbb{R}})|_{\mathbb{R}_+}$. The ideal $\mathcal{K}(L^p(\mathbb{R}_+))$ of all compact operators in $\mathcal{B}(L^p(\mathbb{R}_+))$ is contained in $\widetilde{\mathfrak{A}}_{p,\widetilde{w}_\xi}(\mathbb{R}_+)$ (see, e.g., [21, Theorem 4.1.5]). Since the commutator $[aI, \widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I]$ is a compact operator on the space $L^p(\mathbb{R}_+)$ for every $a \in QC(\mathbb{R}_+)$, it follows that the quotient Banach algebra $\widetilde{\mathfrak{A}}^\pi_{p,\widetilde{w}_\xi}(\mathbb{R}_+) := \widetilde{\mathfrak{A}}_{p,\widetilde{w}_\xi}(\mathbb{R}_+)/\mathcal{K}(L^p(\mathbb{R}_+))$ is commutative.

For $p \in (1,\infty)$, let M_p be the Banach algebra of all Fourier multipliers b on the space $L^p(\mathbb{R})$ with the norm $\|b\|_{M_p} = \|W^0(b)\|_{\mathcal{B}(L^p(\mathbb{R}))}$, where $W^0(b) = \mathcal{F}^{-1} b \mathcal{F}$ is the convolution operator with symbol $b \in M_p$, and let $C_p(\overline{\mathbb{R}})$ be the closure in M_p of the set of all continuous functions on $\overline{\mathbb{R}} := [-\infty, +\infty]$ of bounded total variation. By [27, Lemma 1.2] (also see [21, Proposition 4.2.10]), for every $\nu \in (0,1)$, the Banach algebra $C_p(\overline{\mathbb{R}})$ is generated by the functions 1 and $b_\nu \in V(\mathbb{R})$, where

$$b_\nu(\lambda) := \coth(\pi\lambda + \pi i \nu) \quad \text{for} \quad \lambda \in \mathbb{R}.$$

For every $p \in (1, \infty)$, we introduce the isometric isomorphism

$$E_p : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}), \quad (E_p f)(x) = e^{x/p} f(e^x) \quad (x \in \mathbb{R}). \tag{6.17}$$

A straightforward calculation shows that

$$E_p(\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I) E_p^{-1} = W^0(b_{\nu_\xi}), \tag{6.18}$$

where $b_{\nu_\xi}(\lambda) := \coth(\pi\lambda + \pi i \nu_\xi)$ and $\nu_\xi := 1/p + \delta_\xi \in (0, 1)$. Hence, by (6.18),

$$\mathrm{sp}_{\mathcal{B}^\pi(L^p(\mathbb{R}_+))}[\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I]^\pi = \mathrm{sp}_{\mathcal{B}(L^p(\mathbb{R}_+))}(\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I) = \mathcal{L}_{\nu_\xi}, \tag{6.19}$$

where $\mathrm{sp}_{\mathcal{A}} A$ means the spectrum of an element $A \in \mathcal{A}$ in a unital Banach algebra $\mathcal{A}$, and

$$\mathcal{L}_{\nu_\xi} := b_{\nu_\xi}(\overline{\mathbb{R}}) = \big\{ \coth(\pi\lambda + \pi i \nu_\xi) : \lambda \in \overline{\mathbb{R}} \big\}. \tag{6.20}$$

Consider the commutative Banach subalgebra

$$\mathcal{W}_p^\pi := \big\{ [E_p^{-1} W^0(b) E_p]^\pi : b \in C_p(\overline{\mathbb{R}}) \big\} \tag{6.21}$$

of $\widetilde{\mathfrak{A}}^\pi_{p,\widetilde{w}_\xi}(\mathbb{R}_+)$, where $A^\pi := A + \mathcal{K}(L^p(\mathbb{R}_+))$, E_p is given by (6.17) and $\widetilde{\mathfrak{A}}_{p,\widetilde{w}_\xi}(\mathbb{R}_+)$ is defined by (6.16). As $C_p(\overline{\mathbb{R}})$ is generated by the functions 1 and $b_{\nu_\xi} \in V(\mathbb{R})$, we conclude that the algebra $\mathcal{W}_p^\pi$ is generated by the cosets $[\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I]^\pi$ and I^π. Hence the maximal ideal space $M(\mathcal{W}_p^\pi)$ of the algebra $\mathcal{W}_p^\pi$ is homeomorphic to the spectrum $\mathrm{sp}_{\mathcal{B}^\pi(L^p(\mathbb{R}_+))}[\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I]^\pi$ (see, e.g., [6, Section 1.19]):

$$M(\mathcal{W}_p^\pi) = \mathrm{sp}_{\mathcal{B}^\pi(L^p(\mathbb{R}_+))}[\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I]^\pi = \mathcal{L}_{\nu_\xi}. \tag{6.22}$$

Let $\eta \in M_0^0(QC(\mathbb{R}_+))$ and let $\mathcal{A}^\pi_{p,\eta}(\mathbb{R}_+)$ be the smallest closed subalgebra of $\Lambda^\pi_{p,\eta}(\mathbb{R}_+)$ that contains the cosets $A^\pi_{p,\eta,\mathbb{R}_+} = A^\pi + \mathcal{J}^\pi_{p,\eta,\mathbb{R}_+}$ for all $A \in \widetilde{\mathfrak{A}}_{p,\widetilde{w}_\xi}(\mathbb{R}_+)$, where $\mathcal{J}^\pi_{p,\eta,\mathbb{R}_+}$ is the closed two-sided ideal of the Banach algebra $\Lambda^\pi_p(\mathbb{R}_+)$ generated by the maximal ideal

$$\mathcal{I}^\pi_{p,\eta,\mathbb{R}_+} := \big\{ [aI]^\pi : a \in QC(\mathbb{R}_+),\ a(\eta) = 0 \big\}$$

of the central algebra $\mathcal{Z}^\pi_p(\mathbb{R}_+) := \{[aI]^\pi : a \in QC(\mathbb{R}_+)\}$ of $\Lambda^\pi_p(\mathbb{R}_+)$. Consider the Banach algebra $\widetilde{\mathfrak{A}}^\pi_{p,\widetilde{w}_\xi,\eta} := \widetilde{\mathfrak{A}}^\pi_{p,\widetilde{w}_\xi}(\mathbb{R}_+)/J^\pi_{p,\eta,\mathbb{R}_+}$, where $J^\pi_{p,\eta,\mathbb{R}_+}$ is the closed two-sided ideal of the Banach algebra $\widetilde{\mathfrak{A}}^\pi_{p,\widetilde{w}_\xi}(\mathbb{R}_+)$ generated by the ideal $\mathcal{I}^\pi_{p,\eta,\mathbb{R}_+}$. We save the notation $A^\pi_{p,\eta,\mathbb{R}_+}$ for the cosets $A^\pi + J^\pi_{p,\eta,\mathbb{R}_+}$ in the algebra $\widetilde{\mathfrak{A}}^\pi_{p,\widetilde{w}_\xi,\eta}$.

Theorem 6.2 *If* $p \in (1, \infty)$, $\delta_\xi \in (-1/p, 1/q)$ *and* $\widetilde{w}_\xi(x) = x^{\delta_\xi} \in A_p(\mathbb{R}_+)$, *then for every* $\eta \in M_0^0(QC(\mathbb{R}_+))$,

$$\begin{aligned} \mathrm{sp}_{\Lambda^\pi_{p,\eta}(\mathbb{R}_+)} \big[\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I\big]^\pi_{p,\eta,\mathbb{R}_+} &= \mathrm{sp}_{\mathcal{A}^\pi_{p,\eta}(\mathbb{R}_+)} \big[\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I\big]^\pi_{p,\eta,\mathbb{R}_+} \\ &= \mathrm{sp}_{\widetilde{\mathfrak{A}}^\pi_{p,\widetilde{w}_\xi,\eta}} \big[\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I\big]^\pi_{p,\eta,\mathbb{R}_+} = \mathcal{L}_{\nu_\xi}, \end{aligned} \tag{6.23}$$

where $\mathcal{L}_{\nu_\xi}$ *is defined in* (6.20) *and* $\nu_\xi = 1/p + \delta_\xi$.

Proof Obviously, the invertibility of the coset $A^\pi + J^\pi_{p,\eta,\mathbb{R}_+}$ in the Banach algebra $\widetilde{\mathfrak{A}}^\pi_{p,\widetilde{w}_\xi,\eta}$ implies the invertibility of the coset $A^\pi + \mathcal{J}^\pi_{p,\eta,\mathbb{R}_+}$ in the Banach algebras $\mathcal{A}^\pi_{p,\eta}(\mathbb{R}_+)$ and $\Lambda^\pi_{p,\eta}(\mathbb{R}_+)$. Consequently,

$$\begin{aligned} \mathrm{sp}_{\Lambda^\pi_{p,\eta}(\mathbb{R}_+)} \big[\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I\big]^\pi_{p,\eta,\mathbb{R}_+} &\subset \mathrm{sp}_{\mathcal{A}^\pi_{p,\eta}(\mathbb{R}_+)} \big[\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I\big]^\pi_{p,\eta,\mathbb{R}_+} \\ &\subset \mathrm{sp}_{\widetilde{\mathfrak{A}}^\pi_{p,\widetilde{w}_\xi,\eta}} \big[\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I\big]^\pi_{p,\eta,\mathbb{R}_+}. \end{aligned} \tag{6.24}$$

It follows from [5, Theorems 5.2, 5.3] that

$$\mathrm{sp}_{\widetilde{\mathfrak{A}}^\pi_{p,\widetilde{w}_\xi,\eta}} \big[\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I\big]^\pi_{p,\eta,\mathbb{R}_+} = \mathcal{L}_{\nu_\xi} \quad \text{for all} \quad \eta \in M_0^0(QC(\mathbb{R}_+)). \tag{6.25}$$

Hence, by (6.24),

$$\mathrm{sp}_{\mathcal{A}^\pi_{p,\eta}(\mathbb{R}_+)} \big[\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I\big]^\pi_{p,\eta,\mathbb{R}_+} \subset \mathcal{L}_{\nu_\xi}.$$

Put $A := \widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I$ and suppose that there is a point $\mu \in \mathcal{L}_{\nu_\xi} \setminus \mathrm{sp}_{\mathcal{A}^\pi_{p,\eta}(\mathbb{R}_+)} A^\pi_{p,\eta,\mathbb{R}_+}$. Then the coset $[\mu I - A]^\pi_{p,\eta,\mathbb{R}_+}$ is invertible in the Banach algebra $\mathcal{A}^\pi_{p,\eta}(\mathbb{R}_+)$.

By (6.19), $\mathcal{L}_{\nu_\xi}$ is the spectrum of the coset $A^\pi := [\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I]^\pi$ in the Banach algebra $\mathcal{B}^\pi(L^p(\mathbb{R}_+))$. Consider the Banach algebra $\mathcal{W}^\pi_p$ given by (6.21) and consider the Gelfand transforms of the cosets in $\mathcal{W}^\pi_p$, which are continuous functions defined on $\mathcal{L}_{\nu_\xi}$. Then there exists a coset $B^\pi \in \mathcal{W}^\pi_p$ being a small perturbation of the coset $[\mu I - A]^\pi$ and such that its Gelfand transform vanishes at a neighborhood u_μ of the point μ on $\mathcal{L}_{\nu_\xi}$ (this follows from [27, Lemmas 1.1, 1.2] and [21, Proposition 4.2.10], since the function $\lambda \mapsto \coth(\pi\lambda + \pi i \nu_\xi)$ in (6.20) belongs to M_p), while the coset $B^\pi_{p,\eta,\mathbb{R}_+}$ remains invertible in the quotient algebra $\mathcal{A}^\pi_{p,\eta}(\mathbb{R}_+)$.

Take a coset $D^\pi \in \mathcal{W}^\pi_p$ such that its Gelfand transform has support in u_μ and equals 1 at the point $\mu \in \mathcal{L}_{\nu_\xi}$. Then $B^\pi D^\pi = 0^\pi$, which implies that $B^\pi_{p,\eta,\mathbb{R}_+} D^\pi_{p,\eta,\mathbb{R}_+} = 0^\pi_{p,\eta,\mathbb{R}_+}$ in $\mathcal{A}^\pi_{p,\eta}(\mathbb{R}_+)$. Since the coset $B^\pi_{p,\eta,\mathbb{R}_+}$ is invertible in the Banach algebra $\mathcal{A}^\pi_{p,\eta}(\mathbb{R}_+)$, the coset $D^\pi_{p,\eta,\mathbb{R}_+}$ should be the zero coset in $\mathcal{A}^\pi_{p,\eta}(\mathbb{R}_+)$, which means that $D^\pi \in \mathcal{J}^\pi_{p,\eta,\mathbb{R}_+}$. As $D^\pi \in \widetilde{\mathfrak{A}}^\pi_{p,\widetilde{w}_\xi}(\mathbb{R}_+)$ and $\mathcal{J}^\pi_{p,\eta,\mathbb{R}_+} \cap \widetilde{\mathfrak{A}}^\pi_{p,\widetilde{w}_\xi}(\mathbb{R}_+) = J^\pi_{p,\eta,\mathbb{R}_+}$, it follows that D^π should belong to the ideal $J^\pi_{p,\eta,\mathbb{R}_+}$, which is impossible because the Gelfand transform of the coset $D^\pi + J^\pi_{p,\eta,\mathbb{R}_+}$ in the commutative Banach algebra $\widetilde{\mathfrak{A}}^\pi_{p,\widetilde{w}_\xi,\eta}$ coincides in view of [5] with the Gelfand transform of the coset D^π in the commutative Banach subalgebra $\mathcal{W}^\pi_p$ of $\widetilde{\mathfrak{A}}^\pi_{p,\widetilde{w}_\xi}(\mathbb{R}_+)$, and therefore does not vanish identically on $\mathcal{L}_{\nu_\xi}$. This can be shown by applying the Fredholm symbol calculus for the Banach algebra $\mathrm{alg}\{aI, S_\mathbb{R} : a \in PQC(\mathbb{R})\} \subset \mathcal{B}(L^p(\mathbb{R}, \widetilde{w}_\xi))$ generated by multiplications by all functions in $PQC(\mathbb{R})$ and the Cauchy singular integral operator $S_\mathbb{R}$, which can be constructed by analogy with [5, Theorems 5.6, 7.1 and Subsection 7.4]. Consequently,

$$\mathrm{sp}_{\mathcal{A}^\pi_{p,\eta}(\mathbb{R}_+)}\big[\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I\big]^\pi_{p,\eta,\mathbb{R}_+} = \mathcal{L}_{\nu_\xi} \quad \text{for all} \quad \eta \in M^0_0(QC(\mathbb{R}_+)). \tag{6.26}$$

Finally, since the spectrum $\mathcal{L}_{\nu_\xi}$ of the coset $\big[\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I\big]^\pi_{p,\eta,\mathbb{R}_+}$ in the subalgebra $\mathcal{A}^\pi_{p,\eta}(\mathbb{R}_+)$ of $\Lambda^\pi_{p,\eta}(\mathbb{R}_+)$ has empty interior in the complex plane $\mathbb{C}$ by (6.26), it follows from [22, Corollaries of Theorem 10.18] that

$$\mathrm{sp}_{\Lambda^\pi_{p,\eta}(\mathbb{R}_+)}\big[\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I\big]^\pi_{p,\eta,\mathbb{R}_+} = \mathrm{sp}_{\mathcal{A}^\pi_{p,\eta}(\mathbb{R}_+)}\big[\widetilde{w}_\xi S_{\mathbb{R}_+} \widetilde{w}_\xi^{-1} I\big]^\pi_{p,\eta,\mathbb{R}_+},$$

which completes the proof of (6.23). □

7 Application of the Two Idempotents Theorem

7.1 *Spectra of Cosets $[X_t]^\pi_{p,w,\xi}$ Related to the Two Idempotents Theorem*

If $\xi \in M^0(QC)$, where $M^0(QC)$ is given by (2.3), then it follows from Lemma 5.4(i) that the symbol calculus for the Banach algebra $\mathcal{A}^\pi_{p,w,\xi}$ can be obtained by applying the two idempotents theorem (see, e.g., [3, 21] and the references therein).

For every $t \in \mathbb{T}$ and every $\xi \in M^0_t(QC)$, we consider the operator

$$X_t := I - (\chi_t^+ I - P_+)^2 \in \mathfrak{A}_{p,w} \subset \Lambda_{p,w} \tag{7.1}$$

and the related to the two idempotents theorem coset

$$[X_t]^\pi_{p,w,\xi} := \big[I - (\chi_t^+ I - P_+)^2\big]^\pi + \mathcal{J}^\pi_{p,w,\xi} \in \Lambda^\pi_{p,w,\xi}. \tag{7.2}$$

Theorem 7.1 *Let $p \in (1,\infty)$ and let $w = e^v \in A_p(\mathbb{T})$ be a weight satisfying condition* **(A)**. *If $t \in \mathbb{T}$ and $\xi \in M_t^0(QC)$, then*

$$\mathrm{sp}_{\Lambda^\pi_{p,w,\xi}}[X_t]^\pi_{p,w,\xi} = \mathrm{sp}_{\mathcal{A}^\pi_{p,w,\xi}}[X_t]^\pi_{p,w,\xi} = \widetilde{\mathcal{L}}_{p,w,\nu_\xi}, \tag{7.3}$$

$$\widetilde{\mathcal{L}}_{p,w,\nu_\xi} := \big\{\big(1+\coth[\pi x + \pi i \nu_\xi]\big)/2 :\ x \in \overline{\mathbb{R}}\big\},\quad \nu_\xi = 1/p + \delta_\xi. \tag{7.4}$$

Proof Fix $t \in \mathbb{T}$ and $\xi \in M_t^0(QC)$, where $M_t^0(QC)$ is given by (2.2). Obviously,

$$\mathrm{sp}_{\Lambda^\pi_{p,w,\xi}}[X_t]^\pi_{p,w,\xi} = \mathrm{sp}_{\Lambda^\pi_{p,\xi}}[wX_tw^{-1}I]^\pi_{p,\xi} = \mathrm{sp}_{\Lambda^\pi_{p,\xi}}[\chi_{u_t}wX_tw^{-1}\chi_{u_t}I]^\pi_{p,\xi}, \tag{7.5}$$

where $\mathcal{J}^\pi_{p,\xi} = \big\{[wAw^{-1}I]^\pi : A^\pi \in \mathcal{J}^\pi_{p,w,\xi}\big\}$ and $[(\chi_{u_t}-1)I]^\pi \in \mathcal{J}^\pi_{p,\xi}$. Applying (7.1), we infer from Theorem 6.1 that

$$[\chi_{u_t}wX_tw^{-1}\chi_{u_t}I]^\pi_{p,\xi} = [\chi_{u_t}w_\xi X_t w_\xi^{-1}\chi_{u_t}I]^\pi_{p,\xi}. \tag{7.6}$$

Let $\gamma = [0,\delta]$, where $U_t = [-\delta,\delta]$ and $0 < \delta < \pi/2$. Consider the isometric isomorphism $\Upsilon_t : L^p(u_t) \to L_2^p(\gamma)$ given by (6.3). It follows from (6.6) that

$$\begin{aligned}&\big[\Upsilon_t\big[\chi_{u_t}w_\xi\big(I-(\chi_t^+I-P_+)^2\big)w_\xi^{-1}\chi_{u_t}I\big]\Upsilon_t^{-1}\big]^\pi\\ &= \mathrm{diag}\,\big\{\big[\chi_\gamma 2^{-1}\big(I+\widetilde{w}_\xi S_{\mathbb{R}_+}\widetilde{w}_\xi^{-1}I\big)\chi_\gamma I\big]^\pi, \big[\chi_\gamma 2^{-1}\big(I+\widetilde{w}_\xi S_{\mathbb{R}_+}\widetilde{w}_\xi^{-1}I\big)\chi_\gamma I\big]^\pi\big\}.\end{aligned} \tag{7.7}$$

Identifying the points $\xi \in M_t^0(QC)$ and points $\widetilde{\xi}$ in $M_0^0(QC(\gamma))$ and $M_0^0(QC(\mathbb{R}_+))$, we conclude that if $A^\pi \in \mathcal{J}^\pi_{p,\xi}$, then

$$\Upsilon_t(\chi_{u_t}A^\pi\chi_{u_t}I)\Upsilon_t^{-1} = [\chi_\gamma I]^\pi\big[A^\pi_{i,j}\big]_{i,j=1}^2[\chi_\gamma I]^\pi,$$

where $A^\pi_{i,j} \in \mathcal{J}^\pi_{p,\widetilde{\xi},\gamma}$ for all $i,j = 1,2$. Hence, by (7.5)–(7.7) and the property $[(\chi_\gamma - 1)I]^\pi \in \mathcal{J}^\pi_{p,\widetilde{\xi},\mathbb{R}_+}$, the coset $[X_t]^\pi_{p,w,\xi}$ is invertible in the Banach algebra $\Lambda^\pi_{p,w,\xi}$ if and only if the diagonal matrix

$$\mathrm{diag}\,\Big\{\big[2^{-1}(I+\widetilde{w}_\xi S_{\mathbb{R}_+}\widetilde{w}_\xi^{-1}I)\big]^\pi_{p,\widetilde{\xi},\mathbb{R}_+},\ \big[2^{-1}(I+\widetilde{w}_\xi S_{\mathbb{R}_+}\widetilde{w}_\xi^{-1}I)\big]^\pi_{p,\widetilde{\xi},\mathbb{R}_+}\Big\}$$

with entries in $\Lambda^\pi_{p,\widetilde{\xi}}(\mathbb{R}_+)$ is invertible in the Banach algebra $\Lambda^\pi_{p,\widetilde{\xi}}(\mathbb{R}_+)$. Consequently, making use of the spectral mapping theorem, we obtain

$$\begin{aligned}\mathrm{sp}_{\Lambda^\pi_{p,w,\xi}}[X_t]^\pi_{p,w,\xi} &= \mathrm{sp}_{\Lambda^\pi_{p,\widetilde{\xi}}(\mathbb{R}_+)}\big[2^{-1}(I+\widetilde{w}_\xi S_{\mathbb{R}_+}\widetilde{w}_\xi^{-1}I)\big]^\pi_{p,\widetilde{\xi},\mathbb{R}_+}\\ &= 2^{-1}\big(1+\mathrm{sp}_{\Lambda^\pi_{p,\widetilde{\xi}}(\mathbb{R}_+)}\big[\widetilde{w}_\xi S_{\mathbb{R}_+}\widetilde{w}_\xi^{-1}I\big]^\pi_{p,\widetilde{\xi},\mathbb{R}_+}\big),\end{aligned}$$

which implies in view of Theorem 6.2 that

$$\mathrm{sp}_{\Lambda^\pi_{p,w,\xi}}[X_t]^\pi_{p,w,\xi} = 2^{-1}(1+\mathcal{L}_{\nu_\xi}) = \widetilde{\mathcal{L}}_{p,w,\nu_\xi},$$

where $\widetilde{\mathcal{L}}_{p,w,\nu_\xi}$ and ν_ξ are given by (7.4).

Finally, since the set $\widetilde{\mathcal{L}}_{p,w,\nu_\xi}$ does not separate the complex plane $\mathbb{C}$, the first equality in (7.3) follows from [22, Corollaries of Theorem 10.18]. □

7.2 Corollary of the Two Idempotents Theorem

Since the spectra of the cosets $[X_t]^\pi_{p,w,\xi} = I^\pi_{p,w,\xi} - \big(P^\pi_{p,w,\xi} - Q^\pi_{p,w,\xi}\big)^2$ given by (5.9) and (7.2) for all $\xi \in M^0(QC)$ are described in the Banach algebras $\Lambda^\pi_{p,w,\xi}$ and $\mathcal{A}^\pi_{p,w,\xi}$ by Theorem 7.1, and since the points 0 and 1 are not isolated in $\mathrm{sp}_{\Lambda^\pi_{p,w,\xi}}[X_t]^\pi_{p,w,\xi}$, we can apply to mentioned algebras the version of the two idempotents theorem given in [3, Theorem 8.7] and followed from [10] and [13].

We immediately obtain from Theorem 7.1 and [3, Theorem 8.7] the following.

Theorem 7.2 *Let $p \in (1,\infty)$, let $w = e^v \in A_p(\mathbb{T})$ be a weight satisfying condition* **(A)***, and let $\xi \in M^0_t(QC)$ for $t \in \mathbb{T}$. Then the Banach algebra $\mathcal{A}^\pi_{p,w,\xi}$ is inverse closed in the Banach algebra $\Lambda^\pi_{p,w,\xi}$, and*

(i) *for every $\mu \in \widetilde{\mathcal{L}}_{p,w,\nu_\xi}$, the map $\pi_\mu : \{I^\pi_{p,w,\xi}, P^\pi_{p,w,\xi}, Q^\pi_{p,w,\xi}\} \to \mathbb{C}^{2\times 2}$ given by*

$$\pi_\mu(I^\pi_{p,w,\xi}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \pi_\mu(P^\pi_{p,w,\xi}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \pi_\mu(Q^\pi_{p,w,\xi}) = \begin{bmatrix} \mu & \varrho(\mu) \\ \varrho(\mu) & 1-\mu \end{bmatrix},$$

where $\varrho(\mu) := \sqrt{\mu(1-\mu)}$ is an arbitrary value of the square root, extends to a Banach algebra homomorphism $\pi_\mu : \mathcal{A}^\pi_{p,w,\xi} \to \mathbb{C}^{2\times 2}$;

(ii) *a coset $A^\pi_{p,w,\xi} \in \mathcal{A}^\pi_{p,w,\xi}$ is invertible in the Banach algebra $\Lambda^\pi_{p,w,\xi}$ (equivalently, in the Banach algebra $\mathcal{A}^\pi_{p,w,\xi}$) if and only if $\det\big[\pi_\mu\big(A^\pi_{p,w,\xi}\big)\big] \neq 0$ for all $\mu \in \widetilde{\mathcal{L}}_{p,w,\nu_\xi}$.*

8 The Fredholm Study of the Banach Algebra $\mathfrak{A}_{p,w}$

With the Banach algebra $\mathfrak{A}_{p,w}$, we associate the sets

$$\mathfrak{M} := \bigcup_{t\in\mathbb{T}} \mathfrak{M}_t, \quad \mathfrak{M}_t := \widetilde{M}^-_t(QC) \cup \Big(\bigcup_{\xi\in M^0_t(QC)} \{\xi\} \times \widetilde{\mathcal{L}}_{p,w,\nu_\xi}\Big) \cup \widetilde{M}^+_t(QC),$$

where $\widetilde{\mathcal{L}}_{p,w,\nu_\xi}$ and ν_ξ are given by (7.4). Consider the sets $\widetilde{M}^{\pm}(QC)$ and $M^0(QC)$ defined by (2.3). Let $B(\mathfrak{M}, \mathbb{C}^{2\times 2})$ stand for the C^*-algebra of all bounded matrix functions $G : \mathfrak{M} \to \mathbb{C}^{2\times 2}$.

Applying results of Sects. 4 and 7, we establish the main result of the paper containing a symbol calculus for the Banach algebra $\mathfrak{A}_{p,w}$ and a Fredholm criterion for the operators $A \in \mathfrak{A}_{p,w}$.

Theorem 8.1 *Let $p \in (1,\infty)$ and let a weight $w \in A_p(\mathbb{T})$ satisfy assumption* **(A)**. *Then the map* Sym : $\{aI : a \in PQC\} \cup \{S_{\mathbb{T}}\} \to B(\mathfrak{M}, \mathbb{C}^{2\times 2})$ *given by the matrix functions*

$$(\mathrm{Sym}\, aI)(m) := \begin{cases} \mathrm{diag}\{a(\xi,0), a(\xi,0)\} \;\; \textit{for all} \;\; m = \xi \in \widetilde{M}^-(QC), \\ \begin{bmatrix} a(\xi,1)\mu + a(\xi,0)(1-\mu) & [a(\xi,1)-a(\xi,0)]\varrho(\mu) \\ [a(\xi,1)-a(\xi,0)]\varrho(\mu) & a(\xi,1)(1-\mu)+a(\xi,0)\mu \end{bmatrix} \\ \textit{for all } m = (\xi,\mu) \textit{ with } \xi \in M^0(QC) \textit{ and } \mu \in \widetilde{\mathcal{L}}_{p,w,\nu_\xi}, \\ \mathrm{diag}\{a(\xi,1), a(\xi,1)\} \;\; \textit{for all} \;\; m = \xi \in \widetilde{M}^+(QC), \end{cases} \tag{8.1}$$

$$(\mathrm{Sym}\, S_{\mathbb{T}})(\zeta) := \mathrm{diag}\{1,-1\} \;\; \textit{for all} \;\; m \in \mathfrak{M}, \tag{8.2}$$

where $a(\xi,\mu)$ is the Gelfand transform of a function $a \in PQC$ for $(\xi,\mu) \in M(PQC)$ and $\varrho(\mu) = \sqrt{\mu(1-\mu)}$ for all $\mu \in \bigcup_{\xi \in M^0(QC)} \widetilde{\mathcal{L}}_{p,w,\nu_\xi}$, extends to a Banach algebra homomorphism

$$\mathrm{Sym} : \mathfrak{A}_{p,w} \to B(\mathfrak{M}, \mathbb{C}^{2\times 2})$$

whose kernel contains all compact operators on $L^p(\mathbb{T}, w)$. The Banach algebra $\mathfrak{A}^\pi_{p,w}$ is inverse closed in the Calkin algebra $\mathcal{B}^\pi_{p,w}$, and an operator $A \in \mathfrak{A}_{p,w}$ is Fredholm on the space $L^p(\mathbb{T}, w)$ if and only if

$$\det[(\mathrm{Sym}\, A)(m)] \neq 0 \;\; \textit{for all} \;\; m \in \mathfrak{M}. \tag{8.3}$$

Proof Fix $t \in \mathbb{T}$. For $\xi \in \widetilde{M}^{\pm}_t(QC)$, Theorem 5.3 and Lemma 5.4(ii) imply that

$$\beta_\xi(aI) = a(\xi^{\pm}) I^\pi_{p,w,\xi}, \quad \beta_\xi(S_{\mathbb{T}}) = \left[S_{\mathbb{T}}\right]^\pi_{p,w,\xi} = 2P^\pi_{p,w,\xi} - I^\pi_{p,w,\xi}.$$

Taking the map $\Theta_\xi : \mathcal{A}_{p,w,\xi} \to \mathrm{diag}\{\mathbb{C},\mathbb{C}\}$ defined by (5.10)–(5.11), we conclude that for each $\xi \in \widetilde{M}^{\pm}_t(QC)$ the map $\Theta_\xi \circ \beta_\xi : \mathfrak{A}_{p,w} \to \mathrm{diag}\{\mathbb{C},\mathbb{C}\}$ is a Banach algebra homomorphism whose kernel contains the ideal $\mathcal{K}_{p,w}$. One can easily see from (8.1)–(8.2) for the generators A of the Banach algebra $\mathfrak{A}_{p,w}$ that

$$(\Theta_\xi \circ \beta_\xi)(A) = (\mathrm{Sym}\, A)(m) \;\; \text{for all} \;\; m = \xi \in \widetilde{M}^{\pm}(QC). \tag{8.4}$$

If $\xi \in M_t^0(QC)$ and $\mu \in \widetilde{\mathcal{L}}_{p,w,\nu_\xi}$, then we infer by Theorem 5.3, Lemma 5.4(i) and Theorem 7.2 that the map $\pi_\mu \circ \beta_\xi : \mathfrak{A}_{p,w} \to \mathbb{C}^{2\times 2}$ also is a Banach algebra homomorphism whose kernel contains the ideal $\mathcal{K}_{p,w}$. Applying (5.7), (5.8), Theorem 7.2(i) and (8.1)–(8.2), we again see that for every generator A of the Banach algebra $\mathfrak{A}_{p,w}$,

$$(\pi_\mu \circ \beta_\xi)(A) = (\mathrm{Sym}\, A)(\xi, \mu) \ \text{ for all } \ \xi \in M_t^0(QC) \ \text{ and all } \ \mu \in \widetilde{\mathcal{L}}_{p,w,\nu_\xi}. \tag{8.5}$$

Thus, by (8.4) and (8.5), the map

$$\mathrm{Sym} := \begin{cases} \Theta_\xi \circ \beta_\xi & \text{if} \quad \xi \in \widetilde{M}^-(QC) \cup \widetilde{M}^+(QC), \\ \pi_\mu \circ \beta_\xi & \text{if} \quad \xi \in M^0(QC) \ \text{ and } \ \mu \in \widetilde{\mathcal{L}}_{p,w,\nu_\xi}, \end{cases}$$

is a Banach algebra homomorphism of $\mathfrak{A}_{p,w}$ into $B(\mathfrak{M}, \mathbb{C}^{2\times 2})$.

Moreover, Lemma 5.1 and Theorems 5.5 and 7.2 imply that the Banach algebra $\mathfrak{A}^\pi_{p,w}$ is inverse closed in the Calkin algebra $\mathcal{B}^\pi_{p,w}$, and that an operator $A \in \mathfrak{A}_{p,w}$ is Fredholm on the space $L^p(\mathbb{T}, w)$ if and only if (8.3) holds. □

References

1. M.A. Bastos, C.A. Fernandes, and Yu.I. Karlovich, *C^*-algebras of integral operators with piecewise slowly oscillating coefficients and shifts acting freely.* Integr. Equ. Oper. Theory **55** (2006), 19–67.
2. E. Berkson and T.A. Gillespie, *Multipliers for weighted L^p-spaces, transference, and the q-variation of functions.* Bull. Sci. Math. **122** (1998), 427–454.
3. A. Böttcher and Yu.I. Karlovich, *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators.* Progress in Mathematics **154**, Birkhäuser, Basel, 1997.
4. A. Böttcher, Yu.I. Karlovich, and I.M. Spitkovsky, *Convolution Operators and Factorization of Almost Periodic Matrix Functions.* Operator Theory: Advances and Applications **131**, Birkhäuser, Basel, 2002.
5. A. Böttcher, S. Roch, B. Silbermann, and I.M. Spitkovsky, *A Gohberg-Krupnik-Sarason symbol calculus for algebras of Toeplitz, Hankel, Cauchy, and Carleman operators.* Operator Theory: Advances and Applications **48** (1990), 189–234.
6. A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators.* 2nd edition, Springer, Berlin, 2006.
7. A. Böttcher and I.M. Spitkovsky, *Toeplitz Operators with PQC Symbols on Weighted Hardy Spaces.* J. Funct. Anal. **97** (1991), 194–214.
8. R.G. Douglas, *Banach Algebra Techniques in Operator Theory.* Academic Press, New York, 1972.
9. R.V. Duduchava and A.I. Saginashvili, *Convolution integral equations on a half-line with semi-almost-periodic presymbols.* Differential Equations **17** (1981), 207–216.
10. T. Finck, S. Roch, and B. Silbermann, *Two projection theorems and symbol calculus for operators with massive local spectra.* Math. Nachr. **162** (1993), 167–185.
11. J.B. Garnett, *Bounded Analytic Functions.* Academic Press, New York, 1981.

12. I. Gohberg and N. Krupnik, *One-Dimensional Linear Singular Integral Equations, Vols. I and II.* Birkhäuser, Basel, 1992.
13. I. Gohberg and N. Krupnik, *Extension theorems for invertibility symbols in Banach algebras.* Integr. Equ. Oper. Theory **15** (1992), 991–1010.
14. Yu.I. Karlovich, *An algebra of pseudodifferential operators with slowly oscillating symbols.* Proc. London Math. Soc. (3) **92** (2006), 713–761.
15. Yu.I. Karlovich, *Pseudodifferential operators with compound slowly oscillating symbols.* Operator Theory: Advances and Applications **171** (2007), 189–224.
16. Yu.I. Karlovich, *Nonlocal singular integral operators with slowly oscillating data.* Operator Theory: Advances and Applications **181** (2008), 229–261.
17. Yu.I. Karlovich, *Localization of a class of Muckenhoupt weights by using Mellin pseudo-differential operators.* Advances in Microlocal and Time-Frequency Analysis. Applied Numerical and Harmonic Analysis, Birkhäuser/Springer, Cham, (2020) 281–310.
18. Yu.I. Karlovich and I. Loreto Hernández, *Algebras of convolution type operators with piecewise slowly oscillating data. I: Local and structural study.* Integr. Equ. Oper. Theory **74** (2012), 377–415.
19. Yu.I. Karlovich and I. Loreto Hernández, *On convolution type operators with piecewise slowly oscillating data.* Operator Theory: Advances and Applications **228** (2013), 185–207.
20. M.A. Naimark, *Normed Algebras.* Wolters-Noordhoff, Groningen, 1972.
21. S. Roch, P.A. Santos, and B. Silbermann, *Non-commutative Gelfand Theories. A Tool-kit for Operator Theorists and Numerical Analysts.* Springer, London, 2011.
22. W. Rudin, *Functional Analysis.* 2nd edition, McGraw-Hill Inc., New York, 1991.
23. A.I. Saginashvili, *Singular integral operators with coefficients having semi-almost-periodic discontinuities.* Soobshch. Akad. Nauk Gruzin. SSR **94** (1979), 289–291 (Russian).
24. A.I. Saginashvili, *Singular integral operators with coefficients having semi-almost-periodic type discontinuities.* Soobshch. Akad. Nauk Gruzin. SSR **95** (1979), 541–543 (Russian).
25. D. Sarason, *Functions of vanishing mean oscillation.* Trans. Amer. Math. Soc. **207** (1975), 391–405.
26. D. Sarason, *Toeplitz operators with piecewise quasicontinuous symbols.* Indiana Univ. Math. J. **26** (1977), 817–838.
27. I.B. Simonenko and Chin Ngok Min, *Local Method in the Theory of One-Dimensional Singular Integral Equations with Piecewise Continuous Coefficients. Noetherity.* Rostov Univ. Press, Rostov on Don, 1986 (Russian).

Twisted Dirac Operator on Quantum SU(2) in Disc Coordinates

Ulrich Krähmer and Elmar Wagner

Dedicated to Professor Nikolai Vasilevski on occasion of his 70th birthday

Abstract The quantum disc is used to define a noncommutative analogue of a dense coordinate chart and of left-invariant vector fields on quantum SU(2). This yields two twisted Dirac operators for different twists that are related by a gauge transformation and have bounded twisted commutators with a suitable algebra of differentiable functions on quantum SU(2).

Keywords Dirac operator · Twisted derivation · Quantum SU(2) · Quantum disc

Mathematics Subject Classification (2000) Primary 58B34; Secondary 58B32

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1 Introduction

Connes' noncommutative differential geometry [2, 3] provides in particular a geometric approach to the construction of K-homology classes of a C^*-algebra $\mathcal{A}$: for the commutative C^*-algebra of continuous functions on a compact smooth manifold, the phase $F := \frac{D}{|D|}$ of an elliptic first order differential operator D on a vector bundle defines such a class, and for a noncommutative algebra, the fundamental task is to represent $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and to find a self-adjoint operator D that has compact resolvent (so it is "very" unbounded) but at the same time has bounded commutators with the elements of $\mathcal{A}$.

In classical geometry, equivariant differential operators on Lie groups provide examples that can be described purely in terms of representation theory, so since the discovery of quantum groups, many attempts were made to apply Connes' programme to these C^*-algebras. The fact that their representation theory resembles that of their classical counterparts so closely allows one indeed to define straightforwardly an analogue say of the Dirac operator on a compact simple Lie group, but it turns out to have unbounded commutators with the elements of $\mathcal{A}$.

Many solutions to this conundrum were found and studied, focusing on various approaches and motivations ranging from index theory [1, 4–7, 13, 15, 16] over the theory of covariant differential calculi [10, 17] to the Baum-Connes conjecture [19]. However, it seems fair to say that there is still no sufficient general understanding of how Connes' machinery applies to algebras obtained by deformation quantisation in general and quantum groups in particular.

The aim of the present note is to use the fundamental example of SU(2) for discussing yet another mechanism for obtaining bounded commutators. In a nutshell, the idea is to have a representation of $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ on $\mathcal{H}$ and to use differential operators D with "coefficients" in $\mathcal{A}^{\mathrm{op}}$ to achieve bounded commutators with $\mathcal{A}$. Our starting point is a noncommutative analogue of a dense coordinate chart on SU(2) that is compatible with the symplectic foliation of the quantised Poisson manifold SU(2). The noncommutative analogue is obtained by replacing a complex unit disc by the quantum disc. We use a quantised differential calculus on this chart to define quantisations of left invariant vector fields that act on the function algebra by twisted derivations. This is where it becomes necessary to consider coefficients from $\mathcal{A}^{\mathrm{op}}$.

We then build two twisted Dirac operators using these twisted derivations and show that they are related by a gauge transformation that arises from a rescaling of the volume form. A fruitful direction of further research might be to investigate the spectral and homological properties of these and similar operators.

2 The Dirac Operator on SU(2)

The C^*-algebra $\mathcal{A}$ we are going to consider is a strict deformation quantisation of the algebra of continuous complex-valued functions on the Lie group

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C},\ \alpha\bar{\alpha} + \beta\bar{\beta} = 1 \right\}$$

that we identify as usual with $\mathbb{S}^3 \subset \mathbb{C}^2$, identifying the above matrix with (α, β).

We denote by

$$X_0 := \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad X_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_2 := \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

the standard generators of the Lie algebra su(2) of SU(2), and, by a slight abuse of notation, also the corresponding left invariant vector fields on SU(2).

In this section, we describe the Dirac operator D of SU(2) in local coordinates that are adapted to the quantisation process. To define the local coordinates, consider the map

$$\bar{\mathbb{D}} \times \mathbb{S}^1 \to \mathbb{S}^3, \quad (z, v) \mapsto (z, \sqrt{1 - z\bar{z}}\, v), \tag{2.1}$$

where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disc, $\bar{\mathbb{D}}$ is its closure, and $\mathbb{S}^1 = \partial\mathbb{D}$ is its boundary. Restricting the map to $(\mathbb{D} \times \mathbb{S}^1) \setminus (\mathbb{D} \times \{-1\}) \cong \mathbb{D} \times (-\pi, \pi)$ defines a dense coordinate chart

$$\mathbf{x}\colon \mathbb{D} \times (-\pi, \pi) \longrightarrow \mathbb{S}^3, \quad \mathbf{x}(z, t) := (z, \sqrt{1 - z\bar{z}}\, \mathrm{e}^{\mathrm{i}t}) \tag{2.2}$$

that is compatible with the standard differential structure on $\mathrm{SU}(2) \cong \mathbb{S}^3$. The pull-back of the bi-invariant volume form on $\mathbb{S}^3$ assigns a measure to $\mathbb{D} \times (-\pi, \pi)$ and the resulting Hilbert space of L_2-functions will be denoted by $\mathcal{H}$.

We write $f(z, t) := f \circ \mathbf{x}(z, t)$ for functions f on $\mathbb{S}^3$ and thus identify these with continuous functions on $\bar{\mathbb{D}} \times [-\pi, \pi]$ satisfying the boundary conditions

$$f(u, t) = f(u, 0), \quad f(z, -\pi) = f(z, \pi) \quad \forall\, u \in \mathbb{S}^1,\ z \in \mathbb{D},\ t \in [-\pi, \pi]. \tag{2.3}$$

Let $\Gamma^{(1)}(\mathrm{SU}(2))$ denote the set of $C^{(1)}$-functions (continuously differentiable ones) on $\bar{\mathbb{D}} \times [-\pi, \pi]$ satisfying (2.3). The corresponding functions on $\mathbb{S}^3$ are not necessarily $C^{(1)}$, but absolutely continuous, and can therefore be considered as belonging to the domain of the first order differential operators X_0, X_1 and X_2. Here, derivations are understood to be taken in the weak sense. Therefore we may consider

$$H := -\mathrm{i}X_0, \quad E := \tfrac{1}{2}(X_1 - \mathrm{i}X_2), \quad F := -\tfrac{1}{2}(X_1 + \mathrm{i}X_2)$$

as first order differential operators on $\mathcal{H}$ with domain $\Gamma^{(1)}(\mathrm{SU}(2))$. A direct calculation shows that these operators take in the parametrisation (2.2) the following form:

$$H = z\tfrac{\partial}{\partial z} - \bar{z}\tfrac{\partial}{\partial \bar{z}} + \mathrm{i}\tfrac{\partial}{\partial t}, \tag{2.4}$$

$$E = -\sqrt{1-\bar{z}z}\,\mathrm{e}^{-\mathrm{i}t}\tfrac{\partial}{\partial \bar{z}} - \frac{\mathrm{i}}{2}\frac{z}{\sqrt{1-\bar{z}z}}\mathrm{e}^{-\mathrm{i}t}\tfrac{\partial}{\partial t}, \tag{2.5}$$

$$F = \sqrt{1-\bar{z}z}\,\mathrm{e}^{\mathrm{i}t}\tfrac{\partial}{\partial z} - \frac{\mathrm{i}}{2}\frac{\bar{z}}{\sqrt{1-\bar{z}z}}\mathrm{e}^{\mathrm{i}t}\tfrac{\partial}{\partial t}. \tag{2.6}$$

Since $\mathrm{SU}(2)$ is a Lie group, its tangent bundle is trivial and hence admits a trivial spin structure. We consider $\Gamma^{(1)}(S) := \Gamma^{(1)}(\mathrm{SU}(2)) \oplus \Gamma^{(1)}(\mathrm{SU}(2))$ as a vector space of differentiable sections (in the weak sense) of the associated spinor bundle. The Dirac operator with respect to the bi-invariant metric on $\mathrm{SU}(2)$ is then given by the closure of

$$D := \begin{pmatrix} H-2 & E \\ F & -H-2 \end{pmatrix} : \Gamma^{(1)}(S) \subset \mathcal{H} \oplus \mathcal{H} \longrightarrow \mathcal{H} \oplus \mathcal{H},$$

see e.g. [8].

3 A Representation of Quantum SU(2) by Multiplication Operators

The quantised coordinate ring of $\mathrm{SU}(2)$ at $q \in (0,1)$ is the universal unital *-algebra $\mathcal{O}(\mathrm{SU}_q(2))$ containing elements a, c such that

$$ac = qca, \quad ac^* = qc^*a, \quad cc^* = c^*c,$$
$$aa^* + q^2cc^* = 1, \quad a^*a + cc^* = 1.$$

It admits a faithful Hilbert space representation ρ on $\ell_2(\mathbb{N}) \otimes \ell_2(\mathbb{Z})$ given on orthonormal bases $\{e_n\}_{n\in\mathbb{N}} \subset \ell_2(\mathbb{N})$ and $\{b_k\}_{k\in\mathbb{Z}} \subset \ell_2(\mathbb{Z})$ by

$$\rho(a)(e_n \otimes b_k) = \sqrt{1-q^{2n}}\, e_{n-1} \otimes b_k, \tag{3.1}$$

$$\rho(c)(e_n \otimes b_k) = q^n\, e_n \otimes b_{k-1}. \tag{3.2}$$

The norm closure of the *-algebra generated by $\rho(a), \rho(c) \in B(\ell_2(\mathbb{N}) \otimes \ell_2(\mathbb{Z}))$ is isomorphic to $C(\mathrm{SU}_q(2))$, the universal C*-algebra of $\mathcal{O}(\mathrm{SU}_q(2))$, see e.g. [14].

The starting point of this paper is a quantum counterpart to the chart (2.1). To define it, let $z \in B(\ell_2(\mathbb{N}))$ and $u \in B(\ell_2(\mathbb{Z}))$ be given by

$$z e_n := \sqrt{1 - q^{2n}}\, e_{n-1}, \quad n \in \mathbb{N}, \qquad u b_k := b_{k-1}, \quad k \in \mathbb{Z}, \tag{3.3}$$

and set

$$y := \sqrt{1 - z^* z} \in B(\ell_2(\mathbb{N})). \tag{3.4}$$

Then y is a positive self-adjoint trace class operator on $\ell_2(\mathbb{N})$ acting by

$$y\, e_n = q^n\, e_n$$

and satisfying the relations

$$zy = qyz, \quad yz^* = qz^*y. \tag{3.5}$$

Note that one can now rewrite Eqs. (3.1) and (3.2) as

$$\rho(a) = (z \otimes 1)\,(e_n \otimes b_n), \qquad \rho(c) = (y \otimes u)\,(e_n \otimes b_n). \tag{3.6}$$

The bilateral shift u generates a commutative C*-subalgebra of $B(\ell_2(\mathbb{Z}))$ which is isomorphic to $C(\mathbb{S}^1)$. The operator $z \in B(\ell_2(\mathbb{N}))$ satisfies the defining relation of the quantum disc algebra $\mathcal{O}(\mathbb{D}_q)$,

$$zz^* - q^2 z^* z = 1 - q^2. \tag{3.7}$$

It is known [11] that the universal C*-algebra of the quantum disc $\mathcal{O}(\mathbb{D}_q)$, generated by a single generator and its adjoint satisfying (3.7), is isomorphic to the Toeplitz algebra $\mathcal{T}$ which can also be viewed as the C*-subalgebra of $B(\ell_2(\mathbb{N}))$ generated by the unilateral shift

$$s\, e_n = e_{n+1}, \quad n \in \mathbb{N}. \tag{3.8}$$

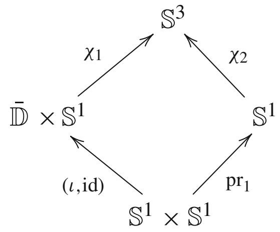
Moreover, the bounded operator z defined in (3.3) also generates the Toeplitz algebra $\mathcal{T} \subset B(\ell_2(\mathbb{N}))$, and the so-called symbol map of the Toeplitz extension [18]

$$0 \longrightarrow \mathcal{K}(\ell_2(\mathbb{N})) \hookrightarrow \mathcal{T} \xrightarrow{\ \tau\ } C(\mathbb{S}^1) \longrightarrow 0 \tag{3.9}$$

can be given by $\tau(z) = \tau(s) = u$, where u denotes the unitary generator of $C(\mathbb{S}^1)$ and $\mathcal{K}(\ell_2(\mathbb{N}))$ stands for the C*-algebra of compact operators on $\ell_2(\mathbb{N})$.

Returning to the map (2.1), observe that $SU(2) \cong \mathbb{S}^3$ is homeomorphic to the topological quotient of $\bar{\mathbb{D}} \times \mathbb{S}^1$ given by shrinking the circle $\mathbb{S}^1$ to a point on the boundary of $\bar{\mathbb{D}}$. This can be visualised by the following push-out diagram:

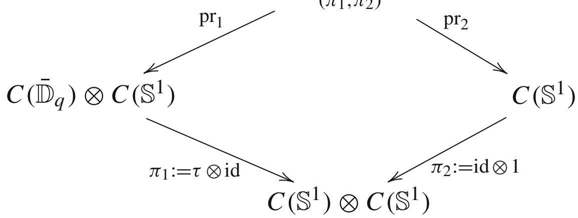
$$\chi_1(z,v) := (z, \sqrt{1 - z\bar{z}}\, v) \qquad \chi_2(u) := (u, 0)$$



$$(\iota, \mathrm{id})(u,v) := (u,v) \qquad \mathrm{pr}_1(u,v) := u.$$

Applying the functor that assigns to a topological space the algebra of continuous functions, we obtain a pull-back diagram of C*-algebras. Quantum SU(2) is now obtained by replacing in this pull-back diagram the C*-algebra $C(\bar{\mathbb{D}})$ by the Toeplitz algebra $C(\bar{\mathbb{D}}_q) := \mathcal{T}$, regarded as the algebra of continuous functions on the quantum disc. The restriction map $\iota^* : C(\bar{\mathbb{D}}) \longrightarrow C(\mathbb{S}^1)$, $\iota^*(f) = f\lceil_{\mathbb{S}^1}$, is replaced by the symbol map $\tau : \mathcal{T} \longrightarrow C(\mathbb{S}^1)$ from (3.9). The resulting pull-back diagram has the following structure:

$$P := \left(C(\bar{\mathbb{D}}_q) \otimes C(\mathbb{S}^1)\right) \underset{(\pi_1,\pi_2)}{\times} C(\mathbb{S}^1)$$



Note that $(t \otimes f, g) \in P$ if and only if $\tau(t) \otimes f = g \otimes 1 \in C(\mathbb{S}^1) \otimes \mathbb{C}1 \subset C(\mathbb{S}^1) \otimes C(\mathbb{S}^1)$. Since $\mathbb{C}1 \cong \mathbb{C}(\{pt\})$, the interpretation of $\tau(t) \otimes f = g \otimes 1$ is that, whenever we evaluate $t \in C(\bar{\mathbb{D}}_q)$ on the boundary, the circle $\mathbb{S}^1$ in $\bar{\mathbb{D}}_q \times \mathbb{S}^1$ collapses to a point.

Moreover, it can be shown [9, Section 3.2] that pr_1 yields an isomorphism of C*-algebras $P \cong \mathrm{pr}_1(P) \cong C(SU_q(2)) \subset B(\ell_2(\mathbb{N}) \otimes \ell_2(\mathbb{Z}))$ such that $\rho(a) = \mathrm{pr}_1((z \otimes 1, u))$ and $\rho(c) = \mathrm{pr}_1((y \otimes u, 0))$, see (3.6).

Viewing $C(SU_q(2))$ as a subalgebra of $C(\bar{\mathbb{D}}_q) \otimes C(\mathbb{S}^1)$ allows us to construct the following faithful Hilbert space representation of the C*-algebra $C(SU_q(2))$ in which it acts by multiplication operators on a noncommutative function algebra. This leads to an interpretation as an algebra of integrable functions on the quantum space $\mathbb{D}_q \times \mathbb{S}^1$. First note that, since $z^*z = 1 - y^2$ and $zz^* = 1 - q^2y^2$, any element

$p \in \mathcal{O}(\mathbb{D}_q)$ can be written as

$$p = \sum_{n=0}^{N} z^{*n} p_n(y) + \sum_{n=1}^{M} p_{-n}(y) z^n, \quad N, M \in \mathbb{N},$$

with polynomials p_n and p_{-n}. Using the functional calculus of the self-adjoint operator y with spectrum $\mathrm{spec}(y) = \{q^n : n \in \mathbb{N}\} \cup \{0\}$, we define

$$\mathcal{F}(\mathbb{D}_q) := \left\{ \sum_{n=0}^{N} z^{*n} f_n(y) + \sum_{n=1}^{M} f_{-n}(y) z^n \, : \, N, M \in \mathbb{N}, \; f_k \in L_\infty(\mathrm{spec}(y)) \right\}.$$

Using the commutation relations

$$z f(y) = f(qy) z, \quad f(y) z^* = z^* f(qy), \quad f \in L_\infty(\mathrm{spec}(y)),$$

one easily verifies that $\mathcal{F}(\mathbb{D}_q)$ is a *-algebra. Let s denote the unilateral shift operator on $\ell_2(\mathbb{N})$ from Eq. (3.8). For all functions $f \in L_\infty(\mathrm{spec}(y))$, it satisfies the commutation relations

$$s^* f(y) = f(qy) s^*, \quad f(y) s = s f(qy).$$

Writing z in its polar decomposition $z = s^* |z| = s^* \sqrt{1 - y^2}$, one sees that

$$\mathcal{F}(\mathbb{D}_q) = \left\{ \sum_{n=0}^{N} s^n f_n(y) + \sum_{n=1}^{M} f_{-n}(y) s^{*n} \, : \, N, M \in \mathbb{N}, \; f_k \in L_\infty(\mathrm{spec}(y)) \right\}.$$

Since $y^\alpha e_n = q^{\alpha n} e_n$, the operator y^α is trace class for all $\alpha > 0$. Therefore the positive functional

$$\int_{\mathbb{D}_q} (\,\cdot\,)\, \mathrm{d}\mu_\alpha \, : \, \mathcal{F}(\mathbb{D}_q) \longrightarrow \mathbb{C}, \qquad \int_{\mathbb{D}_q} f \, \mathrm{d}\mu_\alpha := (1-q) \mathrm{Tr}_{\ell_2(\mathbb{N})}(f y^\alpha). \tag{3.10}$$

is well defined. Explicitly, it is given by

$$\int_{\mathbb{D}_q} \left(\sum_{n=0}^{N} s^n f_n(y) + \sum_{n=1}^{M} f_{-n}(y) s^{*n} \right) \mathrm{d}\mu_\alpha = (1-q) \sum_{n \in \mathbb{N}} f_0(q) q^{\alpha n}.$$

Using $\mathrm{Tr}_{\ell_2(\mathbb{N})}(s^n s^{*k} f(y)y) = 0$ if $k \neq n$, one easily verifies that it is faithful. In terms of the Jackson integral $\int_0^1 f(y)\mathrm{d}_q y = (1-q)\sum_{n\in\mathbb{N}} f(q^n)q^n$, we can write

$$\int_{\mathbb{D}_q} \left(\sum_{n=0}^{N} s^n f_n(y) + \sum_{n=1}^{M} f_{-n}(y)s^{*n}\right) \mathrm{d}\mu_\alpha$$
$$= \int_0^1 \int_{-\pi}^{\pi} \sum_{n=0}^{N} \mathrm{e}^{in\phi} f_n(y) + \sum_{n=1}^{M} f_{-n}(y)\,\mathrm{e}^{-in\phi}\,\mathrm{d}\phi\, y^{\alpha-1}\mathrm{d}_q y. \quad (3.11)$$

Note that the commutation relation between y^α and functions from $\mathcal{F}(\mathbb{D}_q)$ can be expressed by the automorphism $\sigma^\alpha : \mathcal{F}(\mathbb{D}_q) \longrightarrow \mathcal{F}(\mathbb{D}_q)$ given by

$$\sigma^\alpha(s) = q^{-\alpha} s, \quad \sigma^\alpha(s^*) = q^\alpha s^*, \quad \sigma^\alpha(f(y)) = f(y), \quad f \in L_\infty(\mathrm{spec}(y)), \quad (3.12)$$

where $\alpha \in \mathbb{R}$. Then, for all $h, g \in \mathcal{F}(\mathbb{D}_q)$,

$$g\, y^\alpha = y^\alpha \sigma^\alpha(g),$$

and therefore

$$\int_{\mathbb{D}_q} gh\,\mathrm{d}\mu_\alpha = (1-q)\mathrm{Tr}_{\ell_2(\mathbb{N})}(ghy^\alpha) = (1-q)\mathrm{Tr}_{\ell_2(\mathbb{N})}(\sigma^\alpha(h)gy^\alpha)$$
$$= \int_{\mathbb{D}_q} \sigma^\alpha(h)g\,\mathrm{d}\mu_\alpha. \quad (3.13)$$

Note that we also have

$$(\sigma^\alpha(f))^* = \sigma^{-\alpha}(f^*), \quad f \in \mathcal{F}(\mathbb{D}_q). \quad (3.14)$$

We use the faithful positive functional $\int_{\mathbb{D}_q}(\,\cdot\,)\,\mathrm{d}\mu_\alpha$ to define an inner product on $\mathcal{F}(\mathbb{D}_q)$ by

$$\langle f, g\rangle := \int_{\mathbb{D}_q} f^* g\,\mathrm{d}\mu_\alpha.$$

The Hilbert space closure of $\mathcal{F}(\mathbb{D}_q)$ will be denoted by $L_2(\mathbb{D}_q, \mu_\alpha)$. Left multiplication with functions $x \in \mathcal{F}(\mathbb{D}_q)$ defines a faithful *-representation of $\mathcal{F}(\mathbb{D}_q)$ on $L_2(\mathbb{D}_q, \mu_\alpha)$ since

$$\langle xf, g\rangle = \int_{\mathbb{D}_q} f^* x^* g\,\mathrm{d}\mu_\alpha = \langle f, x^* g\rangle.$$

Observe that $\mathcal{F}(\mathbb{D}_q)$ leaves the subspace

$$\mathcal{F}_0(\mathbb{D}_q) := \left\{ \sum_{n=0}^{N} s^n f_n(y) + \sum_{n=1}^{M} f_{-n}(y) s^{*n} \in \mathcal{F}(\mathbb{D}_q) : \text{ supp}(f_k) \text{ is finite} \right\} \tag{3.15}$$

of $L_2(\mathbb{D}_q, \mu_\alpha)$ invariant. Since $\mathcal{F}_0(\mathbb{D}_q)$ contains an orthonormal basis (see [20, Proposition 1]), it is dense in $L_2(\mathbb{D}_q, \mu_\alpha)$. We extend $\mathcal{F}(\mathbb{D}_q)$ by the unbounded element y^{-1} and define $\mathcal{O}^+(\mathbb{D}_q)$ as the *-algebra generated by the operators y^{-1} and all $f \in \mathcal{F}(\mathbb{D}_q)$, considered as operators on $\mathcal{F}_0(\mathbb{D}_q)$. Furthermore, let $\mathcal{O}^+(\mathbb{D}_q)^{\mathrm{op}}$ denote the *-algebra obtained from $\mathcal{O}^+(\mathbb{D}_q)$ by replacing the multiplication with the opposite one, i.e. $a \cdot b := ba$. Then we obtain a representation of $\mathcal{O}^+(\mathbb{D}_q)^{\mathrm{op}}$ on $\mathcal{F}_0(\mathbb{D}_q) \subset L_2(\mathbb{D}_q, \mu_\alpha)$ by right multiplication,

$$a^{\mathrm{op}} f := fa, \quad a \in \mathcal{O}^+(\mathbb{D}_q)^{\mathrm{op}}, \quad f \in \mathcal{F}_0(\mathbb{D}_q).$$

Clearly, this representation commutes with the operators of $\mathcal{O}^+(\mathbb{D}_q)$, as these act by left multiplication. However, it is not a *-representation. More precisely, (3.13) and (3.14) give

$$\langle x^{\mathrm{op}} f, g\rangle = \langle fx, g\rangle = \int_{\mathbb{D}_q} x^* f^* g \, \mathrm{d}\mu_\alpha = \int_{\mathbb{D}_q} f^* g \sigma^{-\alpha}(x^*) \, \mathrm{d}\mu_\alpha = \langle f, (\sigma^\alpha(x)^*)^{\mathrm{op}} g\rangle,$$

therefore

$$(x^{\mathrm{op}})^* = (\sigma^\alpha(x)^*)^{\mathrm{op}}. \tag{3.16}$$

Note that $y > 0$ and $\sigma(y) = y$ imply that the multiplication operators y^β and $(y^\beta)^{\mathrm{op}} = (y^{\mathrm{op}})^\beta$, $\beta \in \mathbb{R}$, determine well defined (unbounded) self-adjoint operators on $L_2(\mathbb{D}_q, \mu_\alpha)$.

Next we use the isomorphism $\ell_2(\mathbb{Z}) \cong L_2(\mathbb{S}^1)$ given by $b_n := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}nt}$ and identify u from (3.3) with the multiplication operator $uf(t) := \mathrm{e}^{\mathrm{i}t} f(t)$. In this way we obtain a faithful *-representation $\tilde{\rho} : \mathcal{O}(\mathrm{SU}_q(2)) \longrightarrow B(L_2(\mathbb{D}_q, \mu_\alpha) \otimes L_2(\mathbb{S}^1))$ by multiplication operators. On generators, it is given by

$$\tilde{\rho}(a)(f \otimes g) := zf \otimes g, \quad \tilde{\rho}(c) := yf \otimes ug.$$

The closure of the image of $\tilde{\rho}$ is again isomorphic to $C(\mathrm{SU}_q(2))$.

4 Quantised Differential Calculi

Taking as its domain the absolutely continuous functions $AC(\mathbb{S}^1)$ with the weak derivative in $L_2(\mathbb{S}^1)$, the partial derivative $\mathrm{i}\frac{\partial}{\partial t}$ becomes a self-adjoint operator on $L_2(\mathbb{S}^1)$ satisfying the Leibniz rule

$$\mathrm{i}\tfrac{\partial}{\partial t}(\varphi g) = (\mathrm{i}\frac{\partial \varphi}{\partial t})g + \varphi(\mathrm{i}\tfrac{\partial}{\partial t}g), \quad \varphi \in C^{(1)}(\mathbb{S}^1), \ \ g \in \mathrm{dom}(\mathrm{i}\tfrac{\partial}{\partial t}).$$

We consider a first order differential *-calculus $\mathrm{d} : \mathcal{O}(\mathbb{D}_q) \longrightarrow \Omega(\mathbb{D}_q)$, where $\Omega(\mathbb{D}_q) = \mathrm{d}z\,\mathcal{O}(\mathbb{D}_q) + \mathrm{d}z^*\mathcal{O}(\mathbb{D}_q)$ with $\mathcal{O}(\mathbb{D}_q)$-bimodule structure given by

$$\mathrm{d}z\, z^* = q^2 z^*\,\mathrm{d}z, \ \ \mathrm{d}z^*\, z = q^{-2} z\,\mathrm{d}z^*, \ \ \mathrm{d}z\, z = q^{-2} z\mathrm{d}z, \ \ \mathrm{d}z^*\, z^* = q^2 z^*\mathrm{d}z^*,$$

see [12] for definitions and background on differential calculi. With σ^α from (3.12), it follows that

$$\mathrm{d}z\, f = \sigma^{-2}(f)\,\mathrm{d}z, \quad \mathrm{d}z^*\, f = \sigma^{-2}(f)\,\mathrm{d}z^*, \quad f \in \mathcal{O}(\mathbb{D}_q).$$

We define partial derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ by

$$\mathrm{d}(f) = \mathrm{d}z\, \tfrac{\partial}{\partial z}(f) + \mathrm{d}z^*\, \tfrac{\partial}{\partial \bar{z}}(f), \quad f \in \mathcal{O}(\mathbb{D}_q).$$

Recall that $y^2 = 1 - z^*z$ and $zz^* - z^*z = (1-q^2)y^2$ by (3.7) and (3.4). Using

$$1 = \tfrac{\partial}{\partial z}(z) = \frac{-1}{1-q^2}\, y^{-2}[z^*, z], \quad 1 = \tfrac{\partial}{\partial \bar{z}}(z^*) = \frac{1}{1-q^2}\, y^{-2}[z, z^*],$$

the Leibniz rule for the commutator and $y^{-2}p = \sigma^2(p)y^{-2}$ for all $p \in \mathcal{O}(\mathbb{D}_q)$, one verifies by direct calculations on monomials $z^n z^{*m}$ that

$$\tfrac{\partial}{\partial z}(p) = \frac{-1}{1-q^2}\, y^{-2}[z^*, p], \quad \tfrac{\partial}{\partial \bar{z}}(p) = \frac{1}{1-q^2}\, y^{-2}[z, p], \quad p \in \mathcal{O}(\mathbb{D}_q).$$

We extend the partial derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ to

$$\mathcal{F}^{(1)}(\mathbb{D}_q) := \{ f \in \mathcal{F}(\mathbb{D}_q) : y^{-2}[z^*, f] \in \mathcal{F}(\mathbb{D}_q), \ \ y^{-2}[z, f] \in \mathcal{F}(\mathbb{D}_q) \}$$

by setting

$$\tfrac{\partial}{\partial z}(f) := \frac{-1}{1-q^2}\, y^{-2}[z^*, f], \quad \tfrac{\partial}{\partial \bar{z}}(f) := \frac{1}{1-q^2}\, y^{-2}[z, f], \quad f \in \mathcal{F}^{(1)}(\mathbb{D}_q).$$

Note that $\mathcal{O}(\mathbb{D}_q) \subset \mathcal{F}^{(1)}(\mathbb{D}_q)$. By the spectral theorem for functions in $y = y^*$, one readily proves that $\mathcal{O}(\mathbb{D}_q)$ is dense in $L_2(\mathbb{D}_q, \mu_\alpha)$. Thus $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ are densely defined linear operators on $L_2(\mathbb{D}_q, \mu_\alpha)$.

Moreover, it is easily seen that the automorphism σ^α from (3.12) preserves $\mathcal{F}^{(1)}(\mathbb{D}_q)$. For instance, $y^{-2}[z^*, \sigma^\alpha(s^n f(y))] = q^{-\alpha n} y^{-2}[z^*, s^n f(y)] \in \mathcal{F}(\mathbb{D}_q)$ for all $s^n f(y) \in \mathcal{F}^{(1)}(\mathbb{D}_q)$. Similarly one shows that $\mathcal{F}^{(1)}(\mathbb{D}_q)$ is a *-algebra. For example,

$$\begin{aligned} y^{-2}[z^*, fg] &= y^{-2}[z^*, f]g + y^{-2} f y^{-2} y^{-2}[z^*, g] \\ &= y^{-2}[z^*, f]g + \sigma^2(f)\, y^{-2}[z^*, g] \in \mathcal{F}(\mathbb{D}_q) \end{aligned}$$

and

$$y^{-2}[z^*, f^*] = -(y^{-2} y^2 [z, f] y^{-2})^* = -q^2 (y^{-2}[z, \sigma^2(f)])^* \in \mathcal{F}(\mathbb{D}_q)$$

for $f, g \in \mathcal{F}^{(1)}(\mathbb{D}_q)$.

5 Twisted Derivations

5.1 Twist: σ^1

Our aim is to replace the first order differential operators H, E and F from (2.4)–(2.6) by appropriate noncommutative versions. First we consider q-analogues of the operators $\sqrt{1 - \bar{z}z}\frac{\partial}{\partial z}$ and $\sqrt{1 - \bar{z}z}\frac{\partial}{\partial \bar{z}}$ and define $T_i : \mathcal{F}^{(1)}(\mathbb{D}_q) \longrightarrow \mathcal{F}(\mathbb{D}_q)$, $i = 1, 2$, by

$$T_1 f := y \tfrac{\partial}{\partial z} f = \frac{-1}{1 - q^2} y^{-1}[z^*, f], \quad T_2 f := y \tfrac{\partial}{\partial \bar{z}} f = \frac{1}{1 - q^2} y^{-1}[z, f]. \tag{5.1}$$

Observe that T_1 and T_2 satisfy a twisted Leibniz rule: for all $f, g \in \mathcal{F}^{(1)}(\mathbb{D}_q)$,

$$\begin{aligned} T_1(fg) &= \frac{-1}{1 - q^2} y^{-1}[z^*, fg] = \frac{-1}{1 - q^2} y^{-1}([z^*, f]g + f y y^{-1}[z^*, g]) \\ &= (T_1 f)g + \sigma^1(f)\, T_1 g \end{aligned} \tag{5.2}$$

and similarly

$$T_2(fg) = (T_2 f)g + \sigma^1(f)\, T_2 g. \tag{5.3}$$

Setting $\hat{T}_1 := T_1 \otimes 1$, $\hat{T}_2 := T_2 \otimes 1$ and $\hat{\sigma}^1 := \sigma^1 \otimes 1$, we get

$$\hat{T}_1(\phi\psi) = (\hat{T}_1\phi)\psi + \hat{\sigma}^1(\phi)\,\hat{T}_1\psi, \quad \hat{T}_2(\phi\psi) = (\hat{T}_2\phi)\psi + \hat{\sigma}^1(\phi)\,\hat{T}_2\psi \tag{5.4}$$

for all $\phi, \psi \in \mathcal{F}^{(1)}(\mathbb{D}_q) \otimes C^{(1)}(\mathbb{S}^1)$ by (5.2) and (5.3).

Next consider the operator $\hat{T}_0 := y^{-1}\frac{\partial}{\partial t}$ on the domain $\mathrm{dom}(y^{-1}) \otimes C^{(1)}(\mathbb{S}^1)$ in $L_2(\mathbb{D}_q, \mu_\alpha) \otimes L_2(\mathbb{S}^1)$. Note that, for all $f, g \in \mathcal{F}(\mathbb{D}_q)$ with $g \in \mathrm{dom}(y^{-1})$, one has $y^{-1}fg = \sigma^1(f)y^{-1}g \in L_2(\mathbb{D}_q, \mu_\alpha)$, hence $fg \in \mathrm{dom}(y^{-1})$. Now, for all $\varphi, \xi \in C^{(1)}(\mathbb{S}^1)$,

$$\begin{aligned}\hat{T}_0(fg \otimes \varphi\xi) &= y^{-1}fg \otimes \tfrac{\partial}{\partial t}(\varphi\xi) \\ &= y^{-1}fg \otimes (\tfrac{\partial}{\partial t}\varphi)\xi + y^{-1}fy\,y^{-1}g \otimes \varphi\,(\tfrac{\partial}{\partial t}\xi) \\ &= \big(\hat{T}_0(f \otimes \varphi)\big)(g \otimes \xi) + (\sigma^1(f) \otimes \varphi)\big(\hat{T}_0(g \otimes \xi)\big).\end{aligned}$$

Therefore, for all $\phi, \psi \in \mathcal{F}(\mathbb{D}_q) \otimes C^{(1)}(\mathbb{S}^1)$ with $\psi \in \mathrm{dom}(y^{-1} \otimes 1)$, we have

$$\hat{T}_0(\phi\psi) = (\hat{T}_0\phi)\psi + \hat{\sigma}^1(\phi)\,(\hat{T}_0\psi). \tag{5.5}$$

As a consequence, $\hat{T}_0$, $\hat{T}_1$ and $\hat{T}_2$ satisfy the same twisted Leibniz rule.

In the definition of the Dirac operator, we will multiply $\hat{T}_0$, $\hat{T}_1$ and $\hat{T}_2$ with multiplication operators from the opposite algebra. The following lemma shows that these multiplication operators do not change the twisted Leibniz rule. Our aim is to prove that the Dirac operator has bounded twisted commutators with functions of an appropriate *-algebra, where the twisted commutator of densely defined operators T on $L_2(\mathbb{D}_q, \mu_\alpha)\bar{\otimes}L_2(\mathbb{S}^1)$ with $\phi \in \mathcal{F}(\mathbb{D}_q) \otimes C(\mathbb{S}^1)$ is defined by

$$[T, \phi]_{\sigma^1} := T\phi - \sigma^1(\phi)T.$$

The purpose of the following lemma is to clarify the setup for the algebraic manipulations to be carried out, and to ensure that these make sense in their Hilbert space realisation.

Lemma 5.1 *Let $\mathcal{A}$ be a unital *-algebra, $\int : \mathcal{A} \to \mathbb{C}$ a faithful positive functional, $\mathcal{H}$ the Hilbert space closure of $\mathcal{A}$ with respect to the inner product $\langle a, b\rangle := \int a^*b$, and assume that left multiplication by an element in $\mathcal{A}$ defines a bounded operator on $\mathcal{H}$. Let T be a densely defined linear operator on $\mathcal{H}$, $\mathcal{A}^1 \subset \mathcal{A}$ a *-subalgebra and $\mathcal{D} \subset \mathcal{H}$ a dense subspace such that $\mathcal{D} + \mathcal{A}^1 \subset \mathrm{dom}(T)$, $T(\mathcal{A}^1) \subset \mathcal{A}$, and T satisfies the twisted Leibniz rule*

$$T(f\psi) = (Tf)\psi + \sigma(f)T\psi, \quad f \in \mathcal{A}^1, \ \ \psi \in \mathcal{D} + \mathcal{A}^1$$

for an automorphism $\sigma : \mathcal{A} \to \mathcal{A}$. Assume finally that $\hat{x}$ is a densely defined linear operator on $\mathcal{H}$ with $\mathcal{D} \subset \mathrm{dom}(\hat{x})$, and that $f\psi \in \mathcal{D}$ and $\hat{x}f\psi = f\hat{x}\psi$ hold for all

$f \in \mathcal{A}$ and $\psi \in \mathcal{D}$. Then

$$\hat{x}T(fg) = \hat{x}(Tf)g + \sigma(f)\hat{x}T(g), \quad f, g \in \mathcal{A}^1 \tag{5.6}$$

as operators on $\mathcal{D}$ and

$$[\hat{x}T, f]_\sigma \psi = \hat{x}(Tf)\psi, \quad f \in \mathcal{A}^1, \ \ \psi \in \mathcal{D}.$$

Proof The only slightly nontrivial statement is that each term in the following algebraic computations is well-defined as an operator on the domain $\mathcal{D}$: Let $\psi \in \mathcal{D}$ and $f, g \in \mathcal{A}^1$. From the twisted Leibniz rule, we get

$$\begin{aligned}(T(fg))\psi &= T(fg\psi) - \sigma(fg)T\psi = (Tf)(g\psi) + \sigma(f)T(g\psi) - \sigma(f)\sigma(g)T\psi \\ &= (Tf)(g\psi) + \sigma(f)(Tg)\psi.\end{aligned}$$

Since $\sigma(f) \in \mathcal{A}$ for all $f \in \mathcal{A}$, it follows that

$$\hat{x}(T(fg))\psi = \big(\hat{x}(Tf)g + \sigma(f)\hat{x}(Tg)\big)\psi,$$

which proves (5.6). As $\hat{x}\,T(f\psi) = \hat{x}\,(Tf)\psi + \hat{x}\,\sigma(f)\,T\psi = \hat{x}\,(Tf)\psi + \sigma(f)\hat{x}\,T\psi$, we also have $[\hat{x}\,T, f]_\sigma\psi = \hat{x}\,(Tf)\psi + \sigma(f)\,\hat{x}\,T\psi - \sigma(f)\,\hat{x}\,T\psi = \hat{x}\,(Tf)\psi$. □

By (5.4) and (5.5), the lemma applies in particular to the operators $\hat{T}_0$, $\hat{T}_1$ and $\hat{T}_2$ with $\mathcal{A} := \mathcal{F}(\mathbb{D}_q)\otimes C(\mathbb{S}^1)$, $\mathcal{A}^1 := \mathcal{F}^{(1)}(\mathbb{D}_q)\otimes C^{(1)}(\mathbb{S}^1)$, $\mathcal{H} := L_2(\mathbb{D}_q, \mu_\alpha)\bar{\otimes} L_2(\mathbb{S}^1)$ (where integration on $\mathbb{S}^1$ is taken with respect to the Lebesgue measure), the automorphism $\hat{\sigma}^1$ and the operators $\hat{x}$ coming from $\mathcal{O}^+(\mathbb{D}_q)^{\mathrm{op}}$. As the dense domain, we may take

$$\mathcal{D} := \mathcal{F}_0(\mathbb{D}_q) \otimes C^{(1)}(\mathbb{S}^1). \tag{5.7}$$

5.2 Twist: σ^2

First we show that $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ satisfy a twisted Leibniz rule for the automorphism σ^2. Let $f, g \in \mathcal{F}^{(1)}(\mathbb{D}_q)$. Then

$$\begin{aligned}\tfrac{\partial}{\partial z}(fg) &= \frac{-1}{1-q^2}\, y^{-2}[z^*, fg] = \frac{-1}{1-q^2}\, y^{-2}([z^*, f]g + fy^2y^{-2}[z^*, g]) \\ &= (\tfrac{\partial}{\partial z} f)g + \sigma^2(f)\,\tfrac{\partial}{\partial z} g\end{aligned} \tag{5.8}$$

and similarly

$$\tfrac{\partial}{\partial \bar{z}}(fg) = (\tfrac{\partial}{\partial \bar{z}} f)g + \sigma^2(f)\,\tfrac{\partial}{\partial \bar{z}} g. \tag{5.9}$$

Setting $\hat{S}_1 := \frac{\partial}{\partial z} \otimes 1$, $\hat{S}_2 := \frac{\partial}{\partial \bar{z}} \otimes 1$ and $\hat{\sigma}^2 := \sigma^2 \otimes 1$, we get for all $\phi, \psi \in \mathcal{F}^{(1)}(\mathbb{D}_q) \otimes C^{(1)}(\mathbb{S}^1)$

$$\hat{S}_1(\phi\psi) = (\hat{S}_1\phi)\psi + \hat{\sigma}^2(\phi)\,\hat{S}_1\psi, \quad \hat{S}_2(\phi\psi) = (\hat{S}_2\phi)\psi + \hat{\sigma}^2(\phi)\,\hat{S}_2\psi$$

by (5.8) and (5.9).

Next consider the operator $\hat{S}_0 := y^{-2}\frac{\partial}{\partial t}$ on the domain $\mathrm{dom}(y^{-2}) \otimes C^{(1)}(\mathbb{S}^1)$ in $L_2(\mathbb{D}_q, \mu_\alpha) \otimes L_2(\mathbb{S}^1)$. Again $fg \in \mathrm{dom}(y^{-2})$ for all $f \in \mathcal{F}(\mathbb{D}_q)$ and $g \in \mathrm{dom}(y^{-2})$ since $y^{-2}fg = \sigma^2(f)y^{-2}g \in L_2(\mathbb{D}_q, \mu_\alpha)$. Now, for all $\varphi,\ \xi \in C^{(1)}(\mathbb{S}^1)$,

$$\begin{aligned}\hat{S}_0(fg \otimes \varphi\xi) &= y^{-2}fg \otimes \tfrac{\partial}{\partial t}(\varphi\xi)\\ &= y^{-2}fg \otimes (\tfrac{\partial}{\partial t}\varphi)\xi + y^{-2}fy^2\,y^{-2}g \otimes \varphi\,(\tfrac{\partial}{\partial t}\xi)\\ &= \big(\hat{S}_0(f \otimes \varphi)\big)(g \otimes \xi) + (\sigma^2(f) \otimes \varphi)\big(\hat{S}_0(g \otimes \xi)\big).\end{aligned}$$

Therefore, for all $\phi, \psi \in \mathcal{F}(\mathbb{D}_q) \otimes C^{(1)}(\mathbb{S}^1)$ with $\psi \in \mathrm{dom}(y^{-1} \otimes 1)$, we have

$$\hat{S}_0(\phi\psi) = (\hat{S}_0\phi)\psi + \hat{\sigma}^2(\phi)\,(\hat{S}_0\psi).$$

As a consequence, $\hat{S}_0$, $\hat{S}_1$ and $\hat{S}_2$ satisfy the same twisted Leibniz rule for the twist $\hat{\sigma}^2$, and so do $x_i^{\mathrm{op}} S_i$ for $x_i^{\mathrm{op}} \in \{y^{\mathrm{op}}, (y^{\mathrm{op}})^2, z^{\mathrm{op}}, z^{*\mathrm{op}}\}$ by Lemma 5.1.

6 Adjoints

6.1 $\alpha = 2$

Set $\alpha = 2$ in (3.10),

$$\mathcal{D}_0 := \{f \in \mathcal{F}^{(1)}(\mathbb{D}_q) \cap \mathrm{dom}(y^{-1}) \cap \mathrm{dom}((y^{-1})^{\mathrm{op}}) : T_1(f), T_2(f) \in \mathrm{dom}((y^{-1})^{\mathrm{op}})\}$$

and

$$\mathcal{D} := \mathcal{D}_0 \otimes C^{(1)}(\mathbb{S}^1) \ \subset\ L_2(\mathbb{D}_q, \mu_\alpha)\ \bar{\otimes}\ L_2(\mathbb{S}^1). \tag{6.1}$$

It follows from $\mathcal{F}_0(\mathbb{D}_q) \subset \mathcal{D}_0$ that $\mathcal{D}$ is dense in $L_2(\mathbb{D}_q, \mu_\alpha)\ \bar{\otimes}\ L_2(\mathbb{S}^1)$. Let T_1 and T_2 be the operators from (5.1) with domain $\mathcal{D}_0$. Using (3.5) and the trace property, we compute for all $f, g \in \mathcal{D}_0$,

$$\langle T_1 f, g\rangle = -\frac{1-q}{1-q^2}\,\mathrm{Tr}_{\ell_2(\mathbb{N})}(f^* z y^{-1} g y^2 - z f^* y^{-1} g y^2)$$

$$= -\frac{1-q}{1-q^2}\,\mathrm{Tr}_{\ell_2(\mathbb{N})}(q^{-1}f^*y^{-1}zgy^2 - q^{-2}f^*y^{-1}gzy^2)$$

$$= \frac{(1-q)(q^{-2}-q^{-1})}{1-q^2}\,\mathrm{Tr}_{\ell_2(\mathbb{N})}(f^*y^{-1}gzy^2)$$

$$-\frac{(1-q)q^{-1}}{1-q^2}\,\mathrm{Tr}_{\ell_2(\mathbb{N})}(f^*y^{-1}zgy^2 - f^*y^{-1}gzy^2)$$

$$= \langle f, (\frac{q^{-2}}{1+q}z^{\mathrm{op}}y^{-1} - q^{-1}T_2)g\rangle,$$

therefore

$$\frac{q^{-2}}{1+q}z^{\mathrm{op}}y^{-1} - q^{-1}T_2 \ \subset\ T_1^*. \tag{6.2}$$

From (3.16), it also follows that

$$\frac{q}{1+q}z^{*\mathrm{op}}y^{-1} - qT_1 \subset T_2^*.$$

Similarly, using $zz^* - z^*z = (1-q^2)y^2$,

$$\langle (zy^{-1})^{\mathrm{op}}T_1 f, g\rangle = -\frac{1-q}{1-q^2}\,\mathrm{Tr}_{\ell_2(\mathbb{N})}(y^{-1}z^*f^*zy^{-1}gy^2 - y^{-1}z^*zf^*y^{-1}gy^2)$$

$$= -\frac{1-q}{1-q^2}\,\mathrm{Tr}_{\ell_2(\mathbb{N})}(y^{-1}(zz^*-z^*z)f^*y^{-1}gy^2 + y^{-1}z^*f^*zy^{-1}gy^2 - y^{-1}zz^*f^*y^{-1}gy^2)$$

$$= -(1-q)\,\mathrm{Tr}_{\ell_2(\mathbb{N})}\big(f^*y^{-1}gyy^2\big) - \frac{1-q}{1-q^2}\,\mathrm{Tr}_{\ell_2(\mathbb{N})}\big(f^*y^{-1}(zg-gz)z^*y^{-1}y^2\big)$$

$$= \langle f, \big(-\sigma^1 - (z^*y^{-1})^{\mathrm{op}}T_2\big)g\rangle,$$

where $\sigma^1(g) = y^{-1}gy = y^{-1}y^{\mathrm{op}}g$ for all $g \in \mathcal{F}(\mathbb{D}_q) \cap \mathrm{dom}(y^{-1})$. Hence

$$-\sigma^1 - (z^*y^{-1})^{\mathrm{op}}T_2 \ \subset \big((zy^{-1})^{\mathrm{op}}T_1\big)^*. \tag{6.3}$$

Since y^{-1} and y^{op} are self-adjoint and thus σ^1 is symmetric, we also get from the above calculations

$$-\sigma^1 - (zy^{-1})^{\mathrm{op}}T_1 \ \subset \big((z^*y^{-1})^{\mathrm{op}}T_2\big)^*. \tag{6.4}$$

Recall that $\mathrm{i}\frac{\partial}{\partial t}$ is a symmetric operator on $C^{(1)}(\mathbb{S}^1) \subset L_2(\mathbb{S}^1)$. Also, for all $\varphi \in C^{(1)}(\mathbb{S}^1)$, we have

$$(\mathrm{e}^{\mathrm{i}t}\mathrm{i}\tfrac{\partial}{\partial t})^*\varphi = \mathrm{i}\tfrac{\partial}{\partial t}(\mathrm{e}^{-\mathrm{i}t}\varphi) = \mathrm{e}^{-\mathrm{i}t}\varphi + \mathrm{e}^{-\mathrm{i}t}\mathrm{i}\tfrac{\partial}{\partial t}\varphi. \tag{6.5}$$

From this, (3.16) and the self-adjointness of y^{-1}, it follows that

$$q^{-2}z^{\mathrm{op}}y^{-1}\mathrm{e}^{-\mathrm{i}t} + q^{-2}z^{\mathrm{op}}y^{-1}\mathrm{e}^{-\mathrm{i}t}\mathrm{i}\tfrac{\partial}{\partial t} \subset (z^{*\mathrm{op}}y^{-1}\mathrm{e}^{\mathrm{i}t}\mathrm{i}\tfrac{\partial}{\partial t})^*,$$

where the left-hand side and $z^{*\mathrm{op}}y^{-1}\mathrm{e}^{\mathrm{i}t}\mathrm{i}\frac{\partial}{\partial t}$ are operators on $\mathcal{D}$. Analogously,

$$-q^2z^{*\mathrm{op}}y^{-1}\mathrm{e}^{\mathrm{i}t} + q^2z^{*\mathrm{op}}y^{-1}\mathrm{e}^{\mathrm{i}t}\mathrm{i}\tfrac{\partial}{\partial t} \subset (z^{\mathrm{op}}y^{-1}\mathrm{e}^{-\mathrm{i}t}\mathrm{i}\tfrac{\partial}{\partial t})^*. \tag{6.6}$$

Now we are in a position to state the main result of this section.

Proposition 6.1 *Consider the following operators on $L_2(\mathbb{D}_q, \mu_\alpha) \bar{\otimes} L_2(\mathbb{S}^1)$ with domain $\mathcal{D}$ defined in (6.1) above:*

$$\hat{H} := (zy^{-1})^{\mathrm{op}}\,y\,\tfrac{\partial}{\partial z} - (zy^{-1})^{*\mathrm{op}}\,y\,\tfrac{\partial}{\partial \bar{z}} + y^{\mathrm{op}}\,y^{-1}\,\mathrm{i}\tfrac{\partial}{\partial t},$$
$$\hat{E} := -\mathrm{e}^{-\mathrm{i}t}\,y\,\tfrac{\partial}{\partial \bar{z}} - \frac{q^{-1}}{1+q}z^{\mathrm{op}}\,y^{-1}\,\mathrm{e}^{-\mathrm{i}t}\,\mathrm{i}\tfrac{\partial}{\partial t},$$
$$\hat{F} := q\,\mathrm{e}^{\mathrm{i}t}\,y\,\tfrac{\partial}{\partial z} - \frac{q}{1+q}z^{*\mathrm{op}}\,y^{-1}\,\mathrm{e}^{\mathrm{i}t}\,\mathrm{i}\tfrac{\partial}{\partial t}.$$

Then $\hat{H} \subset \hat{H}^*$, $\hat{F} \subset \hat{E}^*$ and $\hat{E} \subset \hat{F}^*$.

Proof Since y^{op}, y^{-1} and $\mathrm{i}\frac{\partial}{\partial t}$ are commuting symmetric operators on $\mathcal{D}$, we have $y^{\mathrm{op}}\,y^{-1}\,\mathrm{i}\frac{\partial}{\partial t} \subset (y^{\mathrm{op}}\,y^{-1}\,\mathrm{i}\frac{\partial}{\partial t})^*$. Now it follows from (6.3) and (6.4) that

$$\hat{H}^* \supset -\sigma^1 - (z^*y^{-1})^{\mathrm{op}}\,y\,\tfrac{\partial}{\partial \bar{z}} + \sigma^1 + (zy^{-1})^{\mathrm{op}}\,y\,\tfrac{\partial}{\partial z} + y^{\mathrm{op}}\,y^{-1}\,\mathrm{i}\tfrac{\partial}{\partial t} = \hat{H}.$$

Furthermore, from (6.2) and (6.6), we obtain

$$\hat{F}^* \supset \mathrm{e}^{-\mathrm{i}t}\frac{q^{-1}}{1+q}z^{\mathrm{op}}y^{-1} - \mathrm{e}^{-\mathrm{i}t}\,y\,\tfrac{\partial}{\partial \bar{z}} - \mathrm{e}^{-\mathrm{i}t}\frac{q^{-1}}{1+q}z^{\mathrm{op}}y^{-1} - \frac{q^{-1}}{1+q}z^{\mathrm{op}}y^{-1}\mathrm{e}^{-\mathrm{i}t}\mathrm{i}\tfrac{\partial}{\partial t} = \hat{E}.$$

The last relation also shows that $\hat{F}^*$ is densely defined, thus $\hat{F} \subset \hat{F}^{**} \subset \hat{E}^*$. □

6.2 $\alpha = 1$

Consider now

$$\mathcal{D} := \mathcal{D}_0 \otimes C^{(1)}(\mathbb{S}^1) \subset L_2(\mathbb{D}_q, \mu_\alpha) \bar{\otimes} L_2(\mathbb{S}^1) \tag{6.7}$$

with $\mathcal{D}_0 := \mathcal{F}^{(1)}(\mathbb{D}_q) \cap \mathrm{dom}(y^{-2})$. For all $f, g \in \mathcal{F}^{(1)}(\mathbb{D}_q)$,

$$\begin{aligned}\langle y^{\mathrm{op}} \tfrac{\partial}{\partial z} f, g\rangle &= -\frac{1-q}{1-q^2} \operatorname{Tr}_{\ell_2(\mathbb{N})}(yf^* zy^{-2} gy - yzf^* y^{-2} gy) \\ &= -q^{-2} \frac{1-q}{1-q^2} \operatorname{Tr}_{\ell_2(\mathbb{N})}(f^* y^{-2} zgy^2 - f^* y^{-2} gzy^2) \\ &= -q^{-2} \frac{1-q}{1-q^2} \operatorname{Tr}_{\ell_2(\mathbb{N})}(f^* y^{-2} [z, g] y^2) \\ &= \langle f, -q^{-2} y^{\mathrm{op}} \tfrac{\partial}{\partial \bar{z}} g\rangle,\end{aligned}$$

thus

$$-q^{-1} y^{\mathrm{op}} \tfrac{\partial}{\partial \bar{z}} \subset (q y^{\mathrm{op}} \tfrac{\partial}{\partial z})^* \quad \text{and} \quad q y^{\mathrm{op}} \tfrac{\partial}{\partial z} \subset (q y^{\mathrm{op}} \tfrac{\partial}{\partial z})^{**} \subset (-q^{-1} y^{\mathrm{op}} \tfrac{\partial}{\partial \bar{z}})^*. \tag{6.8}$$

Next,

$$\begin{aligned}\langle z^{\mathrm{op}} \tfrac{\partial}{\partial z} f, g\rangle &= -\frac{1-q}{1-q^2} \operatorname{Tr}_{\ell_2(\mathbb{N})}(z^* f^* zy^{-2} gy - z^* zf^* y^{-2} gy) \\ &= -\frac{1-q}{1-q^2} \operatorname{Tr}_{\ell_2(\mathbb{N})}(q^{-1} f^* y^{-2} zgz^* y - f^* y^{-2} gz^* zy) \\ &= -\frac{1-q}{1-q^2} \operatorname{Tr}_{\ell_2(\mathbb{N})}\big(q^{-1}(f^* y^{-2} zgz^* y - f^* y^{-2} gzz^* y) \\ &\quad + q^{-1} f^* y^{-2} gzz^* y - f^* y^{-2} gz^* zy\big) \\ &= -\langle f, (q^{-1} z^{*\mathrm{op}} \tfrac{\partial}{\partial \bar{z}}) g\rangle - q^{-1} \frac{(1-q)^2}{1-q^2} \operatorname{Tr}_{\ell_2(\mathbb{N})}\big(f^* y^{-2} gy + q f^* y^{-2} gy^2 y\big)\end{aligned} \tag{6.9}$$

and

$$\begin{aligned}\langle (z^{*\mathrm{op}} \tfrac{\partial}{\partial \bar{z}}) f, g\rangle &= \frac{1-q}{1-q^2} \operatorname{Tr}_{\ell_2(\mathbb{N})}(zf^* z^* y^{-2} gy - zz^* f^* y^{-2} gy) \\ &= \frac{1-q}{1-q^2} \operatorname{Tr}_{\ell_2(\mathbb{N})}(q f^* y^{-2} z^* gzy - f^* y^{-2} gzz^* y) \\ &= \frac{1-q}{1-q^2} \operatorname{Tr}_{\ell_2(\mathbb{N})}\big(q(f^* y^{-2} z^* gzy - f^* y^{-2} gz^* zy) + q f^* y^{-2} gz^* zy - f^* y^{-2} gzz^* y\big) \\ &= -\langle f, (q z^{\mathrm{op}} \tfrac{\partial}{\partial z}) g\rangle - \frac{(1-q)^2}{1-q^2} \operatorname{Tr}_{\ell_2(\mathbb{N})}\big(f^* y^{-2} gy + q f^* y^{-2} gy^2 y\big).\end{aligned} \tag{6.10}$$

From (6.9) and (6.10),

$$\langle (qz^{\mathrm{op}}\tfrac{\partial}{\partial z} - z^{*\mathrm{op}}\tfrac{\partial}{\partial \bar{z}})f, g\rangle = \langle f, (qz^{\mathrm{op}}\tfrac{\partial}{\partial z} - z^{*\mathrm{op}}\tfrac{\partial}{\partial \bar{z}})g\rangle \tag{6.11}$$

i.e., the operator $qz^{\mathrm{op}}\frac{\partial}{\partial z} - z^{*\mathrm{op}}\frac{\partial}{\partial \bar{z}}$ is symmetric. As $(y^2)^{\mathrm{op}}y^{-2}\mathrm{i}\frac{\partial}{\partial t}$ is the product of commuting symmetric operators,

$$(y^2)^{\mathrm{op}}y^{-2}\mathrm{i}\tfrac{\partial}{\partial t} \subset \big((y^2)^{\mathrm{op}}y^{-2}\mathrm{i}\tfrac{\partial}{\partial t}\big)^* \tag{6.12}$$

is also symmetric.

By (3.16) and (6.5),

$$z^{\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{-\mathrm{i}t} + z^{\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{-\mathrm{i}t}\mathrm{i}\tfrac{\partial}{\partial t} \subset (z^{*\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{\mathrm{i}t}\mathrm{i}\tfrac{\partial}{\partial t})^*.$$

Similarly,

$$-z^{*\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{\mathrm{i}t} + z^{*\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{\mathrm{i}t}\mathrm{i}\tfrac{\partial}{\partial t} \subset (z^{\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{-\mathrm{i}t}\mathrm{i}\tfrac{\partial}{\partial t})^*.$$

Since $z^{\mathrm{op}}y^{\mathrm{op}} \subset (z^{*\mathrm{op}}y^{\mathrm{op}})^*$ by (3.16),

$$\tfrac{1}{2}z^{\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{-\mathrm{i}t} + z^{\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{-\mathrm{i}t}\mathrm{i}\tfrac{\partial}{\partial t} \subset (-\tfrac{1}{2}z^{*\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{\mathrm{i}t} + z^{*\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{\mathrm{i}t}\mathrm{i}\tfrac{\partial}{\partial t})^*. \tag{6.13}$$

Analogously, using $z^{*\mathrm{op}}y^{\mathrm{op}} \subset (z^{\mathrm{op}}y^{\mathrm{op}})^*$,

$$-\tfrac{1}{2}z^{*\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{\mathrm{i}t} + z^{*\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{\mathrm{i}t}\mathrm{i}\tfrac{\partial}{\partial t} \subset (\tfrac{1}{2}z^{\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{-\mathrm{i}t} + z^{\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{-\mathrm{i}t}\mathrm{i}\tfrac{\partial}{\partial t})^*. \tag{6.14}$$

As in the previous section, we summarise our results in a proposition.

Proposition 6.2 *Let γ_q be a non-zero real number. Consider the following operators on $L_2(\mathbb{D}_q)\,\bar{\otimes}\,L_2(\mathbb{S}^1)$ with domain $\mathcal{D}$ defined in (6.7) above:*

$$\hat{H}_1 := qz^{\mathrm{op}}\tfrac{\partial}{\partial z} - z^{*\mathrm{op}}\tfrac{\partial}{\partial \bar{z}} + (y^2)^{\mathrm{op}}y^{-2}\mathrm{i}\tfrac{\partial}{\partial t},$$

$$\hat{E}_1 := -q^{-1}\mathrm{e}^{-\mathrm{i}t}\,y^{\mathrm{op}}\tfrac{\partial}{\partial \bar{z}} - \gamma_q z^{\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{-\mathrm{i}t}\mathrm{i}\tfrac{\partial}{\partial t} - \frac{\gamma_q}{2}z^{\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{-\mathrm{i}t},$$

$$\hat{F}_1 := q\,\mathrm{e}^{\mathrm{i}t}\,y^{\mathrm{op}}\tfrac{\partial}{\partial z} - \gamma_q z^{*\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{\mathrm{i}t}\mathrm{i}\tfrac{\partial}{\partial t} + \frac{\gamma_q}{2}z^{*\mathrm{op}}y^{\mathrm{op}}y^{-2}\mathrm{e}^{\mathrm{i}t}.$$

Then $\hat{H}_1 \subset \hat{H}_1^*$, $\hat{F}_1 \subset \hat{E}_1^*$ and $\hat{E}_1 \subset \hat{F}_1^*$.

Proof $\hat{H}_1 \subset \hat{H}_1^*$ follows from (6.11) and (6.12), $\hat{E}_1 \subset \hat{F}_1^*$ follows from (6.8) and (6.13), and $\hat{F}_1 \subset \hat{E}_1^*$ follows from (6.8) and (6.14). □

7 The Dirac Operator

Classically, the left invariant vector fields H, E and F act as first order differential operators on differentiable functions on SU(2). In the noncommutative case, we will use the actions of $\hat{H}$, $\hat{E}$ and $\hat{F}$ to define an algebra of differentiable functions.

For $\hat{x}_i = x_i^{\mathrm{op}} \otimes \eta_i \in \mathcal{O}^+(\mathbb{D}_q)^{\mathrm{op}} \otimes C^\infty(\mathbb{S}^1)$, $i = 0, 1, 2$, consider the action on $f \otimes \varphi \in \mathcal{F}^{(1)}(\mathbb{D}_q) \otimes C^{(1)}(\mathbb{S}^1)$ given by

$$\begin{aligned}(\hat{x}_0\hat{T}_0 + \hat{x}_1\hat{T}_1 + \hat{x}_2\hat{T}_2)(f \otimes \varphi)& \\ := y^{-1} f x_0 \otimes \eta_0 \tfrac{\partial}{\partial t}(\varphi) + T_1(f) x_1 \otimes \eta_1 \varphi + T_2(f) x_2 \otimes \eta_2 \varphi,&\end{aligned} \tag{7.1}$$

where the right-hand side of (7.1) is understood as an unbounded operator on $L_2(\mathbb{D}_q, \mu_\alpha) \bar{\otimes} L_2(\mathbb{S}^1)$ with domain of definition containing the subspace $\mathcal{D}$ introduced in (5.7). Note that the operators $\hat{H}$, $\hat{E}$ and $\hat{F}$ are of the form described in (7.1), and that the operators $\hat{x}_i$ and $\hat{T}_i$ satisfy the assumptions of Lemma 5.1. We define

$$\begin{aligned}\Gamma^{(1)}(\mathrm{SU}_q(2)) := \{\phi \in \mathcal{F}^{(1)}(\mathbb{D}_q) \otimes C^{(1)}(\mathbb{S}^1) :& \\ \hat{H}(\phi),\ \hat{E}(\phi),\ \hat{F}(\phi),\ \hat{H}(\phi^*),\ \hat{E}(\phi^*),\ \hat{F}(\phi^*) \text{ are bounded}\}.&\end{aligned} \tag{7.2}$$

From (5.2), (5.3), (5.5) and Lemma 5.1, it follows that

$$\hat{T}(\varphi\psi) = (\hat{T}\varphi)\psi + \hat{\sigma}^1(\varphi)(\hat{T}\psi), \tag{7.3}$$

for all $\varphi, \psi \in \Gamma^{(1)}(\mathrm{SU}_q(2))$ and $\hat{T} \in \{\hat{H}, \hat{E}, \hat{F}\}$. In particular, $\hat{T}(\varphi\psi)$ is again bounded so that $\Gamma^{(1)}(\mathrm{SU}_q(2))$ is a *-algebra.

Finally note that the classical limit of $\hat{H}$, $\hat{E}$ and $\hat{F}$ for $q \to 1$ is formally H, E and F, respectively. This will also be the case if we rescale $\hat{E}$ and $\hat{F}$ by a real number $c = c(q)$ such that $\lim_{q\to 1} c(q) = 1$. Such a rescaling might be useful in later computations of the spectrum of the Dirac operator.

Theorem 7.1 *Let* $\alpha = 2$. *Set* $\mathcal{H} := (L_2(\mathbb{D}_q, \mu_\alpha)\bar{\otimes}L_2(\mathbb{S}^1)) \oplus (L_2(\mathbb{D}_q, \mu_\alpha)\bar{\otimes}L_2(\mathbb{S}^1))$ *and define*

$$\pi : \Gamma^{(1)}(\mathrm{SU}_q(2)) \longrightarrow B(\mathcal{H}), \quad \pi(\phi) := \phi \oplus \phi$$

as left multiplication operators. Then, for any $c \in \mathbb{R}$, *the operator*

$$D := \begin{pmatrix} \hat{H} - 2 & c\hat{E} \\ c\hat{F} & -\hat{H} - 2 \end{pmatrix}$$

is symmetric on $\mathcal{D} \oplus \mathcal{D}$ with $\mathcal{D}$ from (6.1). Furthermore,

$$[D, \pi(\phi)]_{\hat{\sigma}^1} := D\pi(\phi) - \pi(\hat{\sigma}^1(\phi))\,D$$

is bounded for all $\phi \in \Gamma^{(1)}(\mathrm{SU}_q(2))$.

Proof $D \subset D^*$ follows from Proposition 6.1. For all $\phi \in \Gamma^{(1)}(\mathrm{SU}_q(2))$ and $\psi_1 \oplus \psi_2$ in the domain of D,

$$\begin{aligned}[D, \pi(\phi)]_{\hat{\sigma}^1}(\psi_1 \oplus \psi_2) &= \hat{H}(\phi\psi_1) - \hat{\sigma}^1(\phi)\hat{H}(\psi_1) + c(\hat{E}(\phi\psi_2) - \hat{\sigma}^1(\phi)\hat{E}(\psi_2)) \\ &\quad \oplus\ c(\hat{F}(\phi\psi_1) - \hat{\sigma}^1(\phi)\hat{F}(\psi_1)) - (\hat{H}(\phi\psi_2) - \hat{\sigma}^1(\phi)\hat{H}(\psi_2)) \\ &= (\hat{H}\phi)\psi_1 + c(\hat{E}\phi)\psi_2\ \oplus\ c(\hat{F}\phi)\psi_1 - (\hat{H}\phi)\psi_2\end{aligned}$$

by (7.3), so $[D, \pi(\phi)]_{\sigma^1}$ is bounded by the definition of $\Gamma^{(1)}(\mathrm{SU}_q(2))$. □

Theorem 7.2 *Let $\mathcal{H}$ and π be defined as in Theorem 7.1 but with the measure on $\mathbb{D}_q$ given by setting $\alpha = 1$ in (3.10). Let $\hat{H}_1$, $\hat{F}_1$ and $\hat{E}_1$ be defined as in Proposition 6.2 and $\Gamma^{(1)}(\mathrm{SU}_q(2))$ as in (7.2) with $\hat{H}$, $\hat{F}$ and $\hat{E}$ replaced by $\hat{H}_1$, $\hat{F}_1$ and $\hat{E}_1$, respectively. Then, for any $c \in \mathbb{R}$, the operator*

$$D_1 := \begin{pmatrix} \hat{H}_1 - 2 & c\hat{E}_1 \\ c\hat{F}_1 & -\hat{H}_1 - 2 \end{pmatrix}$$

is symmetric on $\mathcal{D} \oplus \mathcal{D}$ with $\mathcal{D}$ from (6.7). Furthermore,

$$[D_1, \pi(\phi)]_{\hat{\sigma}^2} := D_1\pi(\phi) - \pi(\hat{\sigma}^2(\phi))\,D_1$$

is bounded for all $\phi \in \Gamma^{(1)}(\mathrm{SU}_q(2))$.

Proof Using the results for $\hat{S}_0$, $\hat{S}_1$ and $\hat{S}_2$ from Sect. 5.2 and Proposition 6.2, the proof is essentially the same as the proof of the previous theorem. □

To view the Dirac operator of Theorem 7.2 as a deformation of the classical Dirac operator, one may choose a continuously varying positive real number γ_q such that $\lim_{q\to 1}\gamma_q = \frac{1}{2}$. For instance, if $\gamma_q := \frac{q}{1+q}$, then the Dirac operator of Theorem 7.2 resembles the one of Theorem 7.1, the main difference being the additional functions (0-order differential operators) in the definitions of $\hat{E}_1$ and $\hat{F}_1$. In the classical case $q = 1$, the operator D_1 can be obtained from the Dirac operator D in Theorem 7.1 by the "gauge transformation" $D_1 = \sqrt{y}\,D\sqrt{y}^{-1}$.

On the other hand, if one rescales the volume form to $\mathrm{vol}_1 := \frac{1}{y}\mathrm{vol}$ with a non-constant function y without changing the Riemannian metric, then the Dirac operator ceases to be self-adjoint but the above gauge transformed Dirac operator will remedy the problem. To see this, let $f, g, \sqrt{y}^{-1} f, \sqrt{y}^{-1} g \in \mathrm{dom}(D)$. Then

$$\langle \sqrt{y}\, D \sqrt{y}^{-1} f, g\rangle_{L_2(S, \frac{1}{y}\mathrm{vol})} = \int \langle \sqrt{y}\, D \sqrt{y}^{-1} f, g\rangle \frac{1}{y}\,\mathrm{dvol} = \int \langle D \sqrt{y}^{-1} f, \sqrt{y}^{-1} g\rangle \mathrm{dvol}$$

$$= \int \langle f, \sqrt{y} D \sqrt{y}^{-1} g\rangle \frac{1}{y}\mathrm{dvol} = \langle f, \sqrt{y}\, D \sqrt{y}^{-1} g\rangle_{L_2(S, \frac{1}{y}\mathrm{vol})}.$$

For this reason and in view of (3.11), we may regard D_1 as the Dirac operator obtained from D by rescaling the volume form $y\, \mathrm{d}_q y\, \mathrm{d}\phi \;\mapsto\; \mathrm{d}_q y\, \mathrm{d}\phi$.

References

1. P. S. Chakraborty, A. Pal, *Equivariant spectral triples on the quantum SU(2) group.* K-Theory **28** (2003), 107–126.
2. A. Connes, *Noncommutative differential geometry.* IHES Publ. Math. **62** (1985), 257–360.
3. A. Connes, *Noncommutative geometry*. Academic Press, 1994.
4. A. Connes, *Cyclic cohomology, quantum group symmetries and the local index formula for SUq(2).* J. Inst. Math. Jussieu **3** (2004), 17–68.
5. A. Connes, H. Moscovici, *Type III and spectral triples.* Aspects Math. **E38** (2008), 57–71.
6. L. Dabrowski, G. Landi, A. Sitarz, W. van Suijlekom, J. C. Várilly, The Dirac operator on SUq(2). Comm. Math. Phys. **259** (2005), 729–759.
7. L. Dabrowski, G. Landi, A. Sitarz, W. van Suijlekom, J. C. Várilly, *The local index formula for SUq(2).* K-Theory **35** (2005), 375–394
8. T. Friedrich, *Dirac Operators in Riemannian Geometry.* Graduate Studies in Mathematics 25, AMS, 2000.
9. P. M. Hajac, E. Wagner, *The pullbacks of principal coactions.* Doc. Math. **19** (2014), 1025–1060.
10. I. Heckenberger, *Spin geometry on quantum groups via covariant differential calculi.* Adv. Math. **175** (2003), 197–242.
11. S. Klimec, A. Lesniewski, *A two-parameter quantum deformation of the unit disc.* J. Funct. Anal. **155** (1993), 1–23.
12. A. U. Klimyk, K. Schmüdgen, *Quantum Groups and their Representations.* Texts and Monographs in Physics, Springer, 1998.
13. U. Krähmer, A. Rennie, R. Senior, *A residue formula for the fundamental Hochschild 3-cocycle for SUq(2).* J. Lie Theory **22** (2012), 557–585.
14. T. Masuda, Y. Nakagami, J. Watanabe, *Noncommutative Differential Geometry on the Quantum SU(2), I: An Algebraic Viewpoint.* K-Theory **4** (1990), 157–180 .
15. S. Neshveyev, L. Tuset, *The Dirac operator on compact quantum groups.* J. Reine Angew. Math. **641** (2010), 1–20.
16. S. Neshveyev, L. Tuset, *K-homology class of the Dirac operator on a compact quantum group.* Doc. Math. **16** (2011), 767–780.
17. K. Schmüdgen, *Commutator representations of differential calculi on the quantum group SUq(2).* J. Geom. Phys. **31** (1999), 241–264.
18. N. L. Vasilevski, *Commutative Algebras of Toeplitz Operators on the Bergman Space.* Operator Theory: Advances and Applications 185, Birkhäuser, 2008.
19. C. Voigt, *Quantum SU(2) and the Baum-Connes conjecture.* Banach Center Publ. **98** (2012), 417–432.
20. E. Wagner, *Representations of quantum SU(2) operators on a local chart.* J. Phys. Conf. Ser. **670** (2016), 012053.

Toeplitz Algebras on the Harmonic Fock Space

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Dedicated to Nikolai Vasilevski on the occasion of his 70th birthday

Abstract We study Toeplitz operators acting on the harmonic Fock space and consider two classes of symbols: radial and horizontal. Toeplitz operators with radial symbols behave quite similar in both settings, namely, the Fock and the harmonic Fock spaces. In fact, these operators generate a commutative C^*-algebra which is isomorphic to the algebra of uniformly continuous sequences with respect to the square root metric. On the contrary, Toeplitz operators with horizontal symbols on the harmonic Fock space do not commute in general. Nevertheless, up to compact perturbation, they have a similar behavior to the corresponding Toeplitz operators acting on the Fock space. In fact, we prove that the Calkin algebra of the C^*-algebra generated by Toeplitz operators with horizontal symbols is isomorphic to the algebra consisting of bounded uniformly continuous functions with respect to the standard metric on $\mathbb{R}$, which, at the same time, is isomorphic to the C^*-algebra generated by Toeplitz operators with horizontal symbols acting on the Fock space.

Keywords Toeplitz operators · Harmonic function · Fock space · Harmonic Fock space · C^*-algebras of Toeplitz operators

Mathematics Subject Classification (2010) 47B35, 30H20, 31A05, 47L80

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1 Introduction

The Fock (Bargmann-Segal) space was introduced independently by Bargmann and Segal in [2] and [21], respectively. Berger and Coburn in [5] developed the basic theory of Toeplitz operators and applied them to Quantum Mechanics. In [6], the authors described the biggest $*$-subalgebra of $L^\infty(\mathbb{C})$ that generates Toeplitz operators with compact semicommutators.

Concerning the study of C^*-algebras generated by Toeplitz operators acting on the Fock space, in [11] and [12], Esmeral, Maximenko and Vasilevski, described the only two commutative C^*-algebras generated by Toeplitz operators. One of these algebras is generated by Toeplitz operators with radial symbols. These operators turned out to be radial (invariant under rotation operators). The other one is generated by Toeplitz operators with horizontal symbols. These operators are invariant under Weyl operators.

Toeplitz operators with radial symbols behave quite well. In fact, they are diagonal operators with respect to the corresponding monomial basis of the Bergman, the harmonic Bergman, the pluriharmonic Bergman, the Fock and the pluriharmonic Fock spaces (see [1, 13, 15, 18, 19], for example).

Very interesting results related to Toeplitz operators acting on the Fock space are found in the following works [2–6, 8, 10, 22]. As far as we know, the study of Toeplitz operators acting on harmonic Fock spaces began with the work [1], where the author studied algebraic properties of Toeplitz operators with radial and quasi homogeneous symbols on the pluriharmonic Fock space of $\mathbb{C}^n$. If $n = 1$ quasi homogeneous symbols are radial functions.

Inspired by the results obtained for Toeplitz operators acting on the Fock space and considering the differences between Toeplitz operators acting on the Bergman space and Toeplitz operators acting on the harmonic Bergman space [18, 19], in this work we consider Toeplitz operators with radial symbols and Toeplitz operators with horizontal symbols acting on the harmonic Fock space.

The harmonic Fock space is invariant under rotation operators, and we can follow the proofs given in [11] to describe the behavior of Toeplitz operators with radial symbols acting on this space, just with slightly modifications. In both cases, Fock and harmonic Fock settings, the C^*-algebra generated by Toeplitz operators with radial symbols is commutative. Even more, both algebras are isomorphic to the algebra $\mathrm{RO}(\mathbb{Z}_+)$ consisting of all uniformly continuous sequences with respect to the square root metric. Nevertheless, the corresponding Toeplitz operators are not unitarily equivalent. One important result for Toeplitz operators with radial symbols acting on the Fock space is that a Toeplitz operator is radial if and only if its symbol is radial [11]. This result also holds for Toeplitz operators with radial symbols acting on the harmonic Fock space.

On the other hand, a Toeplitz operators acting on the Fock is horizontal if and only if its definig symbol is horizontal. Since the harmonic Fock space is not invariant under Weyl operators, horizontal operators cannot be defined here. However, up to compact perturbation, Toeplitz operators with horizontal symbols have a similar behavior to the corresponding Toeplitz operators acting on the Fock space. In fact, we prove that the Calkin algebra of the C^*-algebra generated by Toeplitz operators with horizontal symbols is isomorphic to the algebra consisting of all bounded uniformly continuous functions with respect to the standard metric on $\mathbb{R}$, which, at the same time, is isomorphic to the C^*-algebra generated by Toeplitz operators with horizontal symbols acting on the Fock space.

In summary the main results of this work are the following:

1. A Toeplitz operator acting on the harmonic Fock space is radial if and only if its symbol is radial. The C^*-algebra generated by Toeplitz operators with radial symbols is isomorphic to the algebra consisting of all uniformly continuous sequences with respect to the square root metric.
2. In general, two Toeplitz operators with horizontal symbols do not commute.
3. The commutator of two Toeplitz operators with horizontal symbols is compact. However, its semicommutator is not compact in general.
4. The Calkin algebra of the C^*-algebra generated by Toeplitz operators with horizontal symbols is isomorphic to the algebra consisting of all bounded uniformly continuous functions with respect to the standard metric on $\mathbb{R}$.

2 The Harmonic Fock Space

Let $L^2(\mathbb{C}, d\lambda)$ be the Hilbert space of all the square integrable functions with respect to the Gaussian measure

$$d\lambda(z) = \frac{1}{\pi} e^{-|z|^2} dA(z), \qquad z \in \mathbb{C},$$

where dA denotes the usual Lebesgue measure in the complex plane $\mathbb{C}$. The Fock space $\mathcal{F}^2$ is the closed subspace of $L^2(\mathbb{C}, d\lambda)$ consisting of all analytic functions, that is, $f : \mathbb{C} \to \mathbb{C}$ is in the Fock space $\mathcal{F}^2$ if $f \in L^2(\mathbb{C}, d\lambda)$ and $\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0$.

A very important result related to the Fock space is its relation with the space $L^2(\mathbb{R}) = L^2(\mathbb{R}, dx)$, where dx is the usual Lebesgue measure. This relation is given through the Bargmann transform $B : L^2(\mathbb{C}, d\lambda) \to L^2(\mathbb{R})$ which was introduced in [2]. The formula for this transform is the following

$$(Bf)(x) = \pi^{-1/4} \int_{\mathbb{C}} f(z) e^{\sqrt{2}x\overline{z} - \frac{x^2}{2} - \frac{\overline{z}^2}{2}} d\lambda(z). \tag{2.1}$$

The Bargmann transform, restricted to the Fock space, is a unitary operator from $\mathcal{F}^2$ onto $L^2(\mathbb{R})$.

It is well known (see for example [5]) that the Fock space $\mathcal{F}^2$ is a Hilbert space. The set consisting of all the functions

$$e_n(z) = \sqrt{\frac{1}{n!}}z^n, \qquad n \in \mathbb{Z}_+, \tag{2.2}$$

where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, forms an orthonormal basis for $\mathcal{F}^2$. The orthogonal projection $P : L^2(\mathbb{C}, d\lambda) \to \mathcal{F}^2$ is given by the integral operator

$$Pf(z) = \int_{\mathbb{C}} e^{z\overline{w}} f(w) d\lambda(w), \qquad f \in L^2(\mathbb{C}, d\lambda). \tag{2.3}$$

It is also true that

$$B^*B = P : L^2(\mathbb{C}, d\lambda) \to \mathcal{F}^2, \text{ and that } BB^* = I : L^2(\mathbb{R}) \to L^2(\mathbb{R}). \tag{2.4}$$

On the other hand, a complex-valued function f is called anti-analytic if $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} = 0$. Notice that $f(z)$ is anti-analytic if and only if $f(\overline{z})$ is an analytic function. Thus, the unitary operator $J : L^2(\mathbb{C}, d\lambda) \to L^2(\mathbb{C}, d\lambda)$ given by

$$Jf(z) = f(\overline{z}), \tag{2.5}$$

maps an analytic function into an anti-analytic function and vice versa. So, the image $J(\mathcal{F}^2) = \overline{\mathcal{F}^2}$ is a closed subspace of $L^2(\mathbb{C}, d\lambda)$ called the anti-Fock space. The orthogonal projection $\overline{P} : L^2(\mathbb{C}, d\lambda) \to \overline{\mathcal{F}^2}$ is given by the integral operator

$$\overline{P}f(z) = \int_{\mathbb{C}} e^{\overline{z}w} f(w) d\lambda(w), \qquad f \in L^2(\mathbb{C}, d\lambda). \tag{2.6}$$

A twice continuously differentiable function $f : \mathbb{C} \to \mathbb{C}$ is harmonic if

$$\Delta f(z) = \frac{\partial^2 f(z)}{\partial x^2} + \frac{\partial^2 f(z)}{\partial y^2} = 0, \qquad z = x + iy \in \mathbb{C}.$$

The harmonic Fock space $\mathcal{H}^2$ is the subspace of $L^2(\mathbb{C}, d\lambda)$ consisting of all complex-valued harmonic functions. Since $\Delta = \frac{1}{4}\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{1}{4}\frac{\partial^2}{\partial \overline{z} \partial z}$, the Fock space $\mathcal{F}^2$ and the anti-Fock space $\overline{\mathcal{F}^2}$ are subspaces of $\mathcal{H}^2$. Then

$$\mathcal{F}^2 + \overline{\mathcal{F}^2} \subset \mathcal{H}^2. \tag{2.7}$$

In fact, the converse inclusion holds (see for example [1]), such as in the case of harmonic spaces in different domains (for example [18, 19]). For completeness, we include the proof of this fact in what follows.

For a function $f = u_1 + iu_2 \in \mathcal{H}^2$, both u_1 and u_2 are real-valued harmonic functions. Define v_1 and v_2 as one of the harmonic conjugate functions of u_1 and u_2, respectively. Then,

$$f = g_1 + \overline{g_2},$$

where $g_1,\ g_2$ are the analytic functions defined by

$$g_1 = \frac{u_1 - v_2 + i(v_1 + u_2)}{2} \quad \text{and} \quad \overline{g_2} = \frac{u_1 + v_2 - i(v_1 - u_2)}{2}.$$

Besides, it is clear that since $f \in \mathcal{H}^2$ then both $u_1,\ u_2$ are in $L^2(\mathbb{C}, d\lambda)$. The upcoming lemma shows that if u_1, u_2 belong to $L^2(\mathbb{C}, d\lambda)$ then their harmonic conjugates v_1, v_2 also belong to $L^2(\mathbb{C}, d\lambda)$. This fact implies that g_1, g_2 are in the Fock space. Thus, using (2.7) we have that

$$\mathcal{H}^2 = \mathcal{F}^2 + \overline{\mathcal{F}^2}. \tag{2.8}$$

In order to prove Lemma 2.1, we adapt the proof of the result given in ([7], p. 53) for bounded disks.

Lemma 2.1 *Let u be a harmonic real-valued function. If $u \in L^2(\mathbb{C}, d\lambda)$ then every conjugate harmonic function v of u is also in $L^2(\mathbb{C}, d\lambda)$.*

Proof Suppose, without loss of generality, that $v(0) = 0$. The function $f = u + iv$ is entire, and therefore, it can be expanded in its power series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

For each $n \in \mathbb{Z}_+$, let a_n and $-b_n$ be the real and imaginary parts of c_n, respectively. Notice that the condition $v(0) = 0$ implies that $b_0 = 0$. In polar coordinates we have

$$f(re^{i\theta}) = a_0 + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta) + i \sum_{n=1}^{\infty} r^n(-b_n \cos n\theta + a_n \sin n\theta).$$

That is, $u(re^{i\theta}) = a_0 + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta)$ and $v(re^{i\theta}) = \sum\limits_{n=1}^{\infty} r^n(-b_n \cos n\theta + a_n \sin n\theta)$. Fixing $r > 0$ and using Parseval's

Theorem we have that for the functions u, v

$$\frac{1}{2\pi}\int_0^{2\pi}|u(re^{i\theta})|^2d\theta = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} r^{2n}(a_n^2+b_n^2),$$

$$\frac{1}{2\pi}\int_0^{2\pi}|v(re^{i\theta})|^2d\theta = \frac{1}{2}\sum_{n=1}^{\infty} r^{2n}(a_n^2+b_n^2),$$

respectively. Thus, for all $r > 0$

$$\int_0^{2\pi}|v(re^{i\theta})|^2d\theta \le \int_0^{2\pi}|u(re^{i\theta})|^2d\theta.$$

Now, since $u \in L^2(\mathbb{C}, d\lambda)$, we have that

$$\frac{1}{\pi}\int_0^{\rho}\int_0^{2\pi} r|v(re^{i\theta})|^2e^{-r^2}d\theta dr \le \frac{1}{\pi}\int_0^{\rho}\int_0^{2\pi} r|u(re^{i\theta})|^2e^{-r^2}d\theta dr \le \|u\|^2,$$

for all $\rho > 0$. Using Tonelli's Theorem and Monotone Convergence Theorem we conclude that

$$\int_{\mathbb{C}}|v(z)|^2d\lambda(z) = \lim_{\rho\to\infty}\frac{1}{\pi}\int_0^{\rho}\int_0^{2\pi} r|v(re^{i\theta})|^2e^{-r^2}d\theta dr \le \|u\|^2.$$

That is, $v \in L^2(\mathbb{C}, d\lambda)$. □

In order to get an orthogonal decomposition of the harmonic Fock space, denote by $\overline{z\mathcal{F}^2}$ the closed subspace of $\overline{\mathcal{F}^2}$ consisting of all functions that vanish at the origin. If f is entire and $f(z)e^{-|z|^2}$ integrable, then from straightforward calculations we conclude that

$$\int_{\mathbb{C}} f(z)d\lambda(z) = f(0).$$

Therefore, the space $\overline{z\mathcal{F}^2}$ is orthogonal to the Fock space $\mathcal{F}^2$. Using (2.8) we lead to the following theorem.

Theorem 2.2 *The harmonic Fock space admits the orthogonal decomposition*

$$\mathcal{H}^2 = \mathcal{F}^2 \oplus \overline{z\mathcal{F}^2}. \tag{2.9}$$

The following results are direct consequences of the last theorem and Eqs. (2.3) and (2.6):

1. The harmonic Fock space $\mathcal{H}^2$ is a closed subspace of $L^2(\mathbb{C}, d\lambda)$.
2. The orthogonal projection $\widetilde{P}$ from $L^2(\mathbb{C}, d\lambda)$ onto $\mathcal{H}^2$ is the integral operator

$$\widetilde{P}f(z) = \int_{\mathbb{C}} \left(e^{z\overline{w}} + e^{\overline{z}w} - 1\right) f(w)d\lambda(w), \qquad f \in L^2(\mathbb{C}, d\lambda).$$

3. The set $\{\eta_n\}_{n\in\mathbb{Z}}$, with

$$\eta_n = \begin{cases} e_n, & n \in \mathbb{Z}_+, \\ \overline{e_{|n|}}, & n \in \mathbb{Z} \setminus \mathbb{Z}_+, \end{cases} \tag{2.10}$$

where e_n is given in (2.2), is an orthonormal basis for the harmonic Fock space $\mathcal{H}^2$.

3 Toeplitz Operators on the Harmonic Fock Space

Denote by $L^\infty(\mathbb{C})$ the algebra of essentially bounded functions on $\mathbb{C}$ with respect to the usual Lebesgue measure. For a function $a \in L^\infty(\mathbb{C})$ define the Toeplitz operators with symbol a, $T_a : \mathcal{F}^2 \to \mathcal{F}^2$ and $\widetilde{T}_a : \mathcal{H}^2 \to \mathcal{H}^2$, as

$$T_a f(z) = P(af)(z) = \int_{\mathbb{C}} e^{z\overline{w}} a(w) f(w)d\lambda(w), \qquad f \in \mathcal{F}^2,$$

$$\widetilde{T}_a h(z) = \widetilde{P}(ah)(z) = \int_{\mathbb{C}} (e^{z\overline{w}} + e^{w\overline{z}} - 1)a(w)h(w)d\lambda(w), \qquad h \in \mathcal{H}^2.$$

It is well known that $T_a = 0$ if and only if $a = 0$ and that every compact supported function generates a compact Toeplitz operator on the Fock space [5, theorems 4 and 5]. The following proposition, which proof is essentially the same of [5, theorems 4 and 5], shows that these facts hold for Toeplitz operators acting on $\mathcal{H}^2$.

Proposition 3.1 *Let $a \in L^\infty(\mathbb{C})$. The following statements hold.*

1 *The Toeplitz operator $\widetilde{T}_a$ is zero if and only if $a(z) = 0$, a.e.*
2 *If a has compact support then the Toeplitz operator $\widetilde{T}_a$ is a compact operator.*

From the decomposition (2.9), a Toeplitz operator acting on the harmonic Fock space $\mathcal{H}^2$ can be represented as a 2×2 matrix-valued operator. Indeed, given a symbol $a \in L^\infty(\mathbb{C})$, the Toeplitz operator $\widetilde{T}_a$ acting on $\mathcal{F}^2 \oplus \overline{z\mathcal{F}^2}$ is the matrix-valued operator

$$\widetilde{T}_a = \begin{pmatrix} Pa|_{\mathcal{F}^2} & Pa|_{\overline{z\mathcal{F}^2}} \\ (\overline{P} - 1 \otimes 1)a|_{\mathcal{F}^2} & (\overline{P} - 1 \otimes 1)a|_{\overline{z\mathcal{F}^2}} \end{pmatrix},$$

where $(1 \otimes 1)f = \langle f, 1 \rangle 1$. Define $\widetilde{J} : L^2(\mathbb{C}, d\lambda) \oplus L^2(\mathbb{C}, d\lambda) \to L^2(\mathbb{C}, d\lambda) \oplus L^2(\mathbb{C}, d\lambda)$ by

$$\widetilde{J} = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix},$$

where J is given in Eq. (2.5). Notice that $\widetilde{J}$ is a unitary operator satisfying $\widetilde{J}^* = \widetilde{J}$. Even more, $\widetilde{J}|_{\mathcal{F}^2 \oplus \overline{z\mathcal{F}^2}} : \mathcal{F}^2 \oplus \overline{z\mathcal{F}^2} \to \mathcal{F}^2 \oplus z\mathcal{F}^2$ is a unitary operator with inverse $\widetilde{J}|_{\mathcal{F}^2 \oplus z\mathcal{F}^2}$. We use $\widetilde{J}$ to denote the operator and its restrictions indistinctly. Using $\widetilde{J}$, the Toeplitz operator $\widetilde{T}_a$ acting on $\mathcal{F}^2 \oplus \overline{z\mathcal{F}^2}$ can be unitarily transformed into an operator acting on $\mathcal{F}^2 \oplus z\mathcal{F}^2$.

Theorem 3.2 *Let $a \in L^\infty(\mathbb{C})$. The Toeplitz operator $\widetilde{T}_a$ is unitarily equivalent to*

$$\widetilde{J}\widetilde{T}_a\widetilde{J} = \begin{pmatrix} T_a & PJ\widehat{a}|_{z\mathcal{F}^2} \\ (PJ - 1 \otimes 1)a|_{\mathcal{F}^2} & (T_{\widehat{a}} - (1 \otimes 1)\widehat{a})|_{z\mathcal{F}^2} \end{pmatrix}$$

acting on $\mathcal{F}^2 \oplus z\mathcal{F}^2$, where $\widehat{a}(z) = a(\overline{z})$ and $(1 \otimes 1)f = \langle f, 1 \rangle 1$.

Proof Since $\overline{P} = JPJ$ and $J^2 = I$, we have

$$\widetilde{J}\widetilde{T}_a\widetilde{J} = \begin{pmatrix} Pa|_{\mathcal{F}^2} & PaJ|_{z\mathcal{F}^2} \\ J(\overline{P} - 1 \otimes 1)a|_{\mathcal{F}^2} & J(\overline{P} - 1 \otimes 1)aJ|_{z\mathcal{F}^2} \end{pmatrix}$$

$$= \begin{pmatrix} Pa|_{\mathcal{F}^2} & PaJ|_{z\mathcal{F}^2} \\ (PJ - J(1 \otimes 1))a|_{\mathcal{F}^2} & (P - J(1 \otimes 1))aJ|_{z\mathcal{F}^2} \end{pmatrix}. \tag{3.1}$$

Now, $J(1 \otimes 1) = (1 \otimes 1)J = (1 \otimes 1)$ and $aJ = J\widehat{a}$, then Eq. (3.1) becomes

$$\widetilde{J}\widetilde{T}_a\widetilde{J} = \begin{pmatrix} Pa|_{\mathcal{F}^2} & PJ\widehat{a}|_{z\mathcal{F}^2} \\ (PJ - (1 \otimes 1))a|_{\mathcal{F}^2} & (P - (1 \otimes 1))\widehat{a}|_{z\mathcal{F}^2} \end{pmatrix},$$

which is what we wanted to prove. □

Similar results for the harmonic Bergman spaces are found in [17, Theorem 2.1] and [19, Theorem 3.1].

4 Toeplitz Operators with Radial Symbols and Radial Toeplitz Operators on the Harmonic Fock Space

A function $a \in L^\infty(\mathbb{C})$ is called radial if $a(z) = a(|z|)$. Denote by $L^\infty(\mathbb{R}_+)$ the algebra of bounded radial functions defined on the complex plane. Toeplitz operators with radial symbols acting on the Fock space are diagonal with respect to the monomial basis (2.2). Indeed, by direct calculations it can be proved that if $a \in L^\infty(\mathbb{R}_+)$ is a radial function, then $T_a(e_n) = \gamma_a(n)e_n$ where

$$\gamma_a(m) = \frac{1}{m!}\int_{\mathbb{R}_+} a(\sqrt{r})e^{-r}r^m dr, \qquad m \in \mathbb{Z}_+. \tag{4.1}$$

(see for example [16]). As a consequence, the C^*-algebra generated by these operators is commutative and isomorphic to a C^*-subalgebra of the algebra of bounded sequences $\ell_+^\infty = \ell^\infty(\mathbb{Z}_+)$.

Theorem 4.1 ([11]) *The C^*-algebra $\mathcal{T}(L^\infty(\mathbb{R}_+))$, generated by Toeplitz operators with bounded radial symbols acting on the Fock space, is C^*-isomorphic to* $\mathrm{RO}(\mathbb{Z}_+)$*, where* $\mathrm{RO}(\mathbb{Z}_+) \subset \ell_+^\infty$ *consists of all the sequences $\gamma = \{\gamma(k)\}_{k\in\mathbb{Z}_+} \in \ell_+^\infty$ such that γ is uniformly continuous with respect to the square root metric $\rho(m,n) = |\sqrt{m} - \sqrt{n}|$.*

The following lemma follows by straightforward calculations. This result can be generalized to higher dimensions, for example, the pluriharmonic Fock space (see [1]).

Lemma 4.2 *Let $a \in L^\infty(\mathbb{R}_+)$. The Toeplitz operator $\widetilde{T}_a$ acting on the harmonic Fock space, is diagonal with respect to the monomial basis* (2.10) *and*

$$\widetilde{T}_a\eta_n = \widetilde{\gamma}_a(n)\eta_n, \qquad n \in \mathbb{Z},$$

where $\widetilde{\gamma}_a(n) = \gamma_a(|n|)$, and γ_a is given by (4.1)*.*

Last lemma implies that the set of eigenvalues of the operator $\widetilde{T}_a$ is

$$\{\gamma_a(m) : m \in \mathbb{Z}_+\},$$

and its spectrum is given by

$$\mathrm{sp}\left(\widetilde{T}_a\right) = \overline{\{\gamma_a(m) : m \in \mathbb{Z}_+\}}.$$

Carrying out a proof that follows the reasoning that concludes in Theorem 3.1 of [16], Lemma 4.2 can be extended to Toeplitz operators acting on the harmonic Fock space with a wider class of symbols. Such class of symbols, $L_1^\infty(\mathbb{R}_+, e^{-r^2})$, consists of all the measurable functions (not necessarily bounded) $a : \mathbb{R}_+ \to \mathbb{C}$ that satisfy

$$\int_{\mathbb{R}_+} |a(r)| r^m e^{-r} dr < \infty,$$

for every $m \in \mathbb{R}_+$. A detailed treatment of this is in [20]. The next theorem summarizes these results.

Theorem 4.3 *Given a symbol $a \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$, the Toeplitz operator $\widetilde{T}_a$ is unitarily equivalent to the multiplication operator $\widetilde{\gamma}_a I : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, where*

$$\widetilde{\gamma}_a(|n|) = \frac{1}{|n|!} \int_{\mathbb{R}_+} a(\sqrt{r}) e^{-r} r^{|n|} dr, \quad n \in \mathbb{Z}$$

Returning to Toeplitz operators with bounded radial symbols, from Lemma 4.2, the C^*-algebra $\widetilde{T}(L^\infty(\mathbb{R}_+))$ is C^*-isomorphic to the C^*-algebra generated by all sequences $\{\gamma_a | a \in L^\infty(\mathbb{R}_+)\}$. Thus, the following result is an immediate consequence of Theorem 4.1.

Theorem 4.4 *The C^*-algebra $\widetilde{\mathcal{T}}(L^\infty(\mathbb{R}_+))$ is C^*-isomorphic to* $\mathrm{RO}(\mathbb{Z}_+)$.

Even when the associated sequence γ_a in $\mathrm{RO}(\mathbb{Z}_+)$ is the same for both operators T_a and $\widetilde{T}_a$, these operators are not unitarily equivalent. Indeed, denote by $\mathrm{Eigen}(A, \lambda)$ the eigenspace of the operator A associated to the eigenvalue λ, and by $\dim \mathrm{Eigen}(A, \lambda)$ its dimension. Thus, it is clear that, if $m \neq 0$,

$$\dim \mathrm{Eigen}(\widetilde{T}_a, \gamma_a(m)) = 2 \dim \mathrm{Eigen}(T_a, \gamma_a(m)).$$

As an example, consider the characteristic function of the interval [0,1], denoted by $\chi_{[0,1]}$. The m-th eigenvalue of the operators $T_{\chi[0,1]}$ and $\widetilde{T}_\chi[0, 1]$ is given by

$$\begin{aligned} \gamma_{\chi[0,1]}(m) &= \frac{1}{m!} \int_0^1 r^m e^{-r} dr \\ &= \frac{1}{m!} \left(m! - \frac{1}{e} \sum_{j=0}^{m} \frac{m!}{(m-j)!} \right). \end{aligned}$$

Since $\gamma_{\chi[0,1]}(m) \neq \gamma_{\chi[0,1]}(n)$ for $n \neq m$, when $m \neq 0$, the eigenspace

$$\mathrm{Eigen}(\widetilde{T}_{\chi[0,1]}, \gamma_{\chi[0,1]}(m))$$

associated to $\gamma_{\chi[0,1]}(m)$ is generated by the vectors η_m, η_{-m}. On the other hand, the corresponding eigenspace $\mathrm{Eigen}(T_{\chi[0,1]}, \gamma_{\chi[0,1]}(m))$ for the operator $T_{\chi[0,1]}$, is generated by the single element e_m given in Eq. (2.2). This implies that the operators $T_{\chi[0,1]}$ and $\widetilde{T}_{\chi[0,1]}$ are not unitarily equivalent.

4.1 Radial Toeplitz Operators

A bounded linear operator is called radial if it commutes with rotation operators. Using the techniques developed by Zorboska in [23] for Bergman spaces, Esmeral and Maximenko in [11] proved that a bounded operator acting on $\mathcal{F}^2$ is radial if and only if it is diagonal with respect to the monomial basis given in Eq. (2.2). Even more, they proved that a bounded Toeplitz operator acting on $\mathcal{F}^2$ is radial if and only if its generating symbol is a radial function. In this section, we prove that this result is also valid for radial Toeplitz operators on $\mathcal{H}^2$.

Recall that for $\theta \in \mathbb{R}$, the rotation operator $U_\theta : L^2(\mathbb{C}, d\lambda) \to L^2(\mathbb{C}, d\lambda)$ is defined by

$$(U_\theta f)(z) = f(ze^{-i\theta}), \qquad z \in \mathbb{C}.$$

Rotation operators $U_\theta, \theta \in \mathbb{R}$, are unitary and satisfy $U_\theta = U_{\theta+2\pi n}$, $n \in \mathbb{Z}$. Besides, for every element on the monomial basis of $\mathcal{H}^2$

$$U_\theta \eta_n(z) = \eta_n(ze^{-i\theta}) = e^{-in\theta}\eta_n(z), \qquad \forall n \in \mathbb{Z}.$$

Therefore, $\mathcal{H}^2$ is invariant under rotation operators and $U_\theta|_{\mathcal{H}^2}$ is unitary. From now on, we write U_θ instead of $U_\theta|_{\mathcal{H}^2}$. A bounded operator $S : \mathcal{H}^2 \to \mathcal{H}^2$ is called *radial* if

$$SU_\theta = U_\theta S, \qquad \text{for all } \theta \in [0, 2\pi).$$

It is easy to see that the set of all radial operators acting on $\mathcal{H}^2$ is a C^*-algebra.

Given a bounded operator $S : \mathcal{H}^2 \to \mathcal{H}^2$, define its *radialization* by

$$\mathrm{Rad}(S) = \frac{1}{2\pi}\int_0^{2\pi} U_{-\theta} S U_\theta d\theta,$$

where the integral is understood in the weak sense, that is, $\mathrm{Rad}(S)f = g$, $f, g \in \mathcal{H}^2$, if

$$\langle g, h\rangle = \int_0^{2\pi} \langle U_{-\theta} S U_\theta f, h\rangle d\theta, \qquad \text{for all } h \in \mathcal{H}^2.$$

Then a bounded operator $S : \mathcal{H}^2 \to \mathcal{H}^2$ is radial if and only if $\mathrm{Rad}(S) = S$. Furthermore, the operator $\mathrm{Rad}(S)$ acts on the monomial basis (2.10) for $\mathcal{H}^2$ in the following way

$$\langle \mathrm{Rad}(S)\eta_n, \eta_m\rangle = \begin{cases} 0, & n \neq m, \\ \langle S\eta_n, \eta_m\rangle, & n = m, \end{cases} \qquad \forall n, m \in \mathbb{Z}.$$

As a consequence, the next lemma follows.

Lemma 4.5 *A bounded operator $S : \mathcal{H}^2 \to \mathcal{H}^2$ is radial if and only if it is diagonal with respect to the monomial basis $\{\eta_n\}_{n\in\mathbb{Z}}$. Thus, the C^*-algebra generated by radial operators is isomorphic to the algebra of bilateral bounded sequences $\ell^\infty(\mathbb{Z})$.*

Radial functions can be characterized similarly. Let a be a bounded function. The *radialization* of a is given by

$$\mathrm{rad}(a)(z) = \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta}z)d\theta.$$

A function $a \in L^\infty(\mathbb{C})$ is radial if and only if $a = \mathrm{rad}(a)$ almost everywhere. The following theorem allows us to use radial Toeplitz operators and Toeplitz operators with radial symbols indistinctly.

Theorem 4.5 *Let $a \in L^\infty(\mathbb{C})$. The Toeplitz operator $\widetilde{T}_a$ is radial if and only if a is a radial function.*

Detailed proofs of Lemma 4.5 and Theorem 4.5 follow from the proofs of [11, theorems 2.4, 2.5 and 2.6] with slightly modifications.

5 Toeplitz Operators with Horizontal Symbols and Horizontal Operators

This section is devoted to study Toeplitz operators on the harmonic Fock space $\mathcal{H}^2$ with horizontal symbols. A function $a \in L^\infty(\mathbb{C})$ is said to be horizontal if $a(z) = a(\mathrm{Re}z)$. We denote the algebra of horizontal functions by $L^\infty(\mathbb{R})$. In [12], Esmeral and Vasilevsli proved that the Toeplitz operator T_a with horizontal symbol $a \in L^\infty(\mathbb{R})$ is unitarily equivalent to the multiplication operator $BT_aB^* = \gamma_a I$ acting on $L^2(\mathbb{R})$, where

$$\gamma_a(x) = \pi^{-1/2}\int_{\mathbb{R}} a\left(\frac{y}{\sqrt{2}}\right)e^{-(x-y)^2}dy, \tag{5.1}$$

and B is the Bargmann transform defined in Eq. (2.1).

Besides, after identifying $BT_aB^* = \gamma_a I$ with the function $\gamma_a \in L^\infty(\mathbb{R})$, they obtained the description of the C^*-algebra $\mathcal{T}(L^\infty(\mathbb{R}))$ generated by Toeplitz operators with horizontal symbols acting on the Fock space.

Theorem 5.1 ([12, Corollary 5.6]) *The C^*-algebra $\mathcal{T}(L^\infty(\mathbb{R}))$ is C^*-isomorphic to the C^*-algebra $C_{b,u}(\mathbb{R})$ of bounded uniformly continuous functions with respect to the standard metric on $\mathbb{R}$.*

Similar to the radial case, Toeplitz operators with horizontal symbols acting on the Fock space are intimately related to horizontal operators (see [12]). We recall that for $\eta \in \mathbb{C}$, the Weyl operator $W_\eta : L^2(\mathbb{C}, d\lambda) \to L^2(\mathbb{C}, d\lambda)$ is defined as

$$W_\eta f(z) = e^{z\overline{\eta} - \frac{|\eta|^2}{2}} f(z-\eta), \qquad f \in L^2(\mathbb{C}, d\lambda).$$

The Weyl operator W_η is a unitary weighted translation operator on $L^2(\mathbb{C}, d\lambda)$.

The Fock space $\mathcal{F}^2$ is an invariant subspace of each Weyl operator and, even more, the restriction $W_\eta|_{\mathcal{F}^2} : \mathcal{F}^2 \to \mathcal{F}^2$ is unitary. A bounded operator $S : \mathcal{F}^2 \to \mathcal{F}^2$ is said to be horizontal if it commutes with Weyl operators W_{it}, for all $t \in \mathbb{R}$. It turns out that a Toeplitz operator is horizontal if and only if its generating symbol is horizontal [12, Proposition 3.10].

Contrary to the radial case, the study of Toeplitz operators with horizontal symbols on $\mathcal{H}^2$ differs a lot from the analytic case. To begin with, Toeplitz operators with horizontal symbols on the harmonic Fock space do not commute in general. For example, consider the following two symbols $a(z) = e^{i\text{Re}z}$, $b(z) = e^{-i\text{Re}z}$ and the constant function $f(z) \equiv 1$. Straightforward calculations, using the formula

$$\int_{\mathbb{R}} e^{-x^2+bx} dx = \sqrt{\pi} e \frac{b^2}{4}.$$

imply that

$$\frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{icx+dy} \left(e^{z(x-iy)} + e^{\overline{z}(x+iy)} - 1 \right) e^{-(x^2+y^2)} dx dy$$
$$= e^{\frac{-c^2+d^2}{4}} \left(e^{\frac{i(c-d)z}{2}} + e^{\frac{i(c-d)\overline{z}}{2}} - 1 \right).$$

Then,

$$\begin{aligned}
\widetilde{T}_a f z &= \frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix} \left(e^{z(x-iy)} + e^{\overline{z}(x-iy)} - 1 \right) e^{-(x^2+y^2)} dx dy \\
&= e^{-frac14} \left(e^{\frac{iz}{2}} + e^{\frac{i\overline{z}}{2}} - 1 \right) \\
\widetilde{T}_b f z &= \frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-ix} \left(e^{z(x-iy)} + e^{\overline{z}(x-iy)} - 1 \right) e^{-(x^2+y^2)} dx dy \\
&= e^{-frac14} \left(e^{\frac{iz}{2}} + e^{\frac{i\overline{z}}{2}} - 1 \right)
\end{aligned}$$

$$\begin{aligned}
\widetilde{T}_b \widetilde{T}_a f(z) = & \frac{e^{-\frac{1}{4}}}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\frac{i}{2}x - \frac{1}{2}y} \left(e^{z(x-iy)} + e^{\overline{z}(x-iy)} - 1 \right) e^{-(x^2+y^2)} dx dy \\
& + \frac{e^{-\frac{1}{4}}}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\frac{i}{2}x + \frac{1}{2}y} \left(e^{z(x-iy)} + e^{\overline{z}(x-iy)} - 1 \right) e^{-(x^2+y^2)} dx dy
\end{aligned}$$

$$-\frac{e^{-\frac{1}{4}}}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-ix}\left(e^{z(x-iy)}+e^{\overline{z}(x-iy)}-1\right)e^{-(x^2+y^2)}dxdy$$
$$= e^{-\frac{1}{2}}\left(\left(e^{\frac{1}{4}}-1\right)\left(e^{\frac{iz}{2}}+e^{\frac{i\overline{z}}{2}}\right)+1\right). \tag{5.2}$$

$$\widetilde{T}_a\widetilde{T}_b f(z) = \frac{e^{-\frac{1}{4}}}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{\frac{ix}{2}+\frac{y}{2}}\left(e^{z(x-iy)}+e^{\overline{z}(x-iy)}-1\right)e^{-(x^2+y^2)}dxdy$$
$$+\frac{e^{-\frac{1}{4}}}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{\frac{ix}{2}-\frac{y}{2}}\left(e^{z(x-iy)}+e^{\overline{z}(x-iy)}-1\right)e^{-(x^2+y^2)}dxdy$$
$$-\frac{e^{-\frac{1}{4}}}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{ix}\left(e^{z(x-iy)}+e^{\overline{z}(x-iy)}-1\right)e^{-(x^2+y^2)}dxdy$$
$$= e^{-\frac{1}{2}}\left(\left(e^{-\frac{1}{4}}-1\right)\left(e^{-\frac{iz}{2}}+e^{\frac{i\overline{z}}{2}}\right)+1\right). \tag{5.3}$$

Evaluating 5.2 and 5.3 at the point $z=\pi$ we obtain,

$$\text{Im}\widetilde{T}_b\widetilde{T}_a f(\pi) = 2e^{-\frac{1}{2}}\left(e^{\frac{1}{4}}-1\right) = -\text{Im}\widetilde{T}_a\widetilde{T}_b f(\pi)$$

Notice that the Weyl operator W_η does not preserve the harmonic Fock space. Indeed, for $f\in\mathcal{H}^2$

$$\Delta W_\eta f(z) = 4\frac{\partial^2}{\partial z\partial\overline{z}}e^{z\overline{\eta}-\frac{|\eta|^2}{2}}f(z-\eta) = 4\overline{\eta}e^{z\overline{\eta}-\frac{|\eta|^2}{2}}\frac{\partial f(z-\eta)}{\partial\overline{z}},$$

which is not always equal to zero. For example, consider the function in $\mathcal{H}^2$, $f(z)=\overline{z}$, then

$$\Delta W_\eta f(z) = 4\overline{\eta}e^{z\overline{\eta}-\frac{|\eta|^2}{2}},$$

and $\Delta W_\eta f(z)$ is not harmonic. In conclusion, horizontal operators are not well defined on the harmonic Fock space $\mathcal{H}^2$ (see also the point 5.6 of the concluding remarks in [9]).

We might try to use the unitary representation of the harmonic Fock space $\widetilde{J}(\mathcal{H}^2)=\mathcal{F}^2\oplus z\mathcal{F}^2$ to define a Weyl type operator . This suggests the next matrix-valued operator

$$\begin{pmatrix} W_\eta & 0\\ 0 & W_\eta\end{pmatrix}: \mathcal{F}^2\oplus z\mathcal{F}^2\to\mathcal{F}^2\oplus z\mathcal{F}^2.$$

However, it is not well defined since W_η does not preserve $z\mathcal{F}^2$. In order to have a well defined operator, this operator must have the following form

$$\begin{pmatrix} W_\eta & (1 \otimes 1)W_\eta \\ 0 & (I - (1 \otimes 1))W_\eta \end{pmatrix}.$$

But this operator is not unitary.

Leaving the horizontal operators aside, we focus on the study of Toeplitz operators with horizontal symbols acting on the harmonic Fock space. To do this, we use the decomposition of the Bargmann transform B given in [12, Theorem 2.1]. Let U_1 be the unitary transformation from $L^2(\mathbb{C}, d\lambda)$ to $L^2(\mathbb{R}^2)$ given by

$$U_1\varphi(z) = \pi^{-1/2} e^{-\frac{x^2+y^2}{2}} \varphi(x, y), \qquad z = x + iy,$$

and $U_2 = I \otimes F : L^2(\mathbb{R}^2) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \to L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$, where $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the Fourier transform

$$F\varphi(y) = (2\pi)^{-1/2} \int_{\mathbb{R}} \varphi(\eta) e^{-\eta y} d\eta.$$

Consider the operator $U_3 : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ given by the formula

$$U_3\varphi(x, y) = \varphi\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right).$$

At last, define the function

$$\ell_0(y) = e^{-\frac{y^2}{2}},$$

Denote by $B_0 : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ the embedding

$$B_0\varphi(x, y) = \varphi(x)\ell_0(y),$$

with adjoint $B_0^* : L^2(\mathbb{R}^2) \to L^2(\mathbb{R})$ given by

$$B_0^*\varphi(x) = \pi^{-1/4} \int_{\mathbb{R}} \varphi(x, y)\ell_0(y) dy.$$

The Bargmann transform is written as

$$B = B_0^* U_3 U_2 U_1.$$

The next diagram summarizes this process.

$$\begin{array}{ccc} L^2(\mathbb{C}, d\lambda) & \xrightarrow{\quad U_3U_2U_1 \quad} & L^2(\mathbb{R}^2) \\ P \downarrow & \searrow^{B} & \downarrow B_0^* \\ \mathcal{F}^2 & \underset{B^*}{\overset{B|_{\mathcal{F}^2}}{\rightleftarrows}} & L^2(\mathbb{R}) \end{array}$$

Following [12], consider the operator $\widetilde{B} : L^2(\mathbb{C}, d\lambda) \oplus L^2(\mathbb{C}, d\lambda) \to L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ defined by

$$\widetilde{B} = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}.$$

Let $L_0 \subset L^2(\mathbb{R})$ be the vector space spanned by ℓ_0. Direct calculations show that ℓ_0 is the image of the constant function $\eta_0(z) = 1$ under the Bargmann transform, that is, $B\eta_0 = \ell_0$. Therefore, the Bargmann transform restricted to $z\mathcal{F}^2$ is a unitary operator from $z\mathcal{F}^2$ onto $L_0^\perp$. In conclusion, the restriction

$$\widetilde{B}\widetilde{J} : \mathcal{F}^2 \oplus z\overline{\mathcal{F}^2} \to L^2(\mathbb{R}) \oplus L_0^\perp \tag{5.4}$$

is a unitary operator from the harmonic Fock space $\mathcal{F}^2 \oplus z\overline{\mathcal{F}^2}$ onto $L^2(\mathbb{R}) \oplus L_0^\perp$. Using the operator $\widetilde{B}\widetilde{J}$ defined in Eq. (5.4) and Theorem 3.2, the next result is obtained.

Theorem 5.2 *Let $a \in L^\infty(\mathbb{R})$ be a horizontal symbol. The Toeplitz operator $\widetilde{T}_a$ acting on the harmonic Fock space $\mathcal{F}^2 \oplus z\overline{\mathcal{F}^2}$ is unitarily equivalent to the operator $\widetilde{B}\widetilde{J}\widetilde{T}_a\widetilde{J}\widetilde{B}^* : L^2(\mathbb{R}) \otimes L_0^\perp \to L^2(\mathbb{R}) \oplus L_0^\perp$ given by the matrix-valued operator*

$$\begin{pmatrix} \gamma_a I & A_a \\ A_a - (\ell_0 \otimes \ell_0)\gamma_a & (I - \ell_0 \otimes \ell_0)\gamma_a I \end{pmatrix},$$

where

$$A_a\varphi(x) = \pi^{-1/2} \int_{\mathbb{R}} a\left(\frac{x+y}{\sqrt{2}}\right) e^{-\frac{x^2+y^2}{2}} \varphi(y)dy,$$

and γ_a is given in Eq. (5.1).

Proof Since a is horizontal then $\widehat{a} = a$, $BT_aB^* = \gamma_a I$ and $BT_{\widehat{a}}B^* = \gamma_a I$, from Theorem 3.2 and Eq. (5.1). Therefore, it suffices to prove that $BPJaB^* = A_a$ and $B(1 \otimes 1)aB^* = (\ell_0 \otimes \ell_0)\gamma_a I$.

First, by direct calculations using Eq. (2.4) we have

$$BPJaB^* = B(B^*B)JaB^* = B_0^*U_3JaU_3^*B_0.$$

Since

$$U_3 J a U_3^* \varphi(x, y) = a\left(\frac{x+y}{\sqrt{2}}\right)\varphi(y, x),$$

then

$$B_0^* U_3 J a U_3^* B_0 \varphi(x) = \pi^{-1/2} \int_{\mathbb{R}} a\left(\frac{x+y}{\sqrt{2}}\right)\varphi(y) e^{-\frac{x^2+y^2}{2}} dy.$$

In conclusion

$$B P J a B^* = \pi^{-1/2} \int_{\mathbb{R}} a\left(\frac{x+y}{\sqrt{2}}\right)\varphi(y) e^{-\frac{x^2+y^2}{2}} dy.$$

On the other hand, we notice that $1 \otimes 1 = P\overline{P} = \overline{P}P$. Thus

$$1 \otimes 1 = PJPJ = JPJP.$$

Therefore,

$$B(1 \otimes 1) a J B^* = BPJPJaB^* = BJPJB^* BPaB^* = BJPJB^* \gamma_a I.$$

Since $B^*\varphi \in \mathcal{F}^2$ for every $\varphi \in L^2(\mathbb{R})$, it follows that $BJPJB^* = BJPJPB^* = B(1 \otimes 1)B^* = \ell_0 \otimes \ell_0$, and then

$$B(1 \otimes 1) a B^* = (\ell_0 \otimes \ell_0)\gamma_a I.$$

Which is what we wanted to prove. □

Theorem 5.1 obviously implies that the commutator of two Toeplitz operators with horizontal symbols acting on the Fock space is compact. The same result holds for Toeplitz operators with this kind of symbols acting on the harmonic Fock space. Indeed, $\ell_0 \otimes \ell_0$ is clearly compact. In addition, the operator A_a is compact since it is an integral operator which kernel is a square integrable function. Therefore, every Toeplitz operator $\widetilde{T}_a$ with horizontal symbol is unitary equivalent to

$$\widetilde{B}\widetilde{J}\widetilde{T}_a\widetilde{J}\widetilde{B}^* = \begin{pmatrix} \gamma_a I & 0 \\ 0 & (I - \ell_0 \otimes \ell_0)\gamma_a \end{pmatrix} + K,$$

where K is a compact operator. Observe that $(I - \ell_0 \otimes \ell_0)$ is the orthogonal projection from $L^2(\mathbb{R})$ onto $L_0^{\perp}$. Thus, up to compact perturbation, the last formula cannot be reduced. Using that the restriction of a compact operator is also a compact operator, it is easy to conclude that the commutator is compact.

Proposition 5.3 *Given two horizontal symbols $a, b \in L^\infty(\mathbb{R})$, the commutator $[\widetilde{T}_a, \widetilde{T}_b] = \widetilde{T}_a\widetilde{T}_b - \widetilde{T}_b\widetilde{T}_a$ is compact. However, the semicommutator $[\widetilde{T}_a, \widetilde{T}_b) = T_aT_b - T_{ab}$ is not compact in general.*

Proof It suffices to give an example of two Toeplitz operators with no compact semicommutator. Let a and b be the characteristic functions of the intervals $[0, \infty)$ and $(-\infty, 0]$, respectively. Since $ab = 0$, then $\gamma_{ab} = 0$, and $\widetilde{T}_{ab} = 0$. By direct calculations we have that $\gamma_a(x) = \frac{1}{2}(1 + \mathrm{erf}(x))$ and $\gamma_b(x) = \frac{1}{2}(1 - \mathrm{erf}(x))$, where $\mathrm{erf}(x)$ is the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Thus $\gamma_a\gamma_b(x) = \frac{1}{4}(1 - \mathrm{erf}^2(x))$. From Theorem 5.2 and the discussion above

$$\begin{aligned} \widetilde{B}\widetilde{J}\widetilde{T}_a\widetilde{T}_b\widetilde{J}\widetilde{B}^* &= \widetilde{B}\widetilde{J}\widetilde{T}_a\widetilde{J}\widetilde{B}^*\widetilde{B}\widetilde{J}\widetilde{T}_b\widetilde{J}\widetilde{B}^* \\ &= \begin{pmatrix} \frac{1}{4}(1 - \mathrm{erf}^2(x))I + K_{1,1} & K_{1,2} \\ K_{2,1} & R + K_{2,2} \end{pmatrix}, \end{aligned} \tag{5.5}$$

where

$$R = (I - \ell_0 \otimes \ell_0)\frac{1}{2}(1 + \mathrm{erf}(x))I(I - \ell_0 \otimes \ell_0)\frac{1}{2}(1 - \mathrm{erf}(x))I,$$

and

$$\begin{aligned} K_{1,1} &= A_a(A_b - (\ell_0 \otimes \ell_0)\gamma_b I), \\ K_{1,2} &= \gamma_a I A_b + A_a(I - \ell_0 \otimes \ell_0)\gamma_b I, \\ K_{2,1} &= (A_a - (\ell_0 \otimes \ell_0)\gamma_a I)\gamma_b I + (I - \ell_0 \otimes \ell_0)\gamma_a I(A_b - (\ell_0 \otimes \ell_0)\gamma_b I), \\ K_{2,2} &= (A_a - (\ell_0 \otimes \ell_0)\gamma_a I)A_b, \end{aligned}$$

are compact operators.

The operator given in Eq. (5.5) cannot be compact since this would imply that the multiplication operator $\frac{1}{4}(1 - \mathrm{erf}^2(x))I$ is compact. □

Throughout this work, we will denote as $\mathcal{K}$ the ideal of compact operators on the Hilbert space under study. To analyze the C^*-algebra $\widetilde{\mathcal{T}}(L^\infty(\mathbb{R}))$ generated by Toeplitz operators with horizontal symbols acting on $\mathcal{H}^2$, we identify it with $\widetilde{B}\widetilde{J}\widetilde{\mathcal{T}}(L^\infty(\mathbb{R}))\widetilde{J}\widetilde{B}^*$. Recall that $C_{b,u}(\mathbb{R})$ is the C^*-algebra of bounded and uniformly continuous functions with respect to the standard metric on $\mathbb{R}$.

Theorem 5.4 *The sequence*

$$0 \to \mathcal{K} \to \widetilde{B}\widetilde{J}\widetilde{\mathcal{T}}(L^2(\mathbb{R}))\widetilde{J}\widetilde{B}^* + \mathcal{K} \xrightarrow{\Phi} C_{b,u}(\mathbb{R}) \to 0,$$

is a short exact sequence. Then, the quotient algebra $(\widetilde{B}\widetilde{J}\widetilde{\mathcal{T}}(L^2(\mathbb{R}))\widetilde{J}\widetilde{B}^* + \mathcal{K})/\mathcal{K}$ *is* C^**-isomorphic to* $C_{b,u}(\mathbb{R})$*, where* Φ *is the Fredholm symbol acting on the generators in the following form*

$$\begin{pmatrix} \gamma_a I & A_a \\ A_a - (\ell_0 \otimes \ell_0)\gamma_a & (I - \ell_0 \otimes \ell_0)\gamma_a \end{pmatrix} \mapsto \gamma_a$$

Proof Denote by $\mathcal{R}_0$ the $*$-algebra generated by the set $\widetilde{B}\widetilde{J}\widetilde{\mathcal{T}}(L^2(\mathbb{R}))\widetilde{J}\widetilde{B}^* + K$: $a \in L^\infty(\mathbb{R})$, $K \in \mathcal{K}\}$. Every element $\Gamma \in \mathcal{R}_0$ has the form

$$\Gamma = \begin{pmatrix} \gamma I & 0 \\ 0 & (I - \ell_0 \otimes \ell_0)\gamma I \end{pmatrix} + \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix},$$

where $\gamma \in C_{b,u}(\mathbb{R})$ and K_j , $j = 1, 2, 3, 4$, are compact operators.

We define Φ on $\mathcal{R}_0$ as follows

$$\Phi(\Gamma) = \gamma.$$

Note that Φ is a $*$-homomorphism from $\mathcal{R}_0$ to $C_{b,u}(\mathbb{R})$.

To prove that Φ is bounded, we notice that for every $\Gamma \in \mathcal{R}_0$,

$$\begin{aligned} \|\Gamma\|^2 &= \left\| \begin{pmatrix} \gamma I + K_1 & K_2 \\ K_3 & (I - \ell_0 \otimes \ell_0)\gamma I + K_4 \end{pmatrix} \right\|^2 \\ &\geq \left\| \begin{pmatrix} \gamma I + K_1 & K_2 \\ K_3 & \gamma I + K_4 \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix} \right\|^2 \\ &= \|(\gamma I + K_1)f\|^2 + \|K_3 f\|^2 \\ &\geq \|(\gamma I + K_1)f\|^2, \end{aligned}$$

for every $f \in L^2(\mathbb{R})$ with $\|f\|= 1$. Therefore,

$$\|\Gamma\| \geq \sup_{\|f\|=1} \|(\gamma I + K_1)f\| = \|\gamma I + K_1\| \geq \inf_{K \in \mathcal{K}} \|\gamma I + K\| = \|\gamma\|_\infty.$$

The last equality is true since the quotient $(C_{b,u}(\mathbb{R}) + \mathcal{K}))/\mathcal{K}$ is C^*-isomorphic to $C_{b,u}(\mathbb{R})$. Thus, $\|\gamma\|_\infty = \inf_{K \in \mathcal{K}} \|\gamma I + K\|$.

In conclusion, $\|\Phi(\Gamma)\| = \|\gamma\|_\infty \leq \|\Gamma\|$, and we can extend Φ to a bounded $*$-homomorphism from $\widetilde{B}\widetilde{J}\widetilde{\mathcal{T}}(L^2(\mathbb{R}))\widetilde{J}\widetilde{B}^* + \mathcal{K}$ to $C_{b,u}(\mathbb{R})$, also denoted by Φ.

Let $\mathcal{A}_0$ be the $*$-algebra generated by $\{\gamma_a : a \in L^\infty(\mathbb{R})\}$. Thus, for every $\gamma_0 \in C_{b,u}(\mathbb{R})$ there is a sequence $\{\gamma_n\}_{n\in\mathbb{N}} \subset \mathcal{A}_0$ such that $\|\gamma_0 - \gamma_n\|_\infty \to 0$, when $n \to \infty$. Note that for every $\gamma \in C_{b,u}(\mathbb{R})$ we have that

$$\left\|\begin{pmatrix}\gamma I & 0\\ 0 & (I-\ell_0\otimes\ell_0)\gamma I\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix}\right\|^2 \le \|\gamma f\|^2 + (\|\gamma g\| + \|(\ell_0\otimes\ell_0)\gamma g\|)^2$$
$$\le 4\|\gamma\|_\infty^2(\|f\|^2+\|g\|^2),$$

which implies that

$$\left\|\begin{pmatrix}\gamma I & 0\\ 0 & (I-\ell_0\otimes\ell_0)\gamma I\end{pmatrix}\right\| \le 2\|\gamma\|_\infty.$$

Therefore

$$\left\|\begin{pmatrix}(\gamma_0-\gamma_n)I & 0\\ 0 & (I-\ell_0\otimes\ell_0)(\gamma_0-\gamma_n)I\end{pmatrix}\right\| \le 2\|\gamma_0-\gamma_n\|_\infty \to 0,$$

when $n \to \infty$. Since for every $n \in \mathbb{N}$

$$\begin{pmatrix}\gamma_n I & 0\\ 0 & (I-\ell_0\otimes\ell_0)\gamma_n I\end{pmatrix}$$

belongs to $\mathcal{R}_0$ then

$$\begin{pmatrix}\gamma_0 I & 0\\ 0 & (I-\ell_0\otimes\ell_0)\gamma_0 I\end{pmatrix}$$

is in $B\widetilde{\mathcal{T}}(L^2(\mathbb{R}))B^* + \mathcal{K}$. In conclusion, $\gamma \in \Phi(\widetilde{B}\widetilde{J}\widetilde{\mathcal{T}}(L^2(\mathbb{R}))\widetilde{J}\widetilde{B}^* + \mathcal{K})$ and Φ is surjective.

Obviously $\mathcal{K} \subseteq \ker(\Phi)$, then it is enough to prove that $\ker(\Phi) \subseteq \mathcal{K}$. Let $\Gamma_0 \in \ker(\Phi)$. Then there is a sequence

$$\Gamma_n = \begin{pmatrix}\gamma_n I & 0\\ 0 & (I-\ell_0\otimes\ell_0)\gamma_n I\end{pmatrix} + K_n \in \mathcal{R}_0,$$

where K_n are compact operators, such that $\Gamma_n \to \Gamma_0$ when $n \to \infty$. Then, to see that $\Gamma_0 \in \mathcal{K}$ it is sufficient to prove that $\gamma_n \to 0$ when $n \to \infty$. But this is clear since by definition,

$$\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \Phi(\Gamma_n)$$
$$= \Phi(\Gamma_0)$$
$$= 0.$$

This completes the proof. □

Corollary 5.5 *The Fredholm symbol algebra of* $\widetilde{\mathcal{T}}(L^\infty(\mathbb{R}))$*, i.e., the image of* $\widetilde{\mathcal{T}}(L^\infty(\mathbb{R}))$ *in the Calkin algebra*

$$\mathrm{Sym}\widetilde{\mathcal{T}}(L^\infty(\mathbb{R})) = (\widetilde{\mathcal{T}}(L^\infty(\mathbb{R})) + \mathcal{K})/\mathcal{K} = \widetilde{\mathcal{T}}(L^\infty(\mathbb{R}))/(\widetilde{\mathcal{T}}(L^\infty(\mathbb{R})) \cap \mathcal{K}),$$

where $\mathcal{K}$ *is the ideal of compact operators, is* C^**-isomorphic to* $C_{b,u}(\mathbb{R})$*. Under this identification the Fredholm symbol map*

$$\mathrm{sym} : \widetilde{\mathcal{T}}(L^\infty(\mathbb{R})) \to C_{(b,u)}(\mathbb{R})$$

is generated by the following map

$$\mathrm{sym}(\widetilde{T}_a) = \gamma_a.$$

As an immediate consequence, we have that the essential spectrum $\mathrm{ess} - \mathrm{sp}(\widetilde{T}_a)$ of a Toeplitz operator $\widetilde{T}_a$ with horizontal symbol a acting on the harmonic Fock space $\mathcal{H}^2$ is given by the next formula

$$\mathrm{ess} - \mathrm{sp}(\widetilde{T}_a) = \overline{\gamma_a(\mathbb{R})}.$$

where γ_a is given in Eq. (5.1).

References

1. D. Agbor, *Algebraic Properties of Toeplitz Operators on the Pluri-harmonic Fock Space.* Journal of Mathematics Research. **9** (2017), 67–81.
2. V. Bargmann, *On a Hilbert Space of Analytic Functions and an Associated Integral Transform Part I.* Comm. Pure Appl. Math. **14** (1961), 187–214.
3. W. Bauer and Y. J. Lee, *Commuting Toeplitz operators on the Segal-Bargmann space.* J. Funct. Anal. **1260** (2011),460–489.
4. W. Bauer and Y. H. Issa, *Commuting Toeplitz operators with quasi-homogeneous symbols on the Segal-Bargmann space.* J. Math. Anal. Appl. **386** (2012), 213–235.
5. C. A. Berger and L. A. Coburn, *Toeplitz operators and quantum mechanics.* J. Funct. Anal. **68** (1986), 273–299.
6. C. A. Berger and L. A. Coburn, *Toeplitz operators on the Segal-Bargmann space.* Trans. Amer. Math. Soc. **301** (1987), 813–829.

7. P. L. Duren, *Theory of H^p Spaces.* Academic press New York, 1970.
8. M. Engliš, *Toeplitz operators and localization operators.* Trans. Amer. Math. Soc. **361** (2009), 1039–1052.
9. M. Engliš, *Berezin Transform in the Harmonic Fock Space.* Journal of Mathematical Analysis and Applications. **367** (2010), 75–97.
10. M. Engliš, *An excursion into Berezin-Toeplitz quantization and related topics.* In Quantization, PDEs, and geometry. (2016) (pp. 69–115). Birkhäuser, Cham.
11. K. Esmeral and E. Maximenko, *Radial Toeplitz operators on the Fock space and square-root-slowly oscillating sequences.* Complex Anal. Oper. Theory. **10** (2016), 1655–1677.
12. K. Esmeral and N. Vasilevski, *C^*-algebra generated by horizontal Toeplitz operators on the Fock space.* Bol. Soc. Mat. Mex. **22** (2016), 567–582.
13. S. Grudsky and N. Vasilevski, *Bergman-Toeplitz operators: Radial component influence.* Integral Equations Operator Theory. **40** (2001), 16–33.
14. S. Grudsky and N. Vasilevski, *Toeplitz operators on the Fock space: radial component effects.* Integral Equations Operator Theory. **44** (2002), 10–37.
15. S. Grudsky, E. Maximenko and N. Vasilevski, *Radial Toeplitz operators on the unit ball and slowly oscillating sequences.* Commun. Math. Anal. **14** (2013), 77–94.
16. S. Grudsky and N. Vasilevski, *Toeplitz operators on the Fock space: radial component effects.* Integral Equations Operator Theory. **44** (2002), 10–37.
17. K. Guo and D. Zheng, *Toeplitz algebra and Hankel algebra on the harmonic Bergman space.* J. Math. Anal. Appl. **276** (2002), 213–230.
18. M. Loaiza and C. Lozano, *On C^*-algebras of Toeplitz operators on the harmonic Bergman space.* Integral Equations Operator Theory. **76** (2013), 105–130.
19. M. Loaiza and C. Lozano, *On Toeplitz operators on the pluriharmonic Bergman space on the Siegel domain.* Bol. Soc. Mat. Mex. **22** (2016), 583–604.
20. J. Macías-Duráan, *Operadores de Toeplitz con símbolo radial en el espacio de Fock armónico.* MSc, thesis, CINVESTAV-IPN, México, 2016.
21. I. Segal, *Lectures at the summer seminar on Applied Math.* Proceedings of the Summer Seminar, Boulder, Colorado, 1960.
22. K. Zhu, *Analysis in Fock spaces.* Grad. Texts in Math., Springer, 2012.
23. N. Zorboska, *The Berezin transform and radial operators.* Proc. Amer. Math. Soc. **131** (2003), 793–800.

Radial Operators on Polyanalytic Bargmann–Segal–Fock Spaces

Egor A. Maximenko and Ana María Tellería-Romero

Dedicated to Nikolai L. Vasilevski, our guide in this area of mathematics, on the occasion of his 70th birthday

Abstract This paper considers bounded linear radial operators on the polyanalytic Fock spaces $\mathcal{F}_n$ and on the true-polyanalytic Fock spaces $\mathcal{F}_{(n)}$. The orthonormal basis of normalized complex Hermite polynomials plays a crucial role in this study; it can be obtained by the orthogonalization of monomials in z and $\overline{z}$. First, using this basis, we decompose the von Neumann algebra of radial operators, acting in $\mathcal{F}_n$, into the direct sum of some matrix algebras, i.e. radial operators are represented as matrix sequences. Secondly, we prove that the radial operators, acting in $\mathcal{F}_{(n)}$, are diagonal with respect to the basis of the complex Hermite polynomials belonging to $\mathcal{F}_{(n)}$. We also provide direct proofs of the fundamental properties of $\mathcal{F}_n$ and an explicit description of the C*-algebra generated by Toeplitz operators in $\mathcal{F}_{(n)}$, whose generating symbols are radial, bounded, and have finite limits at infinity.

Keywords Radial operator · Polyanalytic function · Fock space · von Neumann algebra

Mathematics Subject Classification (2000) Primary 22D25; Secondary 30H20, 47B35

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1 Introduction and Main Results

The theory of bounded linear operators in spaces of analytic functions has been intensively developed since the 1980s. In particular, the general theory of operators on the Bargmann-Segal-Fock space (for the sake of brevity, we will say just "Fock space") is explained in the book of Zhu [36]. Nevertheless, the complete understanding of the spectral properties is achieved only for some special classes of operators, in particular, for Toeplitz operators with generating symbols invariant under some group actions, see Vasilevski [34], Grudsky et al. [11], Dawson et al. [8]. The simplest class of this type consists of Toeplitz operators with bounded radial generating symbols. Various properties of these operators (boundedness, compactness, and eigenvalues) have been studied by many authors, see [13, 20, 24, 37]. The C*-algebra generated by such operators was explicitly described in [12, 32] for the nonweighted Bergman space, in [6, 15] for the weighted Bergman space, and in [10] for the Fock space. Loaiza and Lozano [21, 22] studied radial Toeplitz operators in harmonic Bergman spaces.

The spaces of polyanalytic functions, related with Landau levels, have been used in mathematical physics since 1950s; let us just mention a couple of recent papers: [3, 14]. A connection of these spaces with wavelet spaces and signal processing is shown by Abreu [1] and Hutník [16, 17]. Various mathematicians contributed to the rigorous mathematical theory of square-integrable polyanalytic functions. Our research is based on results and ideas from [2, 4, 5, 30, 33].

Hutník, Hutníková, Ramírez-Ortega, Sánchez-Nungaray, Loaiza, and other authors [18, 19, 23, 26, 29] studied vertical and angular Toeplitz operators in polyanalytic and true-polyanalytic spaces, Bergman and Fock. In particular, vertical Toeplitz operators in the n-analytic Bergman space over the upper half-plane are represented in [26] as $n \times n$ matrices whose entries are continuous functions on $(0, +\infty)$, with some additional properties at 0 and $+\infty$.

Recently, Rozenblum and Vasilevski [27] investigated Toeplitz operators with distributional symbols and showed that Toeplitz operators in true-polyanalytic Fock spaces are equivalent to some Toeplitz operators with distributional symbols in the analytic Fock space.

In this paper, we analyze radial operators in Fock spaces of polyanalytic or true-polyanalytic functions. We denote by μ the Lebesque measure on the complex plane and by γ the Gaussian measure on the complex plane:

$$\mathrm{d}\gamma(z) = \frac{1}{\pi}\,\mathrm{e}^{-|z|^2}\,\mathrm{d}\mu(z).$$

In what follows, we principally work with the space $L^2(\mathbb{C}, \gamma)$ and its subspaces, and denote its norm by $\|\cdot\|$. A very useful orthonormal basis in $L^2(\mathbb{C}, \gamma)$ is formed by complex Hermite polynomials $b_{j,k}$, $j, k \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$; see Sect. 2.

Given n in $\mathbb{N} := \{1, 2, \ldots\}$, let $\mathcal{F}_n$ be the subspace of $L^2(\mathbb{C}, \gamma)$ consisting of all n-analytic functions belonging to $L^2(\mathbb{C}, \gamma)$. It is known that $\mathcal{F}_n$ is a closed subspace of $L^2(\mathbb{C}, \gamma)$; moreover, it is a RKHS (reproducing kernel Hilbert space). We denote by $\mathcal{F}_{(n)}$ the orthogonal complement of $\mathcal{F}_{n-1}$ in $\mathcal{F}_n$.

For every τ in $\mathbb{T} := \{z \in \mathbb{C}\colon\ |z| = 1\}$, let $R_{n,\tau}$ be the rotation operator acting in $\mathcal{F}_n$ by the rule

$$(R_{n,\tau} f)(z) := f(\tau^{-1} z).$$

The family $(R_{n,\tau})_{\tau\in\mathbb{T}}$ is a unitary representation of the group $\mathbb{T}$ in the space $\mathcal{F}_n$. We denote by $\mathcal{R}_n$ the commutant of $\{R_{n,\tau}\colon\ \tau \in \mathbb{T}\}$ in $\mathcal{B}(\mathcal{F}_n)$, i.e. the von Neumann algebra that consists of all bounded linear operators acting in $\mathcal{F}_n$ that commute with $R_{n,\tau}$ for every τ in $\mathbb{T}$. In other words, the elements of $\mathcal{R}_n$ are the operators intertwining the representation $(R_{n,\tau})_{\tau\in\mathbb{T}}$ of the group $\mathbb{T}$. The elements of $\mathcal{R}_n$ are called *radial* operators in $\mathcal{F}_n$.

In a similar manner, we denote by $R_{(n),\tau}$ the rotation operators acting in $\mathcal{F}_{(n)}$ and by $\mathcal{R}_{(n)}$ the von Neumann algebra of radial operators in $\mathcal{F}_{(n)}$.

The principal tool in the study of $\mathcal{R}_n$ is the following orthogonal decomposition of $\mathcal{F}_n$:

$$\mathcal{F}_n = \bigoplus_{d=-n+1} \mathcal{D}_{d,\min\{n,n+d\}}. \tag{1}$$

Here the "truncated diagonal subspaces" $\mathcal{D}_{d,m}$ are defined as the linear spans of $b_{j,k}$ with $j - k = d$ and $0 \le j, k < m$. Another description of $\mathcal{D}_{d,m}$ is given in Proposition 3.7.

The main results of this paper are explicit decompositions of the von Neumann algebras $\mathcal{R}_n$ and $\mathcal{R}_{(n)}$ into direct sums of factors. The symbol $\cong$ means that the algebras are isometrically isomorphic.

Theorem 1.1 *Let $n \in \mathbb{N}$. Then $\mathcal{R}_n$ consists of all operators belonging to $\mathcal{B}(\mathcal{F}_n)$ that act invariantly on the subspaces $\mathcal{D}_{d,\min\{n,n+d\}}$, for $d \ge -n+1$. Furthermore,*

$$\mathcal{R}_n \cong \bigoplus_{d=-n+1}^{\infty} \mathcal{B}(\mathcal{D}_{d,\min\{n,n+d\}}) \cong \bigoplus_{d=-n+1}^{\infty} \mathcal{M}_{\min\{n,n+d\}}.$$

Theorem 1.2 *Let $n \in \mathbb{N}$. Then $\mathcal{R}_{(n)}$ consists of all operators belonging to $\mathcal{B}(\mathcal{F}_{(n)})$ that are diagonal with respect to the orthonormal basis $(b_{p,n-1})_{p=0}^{\infty}$. Furthermore,*

$$\mathcal{R}_{(n)} \cong \ell^{\infty}(\mathbb{N}_0).$$

In particular, Theorems 1.1 and 1.2 imply that the algebra $\mathcal{R}_n$ is noncommutative for $n \ge 2$, whereas $\mathcal{R}_{(n)}$ is commutative for every n in $\mathbb{N}$.

In Sect. 2 we recall the main properties of the complex Hermite polynomials $b_{p,q}$. In Sect. 3 we give direct proofs of the principal properties of the spaces $\mathcal{F}_n$ and $\mathcal{F}_{(n)}$. Section 4 contains some general remarks about unitary representations in RKHS, given by changes of variables. Section 5 deals with radial operators, describes the von Neumann algebra of radial operators in $L^2(\mathbb{C},\gamma)$, and proves Theorems 1.1 and 1.2. Finally, in Sect. 6 we make some simple observations about Toeplitz operators generated by bounded radial functions and acting in the spaces $\mathcal{F}_n$ and $\mathcal{F}_{(n)}$.

Another natural method to prove (1) and Theorems 1.1, 1.2 is to represent $L^2(\mathbb{C},\gamma)$ as the tensor product $L^2(\mathbb{T},\mathrm{d}\mu_T)\otimes L^2([0,+\infty),\mathrm{e}^{-r^2}\,2r\,\mathrm{d}r)$ and to apply the Fourier transform of the group $\mathbb{T}$. We prefer to work with the canonical basis because this method seems more elementary.

Comparing our Theorem 1.1 with the main results of [23, 26, 29], we would like to point out three differences.

1. We study the von Neumann algebra $\mathcal{R}_n$ of *all radial operators*, instead of C*-algebras generated by Toeplitz operators with radial symbols (such C*-algebras can be objects of study in a future).
2. The dual group of $\mathbb{T}$ is the discrete group $\mathbb{Z}$, therefore matrix sequences appear instead of matrix functions.
3. In [23, 26, 29], all matrices have the same order n, whereas in our Theorem 1.1 the matrices have orders $1, 2, \ldots, n-1, n, n, \ldots$.

2 Complex Hermite Polynomials

Most results of Sects. 2 and 3 are well known to experts [2, 5, 33]. Nevertheless, our proofs are more direct than the ideas found in the bibliography.

Given a function $f\colon \mathbb{C}\to\mathbb{C}$, continuously differentiable in the $\mathbb{R}^2$-sense, we define $A^\dagger f$ and $\overline{A}^\dagger f$ by

$$A^\dagger f = \left(\overline{z} - \frac{\partial}{\partial z}\right) f = -\,\mathrm{e}^{z\overline{z}}\,\frac{\partial}{\partial z}\left(\mathrm{e}^{-z\overline{z}} f\right),$$

$$\overline{A}^\dagger f = \left(z - \frac{\partial}{\partial \overline{z}}\right) f = -\,\mathrm{e}^{z\overline{z}}\,\frac{\partial}{\partial \overline{z}}\left(\mathrm{e}^{-z\overline{z}} f\right).$$

The operators $A^\dagger$ and $\overline{A}^\dagger$ are known as the (nonnormalized) creation operators with respect to $\overline{z}$ and z, respectively. For every p,q in $\mathbb{N}_0$, denote by $m_{p,q}$ the monomial function $m_{p,q}(z) := z^p\,\overline{z}^q$. Following Shigekawa [30, Section 7] we define the *normalized complex Hermite polynomials* as

$$b_{p,q} := \frac{1}{\sqrt{p!\,q!}}(A^\dagger)^q(\overline{A}^\dagger)^p m_{0,0} \qquad (p,q\in\mathbb{N}_0). \tag{2}$$

Notice that [30] defines complex Hermite polynomials without the factor $\frac{1}{\sqrt{p!\,q!}}$. These polynomials appear also in Balk [5, Section 6.3]. Let us show explicitly some of them:

$$\begin{aligned}
b_{0,0}(z) &= 1, & b_{0,1}(z) &= \overline{z}, & b_{0,2}(z) &= \tfrac{1}{\sqrt{2}}\overline{z}^2,\\
b_{1,0}(z) &= z, & b_{1,1}(z) &= |z|^2-1, & b_{1,2}(z) &= \tfrac{1}{\sqrt{2}}\,\overline{z}(|z|^2-2),\\
b_{2,0}(z) &= \tfrac{1}{\sqrt{2}}\,z^2, & b_{2,1}(z) &= \tfrac{1}{\sqrt{2}}\,z(|z|^2-2), & b_{2,2}(z) &= \tfrac{1}{2}(|z|^4-4|z|^2+2).
\end{aligned}$$

For every p, α in $\mathbb{N}_0$, we denote by $L_p^{(\alpha)}$ the associated Laguerre polynomial. Recall the Rodrigues formula, the explicit expression, and the orthogonality relation for these polynomials:

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}\,\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{e}^{-x}\;x^{n+\alpha}\right), \tag{3}$$

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{x^k}{k!}, \tag{4}$$

$$\int_0^{+\infty} L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)\,x^\alpha\,\mathrm{e}^{-x}\;\mathrm{d}x = \frac{(n+\alpha)!}{n!}\,\delta_{m,n}. \tag{5}$$

Lemma 2.1 *Let $n, \alpha \in \mathbb{N}$. Then*

$$\mathrm{e}^{xy}\,\frac{\partial^n}{\partial x^n}\left(\mathrm{e}^{-xy}\;x^{n+\alpha}\right) = n!\,x^\alpha L_n^{(\alpha)}(xy). \tag{6}$$

Proof Apply Rodrigues formula (3) and the chain rule:

$$\frac{\partial^n}{\partial x^n}\left(\mathrm{e}^{-xy}(xy)^{n+\alpha}\right) = n!\;\mathrm{e}^{-xy}(xy)^\alpha L_n^{(\alpha)}(xy)\,y^n.$$

Canceling the factor $y^{n+\alpha}$ in both sides yields (6). □

Proposition 2.2 *For every p, q in $\mathbb{N}_0$,*

$$b_{p,q}(z) = \begin{cases}\sqrt{\frac{q!}{p!}}(-1)^q\,z^{p-q}L_q^{(p-q)}(|z|^2), & \text{if } p \geq q;\\ \sqrt{\frac{p!}{q!}}(-1)^p\,\overline{z}^{q-p}L_p^{(q-p)}(|z|^2), & \text{if } p \leq q.\end{cases} \tag{7}$$

In other words,

$$b_{p,q} = \sqrt{\frac{\min\{p,q\}!}{\max\{p,q\}!}}\sum_{s=0}^{\min\{p,q\}}\binom{\max\{p,q\}}{s}\frac{(-1)^s}{(\min\{p,q\}-s)!}m_{p-s,q-s}. \tag{8}$$

Proof Let $p, q \in \mathbb{N}_0$, $p \geq q$. Notice that $\frac{\partial}{\partial z}|z|^2 = \frac{\partial}{\partial z}(z\,\overline{z}) = \overline{z}$. By (2) and (6),

$$\begin{aligned} b_{p,q}(z) &= \frac{(-1)^{p+q}}{\sqrt{p!\,q!}}\,\mathrm{e}^{z\overline{z}}\,\frac{\partial^q}{\partial z^q}\frac{\partial^p}{\partial \overline{z}^p}\;\mathrm{e}^{-z\overline{z}} \\ &= \frac{(-1)^q}{\sqrt{p!\,q!}}\,\mathrm{e}^{z\overline{z}}\,\frac{\partial^q}{\partial z^q}\big(\mathrm{e}^{-z\overline{z}}\,z^p\big) = \sqrt{\frac{q!}{p!}}\,(-1)^q z^{p-q} L_q^{(p-q)}(z\,\overline{z}). \end{aligned}$$

In the case when $p \leq q$, we first notice that the operators $A^\dagger$ and $\overline{A}^\dagger$ commute on the space of polynomial functions. Reasoning as above, but swapping the roles of z and $\overline{z}$, we arrive at the second case of (7). Finally, with the help of (4), we pass from (7) to (8). Formula (8) can also be derived directly from (2), by applying mathematical induction and working with binomial coefficients. □

Denote by $\ell_m^{(\alpha)}$ the *normalized Laguerre function*:

$$\ell_m^{(\alpha)}(t) := \sqrt{\frac{m!}{(m+\alpha)!}}\,t^{\alpha/2}\,\mathrm{e}^{-t/2}\,L_m^{(\alpha)}(t) \qquad (m, \alpha \in \mathbb{N}_0). \tag{9}$$

Corollary 2.3 *For every p, q in $\mathbb{N}_0$,*

$$b_{p,q}(r\tau) = (-1)^{\min\{p,q\}}\tau^{p-q}\,\mathrm{e}^{r^2/2}\,\ell_{\min\{p,q\}}^{(|p-q|)}(r^2) \qquad (r \geq 0,\ \tau \in \mathbb{T}). \tag{10}$$

It is convenient to treat the family $(m_{p,q})_{p,q\in\mathbb{N}_0}$ as an infinite table, and to think in terms of its columns or diagonals (parallel to the main diagonal). Given d in $\mathbb{Z}$ and n in $\mathbb{N}_0$, let $\mathcal{D}_{d,n}$ be the subspace of $L^2(\mathbb{C}, \gamma)$ generated by the first n monomials in the diagonal with index d:

$$\mathcal{D}_{d,n} := \operatorname{span}\{m_{p,q}\colon\ p, q \in \mathbb{N}_0,\ \min\{p,q\} < n,\ p - q = d\}.$$

Proposition 2.4 *The family $(b_{p,q})_{p,q\in\mathbb{N}_0}$ is an orthonormal basis of $L^2(\mathbb{C}, \gamma)$. This family can be obtained from $(m_{p,q})_{p,q=0}^{\infty}$ by applying the Gram–Schmidt orthogonalization.*

Proof

1. The orthonormality is easy to verify by passing to polar coordinates and using (7) with the orthogonality relation (5).
2. Formula (8) tells us that the functions $b_{p,q}$ are linear combinations of $m_{p-s,q-s}$ with $0 \leq s \leq \min\{p,q\}$. Inverting these formulas, $m_{p,q}$ results a linear

combination of $b_{p-s,q-s}$ with $0 \le s \le \min\{p, q\}$. So, for every d in $\mathbb{Z}$ and every n in $\mathbb{N}_0$,

$$\mathcal{D}_{d,n} = \operatorname{span}\{b_{p,q}\colon\ p, q \in \mathbb{N}_0,\ \min\{p, q\} < n,\ p - q = d\}. \tag{11}$$

Jointly with the orthonormality of $(b_{p,q})_{p,q=0}^{\infty}$, this means that the family $(b_{p,q})_{p,q=0}^{\infty}$ is obtained from $(m_{p,q})_{p,q=0}^{\infty}$ by applying the orthogonalization in each diagonal.

3. Due to 2, it is sufficient to prove that the polynomials in z and $\overline{z}$ form a dense subset of $L^2(\mathbb{C}, \gamma)$. Notice that the set of polynomial functions in z and $\overline{z}$ coincides with the set of polynomial functions in $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$. Suppose that $f \in L^2(\mathbb{C}, \gamma)$ and f is orthogonal to the polynomials $\operatorname{Re}(z)^j \operatorname{Im}(z)^k$ for all j, k in $\mathbb{N}_0$. Denote by g the function $g(x, y) = f(x + \mathrm{i}\, y)\, \mathrm{e}^{-x^2-y^2}$ and consider its Fourier transform:

$$\begin{aligned}\widehat{g}(u, v) &= \int_{\mathbb{R}^2} \mathrm{e}^{-2\pi\, \mathrm{i}(xu+yv)}\ f(x + \mathrm{i}\, y)\, \mathrm{e}^{-x^2-y^2}\ \mathrm{d}x\, \mathrm{d}y \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-2\pi\, \mathrm{i}\, u)^j(-2\pi\, \mathrm{i}\, v)^k}{j!\,k!} \int_{\mathbb{R}^2} x^j y^k f(x + \mathrm{i}\, y)\, \mathrm{e}^{-x^2-y^2}\ \mathrm{d}x\, \mathrm{d}y = 0.\end{aligned}$$

By the injective property of the Fourier transform, we conclude that g vanishes a.e. As a consequence, f also vanishes a.e.

□

Remark 2.5 The second part of the proof of Proposition 2.4 implies that for every d in $\mathbb{Z}$, every $q \ge \max\{0, -d\}$ every k in $\mathbb{Z}$ with $\max\{0, -d\} \le k \le q$,

$$\langle m_{d+k,k}, b_{d+q,q}\rangle = \begin{cases} \sqrt{q!\,(d+q)!}, & k = q; \\ 0, & k < q. \end{cases} \tag{12}$$

Formula (11) means that the first n elements in the diagonal d of the table $(b_{p,q})_{p,q\in\mathbb{N}_0}$ generate the same subspace as the first n elements in the diagonal d of the table $(m_{p,q})_{p,q\in\mathbb{N}_0}$. For example,

$$\begin{aligned}\mathcal{D}_{-1,3} &= \operatorname{span}\{m_{0,1}, m_{1,2}, m_{2,3}\} = \operatorname{span}\{b_{0,1}, b_{1,2}, b_{2,3}\}, \\ \mathcal{D}_{2,2} &= \operatorname{span}\{m_{2,0}, m_{3,1}\} = \operatorname{span}\{b_{2,0}, b_{3,1}\}.\end{aligned}$$

In the following tables we show generators of $\mathcal{D}_{2,2}$ (green) and $\mathcal{D}_{-1,3}$ (blue).

$$\begin{matrix} m_{0,0} & m_{0,1} & m_{0,2} & m_{0,3} & m_{0,4} & \ddots \\ m_{1,0} & m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & \ddots \\ m_{2,0} & m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & \ddots \\ m_{3,0} & m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & \ddots \\ m_{4,0} & m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{matrix} \qquad \begin{matrix} b_{0,0} & b_{0,1} & b_{0,2} & b_{0,3} & b_{0,4} & \ddots \\ b_{1,0} & b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & \ddots \\ b_{2,0} & b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & \ddots \\ b_{3,0} & b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & \ddots \\ b_{4,0} & b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{matrix}$$

Given d in $\mathbb{Z}$, we denote by $\mathcal{D}_d$ the closure of the subspace of $L^2(\mathbb{C}, \gamma)$ generated by the monomials $m_{p,q}$, where $p - q = d$:

$$\mathcal{D}_d := \operatorname{clos}\big(\operatorname{span}\{m_{p,q} : \; p, q \in \mathbb{N}_0, \; p - q = d\}\big).$$

Proposition 2.4 implies the following properties of the "diagonal subspaces" $\mathcal{D}_d$, $d \in \mathbb{Z}$.

Corollary 2.6 *The sequence* $(b_{q+d,q})_{q=\max\{0,-d\}}^{\infty}$ *is an orthonormal basis of* $\mathcal{D}_d$.

Corollary 2.7 *The space* $\mathcal{D}_d$ *consists of all functions of the form*

$$f(r\tau) = \tau^d h(r^2) \quad (r \geq 0, \; \tau \in \mathbb{T}), \quad \textit{where} \quad h \in L^2([0, +\infty), \mathrm{e}^{-x}\,\mathrm{d}x). \tag{13}$$

Moreover, $\|f\| = \|h\|_{L^2([0,+\infty),\mathrm{e}^{-x}\,\mathrm{d}x)}$.

Corollary 2.8 *The space* $L^2(\mathbb{C}, \gamma)$ *is the orthogonal sum of the subspaces* $\mathcal{D}_d$*:*

$$L^2(\mathbb{C}, \gamma) = \bigoplus_{d \in \mathbb{Z}} \mathcal{D}_d. \tag{14}$$

Here we show the generators of $\mathcal{D}_1$ (green) and $\mathcal{D}_{-2}$ (blue):

$$\begin{matrix} m_{0,0} & m_{0,1} & m_{0,2} & m_{0,3} & \ddots \\ m_{1,0} & m_{1,1} & m_{1,2} & m_{1,3} & \ddots \\ m_{2,0} & m_{2,1} & m_{2,2} & m_{2,3} & \ddots \\ m_{3,0} & m_{3,1} & m_{3,2} & m_{3,3} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{matrix} \qquad \begin{matrix} b_{0,0} & b_{0,1} & b_{0,2} & b_{0,3} & \ddots \\ b_{1,0} & b_{1,1} & b_{1,2} & b_{1,3} & \ddots \\ b_{2,0} & b_{2,1} & b_{2,2} & b_{2,3} & \ddots \\ b_{3,0} & b_{3,1} & b_{3,2} & b_{3,3} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{matrix}$$

3 Bargmann–Segal–Fock Spaces of Polyanalytic Functions

Fix n in $\mathbb{N}$. Let $\mathcal{F}_n$ be the space of n-polyanalytic functions belonging to $L^2(\mathbb{C}, \gamma)$, and $\mathcal{F}_{(n)}$ be the true-n-polyanalytic Fock space defined in [33] by

$$\mathcal{F}_{(n)} \coloneqq \{f \in \mathcal{F}_n\colon\ f \perp \mathcal{F}_{n-1}\}.$$

Proposition 3.1 *Let $R > 0$. Then there exists a number $C_{n,R} > 0$ such that for every f in $\mathcal{F}_n$ and every z in $\mathbb{C}$ with $|z| \le R$,*

$$|f(z)| \le C_{n,R}\|f\|. \tag{15}$$

Proof Let P_n be the polynomial in one variable of degree $\le n-1$ such that

$$\int_0^1 P_n(x)x^j\,\mathrm{d}x = \delta_{j,0} \qquad (j \in \{0,\dots,n-1\}). \tag{16}$$

The existence and uniqueness of such a polynomial follows from the invertibility of the Hilbert matrix $\left[1/(j+k+1)\right]_{j,k=0}^{n-1}$. Put

$$C_{n,R} \coloneqq \left(\max_{x\in[0,1]} |P_n(x)|\right)\left(\frac{1}{\pi}\int_{(R+1)\mathbb{D}} \mathrm{e}^{|w|^2}\,\mathrm{d}\mu(w)\right)^{1/2}.$$

Let $f \in \mathcal{F}_n$ and $z \in \mathbb{C}$, with $|z| \le R$. It is known [5, Section 1.1] that f can be expanded into a uniformly convergent series of the form

$$f(w) = \sum_{j=0}^{\infty}\sum_{k=0}^{n-1} \alpha_{j,k}(w-z)^j(\overline{w}-\overline{z})^k,$$

where $\alpha_{j,k}$ are some complex numbers. Using the change of variables $w = z + r\,\mathrm{e}^{\mathrm{i}\,\vartheta}$ and the property (16), we obtain the following version of the mean value property of polyanalytic functions:

$$f(z) = \frac{1}{\pi}\int_{z+\mathbb{D}} f(w)P_n(|w-z|^2)\,\mathrm{d}\mu(w). \tag{17}$$

After that, estimating $|P_n|$ by its maximum value, multiplying and dividing by $\mathrm{e}^{|w|^2/2}$, and applying the Schwarz inequality, we arrive at (15). ◻

Remark 3.2 The constant $C_{n,R}$, found in the proof of Proposition 3.1, is not optimal. The exact upper bound for the evaluation functionals in $\mathcal{F}_n$ is given in Corollary 3.16.

Proposition 3.3 *$\mathcal{F}_n$ is a RKHS.*

Proof Let $(g_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{F}_n$. By Proposition 3.1, this sequence converges pointwise on $\mathbb{C}$ and uniformly on compacts to a function f. By [5, Corollary 1.8], the function f is n-analytic. On the other hand, let h be the limit of the sequence $(g_n)_{n\in\mathbb{N}}$ in $L^2(\mathbb{C},\gamma)$. Then for every compact K in $\mathbb{C}$, the sequence of the restrictions $g_n|_K$ converges in the $L^2(K,\gamma)$-norm simultaneously to $f|_K$ and to $h|_K$. Therefore h coincides with f a.e. and $f\in L^2(\mathbb{C},\gamma)$, i.e. $f\in\mathcal{F}_n$. So, $\mathcal{F}_n$ is a Hilbert space. The boundedness of the evaluation functionals is established in Proposition 3.1. □

Proposition 3.4 *The family $(b_{p,q})_{p\in\mathbb{N}_0,q<n}$ is an orthonormal basis of $\mathcal{F}_n$.*

Proof We already know that this family is contained in $\mathcal{F}_n$ and is orthonormal. Let us verify the total property. Our reasoning uses ideas of Ramazanov [25, proof of Theorem 2].

Suppose that $f\in\mathcal{F}_n$ and $\langle f, b_{p,q}\rangle = 0$ for every $p\in\mathbb{N}_0$, $q<n$. We have to show that $f=0$. By the decomposition of polyanalytic functions [5, Section 1.1], there exists a family of numbers $(\alpha_{j,k})_{j\in\mathbb{N}_0,k<n}$ such that

$$f(z)=\sum_{k=0}^{n-1}\sum_{j=0}^{\infty}\alpha_{j,k}m_{j,k}(z),$$

where each of the inner series converges pointwise on $\mathbb{C}$ and uniformly on compacts. For every ν in $\mathbb{N}_0$, we denote by S_ν the partial sum $S_\nu := \sum_{k=0}^{n-1}\sum_{j=0}^{\nu}\alpha_{j,k}m_{j,k}$. Given $r>0$, the sequence $(S_\nu)_{\nu\in\mathbb{N}_0}$ converges to f uniformly on $r\mathbb{D}$. For every p,q in $\mathbb{N}_0$ with $q<n$, using the orthogonality on $r\mathbb{D}$ between $b_{p,q}$ and $m_{j,k}$ with $j-k\neq p-q$, we obtain

$$\int_{r\mathbb{D}} f\,\overline{b_{p,q}}\,\mathrm{d}\gamma=\lim_{\nu\to\infty}\int_{r\mathbb{D}} S_\nu\,\overline{b_{p,q}}\,\mathrm{d}\gamma=\sum_{k=0}^{n-1}\alpha_{k+p-q,k}\int_{r\mathbb{D}} m_{k+p-q,k}\overline{b_{p,q}}\,\mathrm{d}\gamma.$$

The functions $f\,\overline{b_{p,q}}$ and $m_{k+p-q,k}\,\overline{b_{p,q}}$ are integrable on $\mathbb{C}$ with respect to the measure γ. Therefore their integrals over $\mathbb{C}$ are the limits of the corresponding integrals over $r\mathbb{D}$, as r tends to infinity. Since $\langle f, b_{p,q}\rangle = 0$, the coefficients $\alpha_{j,k}$ must satisfy the following infinite system of homogeneous linear equations:

$$\sum_{k=0}^{n-1}\langle m_{k+p-q,k}, b_{p,q}\rangle\alpha_{k+p-q,k}=0\qquad (p\in\mathbb{N}_0,\ 0\le q<n).\tag{18}$$

Now we fix $d > -n$ and restrict ourselves to the equations (18) with $p - q = d$, which yields an $s \times s$ system represented by the matrix M_d, where $s = \min\{n, n+d\}$, and

$$M_d := \left[\langle m_{d+k,k}, b_{d+q,q}\rangle\right]_{q,k=\max\{0,-d\}}^{n-1}.$$

By (12), M_d is an upper triangular matrix with nonzero diagonal entries, hence M_d is invertible. So, all coefficients $\alpha_{j,k}$ are zero. □

Corollary 3.5 *$\mathcal{F}_{(n)}$ is a RKHS, and the sequence $(b_{p,n-1})_{p\in\mathbb{N}_0}$ is an orthonormal basis of $\mathcal{F}_{(n)}$.*

We denote by P_n and $P_{(n)}$ the orthogonal projections acting in $L^2(\mathbb{C}, \gamma)$, whose images are $\mathcal{F}_n$ and $\mathcal{F}_{(n)}$, respectively. They can be explicitly defined in terms of the corresponding reproducing kernels:

$$(P_n f)(z) = \langle f, K_{n,z}\rangle, \qquad (P_{(n)} f)(z) = \langle f, K_{(n),z}\rangle.$$

Corollary 3.6 *If $f \in \mathcal{F}_n$, then*

$$f = \sum_{j=0}^{\infty}\sum_{k=0}^{n-1}\langle f, b_{j,k}\rangle b_{j,k},$$

where the series converges in the $L^2(\mathbb{C}, \gamma)$-norm and uniformly on compact sets. In particular, if $f \in \mathcal{F}_{(n)}$, then

$$f = \sum_{j=0}^{\infty}\langle f, b_{j,n-1}\rangle b_{j,n-1}. \tag{19}$$

For example, $(b_{p,2})_{p\in\mathbb{N}_0}$ is an orthonormal basis of $\mathcal{F}_{(3)}$, and $(b_{p,q})_{p\in\mathbb{N}_0, q<3}$ is an orthonormal basis of $\mathcal{F}_3$:

$$\begin{array}{ccccc} b_{0,0} & b_{0,1} & \colorbox{green}{$b_{0,2}$} & b_{0,3} & \cdots \\ b_{1,0} & b_{1,1} & \colorbox{green}{$b_{1,2}$} & b_{1,3} & \cdots \\ b_{2,0} & b_{2,1} & \colorbox{green}{$b_{2,2}$} & b_{2,3} & \cdots \\ b_{3,0} & b_{3,1} & \colorbox{green}{$b_{3,2}$} & b_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \qquad \begin{array}{ccccc} \colorbox{green}{$b_{0,0}$} & \colorbox{green}{$b_{0,1}$} & \colorbox{green}{$b_{0,2}$} & b_{0,3} & \cdots \\ \colorbox{green}{$b_{1,0}$} & \colorbox{green}{$b_{1,1}$} & \colorbox{green}{$b_{1,2}$} & b_{1,3} & \cdots \\ \colorbox{green}{$b_{2,0}$} & \colorbox{green}{$b_{2,1}$} & \colorbox{green}{$b_{2,2}$} & b_{2,3} & \cdots \\ \colorbox{green}{$b_{3,0}$} & \colorbox{green}{$b_{3,1}$} & \colorbox{green}{$b_{3,2}$} & b_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Using Proposition 3.4, Corollary 2.6, and formula (11) gives

$$\mathcal{D}_d \cap \mathcal{F}_n = \begin{cases} \mathcal{D}_{d,\min\{n,n+d\}}, & d \geq -n+1; \\ \{0\}, & d < -n+1. \end{cases} \tag{20}$$

Here is a description of the subspaces $\mathcal{D}_{d,m}$ in terms of the polar coordinates.

Proposition 3.7 *For every m in $\mathbb{N}_0$ and every d in $\mathbb{Z}$ with $d \ge -m+1$, the space $\mathcal{D}_{d,m}$ consists of all functions of the form*

$$f(r\tau) = \tau^d r^{|d|} Q(r^2) \qquad (r \ge 0,\ \tau \in \mathbb{T}),$$

where Q is a polynomial of degree $\le m-1$. Moreover,

$$\|f\| = \|Q\|_{L^2([0,+\infty), x^{|d|}\,\mathrm{e}^{-x}\,\mathrm{d}x)}.$$

Proof Apply formula (11) and the orthonormality of the polynomials $L_k^{(|d|)}$ in the space $L^2([0,+\infty), x^{|d|}\,\mathrm{e}^{-x}\,\mathrm{d}x)$. □

The decomposition of $\mathcal{F}_n$ into a direct sum of "truncated diagonals" shown below follows from Proposition 3.4 and plays a crucial role in the study of radial operators.

Proposition 3.8

$$\mathcal{F}_n = \bigoplus_{d=-n+1}^{\infty} \mathcal{D}_{d,\min\{n,n+d\}}. \tag{21}$$

Let us illustrate Proposition 3.8 for $n = 3$ with a table (we have marked in different shades of blue the basic functions that generate each truncated diagonal):

$b_{0,0}$	$b_{0,1}$	$b_{0,2}$	$b_{0,3}$	$\dots$
$b_{1,0}$	$b_{1,1}$	$b_{1,2}$	$b_{1,3}$	$\dots$
$b_{2,0}$	$b_{2,1}$	$b_{2,2}$	$b_{2,3}$	$\dots$
$b_{3,0}$	$b_{3,1}$	$b_{3,2}$	$b_{3,3}$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The upcoming fact was proved by Vasilevski [33]. We obtain it as a corollary from Proposition 2.4 and Corollary 3.5.

Corollary 3.9 *The space $L^2(\mathbb{C}, \gamma)$ is the orthogonal sum of the subspaces $\mathcal{F}_{(m)}$, $m \in \mathbb{N}$:*

$$L^2(\mathbb{C}, \gamma) = \bigoplus_{m \in \mathbb{N}} \mathcal{F}_{(m)}.$$

For every f in $\mathcal{F}_{(n)}$, define $A_n^\dagger f$ by

$$(A_n^\dagger f)(z) = \frac{1}{\sqrt{n}} (A^\dagger f)(z) = \frac{1}{\sqrt{n}} \left(\overline{z} - \frac{\partial}{\partial z} \right) f(z).$$

Definition (2) of the family $(b_{p,q})_{p,q\in\mathbb{N}_0}$ implies that

$$A_n^\dagger b_{p,n-1} = b_{p,n}. \tag{22}$$

The next picture shows the action of $A_2^\dagger$ on basic elements:

$$\begin{array}{ccccc} b_{0,0} & b_{0,1} \mapsto & b_{0,2} & b_{0,3} & \dots \\ b_{1,0} & b_{1,1} \mapsto & b_{1,2} & b_{1,3} & \dots \\ b_{2,0} & b_{2,1} \mapsto & b_{2,2} & b_{2,3} & \dots \\ b_{3,0} & b_{3,1} \mapsto & b_{3,2} & b_{3,3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Proposition 3.10 *$A_n^\dagger$ is an isometric isomorphism from $\mathcal{F}_{(n)}$ onto $\mathcal{F}_{(n+1)}$.*

Proof Vasilevski [33] proved this fact by using the Fourier transform. Here we give another proof. Write f as in (19). It is known [5, Corollary 1.9] that the derivative $\frac{\partial}{\partial \overline{z}}$ can be applied to the each term of the series. Therefore

$$A_n^\dagger f = \sum_{j=0}^{\infty} \langle f, b_{j,n-1} \rangle \, A_n^\dagger b_{j,n-1} = \sum_{j=0}^{\infty} \langle f, b_{j,n-1} \rangle \, b_{j,n},$$

and $\|A_n^\dagger f\| = \|f\|$. Also, using the decomposition into series, we see that $A_n^\dagger$ is surjective. □

Now we are going to prove explicit formulas (26) and (27) for the reproducing kernels of $\mathcal{F}_{(n)}$ and $\mathcal{F}_n$, respectively. These formulas were published by Balk [5, Section 6.3], without using the terminology of Laguerre polynomials, and by Askour et al. [4], though they defined the space $\mathcal{F}_{(n)}$ in a different (but equivalent) way. Our proof uses the operators $A_n^\dagger$ and thereby continues the work of Vasilevski [33].

Lemma 3.11 *Let H be a RKHS and $(e_j)_{j=0}^{\infty}$ be an orthonormal sequence in H. Then the series $\sum_{j=0}^{\infty} |e_j(z)|^2$ converges.*

Proof Denote by $K_{H,z}$ the reproducing kernel of H. From the reproducing property and Bessel's inequality,

$$\sum_{j=0}^{\infty} |e_j(z)|^2 = \sum_{j=0}^{\infty} |\langle K_{H,z}, e_j \rangle|^2 \le \|K_{H,z}\|^2 = K_{H,z}(z) < \infty. \qquad \square$$

Lemma 3.12 *For every n in $\mathbb{N}_0$ and every z, w in $\mathbb{C}$,*

$$K_{(n+1),z}(w) = \frac{1}{n}\left(z - \frac{\partial}{\partial \overline{z}}\right)\left(\overline{w} - \frac{\partial}{\partial w}\right) K_{(n),z}(w). \tag{23}$$

Proof It is well known that the reproducing kernel of a RKHS H with an orthonormal basis $(e_j)_{j\in\mathbb{N}_0}$ can be derived from the series

$$K_{H,z}(w) = \sum_{j=0}^{\infty} \overline{e_j(z)} e_j(w). \tag{24}$$

In our case, we use the orthonormal basis $(b_{p,n})_{p\in\mathbb{N}_0}$ of the space $\mathcal{F}_{(n+1)}$. For a fixed z in $\mathbb{C}$, put $\alpha_p = \overline{b_{p,n}(z)}$. So,

$$K_{(n+1),z} = \sum_{p=0}^{\infty} \overline{b_{p,n}(z)} b_{p,n} = \sum_{p=0}^{\infty} \alpha_p b_{p,n} = \sum_{p=0}^{\infty} \alpha_p A_{n-1}^{\dagger} b_{p,n-1}.$$

From Lemma 3.11 we know that $(\alpha_p)_{p=0}^{\infty} \in \ell^2$, thus the series $\sum_{p=0}^{\infty} \alpha_p b_{p,n-1}$ converges in $\mathcal{F}_{(n)}$. Since $A_{n-1}^{\dagger}$ is a bounded operator in $\mathcal{F}_{(n)}$, we can interchange it with the sum operator. Therefore

$$K_{(n+1),z}(z) = \frac{1}{\sqrt{n}}\left(\overline{w} - \frac{\partial}{\partial w}\right) \sum_{p=0}^{\infty} \overline{b_{p,n}(z)} b_{p,n-1}(w).$$

Now we fix w in $\mathbb{C}$, write $b_{p,n}$ as $A_{n-1}^{\dagger} b_{p,n-1}$, and use the fact that the series $\sum_{p=0}^{\infty} |b_{p,n-1}(w)|^2$ converges. Following the same ideas as above, but swapping the roles of z and w, we factorize $\left(z - \frac{\partial}{\partial \overline{z}}\right)$ from the series:

$$K_{(n+1),z}(w) = \frac{1}{n}\left(z - \frac{\partial}{\partial \overline{z}}\right)\left(\overline{w} - \frac{\partial}{\partial w}\right) \sum_{p=0}^{\infty} \overline{b_{p,n-1}(z)} b_{p,n-1}(w).$$

The last sum equals $K_{(n),z}(w)$, which yields (23). □

Corollary 3.13 *For every n in $\mathbb{N}_0$ and every z, w in $\mathbb{C}$,*

$$K_{(n),z}(w) = \frac{1}{(n-1)!}\left(z - \frac{\partial}{\partial \overline{z}}\right)^{n-1}\left(\overline{w} - \frac{\partial}{\partial w}\right)^{n-1} e^{\overline{z}w}. \tag{25}$$

Proposition 3.14 *The reproducing kernel of $\mathcal{F}_{(n)}$ is given by*

$$K_{(n),z}(w) = e^{\overline{z}w} L_{n-1}(|w-z|^2). \tag{26}$$

Proof Using the definition of creation operators, formula (25) and identity (6) for Laguerre polynomials we have

$$\begin{aligned} K_{(n),z}(w) &= \frac{e^{z\overline{z}}}{(n-1)!} \frac{\partial^{n-1}}{\partial \overline{z}^{n-1}} \left(e^{-z\overline{z}} e^{w\overline{w}} \frac{\partial^{n-1}}{\partial w^{n-1}} (e^{-w\overline{w}} e^{w\overline{z}}) \right) \\ &= \frac{e^{z\overline{z}}}{(n-1)!} \frac{\partial^{n-1}}{\partial \overline{z}^{n-1}} \left(e^{-\overline{z}(z-w)} (\overline{z} - \overline{w})^{n-1} \right) \\ &= e^{\overline{z}w} \frac{e^{(z-w)(\overline{z}-w)}}{(n-1)!} \frac{\partial^{n-1}}{\partial (\overline{z} - \overline{w})^{n-1}} \left(e^{-(\overline{z}-\overline{w})(z-w)} (\overline{z} - \overline{w})^{n-1} \right) \\ &= e^{\overline{z}w} L_{n-1}(|z-w|^2). \end{aligned}$$

□

Corollary 3.15 *The reproducing kernel of $\mathcal{F}_n$ is*

$$K_{n,z}(w) = e^{\overline{z}w} L^{(1)}_{n-1}(|w-z|^2). \tag{27}$$

Proof Use (26) and the formula $L^{(1)}_m(x) = \sum_{k=0}^{m-1} L_k(x)$. □

Corollary 3.16 *For every f in $\mathcal{F}_n$ and every z in $\mathbb{C}$,*

$$|f(z)| \leq \sqrt{n}\ \mathrm{e}^{\frac{|z|^2}{2}}\ \|f\|. \tag{28}$$

The equality is achieved when $f = K_{n,z}$.

Proof Indeed, $\|K_{n,z}\|^2 = K_{n,z}(z) = \mathrm{e}^{|z|^2}\, L^{(1)}_{n-1}(0) = n\, \mathrm{e}^{|z|^2}$. □

We finish this section with a couple of simple results about the Berezin transform and Toeplitz operators in $\mathcal{F}_n$. Given a RKHS H over a domain Ω with a reproducing kernel $(K_z)_{z\in\Omega}$, the corresponding Berezin transform Ber_H acts from $\mathcal{B}(H)$ to the space $B(\Omega)$ of bounded functions by the rule

$$\mathrm{Ber}_H(S)(z) = \frac{\langle SK_z, K_z\rangle_H}{\langle K_z, K_z\rangle_H} = \frac{(SK_z)(z)}{K_z(z)}.$$

Stroethoff proved [31] that Ber_H is injective for various RKHS of analytic functions, in particular, for $H = \mathcal{F}_1$. Engliš noticed [9, Section 2] that Ber_H is not injective for various RKHS of harmonic functions. The reasoning of Engliš can be applied without any changes to n-analytic functions with $n \geq 2$.

Proposition 3.17 *Let $n \geq 2$. Then $\mathrm{Ber}_{\mathcal{F}_n}$ is not injective.*

Proof Let u and v be some linearly independent elements of $\mathcal{F}_n$ such that $\overline{f}, \overline{g} \in \mathcal{F}_n$. For example, $u(z) = b_{0,0}(z) = 1$ and $v(z) = b_{1,0}(z) = z$. Following [9, Section 2], consider $S \in \mathcal{B}(\mathcal{F}_n)$ given by

$$Sf := \langle f, \overline{u} \rangle v - \langle f, \overline{v} \rangle u. \tag{29}$$

With the help of the reproducing property we easily see that the function $\mathrm{Ber}_{\mathcal{F}_n}(S)$ is the zero constant, although the operator S is not zero. □

Given a measure space Ω and a function g in $L^\infty(\Omega)$, we denote by M_g the multiplication operator defined on $L^2(\Omega)$ by $M_g f := gf$. If H is a closed subspace of $L^2(\Omega)$, then the *Toeplitz operator* $T_{H,g}$ is defined on H by

$$T_{H,g}(f) := P_H(gf) = P_H M_g f.$$

For $H = \mathcal{F}_n$ and $H = \mathcal{F}_{(n)}$, we write just $T_{n,g}$ and $T_{(n),g}$, respectively.

Proposition 3.18 *Let $g \in L^\infty(\mathbb{C})$ and $T_{n,g} = 0$. Then $g = 0$ a.e.*

Proof For $n = 1$, this result was proven in [7, Theorem 4]. Let us recall that proof which also works for $n \geq 2$. The condition $T_{n,g} = 0$ implies that for all j, k in $\mathbb{N}_0$

$$\langle g, m_{j,k} \rangle = \int_{\mathbb{C}} g(z)\, \overline{z}^j z^k \,\mathrm{d}\gamma(z) = \langle g m_{k,0}, m_{j,0} \rangle = \langle T_{n,g} m_{k,0}, m_{j,0} \rangle = 0.$$

Since $\{m_{j,k} \colon\ j, k \in \mathbb{N}_0\}$ is a dense subset of $L^2(\mathbb{C}, \gamma)$, $g = 0$ a.e. □

4 Unitary Representations Defined by Changes of Variables

This section states some simple general facts about unitary group representations in RKHS, defined by changes of variables. Suppose that (Ω, ν) is a measure space, H is a RKHS over Ω, with the inner product inherited from $L^2(\Omega)$, $(K_z)_{z\in\Omega}$ is the reproducing kernel of H, and $P_H \in \mathcal{B}(L^2(\Omega))$ is the orthogonal projection whose image is H:

$$(P_H f)(z) = \langle f, K_z \rangle_{L^2(\Omega)}.$$

Furthermore, let G be a locally compact group, and α be a group action in Ω. So, for every τ in G we have a "change of variables" $\alpha(\tau)\colon \Omega \to \Omega$, which satisfies $\alpha(\tau_1\tau_2) = \alpha(\tau_1) \circ \alpha(\tau_2)$. Suppose that the function ρ, defined by the following rule, is a strongly continuous unitary representation of the group G in the space $L^2(\Omega)$:

$$\rho(\tau) f := f \circ \alpha(\tau^{-1}) \qquad (f \in L^2(\Omega),\ \tau \in G).$$

In other words, we suppose that $\rho(\tau)f \in L^2(\Omega)$, $\|\rho(\tau)f\|_{L^2(\Omega)} = \|f\|_{L^2(\Omega)}$, and $\rho(\tau)f$ depends continuously on τ.

Proposition 4.1 *The following conditions are equivalent.*

(a) $\rho(\tau)(H) \subseteq H$ *for every* τ *in* G.
(b) $\rho(\tau)P_H = P_H\rho(\tau)$ *for every* τ *in* G.
(c) *The reproducing kernel is invariant under simultaneous changes of variables in both arguments:*

$$K_{\alpha(\tau)(z)}(\alpha(\tau)(w)) = K_z(w) \qquad (\tau \in G,\ z, w \in \Omega).$$

(d) $\rho(\tau)K_z = K_{\alpha(\tau)(z)}$ *for every* z *in* Ω *and every* τ *in* G.

Proof Obviously, (a) is equivalent to (b). Suppose (a) and prove (c):

$$\begin{aligned} K_{\alpha(\tau)(z)}(\alpha(\tau)(w)) &= \langle \rho(\tau^{-1})K_{\alpha(\tau)(z)}, K_w\rangle_{L^2(\Omega)} = \langle K_{\alpha(\tau)(z)}, \rho(\tau)K_w\rangle_{L^2(\Omega)} \\ &= \overline{(\rho(\tau)K_w)(\alpha(\tau)(z))} = \overline{K_w(z)} = K_z(w). \end{aligned}$$

Suppose (c) and prove (d):

$$(\rho(\tau)K_z)(w) = K_z(\alpha(\tau^{-1})(w)) = K_{\alpha(\tau)(z)}(\alpha(\tau)(\alpha(\tau^{-1})(w))) = K_{\alpha(\tau)(z)}(w).$$

Suppose (d) and prove (a). Let $f \in H$. Then

$$\begin{aligned} (\rho(\tau)f)(z) &= f(\alpha(\tau^{-1})(z)) = \langle f, K_{\alpha(\tau^{-1})(z)}\rangle_{L^2(\Omega)} \\ &= \langle \rho(\tau)f, \rho(\tau)K_{\alpha(\tau^{-1})(z)}\rangle_{L^2(\Omega)} = \langle \rho(\tau)f, K_z\rangle_{L^2(\Omega)}. \end{aligned}$$ □

Suppose that conditions (a)–(d) of Proposition 4.1 are fulfilled. For every τ in G we denote by $\rho_H(\tau)$ the compression of $\rho(\tau)$ to the invariant subspace H. Then ρ_H is a unitary representation of G in H. Let us relate this unitary representation with the Berezin transform of operators.

Proposition 4.2 *Let* $S \in \mathcal{B}(H)$ *and* $\tau \in G$. *Then*

$$\operatorname{Ber}_H(\rho_H(\tau^{-1})S\rho_H(\tau))(z) = \operatorname{Ber}_H(S)(\alpha(\tau)(z)) \qquad (z \in \Omega). \tag{30}$$

Proof

$$\begin{aligned} \operatorname{Ber}_H(\rho_H(\tau^{-1})S\rho_H(\tau))(z) &= \frac{(\rho_H(\tau^{-1})S\rho_H(\tau)K_z)(z)}{K_z(z)} \\ &= \frac{(SK_{\alpha(\tau)(z)}(\alpha(\tau)(z))}{K_{\alpha(\tau)(z)}(\alpha(\tau)(z))} = \operatorname{Ber}_H(S)(\alpha(\tau)(z)). \end{aligned}$$ □

Corollary 4.3 *Let $S \in \mathcal{B}(H)$ such that $S\rho(\tau) = \rho(\tau)S$ for every τ in G. Then the function* $\operatorname{Ber}_H(S)$ *is invariant under α, i.e.* $\operatorname{Ber}_H(S) \circ \alpha(\tau) = \operatorname{Ber}_H(S)$ *for every τ in G.*

If Ber_H is injective, then the inverse of the Corollary 4.3 is also true.

The rest of this section does not assume that H has a reproducing kernel; it can be just a closed subspace of $L^2(\Omega)$.

We are going to state some elementary results about the interaction of ρ_H with Toeplitz operators. These results are well known for many particular cases; see [8, Lemma 3.2 and Corollary 3.3] for the case when H is a Bergman space of analytic functions. Recall that T_g is defined on H by $T_g f = P_H(gf)$.

Lemma 4.4 *Let $g \in L^\infty(\Omega)$ and $\tau \in G$. Then*

$$M_g \rho(\tau) = \rho(\tau) M_{g\circ\alpha(\tau)}.$$

Proof Put $u := g \circ \alpha(\tau)$. Given f in $L^2(\Omega)$,

$$M_g \rho(\tau) f = (u \circ \alpha(\tau^{-1}))\,(f \circ \alpha(\tau^{-1})) = (uf) \circ \alpha(\tau^{-1}) = \rho(\tau) M_u f. \qquad \square$$

Proposition 4.5 *Let $g \in L^\infty(\Omega)$ and $\tau \in G$. Then*

$$T_g \rho_H(\tau) = \rho_H(\tau) T_{g\circ\alpha(\tau)}. \tag{31}$$

Proof Use Lemma 4.4 and the assumption $P_H \rho(\tau) = \rho(\tau) P_H$:

$$\begin{aligned} T_g \rho_H(\tau) f &= P_H M_g \rho(\tau) f = P_H \rho(\tau) M_{g\circ\alpha(\tau)} f \\ &= \rho(\tau) P_H M_{g\circ\alpha(\tau)} f = \rho_H(\tau) T_{g\circ\alpha(\tau)} f. \end{aligned} \qquad \square$$

Corollary 4.6 *Let $g \in L^\infty(\Omega)$ such that $g \circ \alpha(\tau) = g$ for every τ in G. Then T_g commutes with $\rho_H(\tau)$ for every τ in G.*

Corollary 4.7 *Suppose that the mapping $L^\infty(\Omega) \to \mathcal{B}(H)$ defined by $a \mapsto T_a$ is injective. Let $g \in L^\infty(X)$ such that T_g commutes with $\rho_H(\tau)$ for every τ in G. Then $g \circ \alpha(\tau)$ and g coincide a.e. for every τ in G.*

5 Von Neumann Algebras of Radial Operators

The methods of this section are similar to ideas from [12, 24, 37]. We start with two simple general schemes, stated in the context of von Neumann algebras, and then apply them to radial operators in $L^2(\Omega, \gamma)$, in $\mathcal{F}_n$, and in $\mathcal{F}_{(n)}$. Proposition 5.2 uses the concept of the (bounded) direct sum of von Neumann algebras [28, Definition 1.1.5].

Definition 5.1 Let H be a Hilbert space, $\mathcal{U}$ be a self-adjoint subset of $\mathcal{B}(H)$, and $(W_j)_{j\in J}$ be a finite or countable family of nonzero closed subspaces of H such that $H = \bigoplus_{j\in J} W_j$. We say that this family *diagonalizes* $\mathcal{U}$ if the following two conditions are satisfied.

1. For each j in J and each U in $\mathcal{U}$, there exists $\lambda_{U,j}$ in $\mathbb{C}$ such that $W_j \subseteq \ker(\lambda_{U,j} I - U)$, i.e. $U(v) = \lambda_{U,j} v$ for every v in W_j.
2. For every j, k in J with $j \neq k$, there exists U in $\mathcal{U}$ such that $\lambda_{U,j} \neq \lambda_{U,k}$.

Proposition 5.2 *Let $H, \mathcal{U}$, and $(W_j)_{j\in J}$ be like in Definition 5.1. Denote by $\mathcal{A}$ the commutant of $\mathcal{U}$. Then*

$$\mathcal{A} = \{S \in \mathcal{B}(H)\colon \quad \forall j \in J \quad S(W_j) \subseteq W_j\}, \tag{32}$$

and $\mathcal{A}$ is isometrically isomorphic to $\bigoplus_{j\in J} \mathcal{B}(W_j)$.

Proof

1. Since $\mathcal{U}$ is a self-adjoint subset of $\mathcal{B}(H)$, its commutant $\mathcal{A}$ is a von Neumann algebra [35, Proposition 18.1].
2. Notice that if $U \in \mathcal{U}$ and $j \in J$, then $\lambda_{U^*,j} = \overline{\lambda_{U,j}}$. Indeed, for every v in $W_j \setminus \{0\}$

$$\lambda_{U,j} \|v\|_H^2 = \langle \lambda_{U,j} v, v\rangle_H = \langle Uv, v\rangle_H = \langle v, U^* v\rangle_H = \langle v, \lambda_{U^*,j} v\rangle_H = \overline{\lambda_{U^*,j}}\, \|v\|_H^2.$$

3. Let $S \in \mathcal{A}$, $j \in J$, $f \in W_j$. We are going to prove that $Sf \in W_j$. If $k \in J \setminus \{j\}$ and $g \in W_k$, then there exists U in $\mathcal{U}$ such that $\lambda_{U,j} \neq \lambda_{U,k}$, and

$$\lambda_{U,j} \langle Sf, g\rangle_H = \langle SUf, g\rangle_H = \langle USf, g\rangle_H = \langle Sf, U^* g\rangle_H = \lambda_{U,k} \langle Sf, g\rangle_H.$$

which implies that $\langle Sf, g\rangle_H = 0$. Since $H = \bigoplus_{k\in H} W_j$, the vector Sf expands into the series of the form $Sf = \sum_{q\in J} h_q$ with $h_k \in W_k$. For every k in $J \setminus \{j\}$,

$$0 = \langle Sf, h_k\rangle_H = \langle h_k, h_k\rangle_H + \sum_{q\in J\setminus\{k\}} \langle h_q, h_k\rangle_H = \|h_k\|_H^2.$$

Thus, $Sf = h_j \in W_j$.

4. Now suppose that $S \in \mathcal{B}(H)$ and $S(W_j) \subseteq W_j$ for every $j \in J$. Then for every U in $\mathcal{U}$, j in J, and g in W_j,

$$USg = U(Sg) = \lambda_{U,j} Sg = S(\lambda_{U,j} g) = SUg.$$

In general, if f in H, then $f = \sum_{j\in J} g_j$ with some g_j in W_j, and

$$USf = \sum_{j\in J} USg_j = \sum_{j\in J} SUg_j = SUf.$$

5. Using (32) we are going to prove that $\mathcal{A}$ is isometrically isomorphic to the direct sum $\bigoplus_{j\in J}\mathcal{B}(W_j)$. Given S in $\mathcal{A}$, for every j in J we denote by A_j the compression of S onto the invariant subspace W_j. Then the family $(A_j)_{j\in J}$ belongs to $\bigoplus_{d\in J}\mathcal{B}(W_d)$, and $\|S\| = \sup_{j\in J}\|A_j\|$.

   Conversely, given a bounded sequence $(A_j)_{j\in J}$ with A_j in $\mathcal{B}(W_j)$, we put

$$S\left(\sum_{j\in J} g_j\right) = \sum_{j\in J} A_j g_j \qquad (g_j \in W_j).$$

   Then $S(W_j) \subseteq W_j$ for every j in J, thus $S \in \mathcal{A}$. Thereby we have constructed isometrical isomorphisms between $\mathcal{A}$ and $\bigoplus_{j\in J}\mathcal{B}(W_j)$. □

Proposition 5.2 implies that the von Neumann algebra generated by $\mathcal{U}$ consists of all operators that act as scalar operators on each W_j, and can be naturally identified with $\bigoplus_{j\in J}\mathbb{C}I_{W_j}$.

Proposition 5.3 *Let H, $\mathcal{U}$, and $(W_j)_{j\in J}$ be like in Definition 5.1, and H_1 be a closed subspace of H invariant under $\mathcal{U}$. For every U in $\mathcal{U}$, denote by U_1 the compression of U onto the invariant subspace H_1, and put*

$$\mathcal{U}_1 := \{U_1\colon\ U \in \mathcal{U}\}, \qquad J_1 := \{j \in J\colon\ W_j \cap H_1 \neq \{0\}\}.$$

Then

$$H_1 = \bigoplus_{j\in J_1}(W_j \cap H_1), \tag{33}$$

and the family $(W_j \cap H_1)_{j\in J}$ diagonalizes $\mathcal{U}_1$.

Proof Denote by P_1 the orthogonal projection that acts in H and has image H_1. The condition that H_1 is invariant under $\mathcal{U}$ means that $P_1 \in \mathcal{A}$. By (32), for every j in J the subspace $P_1(W_j)$ is contained in W_j and therefore coincides with $W_j \cap H_1$. This easily implies (33).

If $U \in \mathcal{U}$ and $j \in J$, then $W_j \cap H_1 \subseteq \ker(\lambda_{U,j} I_{H_1} - U_1)$. So, the eigenvalues $\lambda_{U_1,j}$ coincide with $\lambda_{U,j}$ for every j in J_1.

If $j, k \in J_1$ and $j \neq k$, then there exists U in $\mathcal{U}$ such that $\lambda_{U,j} \neq \lambda_{U,k}$, which means that $\lambda_{U_1,j} \neq \lambda_{U_1,k}$. □

5.1 Radial Operators in $L^2(\mathbb{C}, \gamma)$

For each τ in $\mathbb{T}$, denote by R_τ the rotation operator acting in $L^2(\mathbb{C}, \gamma)$:

$$(R_\tau f)(z) = f(\tau^{-1} z). \tag{34}$$

The family $(R_\tau)_{\tau\in\mathbb{T}}$ is a unitary representation of the group $\mathbb{T}$ in $L^2(\mathbb{C},\gamma)$. Notice that we are in the situation of Sect. 4, with $\Omega=\mathbb{C}$, $\nu=\gamma$, $G=\mathbb{T}$, $\alpha(\tau)(z)=\tau z$, $\rho(\tau)=R_\tau$.

Denote by $\mathcal{R}$ the set of all radial operators acting in $L^2(\mathbb{C},\gamma)$:

$$\mathcal{R}=\{S\in\mathcal{B}(L^2(\mathbb{C},\gamma)):\quad \forall\tau\in\mathbb{T}\quad R_\tau S=SR_\tau\}.$$

Since the set $\{R_\tau:\ \tau\in\mathbb{T}\}$ is an autoadjoint subset of $\mathcal{B}(L^2(\mathbb{C},\gamma))$, its commutant $\mathcal{R}$ is a von Neumann algebra.

Lemma 5.4 *The family $(\mathcal{D}_d)_{d\in\mathbb{Z}}$ diagonalizes the collection $\{R_\tau:\ \tau\in\mathbb{T}\}$ in the sense of Definition 5.1.*

Proof If $\tau\in\mathbb{T}$ and $d\in\mathbb{Z}$, then

$$\mathcal{D}_d\subseteq\ker(\tau^{-d}I-R_\tau). \tag{35}$$

Indeed, for every $p,q\in\mathbb{Z}$ with $p-q=d$ the basic function $b_{p,q}$ is an eigenfunction of R_τ associated to the eigenvalue τ^{-d}:

$$R_\tau b_{p,q}=\tau^{q-p}b_{p,q}=\tau^{-d}b_{p,q}, \tag{36}$$

and by Corollary 2.6 the functions $b_{p,q}$ with $p-q=d$ form an orthonormal basis of $\mathcal{D}_d$. Another way to prove (35) is to use Corollary 2.7.

If $d_1,d_2\in\mathbb{Z}$ and $d_1\neq d_2$, then $\tau^{-d_1}\neq\tau^{-d_2}$ for many values of τ, for example, for $\tau=\mathrm{e}^{\frac{\mathrm{i}\pi}{d_1-d_2}}$ or for $\tau=\mathrm{e}^{2\pi\,\mathrm{i}\,\vartheta}$ with any irrational ϑ. □

Proposition 5.5 *The von Neumann algebra $\mathcal{R}$ consists of all operators that act invariantly on $\mathcal{D}_d$ for every d in $\mathbb{Z}$, and is isometrically isomorphic to $\bigoplus_{d\in\mathbb{Z}}\mathcal{B}(\mathcal{D}_d)$.*

Proof This is a consequence of Proposition 5.2 and Lemma 5.4. □

Now we will describe all radial operators of finite rank.

Remark 5.6 It is well known that every linear operator of finite rank m, acting in a Hilbert space H, can be written in the form

$$Sf=\sum_{k=1}^m \xi_k\langle f,u_k\rangle_H v_k, \tag{37}$$

where $\xi_1,\dots,\xi_m\in\mathbb{C}\setminus\{0\}$, $u_1,\dots,u_m$ and $v_1,\dots,v_m$ are some orthonormal lists of vectors in H.

Corollary 5.7 *Let $m\in\mathbb{N}$ and $S\in\mathcal{B}(L^2(\mathbb{C},\gamma))$ such that the rank of S is m. Then S is radial if and only if there exist $d_1,\dots,d_m$ in $\mathbb{Z}$ such that S has the form (37), where u_j, v_j, ξ_j are like in Remark 5.6, and additionally $u_j,v_j\in\mathcal{D}_{d_j}$ for every j in $\{1,\dots,m\}$.*

Proof This is a simple consequence of Proposition 5.5. Suppose that S is radial. For every d in $\mathbb{Z}$ let A_d be the compression of S to $\mathcal{D}_d$. There is only a finite set of d such that $A_d \neq 0$. Apply Remark 5.6 to each of the nonzero operators A_d and join the obtained decompositions. □

Following Zorboska [37], we will describe radial operators in term of the "radialization" $\operatorname{Rad}\colon \mathcal{B}(L^2(\mathbb{C},\gamma)) \to \mathcal{B}(L^2(\mathbb{C},\gamma))$ defined by

$$\operatorname{Rad}(S) := \int_{\mathbb{T}} R_\tau S R_{\tau^{-1}} \, \mathrm{d}\mu_{\mathbb{T}}(\tau),$$

where $\mu_{\mathbb{T}}$ is the normalized Haar measure on $\mathbb{T}$. The integral is understood in the weak sense, i.e. the operator $\operatorname{Rad}(S)$ is actually defined by the equality of the corresponding sesquilinear forms:

$$\langle \operatorname{Rad}(S) f, g\rangle = \int_{\mathbb{T}} \langle R_\tau S R_{\tau^{-1}} f, g\rangle \, \mathrm{d}\mu_{\mathbb{T}}(\tau).$$

Making an appropriate change of variables in the integral and using the invariance of the measure $\mu_{\mathbb{T}}$, we see that $\operatorname{Rad}(S) \in \mathcal{R}$. This immediately implies the following criterion of radial operators in terms of the radialization.

Proposition 5.8 *Let $S \in \mathcal{B}(L^2(\mathbb{C},\gamma))$. Then $S \in \mathcal{R}$ if and only if* $\operatorname{Rad}(S) = S$.

5.2 Radial Operators in $\mathcal{F}_n$

Let $n \in \mathbb{N}$. Obviously, the reproducing kernel of $\mathcal{F}_n$, given by (27), is invariant under simultaneous rotations in both arguments:

$$K_{n,\tau z}(\tau w) = K_{n,z}(w) \qquad (z, w \in \mathbb{C},\ \tau \in \mathbb{T}). \tag{38}$$

Therefore, by Proposition 4.1, $\mathcal{F}_n$ is invariant under rotations, and $P_n \in \mathcal{R}$. For every τ in $\mathbb{T}$, we denote by $R_{n,\tau}$ the compression of R_τ onto the space $\mathcal{F}_n$. In other words, the operator $R_{n,\tau}$ acts in $\mathcal{F}_n$ and is defined by (34). The family $(R_{n,\tau})_{\tau\in\mathbb{T}}$ is a unitary representation of $\mathbb{T}$ in $\mathcal{F}_n$. Let $\mathcal{R}_n$ be the von Neumann algebra of all bounded linear radial operators acting in $\mathcal{F}_n$.

Denote by $\mathfrak{M}_n$ the following direct sum of matrix algebras:

$$\mathfrak{M}_n := \bigoplus_{d=-n+1}^{\infty} \mathcal{M}_{\min\{n,n+d\}} = \left(\bigoplus_{d=-n+1}^{-1} \mathcal{M}_{n+d} \right) \oplus \left(\bigoplus_{d=0}^{\infty} \mathcal{M}_n \right).$$

The elements of $\mathfrak{M}_n$ are matrix sequences of the form $A = (A_d)_{d=-n+1}^{\infty}$, where $A_d \in \mathcal{M}_{n+d}$ if $d < 0$, $A_d \in \mathcal{M}_n$ if $d \geq 0$, and

$$\sup_{d \geq -n+1} \|A_d\| < +\infty.$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 By Propositions 5.2, 5.3 and formula (20), $\mathcal{R}_n$ is isometrically isomorphic to the direct sum of $\mathcal{B}(\mathcal{D}_{d,\min\{n,n+d\}})$, with $d \geq -n+1$. Using the orthonormal basis $(b_{d+k,k})_{k=\max\{0,-d\}}^{n-1}$ of the space $\mathcal{D}_{d,\min\{n,n+d\}}$, we represent linear operators on this space as matrices. Define $\Phi_n \colon \mathcal{R}_n \to \mathfrak{M}_n$ by

$$\Phi_n(S) = \left(\left[\langle S b_{d+k,k}, b_{d+j,j} \rangle \right]_{j,k=\max\{0,-d\}}^{n-1} \right)_{d=-n+1}^{\infty}. \tag{39}$$

Then Φ_n is an isometrical isomorphism. □

Similarly to Corollary 5.7, there is a simple description of radial operators of finite rank acting in $\mathcal{F}_n$. Of course, now $d_1, \ldots, d_m \geq -n+1$.

By Corollary 4.3, if $S \in \mathcal{R}_n$, then $\operatorname{Ber}_{\mathcal{F}_n}(S)$ is a radial function. For $n = 1$, the Berezin transform $\operatorname{Ber}_{\mathcal{F}_1}$ is injective. So, if $S \in \mathcal{B}(\mathcal{F}_1)$ and the function $\operatorname{Ber}_{\mathcal{F}_1}(S)$ is radial, then $S \in \mathcal{R}_1$. For $n \geq 2$, there are nonradial operators S with radial Berezin transforms.

Example 5.9 Let $n \geq 2$. Define u, v, and S like in the proof of Proposition 3.17. Then $\operatorname{Ber}(S)$ is the zero constant. In particular, $\operatorname{Ber}(S)$ is a radial function. On the other hand, $Sb_{0,0} = b_{1,0}$, the subspace $\mathcal{D}_0$ is not invariant under S, and thus S is not radial.

5.3 Radial Operators in $\mathcal{F}_{(n)}$

Let $n \in \mathbb{N}$. By Proposition 4.1 and formula (26), the subspace $\mathcal{F}_{(n)}$ is invariant under the rotations R_τ for all τ in $\mathbb{T}$. Denote the corresponding compression of R_τ by $R_{(n),\tau}$. Let $\mathcal{R}_{(n)}$ be the von Neumann algebra of all radial operators in $\mathcal{F}_{(n)}$.

Proof of Theorem 1.2 Corollaries 2.6 and 3.5 give

$$\mathcal{D}_d \cap \mathcal{F}_{(n)} = \begin{cases} \mathbb{C} b_{d+n-1,n-1}, & d \geq -n+1, \\ \{0\}, & d < -n+1. \end{cases} \tag{40}$$

By Propositions 5.2, 5.3 and formula (40), $\mathcal{R}_{(n)}$ consists of the operators that act invariantly on $\mathbb{C}b_{d+n-1,n-1}$, $d \geq -n+1$, i.e. are diagonal with respect to the basis $(b_{p,n-1})_{p=0}^{\infty}$. Therefore the function $\Phi_{(n)}\colon \mathcal{R}_{(n)} \to \ell^{\infty}(\mathbb{N}_0)$, defined by

$$\Phi_{(n)}(S) = \big(\langle Sb_{p,n-1}, b_{p,n-1}\rangle\big)_{p=0}^{\infty}, \tag{41}$$

is an isometric isomorphism. □

Similarly to Corollary 5.7, there is a simple description of radial operators of finite rank acting in $\mathcal{F}_{(n)}$.

6 Radial Toeplitz Operators in Polyanalytic Spaces

A measurable function $g\colon \mathbb{C} \to \mathbb{C}$ is called *radial* if for every τ in $\mathbb{T}$ the equality $g(\tau z) = g(z)$ is true for a.e. z in $\mathbb{C}$. If $g \in L^2(\mathbb{C}, \gamma)$, then this condition means that $R_\tau g = g$ for every τ in $\mathbb{T}$.

Given a function a in $L^{\infty}([0, +\infty))$, let $\widetilde{a}$ be its extension defined on $\mathbb{C}$ as

$$\widetilde{a}(z) := a(|z|) \qquad (z \in \mathbb{C}).$$

It is easy to see that a function g in $L^{\infty}(\mathbb{C})$ is radial if and only if there exists a in $L^{\infty}([0, +\infty))$ such that $g = \widetilde{a}$.

By Lemma 4.4, the multiplication operator $M_{\widetilde{a}}$, acting in $L^2(\mathbb{C}, \gamma)$, is radial. Let us compute the matrix of this operator with respect to the basis $(b_{p,q})_{p,q\in\mathbb{N}_0}$. Put

$$\beta_{a,d,j,k} := \langle \widetilde{a} b_{j+d,j}, b_{k+d,k}\rangle \qquad (d \in \mathbb{Z},\ \ j, k,\ j+d, k+d \in \mathbb{N}_0).$$

Passing to the polar coordinates and using (10) we get

$$\beta_{a,d,j,k} = \int_0^{+\infty} a(\sqrt{t})\, \ell_{\min\{j,j+d\}}^{(|d|)}(t) \ell_{\min\{k,k+d\}}^{(|d|)}(t)\, \mathrm{d}t. \tag{42}$$

Proposition 6.1 *Let $a \in L^{\infty}([0, +\infty))$. Then $M_{\widetilde{a}} \in \mathcal{R}$, and*

$$\langle M_{\widetilde{a}} b_{p,q}, b_{j,k}\rangle = \langle \widetilde{a} b_{p,q}, b_{j,k}\rangle = \delta_{p-q,j-k}\beta_{a,p-q,q,k}.$$

Proof Use the fact that $M_{\widetilde{a}}$ is radial and the orthogonality of the "diagonal subspaces". Then apply the definition of $\beta_{a,d,j,k}$. □

Denote $T_{n,g}$ the Toeplitz operator acting in F_n with generating symbol g and $T_{(n),g}$ the Toeplitz operator acting on $F_{(n)}$ with generating symbol g.

Proposition 6.2 *Let $g \in L^{\infty}(\mathbb{C})$. Then the opeator $T_{n,g}$ is radial if and only if the function g is radial.*

Proof Apply Proposition 3.18 and Corollaries 4.6, 4.7. □

Proposition 6.3 *Let* $a \in L^\infty([0,+\infty))$. *Then* $T_{(n),\widetilde{a}} \in \mathcal{R}_{(n)}$, *the operator* $T_{(n),\widetilde{a}}$ *is diagonal with respect to the orthonormal basis* $(b_{p,n-1})_{p=0}^{\infty}$, *and the sequence* $\lambda_{a,n}$ *of the corresponding eigenvalues can be computed by*

$$\lambda_{a,n}(p) = \beta_{a,p-n+1,n-1,n-1} = \int_0^{+\infty} a(\sqrt{t})\left(\ell_{\min\{p,n-1\}}^{(|p-n+1|)}(t)\right)^2 \mathrm{d}t \qquad (p \in \mathbb{N}_0). \tag{43}$$

Proof From Corollary 4.6 we get $T_{(n),\widetilde{a}} \in \mathcal{R}_{(n)}$. Due to Proposition 6.1 and Theorem 1.2,

$$\lambda_{a,n}(p) = (\Phi_{(n)}(T_{(n),\widetilde{a}}))_p = \langle T_{(n),\widetilde{a}} b_{p,n-1}, b_{p,n-1}\rangle = \beta_{a,p-n+1,n-1,n-1}. \qquad \square$$

Given a class $G \subseteq L^\infty(\mathbb{C})$ of generating symbols, we denote by $\mathcal{T}_{(n)}(G)$ the C*-subalgebra of $\mathcal{B}(\mathcal{F}_{(n)})$ generated by the set $\{T_{(n),g}\colon\ g \in G\}$. Let RB be the space of all radial bounded functions on $\mathbb{C}$, and RBC be the space of all radial bounded functions on $\mathbb{C}$ having a finite limit at infinity.

We are going to describe the algebra $\mathcal{T}_{(n)}(\mathrm{RBC})$.

Lemma 6.4 *Let* $m \in \mathbb{N}_0$ *and* $x > 0$. *Then*

$$\lim_{d\to\infty} \sup_{0\le t\le x} |\ell_m^{(d)}(t)| = 0.$$

Proof For each $t < x$, we write $\ell_m^{(d)}(t)$ explicitly by (9) and (4), then apply simple upper bounds:

$$\begin{aligned} |\ell_m^{(d)}(t)| &= \sqrt{\frac{m!}{(m+d)!}}\,|t^{\frac{d}{2}}\,\mathrm{e}^{-\frac{t}{2}}\,L_m^{(d)}(t)| \le \sqrt{\frac{m!}{(m+d)!}}\,\mathrm{e}^{-t/2}\sum_{j=0}^{m}\binom{m+d}{m-j}\frac{t^{j+\frac{d}{2}}}{j!} \\ &\le \sqrt{\frac{m!}{(m+d)!}}\sum_{j=0}^{m}\frac{(m+d)!}{(d+j)!}t^{j+\frac{d}{2}} \le (m+1)\sqrt{m!}\,\frac{(m+d)^m\,(1+t)^{m+\frac{d}{2}}}{\sqrt{(m+d)!}}. \end{aligned}$$

Then,

$$\sup_{0\le t\le x}|\ell_m^{(d)}(t)| \le \frac{\sqrt{m!}\,(m+1)\,(m+d)^m\,(1+x)^{m+\frac{d}{2}}}{\sqrt{(m+d)!}},$$

and the last expression tends to 0 as d tends to ∞. □

The following lemma and proposition are similar to [34, Lemma 7.2.3 and Theorem 7.2.4].

Lemma 6.5 *Let $a \in L^\infty([0,+\infty))$, $v \in \mathbb{C}$, and $\lim_{r\to+\infty} a(r) = v$. Then*

$$\lim_{d\to+\infty} \beta_{a,d,j,k} = \delta_{j,k} v \qquad (j,k \in \mathbb{N}_0). \tag{44}$$

In particular,

$$\lim_{p\to\infty} \lambda_{a,n}(p) = v \qquad (n \in \mathbb{N}). \tag{45}$$

Proof

1. First, suppose that $v = 0$ and $j = k$. For every $x > 0$ and $d \ge 0$,

$$|\beta_{a,d,j,j}| \le \int_0^x |a(\sqrt{t})| \left(\ell_j^{(d)}(t)\right)^2 \mathrm{d}t + \int_x^{+\infty} |a(\sqrt{t})| \left(\ell_j^{(d)}(t)\right)^2 \mathrm{d}t$$
$$\le x\|a\|_\infty \left(\sup_{0\le t\le x} |\ell_j^{(d)}(t)|\right)^2 + \sup_{t>x} |a(\sqrt{t})|.$$

Let $\varepsilon > 0$. Using the assumption that $a(r) \to 0$ as $r \to +\infty$, we choose x such that the second summand is less than $\varepsilon/2$. After that, applying Lemma 6.4 with this fixed x, we make the first summand less than $\varepsilon/2$.
2. If $v = 0$, $j,k \in \mathbb{N}_0$, then we obtain $\lim_{d\to+\infty} \beta_{a,d,j,k} = 0$ by applying the Schwarz inequality and the first part of this proof.
3. For a general v in $\mathbb{C}$, we rewrite a in the form $(a - v1_{(0,+\infty)}) + v1_{(0,+\infty)}$. Since

$$\beta_{1_{(0,+\infty)},d,j,k} = \int_0^{+\infty} \ell_j^{(d)}(t)\ell_k^{(d)}(t)\,\mathrm{d}t = \delta_{j,k},$$

the limit relation (44) follows from the second part of this proof. □

Proposition 6.6 *The C*-algebra $\mathcal{T}_{(n)}(\mathrm{RBC})$ is isometrically isomorphic to $c(\mathbb{N}_0)$.*

Proof Recall that $\Phi_{(n)}$ is an isometrical isomorphism $\mathcal{R}_{(n)} \to \ell^\infty(\mathbb{N}_0)$ defined by (41). By Proposition 6.3, $\Phi_{(n)}(\{T_b\colon\ b \in \mathrm{RBC}\}) = \mathfrak{L}$, where

$$\mathfrak{L} := \{\lambda_{a,n}\colon\ a \in L^\infty([0,+\infty)),\ \exists v \in \mathbb{C}\ \lim_{r\to+\infty} a(r) = v\}.$$

So, $\mathcal{T}_{(n)}(\mathrm{RBC})$ is isometrically isomorphic to the C*-subalgebra of $\ell^\infty(\mathbb{N}_0)$ generated by the set $\mathfrak{L}$. By Lemma 6.5, $\mathfrak{L} \subseteq c(\mathbb{N}_0)$. Our objective is to show that the C*-subalgebra of $c(\mathbb{N}_0)$ generated by $\mathfrak{L}$ coincides with $c(\mathbb{N}_0)$. The space $c(\mathbb{N}_0)$ may be viewed as the C*-algebra of the continuous functions on the compact $\mathbb{N}_0 \cup \{+\infty\}$. The set $\mathfrak{L}$ is a vector subspace of $c(\mathbb{N}_0)$ which contains the constants and is closed under the pointwise conjugation. In order to apply the Stone–Weierstrass theorem, we have to prove that the set $\mathfrak{L}$ separates the points of $\mathbb{N}_0 \cup \{+\infty\}$. For every u in $(0, +\infty]$, define a_u to be the characteristic function $1_{(0,u)}$. Then

$$\lambda_{a_u,n}(p) = \int_0^{u^2} \left(\ell_{\min\{p,n-1\}}^{(|p-n+1|)}(t)\right)^2 \mathrm{d}t.$$

Let $p, q \in \mathbb{N}_0$, $p \neq q$. If $\lambda_{a_u,n}(p) = \lambda_{a_u,n}(q)$ for all $u > 0$, then for all $t > 0$

$$\left(\ell_{\min\{p,n-1\}}^{(|p-n+1|)}(t)\right)^2 = \left(\ell_{\min\{q,n-1\}}^{(|q-n+1|)}(t)\right)^2,$$

which is not true. So, the set $\mathfrak{L}$ separates p and q.

Now let $p \in \mathbb{N}_0$ and $q = +\infty$. Put $u = 1$. Then $\lambda_{a_1,n}(p) > 0$, but $\lambda_{a_1,n}(+\infty) = \lim_{r\to+\infty} a_1(r) = 0$. So, the set $\mathfrak{L}$ separates p and $+\infty$. □

Recall that $\Phi_n : \mathcal{R}_n \to \mathfrak{M}_n$ is defined by (39).

Proposition 6.7 *Let $a \in L^\infty([0, +\infty))$. Then $T_{n,\widetilde{a}} \in \mathcal{R}_n$, and the d-th component of the sequence $\Phi_n(T_{n,\widetilde{a}})$ is the matrix*

$$\Phi_n(T_{n,\widetilde{a}})_d = \left[\beta_{a,d,j,k}\right]_{j,k=\max\{0,-d\}}^{n-1}.$$

Proof Apply Corollary 4.6 and Proposition 6.1. □

Let $\mathfrak{C}_n$ be the C*-subalgebra of $\mathfrak{M}_n$ that consists of all matrix sequences that have scalar limits:

$$\mathfrak{C}_n := \{A \in \mathfrak{M}_n : \quad \exists v \in \mathbb{C} \quad \lim_{d\to+\infty} A_d = v\, I_n\}.$$

Proposition 6.8 $\Phi_n(\mathcal{T}_n(\mathrm{RBC})) \subseteq \mathfrak{C}_n$.

Proof Follows from Lemma 6.5. □

We finish this section with a couple of conjectures.

Conjecture 6.9 The C*-algebra $\mathcal{T}_{(n)}(\mathrm{RB})$ is isometrically isomorphic to the C*-algebra of bounded square-root-oscillating sequences.

The concept of square-root-oscillating sequences and a proof of Conjecture 6.9 for $n = 1$ can be found in [10].

Conjecture 6.10 $\Phi_n(\mathcal{T}_n(\mathrm{RBC})) = \mathfrak{C}_n$.

Various results, similar to Conjecture 6.10, but for Toeplitz operators in other spaces of functions or with generating symbols invariant under other group actions, were proved by Loaiza, Lozano, Ramírez-Ortega, Sánchez-Nungaray, González-Flores, López-Martínez, and Arroyo-Neri [22, 23, 26, 29].

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References

1. L.D. Abreu, On the structure of Gabor and super Gabor spaces, *Monatsh. Math.* **161** (2010), 237–253, https://doi.org/10.1007/s00605-009-0177-0.
2. L.D. Abreu and H.G. Feichtinger, Function spaces of polyanalytic functions, *Harmonic and Complex Analysis and its Applications*, Birkhäuser, 2014, 1–38, https://doi.org/10.1007/978-3-319-01806-5_1.
3. A.T. Ali, F. Bagarello, and J.P. Gazeau, D-pseudo-bosons, complex Hermite polynomials, and integral quantization, *Symmetry Integr. Geom.* **11** (2015), 078, 23 pages, https://doi.org/10.3842/SIGMA.2015.078.
4. N. Askour, A. Intissar, and Z. Mouayn, Explicit formulas for reproducing kernels of generalized Bargmann spaces, *C. R. Acad. Sci. Paris, Ser. I* **325** (1997), 707–712, https://doi.org/10.1016/S0764-4442(97)80045-6.
5. M.B. Balk, *Polyanalytic Functions*, Akad.-Verl., 1991.
6. W. Bauer, C. Herrera Yañez, and N. Vasilevski, Eigenvalue characterization of radial operators on weighted Bergman spaces over the unit ball, *Integr. Equat. Oper. Th.* **78** (2014), 1–30, https://doi.org/10.1007/s00020-013-2101-1.
7. C.A. Berger, L.A. Coburn, Toeplitz operators and quantum mechanics, *J. Funct. Anal.* **68** (1986), 273–299, https://doi.org/10.1016/0022-1236(86)90099-6.
8. M. Dawson, G. Ólafsson, and R. Quiroga-Barranco, Commuting Toeplitz operators on bounded symmetric domains and multiplicity-free restrictions of holomorphic discrete series, *J. Funct. Anal.* **268** (2015), 1711–1732, https://doi.org/10.1016/j.jfa.2014.12.002.
9. M. Engliš, Berezin and Berezin-Toeplitz quantizations for general function spaces, *Rev. Mat. Complut.* **19** (2006), 385–430, http://eudml.org/doc/41908.
10. K. Esmeral and E. Maximenko, Radial Toeplitz operators on the Fock space and square-root-slowly oscillating sequences, *Complex Anal. Oper. Th.* **10** (2016), 1655–1677, https://doi.org/10.1007/s11785-016-0557-0.
11. S. Grudsky, R. Quiroga-Barranco, and N. Vasilevski, Commutative C*-algebras of Toeplitz operators and quantization on the unit disk, *J. Funct. Anal.* **234** (2006), 1–44, https://doi.org/10.1016/j.jfa.2005.11.015.
12. S.M. Grudsky, E.A. Maximenko, and N.L. Vasilevski, Radial Toeplitz operators on the unit ball and slowly oscillating sequences, *Commun. Math. Anal.* **14**:2 (2013), 77–94, https://projecteuclid.org/euclid.cma/1356039033.
13. S. Grudsky and N. Vasilevski, Toeplitz operators on the Fock space: Radial component effects, *Integr. Equat. Oper. Th.* **44** (2002), 10–37, https://doi.org/10.1007/BF01197858.
14. A. Haimi and H. Hedenmalm, The polyanalytic Ginibre ensembles, *J. Stat. Phys.* **153** (2013), 10–47, https://doi.org/10.1007/s10955-013-0813-x.
15. C. Herrera Yañez, N. Vasilevski, and E.A. Maximenko, Radial Toeplitz operators revisited: Discretization of the vertical case *Integr. Equat. Oper. Th.* **83** (2015), 49–60, https://doi.org/10.1007/s00020-014-2213-2.
16. O. Hutník, On the structure of the space of wavelet transforms, *C. R. Acad. Sci. Paris, Ser. I* **346** (2008), 649–652, https://doi.org/10.1016/j.crma.2008.04.013.
17. O. Hutník, A note on wavelet subspaces, *Monatsh. Math.* **160** (2010), 59–72, https://doi.org/10.1007/s00605-008-0084-9.

18. O. Hutník, E. Maximenko, and A. Mišková, Toeplitz localization operators: spectral functions density, *Complex Anal. Oper. Th.* **10** (2016), 1757–1774, https://doi.org/10.1007/s11785-016-0564-1.
19. O. Hutník and M. Hutníková, Toeplitz operators on poly-analytic spaces via time-scale analysis, *Oper. Matrices* **8** (2015), 1107–1129, https://doi.org/10.7153/oam-08-62.
20. B. Korenblum and K. Zhu, An application of Tauberian theorems to Toeplitz operators, *J. Oper. Th.* **33** (1995), 353–361, https://www.jstor.org/stable/24714916.
21. M. Loaiza and C. Lozano, On C*-algebras of Toeplitz operators on the harmonic Bergman space, *Integr. Equat. Oper. Th.* **76** (2013), 105–130, https://doi.org/10.1007/s00020-013-2046-4.
22. M. Loaiza and C. Lozano, On Toeplitz operators on the weighted harmonic Bergman space on the upper half-plane, *Complex Anal. Oper. Th.* **9** (2014), 139–165, https://doi.org/10.1007/s11785-014-0388-9.
23. M. Loaiza and J. Ramírez-Ortega, Toeplitz operators with homogeneous symbols acting on the poly-Bergman spaces of the upper half-plane, *Integr. Equat. Oper. Th.* **87** (2017), 391–410, https://doi.org/10.1007/s00020-017-2350-5.
24. R. Quiroga-Barranco, Separately radial and radial Toeplitz operators on the unit ball and representation theory, *Bol. Soc. Mat. Mex.* **22** (2016), 605–623 https://doi.org/10.1007/s40590-016-0111-0
25. A.K. Ramazanov, Representation of the space of polyanalytic functions as a direct sum of orthogonal subspaces. Application to rational approximations, *Math. Notes* **66** (1999), 613–627, https://doi.org/10.1007/BF02674203.
26. J. Ramírez Ortega and A. Sánchez-Nungaray, Toeplitz operators with vertical symbols acting on the poly-Bergman spaces of the upper half-plane, *Complex Anal. Oper. Th.* **9** (2015), 1801–1817, https://doi.org/10.1007/s11785-015-0469-4.
27. G. Rozenblum and N.L. Vasilevski, Toeplitz operators via sesquilinear forms, *Operator Theory: Adv. and Appl.* **262**, Birkhäuser, 2018, 287–304. https://doi.org/10.1007/978-3-319-62527-0_9.
28. S. Sakai, C^*-algebras and W^*-algebras, Springer-Verlag, 1971.
29. A. Sánchez-Nungaray, C. González-Flores, R.R. López-Martínez, and J.L. Arroyo-Neri, Toeplitz operators with horizontal symbols acting on the poly-Fock spaces, *J. Funct. Spaces* **2018** (2018), Article ID 8031259, 8 pages, https://doi.org/10.1155/2018/8031259.
30. I. Shigekawa, Eigenvalue problems for the Schrödinger operator with the magnetic field on a compact Riemannian manifold, *J. Funct. Anal.* **75** (1987), 92–127, https://doi.org/10.1016/0022-1236(87)90108-X.
31. K. Stroethoff, The Berezin transform and operators on spaces of analytic functions, *Banach Center Publ.* **38** (1997), 361–380, https://doi.org/10.4064/-38-1-361-380.
32. D. Suárez, The eigenvalues of limits of radial Toeplitz operators, *Bull. Lond. Math. Soc.* **40** (2008), 631–641, https://doi.org/10.1112/blms/bdn042.
33. N.L. Vasilevski, Poly-Fock spaces, *Operator Theory: Adv. and Appl.* **117**, Birkhäuser, 2000, 371–386, https://doi.org/10.1007/978-3-0348-8403-7_28.
34. N.L. Vasilevski, *Commutative Algebras of Toeplitz Operators on the Bergman Space*, Birkhäuser, 2008, https://doi.org/10.1007/978-3-7643-8726-6.
35. K. Zhu, *An Introduction to Operator Algebras*, CRC Press, 1993.
36. K. Zhu, *Analysis on Fock Spaces*, Springer, 2012, https://doi.org/10.1007/978-1-4419-8801-0.
37. N. Zorboska, The Berezin transform and radial operators, *Proc. Amer. Math. Soc.* **131** (2003), 793–800, https://www.jstor.org/stable/1194482.

Toeplitz C^*-Algebras on Boundary Orbits of Symmetric Domains

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This paper is dedicated to Nikolai Vasilevski on the occasion of his 70th birthday

Abstract We study Toeplitz operators on Hilbert spaces of holomorphic functions on symmetric domains, and more generally on certain algebraic subvarieties, determined by integration over boundary orbits of the underlying domain. The main result classifies the irreducible representations of the Toeplitz C^*-algebra generated by Toeplitz operators with continuous symbol. This relies on the limit behavior of "hypergeometric" measures under certain peaking functions.

Keywords Symmetric domain · Boundary orbit · Algebraic variety · Weighted Bergman spaces · Toeplitz operator · Subnormal operator · C^*-algebra

1991 Mathematics Subject Classification Primary 32M15, 46E22; Secondary 14M12, 17C36, 47B35

1 Introduction

Toeplitz operators and Toeplitz C^*-algebras on Hilbert spaces over bounded symmetric domains $\Omega = G/K$, for a semisimple Lie group G and a maximal compact subgroup K, are a deep and interesting part of multi-variable operator

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theory [22–24], closely related to harmonic analysis (holomorphic discrete series of representations of G) and index theory. In this paper we study Hilbert spaces over non-symmetric G-orbits contained in the **boundary** of Ω. These Hilbert spaces do not belong to the holomorphic discrete series, but the associated Toeplitz operators are still G-homogeneous in the sense of [17]. We study the C^*-algebra generated by these Toeplitz operators on boundary orbits and construct its irreducible representations, similar as in the symmetric case, via a refined analysis of the boundary faces of these orbits. The most interesting discovery is that for the boundary Toeplitz C^*-algebra, the irreducible representations do not always belong to boundary orbits, but comprise also some distinguished parameters in the discrete series (relative to the face).

Recently, certain algebraic varieties in symmetric domains, called **Jordan-Kepler varieties**, have been studied from various points of view [7, 24]. Although these varieties are not homogeneous, there exist natural K-invariant measures giving rise to Hilbert spaces of holomorphic functions and associated Toeplitz operators. In [25] the corresponding Toeplitz C^*-algebra and its representations have been investigated using asymptotic properties of hypergeometric functions. As a second main result of this paper, we combine both settings and treat Kepler-type varieties related to boundary orbits. The associated Toeplitz operators are subnormal, but the explicit description of the underlying boundary measure requires some effort. It seems that our setting is the natural level of generality, where methods of harmonic analysis based on Jordan algebraic concepts still yield a complete structure theory of Toeplitz C^*-algebras.

Compared to the paper [25], to which we frequently refer, the main new result concerns the description of the measures and inner product for the underlying Hilbert space, and the expression of the reproducing kernel in terms of generalized hypergeometric series. For boundary orbits this is not straightforward. Also, the concept of "hypergeometric measure" introduced in Sect. 4 serves to clarify and streamline the exposition, especially in the proof of Theorem 6.2.

2 Subnormal and Homogeneous Operator Tuples

To put the results of this paper in perspective, recall that a commuting n-tuple of operators $\mathbf{S} = (S_1, \ldots, S_n)$ is said to be **subnormal** if it is the restriction of a commuting tuple of normal operators $\mathbf{N}$, acting on a Hilbert space $\mathcal{H}$, to an invariant subspace $\mathcal{H}_0 \subset \mathcal{H}$. There are several intrinsic characterizations of subnormality; the one closest to the spirit of this paper is the following C^***-algebraic characterization**. Let $C^*[\mathbf{S}]$ be the C^*-algebra generated by $\{\mathrm{Id},\ S_1, \ldots, S_n\}$

Theorem 2.1 ([16, Theorem 2]) *A commuting n-tuple of operators* $\mathbf{S}$ *is subnormal if and only if for every subset* $\{T_I : I \in \mathcal{F}\}$ *of* $C^*[\mathbf{S}]$*,* $\mathcal{F}$ *finite, it follows that*

$$\sum_{I,J\in\mathcal{F}} T_I^* \mathbf{S}^{J^*} \mathbf{S}^I T_J \geq 0,$$

where $T_I = T_{i_1} \cdots T_{i_n}$ *and* $\mathbf{S}^I = S_1^{i_1} \cdots S_n^{i_n}$.

An immediate corollary is that if $\mathbf{S}$ is a subnormal commuting n-tuple and π is a $*$-representation of the C^*-algebra $C^*[S]$, then $\pi(\mathbf{S})$ is also subnormal. For $n = 1$, these results were obtained by Bunce and Deddens [5]. Natural examples of subnormal operators are obtained by restricting the multiplication by the coordinate functions on the Hilbert space $L^2(\Omega, m)$ to the subspace of holomorphic functions $H^2(\Omega, m)$, where $\Omega \subset \mathbf{C}^d$ is a bounded domain and m is a finite measure supported in the closure $\overline{\Omega}$ of Ω. Determining when a commuting tuple of operators is subnormal, in general, is not easy. For instance, let Ω be a bounded symmetric domain of genus p, and let B be the Bergman kernel of Ω. Then the set of positive real ν for which $B^{\nu/p}$ remains a positive definite kernel is known (cf. [8]) and is designated the **Wallach set** of Ω. For a fixed but arbitrary ν in the Wallach set, let $\mathcal{H}^{(\nu)}$ denote the Hilbert space determined by $B^{\nu/p}$. The biholomorphic functions of the domain Ω form a group, say G. Thus $g \in G$ acts on Ω via the map $(g, z) \mapsto g(z)$. This action lifts $(g \mapsto U_g,\ g \in G)$ to the Hilbert space $\mathcal{H}^\nu$:

$$\Big(U_{g^{-1}}^{(\nu)} f\Big)(z) = Jg(z)^{\nu/p}\Big(f(g(z))\Big),\ \ g \in G,\ z \in \Omega,\ f \in \mathcal{H}^{(\nu)},$$

where $Jg(z) := \det(Dg(z))$. It is easy to verify, using the transformation rule for the Bergman kernel, that U_g is unitary. The map $g \to U_g^{(\nu)}$ is not a homomorphism, in general, however $U_{gh}^{(\nu)} = c(g, h)U_g^{(\nu)} U_h^{(\nu)}$, where $c : G \times G \to \mathbf{T}$ is a Borel multiplier. Thus U defines a projective unitary representation of the group on $\mathcal{H}^{(\nu)}$.

The automorphism group G admits the structure of a Lie group. Consider the bounded symmetric domain Ω in its Harish-Chandra realization (cf. [11, Section 2.1]). The construction of the **discrete series representations** due to Harish-Chandra is well known, see [12, Theorem 6.6]. The (scalar holomorphic) discrete series representations (when realized as sections of homogeneous holomorphic line bundles) occur among the projective unitary representations $U^{(\nu)}$. Harish-Chandra had determined a cut-off ν_1 such that for all $\nu > \nu_1$, the representation $U^{(\nu)}$ is in the discrete series and the Hilbert space $\mathcal{H}^{(\nu)}$ is realized as the space $H^2(\Omega, dm_\nu)$, where $dm_\nu(z) = B(z, z)^{1-\nu/p}\, dv(z)$, clearly, $\operatorname{supp}(m) = \overline{\Omega}$. However, we also have the so-called limit discrete series representations and their analytic continuation. It is therefore natural to ask if there are other values of ν for which the inner product in the Hilbert space $\mathcal{H}^{(\nu)}$ is given by an integral with respect to a measure supported on possibly some other G-invariant closed subset of $\overline{\Omega}$. The answer to this question involves the G-invariant **boundary strata** of $\overline{\Omega}$ introduced below, namely, $\overline{\Omega}_{k,r}$, $1 \leq k \leq r$, where r is the rank of the bounded symmetric domain Ω. In this notation,

$\Omega_{r,r}$ is the Shilov boundary and $\Omega_{0,r} = \Omega$. For ν in $\{\nu_1, \dots, \nu_r\}$, where

$$\nu_i = \tfrac{d}{r} + \tfrac{a}{2}(r - i),$$

there exists a quasi-invariant measure

$$dm_i(gz) = |Jg(z)|^{\frac{2\nu_i}{p}} dm_i(z), \;\; z \in \Omega, \;\; \mathrm{supp}(m_i) = \overline{\Omega}_{i,r}, \;\; 1 \leq i \leq r,$$

such that $L^2(\Omega_{i,r}, dm_i)$ contains the representation space $\mathcal{H}^{(\nu)}$ as a closed subspace. (Here, with a slight abuse of notation, we let $\Omega_{0,r} = \overline{\Omega}$.) The representation $U^{(\nu)}$ lifts to $\widehat{U^{(\nu)}}$ on $L^2(\Omega_{i,r}, dm_i)$, again, as a multiplier representation, see [2, theorem 6.1]. The existence of the quasi-invariant measure (in the unbounded realization of G/K) is in [13, 20], see also [3, Lemma 5.1]. (The generalization to the case of vector valued holomorphic functions appears in [11, Theorem 4.49].) However, the fact that these are the only quasi-invariant measures with support in $\overline{\Omega}$ was proved for the domains Ω of type $I_{n,m}$, $m \geq n \geq 1$, in [3] and was extended to all bounded symmetric domains in [2]. Furthermore, it can be shown that these are the only commuting tuples of "homogeneous" subnormal operators in the Cowen-Douglas class of rank 1 on Ω.

Thus the commuting tuple $\mathbf{M}^{(\nu)} := (M_1^{(\nu)}, \dots, M_d^{(\nu)})$ of multiplication by the coordinate functions on the Hilbert space $\mathcal{H}^{(\nu)}$ is subnormal if and only if ν is in the set

$$W_{\text{sub}} := \{\nu : \; \nu = \tfrac{d}{r} + \tfrac{a}{2}(r - j), \; 1 \leq j \leq r\} \cup \{\nu : \nu > p - 1\}.$$

For ν as above, this is evident since the Hilbert space $\mathcal{H}^{(\nu)}$ is a closed subspace of the Hilbert space $L^2(dm_\nu)$ for some quasi-invariant measure m_ν. The converse is Theorem 3.1 of [3] for tube type domains and Theorem 5.1 of [2] in general.

The commuting tuple $\widehat{\mathbf{M}}$ of multiplication by the coordinate functions on the Hilbert space $L^2(dm_\nu)$ induces a $*$-homomorphism $\widehat{\Phi}_\nu : \mathcal{C}(\Omega_{i,r}) \to \mathcal{L}(L^2(dm_\nu))$, namely, $\widehat{\Phi}_\nu(f) = f(\widehat{\mathbf{M}})$, $f \in \mathcal{C}(\Omega_{i,r})$, the space of continuous functions on $\Omega_{i,r}$ and $\nu \in W_{\text{sub}}$. The quasi-invariance of the measure m_ν ensures that $\widehat{U}^{(\nu)}$ is unitary and therefore the triple $(L^2(dm_\nu), \widehat{U}^{(\nu)}, \widehat{\Phi}_\nu)$ is a **system of imprimitivity** in the sense of Mackey [26, chapter 6]:

$$(\widehat{U}_g^{(\nu)})^* \widehat{\Phi}_\nu \, \widehat{U}^{(\nu)} = g \cdot \widehat{\Phi}_\nu, \;\; g \in G, \tag{2.1}$$

where $((g \cdot \widehat{\Phi}_\nu) f)(z) = f(g \cdot z)$. Since the representation $\widehat{U}^{(\nu)}$ leaves the subspace $\mathcal{H}^{(\nu)}$ invariant as well, we see that

$$(\mathcal{H}^{(\nu)}, U^{(\nu)}, \Phi_\nu) = (L^2(dm_\nu), \widehat{U}_g^{(\nu)}, \widehat{\Phi}_\nu)_{|\mathcal{H}^{(\nu)}}, \;\; \nu \in W_{\text{sub}},$$

is the restriction of an imprimitivity.

Recall that the $*$-homomorphism $\widehat{\Phi}$ must be given by the formula $\widehat{\Phi}(f) = \widehat{\mathbf{M}}_f = f(\widehat{\mathbf{M}})$, $f \in \mathcal{C}(\Omega_{i,r})$, $0 \leq i \leq r$, via the usual functional calculus. The group G acts on the space of continuous functions via $(g^{-1} \cdot f)(z) = f(g \cdot z) = (f \circ g)(z)$. Therefore,

$$\widehat{\Phi}(g \cdot f) = \widehat{\mathbf{M}}_{f \circ g} = (f \circ g)(\widehat{\mathbf{M}}).$$

Choosing f to be the coordinate functions, we see that the imprimitivity condition (2.1) of Mackey is equivalent to the homogeneity of the commuting tuple $\mathbf{M}$, relative to the group G, of the commuting tuple $\widehat{\mathbf{M}}$, namely,

$$U_g \mathbf{M} U_g^* := (U_g M_1 U_g^*, \ldots, U_g^* M_d U_g) = g \cdot \mathbf{M}, \;\; g \in G, \tag{2.2}$$

where $g \cdot \mathbf{M} = (g_1(\mathbf{M}), \ldots, g_d(\mathbf{M}))$. Here g_i, $1 \leq i \leq d$, are the components of g in G, when it is thought of as an injective biholomorphic map on Ω. This notion for a single operator is from [17] and for a commuting tuple is from [18], see also [3, 4]. For ν in the Wallach set, the multiplication by the coordinate functions acting on the Hilbert space of holomorphic functions $\mathcal{H}^{(\nu)}$ are bounded if and only if $\nu \in (\frac{a}{2}(r-1), \infty)$, the continuous part of the Wallach set, see [2, Theorem 4.1] and [3, Theorem 1.1]. Since the kernel function of the Hilbert space $\mathcal{H}^{(\nu)}$ is a power of the Bergman kernel, it also transforms like the Bergman kernel ensuring that the operator $\mathbf{M}$ on this Hilbert space is G-homogeneous for all ν in the continuous part of the Wallach set. A simple computation involving the curvature shows that these are the only G-homogeneous operators in the Cowen-Douglas class $B_1(\Omega)$. The details are in [18] for the case of rank $r = 1$. The proofs in the general case can be obtained using [2, Proposition 4.4] and spectral mapping properties of the Taylor spectrum of the commuting tuple $\mathbf{M}$.

It is clearly of interest to study homogeneity, or equivalently, imprimitivity relative to **subgroups** of the group G. This already occurs in the study of spherically balanced tuples of operators [6, Definition 1.1]. In this case, the domain is the Euclidean unit ball $\mathbf{B}_d$ and the group is the maximal compact subgroup K of the automorphism group G of $\mathbf{B}_d$. The group K can be identified with the unitary group $U(d)$, it acts on $\mathbf{B}_d$ by the rule: $(U, z) \mapsto U(z)$, $z \in \mathbf{B}_d$, $U \in U(d)$. Let $\mathbf{T}$ be a commuting d-tuple of operators acting on a complex separable Hilbert space $\mathcal{H}$. The usual functional calculus gives

$$U \cdot \mathbf{T} = \Big(\sum_{j=1}^{d} U_{1j} T_j, \ldots, \sum_{j=1}^{d} U_{d,j} T_j\Big), \;\; U \in K.$$

The commuting d-tuple of operators $\mathbf{T}$ is said to be "spherically symmetric", or equivalently, K**-homogeneous** if $\Gamma_U^* \mathbf{T} \Gamma_U = U \cdot \mathbf{T}$ for each U in K and some unitary Γ_U on $\mathcal{H}$. In general, Γ need not be a unitary representation. However, we will assume that a choice of Γ_U exists such that the map $U \to \Gamma_U$ is a unitary homomorphism. What we have said about the Euclidean ball applies equally well to the case of a bounded symmetric domain. So, we speak freely of K-homogeneous operators, where $\Omega = G/K$. To describe this more general situation, we recall some basic notions from the representation theory of the group K.

Let $\boldsymbol{m} \in \mathbf{N}_+^r$ be a partition of length r. Let $\mathcal{P}_{\boldsymbol{m}}$ denote the space of irreducible K-invariant homogeneous polynomials of isotypic type $\boldsymbol{m}$, having total degree $|\boldsymbol{m}|$. These are mutually inequivalent as K-modules and $\mathcal{P} = \sum_{\boldsymbol{m}\in\mathbf{N}_+^r} \mathcal{P}_{\boldsymbol{m}}$ is the Peter-Weyl decomposition of the polynomials $\mathcal{P}$ under the action of the group K. Now, equip the submodules $\mathcal{P}_{\boldsymbol{m}}$ with the Fischer-Fock inner product $(p|q)_{\boldsymbol{m}} = (q^*(\partial)(p))(0)$, where $q^*(z) = \overline{q(\overline{z})}$. Let $E^{\boldsymbol{m}}$ be the reproducing kernel of the finite dimensional space $\mathcal{P}_{\boldsymbol{m}}$. Then the Faraut-Korányi formula for the reproducing kernel $K^{(\nu)}$ of the Hilbert space $\mathcal{H}^{(\nu)}$ is

$$K^{(\nu)} = \sum_{\boldsymbol{m}\in\mathbf{N}_+^r} (\nu)_{\boldsymbol{m}} E^{\boldsymbol{m}}, \tag{2.3}$$

where $(\nu)_{\boldsymbol{m}} := \prod_{j=1}^{r} (\nu - \frac{a}{2}(j-1))_{m_j}$ are the generalized Pochhammer symbols. We have pointed out that the commuting tuple of multiplication operators $\mathbf{M}$ on the Hilbert space $\mathcal{H}^{(\nu)}$ is G-homogeneous, therefore, it is also K-homogeneous. What are the other K-homogeneous operators? Since $\mathcal{P}_{\boldsymbol{m}}$ is a K irreducible module, it follows that the Hilbert space $\mathcal{H}^{(\boldsymbol{a})}$, obtained by setting $K^{(\boldsymbol{a})} = \sum_{\boldsymbol{m}\in\mathbf{N}_+^r} a_{\boldsymbol{m}} E^{\boldsymbol{m}}$ for an arbitrary choice of positive numbers $a_{\boldsymbol{m}}$ is a weighted direct sum of the K modules $\mathcal{P}_{\boldsymbol{m}}$. Hence the commuting tuple of multiplication operators $\mathbf{M}$ on $\mathcal{H}^{(\boldsymbol{a})}$ is K-homogeneous. It is shown in [10], under some additional hypothesis, that these are the only K-homogeneous operators.

If the rank $r = 1$, then a full description of all multi-shifts within the class of spherically symmetric operators is given in [6, Theorem 2.5]. In the present set-up, this characterization amounts to saying that a multi-shift on a Hilbert space $\mathcal{H}$ with reproducing kernel $K : \mathbf{B}_d \times \mathbf{B}_d \to \mathbf{C}$ is spherically symmetric if and only if the kernel is of the form

$$\sum_n a_n \langle \boldsymbol{z}, \boldsymbol{w} \rangle^n$$

for $\boldsymbol{z}, \boldsymbol{w} \in \mathbf{B}_d$. It then follows that several properties of the commuting tuple of multiplication operators $\mathbf{M}$ on the Hilbert space are determined by the ordinary shift with weight sequence $\left\{\left(\frac{a_n}{a_{n+1}}\right)^{1/2}\right\}$, $n \geq 0$, see [6, Theorem 5.1].

3 Spectral Varieties and Boundary Orbits

In this section we describe the Jordan theoretic background needed for the rest of the paper. For details, cf. [9, 15]. Let V be an irreducible hermitian Jordan triple of rank r. Every element $z \in V$ has a **spectral decomposition**

$$z = \sum_{i=1}^{r} \lambda_i c_i$$

where the singular values $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0$ are uniquely determined by z, and $c_1, \ldots, c_r$ is a frame of minimal orthogonal tripotents. The largest singular value $\|z\| := \lambda_1$ defines a (spectral) norm on V and the (open) unit ball

$$\Omega = \{z \in V : \|z\| < 1\}$$

is a bounded symmetric domain. It is a fundamental fact [15] that, conversely, every hermitian bounded symmetric domain can be realized, in an essentially unique way, as the spectral unit ball of a hermitian Jordan triple. In this paper we use the Jordan algebraic approach to study analysis on symmetric domains and related geometric structures.

The compact group K acts transitively on the set of frames. Hence, for fixed $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, the **level set**

$$V(\lambda) := \{z = \sum_{i=1}^{r} \lambda_i c_i : (c_i) \text{ frame}\} \tag{3.1}$$

is a compact K-orbit. As a special case we obtain the compact manifold

$$S_k := V(1^k, 0^{r-k})$$

of all **tripotents** of rank k, where $0 \leq k \leq r$. Every union of such level sets (3.1) is K-invariant but may be an orbit of a larger group. As an example, for $0 \leq \ell \leq r$, the **Jordan-Kepler manifold**

$$\mathring{V}_\ell = \bigcup_{\lambda_1 \geq \ldots \geq \lambda_\ell > 0} V(\lambda_1, \ldots, \lambda_\ell, 0^{r-\ell}),$$

consisting of all elements of rank ℓ, is a complex manifold which is an orbit under the complexified group $K^{\mathbf{C}}$. Its closure

$$V_\ell = \bigcup_{\lambda_1 \geq \cdots \geq \lambda_\ell \geq 0} V(\lambda_1, \ldots, \lambda_\ell, 0^{r-\ell}) = \bigcup_{0 \leq j \leq \ell} \mathring{V}_j$$

consists of all elements of rank $\leq \ell$ and is called the **Jordan-Kepler variety**. Its regular (smooth) part coincides with $\mathring{V}_\ell$. For $\ell = r$ we have $V_r = V$ and $\mathring{V}_r = \mathring{V}$ is an open dense subset, consisting of all elements of maximal rank. As another example the set

$$\Omega_{k,r} = \bigcup_{1>\lambda_{k+1}\geq\ldots\geq\lambda_r\geq 0} V(1^k, \lambda_{k+1}, \ldots, \lambda_r)$$

is an orbit under the identity component G of the biholomorphic automorphism group of Ω. For $k = 0$, we have $\Omega_{0,r} = \Omega$. For $k > 0$ we obtain a **boundary orbit** which is not a complex submanifold. It has the closure

$$\overline{\Omega}_{k,r} = \bigcup_{1\geq\lambda_{k+1}\geq\ldots\geq\lambda_r\geq 0} V(1^k, \lambda_{k+1}, \ldots, \lambda_r) = \bigcup_{i=k}^{r} \Omega_{i,r}.$$

The intersection

$$S_k = \mathring{V}_k \cap \Omega_{k,r}$$

is the common **center** of $\mathring{V}_k$ and $\Omega_{k,r}$. In particular, $S_0 = \{0\}$ is the center of Ω. The triple

$$\mathring{V}_k \supset S_k \subset \Omega_{k,r}$$

is a special case of **Matsuki duality**, which gives a 1–1 correspondence between G-orbits and $K^{\mathbf{C}}$-orbits in a flag manifold (which in our case is the so-called conformal hull of V), determined by the condition that the intersection is a K-orbit. For $k = r$ we obtain the **Shilov boundary**

$$\Omega_{r,r} = S_r =: S$$

which is the only closed stratum of $\partial\Omega$ and is its own center. Generalizing both the Jordan-Kepler varieties and the boundary orbits, we define for $0 \leq k \leq \ell \leq r$ the K-invariant set

$$\mathring{\Omega}_{k,\ell} := \mathring{V}_\ell \cap \Omega_{k,r} = \bigcup_{1>\lambda_{k+1}\geq\ldots\geq\lambda_\ell>0} V(1^k, \lambda_{k+1}, \ldots, \lambda_\ell, 0^{r-\ell}).$$

It has the closure

$$\overline{\Omega}_{k,\ell} = V_\ell \cap \overline{\Omega}_{k,r} = \bigcup_{1\geq\lambda_{k+1}\geq\ldots\geq\lambda_\ell\geq 0} V(1^k, \lambda_{k+1}, \ldots, \lambda_\ell, 0^{r-\ell}) = \bigcup_{k\leq i\leq j\leq\ell} \mathring{\Omega}_{i,j}.$$

We also use the 'partial closure'

$$\Omega_{k,\ell} := V_\ell \cap \Omega_k = \bigcup_{1>\lambda_{k+1}\geq\ldots\geq\lambda_\ell\geq 0} V(1^k, \lambda_{k+1}, \ldots, \lambda_\ell, 0^{r-\ell}) = \bigcup_{j=k}^{\ell} \mathring{\Omega}_{k,j}.$$

Then

$$\Omega_\ell := \Omega_{0,\ell} = V_\ell \cap \Omega$$

is the so-called **Kepler ball**.

Our first goal is to describe a **facial decomposition** of the K-invariant sets $\Omega_{k,\ell}$. For a tripotent c we consider the **Peirce decomposition** [14, 15]

$$V = V_2^c \oplus V_1^c \oplus V_0^c.$$

Define $V^c := V_0^c$ and $\Omega^c := \Omega \cap V^c$. This is itself a bounded symmetric domain of rank $r - k$, when $c \in S_k$.

Proposition 3.1 *There exist fibrations (disjoint union)*

$$\mathring{\Omega}_{k,\ell} = \bigcup_{c\in S_k} c + \mathring{\Omega}^c_{\ell-k} \subset \Omega_{k,\ell} = \bigcup_{c\in S_k} c + \Omega^c_{\ell-k} = \bigcup_{i=k}^{\ell} \mathring{\Omega}_{k,i} \tag{3.2}$$

Proof If $z \in \mathring{\Omega}_{k,\ell}$ then

$$z = c_1 + \ldots + c_k + \sum_{k<i\leq\ell} \lambda_i c_i$$

for some frame (c_i) and $1 > \lambda_{k+1} \geq \ldots \geq \lambda_\ell > 0$. It follows that $c := c_1 + \ldots + c_k \in S_k$ and

$$w := \sum_{k<i\leq\ell} \lambda_i c_i \in \Omega^c \cap \mathring{V}_{\ell-k} = \mathring{\Omega}^c_{\ell-k}.$$

For different tripotents $c, c' \in S_k$ the boundary components $c + \Omega^c$ and $c' + \Omega^{c'}$ are disjoint [15, Section 6]. This proves the first assertion. If $z \in \Omega_{k,\ell}$ then we require only $\lambda_\ell \geq 0$. Therefore

$$w \in \Omega^c \cap V_{\ell-k} = \Omega^c_{\ell-k} = \bigcup_{i=k}^{\ell} \mathring{\Omega}^c_{i-k}.$$

It follows that

$$\Omega_{k,\ell} = \bigcup_{c \in S_k} c + \Omega^c_{\ell-k} = \bigcup_{c \in S_k} \bigcup_{i=k}^{\ell} c + \mathring{\Omega}^c_{i-k} = \bigcup_{i=k}^{\ell} \bigcup_{c \in S_k} c + \mathring{\Omega}^c_{i-k} = \bigcup_{i=k}^{\ell} \mathring{\Omega}_{k,i}.$$

□

For $k \leq i \leq \ell$ the set $\mathring{\Omega}_{k,i}$ is called the **i-th stratum** of $\Omega_{k,\ell}$. In the special case $\ell = r$ we obtain a stratification

$$\Omega_{k,r} = \bigcup_{c \in S_k} c + \Omega^c_{r-k} = \bigcup_{i=k}^{r} \mathring{\Omega}_{k,i}$$

of the boundary G-orbit.

4 Hypergeometric Measures

If V is an irreducible hermitian Jordan triple of rank r, with automorphism group K, define the K-average

$$f^{\natural}(t) := \int_K dk\ f(kt)$$

for $t \in \mathbf{R}^r_{++} := \{t \in \mathbf{R}^r : t_1 \geq \ldots \geq t_r \geq 0\}$. Any K-invariant measure μ on V (or a K-invariant subset) has a **polar decomposition**

$$\int \mu(dz)\ f(z) = \int \tilde{\mu}(dt_1, \ldots, dt_r)\ f^{\natural}(\sqrt{t}_1, \ldots, \sqrt{t}_r)$$

for a uniquely defined measure $\tilde{\mu}$ on $\mathbf{R}^r_{++}$ (or a suitable subset), called the **radial part** of μ. In the following we use various unspecified constants, all of which are explicitly known.

Proposition 4.1 *The Lebesgue measure $dz =: \lambda_r(dz)$, for the normalized K-invariant inner product on V, has the radial part*

$$\tilde{\lambda}_r(dt_1, \ldots, dt_r) = \text{const.} \prod_{i=1}^{r} dt_i\ t_i^b \prod_{1 \leq i < j \leq r} (t_i - t_j)^a \tag{4.1}$$

on $\mathbf{R}^r_{++}$. Here a, b denote the so-called characteristic multiplicities of V [14, Section 17].

Proof We start with the well known formula

$$\int_X dx\ f(x) = \text{const.} \int_{\mathbf{R}_+^r} dt_1 \cdots dt_r \prod_{1\le i<j\le r} (t_i - t_j)^a \int_L dh\ f(ht) \tag{4.2}$$

for a Euclidean Jordan algebra X with automorphism group L [9, Theorem VI.2.3]. Let Λ_e be the symmetric cone of the Peirce 2-space V_2^e for some maximal tripotent $e \in S_r$ [9]. Then

$$\int_V dz\ f(z) = \text{const.} \int_{\Lambda_e} dx\ N_e(x)^b \int_K dk\ f(k\sqrt{x}) \tag{4.3}$$

by [9, Proposition X.3.4] (for the tube domain case $b = 0$) and [1, (2.1.1)] (for the general case). Applying (4.2) to the right hand side of (4.3) we obtain

$$\int_V dz\ f(z) = \text{const.} \int_{\mathbf{R}_{++}^r} \prod_{i=1}^r dt_i\ t_i^b \prod_{1\le i<j\le r} (t_i - t_j)^a\ f^\natural(\sqrt{t}).$$

□

Proposition 4.2 *For $\ell \le r$, consider the map*

$$\alpha : \mathbf{R}_{++}^\ell \to \mathbf{R}_{++}^r, \quad \alpha(t_1, \ldots, t_\ell) := (t_1, \ldots, t_\ell, 0^{r-\ell}).$$

Then the **Riemann measure** *λ_ℓ on the Kepler variety $\mathring{V}_\ell$, induced by the inner product $(z|w)$, has the radial part $\tilde{\lambda}_\ell = \alpha_* \hat{M}_\ell$, where*

$$\hat{M}_\ell(dt_1, \ldots, dt_\ell) := \text{const.} \prod_{i=1}^\ell dt_i\ t_i^{d_1^c/\ell} \prod_{1\le i<j\le\ell} (t_i - t_j)^a \tag{4.4}$$

and $d_1^c/\ell = b + a(r-\ell)$. If $\ell = r$, then $d_1^e = rb$ and (4.4) *reduces to* (4.1).

Proof By [7, Theorem 3.4] we have

$$\int_{\mathring{V}_\ell} \lambda_\ell(dz)\ f(z) = \text{const.} \int_{\Lambda_c^2} dx\ N_c(x)^{d_1^c/\ell}\ f^\natural(\sqrt{x})$$

$$= \text{const.} \int_{\mathbf{R}_{++}^\ell} \prod_{i=1}^\ell dt_i\ t_i^{d_1^c/\ell} \prod_{1\le i<j\le\ell} (t_i - t_j)^a\ f^\natural(\sqrt{t})$$

by applying (4.2) to the Peirce 2-space V_2^c and its positive cone Λ_c. □

Let $\mathcal{P}(V)$ denote the polynomial algebra of a hermitian Jordan triple V, endowed with the Fischer-Fock inner product $(p|q)_V$ for the normalized K-invariant inner product $(z|w)$ on V. Let

$$\mathcal{P}(V) = \sum_{\boldsymbol{m}} \mathcal{P}_{\boldsymbol{m}}(V)$$

be the **Peter-Weyl decomposition** of $\mathcal{P}(V)$ under the group K [8, Theorem 2.1]. Here $\boldsymbol{m}$ runs over the set $\mathbf{N}^r_+$ of all integer **partitions**

$$\boldsymbol{m} = (m_1 \geq \ldots \geq m_r)$$

of length $\leq r$. For a complex parameter ν let

$$(\nu)_{\boldsymbol{m}} = \prod_{j=1}^{r} (\nu - \frac{a}{2}(j-1))_{m_j}$$

denote the multivariate **Pochhammer symbol**. Then the identity

$$(\nu)_{\boldsymbol{m}+n} = (\nu + n)_{\boldsymbol{m}} \, (\nu)_n \tag{4.5}$$

holds for any integer $n \geq 0$.

Let $x_1, \ldots x_h, y_0, \ldots y_h$ be positive parameters. We say that a K-invariant measure μ supported on $\overline{\Omega}$ (or a K-invariant subset) is **hypergeometric of type** $\binom{y_0,\ldots,\, y_h}{x_1,\ldots,\, x_h}$ if

$$(p|q)_\mu := \int \mu(dz) \, \overline{p(z)} \, q(z) = \frac{\prod_{i=1}^{h} (x_i)_{\boldsymbol{m}}}{\prod_{i=0}^{h} (y_i)_{\boldsymbol{m}}} \, (p|q)_V \tag{4.6}$$

for all $\boldsymbol{m} \in \mathbf{N}^r_+$ and $p, q \in \mathcal{P}_{\boldsymbol{m}}(V)$. More generally, for $\ell \leq r$, a K-invariant measure μ supported on $\overline{\Omega}_\ell$ (or a K-invariant subset) is **ℓ-hypergeometric** if (4.6) holds for all partitions $\boldsymbol{m} \in \mathbf{N}^\ell_+$ of length $\leq \ell$. By the Stone-Weierstrass approximation theorem and K-invariance, the condition (4.6) determines the measure μ uniquely, but not every choice of parameters defines such a measure (a kind of multi-variate moment problem).

Let $\Delta(z, w)$ be the Jordan triple determinant [8].

Proposition 4.3 *Let $p := 2 + a(r-1) + b$ be the genus of Ω, and let $\nu > p-1$. Then the probability measure M_ν on Ω, defined by*

$$\int_\Omega M_\nu(dz)\, f(z) = \text{const.} \int_\Omega d\zeta\ \Delta(\zeta,\zeta)^{\nu-p}\ f(\zeta) \tag{4.7}$$

is hypergeometric of type $\binom{\nu}{}$.

Proof This follows from the Faraut-Korányi binomial formula (2.3) proved in [8]. □

Proposition 4.4 *For $1 \le k \le r$ let $p_k := 2 + a(r-k-1) + b$ be the genus for rank $r-k$, and put*

$$\nu_k := \frac{d}{r} + \frac{a}{2}(r-k) = p - 1 - \frac{a}{2}(k-1) = 1 + b + \frac{a}{2}(2r-k-1)$$
$$= p_k + \frac{a}{2}(k+1) - 1. \tag{4.8}$$

Then the probability measure $M_{k,r}$ on the k-th boundary orbit $\Omega_{k,r}$, defined in terms of the fibration (3.2) by

$$\int_{\Omega_{k,r}} M_{k,r}(dz)\, f(z) = \text{const.} \int_{S_k} dc \int_{\Omega^c} d\zeta\ \Delta(\zeta,\zeta)^{\nu_k - p_k}\ f(c+\zeta) \tag{4.9}$$

is hypergeometric of type $\binom{\nu_k}{}$.

Proof For the special case $a = 2$, corresponding to the matrix Jordan triple $V = \mathbf{C}^{r\times s}$, this is proved in [3] using combinatorial properties of Schur polynomials. The general case [1, Theorems 6.7 and 6.8] uses transformation properties under certain non-unimodular groups acting on the boundary. □

For the Shilov boundary $k = r$ $M_{r,r}(dz)$ is the unique K-invariant probability measure on $\Omega_{r,r} = S$, since $c + \Omega^c = \{c\}$ is a singleton for each $c \in S = S_r$. For $k = 0$ we have $\Omega_{0,r} = \Omega$ and $p_0 = p$. In this case (4.9) reduces to (4.7) for $\nu_0 = p - 1 + \frac{a}{2}$. However, in this case we may take any parameter $\nu > p-1$. Given a frame of minimal orthogonal tripotents $e_1, \ldots, e_r$ of V put

$$c_k := e_1 + \ldots + e_k.$$

Define

$$\mathbf{I}_+^r := \{s \in \mathbf{R}^r :\ 1 \ge s_1 \ge \ldots \ge s_r \ge 0\}.$$

The explicit realization (4.9) of $M_{k,r}$ implies the following proposition:

Proposition 4.5 *For $1 \le k \le r$ consider the map*

$$\beta : \mathbf{I}_+^{r-k} \to \mathbf{I}_+^r, \quad \beta(t_{k+1}, \ldots, t_r) := (1^k, t_{k+1}, \ldots, t_r).$$

Then the K-invariant measure $M_{k,r}$ on Ω_k has the radial part $\tilde{M}_{k,r} = \beta_* \tilde{M}_{\nu_k}^{c_k}$, where

$$\tilde{M}_{\nu_k}^{c_k}(dt_{k+1}, \ldots, dt_r) = \text{const.} \prod_{i=k+1}^{r} t_i^b (1-t_i)^{\nu_k - p_k}\, dt_i \prod_{k<i<j\le r} (t_i - t_j)^a \tag{4.10}$$

is the radial part, relative to the Peirce 0-space V^{c_k} of rank $r-k$, of the weighted Bergman measure $M_{\nu_k}^{c_k}$ for parameter ν_k. Thus

$$\begin{aligned}\int_{\Omega_{k,r}} M_{k,r}(dz)\ f(z) &= \int_{\mathbf{I}_+^r} \tilde{M}_{k,r}(dt_1, \ldots, dt_r)\ f^\natural(\sqrt{t_1}, \ldots, \sqrt{t_r}) \\ &= \int_{\mathbf{I}_+^r} (\beta_* \tilde{M}_{\nu_k}^{c_k})(dt_1, \ldots, dt_r)\ f^\natural(\sqrt{t_1}, \ldots, \sqrt{t_r}) \\ &= \int_{\mathbf{I}_+^{r-k}} \tilde{M}_{\nu_k}^{c_k}(dt_{k+1}, \ldots, dt_r)\ f^\natural(1^k, \sqrt{t_{k+1}}, \ldots, \sqrt{t_r})\end{aligned}$$

$$= \text{const.} \int_{\mathbf{I}_+^{r-k}} \prod_{i=k+1}^{r} t_i^b\ (1-t_i)^{\nu_k - p_k}\ dt_i \prod_{k<i<j\le r} (t_i - t_j)^a\ f^\natural(1^k, \sqrt{t_{k+1}}, \ldots, \sqrt{t_r}).$$

Now let $\ell \le r$. For $\nu > p - 1$ define the probability measure

$$M_{\nu,\ell}(dz) := \text{const.}\ \Delta(z,z)^{\nu - p}\ \lambda_\ell(dz) \tag{4.11}$$

on the Kepler ball Ω_ℓ. For $\ell = r$ we have $\Omega_r = \Omega$ and recover the "full" measure $M_{\nu,r} = M_\nu$. Finally, combining boundary orbits and Kepler varieties, we define the probability measure

$$\begin{aligned}\int_{\Omega_{k,\ell}} M_{k,\ell}(dz)\ f(z) &= \int_{S_k} dc \int_{\Omega_{\ell-k}^c} M_{\nu_k,\ell-k}^c(d\zeta)\ f(c+\zeta) \\ &= \text{const.} \int_{S_k} dc \int_{\Omega_{\ell-k}^c} \lambda_{\ell-k}^c(d\zeta)\ \Delta(\zeta,\zeta)^{\nu_k - p_k}\ f(c+\zeta)\end{aligned} \tag{4.12}$$

on $\Omega_{k,\ell}$, written in terms of the fibration (3.2). Here $\lambda^c_{\ell-k}$ is the Riemann measure on the 'little' Kepler ball $\Omega^c_{\ell-k} = \Omega^c \cap V_{\ell-k}$ induced by the hermitian metric $(z|w)$ restricted to V^c.

Consider the commuting diagram

$$\begin{array}{ccc} \mathbf{I}_+^{\ell-k} & \xrightarrow{\beta'} & \mathbf{I}_+^{\ell} \\ {\scriptstyle\alpha'}\downarrow & \searrow^{\gamma} & \downarrow{\scriptstyle\alpha} \\ \mathbf{I}_+^{r-k} & \xrightarrow[\beta]{} & \mathbf{I}_+^{r} \end{array}$$

where

$$\alpha'(t_{k+1},\ldots,t_\ell) := (1^k, t_{k+1},\ldots,t_\ell)$$

$$\beta'(t_{k+1},\ldots,t_\ell) := (t_{k+1},\ldots,t_\ell, 0^{r-\ell})$$

$$\gamma(t_{k+1},\ldots,t_\ell) = (1^k, t_{k+1},\ldots,t_\ell, 0^{r-\ell}).$$

Proposition 4.6 *The K-invariant measure $M_{k,\ell}$ on $\Omega_{k,\ell}$ has the radial part $\tilde{M}_{k,\ell} = \gamma_* \hat{M}_{k,\ell}$, for the measure*

$$\hat{M}_{k,\ell}(dt_{k+1},\ldots,dt_\ell) := \text{const.} \prod_{i=k+1}^{\ell} dt_i\ (1-t_i)^{\nu_k - p_k}\ t_i^{d_1^c/\ell} \prod_{k<i<j\le\ell} (t_i - t_j)^a \tag{4.13}$$

on $\mathbf{I}_+^{\ell-k}$. Thus

$$\int_{\Omega_{k,\ell}} M_{k,\ell}(dz)\ f(z) = \int_{\mathbf{I}_+^r} \tilde{M}_{k,\ell}(dt)\ f^\natural(\sqrt{t}) = \int_{\mathbf{I}_+^r} (\gamma_*\hat{M}_{k,\ell})(dt)\ f^\natural(\sqrt{t})$$

$$= \int_{\mathbf{I}_+^{\ell-k}} \hat{M}_{k,\ell}(dt_{k+1},\ldots,dt_\ell)\ f^\natural(1^k, \sqrt{t_{k+1}},\ldots,\sqrt{t_\ell}, 0^{r-\ell})$$

$$= \text{const.} \int_{\mathbf{I}_+^{\ell-k}} \prod_{i=k+1}^{\ell} dt_i\ (1-t_i)^{\nu_k-p_k}\ t_i^{d_1^c/\ell} \prod_{k<i<j\le\ell} (t_i-t_j)^a\ f^\natural(1^k, \sqrt{t_{k+1}},\ldots,\sqrt{t_\ell}, 0^{r-\ell})$$

Consider the Fischer-Fock kernel $E^{\boldsymbol{m}}(z,w) = E^{\boldsymbol{m}}_w(z)$ of $\mathcal{P}_{\boldsymbol{m}}(V)$. Then

$$(E^{\boldsymbol{m}}_z | E^{\boldsymbol{m}}_w)_V = E^{\boldsymbol{m}}(z,w).$$

Define $d_{\boldsymbol{m}} = \dim \mathcal{P}_{\boldsymbol{m}}(V)$.

Lemma 4.7 *For all $t \in \mathbf{I}_+^r$ and $w \in V$ we have*

$$(|E_w^{\boldsymbol{m}}|^2)^\natural(\sqrt{t}) = \frac{E^{\boldsymbol{m}}(w,w)}{d_{\boldsymbol{m}}}\ E_e^{\boldsymbol{m}}(t).$$

Proof Schur orthogonality implies

$$(|E_w^{\boldsymbol{m}}|^2)^\natural(\sqrt{t}) = \int_K dk\ |E^{\boldsymbol{m}}(k\sqrt{t},w)|^2 = \int_K dk\ |(E_{k\sqrt{t}}^{\boldsymbol{m}}|E_w^{\boldsymbol{m}})_V|^2$$

$$= \int_K dk\ |(k\cdot E_{\sqrt{t}}^{\boldsymbol{m}}|E_w^{\boldsymbol{m}})_V|^2 = \frac{\|E_w^{\boldsymbol{m}}\|_V^2\ \|E_{\sqrt{t}}^{\boldsymbol{m}}\|_V^2}{d_{\boldsymbol{m}}}$$

Since $\|E_w^{\boldsymbol{m}}\|_V^2 = E^{\boldsymbol{m}}(w,w)$ and $\|E_{\sqrt{t}}^{\boldsymbol{m}}\|_V^2 = E^{\boldsymbol{m}}(\sqrt{t},\sqrt{t}) = E^{\boldsymbol{m}}(t,e)$, the assertion follows. □

Proposition 4.8

$$\int_{\mathbf{I}_+^{r-k}} \prod_{i=k+1}^r t_i^b(1-t_i)^{\nu_k-p_k}\ dt_i \prod_{k<i<j\le r}(t_i-t_j)^a\ E_e^{\boldsymbol{m}}(1^k,t_{k+1},\ldots,t_r) = \frac{d_{\boldsymbol{m}}}{(\nu_k)_{\boldsymbol{m}}}. \tag{4.14}$$

Proof From (4.10) it follows that

$$\int_{\mathbf{I}_+^{r-k}} \prod_{i=k+1}^r t_i^b(1-t_i)^{\nu_k-p_k}\ dt_i \prod_{k<i<j\le r}(t_i-t_j)^a\ E_e^{\boldsymbol{m}}(1^k,t_{k+1},\ldots,t_r)$$

$$= \int_{\mathbf{I}_+^r} \tilde{M}_{k,r}(dt)\ E^{\boldsymbol{m}}(t,e)$$

$$= \frac{d_{\boldsymbol{m}}}{E^{\boldsymbol{m}}(e,e)}\int_{\mathbf{I}_+^r} \tilde{M}_{k,r}(dt)\ (|E_e^{\boldsymbol{m}}|^2)^\natural(\sqrt{t}) = \frac{d_{\boldsymbol{m}}}{E^{\boldsymbol{m}}(e,e)}\int_{\Omega_k} M_{k,r}(dz)\ |E_e^{\boldsymbol{m}}(z)|^2$$

$$= \frac{d_{\boldsymbol{m}}}{\|E_e^{\boldsymbol{m}}\|_V^2}\ \|E_e^{\boldsymbol{m}}\|_{\nu_k}^2 = \frac{d_{\boldsymbol{m}}}{(\nu_k)_{\boldsymbol{m}}}.$$

□

Remark 4.9 In the special case $V = \mathbf{C}^{r\times s}$ the polynomials $E_e^{\boldsymbol{m}}$ are proportional to the Schur polynomials, and the identity (4.14) was shown directly in [3]. A direct proof of (4.14) in the general case would be of interest.

The following theorem is our first main result.

Theorem 4.10 *For* $1 \le k \le \ell \le r$ *the probability measure* $M_{k,\ell}$ *on* $\Omega_{k,\ell}$ *is* ℓ-*hypergeometric of type* $\binom{\frac{d}{r},\ r\frac{a}{2},\ \nu_k}{\ell\frac{a}{2},\ \nu_\ell}$.

Proof Let $c = c_\ell$. Put $h := d_1^c/\ell = b + a(r-\ell)$. Applying (4.14) to the Jordan triple V_2^c (of tube type) we obtain for $\boldsymbol{m} \in \mathbf{N}_+^\ell$, putting $d_{\boldsymbol{m}}^c = \dim \mathcal{P}_{\boldsymbol{m}}(V_2^c)$,

$$\text{const.} \int_{\mathbf{I}_+^{\ell-k}} \prod_{i=k+1}^{\ell} (1-t_i)^{\nu_k - p_k}\, dt_i \prod_{k<i<j\le\ell} (t_i - t_j)^a\, E_{c_\ell}^{\boldsymbol{m}}(1^k, t_{k+1}, \dots, t_\ell)$$

$$= \frac{d_{\boldsymbol{m}}^c}{(1 + \frac{a}{2}(2\ell - k - 1))_{\boldsymbol{m}}} = \frac{d_{\boldsymbol{m}}^c}{(\nu_k - h)_{\boldsymbol{m}}}$$

since

$$1 + \frac{a}{2}(2\ell - k - 1) + h = 1 + \frac{a}{2}(2\ell - k - 1) + b + a(r-\ell) = 1 + b + \frac{a}{2}(2 \mathring{-} k - 1) = \nu_k$$

For $z \in V_2^c$ we have $E_c^{\boldsymbol{m}}(z) = E^{\boldsymbol{m}}(c,c)\ \Phi_{\boldsymbol{m}}^c(z)$, where $\Phi_{\boldsymbol{m}}^c \in \mathcal{P}_{\boldsymbol{m}}(V_2^c)$ is the spherical polynomial normalized by $\Phi_{\boldsymbol{m}}^c(c) = 1$. Therefore

$$N_c(z)^h\ E_c^{\boldsymbol{m}}(z) = E^{\boldsymbol{m}}(c,c)\ N_c(z)^h\ \Phi_{\boldsymbol{m}}^c(z) = E^{\boldsymbol{m}}(c,c)\ \Phi_{\boldsymbol{m}+h}^c(z) = \frac{E^{\boldsymbol{m}}(c,c)}{E^{\boldsymbol{m}+h}(c,c)}\ E_c^{\boldsymbol{m}+h}(z).$$

We have

$$E^{\boldsymbol{m}}(c,c) = \frac{d_{\boldsymbol{m}}^c}{(1 + \frac{a}{2}(\ell - 1))_{\boldsymbol{m}}}$$

and, similarly,

$$E^{\boldsymbol{m}+h}(c,c) = \frac{d_{\boldsymbol{m}+h}^c}{(1 + \frac{a}{2}(\ell - 1))_{\boldsymbol{m}+h}} = \frac{d_{\boldsymbol{m}}^c}{(\nu_\ell - h)_{\boldsymbol{m}+h}},$$

since

$$1 + \frac{a}{2}(\ell - 1) + h = 1 + \frac{a}{2}(\ell - 1) + b + a(r-\ell) = 1 + b + \frac{a}{2}(2 \mathring{-} l - 1) = \nu_\ell.$$

It follows that

$$\prod_{i=k+1}^{\ell} t_i^h\ (|E_c^{\boldsymbol{m}}|^2)^\natural(1^k, \sqrt{t_{k+1}}, \dots, \sqrt{t_\ell}, 0^{r-\ell})$$

$$= \frac{E^{\boldsymbol{m}}(c,c)}{d_{\boldsymbol{m}}} N_c(1^k, t_{k+1}, \dots, t_\ell)^h\ E_c^{\boldsymbol{m}}(1^k, t_{k+1}, \dots, t_\ell)$$

$$= \frac{E^{\boldsymbol{m}}(c,c)}{d_{\boldsymbol{m}}} \frac{E^{\boldsymbol{m}}(c,c)}{E^{\boldsymbol{m}+h}(c,c)} E_c^{\boldsymbol{m}+h}(1^k, t_{k+1}, \ldots, t_\ell)$$
$$= \frac{E^{\boldsymbol{m}}(c,c)}{d_{\boldsymbol{m}}} \frac{(\nu_\ell - h)_{\boldsymbol{m}+h}}{(\nu_\ell - h)_{\boldsymbol{m}}} E_c^{\boldsymbol{m}+h}(1^k, t_{k+1}, \ldots, t_\ell).$$

Applying (4.14) to $\boldsymbol{m} + h \in \mathbf{N}_+^\ell$ we obtain

$$\frac{1}{\text{const.}} \| E_c^{\boldsymbol{m}} \|^2_{\nu_k,\ell} = \frac{1}{\text{const.}} \int\limits_{\Omega_{k,\ell}} M_{k,\ell}(dz)\, |E_c^{\boldsymbol{m}}(z)|^2$$

$$= \int\limits_{\mathbf{I}_+^{\ell-k}} \prod_{i=k+1}^{\ell} t_i^h (1-t_i)^{\nu_k - p_k}\, dt_i \prod_{k<i<j\le\ell} (t_i - t_j)^a \, (|E_c^{\boldsymbol{m}}|^2)^\natural (1^k, \sqrt{t_{k+1}}, \ldots, \sqrt{t_\ell}, 0^{r-\ell})$$

$$= \frac{E^{\boldsymbol{m}}(c,c)}{d_{\boldsymbol{m}}} \frac{(\nu_\ell - h)_{\boldsymbol{m}+h}}{(\nu_\ell - h)_{\boldsymbol{m}}} \int\limits_{\mathbf{I}_+^{\ell-k}} \prod_{i=k+1}^{\ell} (1-t_i)^{\nu_k - p_k}\, dt_i$$
$$\prod_{k<i<j\le\ell} (t_i - t_j)^a \, E_c^{\boldsymbol{m}+h}(1^k, t_{k+1}, \ldots, t_\ell)$$

$$= \frac{E^{\boldsymbol{m}}(c,c)}{d_{\boldsymbol{m}}} \frac{(\nu_\ell - h)_{\boldsymbol{m}+h}}{(\nu_\ell - h)_{\boldsymbol{m}}} \frac{d^c_{\boldsymbol{m}+h}}{(\nu_k - h)_{\boldsymbol{m}+h}}$$
$$= \frac{E^{\boldsymbol{m}}(c,c)}{(\nu_k - h)_{\boldsymbol{m}+h}} \frac{(\nu_\ell - h)_{\boldsymbol{m}+h}}{(\nu_\ell - h)_{\boldsymbol{m}}} \frac{(a\ell/2)_{\boldsymbol{m}}}{(ar/2)_{\boldsymbol{m}}} \frac{(\nu_\ell - h)_{\boldsymbol{m}}}{(d/r)_{\boldsymbol{m}}}$$

using the identity

$$\frac{d^c_{\boldsymbol{m}+h}}{d_{\boldsymbol{m}}} = \frac{d^c_{\boldsymbol{m}}}{d_{\boldsymbol{m}}} = \frac{(a\ell/2)_{\boldsymbol{m}}}{(ar/2)_{\boldsymbol{m}}} \frac{(1 + \frac{a}{2}(\ell-1))_{\boldsymbol{m}}}{(d/r)_{\boldsymbol{m}}} = \frac{(a\ell/2)_{\boldsymbol{m}}}{(ar/2)_{\boldsymbol{m}}} \frac{(\nu_\ell - h)_{\boldsymbol{m}}}{(d/r)_{\boldsymbol{m}}}$$

as computed in the proof of [7, Theorem 5.1]. Simplifying and using (4.5) we finally obtain

$$\| E_c^{\boldsymbol{m}} \|^2_{k,\ell} = E^{\boldsymbol{m}}(c,c) \frac{(\nu_\ell)_{\boldsymbol{m}}}{(\nu_k)_{\boldsymbol{m}}\, (d/r)_{\boldsymbol{m}}} \frac{(a\ell/2)_{\boldsymbol{m}}}{(ar/2)_{\boldsymbol{m}}}$$

since $M_{k,\ell}$ is a probability measure. It follows that for $\boldsymbol{m} \in \mathbf{N}_+^\ell$ and $p, q \in \mathcal{P}_{\boldsymbol{m}}(V)$ we have

$$(p|q)_{k,\ell} := \int\limits_{\Omega_{k,\ell}} M_{k,\ell}(dz)\, \overline{p(z)}\, q(z) = (p|q)_V \frac{(\nu_\ell)_{\boldsymbol{m}}}{(\nu_k)_{\boldsymbol{m}}\, (d/r)_{\boldsymbol{m}}} \frac{(a\ell/2)_{\boldsymbol{m}}}{(ar/2)_{\boldsymbol{m}}}.$$

□

5 Holomorphic Function Spaces and Toeplitz Operators

We now define Hilbert spaces of holomorphic functions and Toeplitz type operators associated with hypergeometric measures of rank $\ell \leq r$, keeping in mind the examples $M_{k,\ell}$ on $\overline{\Omega}_{k,\ell}$ constructed above. For $\ell \leq r$ define

$$\mathcal{P}^\ell(V) = \sum_{\boldsymbol{m} \in \mathbf{N}_+^\ell} \mathcal{P}_{\boldsymbol{m}}(V),$$

involving only partitions of length $\leq \ell$. Then the restriction map $p \mapsto p|_{V_\ell}$ is injective and yields a linear isomorphism between $\mathcal{P}^\ell(V)$ and the regular functions on the Kepler variety V_ℓ. For a K-invariant ℓ-hypergeometric measure μ on $\overline{\Omega}_{k,\ell}$ let $\mathcal{H}_{\mu,\ell}$ denote the Hilbert space of all holomorphic functions on the Kepler ball Ω_ℓ which are square-integrable under the measure μ. This is the completion of $\mathcal{P}^\ell(V)$, restricted to Ω_ℓ, for the measure μ.

This general definition covers all classical examples. Consider first the "full" case $\ell = r$. For a discrete series Wallach parameter $\nu > p-1$, the **weighted Bergman space** $\mathcal{H}_\nu$ consists of all holomorphic functions on Ω which are square-integrable under the measure M_ν. For $1 \leq k \leq r$ the **embedded Wallach parameters** ν_k defined in (4.8) belong to the continuous Wallach set

$$\nu > \frac{a}{2}(r-1) \tag{5.1}$$

but not to the discrete series since $k \geq 1$ implies $\nu_k \leq 1+b+\frac{a}{2}(2r-2) = p-1$. The associated **Hardy type spaces** $\mathcal{H}_{k,r}$ consist of all holomorphic functions on Ω which are square-integrable under the measure $M_{k,r}$. Then $\nu_r = \frac{d}{r}$ is the "true" Hardy space parameter, corresponding to the Shilov boundary $S = \Omega_{r,r}$. The left endpoint $\nu_1 = p-1$ of the holomorphic discrete series corresponds to the probability measure $M_{1,r}$ on the dense open boundary orbit $\Omega_{1,r}$. As explained in Sect. 3, the parameters ν_k are of special importance for subnormal G-homogeneous Toeplitz operators. By Propositions 4.3 and 4.4, these measures are of hypergeometric type.

Now consider the "partial" case $\ell \leq r$. If $\nu > p-1$, the **partial weighted Bergman space** $\mathcal{H}_{\nu,\ell}$ consists of all holomorphic functions on the Kepler ball Ω_ℓ which are square-integrable for the probability measure $M_{\nu,\ell}$. The inner product is

$$(\phi|\psi)_{\nu,\ell} := \int_{\Omega_\ell} M_{\nu,\ell}(dz)\, \overline{\phi(z)}\, \psi(z) = \text{const.} \int_{\Omega_\ell} \lambda_\ell(dz)\, \Delta(z,z)^{\nu-p}\, \overline{\phi(z)}\, \psi(z).$$

For $\ell = r$ we have $\Omega_r = \Omega$ and $M_{\nu,r} = M_\nu$. Thus we recover the 'full' weighted Bergman space $\mathcal{H}_{\nu,r} = \mathcal{H}_\nu$. For $1 \leq k \leq \ell \leq r$, the **partial Hardy type space** $\mathcal{H}_{k,\ell}$ consists of all holomorphic functions on the Kepler ball Ω_ℓ which are square-integrable for the probability measure $M_{k,\ell}$. The inner product is

$$(\phi|\psi)_{k,\ell} := \int\limits_{\Omega_{k,\ell}} M_{k,\ell}(dz)\, \overline{\phi(z)}\, \psi(z) = \int\limits_{S_k} dc \int\limits_{\Omega^c_{\ell-k}} \lambda^c_{\ell-k}(d\zeta)\, \Delta(\zeta,\zeta)^{\nu_k - p_k}\, (\overline{\phi}\psi)(c+\zeta).$$

Putting $\ell = r$ we recover the inner product (4.14) since $\Omega^c_{r-k} = \Omega^c$ and $M^c_{r-k}(d\zeta) = d\zeta$ is the Lebesgue measure on V^c. For $k=0$ we have $c=0,\ V^0 = V,\ \Omega^0_\ell = \Omega_\ell = \Omega \cap V_\ell,\ M^0_\ell = M_\ell$ and $p_0 = p$. Thus we recover the M_ℓ-inner product.

In summary, we obtain examples of type $\binom{\frac{d}{r},\, r\frac{a}{2},\, \nu_k}{\ell\frac{a}{2},\, \nu_\ell}$ for $0 \le k \le \ell \le r$. For fixed ℓ we have as special cases the partial weighted Bergman spaces of type $\binom{\frac{d}{r},\, r\frac{a}{2},\, \nu}{\ell\frac{a}{2},\, \nu_\ell}$, corresponding to $k=0$, and the partial Hardy space of type $\binom{\frac{d}{r},\, r\frac{a}{2}}{\nu_\ell}$ corresponding to maximal $k=\ell$. For $\ell = r$ we obtain the full type $\binom{\nu_k}{}$, since $\nu_r = \frac{d}{r}$, specializing to the full weighted Bergman spaces of type $\binom{\nu}{}$ if $k=0$ and the full Hardy space of type $\binom{\frac{d}{r}}{}$ if $k=r$. It would be interesting to construct natural examples of more complicated hypergeometric type.

We now introduce Toeplitz operators in our setting. For the 'full' Hilbert space $\mathcal{H}_\mu$ over Ω we denote by $P_\mu : L^2(\overline{\Omega}, \mu) \to \mathcal{H}_\mu$ the orthogonal projection and define the "full" Toeplitz operator $T_\mu(f)$, with symbol function $f \in L^\infty(\overline{\Omega})$, by

$$T_\mu(f) = P_\mu\ f\ P_\mu.$$

Restricting to continuous symbols we obtain the "full" Toeplitz C^*-algebra

$$\mathcal{T}_\mu = C^*(T_\mu(f) :\ f \in \mathcal{C}(\overline{\Omega})).$$

As special cases, we obtain the "full" Bergman-Toeplitz operators $T_{\nu,r}(f)$ ($\nu > p-1$) and the "full" Hardy type Toeplitz operators $T_{k,r}(f)$ ($1 \le k \le r$) associated with the hypergeometric measures $M_{\nu,r}$ on Ω and $M_{k,r}$ on $\Omega_{k,r}$, respectively. The corresponding Toeplitz C^*-algebras are denoted by $\mathcal{T}_{\nu,r}$ and $\mathcal{T}_{k,r}$, respectively.

In the more general setting of the "partial" Hilbert space $\mathcal{H}_{\mu,\ell}$ over Ω_ℓ, associated with a K-invariant ℓ-hypergeometric measure μ ($\ell \le r$), denote by $P_{\mu,\ell} : L^2(\overline{\Omega}_\ell, \mu) \to \mathcal{H}_{\mu,\ell}$ the orthogonal projection and define the "partial" Toeplitz operator $T_{\mu,\ell}(f)$, with symbol function $f \in L^\infty(\overline{\Omega}_\ell)$, by

$$T_{\mu,\ell}(f) = P_{\mu,\ell}\ f\ P_{\mu,\ell}.$$

Restricting to continuous symbols we obtain the "partial" Toeplitz C^*-algebra

$$\mathcal{T}_{\mu,\ell} = C^*(T_{\mu,\ell}(f) :\ f \in \mathcal{C}(\overline{\Omega}_\ell)).$$

As special cases, we obtain the “partial” Bergman-Toeplitz operators $T_{\nu,\ell}(f)$ ($\nu > p-1$) and the “partial” Hardy type Toeplitz operators $T_{k,\ell}(f)$ ($1 \le k \le \ell$) associated with the ℓ-hypergeometric measures $M_{\nu,\ell}$ on Ω_ℓ and $M_{k,\ell}$ on $\Omega_{k,\ell}$, respectively. The corresponding Toeplitz C^*-algebras are denoted by $\mathcal{T}_{\nu,\ell}$ and $\mathcal{T}_{k,\ell}$, respectively.

Lemma 5.1 *Let $p, q \in \mathcal{P}(V)$. Then the Toeplitz type operators satisfy*

$$T_{\mu,\ell}(p)\, T_{\mu,\ell}(q) = T_{\mu,\ell}(pq).$$

Proof Since $\mathcal{P}^\ell(V)^\perp$ is an ideal in $\mathcal{P}(V)$ it follows that

$$T_{\mu,\ell}(pq)\phi = P_{\mu,\ell}(pq\phi) = P_{\mu,\ell}(p(P_{\mu,\ell} + P^\perp_{\mu,\ell})(q\phi))$$

$$= P_{\mu,\ell}(p\ P_{\mu,\ell}(q\phi)) + P_{\mu,\ell}(p\ P^\perp_{\mu,\ell}(q\phi)) = P_{\mu,\ell}(p\ T_{\mu,\ell}(q)\phi) = T_{\mu,\ell}(p)(T_{\mu,\ell}(q)\phi).$$

□

It follows that $\mathcal{T}_{\mu,\ell}$ is generated by Toeplitz type operators with linear symbols and their adjoints.

Remark 5.2 A standard reproducing kernel argument (carried out in [25, Proposition 4.2]) shows, at least for the ‘concrete’ hypergeometric measures described above (where the support is connected), that the C^*-algebra $\mathcal{T}_{\mu,\ell}$ acts **irreducibly** on $\mathcal{H}_{\mu,\ell}$.

For any $v \in V$ let

$$v^*(z) := (z|v)$$

denote the associated linear form. Its conjugate is $\overline{v}^*(z) = \overline{(z|v)} = (v|z)$. Let $\partial_v p(z) := p'(z)v$ denote the directional derivative. Put

$$\varepsilon_j := (0, \ldots, 0, 1, 0, \ldots, 0)$$

with 1 at the j-th place. It is shown in [22, Corollary 2.10] that

$$v^* p \in \sum_{j=1}^r \mathcal{P}_{\boldsymbol{m}+\varepsilon_j}(V), \quad \partial_v p \in \sum_{j=1}^r \mathcal{P}_{\boldsymbol{m}-\varepsilon_j}(V) \tag{5.2}$$

for all $p \in \mathcal{P}_{\boldsymbol{m}}(V)$, with zero-component if $\boldsymbol{m} \pm \varepsilon_j$ is not a partition. Let $q \mapsto q_{\boldsymbol{m}} \in \mathcal{P}_{\boldsymbol{m}}(V)$ denote the $\boldsymbol{m}$-th isotypic projection.

The next result determines the fine structure of the adjoint Toeplitz type operator $T_{\mu,\ell}(v^*)^* = T_{\mu,\ell}(\overline{v}^*)$.

Proposition 5.3 *Let μ be a ℓ-hypergeometric measure on $\overline{\Omega}_\ell$. Let $v \in V$. Then*

$$T_{\mu,\ell}(\overline{v}^*)p = \sum_{j=1}^{\ell} \frac{\prod_{i=1}^{h}(x_i - \frac{a}{2}(j-1) + m_j - 1)}{\prod_{i=0}^{h}(y_i - \frac{a}{2}(j-1) + m_j - 1)} (\partial_v p)_{\boldsymbol{m}-\varepsilon_j}$$

for all $\boldsymbol{m} \in \mathbf{N}_+^\ell$ and $p \in \mathcal{P}_{\boldsymbol{m}}(V)$.

Proof Let $q \in \mathcal{P}_{\mathbf{n}}(V),\ \mathbf{n} \in \mathbf{N}_+^\ell$, satisfy $(T_{\mu,\ell}(\overline{v}^*)p|q)_{\mu,\ell} \neq 0$. Then

$$(p|v^*q)_{\mu,\ell} = (T_{\mu,\ell}(\overline{v}^*)p|q)_{\mu,\ell} \neq 0.$$

With (5.2) it follows that $\boldsymbol{m} = \mathbf{n} + \varepsilon_j$ for some $j \leq \ell$ and hence $\mathbf{n} = \boldsymbol{m} - \varepsilon_j$. Since μ is ℓ-hypergeometric, it follows that

$$(T_{\mu,\ell}(\overline{v}^*)p|q)_\mu = (p|v^*q)_\mu = \frac{\prod_{i=1}^{h}(x_i)_{\boldsymbol{m}}}{\prod_{i=0}^{h}(y_i)_{\boldsymbol{m}}} (p|v^*q)_V = \frac{\prod_{i=1}^{h}(x_i)_{\boldsymbol{m}}}{\prod_{i=0}^{h}(y_i)_{\boldsymbol{m}}} (\partial_v p|q)_V$$

$$= \frac{\prod_{i=1}^{h}(x_i)_{\boldsymbol{m}}/(x_i)_{\boldsymbol{m}-\varepsilon_j}}{\prod_{i=0}^{h}(y_i)_{\boldsymbol{m}}/(y_i)_{\boldsymbol{m}-\varepsilon_j}} (\partial_v p|q)_\mu.$$

Since q is arbitrary, it follows that

$$T_\mu^\ell(\overline{v}^*)p = \sum_{j=1}^{\ell} \frac{\prod_{i=1}^{h}(x_i)_{\boldsymbol{m}}/(x_i)_{\boldsymbol{m}-\varepsilon_j}}{\prod_{i=0}^{h}(y_i)_{\boldsymbol{m}}/(y_i)_{\boldsymbol{m}-\varepsilon_j}} (\partial_v p)_{\boldsymbol{m}-\varepsilon_j}.$$

Now the assertion follows from

$$\frac{(\lambda)_{\boldsymbol{m}}}{(\lambda)_{\boldsymbol{m}-\varepsilon_j}} = \frac{(\lambda - \frac{a}{2}(j-1))_{m_j}}{(\lambda - \frac{a}{2}(j-1))_{m_j-1}} = \lambda - \frac{a}{2}(j-1) + m_j - 1.$$

□

6 Limit Measures

The basic result concerning Toeplitz C^*-algebras on bounded symmetric domains states that every irreducible representation is realized on a unique boundary component Ω^c, for any tripotent c. This was carried out in full detail for the Hardy space in [22, 23] and its generalization to weighted Bergman spaces was described in [24]. Here a crucial step, which was indicated in [24] and proved in detail in the recent paper [25], is the limit behavior of the underlying measures under certain **peaking functions**. In the present paper, this crucial result will be generalized to the boundary orbits $\Omega_{k,\ell}$, and their intersection with Kepler varieties. This is not completely straightforward, since the assignment $f^{(c)}(\zeta) := f(c+\zeta)$ is not compatible with the Peter-Weyl decomposition of $\mathcal{P}(V)$.

Let $c \in S_i$ with $i \le \ell$. Since $V_2^c = P_2^c V$ has rank $i \le \ell$ and $(z|c)^n = (P_2^c z|c)^n$, where P_2^c denotes the Peirce 2-projection, it follows that

$$(z|c)^n \in \mathcal{P}(V_2^c) \subset \mathcal{P}^i(V) \subset \mathcal{P}^\ell(V).$$

Restricting (injectively) to Ω_ℓ, the holomorphic function

$$H_c(z) := \exp(z|c) = \sum_{n=0}^{\infty} \frac{(z|c)^n}{n!} \tag{6.1}$$

on Ω_ℓ can be regarded as an element of the Hilbert completion $\mathcal{H}_{\mu,\ell}$ of $\mathcal{P}^\ell(V)$ under μ. This applies in particular to $i = 1$.

Let $0 \le i \le \ell \le r$ and $c \in S_i$. Then $c + \overline{\Omega}^c \subset \overline{\Omega}$. For functions $f \in \mathcal{C}(\overline{\Omega}_\ell)$ we define $f^{(c)} \in \mathcal{C}(\overline{\Omega}^c_{\ell-i})$ by

$$f^{(c)}(\zeta) := f(c+\zeta) \quad (\zeta \in \overline{\Omega}^c_{\ell-i}). \tag{6.2}$$

Lemma 6.1 *Let μ be an ℓ-hypergeometric measure on $\overline{\Omega}_\ell$. Let $0 \le i \le \ell$ and $c \in S_i$. Then*

$$\lim_{n\to\infty} \int\limits_{\overline{\Omega}_\ell} \mu(dz)\, \frac{|H_c^n(z)|^2}{\|H_c^n\|_\mu^2}\, f(z) = 0$$

for all $f \in \mathcal{C}(\overline{\Omega}_\ell)$ satisfying $f^{(c)} = 0$.

Proof By assumption, for every $\varepsilon > 0$ there is an open neighborhood $U \subset \overline{\Omega}_\ell$ of $c + \Omega^c_{\ell-i}$ satisfying $\sup |f(U)| \le \varepsilon$. By [15, Lemma 6.2] we have $|(z|c)| < (c|c)$ for all $z \in \overline{\Omega} \setminus \overline{\Omega}^c$. Peirce orthogonality implies $(z|c) = (c|c)$ for all $z \in c + \overline{\Omega}^c$. Therefore $|H_c| < H_c(c)$ on $\overline{\Omega}_\ell \setminus U$, and a compactness argument shows that there exists an open neighborhood $V \subset U \subset \overline{\Omega}_\ell$ of $c + \overline{\Omega}^c_{\ell-i}$ such that

$$q := \frac{\sup_{\overline{\Omega}_\ell \setminus U} |H_c|}{\inf_V |H_c|} < 1.$$

Therefore

$$\int\limits_{\overline{\Omega}_\ell} \mu(dz) \, \frac{|H_c^n(z)|^2}{\|H_c^n\|_\mu^2} \, f(z) = \int\limits_U \mu(dz) \, \frac{|H_c^n(z)|^2}{\|H_c^n\|_\mu^2} \, f(z) + \int\limits_{\overline{\Omega}_\ell \setminus U} \mu(dz) \, \frac{|H_c^n(z)|^2}{\|H_c^n\|_\mu^2} \, f(z)$$

$$\leq \sup_U |f| + \sup_{\overline{\Omega}_\ell} |f| \cdot \frac{\int\limits_{\overline{\Omega}_\ell \setminus U} \mu(dz) \, |H_c^n(z)|^2}{\int\limits_V \mu(dz) \, |H_c^n(z)|^2} \leq \varepsilon + \sup_{\overline{\Omega}_\ell} |f| \cdot q^{2n} \, \frac{\mathrm{Vol}_\mu(\overline{\Omega}_\ell \setminus U)}{\mathrm{Vol}_\mu(V)}.$$

Since $q^{2n} \to 0$ it follows that

$$\limsup_{n \to \infty} \int\limits_{\overline{\Omega}_\ell} \mu(dz) \, \frac{|H_c^n(z)|^2}{\|H_c^n\|_\mu^2} \, f(z) \leq \varepsilon.$$

□

Now consider the special case $i = 1$. For $c = e_1 \in S_1$, let $\alpha := (\alpha_1, \ldots, \alpha_{\ell-1}) \in \mathbf{N}_+^{\ell-1}$ be a partition of length $\ell - 1$. Define

$$\alpha^+ := (\alpha_1, \alpha) \in \mathbf{N}_+^\ell \tag{6.3}$$

and consider the conical function

$$N_{\alpha^+} = N_2^{\alpha_1 - \alpha_2} \, N_3^{\alpha_2 - \alpha_3} \cdots N_\ell^{\alpha_{\ell-1}},$$

where $N_1, \ldots, N_r$ are the Jordan theoretic minors [21]. Then the conical function N_α^c relative to V^c for the partition α satisfies

$$N_{\alpha^+}^{(c)} = N_\alpha^c.$$

The asymptotic expansion of generalized hypergeometric series

$${}_pF_q\,(z) = \sum_{n=0}^\infty \frac{\prod\limits_{r=1}^p \Gamma(n + \beta_r)}{\prod\limits_{r=1}^q \Gamma(n + \mu_r)} \, \frac{z^n}{n!} \tag{6.4}$$

in one variable z has been determined in [27]. Put $\kappa := 1 + q - p$ and

$$\vartheta := \frac{q-p}{2} + \beta_1 + \ldots + \beta_p - \mu_1 - \ldots - \mu_q.$$

As a special case $M = 1$ of [27, Theorem 1], using [27, Lemma 1], one obtains

$$\lim_{x \to +\infty} X^{-\vartheta}\, e^{-X}\, {}_pF_q\,(x) = A_0 = (2\pi)^{(p-q)/2}\, \kappa^{\frac{1}{2}-\vartheta},$$

where $X := \kappa\, x^{1/\kappa}$. If $q = p + 1$, this simplifies to $\kappa = 2$, $X = 2\sqrt{x}$ and $A_0 = (2\pi)^{-1/2}\, 2^{\frac{1}{2}-\vartheta} = \pi^{-1/2}\, 2^{-\vartheta}$. Therefore

$$\lim_{x \to \infty} x^{-\vartheta/2}\, e^{-2\sqrt{x}}\, {}_pF_q\,(x) = \frac{1}{\sqrt{\pi}}. \tag{6.5}$$

Theorem 6.2 *Let μ be a K-invariant ℓ-hypergeometric probability measure of type $\binom{y_0,\ldots,y_h}{x_1;\ldots,x_h}$ on $\overline{\Omega}_\ell$. Then for each $c \in S_1$ there exists a unique K^c-invariant $(\ell-1)$-hypergeometric probability measure $\mu^{(c)}$ of type $\binom{y_0-\frac{a}{2},\ldots,y_h-\frac{a}{2}}{x_1-\frac{a}{2},\ldots,x_h-\frac{a}{2}}$ on $\overline{\Omega}^c_{\ell-1}$ such that for all continuous functions f we have*

$$\lim_{n \to \infty} \int\limits_{\overline{\Omega}_\ell} \mu(dz)\, \frac{|H^n_c(z)|^2}{\|H^n_c\|^2_\mu}\, f(z) = \int\limits_{\overline{\Omega}^c_{\ell-1}} \mu^{(c)}(d\zeta)\;\; f^{(c)}(\zeta). \tag{6.6}$$

Proof By K-invariance, we may assume that $c = e_1$. By Lemma 6.1 each weak cluster point μ' of the sequence of probability measures on the left of (6.6) is supported on the closure $\overline{\Omega}^c_{\ell-1}$ and is invariant under K^c. Thus it suffices to compute the μ'-inner product for α-homogeneous polynomials on V^c, where $\alpha \in \mathbf{N}^{\ell-1}_+$ is arbitrary. By irreducibility, it is enough to consider the conical functions N^c_α relative to V^c. Defining $\alpha^+ \in \mathbf{N}^\ell_+$ as in (6.3), we consider for any $s \in \mathbf{N}$ the conical function

$$(z|e_1)^s\; N_{\alpha^+} = N^s_1\; N_{\alpha^+} = N_{\boldsymbol{m}},$$

where $\boldsymbol{m} = (m_1, \alpha_1, \ldots, \alpha_{\ell-1}, 0^{r-\ell})$ and $m_1 = s + \alpha_1$. In the proof of [25, Theorem 5.5] it was shown that the respective Fock inner products are related by

$$\begin{aligned}\frac{\|N_{\boldsymbol{m}}\|^2_V}{\|N^c_\alpha\|^2_{V^c}} &= \frac{(1+\frac{a}{2}(\ell-1))_{\boldsymbol{m}}}{(1+\frac{a}{2}(\ell-2))_\alpha} \prod_{1 \le j < \ell} \frac{(1+\frac{a}{2}(j-1))_{m_1-\alpha_j}}{(1+\frac{a}{2}j)_{m_1-\alpha_j}} \\ &= (1+\frac{a}{2}(\ell-1))_{m_1} \prod_{1 \le j < \ell} \frac{(1+\frac{a}{2}(j-1))_{m_1-\alpha_j}}{(1+\frac{a}{2}j)_{m_1-\alpha_j}}.\end{aligned}$$

For any $\lambda \in \mathbf{C}$ we have

$$\frac{(\lambda)_{\boldsymbol{m}}}{(\lambda - \frac{a}{2})_\alpha} = (\lambda)_{m_1} \prod_{1<j\le\ell} \frac{(\lambda - \frac{a}{2}(j-1))_{m_j}}{(\lambda - \frac{a}{2} - \frac{a}{2}(j-2))_{\alpha_{j-1}}} = (\lambda)_{m_1}.$$

It follows that

$$\frac{\|N_{\boldsymbol{m}}\|_\mu^2}{\|N_\alpha^c\|_{V^c}^2} = \frac{\|N_{\boldsymbol{m}}\|_V^2}{\|N_\alpha^c\|_{V^c}^2} \frac{\prod_{i=1}^h (x_i)_{\boldsymbol{m}}}{\prod_{i=0}^h (y_i)_{\boldsymbol{m}}}$$

$$= \frac{\prod_{i=1}^h (x_i)_{\boldsymbol{m}}}{\prod_{i=0}^h (y_i)_{\boldsymbol{m}}} (1 + \frac{a}{2}(\ell-1))_{m_1} \prod_{1\le j<\ell} \frac{(1+\frac{a}{2}(j-1))_{m_1-\alpha_j}}{(1+\frac{a}{2}j)_{m_1-\alpha_j}}$$

$$= \frac{\prod_{i=1}^h (x_i - \frac{a}{2})_\alpha}{\prod_{i=0}^h (y_i - \frac{a}{2})_\alpha} \frac{(1+\frac{a}{2}(\ell-1))_{m_1} \prod_{i=1}^h (x_i)_{m_1}}{\prod_{i=0}^h (y_i)_{m_1}} \prod_{1\le j<\ell} \frac{(1+\frac{a}{2}(j-1))_{m_1-\alpha_j}}{(1+\frac{a}{2}j)_{m_1-\alpha_j}}$$

$$= A \frac{\prod_{i=1}^h (x_i - \frac{a}{2})_\alpha}{\prod_{i=0}^h (y_i - \frac{a}{2})_\alpha} B(m_1),$$

where A is independent of α and s, and

$$B(t) := \frac{\Gamma(t+1+\frac{a}{2}(\ell-1)) \prod_{i=1}^h \Gamma(t+x_i)}{\prod_{i=0}^h \Gamma(t+y_i)} \prod_{1\le j<\ell} \frac{\Gamma(t+1+\frac{a}{2}(j-1)-\alpha_j)}{\Gamma(t+1+\frac{a}{2}j-\alpha_j)}.$$

For $(e^{(z|e_1)})^n = e^{n(z|e_1)}$ we obtain by orthogonality

$$\frac{1}{\|N_\alpha^c\|_{V^c}^2} \int_{\overline{\Omega}^\ell} \mu(dz)\, |e^{(z|e_1)}|^{2n}\, |N_{\alpha^+}(z)|^2$$

$$= \sum_{s\ge 0} \frac{n^{2s}}{(s!)^2} \frac{1}{\|N_\alpha^c\|_{V^c}^2} \int_{\overline{\Omega}^\ell} \mu(dz)\, |(z|e_1)|^{2s}\, |N_{\alpha^+}(z)|^2$$

$$= \sum_{s\geq 0} \frac{n^{2s}}{(s!)^2} \frac{\|N_{\boldsymbol{m}}\|^2_\mu}{\|N^c_\alpha\|^2_{V^c}} = A\, \frac{\prod_{i=1}^h (x_i - \frac{a}{2})_\alpha}{\prod_{i=0}^h (y_i - \frac{a}{2})_\alpha} \sum_{s\geq 0} \frac{n^{2s}}{(s!)^2} B(\alpha_1 + s) = A\, \frac{\prod_{i=1}^h (x_i - \frac{a}{2})_\alpha}{\prod_{i=0}^h (y_i - \frac{a}{2})_\alpha}\, F_\alpha(n^2),$$

where $F_\alpha(X)$ is a hypergeometric series in the sense of (6.4), with parameters

$$\alpha_1+x_1, \ldots, \alpha_1+x_h,\ \alpha_1+1+\frac{a}{2}(\ell-1),\ \alpha_1-\alpha_2+1+\frac{a}{2}, \ldots,\ \alpha_1-\alpha_{\ell-1}+1+\frac{a}{2}(\ell-2)$$

in the numerator and

$$\alpha_1 + y_0, \ldots\ \alpha_1 + y_h,\ 1 + \frac{a}{2},\ \alpha_1 - \alpha_2 + 1 + \frac{a}{2}\,2, \ldots,\ \alpha_1 - \alpha_{\ell-1} + 1 + \frac{a}{2}(\ell - 1)$$

in the denominator. One power of $s!$ cancels against the numerator term $\Gamma(1+\frac{a}{2}(j-2)+\alpha_1-\alpha_{j-1}+s)$ for $j=2$. The crucial parameter ϑ in (6.5) is computed as

$$\vartheta = \frac{1}{2} + \sum_{i=1}^h(\alpha_1+x_i) + \Big(\alpha_1+1+\frac{a}{2}(\ell-1)\Big) + \Big(\alpha_1-\alpha_2+1+\frac{a}{2}\Big) + \ldots + \Big(\alpha_1-\alpha_{\ell-1}+1+\frac{a}{2}(\ell-2)\Big)$$

$$-\sum_{i=0}^h(\alpha_1+y_i) - \Big(1+\frac{a}{2}\Big) - \Big(\alpha_1-\alpha_2+1+\frac{a}{2}\,2\Big) - \ldots - \Big(\alpha_1-\alpha_{\ell-1}+1+\frac{a}{2}(\ell-1)\Big)$$

$$= \frac{1}{2} + \sum_{i=1}^h x_i - \sum_{i=0}^h y_i + \Big(1+\frac{a}{2}(\ell-1)\Big) - \Big(1+\frac{a}{2}\Big) - \frac{a}{2}(\ell-2) = \frac{1}{2} + \sum_{i=1}^h x_i - \sum_{i=0}^h y_i.$$

Putting $x = n^2$, (6.5) implies

$$\lim_{n\to\infty} n^{-\vartheta}\ e^{-2n}\ F_\alpha(n^2) = \frac{1}{\sqrt{\pi}}.$$

Since ϑ is independent of α, the same limit holds for $\alpha = 0$. Thus we obtain

$$\lim_{n\to\infty} \frac{F_\alpha(n^2)}{F_0(n^2)} = 1.$$

Passing to the probability measure cancels the constant A and we obtain

$$\frac{1}{\|N^c_\alpha\|^2_{V^c}} \int_{\Omega^\ell} \mu(dz)\ \frac{|e^{(z|e_1)}|^{2n}}{\|(e^{(z|e_1)})^n\|^2_\mu}\ |N_{\alpha^+}(z)|^2 \to \frac{\prod_{i=1}^h (x_i - \frac{a}{2})_\alpha}{\prod_{i=0}^h (y_i - \frac{a}{2})_\alpha}.$$

Hence any cluster point μ' is an $(\ell-1)$-hypergeometric probability measure of the same type $\binom{y_0-\frac{a}{2},\ldots,y_h-\frac{a}{2}}{x_1-\frac{a}{2},\ldots,x_h-\frac{a}{2}}$ on $\overline{\Omega}^c_{\ell-1}$. In view of Lemma 6.1 this determines the limit measure on each irreducible K^c-type, which, as explained above, implies the assertion. □

Remark 6.3 For the "concrete" ℓ-hypergeometric measures $M_{\nu,\ell}$ $(k=0)$ and $M_{k,\ell}$ $(k>0)$ constructed in Sect. 4 we obtain as limit measures

$$M^{(c)}_{\nu,\ell} = M^c_{\nu-\frac{a}{2},\ell-1}$$

$$M^{(c)}_{k,\ell} = M^c_{k-1,\ell-1},$$

where the superscript c refers to the Peirce 0-space V^c. In the second case this follows from

$$\nu_k - \frac{a}{2} = \nu^c_{k-1}.$$

If $k=0$ then $\nu > p-1$ is any parameter in the discrete series, in which case $\nu - \frac{a}{2} > p^c - 1$ belongs to the discrete series of Ω^c. As special cases $(\ell = r)$ we have

$$M^{(c)}_{\nu} = M^c_{\nu-\frac{a}{2}}$$

$$M^{(c)}_{k,r} = M^c_{k-1,r-1}$$

for the "full" measures. Here for $k \geq 2$ and rank $\Omega^c = r-1$ the value

$$\nu^c_{k-1} = 1+b+\frac{a}{2}(2(r-1)-(k-1)-1) = 1+b+\frac{a}{2}(2r-k-1)-\frac{a}{2} = \nu^r_k - \frac{a}{2}$$

is again a boundary parameter for Ω^c, whereas for $k=1$ the parameter

$$\nu_0 = \nu_1 - \frac{a}{2} = p-1-\frac{a}{2} = 1+b+a(r-1)-\frac{a}{2} > 1+b+a(r-2) = p_{r-1}-1$$

belongs to the discrete series of Ω^c. Understanding this "disappearing boundary orbit" in the limit was one of the original motivations for the current paper.

7 Boundary Representations

The (unital) Toeplitz C^*-algebra $\mathcal{T}$ associated with a bounded domain $\Omega \subset \mathbf{C}^d$ can be regarded as a deformation of $\mathcal{C}(\overline{\Omega})$ in the sense of "non-commutative geometry". Thus the spectrum of $\mathcal{T}$, consisting of all irreducible $*$-representations, is a 'non-commutative' (non-Hausdorff) compactification of Ω, involving the geometry of the boundary. In this section we carry out this program for Toeplitz operators over boundary orbits and algebraic varieties, using the boundary stratification described in Proposition 3.1. For each $0 \le j < k$ the partial closures satisfy

$$\Omega_{k,r} = \bigcup_{c \in S_j} c + \Omega^c_{k-j,r-j}.$$

as a non-disjoint union.

For two sequences $(f_n), (g_n)$ in $\mathcal{H}_{\mu,\ell}$ we put

$$f_n \sim g_n$$

if $\lim_{n\to\infty} \|f_n - g_n\|_{\mu,\ell} = 0$. For any $c \in S_i$ put

$$h^n_c(z) := H^n_c(z)/\|H^n_c\|_{\mu,\ell}.$$

In the following we embed $\mathcal{P}(V^c) \subset \mathcal{P}(V)$ via the Peirce projection $V \to V^c$.

Lemma 7.1 *Let $p \in \mathcal{P}^\ell(V)$ and $q \in \mathcal{P}^{\ell-1}(V^c) \subset \mathcal{P}^\ell(V)$. Then*

$$T_{\mu,\ell}(p)(h^n_c\, q) \sim h^n_c\, T_{\mu^c,\ell-1}(p^{(c)})q$$

for all $c \in S_1$

Proof Since $p - p^{(c)}$ vanishes on $c + \Omega^c_{\ell-1}$, Lemma 6.1 implies

$$\|\frac{H^n_c}{\|H^n_c\|_{\mu,\ell}}\, p - \frac{H^n_c}{\|H^n_c\|_{\mu,\ell}}\, p^{(c)}\|^2_{\mu,\ell} = \int\limits_{\overline{\Omega}_\ell} \mu(dz)\, \frac{|H^n_c(z)|^2}{\|H^n_c\|^2_{\mu,\ell}}\, |p(z) - p^{(c)}(z)|^2 \to 0.$$

It follows that

$$T_{\mu,\ell}(p)(h^n_c\, q) = p(h^n_c\, q) \sim h^n_s(p^{(c)}\, q) \sim h^n_c\, T^c_{\mu^c,\ell-1}(p^{(c)})q.$$

□

The adjoint operators $T_{\mu,\ell}(\overline{p})$ are more difficult to handle. For a partition $\alpha = (\alpha_1, \dots, \alpha_{\ell-1}) \in \mathbf{N}^{\ell-1}_+$ consider the orthogonal projection

$$\pi^\ell_\alpha : \mathcal{P}^\ell(V) \to \sum_{m_1 \ge \alpha_1} \mathcal{P}_{m_1,\alpha}(V) \subset \mathcal{P}^\ell(V),$$

with $(m_1, \alpha) \in \mathbf{N}^\ell_+ \subset \mathbf{N}^r_+$. Then $\sum\limits_{\alpha \in \mathbf{N}^{\ell-1}_+} \pi^\ell_\alpha = \mathrm{Id}$ on $\mathcal{P}^\ell(V)$.

Lemma 7.2 *Let $p \in \mathcal{P}(V_2^c) \subset \mathcal{P}^\ell(V)$ and $v \in V^c$, where $c = e_1$. Then we have for every $\alpha \in \mathbf{N}_+^{\ell-1}$*

$$p\ N_{\alpha^+} \in \mathrm{Ran}(\pi_\alpha^\ell) \tag{7.1}$$

$$T_{\mu,\ell}(\overline{v}^*)(p\ N_{\alpha^+}) = \sum_{j=1}^{\ell-1} \frac{\prod_{i=1}^{h}(x_i - \frac{a}{2} - \frac{a}{2}(j-1) + \alpha_j - 1)}{\prod_{i=0}^{h}(y_i - \frac{a}{2} - \frac{a}{2}(j-1) + \alpha_j - 1)}\ \pi_{\alpha-\varepsilon_j}^\ell(p \cdot \partial_v N_{\alpha^+}). \tag{7.2}$$

Proof The first assertion is proved in [22, Lemma 3.5]. By [22, Lemma 2.9] we have

$$\partial_v N_{\boldsymbol{m}} \in \sum_{j=2}^{\ell} \mathcal{P}_{\boldsymbol{m}-\varepsilon_j}(V).$$

Since $v \in V^c$ implies $\partial_v p = 0$, we have $\partial_v N_{\boldsymbol{m}} = p \cdot \partial_v N_{\alpha^+}$, and Proposition 5.3 yields

$$T_{\mu,\ell}(\overline{v}^*)(p\ N_{\alpha^+}) = T_{\mu,\ell}(\overline{v}^*)\ N_{\boldsymbol{m}} = \sum_{j=2}^{\ell} \frac{\prod_{i=1}^{h}(x_i - \frac{a}{2}(j-1) + m_j - 1)}{\prod_{i=0}^{h}(y_i - \frac{a}{2}(j-1) + m_j - 1)}\ (\partial_v N_{\boldsymbol{m}})_{\boldsymbol{m}-\varepsilon_j}$$

$$= \sum_{j=2}^{\ell} \frac{\prod_{i=1}^{h}(x_i - \frac{a}{2}(j-1) + \alpha_j^+ - 1)}{\prod_{i=0}^{h}(y_i - \frac{a}{2}(j-1) + \alpha_j^+ - 1)}\ (p \cdot \partial_v N_{\alpha^+})_{\boldsymbol{m}-\varepsilon_j}.$$

Shifting $j \mapsto j-1$ and using $\mathcal{P}_{\boldsymbol{m}-\varepsilon_j}(V) \subset \mathrm{Ran}(\pi_{\alpha-\varepsilon_{j-1}}^\ell)$ for all $1 < j \le \ell$, the assertion follows. □

Lemma 7.3 *Let $q \in \mathcal{P}^{\ell-1}(V^c)$ and $\alpha \in \mathbf{N}_+^{\ell-1}$. Then*

$$\pi_\alpha^\ell(h_c^n\ q) \sim h_c^n\ q_\alpha.$$

Proof We may assume that $q \in \mathcal{P}_\beta(V^c)$ for some partition $\beta \in \mathbf{N}_+^{\ell-1}$. Every $\gamma \in K^c$ has an extension $g \in K$ satisfying $gc = c$ (see the proof of [25, Lemma 6.2]). Since h_c^n is fixed under the action of g, we may assume that $q = N'_\beta$ is the conical polynomial in V^c of type β. Then $N_{\beta^+} - q$ vanishes on $c + \Omega_{\ell-1}^c$, and

Lemma 6.1 implies

$$h_c^n\, q \sim h_c^n\, N_{\beta+}. \tag{7.3}$$

Since the projection π_α^ℓ has a continuous extension to $\mathcal{H}_{\mu,\ell}$ it follows that

$$\pi_\alpha^\ell(h_c^n\, q) \sim \pi_\alpha^\ell(h_c^n\, N_{\beta+}).$$

Since h_c^n belongs to the closure of $\mathcal{P}(V_2^c)$ in $\mathcal{H}_{\mu,\ell}$, (7.1) implies $h_c^n\, N_{\beta+} \in \mathrm{Ran}(\pi_\beta^\ell)$. Therefore orthogonality implies

$$\pi_\alpha^\ell(h_c^n\, N_{\beta+}) = \delta_{\alpha,\beta}\, h_c^n\, N_{\beta+} \sim \delta_{\alpha,\beta}\, h_c^n\, q = h_c^n\, q_\alpha.$$

□

Proposition 7.4 *Let $p \in \mathcal{P}^\ell(V)$ and $q \in \mathcal{P}^{\ell-1}(V^c) \subset \mathcal{P}^\ell(V)$. Then the adjoint Toeplitz operators satisfy*

$$T_{\mu,\ell}(\overline{p})(h_c^n\, q) \sim h_c^n\, T^c_{\mu^{(c)},\ell-1}(\overline{p}^{(c)})q$$

for all $c \in S_1$.

Proof Assume first that $p(z) = (z|v)$ is linear. If $v \in V_2^c \oplus V_1^c$, then $p^{(c)}$ is constant and Lemma 7.1 implies

$$\begin{aligned}T_{\mu,\ell}(\overline{p})(h_c^n\, q) &= P_{\mu,\ell}(\overline{p}\, h_c^n\, q) \sim P_{\mu,\ell}(\overline{p}^{(c)}\, h_c^n\, q)\\ &= \overline{p}^{(c)}\, h_c^n\, q = h_c^n\, T^c_{\mu^{(c)},\ell-1}(\overline{p}^{(c)})q\end{aligned}$$

since the orthogonal projection $P_{\mu,\ell}$ is continuous. If $v \in V^c$, we may assume as in the proof of Lemma 7.3 that $q = N'_\alpha$ is the conical polynomial in $\mathcal{P}_\alpha(V^c)$ for some partition $\alpha \in \mathbf{N}_+^{\ell-1}$. Then $N_{\alpha+} - q$ vanishes on $c + \Omega^c_{\ell-1}$. Since v is tangent to V^c it follows that $(\partial_v N_{\alpha+})^c = \partial_v N_\alpha^c$. Hence $\partial_v(N_{\alpha+} - q)$ vanishes on $c + \Omega^c_{\ell-1}$ as well. Applying (7.3), Lemmas 7.2 and 7.3, we obtain

$$\begin{aligned}&T_{\mu,\ell}(\overline{v}^*)(h_c^n\, q) \sim T_{\mu,\ell}(\overline{v}^*)(h_c^n\, N_{\alpha+})\\ &= \sum_{j=2}^{\ell} \frac{\prod_{i=1}^{h}(x_i - \frac{a}{2} - \frac{a}{2}(j-1) + \alpha_j - 1)}{\prod_{i=0}^{h}(y_i - \frac{a}{2} - \frac{a}{2}(j-1) + \alpha_j - 1)}\, \pi^\ell_{\alpha-\varepsilon_{j-1}}(h_c^n \cdot \partial_v N_{\alpha+})\\ &\sim \sum_{j=2}^{\ell} \frac{\prod_{i=1}^{h}(x_i - \frac{a}{2} - \frac{a}{2}(j-1) + \alpha_j - 1)}{\prod_{i=0}^{h}(y_i - \frac{a}{2} - \frac{a}{2}(j-1) + \alpha_j - 1)}\, \pi^\ell_{\alpha-\varepsilon_{j-1}}(h_c^n \cdot \partial_v q)\end{aligned}$$

$$\sim h_c^n \sum_{j=2}^{\ell} \frac{\prod_{i=1}^{h} (x_i - \frac{a}{2} - \frac{a}{2}(j-1) + \alpha_j - 1)}{\prod_{i=0}^{h} (y_i - \frac{a}{2} - \frac{a}{2}(j-1) + \alpha_j - 1)} (\partial_v q)_{\alpha - \varepsilon_{j-1}} = h_c^n \, T^c_{\mu^{(c)},\ell-1}(\overline{p}^{(c)})(q),$$

since $r - j = (r-1) - (j-1)$ and $\ell - j = (\ell - 1) - (j-1)$. The last identity follows from Proposition 5.3 and the fact that $p^{(c)} = p$ if $v \in V^c$. This proves the assertion for linear symbol functions.

Now suppose that the assertion holds for polynomials ϕ, ψ up to a certain degree. Since μ^c is again a $(\ell - 1)$-hypergeometric measure for V^c and $\phi^{(c)}$ has degree $\leq \deg \phi$, we may apply this assumption to q and $T^c_{\mu^c,\ell-1}(\overline{\phi}^{(c)})q \in \mathcal{P}^{\ell-1}(V^c)$ to obtain

$$T_{\mu,\ell}(\overline{\phi\psi})(h_c^n \, q) = T_{\mu,\ell}(\overline{\psi}) \, T_{\mu,\ell}(\overline{\phi})(h_c^n \, q) \sim T_{\mu,\ell}(\overline{\psi})(h_c^n \, T^c_{\mu^{(c)},\ell-1}(\overline{\phi}^{(c)})q)$$

$$\sim h_c^n \, T^c_{\mu^c,\ell-1}(\overline{\psi}^s) \, T^c_{\mu^{(c)},\ell-1}(\overline{\phi}^{(c)})q = h_c^n \, T^c_{\mu^{(c)},\ell-1}(\overline{\phi\psi}^c)q.$$

Thus the assertion holds for $\phi\psi$. Since the assertion holds for linear forms, the proof is complete. □

The following is our main result.

Theorem 7.5 *Let $0 \leq i \leq \ell$ and let $c \in S_i$ be arbitrary. Then the Toeplitz C^*-algebra $\mathcal{T}_{k,\ell}$ has an irreducible $*$-representation*

$$\sigma^{(c)}_{k,\ell} : \mathcal{T}_{k,\ell} \to \mathcal{T}^c_{k\backslash i,\ell-i}$$

which is uniquely determined by the property

$$\sigma^{(c)}_{k,\ell} T_{k,\ell}(f) = T^c_{k\backslash i,\ell-i}(f^{(c)}) \tag{7.4}$$

for all $f \in \mathcal{C}(\Omega_{k,\ell})$, with $f^{(c)} \in \mathcal{C}(\Omega^c_{k\backslash i,\ell-i})$ defined by (6.2). Here we define

$$k \setminus i := \begin{cases} k - i & i < k \\ 0 & k \leq i \leq \ell \end{cases}.$$

In the first case the Toeplitz operator $T^c_{k-i,\ell-i}$ acts on a boundary orbit of the "little" Kepler ball $\Omega^c_{\ell-i}$. In the second case the Toeplitz operator $T^c_{0,\ell-i} = T^c_{\ell-i}$ acts on $\Omega^c_{\ell-i} = \Omega^c_{0,\ell-i}$ with discrete series parameter $\nu_k - i\frac{a}{2}$.

Proof For orthogonal tripotents $c \in S_i$, $d \in S_j^c$, the defining property (7.4) yields a commuting diagram

$$\begin{array}{ccc} & & \mathcal{T}^c_{k\backslash i,\ell-i} \\ & \overset{\sigma^{(c)}_{k,\ell}}{\nearrow} & \\ \mathcal{T}_{k,\ell} & & \Big\downarrow (\sigma^c)^{(d)}_{k\backslash i,\ell-i} \\ & \underset{\sigma^{(c+d)}_{k,\ell}}{\searrow} & \\ & & \mathcal{T}^{c+d}_{k\backslash(i+j),\ell-(i+j)} \end{array} .$$

Since every tripotent is the orthogonal sum of minimal tripotents, it therefore suffices to consider minimal tripotents $c \in S_1$. We may also assume $k \geq 1$, since the Kepler ball case $k = 0$ has been proven in [25].

Let $\mathcal{A}$ denote the set of all operators A in the $*$-subalgebra $\mathcal{T}_0 \subset \mathcal{T}_{k,\ell}$ generated by polynomial symbols, such that there exists an operator A_c acting on $\mathcal{P}(V^c)$ which satisfies

$$\lim_{n\to\infty} \|A(h_c^n\, q) - h_c^n\, (A_c q)\|_{k,\ell} = 0 \tag{7.5}$$

for all $q \in \mathcal{P}(V^c) \subset \mathcal{P}^\ell(V)$. Theorem 6.2 implies that A_c is uniquely determined by A and

$$\|A_c\| \leq \|A\| \tag{7.6}$$

for the respective operator norms. By definition, $\mathcal{A}$ is an algebra and (7.6) implies that $A \mapsto A_c$ has an extension $\mathcal{A} \to \mathcal{B}(\mathcal{H}^c_{k-1,\ell-1})$ (bounded operators) which is an algebra homomorphism. For every $p \in \mathcal{P}(V)$, it follows from Lemma 7.1 that $T_{k,\ell}(p) \in \mathcal{A}$ and $(T_{k,\ell}\,p)_c = T^c_{k-1,\ell-1}\overline{p}^{(c)}$. The corresponding statement $T_{k,\ell}(\overline{p}) \in \mathcal{A}$ and $T_{k,\ell}(\overline{p})_c = \overline{T^c_{k-1,\ell-1}\,p^{(c)}}$ for the adjoint operator follows from the deeper Proposition 7.4. Thus we have $\mathcal{A} = \mathcal{T}_0$ and, by (7.6), $A \mapsto A_c$ has a unique C^*-extension, denoted by $\sigma^{(c)}_{k,\ell}$ to the closure $\mathcal{T}_{k,\ell}$ of $\mathcal{T}_0$. This extension satisfies (7.5) for all continuous symbols f, since this property holds for polynomials and their conjugates. Thus we obtain a C^*-homomorphism

$$\sigma^{(c)}_{k,\ell} : \mathcal{T}_{k,\ell} \to \mathcal{T}^c_{k-1,\ell-1}.$$

As mentioned above, the case for arbitrary tripotents follows by iteration. The irreducibility of these representations follows from Remark 6.1 applied to $M^c_{k\backslash i,\ell-i}$. □

Remark 7.6 For different tripotents $c \in S_i$ and $d \in S_j$ the representations $\sigma^{(c)}$ and $\sigma^{(d)}$ are inequivalent. This follows from Urysohn's Lemma since there exists

$f \in \mathcal{C}(\Omega_{k,\ell})$ which vanishes on $c+\Omega^c_{k\setminus i,\ell-i}$ but not on $d+\Omega^d_{k\setminus j,\ell-j}$. Hence $T_{k,\ell}(f)$ belongs to $\mathrm{Ker}(\sigma^{(c)})$ but not to $\mathrm{Ker}(\sigma^{(d)})$. With more effort one can show that the full spectrum of $\mathcal{T}_{k,\ell}$ is given by the representations constructed above.

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References

1. J. Arazy, H. Upmeier, *Boundary measures for symmetric domains and integral formulas for the discrete Wallach points*, Int. Eq. Op. Th. **47** (2003), 375–434.
2. J. Arazy and G. Zhang, *Homogeneous multiplication operators on bounded symmetric domains*, J. Functional Anal. **202** (2003), 44–66.
3. B. Bagchi, G. Misra, *Homogeneous tuples of multiplication operators on twisted Bergman spaces*, J. Functional Anal. **136** (1996), 171–213.
4. ———, *Homogeneous tuples of operators and systems of imprimitivity, Contemp. Math.*, **185** (1995), 67–76.
5. J. Bunce, J. Deddens, *On the normal spectrum of a subnormal operator*, Proc. Amer. Math. Soc. **63** (1977), 107–110.
6. S. Chavan, D. Yakubovich, *Spherical Tuples of Hilbert Space Operators*, Indiana Univ. Math. J. **64** (2015), 577–612.
7. M. Englis, H. Upmeier, **Reproducing kernel functions and asymptotic expansions on Jordan-Kepler varieties**, Advances in Math. **347** (2019), 780–826
8. J. Faraut, A. Korányi, *Function spaces and reproducing kernels on bounded symmetric domains*, J. Funct. Anal. **88** (1990), 64–89.
9. J. Faraut, A. Korányi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford (1994).
10. S. Ghara, S. Kumar, and P. Pramanick, **K**-*homogeneous tuple of operators on bounded symmetric domains*, Preprint (arXiv:2002.01298).
11. T. Inoue, *Unitary representations and kernel functions associated with the boundaries of a bounded symmetric domain*, Hiroshima Math. J. **10** (1980), 75–140.
12. A. Knapp, *Representation Theory of Semisimple Groups: An Overview Based on Examples*, Princeton Univ. Press, 1986.
13. M. Lassalle, *Algèbres de Jordan et ensemble de Wallach*, Invent. Math. **89** (1987), 375–393.
14. O. Loos, *Jordan Pairs*, Springer Lect. Notes in Math. **460** (1975).
15. O. Loos, *Bounded Symmetric Domains and Jordan Pairs*, Univ. of California, Irvine (1977).
16. A.R. Lubin, *Spectral inclusion and C.N.E.*, Can. J. Math. **34** (1982), 883–887.
17. G. Misra, *Curvature and the backward shift operator*, Proc. Amer. Math. Soc. **91** (1984), 105–107.
18. G. Misra, N.S.N. Sastry, *Homogeneous tuples of operators and holomorphic discrete series representation of some classical groups*, J. Operator Th. **24** (1990), 23–32.
19. G. Misra, H. Upmeier, *Singular Hilbert modules on Jordan-Kepler varieties*, Proc. IWOTA (Shanghai 2018), to appear.
20. H. Rossi, M. Vergne, *Analytic continuation of the holomorphic Discrete series of a semi-simple Lie group*, Acta. Math. **136** (1976), 1–59.
21. H. Upmeier, *Jordan algebras and harmonic analysis on symmetric spaces*, Amer. J. Math. **108** (1986), 1–25.
22. H. Upmeier, *Toeplitz operators on bounded symmetric domains*, Trans. Amer. Math. Soc. **280** (1983), 221–237.

23. H. Upmeier, *Toeplitz C^*-algebras on bounded symmetric domains*, Annals of Math. **119** (1984), 549–576.
24. H. Upmeier, *Toeplitz Operators and Index Theory in Several Complex Variables*, Operator Theory: Advances and Applications, Vol. 81, Birkhäuser Verlag, Basel-Boston-Berlin, 1996.
25. H. Upmeier, *Toeplitz operators on Jordan-Kepler varieties*. Banach J. of Math. Anal. **13** (2019), 314–337.
26. V.S. Varadarajan, *Geometry of Quantum Theory*, Springer Verlag, New York, 1985.
27. E.M. Wright, *The asymptotic expansion of the generalized hypergeometric function*. Proc. London Math. Soc. **46** (1940), 389–408.

A Gaussian Analytic Function

Carlos G. Pacheco

We dedicate this article to Prof. Nikolai Vasilevsky, good colleague and friend.

Abstract We give a closed expression of a random analytic power series as the stochastic integral of a Möbius transformation. The coefficients of the random series are Gaussian random variables, and the closed expression is a stochastic integral with respect to Brownian motion. As a corollary, the set of zeros of the stochastic integral turns out to be what is known as a determinantal point process with the Bergman kernel.

Keywords Random series · Stochastic integral · Determinantal point process · Bergman kernel

Mathematics Subject Classification (2010) Primary 60H99; Secondary 47H40

1 Introduction

Random polynomials and random power series have been studied for a long time in the mathematics and physics communities; as an introduction the reader might find the book by Jean-Pierre Kahane [2] useful.

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The random power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \ z \in \mathbb{C}, \tag{1.1}$$

show particularly interesting properties when $a_0, a_1, \ldots$ are independent identically distributed complex valued random variables. As mentioned in [2, Chapter 4], the idea of considering such a series goes back to Emile Borel in 1896; it was Hugo Steinhaus who gave rigorous arguments regarding the radius of convergence in 1929. An interesting question that has brought attention is to analyze the level sets, that is $\{z : f(z) = c\}$ for a fixed constant c. In particular when $c = 0$, one studies the behaviour of the zeros. The first studies about this set are the works of J.E. Littlewood and A.C. Offord in 1948, see e.g. [3]; before that, in 1929, Geoge Pólya already addressed this kind of question, see [6]. Consult also [4] for more historical information.

In this article we consider the case where a_n is complex Gaussian. The very first thing that one knows about f is that its radius of convergence is 1 almost surely, with no analytic extension outside the unit disk $\{|z| \leq 1\}$. Thus, its zero set $f^{-1}(0)$ lies inside the open unit disk. Among the interesting properties of the zero set of f, one knows that almost surely it is a countable set of points, all of them isolated, and they form a specific structure called a *determinantal point process*, which is in this case characterized by the so-called *Bergman kernel*. This and more information is excitingly developed in Peres and Virág [5]; the book [1] also presents the results.

Another intriguing conexion is with the *Poisson kernel*. It is known, see [5] as well, that the real part $real(f)$ of f can be expressed as a linear transformation of the stochastic integral

$$u(z) = \int_0^{2\pi} P(z, e^{it}) dB_t, \tag{1.2}$$

where $P(z, e^{it})$ is the Poisson kernel and B_t is one-dimensional real-valued Brownian motion. Briefly speaking, the equality

$$real(f)(z) \stackrel{d}{=} cu(z) + \xi; \tag{1.3}$$

holds, where c is a constant and ξ is a normal random variable.

Since the Poisson kernel is the real part of the Möbius transformation $\frac{e^{it}+z}{e^{it}-z}$, one can ask whether f can be expressed as an stochastic integral of it. The answer is affirmative and is Theorem 3.1, together with some consequences in Sect. 4.

2 Preliminaries

Let us give a more specific description of the model and some notation we are using.

Let $a_0, a_1, \ldots$ be a sequence of independent identically distributed (i.i.d.) random variables (r.v.s) which are Gaussian complex-valued. This means that the probability density is given by

$$\frac{1}{\pi}e^{-|z|^2}, \ z \in \mathbb{C}. \tag{2.1}$$

This is equivalent to saying that the variables a_n have the same distribution as $X + iY$, where X and Y are two independent real-valued normally distributed with mean zero and variance $1/2$. The above is written in symbols as: $a_n \stackrel{d}{=} X + iY$ with $X \perp Y$ and $X, Y \sim N(0, 1/2)$.

With the described sequence one can form the following random power series, which is known to have radius of convergence 1, see e.g. [1],

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \ z \in \mathbb{D}, \tag{2.2}$$

where $\mathbb{D} = \{z : |z| < 1\}$ is the open unit disk.

In [5] it is proved that the real part of f coincides in distribution with a linear transformation of a stochastic integral of the Poisson kernel. Let us explain. Consider the Poisson kernel $P(z, w)$ as the real part of $\frac{1}{2\pi}\frac{1+z\overline{w}}{1-z\overline{w}}$, and recall the function $u(z)$ in (1.2) Then, the real part of f is the same, in distribution, as the function $z \mapsto \sqrt{\frac{\pi}{2}}u(z) + \xi/2$, where $\xi \sim N(0, 1)$ is independent of B.

3 Main Result

As mentioned in the introduction, after seeing (1.2) one can suspect that the whole random series (2.2) can be expressed as an stochastic integral as well. This is indeed the case.

Define

$$H(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} dB_t. \tag{3.1}$$

Then we have that

Theorem 3.1 *The random power series f in (2.2) is the same in distribution as the random function*

$$z \mapsto \frac{H(z)}{2\sqrt{2\pi}} + \frac{\xi}{\sqrt{2}} + i\eta, \tag{3.2}$$

where $\xi \sim N(0, 1/2)$ and $\eta \sim N(0, 1/2)$ are two independent r.v.s, independent also from B.

The idea of the proof is the following. After splitting into the real and imaginary parts, we expand (2.2) and (3.2) into trigonometric series to check that term by term they are the same.

Proof Here $\{X_n, Y_n,\ n = 0, 1, \ldots\}$ are i.i.d. $N(0, 1/2)$ r.v.s.

Let us first expand f by using $a_n = X_n + iY_n$ and $z = re^{i\theta}$;

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}(X_n + iY_n)r^n e^{in\theta} \\ &= \sum_{n=0}^{\infty}(X_n + iY_n)r^n(\cos(n\theta) + i\sin(n\theta)) \\ &= \sum_{n=0}^{\infty} r^n[X_n\cos(n\theta) - Y_n\sin(n\theta)] \\ &\quad + i\sum_{n=0}^{\infty} r^n[X_n\sin(n\theta) + Y_n\cos(n\theta)]. \end{aligned}$$

On the other hand, to analyze $H(z)$ we have that

$$\begin{aligned} \frac{e^{it} + z}{e^{it} - z} &= \frac{1 + ze^{-it}}{1 - ze^{-it}} = \frac{1}{1 - ze^{-it}} + \frac{ze^{-it}}{1 - ze^{-it}} \\ &= \sum_{n=0}^{\infty} e^{-int}z^n + e^{-it}z\sum_{n=0}^{\infty} e^{-int}z^n = 1 + 2\sum_{n=1}^{\infty} e^{-int}z^n \end{aligned}$$

$$= 1 + 2\sum_{n=1}^{\infty} r^n[\cos(nt)\cos(n\theta) + \sin(nt)\sin(n\theta)]$$

$$+2i\sum_{n=1}^{\infty} r^n[\cos(nt)\sin(n\theta) - \sin(nt)\cos(n\theta)].$$

The linearity and the continuity of the stochastic integral yield

$$H(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} dB_t = \int_0^{2\pi} \left[1 + 2 \sum_{n=1}^{\infty} e^{-int} z^n \right] dB_t$$

$$= B_{2\pi} + 2\sqrt{2\pi} \sum_{n=1}^{\infty} r^n [\tilde{X}_n \cos(n\theta) - \tilde{Y}_n \sin(n\theta)]$$

$$+2i\sqrt{2\pi} \sum_{n=1}^{\infty} r^n [\tilde{X}_n \sin(n\theta) + \tilde{Y}_n \cos(n\theta)],$$

where

$$\tilde{X}_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \cos(nt) dB_t \text{ and } \tilde{Y}_n = -\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \sin(nt) dB_t.$$

Now, to reproduce the structure of f, let us see how $\{\tilde{X}_n, \tilde{Y}_n,\ n = 0, 1, \ldots\}$ form a sequence of i.i.d. $N(0, 1/2)$ r.v.s. By properties of the stochastic integral, each $\tilde{X}_n$, and also each $\tilde{Y}_n$, is normally distributed with mean and variance given respectively by

$$E[\tilde{X}_n] = 0 \text{ and } E[\tilde{X}_n^2] = E\left[\frac{1}{2\pi} \int_0^{2\pi} \cos^2(nt) dt \right] = \frac{1}{2}.$$

Let us see the independence. Notice first that $(\tilde{X}_n, \tilde{Y}_m)$ is a Gaussian vector for each pair (n, m). Then, to know that $\tilde{X}_n \perp \tilde{Y}_m$, it is enough to see that the covariance is naught. Indeed, this can be seen by using the isometry property of the stochastic integral:

$$E[\tilde{X}_n \tilde{Y}_m] = \frac{1}{2\pi} \int_0^{2\pi} \cos(nt) \sin(mt) dt = 0.$$

And the same is true for the vectors $(\tilde{X}_n, \tilde{X}_m)$ and $(\tilde{Y}_n, \tilde{Y}_m)$ with $n \neq m$. For $n \geq 1$, notice that $\tilde{X}_n$ and $\tilde{Y}_n$ are also independent of $B_{2\pi}$ because $B_{2\pi} = \int_0^{2\pi} \cos(0 \times t) dB_t$.

Therefore, since the coefficients are the same in distribution, we can see that f and H share almost the same trigonometric expansion. However, to be exactly the same, one needs some small amendments. Take independent r.v.s $\xi \sim N(0, 1/2)$ and $\eta \sim N(0, 1/2)$, and form

$$\tilde{H}(z) = \frac{1}{2\sqrt{2\pi}} H(z) + \frac{\xi}{\sqrt{2}} + i\eta \tag{3.3}$$

which reproduces $f(z)$. The proof is finished. □

4 The Zero Set

It turns out, see [5], that the set of zeros $\mathcal{Z} = f^{-1}(0)$ is almost surely a countable set of isolated points in $\mathbb{D}$. Moreover, they form an object called a determinantal point process with Bergman kernel. This means that the statistical distribution of the points in $\mathcal{Z}$ obey a specific structure. Let us elaborate. The following is the key description of such an structure.

Take k different points $z_1, \ldots, z_k$ in $\mathbb{D}$, and define $p_\epsilon(z_1, \ldots, z_k)$ as the probability that there is one zero in each ball $B_\epsilon(z_i) = \{z : |z - z_i| < \epsilon\}$. For the balls $B_\epsilon(z_i)$ to have a single zero, ϵ needs to be sufficiently small.

Then, asymptotically when $\epsilon \to 0$, the probability of finding zeros inside the balls $B_\epsilon(z_i)$ is described by a determinant of a matrix formed with the Bergman kernel. More precisely,

$$\lim_{\epsilon \to 0} \frac{p_\epsilon(z_1, \ldots, z_k)}{\pi^k \epsilon^{2k}} = \lim_{\epsilon \to 0} \frac{P(B_\epsilon(z_i) \bigcap \mathcal{Z} = 1,\ i = 1, \ldots, k)}{\pi^k \epsilon^{2k}}$$
$$= det[K(z_i, z_j)]_{i,j=1}^k,$$

where $K(z, w) = \pi^{-1}(1 - z\overline{w})^{-2}$. Notice that this limit is in a sense a density, that is why people call it the *joint intensity*, but it is also called the *k-point correlation function*.

By Theorem 3.1 the first thing we know is that $\tilde{H}$ behaves as f. Thus

Corollary 4.1

(i) The function $\tilde{H}$ in (3.3) is analytic in $\mathbb{D}$.
(ii) The zeros of $\tilde{H}$ form a determinantal point process in $\mathbb{D}$ with the Bergman kernel.

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References

1. J.B. Hough, M. Krishnapur, Y. Peres, and B. Virág (2009). *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*, University Lecture Series Vol. 51. American Mathematical Society.
2. J.-P. Kahane (1985). *Some random series of functions*, second edition. Cambridge University Press.
3. J.E. Littlewood and A.C. Offord (1948). On the distribution of zeros and a-values of a random integral function (II). *Annals of Mathematics* **49**(4), pp. 885–952.
4. A.C. Offord (1995). The distribution of the values of an entire function whose coefficients are independent random variables. *Math. Proc. Camb. Phil. Soc.* **118**, pp. 527–542.
5. Y. Peres and B. Virág (2005). Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process *Acta Math.* **194**, pp. 1–35.
6. G. Polya (1929). Untersuchengen über Lücken und Singularitäten von Potenzreihen. *Math. Zeitschrift* **29**, pp. 549–640.

Dirac Operators on $\mathbb{R}$ with General Point Interactions

Vladimir Rabinovich

Dedicated to Professor Nikolai Vasilevski on the occasion of his 70-th birthday anniversary

Abstract We consider the Dirac operator on $\mathbb{R}$ of the form

$$\mathfrak{D}u(x)=\left(J\frac{d}{dx}+Q+Q_s\right)u(x),\, x\in\mathbb{R}$$

where $J=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

$$Q(x)=\begin{pmatrix} p(x)+r(x) & q(x) \\ q(x) & -p(x)+r(x) \end{pmatrix},\, p,q,r\in L^{\infty}(\mathbb{R})$$

is the regular potential, and

$$Q_s(x)=\sum_{y\in\mathbb{Y}}\Gamma\left(y\right)\delta(x-y) \tag{1}$$

is the singular potential, δ is the Dirac delta-function, $\Gamma\left(y\right)=(\gamma_{ij}(y))_{i,j=1,2}$ is a 2×2-matrix with elements $\gamma_{ij}(y)\in l^{\infty}(\mathbb{Y}), i,j=1,2$, $\mathbb{Y}\subset\mathbb{R}$ is an infinite or finite discrete set.

We associate with the formal Dirac operator $\mathfrak{D}$ the unbounded operator $\mathrm{D}_{Q,A,B}$ in $L^2(\mathbb{R},\mathbb{C}^2)$ defined by the operator $J\frac{d}{dx}+Q$ with regular potential Q and the point interaction conditions: $A(y)u(y+0)=B(y)u(y-0)$, $y\in\mathbb{Y}$ where $A(y)=\frac{1}{2}\Gamma(y)-J$, $B(y)=-(\frac{1}{2}\Gamma(y)+J)$.

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We study the self-adjointness of $\mathrm{D}_{Q,A,B}$ in $L^2(\mathbb{R},\mathbb{C}^2)$, its Fredholm properties, and the essential spectrum. We consider also the influence of slowly oscillating perturbations of regular potentials of periodic Dirac operators to his essential spectrum.

Keywords Dirac operators · Point interactions · Self-adjointness · Fredholmness · Essential spectrum

Mathematics Subject Classification 34L40, 47E05, 47B25, 81Q10

1 Introduction

There is an extensive literature devoted to physical and mathematical aspects related to Schrödinger operators with singular potentials (see [3–15] and extensive list of references therein). In the $1-D$ case the most known and interesting are the formal Schrödinger operators

$$\mathcal{L} = -\frac{d^2}{dx^2} + \sum_{j=1}^{\infty} \alpha_j \delta(x - y_j), \mathcal{L}' = -\frac{d^2}{dx^2} + \sum_{j=1}^{\infty} \alpha_j \delta'(x - y_j), \tag{2}$$

where δ is the Dirac delta-function, and δ' is the derivative of δ, $\mathbb{Y} = \{y_1, y_2, \ldots, y_j, \ldots\}$ is a discrete set in $\mathbb{R}, \alpha_j$ are real-valued coefficients called the strength of interactions. If $\alpha_j = \alpha$ and the sequence $\mathbb{Y}$ is periodic the physical model described by Eqs. (2) is called the "Kronig–Penney Hamiltonian" [20]. This is a simplest model of electron moving in a $1-D$ crystal. The rigorous consideration of operators $\mathcal{L}, \mathcal{L}'$ leads to the study of unbounded operators in $L^2(\mathbb{R})$ generated by $-\frac{d^2}{dx^2}$ and so-called interaction conditions at the points of sequence $\mathbb{Y}$ (see for instance [3, 4, 14, 15, 19, 22], and reference cited there). Some numerical aspects of calculation of discrete and essential spectra of operators $\mathcal{L}, \mathcal{L}'$ and more general given in paper [35]. Relativistic operators with $\delta-$interactions have received a lot of attention recently (see for instance [3, 5, 12, 17, 18, 24], and references cited there).

We consider here the $1-D$ Dirac operator

$$\mathfrak{D}_{Q,Q_s} u(x) = J\frac{du(x)}{dx} + Q(x)u(x) + Q_s(x)u(x), x \in \mathbb{R} \tag{3}$$

acting in the space two-dimensional vector valued distributions $u(x) = \begin{pmatrix} u^1(x) \\ u^2(x) \end{pmatrix}$ where Q is a regular potential and Q_s is a singular potential. We assume that

$$Q(x) = q_{am}(x)\sigma_1 + (m + q_{sc}(x))\sigma_3 + q_{el}(x)\mathbb{I}$$

where

$$\mathbb{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix}, \sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}, J=i\sigma_2=\begin{pmatrix}0&-1\\1&0\end{pmatrix}, \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

q_{el} is an electrostatic potential, q_{am} is an anomalous magnetic moment, q_{sc} is a scalar potential, and $m \geq 0$ is a mass of particle. The regular potential Q can be written in the matrix form

$$Q(x)=\begin{pmatrix}p(x)+r(x) & q(x)\\ q(x) & -p(x)+r(x)\end{pmatrix}$$

where $p = m + q_{sc}, q = q_{am}, r = q_{sc} \in L^{\infty}(\mathbb{R}^3)$. We consider the singular potential Q_s of the form

$$Q_s(x)=\sum_{y\in\mathbb{Y}}\Gamma\left(y\right)\delta(x-y) \tag{4}$$

where

$$\Gamma\left(y\right)=\left(\gamma_{ij}(y)\right)_{ij=1}^{2}, y\in\mathbb{Y}$$

is a 2×2 matrix with elements $\gamma_{ij}(y)\in l^{\infty}(\mathbb{Y}), i, j=1,2, \mathbb{Y}\subset\mathbb{R}$ is an infinite or finite discrete set. If $\mathbb{Y}=\left\{y_j\right\}_{j\in\mathbb{Z}}$ is the infinite set we assume that

$$0<\inf_{j\in\mathbb{Z}}(y_{j+1}-y_j)\leq\sup_{j\in\mathbb{Z}}(y_{j+1}-y_j)<\infty. \tag{5}$$

We associate with formal Dirac operator (3) the unbounded operator $\mathrm{D}_{Q,A,B}$ in the Hilbert space $L^2(\mathbb{R},\mathbb{C}^2)$ defined by the Dirac operator $\mathfrak{D}_Q = J\frac{d}{dx}+Q$ with interaction conditions

$$A(y)u(y+0)=B(y)u(y-0), y\in\mathbb{Y}. \tag{6}$$

The matrices $A(y), B(y)$ are defined by the formulas

$$A(y)=\frac{1}{2}\Gamma(y)-J, B(y)=-\left(\frac{1}{2}\Gamma(y)+J\right). \tag{7}$$

It is convenient to change sometime interaction conditions (6) as follows

$$u(y+0)=C(y)u(y-0), y\in\mathbb{Y} \tag{8}$$

with $C(y)=A^{-1}(y)B(Y)$ if the matrices $A(y)$ are invertible.

The main results of the paper are following.

1^0. We obtained the sufficient conditions for the operator $\mathrm{D}_{Q,\mathbb{I},C}$ to be self-adjoint. It should be noted that self-adjointness of operators $\mathrm{D}_{Q,\mathbb{I},C}$ associated with some interaction conditions have been studied in [16]. The authors of this paper used the boundary triplets technique and the corresponding Weyl functions. Our approach to the self-adjointness is different and closed to the study of self-adjointness of realizations of elliptic formally self-adjoint differential operators (see for instance [2, Chap. 4]).

2^0. We study the *essential spectra* of operators $\mathrm{D}_{Q,A,B}$ for finite set $\mathbb{Y}$ of interactions applying the limit operators method. This method and its applications to the operator theory are presented in the book [26]. It was applied for the study of Fredholm properties and essential spectra of different operators of Mathematical Physics, in particular, electromagnetic Schrödinger and Dirac operators on $\mathbb{R}^n$ for wide classes of potentials [27], discrete Schrödinger and Dirac operators on $\mathbb{Z}^n$, and on periodic combinatorial graphs (see [28, 29]), and on quantum waveguides (see [30]), and Schrödinger operators on $\mathbb{R}^n$ with singular potentials supported on unbounded hypersurfaces in $\mathbb{R}^n$ (see [33, 34]). Note that the method of limit operators has been applied to the investigation of the essential spectrum of the Schrödinger operators on periodic graphs (see [31, 32].)

Under assumption that the function p, q, r are uniformly continuous at infinity we introduce limit operators $\mathrm{D}^h_{Q,A,B}$ for the operators $\mathrm{D}_{Q,A,B}$ defined by the sequences $h = (h_m), h_m \in \mathbb{Z}$ tending to infinity, and we prove that the essential spectrum $sp_{ess}\mathrm{D}_{Q,A,B}$ of $\mathrm{D}_{Q,A,B}$ is defined as

$$sp_{ess}\mathrm{D}_{Q,A,B} = \bigcup_{\mathrm{D}^h_{Q,A,C} \in Lim\mathrm{D}_{Q,A,C}} sp\mathrm{D}^h_{Q,A,B} \tag{9}$$

where $Lim\mathrm{D}_{Q,A,B}$ is the set of all limit operators of $\mathrm{D}_{Q,A,B}$.

Moreover the essential spectra of operators $\mathrm{D}_{Q,A,B}$ are independent of singular potentials with finite supports.

Further we show that if the functions p, q, r are slowly oscillating at infinity (see [26], page 88) the spectra of limit operators are defined in explicit forms and formula (9) gives an effective description of $sp_{ess}\mathrm{D}_{Q,A,B}$.

3^0. We consider also the essential spectrum of $\mathrm{D}_{Q,\mathbb{I},C}$ for infinite periodic set $\mathbb{Y} = \mathbb{Y}_0 + l\mathbb{Z}, l \in \mathbb{R}_+$ where $\mathbb{Y}_0 \subset \mathbb{R}$ is a finite set. We assume that the coefficient C in interaction conditions (8) is l-periodic $2 \times 2-$matrix-valued function, the functions $p, q, r \in L^\infty(\mathbb{R})$ and are uniformly continuous at infinity. The limit operators $\mathrm{D}_{Q^h,\mathbb{I},C}$ for $\mathrm{D}_{Q,\mathbb{I},C}$ are defined by sequences $h = (h_m), h_m \in l\mathbb{Z}, h_m \to \infty$, and we obtain a formula for the essential spectrum $\mathrm{D}_{Q,\mathbb{I},C}$ similar to formula (9).

4^0. As an application we study also the slowly oscillating perturbations of periodic potentials. Let p, q, r be continuous real-valued periodic functions with respect to the group $l\mathbb{Z}$, and the coefficient C in interaction conditions (8) is a real-valued periodic matrix-function with respect to $l\mathbb{Z}$, such that $\det C(y) = 1$ for every $y \in \mathbb{Y}_0$. Then the operator $D_{Q,\mathbb{I},C}$ is self-adjoint and periodic, that is

$$V_{-g} D_{Q,\mathbb{I},C} V_g = D_{Q,\mathbb{I},C} \text{ for every } g \in l\mathbb{Z},$$

where V_g is the shift operator, that is $V_g u(x) = u(x-g)$, $x \in \mathbb{R}$, $g \in l\mathbb{Z}$. Then all limit operators $D_{Q^h,\mathbb{I},C}$ coincide with $D_{Q,\mathbb{I},C}$, and $sp_{ess} D_{Q,\mathbb{I},C} = sp D_{Q,\mathbb{I},C}$.

Moreover, the spectrum of periodic operator $D_{Q,\mathbb{I},C}$ has a band-gap structure (see for instance, [21, 38])

$$sp D_{Q,\mathbb{I},C} = \bigcup_{j=1}^{\infty} \left[a_j, b_j\right].$$

We consider the perturbation $D_{\tilde{Q},A,C}$ of operator $D_{Q,A,C}$ by addition a term $r_1 \mathbb{I}$ to the periodic electrostatic potential $r\mathbb{I}$ where r_1 is a slowly oscillating at infinity function. Applying formula (9) we obtain the description of essential spectrum of operator $D_{\tilde{Q},\mathbb{I},C}$

$$sp_{ess} D_{\tilde{Q},\mathbb{I},C} = \bigcup_{j=1}^{\infty} \left[a_j + m(r_1), b_j + M(r_1)\right] \tag{10}$$

where $m(r_1) = \liminf_{x\to\infty} r_1(x)$, $M(r_1) = \limsup_{x\to\infty} r_1(x)$. Formula (10) yields that some spectral bands of $sp_{ess} D_{\tilde{Q},\mathbb{I},C}$ may overlap depending on the intensity of the perturbation r_1. This can lead to the closure of some and possibly all gaps in the spectrum of unperturbed periodic operator.

1.1 Notations

- If X, Y are Banach spaces then we denote by $\mathcal{B}(X, Y)$ the space of bounded linear operators acting from X into Y with the uniform operator topology, and by $\mathcal{K}(X, Y)$ the subspace of $\mathcal{B}(X, Y)$ of all compact operators. In the case $X = Y$ we write shortly $\mathcal{B}(X)$ and $\mathcal{K}(X)$.
- An operator $A \in \mathcal{B}(X, Y)$ is called a Fredholm operator if

  $$ker A = \{x \in X : Ax = 0\}, coker A = Y/\Im(A)$$

  are finite dimensional spaces. Let $\mathcal{A}$ be a closed unbounded operator in a Hilbert space $\mathcal{H}$ with a dense in $\mathcal{H}$ domain $dom\mathcal{A}$. Then $\mathcal{A}$ is called a Fredholm operator if $ker\ \mathcal{A} = \left\{x \in dom\mathcal{A} : \mathcal{A}x = 0\right\}$ and $coker A = \mathcal{H}/\Im(\mathcal{A})$ where $\Im(\mathcal{A}) = \left\{y \in \mathcal{H} : y = \mathcal{A}x, x \in dom\mathcal{A}\right\}$ are finite-dimensional spaces. Note that $\mathcal{A}$ is a Fredholm operator as unbounded operator in $\mathcal{H}$ if and only if $\mathcal{A} : dom_{\mathcal{A}} \to \mathcal{H}$

is a Fredholm operator as a bounded operator where $dom\mathcal{A}$ is equipped by the graph norm

$$\|u\|_{dom\mathcal{A}} = \left(\|u\|^2_{\mathcal{H}} + \|\mathcal{A}u\|^2_{\mathcal{H}}\right)^{1/2}, u \in dom\mathcal{A}$$

(see for instance [1, Chap. 2]).

- The essential spectrum $sp_{ess}\mathcal{A}$ of an unbounded operator $\mathcal{A}$ is a set of $\lambda \in \mathbb{C}$ such that $\mathcal{A} - \lambda I$ is not Fredholm operator as unbounded operator, and the discrete spectrum $sp_{dis}\mathcal{A}$ of $\mathcal{A}$ is a set of isolated eigenvalues of finite multiplicity. It is well known that if $\mathcal{A}$ is a self-adjoint operator then $sp_{dis}\mathcal{A}=sp\mathcal{A}\diagdown sp_{ess}\mathcal{A}$. (see for instance [1, Chap. 2]).
- We denote by $L^2(\mathbb{R}, \mathbb{C}^2)$ the Hilbert space of 2-dimensional vector-functions $u(x) = (u^1(x), u^2(x))$, $x \in \mathbb{R}$ with the scalar product

$$\langle u, v\rangle = \int_{\mathbb{R}} \left(u(x), v(x)\right)_{\mathbb{C}^2} dx.$$

- We denote by $H^s(\mathbb{R}, \mathbb{C}^2)$ the Sobolev space on $\mathbb{R}$ of two-dimensional vector-valued functions, that is the space of distributions $u \in \mathcal{D}'(\mathbb{R}, \mathbb{C}^2)$ such that

$$\|u\|_{H^s(\mathbb{R},\mathbb{C}^2)} = \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s \, \|\hat{u}(\xi)\|^2_{\mathbb{C}^2} \, d\xi\right)^{1/2} < \infty, s \in \mathbb{R}$$

where $\hat{u}$ is the Fourier transform of u. If (a, b) is an interval in $\mathbb{R}$ then $H^s((a, b), \mathbb{C}^2)$ is the space of restrictions of $u \in H^s(\mathbb{R}, \mathbb{C}^2)$ on (a, b) with the standard norm.
- We denote by $C_b^m(\mathbb{R})$ the class of functions on $\mathbb{R}$ with m bounded continuous derivatives, $C_b(\mathbb{R}) = \mathbb{C}_b^0(\mathbb{R})$ is the class of bounded continuous functions on $\mathbb{R}$, and $C_b^\infty(\mathbb{R}) = \cap_{m\geq 0} C_b^m(\mathbb{R})$.

1.2 Dirac Operators on $\mathbb{R}$

The Dirac operator $\mathfrak{D}_Q$ on $\mathbb{R}$ with regular potential Q can be written as

$$\mathfrak{D}_Q = J\frac{d}{dx} + QI, \tag{11}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and the potential Q has the matrix form

$$Q = \begin{pmatrix} p+r & q \\ q & -p+r \end{pmatrix}$$

with $p, q, r \in L^\infty(\mathbb{R})$.

Note that $\mathfrak{D}_Q$ is a bounded operator from $H^1\left(\mathbb{R}, \mathbb{C}^2\right)$ into $L^2(\mathbb{R}, \mathbb{C}^2)$. Moreover, it is well-known that in the case of constant $p, q, r \in \mathbb{R}$ the operator $\mathfrak{D}_Q$ with domain $H^1(\mathbb{R}, \mathbb{C}^2)$ is self-adjoint in $L^2(\mathbb{R}, \mathbb{C}^2)$.

Let $p, q \in \mathbb{R}$, $r = 0$. Then

$$\mathfrak{D}_Q^2 = \left(-\frac{d^2}{dx^2} + p^2 + q^2\right)\mathbb{I}, \tag{12}$$

and for every $\lambda \in \mathbb{C}$

$$\left(\mathfrak{D}_Q - \lambda I\right)\left(\mathfrak{D}_Q + \lambda I\right) = \left(\mathfrak{D}_Q + \lambda I\right)\left(\mathfrak{D}_Q - \lambda I\right) = \left(-\frac{d^2}{dx^2} + \left(p^2 + q^2 - \lambda^2\right)\right)\mathbb{I}. \tag{13}$$

Formula (13) yields that

$$sp\mathfrak{D}_Q = \left(-\infty, -\sqrt{p^2+q^2}\right]\bigcup\left[\sqrt{p^2+q^2}, +\infty\right). \tag{14}$$

Let $\lambda \notin sp\mathfrak{D}_Q$. Then the operators $\mathfrak{D}_Q - \lambda I, \mathfrak{D}_Q + \lambda I$ are invertible from $H^1(\mathbb{R}, \mathbb{C}^2)$ into $L^2(\mathbb{R}, \mathbb{C}^2)$, and

$$\begin{aligned}\left(\mathfrak{D}_Q \pm \lambda I\right)^{-1} u(x) &= \left(\mathfrak{D}_Q \mp \lambda I\right)\left(-\frac{d^2}{dx^2} + p^2 + q^2 - \lambda^2\right)^{-1} u(x) \\ &= \left(\mathfrak{D}_Q \mp \lambda I\right)\int_{\mathbb{R}} \frac{ie^{-k(\lambda)|x-y|}u(y)dy}{2k(\lambda)}, u \in L^2(\mathbb{R}, \mathbb{C}^2),\end{aligned}$$

where $k(\lambda) = \sqrt{p^2+q^2-\lambda^2}$ and the branch of root is chosen such that $\sqrt{p^2+q^2-\lambda^2} > 0$ for $\lambda \in \mathbb{R} : |\lambda| < \sqrt{p_0^2+q_0^2}$.

2 Dirac Operator on $\mathbb{R}$ with Singular Potentials

2.1 Realization of Formal Dirac Operator with Singular Potential

We denote by $H^1(\mathbb{R}\backslash\mathbb{Y},\mathbb{C}^2)=\oplus_{j=-\infty}^{+\infty}H^1\left((y_j,y_{j+1}),\mathbb{C}^2\right)$ where $H^1\left((y_j,y_{j+1}),\mathbb{C}^2\right)$ is the Sobolev space on the interval (y_j,y_{j+1}). The norm in $H^1(\mathbb{R}\backslash\mathbb{Y},\mathbb{C}^2)$ is introduced as

$$\|u\|_{H^1(\mathbb{R}\backslash\mathbb{Y})}=\left(\sum_{j=1}^{\infty}\|u_j\|^2_{H^1((y_j,y_{j+1}),\mathbb{C}^2)}\right)^{\frac{1}{2}},u_j=u\mid_{(y_j,y_{j+1})}.$$

The functions $u\in H^1(\mathbb{R}\diagdown\mathbb{Y},\mathbb{C}^2)$ have the one-side limits at the points $y\in\mathbb{Y}$

$$u_\pm(y)=u(y\pm 0)=\lim_{x\to y\pm 0}u(x).$$

The action of singular potential $\Gamma(y)\delta(x-y)$ on functions $u\in H^1(\mathbb{R}\diagdown\mathbb{Y},\mathbb{C}^2)$ is defined as

$$\Gamma(y)\delta(x-y)u(x)=\frac{1}{2}\Gamma(y)(u_+(y)+u_-(y))\delta(x-y)$$

(see for instance [22].)

Applying the operator $\mathfrak{D}_{Q,Q_s}$ to a function $u\in H^1(\mathbb{R}\diagdown\mathbb{Y},\mathbb{C}^2)$ as a distribution in $\mathcal{D}'(\mathbb{R},\mathbb{C}^2)$ we obtain that

$$\mathfrak{D}_{Q,Q_s}u=\mathfrak{D}_Qu+\sum_{y\in\mathbb{Y}}\left(-J(u_+(y)-u_-(y))+\frac{1}{2}\Gamma(y)(u_+(y)+u_-(y))\right)\delta(x-y) \tag{15}$$

where $\mathfrak{D}_Qu$ is a regular distribution defined as

$$\mathfrak{D}_Qu(x)=J\frac{du(x)}{dx}+Q(x)u(x),x\in\mathbb{R}\diagdown\mathbb{Y}. \tag{16}$$

Since we are going to consider $\mathfrak{D}_{Q,Q_s}$ as unbounded operator in $L^2(\mathbb{R},\mathbb{C}^2)$ the singular terms have to disappear in the right hand side in (15). Hence the following conditions should be satisfied at every point $y\in\mathbb{Y}$

$$\frac{1}{2}\Gamma(y)(u_+(y)+u_-(y))=J(u_+(y)-u_-(y)). \tag{17}$$

Conditions (17) can be written as

$$A(y)u_+(y) = B(y)u_-(y),\ y \in \mathbb{Y} \tag{18}$$

with

$$A(y) = \frac{1}{2}\Gamma(y) - J,\ B(y) = -\left(\frac{1}{2}\Gamma(y) + J\right). \tag{19}$$

We set

$$H^1_{A,B}(\mathbb{R}\diagdown\mathbb{Y}, \mathbb{C}^2) = \left\{u \in H^1(\mathbb{R}\diagdown\mathbb{Y}, \mathbb{C}^2) : A(y)u_+(y) = B(y)u_-(y),\ y \in \mathbb{Y}\right\}. \tag{20}$$

Assume that there exist inverse matrices $A^{-1}(y)$ for every $y \in \mathbb{Y}$. Then the interaction condition (18) can be written as

$$u_+(y) = C(y)u_-(y),\ C(y) = A^{-1}(y)B(y),\ y \in \mathbb{Y}, \tag{21}$$

and we use the notation

$$H^1_{\mathbb{I},C}(\mathbb{R}\diagdown\mathbb{Y}, \mathbb{C}^2) = \left\{u \in H^1(\mathbb{R}\diagdown\mathbb{Y}, \mathbb{C}^2) : u_+(y) = C(y)u_-(y),\ y \in \mathbb{Y}\right\}.$$

Let $u, v \in H^1_{\mathbb{I},C}(\mathbb{R}\diagdown\mathbb{Y}, \mathbb{C}^2)$. Then integrating by parts we obtain

$$\left(\left(J\frac{d}{dx} + Q\right)u, v\right) = \left(u, \left(J\frac{d}{dx} + \bar{Q}\right)v\right) - \sum_{j=1}^{\infty}\left(Ju_+(y_j)\cdot v_+(y_j) - Ju_-(y_j)\cdot v_-(y_j)\right) \tag{22}$$

where

$$x \cdot y = x^1\bar{y}^1 + x^2\bar{y}^2.$$

Note that

$$\begin{aligned} &Ju_+(y)\cdot v_+(y) - Ju_-(y)\cdot v_-(y) \qquad (23)\\ &= \left(-u^2_+(y)\bar{v}^1_+(y) + u^1_+(y)\bar{v}^2_+(y)\right) - \left(-u^2_-(y)\bar{v}^1_-(y) + u^1_-(y)\bar{v}^2_-(y)\right) = \\ &\det\begin{pmatrix} u^1_+(y) & \bar{v}^1_+(y) \\ u^2_+(y) & \bar{v}^2_+(y)\end{pmatrix} - \det\begin{pmatrix} u^1_-(y) & \bar{v}^1_-(y) \\ u^2_-(y) & \bar{v}^2_-(y)\end{pmatrix}\end{aligned}$$

$$= \det\left(C(y)\begin{pmatrix} u_-^1(y) \\ u_-^2(y) \end{pmatrix}, C(y)\begin{pmatrix} \bar{v}_-^1(y) \\ \bar{v}_-^2(y) \end{pmatrix}\right) - \det\begin{pmatrix} u_-^1(y) & \bar{v}_-^1(y) \\ u_-^2(y) & \bar{v}_-^2(y) \end{pmatrix}$$

$$= \big(\det C(y) - 1\big) \det\begin{pmatrix} u_-^1(y) & \bar{v}_-^1(y) \\ u_-^2(y) & \bar{v}_-^2(y) \end{pmatrix}.$$

Hence

$$\left(\left(J\frac{d}{dx} + Q\right)u, v\right) = \left(u, \left(J\frac{d}{dx} + \bar{Q}\right)v\right) \tag{24}$$

if and only if the matrix $C(y)$ is real-valued and

$$\det C(y) = 1 \text{ for every } y \in \mathbb{Y}. \tag{25}$$

Hence $\mathfrak{D}_{Q,\mathbb{I},C}$ is symmetric (formally self-adjoint) operator if the potential Q, and the matrix C are real-valued, and condition (25) holds.

Example 1 Let

$$\Gamma(y) = \begin{pmatrix} 2\alpha(y) & 0 \\ 0 & 0 \end{pmatrix}, \alpha(y) \in \mathbb{R}, y \in \mathbb{Y}.$$

Then

$$A(y) = \frac{1}{2}\Gamma(y) - J = \begin{pmatrix} \alpha(y) & 1 \\ -1 & 0 \end{pmatrix}, B(y) = -\left(\frac{1}{2}\Gamma + J\right) = \begin{pmatrix} -\alpha(y) & 1 \\ -1 & 0 \end{pmatrix},$$

and

$$C(y) = \begin{pmatrix} 1 & 0 \\ -2\alpha(y) & 1 \end{pmatrix}. \tag{26}$$

Example 2 Let

$$\Gamma(y) = \begin{pmatrix} 0 & 0 \\ 0 & 2\beta(y) \end{pmatrix}, \beta(y) \in \mathbb{R}, y \in \mathbb{Y}.$$

Then as above we obtain that

$$C(y) = \begin{pmatrix} 1 & -2\beta(y) \\ 0 & 1 \end{pmatrix}, y \in \mathbb{Y}. \tag{27}$$

2.2 A Priori Estimate for Dirac Operators

Proposition 3 *Let $p, q, r \in L^\infty(\mathbb{R})$. Then there exists a constant $C > 0$ such that for every $u \in H^1_{A,B}(\mathbb{R}\diagdown\mathbb{Y}, \mathbb{C}^2)$*

$$\|u\|_{H^1(\mathbb{R},\mathbb{C}^2)} \le C\left(\left\|\mathfrak{D}_Q u\right\|_{L^2(\mathbb{R},\mathbb{C}^2)} + \|u\|_{L^2(\mathbb{R},\mathbb{C}^2)}\right). \tag{28}$$

Proof The proof is similar to the proof of a priori estimates for solutions of boundary value problems for elliptic partial differential equations (see for instance [2, Chap. 2]). Let $I_\varepsilon(x_0) = \left\{x \in \mathbb{R} : |x - x_0| < \varepsilon\right\}$ and $x_j \in \mathbb{R}\diagdown\mathbb{Y}$. It follows from the ellipticity of $\mathfrak{D}_Q$ that there exists $\varepsilon > 0$ such that $I_\varepsilon(x_j) \cap \mathbb{Y} = \varnothing$ and an operator $L^\varepsilon_{x_j} \in \mathcal{B}(L^2(\mathbb{R}, \mathbb{C}^2), H^1(\mathbb{R}, \mathbb{C}^2))$ such that

$$L_{x_j}\mathfrak{D}_Q\chi_{x_j}I = \chi_{x_j}I + T_{x_j}\chi_{x_j}I \tag{29}$$

for every $\chi_{x_j} \in C_0^\infty(I_\varepsilon(x_j))$, $\varepsilon > 0$ where $T_{x_j} \in \mathcal{B}(L^2(\mathbb{R}, \mathbb{C}^2), H^1(\mathbb{R}, \mathbb{C}^2))$. Let $x_j \in \mathbb{Y}$, and $\mathfrak{D}^0 = J\frac{d}{dx}$ on $I_\varepsilon(x_j)$. Then the operator $\mathfrak{D}^0_{x_j} : H^1_{A(x_j),B(x_j)}(I_\varepsilon(x_j)) \to L^2(\mathbb{R}, \mathbb{C}^2)$ is surjective and it has a kernel of the dimension less or equal 2. It implies that there exists a right locally inverse operator $R^0_{x_j}$ for the operator $\mathfrak{D}^0_{x_j}$. That is

$$R^0_{x_j}\mathfrak{D}^0_{x_j}\chi_{x_j}I = \chi_{x_j}I + P_j\chi_{x_j}I$$

where P_j is the projector on the kernel of operator $\mathfrak{D}^0_{x_j}$ with interaction conditions

$$u_+(y) = C(y)u_-(y).$$

Note that the projector P_j is the integral operator

$$P_j u(x) = \int_{I_\varepsilon(x_j)} k_{P_j}(x, y)u(y)dy, \, x \in I_\varepsilon(x_j)$$

where $k_{P_j}(x, y) \in C^\infty(\overline{I_\varepsilon(x_j)} \times \overline{I_\varepsilon(x_j)})$. It implies that for an enough small $\varepsilon > 0$ there exits a left local regularizator $L^\varepsilon_{x_j}$ such that

$$L^\varepsilon_{x_j}\mathfrak{D}_Q\chi_{x_j}I = \chi_{x_j}I + T_{x_j}\chi_{x_j}I \tag{30}$$

for every $\chi_{x_j} \in C_0^\infty(I_\varepsilon(x_j))$. Thus there exists a countable covering $\left\{I_\varepsilon(x_j)\right\}_{j=1}^\infty$ of $\mathbb{R}$ by the intervals $I_\varepsilon(x_j)$ of a finite multiplicity N, and operators L^ε_j such that

$$L^\varepsilon_j\mathfrak{D}_Q\chi_{x_j}I = \chi_{x_j}I + T^\varepsilon_j\chi_{x_j}I$$

where $\chi_{x_j} \in C_0^\infty(I_\varepsilon(x_j))$, and

$$\sup_j \left\| L_j^\varepsilon \right\|_{\mathcal{B}L^2(\mathbb{R},\mathbb{C}^2),H^1(\mathbb{R},\mathbb{C}^2))} < \infty, \tag{31}$$

and

$$\sup_j \left\| T_j^\varepsilon \right\|_{\mathcal{B}L^2(\mathbb{R},\mathbb{C}^2),H^1(\mathbb{R},\mathbb{C}^2))} < \infty. \tag{32}$$

Let $\left\{\varphi_j\right\}_{j=1}^\infty, \varphi_j \in C_0^\infty(I_\varepsilon(x_j))$ be a partition of unity

$$\sum_{j=1}^\infty \varphi_j(x) = 1, x \in \mathbb{R}$$

subordinate to the covering $\left\{I_\varepsilon(x_j)\right\}_{j=1}^\infty$. Let $\psi_j, \phi_j \in C_0^\infty(I_{x_j}^\varepsilon)$ and $\varphi_j\psi_j = \varphi_j, \psi_j\phi_j = \psi_j, j \in \mathbb{N}$. We set

$$Lw = \sum_{j=1}^\infty \varphi_j L_j^\varepsilon \psi_j w \tag{33}$$

where $w \in C_0^\infty(\mathbb{R})$. Then the right part side in (33) is a finite sum, hence $Lw \in H^1(\mathbb{R}, \mathbb{C}^2)$. Applying the finite multiplicity of the covering $\left\{I_\varepsilon(x_j)\right\}_{j=1}^\infty$ we obtain (see for instance [26], Proposition 2.2.2) the inequality

$$\|Lw\|_{\mathcal{B}L^2(\mathbb{R},\mathbb{C}^2),H^1(\mathbb{R},\mathbb{C}^2))} \leq C \sup_{j\in\mathbb{N}} \left\| L_j^\varepsilon \right\|_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2),H^1(\mathbb{R},\mathbb{C}^2))} \|w\|_{L^2(\mathbb{R},\mathbb{C}^2)}, w \in C_0^\infty(\mathbb{R}).$$

Hence the operator L can be continued to a bounded operator from $L^2(\mathbb{R}, \mathbb{C}^2)$ into $H^1(\mathbb{R}, \mathbb{C}^2)$. Then

$$\begin{aligned} L\mathfrak{D}_Q &= \sum_{j=1}^\infty \varphi_j L_j^\varepsilon \psi_j \mathfrak{D}_Q = \sum_{j=1}^\infty \varphi_j L_j^\varepsilon \psi_j \mathfrak{D}_Q \phi_j I \\ &= \sum_{j=1}^\infty \varphi_j L_j^\varepsilon \mathfrak{D}_Q \psi_j I + \sum_{j=1}^\infty \varphi_j L_j^\varepsilon \left[\psi_j I, \mathfrak{D}_Q\right] \phi_j I \\ &= I + \sum_{j=1}^\infty \varphi_j T_j^\varepsilon \psi_j I + \sum_{j=1}^\infty \varphi_j L_j^\varepsilon \left[\psi_j I, \mathfrak{D}_Q\right] \phi_j I. \end{aligned}$$

By estimates (31) and (32) and the finite multiplicity N of the covering $\left\{I_\varepsilon(x_j)\right\}_{j=1}^{\infty}$ we obtain that

$$\left\| \sum_{j=1}^{\infty} \varphi_j T_j^{\varepsilon} \psi_j I \right\|_{\mathcal{B}L^2(\mathbb{R},\mathbb{C}^2)\ ,H^1(\mathbb{R},\mathbb{C}^2))} \leq C \sup_j \left\| T_j \right\|_{\mathcal{B}L^2(\mathbb{R},\mathbb{C}^2)\ ,H^1(\mathbb{R},\mathbb{C}^2))} \leq C_1$$

and

$$\left\| \sum_{j=1}^{\infty} \varphi_j L_j^{\varepsilon} \left[\psi_j I, \mathfrak{D}_Q\right] \phi_j \right\|_{\mathcal{B}L^2(\mathbb{R},\mathbb{C}^2)\ ,H^1(\mathbb{R},\mathbb{C}^2))}$$
$$\leq C \sup_j \left\| L_j^{\varepsilon} \right\| \sup_j \left\| \left[\psi_j I, \mathfrak{D}_Q\right] \right\|_{\mathcal{B}L^2(\mathbb{R},\mathbb{C}^2)\ ,H^1(\mathbb{R},\mathbb{C}^2))} \leq C_2.$$

Hence

$$L\mathfrak{D}_Q = I + T \tag{34}$$

where $T \in \mathcal{B}L^2(\mathbb{R}, \mathbb{C}^2)\ , H^1(\mathbb{R}, \mathbb{C}^2))$. Equality (34) implies a priori estimate (28). □

2.3 Parameter Dependent Dirac Operators

Let $y \in \mathbb{Y}$, $\mathbb{R}_y^{\pm} = \left\{x \in \mathbb{R} : x \lessgtr y\right\}$, and

$$H^1_{\mathbb{I},C(y)}\left(\mathbb{R}, \mathbb{C}^2\right)$$
$$= \left\{u \in H^1\left(\mathbb{R} \diagdown \{y\}, \mathbb{C}^2\right) = H^1\left(\mathbb{R}_y^{+}, \mathbb{C}^2\right) \oplus H^1\left(\mathbb{R}_y^{-}, \mathbb{C}^2\right) : u_+(y) = C(y)u_-(y)\right\}.$$

We denote by

$$\varphi_{\pm}(x) = \begin{pmatrix} 1 \\ \pm i \end{pmatrix} e^{\pm|\mu|(x-y)}, x \in \mathbb{R}$$

two linearly independent solutions of equation

$$\mathfrak{D}_0(i\mu)\varphi(x) = 0, x \in \mathbb{R}, \mu \in \mathbb{R} \diagdown \{0\}. \tag{35}$$

in the space $H^1(\mathbb{R}, \mathbb{C}^2)$. Then the subspace of solutions of Eq. (35) in $H^1_{\mathbb{I},C(y)}\left(\mathbb{R}, \mathbb{C}^2\right)$ is one-dimensional, and generated by the function

$$\varphi(x,\eta) = \begin{cases} \eta\varphi_-(x), x > y \\ \varphi_+(x), x < y \end{cases},$$

where $\eta \in \mathbb{C}$ is such that the interaction condition $\varphi_+(x,\eta) = C(y)\varphi_+(x,\eta)$ holds. Then we obtain that $\eta \in \mathbb{C}$ have to satisfy the equality

$$C(y)\begin{pmatrix} 1 \\ -i \end{pmatrix} = \eta \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Proposition 4 *Let the matrix $C(y)$ in the interaction conditions be such that the vectors $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $C(y)\begin{pmatrix} 1 \\ -i \end{pmatrix}$ are linearly independent. Then the operator*

$$\mathfrak{D}_0(i\mu) : H^1_{\mathbb{I},C(y)}(\mathbb{R}\diagdown\left\{y\right\}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$$

is invertible for every $\mu \in \mathbb{R}\diagdown\{0\}$ and

$$\left\|\mathfrak{D}_0^{-1}(i\mu)\right\|_{\mathcal{B}L^2(\mathbb{R},\mathbb{C}^2),H^1_{\mathbb{I},C(Y)}(\mathbb{R}\diagdown\{y\},\mathbb{C}^2))} \leq C\left(1+\left|\mu\right|\right)^{-1}, C > 0. \tag{36}$$

Proof We consider the equation

$$\mathfrak{D}_0(i\mu)u(x) = f(x), x \in \mathbb{R}\diagdown\left\{y\right\}, f \in L^2(\mathbb{R}, \mathbb{C}^2), \mu \in \mathbb{R}\diagdown\{0\} \tag{37}$$

where $u \in H^1(\mathbb{R}\diagdown\left\{y\right\}, \mathbb{C}^2)$ and u satisfies the interaction conditions

$$u_+(y) = C(y)u_-(y). \tag{38}$$

The general solution of Eq. (72) in the space $H^1(\mathbb{R}\diagdown\left\{y\right\}, \mathbb{C}^2)$ is

$$u(x) = \begin{cases} u_+(x) = \mathfrak{D}_0^{-1}(i\mu)f^+(x) + \gamma_+\begin{pmatrix} 1 \\ i \end{pmatrix}e^{|\mu|x}, x > 0 \\ u_-(x) = \mathfrak{D}_0^{-1}(i\mu)f^-(x) + \gamma_-\begin{pmatrix} 1 \\ -i \end{pmatrix}e^{|\mu|x}, x < 0 \end{cases} \tag{39}$$

where $f^+ = \theta_+ f$, $f^- = \theta_- f$, $\theta^{\pm}$ are characteristic functions of $\mathbb{R}_{\pm}$,and $\gamma_{\pm} \in \mathbb{C}$.

Condition (38) implies that

$$\left(\mathfrak{D}_0^{-1}(i\mu)f^+\right)_+(y)+\gamma_+\begin{pmatrix}1\\ i\end{pmatrix}=C(y)\left(\mathfrak{D}_0^{-1}(i\mu)f^-\right)_-(y)+\gamma_-C(y)\begin{pmatrix}1\\ -i\end{pmatrix}. \tag{40}$$

We set

$$\psi(\mu,y)=C(y)(\mathfrak{D}_0^{-1}(i\mu)f^-)_-(y)-(\mathfrak{D}_0^{-1}(i\mu)f^+)_+(y).$$

Then we obtain the system of linear equations for the definition of $\gamma_\pm$

$$\gamma_+\begin{pmatrix}1\\ i\end{pmatrix}-\gamma_-C(y)\begin{pmatrix}1\\ -i\end{pmatrix}=\psi(\mu,y). \tag{41}$$

It follows from the condition of Proposition 4 that the vectors $\begin{pmatrix}1\\ i\end{pmatrix}$ and $C(y)\begin{pmatrix}1\\ -i\end{pmatrix}$ are linear independent. Hence the system (41) has the unique solution $\gamma_\pm=\gamma_\pm(f)$ and the operator $\mathfrak{D}_0(i\mu):H^1_{\mathbb{I},C(y)}\left(\mathbb{R},\mathbb{C}^2\right)\to L^2\left(\mathbb{R},\mathbb{C}^2\right)$ is invertible, and

$$\begin{aligned}&\mathfrak{D}^{-1}_{0,C(y)}(i\mu)f(x)\\ &=P_+\mathfrak{D}_0^{-1}(i\mu)f^+(x)+P_-\mathfrak{D}_0^{-1}(i\mu)f^-(x)\\ &+\gamma_+(f)\theta_+(x)e^{-|\mu|x}+\gamma_-(f)\theta_-(x)e^{-|\mu|x},\ x\in\mathbb{R}\end{aligned} \tag{42}$$

where $P_\pm:H^1(\mathbb{R},\mathbb{C}^2)\to H^1(\mathbb{R}_\pm,\mathbb{C}^2)$ are the restriction operators. Estimate (36) follows from formula (42). □

Proposition 5 *Let the vectors* $\begin{pmatrix}1\\ i\end{pmatrix}$ *and* $C(y)\begin{pmatrix}1\\ -i\end{pmatrix}$ *be linearly independent for every* $y\in\mathbb{Y}$. *Then there exists* $\rho>0$ *such that the operator* $\mathfrak{D}_0(i\mu)$: $H^1_{\mathbb{I},C}(\mathbb{R}\backslash\mathbb{Y},\mathbb{C}^2)\to L^2(\mathbb{R},\mathbb{C}^2)$ *is invertible for all* $\mu\in\mathbb{R}:\left|\mu\right|>\rho$.

Proof The proof is similar to the proof of invertibility of operators of elliptic boundary value problems with parameter (see for instance [2, Chap. 3]). We introduce the Sobolev space $H^1\left(\mathbb{R},\mathbb{C}^2,\mu\right)$ with norm depending on the parameter $\mu\in\mathbb{R}$

$$\|u\|_{H^1(\mathbb{R},\mathbb{C}^2}=\int_{\mathbb{R}}(1+\mu^2+\xi^2)\|\hat{u}(\xi)\|^2_{\mathbb{C}^2}d\xi)^{1/2}. \tag{43}$$

Note that the norm $\|u\|_{H^1(\mathbb{R},\mathbb{C}^2,\mu)}$ is equivalent to the norm $\|u\|_{H^1(\mathbb{R},\mathbb{C}^2)}$ for every $\mu \in \mathbb{R}$. The similar way we define the space $H^1_{\mathbb{I},C(y)}\left(\mathbb{R}\diagdown\{y\},\mathbb{C}^2,\mu\right)$ depending on the parameter $\mu \in \mathbb{R}$. Conditions of Proposition 5 yield that the operators

$$\mathfrak{D}_0(i\mu) : H^1_{\mathbb{I},C(y)}\left(\mathbb{R}\diagdown\{y\},\mathbb{C}^2,\mu\right) \to L^2(\mathbb{R},\mathbb{C}^2)$$

are invertible, and there exists $C > 0$ such that

$$\left\|\mathfrak{D}_0(i\mu)^{-1}\right\|_{\mathcal{B}\left(L^2(\mathbb{R},\mathbb{C}^2),H^1(\mathbb{R},\mathbb{C}^2,\mu)\right)} \le C, \tag{44}$$

and

$$\left\|\mathfrak{D}_0(i\mu)^{-1}\right\|_{\mathcal{B}\left(L^2(\mathbb{R},\mathbb{C}^2),H^1_{\mathbb{I},C(y)}(\mathbb{R}\diagdown\{y\},\mathbb{C}^2,\mu)\right)} \le C \tag{45}$$

for every $y \in \mathbb{Y}$. The standard arguments of perturbation theory imply that there exists $\varepsilon > 0$, $\rho_0 > 0$ and a countable covering $\cup_{j=1}^{\infty} I_\varepsilon(x_j)$ of $\mathbb{R}$ by open intervals $I_\varepsilon(x_j) = \left\{x \in \mathbb{R} : \left|x - x_j\right| < \varepsilon\right\}$ and a system of uniformly bounded with respect to j and $\mu : |\mu| > \rho_0$ operators $R_{x_j}(\mu), L_{x_j}(\mu) \in \mathcal{B}(L^2(\mathbb{R}), H^1_{x_j}\left(\mathbb{R},\mathbb{C}^2,\mu\right))$ where

$$H^1_{x_j}\left(\mathbb{R},\mathbb{C}^2,\mu\right)) = \begin{cases} H^1\left(\mathbb{R},\mathbb{C}^2,\mu\right)), x_j \notin \mathbb{Y} \\ H^1\left(\mathbb{R}\diagdown\{x_j\},\mathbb{C}^2,\mu\right)), x_j \in \mathbb{Y}. \end{cases}$$

such that

$$L_{x_j}(\mu)\mathfrak{D}_0(i\mu)\eta_j I = \eta_j I$$

$$\eta_j \mathfrak{D}_0(i\mu) R_{x_j}(\mu) = \eta_j I$$

for every function $\eta_j \in C_0^\infty(I_\varepsilon(x_j))$. Let

$$\sum_{j\in\mathbb{Z}} \varphi_{j,\varepsilon}(x) = 1, x \in \mathbb{R} \tag{46}$$

be the partition of unity subordinated to the covering $\cup_{j=1}^{\infty} I_\varepsilon(x_j)$ where $\varphi_{j,\varepsilon} \in C_0^\infty(I_\varepsilon(x_j))$. Let $\psi_{j,\varepsilon} \in C_0^\infty(I_\varepsilon(x_j))$ be such that $\varphi_{j,\varepsilon}\psi_{j,\varepsilon} = \varphi_{j,\varepsilon}$.

Then we set

$$L^{\varepsilon}(\mu)f = \sum_{j\in\mathbb{N}} \varphi_{j,\varepsilon} L_{x_j}(\mu)\psi_{j,\varepsilon} f\ ,\ \ f \in C_0^{\infty}(\mathbb{R}, \mathbb{C}^2). \tag{47}$$

As in the proof of Proposition 3 we obtain that

$$\left\| L^{\varepsilon}(\mu) f \right\|_{H^1(\mathbb{R},\mathbb{C}^2,\mu)} \le C \left\| f \right\|_{L^2(\mathbb{R},\mathbb{C}^2)},\ f \in C_0^{\infty}(\mathbb{R}, \mathbb{C}^2). \tag{48}$$

Estimate (48) yields that the operator $L^{\varepsilon}(\mu)$ is continued to a bounded operator from $L^2(\mathbb{R}, \mathbb{C}^2)$ into $H^1\left(\mathbb{R}, \mathbb{C}^2, \mu\right)$. Let $\phi_{j,\varepsilon} \in C_0^{\infty}(I_{\varepsilon}(x_j))$ and $\psi_{j,\varepsilon}\phi_{j,\varepsilon} = \psi_{j,\varepsilon}$. Hence $supp\psi_{j,\varepsilon} \cap\ supp(1 - \phi_{j,\varepsilon}) = \varnothing$. The definition of $\mathfrak{D}_0(i\mu)$ yields that $\psi_{j,\varepsilon}\mathfrak{D}_0(i\mu)(1 - \phi_{j,\varepsilon}) = 0$. Hence $\psi_{j,\varepsilon}\mathfrak{D}_0(i\mu) = \psi_{j,\varepsilon}\mathfrak{D}_0(i\mu)\phi_{j,\varepsilon}I$. Then

$$\begin{aligned} L^{\varepsilon}(\mu)\mathfrak{D}_0(i\mu) = \sum_{j=-\infty}^{\infty} \varphi_{j,\varepsilon} L_{x_j}(\mu)\psi_{j,\varepsilon}\mathfrak{D}_0(i\mu) = \sum_{j=-\infty}^{\infty} \varphi_{j,\varepsilon} L_{x_j}(\mu)\psi_{j,\varepsilon}\mathfrak{D}_0(i\mu)\phi_{j,\varepsilon} I \\ = \sum_{j=1}^{\infty} \varphi_{j,\varepsilon} L_{x_j}(\mu)\psi_{j,\varepsilon}\mathfrak{D}_0(i\mu)\psi_{j,\varepsilon} I + \sum_{j=1}^{\infty} \varphi_{j,\varepsilon} L_{x_j}(\mu)\left[\psi_{j,\varepsilon}, \mathfrak{D}_0(i\mu)\right]\phi_{j,\varepsilon} I \\ = I + T^{\varepsilon}(\mu), \end{aligned} \tag{49}$$

where

$$T^{\varepsilon}(\mu) = \sum_{j=1}^{\infty} \varphi_{j,\varepsilon} L_{x_j}(\mu)\left[\psi_{j,\varepsilon}, \mathfrak{D}(i\mu)\right]\phi_{j,\varepsilon} I.$$

Applying the finite multiplicity of covering $\left\{I_{\varepsilon}(x_j)\right\}_{j=-N}^{\infty}$ we obtain that

$$\begin{aligned} \left\| T^{\varepsilon}(\mu) \right\|_{\mathcal{B}H^1(\mathbb{R}\setminus\mathbb{Y},\mathbb{C}^2,\mu))} &\le C \left\| \left[\psi_{j,\varepsilon}, \mathfrak{D}_0(i\mu)\right] \right\|_{\mathcal{B}H^1(\mathbb{R}\setminus\mathbb{Y},\mathbb{C}^2,\mu),L^2(\mathbb{R},\mathbb{C}^2))} \\ &\le C_1 \left(1 + \left|\mu\right|\right)^{-1}. \end{aligned} \tag{50}$$

The estimate (50) implies that there exists $\rho > \rho_0$ such that

$$\left\| T^{\varepsilon}(\mu) \right\|_{\mathcal{B}H^1_{\mathbb{I},C}(\mathbb{R}\setminus\mathbb{Y},\mathbb{C}^2,\mu))} < 1/2$$

for all $\mu \in \mathbb{R} : |\mu| > \rho$. Hence there exist a left inverse operator $\mathcal{L}^{\varepsilon}(\mu) = \left(I + T^{\varepsilon}(\mu)\right)^{-1} L^{\varepsilon}(\mu)$ of $\mathfrak{D}_0(i\mu) : H^1_{\mathbb{I},C}(\mathbb{R}\setminus\mathbb{Y}, \mathbb{C}^2, \mu) \to L^2(\mathbb{R}, \mathbb{C}^2)$ for $\left|\mu\right| > \rho$ where $\rho > 0$ is large enough. Since the norm in $H^1(\mathbb{R}\setminus\mathbb{Y}, \mathbb{C}^2, \mu)$ is equivalent to the norm in $H^1(\mathbb{R}\backslash\mathbb{Y},\mathbb{C}^2)$ we obtain that the operator $\mathfrak{D}_0(i\mu):H^1_{\mathbb{I},C}(\mathbb{R}\backslash\mathbb{Y},\mathbb{C}^2)\to L^2(\mathbb{R},\mathbb{C}^2)$ is invertible for all $\mu\in\mathbb{R}:|\mu|>\rho$. □

Corollary 6 *Let $p,q,r\in L^\infty(\mathbb{R})$ and conditions of Proposition 5 hold. Then there exists $\rho>0$ such that the operator $\mathfrak{D}_Q(i\mu)=\mathfrak{D}_0(i\mu)+QI:H^1_{\mathbb{I},C}\left(\mathbb{R}\backslash\mathbb{Y},\mathbb{C}^2\right)\to L^2(\mathbb{R},\mathbb{C}^2)$ is invertible for all $\mu\in\mathbb{R}:|\mu|>\rho$.*

Proof Note that

$$\|QI\|_{\mathcal{B}H^1(\mathbb{R}\backslash\mathbb{Y},\mathbb{C}^2,\mu),L^2(\mathbb{R},\mathbb{C}^2))}\leq C(1+|\mu|)^{-1}. \tag{51}$$

Hence Proposition 5 and estimate (51) yield the invertibility of $\mathfrak{D}_Q(i\mu)$ for all $\mu\in\mathbb{R}$, with $|\mu|$ is large enough. □

2.4 Self-adjointness of Dirac Operators with Interactions

We denote by $\mathrm{D}_{Q,\mathbb{I},C}$ the unbounded operator in $L^2(\mathbb{R})$ defined by the Dirac operator $\mathfrak{D}_Q$ with dense in $L^2(\mathbb{R},\mathbb{C}^2)$ domain $dom\mathrm{D}_{Q,\mathbb{I},C}=H^1_{\mathbb{I},C}(\mathbb{R}\diagdown\mathbb{Y},\mathbb{C}^2)$.

Theorem 7 *Let $p,q,r\in L^\infty(\mathbb{R})$ be real-valued functions and the matrix $C(y)$ in the interaction conditions $u_+(y)=C(y)u_-(y)$, $y\in\mathbb{Y}$ satisfies conditions of Proposition 5. Moreover we assume that $C(y)$ is a real matrix , and* $\det C(y)=1$ *for every $y\in\mathbb{Y}$. Then the unbounded operator $\mathrm{D}_{Q,\mathbb{I},C}$ with domain $H^1_{\mathbb{I},C}\left(\mathbb{R}\backslash\mathbb{Y},\mathbb{C}^2\right)$ is self-adjoint in $L^2(\mathbb{R},\mathbb{C}^2)$.*

Proof It follows from a priori estimate (26) that the operator $\mathrm{D}_{Q,\mathbb{I},C}$ with domain $H^1_{\mathbb{I},C}\left(\mathbb{R}\backslash\mathbb{Y},\mathbb{C}^2\right)$ is closed in $L^2(\mathbb{R},\mathbb{C}^2)$. Since p,q,r and the entries of the matrix C are real-valued functions, and $\det C(y)=1$, for every $y\in\mathbb{Y}$ the equality

$$\left(\left(J\frac{d}{dx}+Q\right)u,v\right)_{L^2(\mathbb{R},\mathbb{C}^2)}=\left(u,\left(J\frac{d}{dx}+Q\right)v\right)_{L^2(\mathbb{R},\mathbb{C}^2)};u,v\in H^1_{\mathbb{I},C}(\mathbb{R}\backslash\mathbb{Y},\mathbb{C}^2)$$

holds. Hence the operator $\mathrm{D}_{Q,\mathbb{I},C}$ is symmetric. Moreover, Corollary 6 yields that the deficiency indices $N_\pm(\mathrm{D}_{Q,\mathbb{I},C})$ of operator $\mathrm{D}_{Q,\mathbb{I},C}$ equal zero. Hence (see for instance [6], page 100) the operator $\mathrm{D}_{Q,\mathbb{I},C}$ is self-adjoint. □

Example 8 Let the interaction conditions be the form

$$u_+(y)=C(y)u_-(y)$$

where $C(y) = \begin{pmatrix} 1 & 2\alpha(y) \\ 0 & 1 \end{pmatrix}$, $\alpha(y)$ is real-valued function. It is easy to check the vectors $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $C(y)\begin{pmatrix} 1 \\ -i \end{pmatrix}$ are linearly independent.Hence for the real-valued potential Q the operator $\mathrm{D}_{Q,\mathbb{I},C}$ is self-adjoint. In the same way one can prove if $C(y) = \begin{pmatrix} 1 & 0 \\ 2\beta(y) & 1 \end{pmatrix}$, $\beta(y) \in \mathbb{R}$ the operator $\mathrm{D}_{Q,\mathbb{I},C}$ is self-adjoint.

3 Fredholm Theory and Essential Spectrum of Dirac Operators with Delta-Interactions

3.1 Local Principle in the Fredholm Theory of Dirac Operators

We consider the Fredholm property of Dirac operator $\mathfrak{D}_Q$ as a bounded operator acting from $H^1_{A,B}(\mathbb{R}\setminus\mathbb{Y}, \mathbb{C}^2)$ into $L^2(\mathbb{R}, \mathbb{C}^2)$. For the investigation of the Fredholm property of $\mathfrak{D}_Q$ we will use the Simonenko local principle [37] modified for the differential and pseudodifferential operators in [36].

We denote by $\dot{\mathbb{R}}$ the compactification of $\mathbb{R}$ obtained by joining to $\mathbb{R}$ the infinitely distant point ∞.

Definition 9 We say that the operator $\mathfrak{D}_Q : H^1_{A,B}(\mathbb{R}\setminus\mathbb{Y}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$ is locally Fredholm at the point $x \in \mathbb{R}$ if there exists a neighborhood $I_\varepsilon(x)$ and an operators $L^\varepsilon_x, R^\varepsilon_x \in \mathcal{B}(L^2(\mathbb{R}, \mathbb{C}^2), H^1_{A,B}(\mathbb{R}, \mathbb{C}^2))$ such that

$$L^\varepsilon_x \mathfrak{D}_Q \chi_x I = \chi_x I + T'_x \chi_x I, \chi_x \mathfrak{D}_Q R^\varepsilon_x = \chi_x I + T''_x \chi_x I, \tag{52}$$

for every $\chi_x \in C^\infty_0(I_\varepsilon(x))$, $\varepsilon > 0$ where $T'_{x_0} \in \mathcal{K}(H^1(\mathbb{R}, \mathbb{C}^2)), T''_x \in \mathcal{K}(L^2(\mathbb{R}, \mathbb{C}^2))$.

Definition 10 Let $\varphi \in C^\infty(\mathbb{R})$ and $\varphi(x) = 1$ for $|x| \geq 1$ and $\varphi(x) = 0$ for $|x| \leq 1/2$, $\varphi_R(x) = \varphi(\frac{x}{R})$, $R > 0$. We say that the operator $\mathfrak{D}_Q : H^1_{A,B}(\mathbb{R}\setminus\mathbb{Y}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$ is locally invertible at the point ∞ if there exists $R > 0$ and operators $\mathcal{L}_R, \mathcal{R}_R \in \mathcal{B}L^2(\mathbb{R}, \mathbb{C}^2), H^1_{A,B}(\mathbb{R}, \mathbb{C}^2))$ such that

$$\mathcal{L}_R \mathfrak{D}_Q \varphi_R I = \varphi_R I, \varphi_R \mathfrak{D}_Q \mathcal{R}_R = \varphi_R I.$$

Proposition 11 *The operator $\mathfrak{D}_{Q,A,B} : H^1_{A,B}(\mathbb{R}\setminus\mathbb{Y}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$ is Fredholm if and only if $\mathfrak{D}_{Q,A,B}$ is locally Fredholm at every point $x \in \mathbb{R}$ and locally invertible at infinitely distant point ∞.*

The proof of Proposition 11 follows from the local principle [36, 37].

3.2 Finite Set $\mathbb{Y}$ of Interactions

3.2.1 Fredholm Theory

Let $\mathbb{Y} = \left\{y_1, y_2, \ldots, y_n\right\}$ where $y_1 < y_2 < \ldots . < y_n$.

We say that a function $a \in L^\infty(\mathbb{R})$ is uniformly continuous at infinity if there exists $R > 0$ such that a is a uniformly continuous function on $\mathbb{R} \diagdown (-R, R)$.

Let $p, q, r \in L^\infty(\mathbb{R})$ and be uniformly continuous functions at infinity. Then the Arcela-Ascoli Theorem implies that for every sequence $\mathbb{Z} \ni g_m \to \infty$ there exists a subsequence $h_m \to \infty$ and limit functions $p_h, q_h, r_h \in C_b(\mathbb{R})$ such that

$$\lim_{m\to\infty} \sup_{x\in K} \left|p(x + h_m) - p_h(x)\right| = 0, \tag{53}$$

$$\lim_{m\to\infty} \sup_{x\in K} \left|q(x + h_m) - q_h(x)\right| = 0,$$

$$\lim_{m\to\infty} \sup_{x\in K} \left|r(x + h_m) - r_h(x)\right| = 0$$

for every compact set $K \subset \mathbb{R} \diagdown (-R, R)$. The Dirac operator

$$\mathfrak{D}_{Q^h} = J\frac{d}{dx} + Q^h(x) \tag{54}$$

with

$$Q^h(x) = \begin{pmatrix} p_h(x) + r_h(x) & q_h(x) \\ q_h(x) & -p_h(x) + r_h(x) \end{pmatrix} \tag{55}$$

is called the limit operator of $\mathfrak{D}_Q$ generated by the sequence $\mathbb{Z} \ni h_m \to \infty$. We denote by $Lim\mathfrak{D}_Q$ the set of all limit operators of $\mathfrak{D}_Q$ generated by the sequence $\mathbb{Z} \ni h_m \to \infty$.

Theorem 12 *Let $p, q, r \in L^\infty(\mathbb{R})$ be uniformly continuous functions at ∞. Then*

$$\mathfrak{D}_Q : H^1_{A,B}\left(\mathbb{R}, \mathbb{C}^2\right) \to L^2(\mathbb{R}, \mathbb{C}^2)$$

is a Fredholm operator if and only if all limit operators $\mathfrak{D}_{Q^h} \in Lim\mathfrak{D}_Q$ are invertible from $H^1(\mathbb{R}, \mathbb{C}^2)$ into $L^2(\mathbb{R}, \mathbb{C}^2)$.

Proof The operator $\mathfrak{D}_Q$ is locally Fredholm at every point $x \in \mathbb{R}$ since $\mathfrak{D}_Q$ is the elliptic operator on $\mathbb{R}$ (see the proof of Proposition 3). Hence by Proposition 11 the operator $\mathfrak{D}_Q$ is a Fredholm operator if and only if $\mathfrak{D}_Q$ is locally invertible at the infinitely distant points. It follows from [23, 27] the operator $\mathfrak{D}_Q$ is locally invertible at the infinitely distant point if and only if all limit operators $\mathfrak{D}_{Q^h} \in Lim\mathfrak{D}_Q$ are invertible from $H^1\left(\mathbb{R}, \mathbb{C}^2\right)$ in $L^2(\mathbb{R}, \mathbb{C}^2)$. □

Theorem 13 *Let $p, q, r \in L^{\infty}(\mathbb{R})$ and uniformly continuous at infinity. Then the essential spectrum of unbounded operator $\mathrm{D}_{Q,A,B}$ is given by the formula*

$$sp_{ess}\mathrm{D}_{Q,A,B} = \bigcup_{\mathrm{D}_{Q^h} \in \mathrm{Lim}\mathrm{D}_Q} sp\mathrm{D}_{Q^h} \tag{56}$$

where D_{Q^h} are unbounded operators generated by

$$\mathfrak{D}_{Q^h} = J\frac{d}{dx} + Q^h$$

with domain $H^1\left(\mathbb{R}, \mathbb{C}^2\right)$, and $Lim\mathrm{D}_Q$ is the set of all such operators.

This theorem is immediate corollary of Theorem 12.

Remark 14 Formula (56) yields that the addition of singular potential with support on a finite set of points does not change the essential spectrum of Dirac operator with regular potential.

3.3 Slowly Oscillating at Infinity Potentials

Definition 15 We say that a function $a \in L^{\infty}(\mathbb{R})$ is slowly oscillating at infinity and belongs to the class $SO_{\infty}(\mathbb{R})$ if there exists $R > 0$ such that

$$\lim_{x\to\infty} \sup_{y\in K} \left|a(x+y) - a(x)\right| = 0 \tag{57}$$

for every compact set $K \subset \left\{x \in \mathbb{R} : |x| > R\right\}$.

Note that if $a \in SO_{\infty}(\mathbb{R})$ then all limit functions defined by the sequences $\mathbb{Z} \ni h_m \to \infty$ are constant (see [26], page 88.)

Theorem 16 *Let $p, q, r \in SO_{\infty}(\mathbb{R})$ be real-valued functions. Then*

$$\mathfrak{D}_{Q,A,B} : H^1_{A,B}\left(\mathbb{R}, \mathbb{C}^2\right) \to L^2(\mathbb{R}, \mathbb{C}^2)$$

is a Fredholm operator if and only if

$$0 \in \left(M_-(Q), M_+(Q)\right)$$

where

$$M_-(Q)=\limsup_{x\to\infty}\left(r(x)-\sqrt{p^2(x)+q^2(x)}\right), \tag{58}$$

$$M_+(Q)=\liminf_{x\to\infty}\left(r(x)+\sqrt{p^2(x)+q^2(x)}\right).$$

Proof Let $p, q, r \in SO_\infty(\mathbb{R})$ be real-valued functions, $\mathbb{Z} \ni h_m \to \infty$, and there exist limits $p_h, q_h, r_h \in \mathbb{R}$ in the sense of formula (53). Then the limit operator $\mathcal{D}_Q^h$ is of the form

$$\mathfrak{D}_Q^h = J\frac{d}{dx} + \begin{pmatrix} p_h + r_h & q_h \\ q_h & -p_h + r_h \end{pmatrix}. \tag{59}$$

It follows from formula (14) that the operator $\mathfrak{D}_Q^h : H^1(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$ is invertible if and only if

$$0 \in \left(r_h - \sqrt{p_h^2 + q_h^2}, r_h + \sqrt{p_h^2 + q_h^2}\right). \tag{60}$$

Then Theorem 12 yields the statement of Theorem 16. □

Corollary 17 *Let $p, q, r \in SO_\infty(\mathbb{R})$ be real-valued functions. Then*

$$sp_{ess}\mathrm{D}_{Q,A,B} = (-\infty, M_-(Q)]\bigcup[M_+(Q), +\infty) \tag{61}$$

Note that $sp_{ess}\mathrm{D}_{Q,A,B} = \mathbb{R}$ if $M_-(\mathfrak{D}_Q) \geq M_+(\mathfrak{D}_Q)$.

3.4 Infinite Set of Interactions

3.4.1 Periodic Dirac Operators

We assume that the set $\mathbb{Y}$ is l-periodic that is $\mathbb{Y} = \mathbb{Y}_0 + l\mathbb{Z}$, $\mathbb{Y}_0 = \left\{0 < y_1 < y_2 < \ldots < y_n < l\right\}$, $\mathfrak{D}_Q$ is the Dirac operator with potential

$$Q(x) = \begin{pmatrix} p(x) + r(x) & q(x) \\ q(x) & -p(x) + r(x) \end{pmatrix}$$

where $p, q, r \in C(\mathbb{R})$ are $l-$periodic functions, the matrix $C(y)$ in the interaction condition is also $l-$ periodic. Let $\mathrm{D}_{Q,\mathbb{I},C}$ be unbounded operator associated with $\mathfrak{D}_Q$ and the interaction conditions $u_+(y) = C(y)u_-(y)$.

The Floquet transform is defined for vector-functions $f \in C_0^\infty(\mathbb{R}, \mathbb{C}^2)$ (see for instance [21]) as

$$\left(\mathcal{F} f\right)(x,\theta) = \widetilde{f}(x,\theta) := \frac{1}{\sqrt{2\pi}} \sum_{\alpha \in l\mathbb{Z}} f(x-\alpha)\, e^{i\alpha\theta}, \quad x \in \mathbb{R},\ \theta \in \left[-\frac{\pi}{l}, \frac{\pi}{l}\right]$$

The operator $\mathcal{F}$ is continued to a unitary operator from $L^2\left(\mathbb{R}, \mathbb{C}^2\right)$ into $\mathcal{H} = L^2\left((0,l), L^2(\left[-\frac{\pi}{l}, \frac{\pi}{l}\right], \mathbb{C}^2\right))$ of vector-valued functions on the interval $(0,l)$ with values in $L^2\left(\left[-\frac{\pi}{l}, \frac{\pi}{l}\right]\right), \mathbb{C}^2)$ with the norm

$$\|u\|_{\mathcal{H}} = \left(\int_0^l \left\|u(x,\ \cdot)\right\|^2_{L^2\left(\left[-\frac{\pi}{l}, \frac{\pi}{l}\right], \mathbb{C}^2\right)} dx\right)^{1/2}.$$

The inverse operator $\mathcal{F}^{-1}$ to the Floquet transform is

$$\left(\mathcal{F}^{-1}\widetilde{f}\right)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{l}}^{\frac{\pi}{l}} \widetilde{f}(x,\theta)\, d\theta.$$

Applying the Floquet transform to the Dirac operator $\mathrm{D}_{Q,\mathbb{I},C}$ we obtain (see for instance [21]) that

$$\mathcal{F}^{-1}\mathrm{D}_{Q,\mathbb{I},C}\mathcal{F} = \bigoplus_{\theta \in \left[-\frac{\pi}{l}, \frac{\pi}{l}\right]} \mathrm{D}^{\theta}_{\mathrm{D}_{Q,\mathbb{I},C}} \tag{62}$$

where $\mathrm{D}^{\theta}_{\mathrm{D}_{Q,\mathbb{I},C}}$ are unbounded operators in $L^2((0,l), \mathbb{C}^2)$ generated by the Dirac operator $\mathfrak{D}_Q$ on the interval $(0,l)$ with the domain

$$dom \mathrm{D}^{\theta}_{\mathrm{D}_{Q,\mathbb{I},C}} = \left\{u \in H^1(0,l) \diagdown \mathbb{Y}_0\right\} : u_+(y,\theta) = C(y) u_-(y,\theta),\ y \in \mathbb{Y}_0$$
$$u(l,\theta) = e^{il\theta} u(0,\theta).$$

Note that the operator $\mathrm{D}^{\theta}_{\mathrm{D}_{Q,\mathbb{I},C}}$ is self-adjoint in $L^2((0,l), \mathbb{C}^2)$ and it has a discrete spectrum

$$\lambda_1(\theta) < \lambda_2(\theta) < \ldots < \lambda_j(\theta) < \ldots; \theta \in \left[-\frac{\pi}{l}, \frac{\pi}{l}\right]$$

where $\lambda_j(\theta)$, $j = 1, 2, \ldots, \ldots$ are continuous functions on $\left[-\frac{\pi}{l}, \frac{\pi}{l}\right]$. Formula (62) yields that

$$spD_{Q,\mathbb{I},C} = \bigcup_{j=1}^{\infty} \left[a_j, b_j\right] \tag{63}$$

where $\left[a_j, b_j\right] = \{\lambda \in \mathbb{R} : \lambda = \lambda_j(\theta), \theta \in [-\frac{\pi}{l}, \frac{\pi}{l}]\}$ (see for instance [38]).

For each $\theta \in \left[-\frac{\pi}{l}, \frac{\pi}{l}\right]$ we consider the spectral problem

$$\begin{aligned} \mathfrak{D}_Q u\,(x, \theta, \lambda) &= \lambda u\,(x, \theta)\,, x \in (0, l) \diagdown \mathbb{Y}_0, \\ u_+\left(y, \theta, \lambda\right) &= C(y) u_-\left(y, \theta, \lambda\right), y \in \mathbb{Y}_0 \\ u(0, \theta, \lambda) &= e^{il\theta} u(l, \theta, \lambda), \theta \in \left[-\frac{\pi}{l}, \frac{\pi}{l}\right] \end{aligned}$$

Solutions of this problem are sought of the form

$$u\,(x, \theta, \lambda) = a_1\,(\theta, \lambda)\,\varphi_1\,(x, \lambda) + a_2\,(\theta, \lambda)\,\varphi_2\,(x, \lambda)\,,$$

where a_1, a_2 are arbitrary coefficients, and φ_1, φ_2 are linearly independent solutions of the Dirac equation

$$\mathfrak{D}_Q \varphi(x, \lambda) = \lambda \varphi(x, \lambda), \quad x \in (0, l) \diagdown \mathbb{Y}_0$$

satisfying the interaction condition

$$\varphi_+\left(y, \lambda\right) = C(y)\varphi_-\left(y, \lambda\right), y \in \mathbb{Y}_0$$

as well as the initial conditions

$$\begin{aligned} \varphi_1^1\,(0, \lambda) &= 1, \varphi_1^2\,(0, \lambda) = 0, \\ \varphi_2^1\,(0, \lambda) &= 0, \varphi_2^2\,(0, \lambda) = 1. \end{aligned}$$

From the quasi-periodic conditions

$$u\,(l, \theta, \lambda) = e^{il\theta} u\,(0, \theta, \lambda)\,, \theta \in \left[-\frac{\pi}{l}, \frac{\pi}{l}\right]$$

we obtain the system of equations

$$\begin{aligned} a_1\,(\theta, \lambda)\,\varphi_1^1\,(l, \lambda) + a_2\,(\theta, \lambda)\,\varphi_2^1\,(l, \lambda) &= e^{il\theta} a_1\,(\theta, \lambda) \\ a_1\,(\theta, \lambda)\,\varphi_1^2\,(l, \lambda) + a_2\,(\theta, \lambda)\,\varphi_2^2\,(l, \lambda) &= e^{il\theta} a_2\,(\theta, \lambda)\,, \end{aligned} \tag{64}$$

with respect to $a_1(\theta,\lambda), a_2(\theta,\lambda)$. System (64) implies that $\begin{pmatrix} a_1(\theta,\lambda) \\ a_2(\theta,\lambda) \end{pmatrix}$ is an eigenvector of the monodromy matrix

$$\mathbf{M}(\lambda) = \begin{pmatrix} \varphi_1^1(l,\lambda) & \varphi_2^1(l,\lambda) \\ \varphi_1^2(l,\lambda) & \varphi_2^2(l,\lambda) \end{pmatrix}$$

associated to the eigenvalue $\mu := e^{il\theta}$. System (64) possesses non-trivial solutions if and only if

$$\det \begin{pmatrix} \varphi_1^1(l,\lambda) - \mu & \varphi_2^1(l,\lambda) \\ \varphi_1^2(l,\lambda) & \varphi_2^2(l,\lambda) - \mu \end{pmatrix} = 0.$$

Taking into account that

$$\det \begin{pmatrix} \varphi_1^1(l,\mu) & \varphi_2^1(l,\mu) \\ \varphi_1^2(l,\mu) & \varphi_2^2(l,\mu) \end{pmatrix} = \det \begin{pmatrix} \varphi_1^1(0,\mu) & \varphi_2^1(0,\mu) \\ \varphi_1^2(0,\mu) & \varphi_2^2(0,\mu) \end{pmatrix} = 1$$

we obtain the dispersion equation

$$\mu^2 - 2\mu D(\lambda) + 1 = 0, \tag{65}$$

where

$$D(\lambda) := \frac{1}{2}\left(\varphi_1^1(l,\lambda) + \varphi_2^2(l,\lambda)\right).$$

Equation (65) has solutions of the form $\mu := e^{i\theta}$, $\theta \in [0, 2\pi]$ if and only if $|D(\lambda)| \leq 1$. Hence

$$sp\mathrm{D}_{Q,\mathbb{I},C} = \left\{\lambda \in \mathbb{R} : |D(\lambda)| \leq 1\right\}$$

and the edges of the spectral bands of $sp\mathrm{D}_{Q,\mathbb{I},C}$ are solutions of the equation

$$|D(\lambda)| = 1.$$

3.5 Fredholm Theory of Dirac Operators with Periodic Set of Interactions

We assume that:

(a) As above $\mathbb{Y} = \mathbb{Y}_0 + l\mathbb{Z}$ is the periodic set.
(b) The interaction matrix $C : \mathbb{Y} \to \mathcal{B}(\mathbb{R}^2)$ is a real-valued l-periodic matrix-function;
(c) $\det C(y) = 1$ for every $y \in \mathbb{Y}_0$, and the vectors $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $C(y)\begin{pmatrix} 1 \\ -i \end{pmatrix}$ are linearly independent for any $y \in \mathbb{Y}_0$.
(d) $p, q, r \in L^\infty(\mathbb{R})$ and are uniformly continuous functions at infinity.

Let the sequence $l\mathbb{Z} \ni g_m \to \infty$. Then there exists a subsequence h_m of g_m such that there exist limit functions p_h, q_h, r_h in the sense of formulas (53). Then the Dirac operator

$$\mathfrak{D}_{Q^h} = J\frac{d}{dx} + Q^h(x), \tag{66}$$

with

$$Q^h(x) = \begin{pmatrix} p_h(x) & q_h(x) \\ q_h(x) & -p_h(x) \end{pmatrix} + r_h(x)\mathbb{I} \tag{67}$$

is called the limit operator of $\mathfrak{D}_Q$ defined by the sequence $l\mathbb{Z} \ni h_m \to \infty$. We denote by $Lim\mathfrak{D}_Q$ the set of all limit operators of $\mathfrak{D}_Q$.

Theorem 18 *Let conditions (a), (b), (c), (d) be satisfied. Then the operator*

$$\mathfrak{D}_Q : H^1_{\mathbb{I},C}(\mathbb{R}\diagdown\mathbb{Y}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$$

is Fredholm if and only if all limit operators $\mathfrak{D}_{Q^h} \in Lim(\mathfrak{D}_Q)$ *acting from* $H^1_{\mathbb{I},C}(\mathbb{R}\diagdown\mathbb{Y}, \mathbb{C}^2)$ *into* $L^2(\mathbb{R}, \mathbb{C}^2)$ *are invertible.*

Proof Since the operator $\mathfrak{D}_Q$ is elliptic the operator $\mathfrak{D}_Q$ is locally Fredholm at every point $x \in \mathbb{R}$. Then Proposition 11 yields that $\mathfrak{D}_Q : H^1_{\mathbb{I},C}(\mathbb{R}\diagdown\mathbb{Y}) \to L^2(\mathbb{R}, \mathbb{C}^2)$ is a Fredholm operator if and only if $\mathfrak{D}_Q$ is locally invertible at infinity. It follows from Proposition 5 that there exist $\mu > 0$ such that the operator

$$\mathfrak{D}_0(\mu) = J\frac{d}{dx} - i\mu\mathbb{I} : H^1_{\mathbb{I},C}(\mathbb{R}\diagdown\mathbb{Y}, \mathbb{C}^2) \to L^2(\mathbb{R}.\mathbb{C}^2)$$

is an isomorphism. We set

$$\mathcal{A} = \mathfrak{D}_Q\mathfrak{D}_0^{-1}(\mu) : L^2(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2). \tag{68}$$

It is easy to prove that $\mathfrak{D}_Q : H^1_{\mathbb{I},C}(\mathbb{R}\diagdown\mathbb{Y}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$ is locally invertible at infinity if and only if $\mathcal{A} : L^2(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$ is locally invertible at infinity. For the study of local invertibility at infinity we use the results of the book [26] and papers [23, 25].

Let $\phi \in C_b^\infty(\mathbb{R})$ be an ˇarbitrary function, $\phi_t(x) = \phi(tx), t \in \mathbb{R}$. Then it is easy to prove that

$$\left\| [\phi_t, \mathcal{A}] \right\| = \lim_{t\to 0} \left\| \phi_t \mathcal{A} - \mathcal{A}\phi_t I \right\| = 0, \tag{69}$$

That is $\mathcal{A}$ belongs to the C^*-algebra of so-called band-dominated operators in $L^2(\mathbb{R}, \mathbb{C}^2)$ (See for instance [25]).

We introduce the limit operators of $\mathcal{A}$ as follows. Let $l\mathbb{Z} \ni h_m \to \infty$, and $V_{h_m}u(x) = u(x - h_m)$ be the sequence of shift operators in $L^2(\mathbb{R}, \mathbb{C}^2)$. We say that $\mathcal{A}^h$ be a limit operator defined by the sequence (h_m) if

$$\lim_{m\to\infty} \left\| \left(V_{-h_m} \mathcal{A} V_{h_m} - \mathcal{A}^h \right) \varphi I \right\|_{\mathcal{B}(L^2(\mathbb{R}), \mathbb{C}^2)} = 0$$

$$\lim_{m\to\infty} \left\| \varphi \left(V_{-h_m} \mathcal{A} V_{h_m} - \mathcal{A}^h \right) \right\|_{\mathcal{B}(L^2(\mathbb{R}), \mathbb{C}^2)} = 0$$

for every $\varphi \in C_0^\infty(\mathbb{R})$. One can see that

$$V_{-h_m} \mathcal{A} V_{h_m} = V_{-h_m} \mathfrak{D}_Q V_{h_m} V_{-h_m} \mathfrak{D}_0^{-1}(\mu) V_{h_m} = V_{-h_m} \mathfrak{D}_Q V_{h_m} \mathfrak{D}_0^{-1}(\mu) \tag{70}$$

Condition (d) for functions p, q, r implies that for every $l\mathbb{Z} \ni g_m \to \infty$ there exists a subsequence (h_m) of (g_m) defining the limit operator $\mathfrak{D}_{Q^h}$. Hence the sequence (h_m) defines the limit operator $\mathcal{A}^h$ of $\mathcal{A}$ by formula

$$\mathcal{A}^h = \mathfrak{D}_{Q^h} \mathfrak{D}_0^{-1}(\mu). \tag{71}$$

It follows from results [23, 25] the operator $\mathcal{A}$ is locally invertible at infinity if and only if all limit operators $\mathcal{A}^h$ are invertible. Formula (71) yields that the invertibility of all limit operators $\mathcal{A}^h : L^2(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$ is equivalent to invertibility of all limit operators $\mathfrak{D}_{Q^h} : H^1_{\mathbb{I},B}(\mathbb{R}\diagdown\mathbb{Y}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$ of $\mathfrak{D}_Q$ □

- Let conditions (a), (b), (c), (d) be satisfied. We denote by $\mathrm{D}_{Q,\mathbb{I},C}$,$\mathrm{D}_{Q^h,\mathbb{I},C}$ the unbounded closed operators in $L^2(\mathbb{R}, \mathbb{C}^2)$ with domain $H^1_{\mathbb{I},C}(\mathbb{R}\diagdown\mathbb{Y}, \mathbb{C}^2)$ associated with the operators $\mathfrak{D}_Q$, $\mathfrak{D}_{Q^h}$, respectively. and we denote by $Lim\mathrm{D}_{Q,\mathbb{I},C}$ the set of all limit operators of $\mathrm{D}_{Q,\mathbb{I},C}$.

As a corollary of Theorem 18 we obtain the following results.

Theorem 19 *Let conditions (a), (b), (c), (d) hold. Then*

$$sp_{ess}\mathrm{D}_{Q,\mathbb{I},C} = \bigcup_{\mathrm{D}_{Q^h,\mathbb{I},C}\in Lim\mathrm{D}_{Q,\mathbb{I},C}} sp\mathrm{D}_{Q^h,\mathbb{I},C}. \tag{72}$$

Hence if conditions (a), (b), (c) hold, p, q, r are $l-$periodic continuous functions on $\mathbb{R}$. Then

$$sp_{ess}\mathrm{D}_{Q,\mathbb{I},C} = sp\mathrm{D}_{Q,\mathbb{I},C}.$$

3.6 Slowly Oscillating at Infinity Perturbations of Electrostatic Potentials

Let $\mathrm{D}_{Q,\mathbb{I},C}$ be the above introduced periodic self-adjoint Dirac operator. We consider the essential spectrum of the operator $\mathrm{D}_{\tilde{Q},\mathbb{I},C}$ with perturbed electrostatic potential $\tilde{r}\mathbb{I} = (r + r_1)\,\mathbb{I}$ where r_1 is a real-valued function belongs to the class $SO_\infty(\mathbb{R})$.

The unperturbed periodic Dirac operator $\mathrm{D}_{Q,\mathbb{I},C}$ has a band-gap spectrum (see formula (63))

$$sp_{ess}\mathrm{D}_{Q,\mathbb{I},C} = sp\mathrm{D}_{Q,\mathbb{I},C} = \bigcup_{j=1}^{\infty}\left[a_j, b_j\right].$$

Applying Theorem 19 we investigate the essential spectrum of perturbed operator $\mathrm{D}_{\tilde{Q},\mathbb{I},C}$. Note that the limit operators of $\mathrm{D}_{\tilde{Q},\mathbb{I},C}$ defined by the sequences $l\mathbb{Z} \ni h_m \to \infty$ are

$$\mathrm{D}_{\tilde{Q}^h,\mathbb{I},C} = \mathrm{D}_{Q,\mathbb{I},C} + r_1^h I$$

where

$$r_1^h = \lim_{m\to\infty} r_1(h_m) \in \mathbb{R}. \tag{73}$$

Formula (72) yields that

$$sp_{ess}\mathrm{D}_{\tilde{Q}^h,\mathbb{I},C} = \bigcup_{j=1}^{\infty}\bigcup_{h}[a_j + r_1^h, b(j) + r_1^h]$$

where the union $\bigcup\limits_{h}$ is taken with respect to all sequences (h_n) for which there exist limits (73). Hence

$$sp_{ess}\mathrm{D}_{\tilde{Q},\mathbb{I},C} = \bigcup_{j=1}^{\infty}[a_j + m(r_1), b_j + M(r_1)] \tag{74}$$

where $m(r_1) = \liminf_{x\to\infty} r_1(x)$, $M(r_1) = \limsup_{x\to\infty} r_1(x)$.

Formula (74) implies that some spectral bands of $sp_{ess}\mathrm{D}_{\tilde{Q},\mathbb{I},C}$ may overlap depending on the intensity of the perturbation r_1. Let (b_j, a_{j+1}), $j \in \mathbb{N}$, be a gap of $sp\mathrm{D}_{Q,\mathbb{I},C}$, hence if the relation

$$M(r_1) - m(r_1) > a_{j+1} - b_j$$

holds this gap will disappear due to the merging of the adjacent bands.

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References

1. Agranovich, M.S.: Elliptic operators on closed manifolds. In: Partial Differential Equations, VI, Volume 63 of the Series Encyclopaedia of Mathematical Sciences, Yu.V. Egorov, M.A. Shubin (eds.), pp. 1–130 Springer-Verlag, Berlin, Heidelberg (1994)
2. Agranovich, M.S.: Elliptic boundary problems, in Partial Differential Equations, IX, Agranovich, M.S., Egorov, Y.V., Shubin, M.A. (Eds.) Springer, Berlin-Heidelberg-New York (2010).
3. Albeverio, S., Gesztesy, F., Hoegh-Krohn, F., Holden, H.: Solvable Models in Quantum Mechanics, with an Appendix by Pavel Exner, 2nd edition, AMS Chelsea Publishing, Providence, RI (2005).
4. Albeverio, S., Kurasov, P.: Singular Perturbations of Differential Operators and Schrödinger Type Operators, Cambridge Univ. Press (2000).
5. Avron, J.E., Grosman, A.: The relativistic Kronig–Penney Hamiltonian, Phys. Lett. A 56, 5–57 (1976).
6. Birman, M.Sh., Solomjak, M.Sh.: Spectral Theory of Selfadjoint Operators in Hilbert Spaces. Reidel, Dordrecht (1987).
7. Bruening, J., Geyler, V., Pankrashkin, K.: Spectra of self-adjoint extensions and applications to solvable Schrödinger operators, Rev. Math. Phys. 20, 1–70 (2008).
8. Brasche, J.F., Exner, P., Kuperin, Yu.A., and Šeba P.: Schrödinger operators with singular interactions. J. Math. Anal. Appl. 184(1):112–139 (1994).
9. Behrndt, J., Exner, P., Lotoreichik, V.: Essential spectrum of Schrödinger operators with δ-interactions on the union of compact Lipschitz hypersurfaces. PAMM Proc. Appl. Math. Mech.; 13: 523–524 (2013).
10. Behrndt, J., Langer, M., Lotoreichik, V.: Schrödinger operators with δ and δ'-potentials supported on hypersurfaces. Ann. Henri Poincaré, 14: 385–423 (2013).

11. Behrndt, J., Exner, P., Lotoreichik, V.: Schrödinger operators with $\delta-$ and $\delta'-$ interactions on Lipschitz surfaces and chromatic numbers of associated partitions. Rev. Math. Phys. 26 (1450015) [43 pages] (2014).
12. Behrndt, J., Exner, P., Holzmann, M., Lotoreichik, V.: On the spectral properties of Dirac operators with electrostatic δ-shell interactions, J. Math. Pures Appl. 111, 47–78 (2018).
13. Behrndt, J., Exner, P., Holzmann, M., Lotoreichik, V: On Dirac operators in $\mathbb{R}^3$ with electrostatic and Lorentz scalar δ-shell interactions, Quantum Stud.: Math. Found., https://doi.org/10.1007/s40509-019-00186-6 (2019).
14. Berezin, F.A., Faddeev, L.D.: A remark on Schrödinger operators with a singular potentials. Soviet Math. Dokl.;137: 1011–1014 (1961).
15. Buschmann, D., Stolz, G., Weidmann, J.: One-dimensional Schrödinger operators with local point interactions, J. Reine Angew. Math. 467, 169–186 (1995).
16. Carlone, R., Malamud, M., Posilicano, A.: On the spectral theory of Gesztesy–Šeba realizations of $1-D$ Dirac operators with point interactions on a discrete set, J. Differential Equations 254, 3835–3902,A. (2013).
17. Hounkonnou, M.N., Avossevou, G.Y.H.: Spectral and resonance properties of δ and δ'-type interactions in relativistic quantum mechanics, J. Math. Phys. 42 (1), 30–51 (2001).
18. Hughes, R.J.: Relativistic Kronig–Penney- type Hamiltonians, Integral Equations Operator Theory 31 (4), 436–448 (1998).
19. Kostenko, A.S., Malamud, M.M.: $1-D$ Schrödinger operators wih local point interactions on a discrete set, J. Differential Equations 249, 253–304 (2010).
20. Kronig, R.L., Penney, W.G.: Quantum mechanics of electrons in crystal lattices, Proc. R. Soc. Lond. A 130, 499–513 (1931).
21. Kuchment, P.: An overview of periodic elliptic operators. Bull Amer Math Soc. 53:343–414, https://doi.org/10.1090/bull/1528 (2016).
22. Kurasov, P.: Distribution theory for discontinuous test functions and differential operators with generalized coefficients. Journal of Math. Anal. and Appl. 201, 297–323 (1996).
23. Lindner, M., Seidel, M.: An affirmative answer to a core issue on limit operators. Journal of Functional Analysis, V. 267, Iss. 3, 901–917 (2014).
24. Ourmieres-Bonafos, Th., Vega, L.: A strategy for self-adjointness of Dirac operators: Applications to the MIT BAG model and shell interactions. Publ. Mat. 62, 397–437 (2018).
25. Rabinovich, V.S., Roch, S., Silbermann, B.: Band-dominated operators with operator-valued coefficients, their Fredholm properties and finite sections, Integr. Equ. Oper. Theory, 40:3, 342–381 (2001).
26. Rabinovich, V.S., Roch, S., Silbermann, B.: Limit Operators and their Applications in Operator Theory, In ser. Operator Theory: Advances and Applications, vol 150, Birkhäuser Verlag (2004).
27. Rabinovich, V.S.: Essential spectrum of perturbed pseudodifferential operators. Applications to the Schrödinger, Klein-Gordon, and Dirac operators, Russ. J. Math. Physics, 12:1, 62–80 (2005).
28. Rabinovich, V.S., Roch, S.: The essential spectrum of Schrödinger operators on lattices, J. of Physics A, Math. Theor. 39, 8377–8394 (2006).
29. Rabinovich, V.S., Roch, S.: Essential spectra of difference operators on $\mathbb{Z}^n$-periodic graphs, J. of Physics A: Math. Theor. 40, 10109–10128 (2007).
30. Rabinovich, V.S., Castillo-Pérez, R., Urbano-Altamirano, F.: On the essential spectrum of quantum waveguides; Math. Meth. Appl. Sci., 36:7, 761–772 (2013).
31. Rabinovich, V.S.: On the essential spectrum of quantum graphs, Integr. Equ. Oper. Theory 88, 339–362 (2017).
32. Rabinovich, V.S.: Electromagnetic Schrödinger operators on periodic graphs with general conditions at vertices, Russian Journal of Mathematical Physics, Vol. 26, No. 2, pp. 185–205 (2019).
33. Rabinovich, V.S.: Essential spectrum of Schroödinger operators with $\delta-$interactions on unbounded hypersurfaces, Math. Notes, 102: 5, 698–709 (2017).

34. Rabinovich, V.S.: Schrödinger operators with interactions on unbounded hypersurfaces: Math. Meth. Appl. Sci.; 0:1–18, https://doi.org/10.1002/mma.5083 (2018).
35. Rabinovich, V.S., Barera-Figueroa, V, and Olivera-Ramírez, L.: On the spectra of one-dimensional Schrödinger operators with singular potentials, Frontiers in Physics, www.frontiersin.org, published: 16 April 2019. https://doi.org/10.3389/fphy.2019.00057
36. Rabinovich, V.S.: Pseudodifferential operators on a class of noncompact manifolds, Math. USSR-Sb., 18:1, 45–59 (1972).
37. Simonenko I.B.: Operators of convolution type in cones, Math. USSR-Sb., 3:2, 279–293 (1967).
38. Yoshitomi, K.: Dirac operators with periodic δ-interactions: spectral gaps and inhomogeneous Diophantine approximation, Michigan Math. J. 58, 363–384 (2009).

Algebra Generated by a Finite Number of Toeplitz Operators with Homogeneous Symbols Acting on the Poly-Bergman Spaces

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Dedicated to Nikolai L. Vasilevski on the occasion of his 70th birthday.

Abstract In this work Toeplitz operators with bounded homogeneous symbols and acting on the n-poly-Bergman space $\mathcal{A}_n^2(\Pi)$ are studied, where $\Pi \subset \mathbb{C}$ is the upper half-plane. Here we consider homogeneous symbols of exponential type $a(z) = e^{N\theta i}$, where N is integer and $\theta = \arg z$. We show that the C^*-algebra generated by a finite number of Toeplitz operators on $\mathcal{A}_n^2(\Pi)$, with homogeneous symbols of exponential type, is isomorphic and isometric to the C^*-algebra consisting of all the matrix-valued functions $M(x) \in M_n(\mathbb{C}) \otimes C[-\infty, +\infty]$ such that $M(-\infty)$ and $M(+\infty)$ are scalar matrices. The C^*-algebra generated by a finite number of Toeplitz operators acting on the n-poly-harmonic Bergman space of Π is also studied.

Keywords Bergman space · Harmonic function · Toeplitz operator

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1 Introduction

Recall that the poly-Bergman space $\mathcal{A}_n^2(D) \subset L_2(D, dxdy)$ consists of all n-analytic functions $\varphi = \varphi(x, y)$ on $D \subset \mathbb{C}$, that is, those square-integrable functions on D satisfying the equation $\left(\frac{\partial}{\partial \overline{z}}\right)^n = 0$, where $dxdy$ is the usual Lebesgue measure. The orthogonal complement $\mathcal{A}_{(n)}^2(D) := \mathcal{A}_n^2(D) \ominus \mathcal{A}_{n-1}^2(D)$ is called the true-poly-Bergman space, and it consists of all true-n-analytic functions. For convenience, we define $\mathcal{A}_{(0)}^2(D) = 0$. Of course, $\mathcal{A}_1^2(D)$ is the usual Bergman space on D, which is simply denoted by $\mathcal{A}^2(D)$. Analogously, introduce the spaces $\widetilde{\mathcal{A}}_n^2(D)$ and $\widetilde{\mathcal{A}}_{(n)}^2(D)$ of all n-anti-analytic and true-n-anti-analytic functions, respectively. In fact, each n-anti-analytic function is just the complex conjugation of an n-analytic function. For the upper half-plane $\Pi = \{z : \operatorname{Im} z > 0\}$, N. L. Vasilevski [20, 21] proved that $L_2(\Pi)$ has a decomposition as a direct sum of the true-n-analytic and true-n-anti-analytic function spaces:

$$L_2(\Pi) = \bigoplus_{k=1}^{\infty} \mathcal{A}_{(n)}^2(\Pi) \oplus \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{A}}_{(n)}^2(\Pi).$$

Moreover, the author gave explicit expressions for the reproducing kernels of all these function spaces, and he found an isometric isomorphism from $L_2(\mathbb{R}_+)$ onto the true-poly-Bergman space $\mathcal{A}_{(n)}^2(\Pi)$.

In [6] the authors characterized all the commutative C^*-algebras of Toeplitz operators acting on the Bergman space of the unit disk $\mathbb{D}$ (or equivalently, in Π). Actually, there exist three types of maximal abelian groups of Möbius transformations on Π, and their corresponding classes of symbols invariant under the action of such groups. For every class of these symbols we have a commutative C^*-algebra of Toeplitz operators acting on $\mathcal{A}^2(\Pi)$. The first class of symbols consists of all vertical functions, which depend only on $y = \operatorname{Im} z$. The second class is the set of homogeneous symbols, which are functions depending only on $\theta = \arg z$. The third class of symbols can be easily described in the unit disk as the family of all functions depending only on $r = |z|$.

With every class of symbols just mentioned above, the Toeplitz operators acting on the true-poly-Bergman space $\mathcal{A}_{(n)}^2(\Pi)$ generate a commutative C^*-algebra, this fact can be proved using the techniques due to N. L. Vasilevski [21]. The C^*-algebra generated by Toeplitz operators acting on $\mathcal{A}_n^2(\Pi)$ is noncommutative, nevertheless, it is isomorphic to a C^*-algebra of continuous matrix-valued functions in the cases of vertical and homogeneous symbols [17, 19]. Recently, it was proved in [18] that the C^*-algebra generated by all the Toeplitz operators on $\mathcal{A}_n^2(\Pi)$ with vertical symbols can be generated by a finite number of Toeplitz operators. In such work, the representation of the affine group on $L^2(\mathbb{R})$ was an important tool, where poly-Bergman spaces are identified with wavelet spaces. Certainly, the wavelet transform has multiple applications, say, in signal processing and quantum mechanics, cf. [1, 4]. Of course, a pioneering contribution on the one-dimensional wavelet analysis was made by A. Grossmann and J. Morlet [5]. See also [10] for this matter, and see [7–9] for the study of Toeplitz operators on the true-poly-Bergman spaces and their analogous wavelet spaces.

In a similar way, the study of Toeplitz operators can be carried out on the poly-harmonic spaces of Π. For example, in [14, 15] the authors used the three classes of symbols and studied the corresponding Toeplitz operator algebra for the harmonic Bergman space $\mathcal{H}_1^2(\Pi) := \mathcal{A}_1^2(\Pi) \oplus \widetilde{\mathcal{A}}_1^2(\Pi)$. In [16] the authors used homogeneous symbols of the form $a(\theta) = \chi_{[0,\alpha]}(\theta)$, and they described the C^*-algebra generated by all the Toeplitz operators acting on $\mathcal{H}_n^2(\Pi) := \mathcal{A}_n^2(\Pi) \oplus \widetilde{\mathcal{A}}_n^2(\Pi)$.

Using polar coordinates in Π and the Mellin transform, each true-poly-Bergman space $\mathcal{A}_{(k)}^2(\Pi)$ can be identified with $L_2(\mathbb{R})$ through a Bargmann type transform [17]. This point of view fits to the study of Toeplitz operators with homogeneous symbols. Since $\mathcal{A}_n^2(\Pi) = \oplus_{k=1}^n \mathcal{A}_{(k)}^2(\Pi)$, the poly-Bergman space $\mathcal{A}_n^2(\Pi)$ is isomorphic to $(L_2(\mathbb{R}))^n$. Thus, for each homogeneous symbol a, the Toeplitz operator $T_{n,a}$ acting on $\mathcal{A}_n^2(\Pi)$ is unitary equivalent to a multiplication operator $\gamma^{n,a}(x)I$ acting on $(L_2(\mathbb{R}))^n$, where $\gamma^{n,a}$ is a matrix-valued function continuous on $(-\infty, +\infty)$, cf. [17]. Consequently, the C^*-algebra generated by all the Toeplitz operators $T_{n,a}$ is isomorphic to the C^*-algebra generated by all the functions $\gamma^{n,a}$. We confine ourselves to homogeneous symbols $a(\theta)$ having limits values at $\theta = 0, \pi$. The main result in this work says that the C^*-algebra generated by all the Toeplitz operators with bounded homogeneous symbols and acting on $\mathcal{A}_n^2(\Pi)$ can also be generated by a finite number of Toeplitz operators with symbols of exponential type.

This paper is organized as follows. In Sect. 2 we introduce preliminary results about the poly-analytic function spaces $\mathcal{A}_n^2(\Pi)$ and their relationship with certain class of orthogonal functions. In Sect. 3 we recall how a Toeplitz operator $T_{n,a}$ on $\mathcal{A}_n^2(\Pi)$, with homogeneous symbol a, is unitary equivalent to a multiplication operator $\gamma^{n,a}I$. We establish a convenient factorization of $\gamma^{n,a}$ in order to carry out the separation of the pure states of the corresponding C^*-algebra generated by these functions. In Sect. 4 we describe the C^*-algebra generated by the Toeplitz operators $T_{n,a}$, and we prove that such C^*-algebra can be generated by a finite number of Toeplitz operators with symbols of exponential type. Finally, in Sect. 5, we describe the C^*-algebra generated by all the Toeplitz operators acting on $\mathcal{H}_n^2(\Pi)$ with homogeneous symbols, and we prove that such algebra can also be generated by a finite number of Toeplitz operators with symbols of exponential type.

2 Bergman and Poly-Bergman Spaces

The results that will appear in this section can be found in [17]. Let Π be the upper half-plane in $\mathbb{C}$, and consider the space $L_2(\Pi) = L_2(\Pi, dxdy)$, where $dxdy$ is the usual Lebesgue measure. Let $\mathcal{A}_n^2(\Pi)$ be the poly-Bergman space that consists of all functions in $L_2(\Pi)$ satisfying the equation $(\partial/\partial\overline{z})^n\varphi = 0$. Introduce the true-poly-Bergman spaces as follows:

$$\mathcal{A}^2_{(n)}(\Pi) = \mathcal{A}^2_n(\Pi) \ominus \mathcal{A}^2_{n-1}(\Pi), \quad n = 1, 2, \ldots$$

where $\mathcal{A}^2_0(\Pi) = \{0\}$. Of course $\mathcal{A}^2_1(\Pi) = \mathcal{A}^2_{(1)}(\Pi)$ is the usual Bergman space.

By representing scalars in Π with respect to polar coordinates we get the tensor decomposition

$$L_2(\Pi) = L_2(\mathbb{R}_+, rdr) \otimes L_2([0, \pi], d\theta).$$

Let $M : L_2(\mathbb{R}_+, rdr) \longrightarrow L_2(\mathbb{R}, dx)$ be the Mellin type transform given by the rule

$$(Mg)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} r^{-ix} g(r) dr.$$

It is well known that M is an isometric isomorphism. Then,

$$U = M \otimes I$$

is a unitary operator from $L_2(\Pi)$ onto $L_2(\mathbb{R}, dx) \otimes L_2([0, \pi], d\theta)$. Let us see how the poly-Bergman space $\mathcal{A}^2_n(\Pi)$ can be identified with $(L_2(\mathbb{R}, dx))^n$. We have $2\frac{\partial}{\partial\overline{z}} = ED$, where

$$E = \frac{e^{i\theta}}{r} I, \qquad D = r\frac{\partial}{\partial r} + i\frac{\partial}{\partial\theta}.$$

Since $DE = E(D-2)$ and $M(r\frac{d}{dr})M^* = (ix-1)I$, the space $A^2_n := U(\mathcal{A}^2_n(\Pi)) \subset L_2(\mathbb{R}, dx) \otimes L_2([0, \pi], d\theta)$ consists of all functions $f(x, \theta)$ satisfying the equation

$$\left(ix - 1 - 2[n-1] + i\frac{\partial}{\partial\theta}\right) \cdots \left(ix - 1 + i\frac{\partial}{\partial\theta}\right) f = 0.$$

Thus,

$$\begin{aligned} A^2_{(n)} &= A^2_n \ominus A^2_{n-1} \\ &= \{f(x) p_{n-1}(x, \theta) \psi(x) e^{-x\theta - \theta i} \mid f(x) \in L_2(\mathbb{R}, dx)\}, \quad n = 1, 2, \ldots \end{aligned}$$

where $p_n(x, \theta)$ is a polynomial with respect to $z = e^{-2\theta i}$ given by

$$p_n(x, \theta) = \sum_{k=0}^{n} (-1)^k b_{nk}(x) e^{-2k\theta i}, \quad n = 0, 1, 2, \ldots \tag{2.1}$$

with $b_{00}(x) = 1$, $b_{nn}(x) = (n!)^{-1}\sqrt{(x^2+1^2)(x^2+2^2)\cdots(x^2+n^2)}$,

$$b_{nk}(x) = \binom{n}{k} b_{nn}(x) \prod_{j=0}^{n-1} \frac{x - ji + ki}{x - ji + ni},$$

and

$$\psi(x) = \sqrt{\frac{2x}{1 - e^{-2\pi x}}}, \quad \psi(0) = \frac{1}{\sqrt{\pi}}.$$

Actually, $\{p_n(x,\theta) \mid n = 0, 1, 2, \ldots\}$ is an orthonormal set in the Hilbert space $L_2([0,\pi], (\psi(x))^2 e^{-2x\theta} d\theta)$ for each $x \in \mathbb{R}$. On the other hand, $A^2_{(n)}$ is the image of the true-poly-Bergman space $\mathcal{A}^2_{(n)}(\Pi)$ under the operator U. Let $P_{(n)}$ be the orthogonal projection from $L_2(\mathbb{R}, dx) \otimes L_2([0,\pi], d\theta)$ onto the space $A^2_{(n)}$. This projection is given by

$$(P_{(n)} f)(x,\theta) = (\psi(x))^2 p_{n-1}(x,\theta) e^{-x\theta - \theta i} \int_0^{\pi} f(x,\varphi) \overline{p_{n-1}(x,\varphi)} e^{-x\varphi + \varphi i} d\varphi.$$

Of course, $P_n := \sum_{k=1}^{n} P_{(k)}$ is the orthogonal projection from the space $L_2(\mathbb{R}, dx) \otimes L_2([0,\pi], d\theta)$ onto A^2_n.

Let $B_{\Pi,(n)}$ and $B_{\Pi,n}$ denote the orthogonal projections from $L_2(\Pi)$ onto $\mathcal{A}^2_{(n)}(\Pi)$ and $\mathcal{A}^2_n(\Pi)$, respectively.

Theorem 2.1 *The unitary operator U gives an isometric isomorphism from the space $L_2(\Pi)$ onto $L_2(\mathbb{R}, dx) \otimes L_2([0,\pi], d\theta)$, under which*

1. *the poly-Bergman spaces $\mathcal{A}^2_{(n)}(\Pi)$ and $\mathcal{A}^2_n(\Pi)$ are mapped onto $A^2_{(n)}$ and A^2_n, respectively,*
2. *the poly-Bergman projections $B_{\Pi,(n)}$ and $B_{\Pi,n}$ are unitary equivalent to $P_{(n)}$ and P_n, respectively. That is,*

$$U B_{\Pi,(n)} U^* = P_{(n)}, \quad U B_{\Pi,n} U^* = P_n.$$

Introduce the isometric embedding

$$R_{0,(n)} : L_2(\mathbb{R}, dx) \longrightarrow L_2(\mathbb{R}, dx) \otimes L_2([0,\pi], d\theta)$$

by the rule

$$(R_{0,(n)} f)(x,\theta) = f(x)\, p_{n-1}(x,\theta) \psi(x) e^{-x\theta - \theta i}.$$

Then, the operator

$$R_{(n)} = R^*_{0,(n)} U$$

maps $L_2(\Pi)$ onto $L_2(\mathbb{R}, dx)$, and its restriction to $\mathcal{A}^2_{(n)}(\Pi)$ is an isometric isomorphism onto $L_2(\mathbb{R}, dx)$. Thus, $R^*_{(n)} R_{(n)} = B_{\Pi,(n)}$ and $R_{(n)} R^*_{(n)} = I$. Let us consider the isometric embedding

$$R_{0,n} : (L_2(\mathbb{R}, dx))^n \longrightarrow L_2(\mathbb{R}, dx) \otimes L_2([0, \pi], d\theta)$$

given by the rule $(R_{0,n} f)(x, \theta) = H_n(x, \theta)^t f(t)$, where $f = (f_1, \ldots, f_n)^t$, and

$$H_n(x, \theta) = \psi(x) e^{-x\theta - i\theta} (p_0(x, \theta), \ldots, p_{n-1}(x, \theta))^t. \tag{2.2}$$

Finally, we define the operator $R_n : L_2(\Pi) \longrightarrow (L_2(\mathbb{R}, dx))^n$ by the formula

$$R_n := R^*_{0,n} U.$$

The operator R_n maps $L_2(\Pi)$ onto $(L_2(\mathbb{R}, dx))^n$, and its restriction to $\mathcal{A}^2_n(\Pi)$ is an isometric isomorphism. Moreover,

$$R^*_n R_n = B_{\Pi,n} : L_2(\Pi) \longrightarrow \mathcal{A}^2_n(\Pi),$$
$$R_n R^*_n = I : (L_2(\mathbb{R}, dx))^n \longrightarrow (L_2(\mathbb{R}, dx))^n.$$

3 Toeplitz Operators with Homogeneous Symbols

In this section we study Toeplitz operators acting on the poly-Bergman spaces on Π, and with homogeneous symbols. Let $L^{\{0,\pi\}}_\infty$ stand for the subalgebra of $L_\infty[0, \pi]$ consisting of all functions having limit values at 0 and π. We shall say that $a \in L^{\{0,\pi\}}_\infty$ is a homogeneous symbol, and we write

$$a(0) := \lim_{\theta \longrightarrow 0^+} a(\theta) \quad \text{and} \quad a(\pi) := \lim_{\theta \longrightarrow \pi^-} a(\theta).$$

We will identify $a \in L^{\{0,\pi\}}_\infty$ with the function $a(z) = a(\theta)$ defined on the upper half-plane Π, where $\theta = \arg z$. For $a \in L^{\{0,\pi\}}_\infty$, the Toeplitz operator $T_{n,a}$, acting on $\mathcal{A}^2_n(\Pi)$, is the operator defined by

$$T_{n,a} : \mathcal{A}^2_n(\Pi) \ni \varphi \longmapsto B_{\Pi,n}(a\varphi) \in \mathcal{A}^2_n(\Pi).$$

In [17], the authors proved that $T_{n,a}$ is unitary equivalent to the multiplication operator $\gamma^{n,a}(x) I = R_n T_{n,a} R^*_n$ acting on $(L_2(\mathbb{R}, dx))^n$, where $\gamma^{n,a}(x)$ is the continuous matrix-valued function

$$\gamma^{n,a}(x) = \int_0^\pi a(\theta)\overline{H_n(x,\theta)}[H_n(x,\theta)]^t d\theta, \tag{3.1}$$

which satisfies

$$a(\pi)I = \lim_{x\longrightarrow-\infty} \gamma^{n,a}(x), \qquad a(0)I = \lim_{x\longrightarrow+\infty} \gamma^{n,a}(x). \tag{3.2}$$

Let $\mathcal{T}^n_{-\infty\infty}$ be the C^*-algebra generated by all the Toeplitz operators $T_{n,a}$ acting on the poly-Bergman space $\mathcal{A}^2_n(\Pi)$, with $a \in L^{\{0,\pi\}}_\infty$. It was proved in [17] that $\mathcal{T}^n_{-\infty\infty}$ is isomorphic and isometric to the C^*-algebra

$$\mathfrak{C}_n = \{M \in M_n(\mathbb{C}) \otimes C[-\infty,\infty] : M(-\infty), M(\infty) \in \mathbb{C}I\}, \tag{3.3}$$

where $M_n(\mathbb{C})$ denotes the algebra of all $n \times n$ matrices with complex entries. Of course, $\mathfrak{C}_n$ is a C^*-bundle, where each fiber $\mathfrak{C}_n(x) = \{M(x) \; : \; M \in \mathfrak{C}_n\}$ is a C^*-subalgebra of $M_n(\mathbb{C})$. Our main result in this section asserts that $\mathcal{T}^n_{-\infty\infty}$ can be generated using only $4n-1$ Toeplitz operators $T_{n,a}$, with homogeneous symbols of exponential type $a(\theta) = e^{N\theta i}$, where N is integer. Actually, we will prove that the C^*-algebra $\mathfrak{B}$ generated by certain matrix-valued functions $\gamma^{n,a_1}(x),\ldots,\gamma^{n,a_{4n-1}}(x)$ separates all the pure states of $\mathfrak{C}_n$, each of which has the form

$$f_{x_0,v}(M) = \langle M(x_0)v, v\rangle, \quad M \in \mathfrak{C}_n, \tag{3.4}$$

where $x_0 \in [-\infty,\infty]$, and $v \in \mathbb{C}^n$ is a unit vector [3, 12, 13]. In particular, there exists only one pure state corresponding to each point $x = \infty$ and $x = -\infty$. In fact, we can take $v = (1,0,\ldots,0)^t$ in these two cases, and

$$f_{\pm\infty,v}(M) = c_{\pm\infty}, \quad M \in \mathfrak{C}_n,$$

where $M(\pm\infty) = c_{\pm\infty}I$.

Let us start with a convenient factorization of the function $\gamma^{n,a}(x)$ given in (3.1). Let

$$L_n(e^{-2i\theta}) = (1, e^{-2i\theta}, \ldots, (e^{-2i\theta})^{n-1})^t$$

and

$$\Phi_n(x) := \left((-1)^{k-1} b_{(j-1)(k-1)}(x)\right)^n_{j,k=1},$$

where $b_{jk}(x) = 0$ if $j < k$. Then $\Phi_n(x)L_n(e^{-2\theta i}) = (p_0(x,\theta), \dots, p_{n-1}(x,\theta))^t$, and the Eq. (2.2) can be written as

$$H_n(x,\theta) = \psi(x)e^{-x\theta - i\theta}\Phi_n(x)L_n(e^{-2\theta i}). \tag{3.5}$$

Lemma 3.1 *The function $\gamma^{n,a}(x)$ can be written as*

$$\gamma^{n,a}(x) = \overline{\Phi_n(x)}M^{n,a}(x)[\Phi_n(x)]^t, \tag{3.6}$$

where

$$M^{n,a}(x) = (\psi(x))^2 \int_0^\pi a(\theta)e^{-2x\theta}\overline{L_n(e^{-2\theta i})}[L_n(e^{-2\theta i})]^t d\theta.$$

The matrix-valued function $\Phi_n(x)$ satisfies the equation

$$\overline{\Phi_n(x)}T(x)[\Phi_n(x)]^t = I,$$

where $\Phi_n(0) = I$, $T(0) = I$ and

$$T(x) = (T_{jk}(x))_{j,k=1,\dots,n} = \left(\frac{x}{x-(j-k)i}\right)_{j,k=1,\dots,n}, \quad x \neq 0. \tag{3.7}$$

Proof According to formulas (3.1) and (3.5),

$$\begin{aligned}
\gamma^{n,a}(x) &= \int_0^\pi a(\theta)\overline{\psi(x)e^{-x\theta-i\theta}\Phi_n(x)L_n(e^{-2\theta i})}[\psi(x)e^{-x\theta-i\theta}\Phi_n(x)L_n(e^{-2\theta i})]^t d\theta \\
&= \overline{\Phi_n(x)}\left((\psi(x))^2\int_0^\pi a(\theta)e^{-2x\theta}\overline{L_n(e^{-2\theta i})}[L_n(e^{-2\theta i})]^t d\theta\right)(\Phi_n(x))^t \\
&= \overline{\Phi_n(x)}M^{n,a}(x)[\Phi_n(x)]^t.
\end{aligned}$$

Now, if $a(z) = 1$, then $\gamma^{n,1}(x) = I$. Thus $I = \overline{\Phi_n(x)}T(x)[\Phi_n(x)]^t$, where $T(x) = M^{n,1}(x)$ and

$$M^{n,1}(x) = (\psi(x))^2 \int_0^\pi e^{-2x\theta}\overline{L_n(e^{-2\theta i})}[L_n(e^{-2\theta i})]^t d\theta.$$

A direct computation of this integral leads to (3.7). □

Let us see how $M^{n,a}$ is related to a Toeplitz matrix. Consider the measurable function $\pi a(\theta/2)(\psi(x))^2 e^{-x\theta} \in L_\infty([0, 2\pi])$, and its Fourier coefficients

$$A_m^a(x) := \pi(\psi(x))^2 \int_0^{2\pi} a(\theta/2)e^{-x\theta}e^{-m\theta i}\,\frac{d\theta}{2\pi}. \tag{3.8}$$

Then $M^{n,a}(x)$ is the transpose truncated matrix of the infinite Toeplitz matrix $(A_{j-k}(x))_{j,k}$, that is, $M^{n,a}(x) = (A^a_{k-j}(x))^n_{j,k=1}$. Let $J = (J_{jk}) \in M_n(\mathbb{C})$ be the Jordan matrix of order n, that is, $J_{k,k+1} = 1$, and $J_{jk} = 0$ elsewhere. Then

$$M^{n,a}(x) = \sum_{m=-n+1}^{n-1} A^a_m(x) J_m, \tag{3.9}$$

where $J_m = J^m$ and $J_{-m} = (J^t)^m$ for $m = 0, \ldots, n-1$.

4 C^*-Algebra Generated by Toeplitz Operators

Recall that $\mathcal{T}^n_{-\infty\infty}$ denotes the C^*-algebra generated by all the Toeplitz operators $T_{n,a}$ acting on $\mathcal{A}^2_n(\Pi)$, with homogeneous symbols. We have the unitary equivalence $R_n T_{n,a} R_n^* = \gamma^{n,a}(x)I$, where $\gamma^{n,a}$ is given in Eq. (3.1). In this section we prove that $\mathcal{T}^n_{-\infty\infty}$ can be generated by a finite number of Toeplitz operators.

Let N be an odd integer, and take distinct integers $N_{-n+1}, \ldots, N_{3n-2}$ such that

$$N_k = k, \quad \forall k = -n+1, \ldots, n-1.$$

Theorem 4.1 *The C^*-algebra $\mathcal{T}^n_{-\infty\infty}$ is generated by the Toeplitz operators $T_{n,a_{N/2}}$, $T_{n,a_{-n+1}}, \ldots, T_{n,a_{3n-2}}$, where $a_{N/2}(\theta) = e^{N\theta i}$ and $a_k(\theta) = e^{2N_k\theta i}$ for every $k = -n+1, \ldots, 3n-2$. Equivalently, the C^*-algebra $\mathfrak{C}_n$ is generated by the matrix-valued functions*

$$\gamma^{n,a_{N/2}}(x), \gamma^{n,a_{-n+1}}(x), \ldots, \gamma^{n,a_{3n-2}}(x). \tag{4.1}$$

Moreover, the map $\mathcal{T}^n_{-\infty\infty} \ni T \mapsto R_n T R_n^* \in \mathfrak{C}_n$ is an isometric isomorphism of C^*-algebras, where

$$T_{n,a} \longmapsto \gamma^{n,a}(x).$$

Proof It is known that $T \mapsto R_n T R_n^*$ is an isometric isomorphism [17]. Note that $\mathfrak{C}_n$ is a type I C^*-algebra. Let $\mathfrak{B}$ be the C^*-subalgebra of $\mathfrak{C}_n$ generated by the matrix-valued functions given in (4.1). Then, $\mathfrak{B}$ separates all the pure states of $\mathfrak{C}_n$ as shown in Lemmas 4.2 and 4.4 below. By *the noncommutative Stone-Weierstrass conjecture* proved by I. Kaplansky [12] for type I or GCR C^*-algebras, we have that $\mathfrak{C}_n = \mathfrak{B}$. □

Take a symbol $a_N = e^{2N\theta i}$, with N an integer. For the matrix-valued function $\gamma^{n,a_N}(x)$ given by formulas (3.6), (3.8) and (3.9), we have

$$A^{a_N}_j(x) = \frac{xi}{N - j + xi}, \quad j = -n+1, \ldots, n-1 \tag{4.2}$$

where $A_j^{a_N}(0) = \delta_{jN}$, and δ_{jk} is the Kronecker delta function. On the other hand, if $a_{\frac{N}{2}}(\theta) = e^{N\theta i}$, with N an odd integer, then

$$A_j^{a_{N/2}}(x) = i(\psi(x))^2(1+e^{-2\pi x})\frac{1}{N-(2j-2xi)}. \tag{4.3}$$

The value $\overline{A_j^{\overline{a_{N/2}}}(-x)}$ will be needed in Sect. 5. From $\psi(-x) = e^{-\pi x}\psi(x)$ it follows that

$$\overline{A_j^{\overline{a_{N/2}}}(-x)} = i(\psi(x))^2(1+e^{-2\pi x})\frac{1}{N-(-2j+2xi)}. \tag{4.4}$$

Let E_{jk} denote the $n \times n$ matrix with 1 in the (j,k)-entry, and 0 elsewhere. The next lemma asserts that two pure states of $\mathfrak{C}_n$ supported at the same fiber can be separated.

Lemma 4.2 *Take $a_k(z) = e^{2N_k\theta i}$ for $k = -n+1, \ldots, n-1$, where the integers N_k's are distinct from each other. For $x_0 \in (-\infty, \infty) \setminus \{0\}$ fixed, the C^*-algebra generated by the matrices $\gamma^{n,a_{-n+1}}(x_0), \ldots, \gamma^{n,a_{n-1}}(x_0)$ is equal to $M_n(\mathbb{C})$. In the case $x_0 = 0$, the matrices $\gamma^{n,a_{-n+1}}(x_0), \ldots, \gamma^{n,a_{n-1}}(x_0)$ generate $M_n(\mathbb{C})$ if $N_k = k$ for $k = -n+1, \ldots, n-1$.*

Proof Take $x_0 \in (-\infty, \infty)$. We have $\gamma^{n,a}(x_0) = \overline{\Phi_n(x_0)}M^{n,a}(x_0)[\Phi_n(x_0)]^t$ for any homogeneous symbol $a(\theta)$. Suppose that

$$0 = \sum_{k=-n+1}^{n-1} c_k\gamma^{n,a_k}(x_0) = \overline{\Phi_n(x_0)}\left(\sum_{k=-n+1}^{n-1} c_k M^{n,a_k}(x_0)\right)[\Phi_n(x_0)]^t.$$

By (3.9),

$$\begin{aligned} 0 &= \sum_{k=-n+1}^{n-1} c_k M^{n,a_k}(x_0) \\ &= \sum_{k=-n+1}^{n-1} c_k \left[\sum_{j=-n+1}^{n-1} A_j^{a_k}(x_0)J_j\right] \\ &= \sum_{j=-n+1}^{n-1}\left[\sum_{k=-n+1}^{n-1} c_k A_j^{a_k}(x_0)\right] J_j. \end{aligned}$$

Since $\mathcal{J} = \{J_j\}_{j=-n+1}^{n-1}$ is linearly independent, we have

$$\sum_{k=-n+1}^{n-1} c_k A_j^{a_k}(x_0) = 0, \quad j = -n+1, \ldots, n-1. \tag{4.5}$$

We will show that $S := \{\gamma^{n,a_{-n+1}}(x_0), \ldots, \gamma^{n,a_{n-1}}(x_0)\}$ is linearly independent. For the time being suppose that $x_0 \neq 0$. Thus, S is linearly independent if only if the determinant of $(A_j^{a_k}(x_0))_{j,k=-n+1}^{n-1}$ is nonzero. By formula (4.2),

$$\det(A_j^{a_k}(x_0))_{j,k=-n+1}^{n-1} = \det\left(\frac{x_0 i}{-j + x_0 i + N_k}\right)_{j,k=-n+1}^{n-1}.$$

Define $\alpha_j = -j + x_0 i$ and $\beta_k = N_k$. Note that $\alpha_k \neq \alpha_l$ and $\beta_k \neq \beta_l$ for $k \neq l$. Then, the determinant of $(A_j^{a_k}(x_0))_{j,k=-n+1}^{n-1}$ is computed by the Cauchy double alternant, cf. [2, 11]. That is,

$$\begin{aligned}\det(A_j^{a_k}(x_0))_{j,k=-n+1}^{n-1} &= (x_0 i)^{2n-1} \det\left(\frac{1}{\alpha_j + \beta_k}\right)_{j,k=-n+1}^{n-1} \\ &= (x_0 i)^{2n-1} \frac{\prod_{-n+1\le j<k\le n-1}(\alpha_k - \alpha_j)(\beta_k - \beta_j)}{\prod_{j,k=-n+1}^{n-1}(\alpha_j + \beta_k)}.\end{aligned}$$

This proves that S is linearly independent in the case $x_0 \neq 0$. Then, we have span $S = \overline{\Phi_n(x_0)}(\text{span}\,\mathcal{J})\Phi_n(x_0)^t$. Let $\mathcal{B}$ be the C^*-algebra generated by S. Of course, $\overline{\Phi_n(x_0)} J^0 \Phi_n(x_0)^t$ belongs to $\mathcal{B}$. Therefore $(\Phi_n(x_0)^t)^{-1}(\overline{\Phi_n(x_0)})^{-1} \in \mathcal{B}$. Consequently, $\overline{\Phi_n(x_0)}(\text{span}\,\mathcal{J})(\overline{\Phi_n(x_0)})^{-1} \subset \mathcal{B}$. On the other hand, $(J^t)^{j-1}J^{n-1} = E_{jn}$ and $(J^t)^{n-1}J^{j-1} = E_{nj}$ for $j = 1, \ldots, n$. Further $E_{jk} = E_{jn}E_{nk}$. Thus, $\mathcal{B} = M_n(\mathbb{C})$.

Now suppose that $x_0 = 0$, and $N_k = k$ for $k = -n+1, \ldots, n-1$. Thus $A_j^{a_k}(0) = \delta_{jk}$ for $j, k = -n+1, \ldots, n-1$. Actually, $\Phi_n(0) = I$ and $M^{n,a_k}(0) = J_k$. Hence, $\gamma^{n,a_k}(0) = J_k$ for $k = -n+1, \ldots, n-1$. We know that $\mathcal{J}$ generates $M_n(\mathbb{C})$. □

We proceed to separate two pure states of $\mathfrak{C}_n$ supported at different fibers.

Lemma 4.3 *Let $R(z)$ be a rational function of the form*

$$R(z) = \sum_{k=1}^{m} \frac{c_k}{z - b_k}, \tag{4.6}$$

where c_k and b_k are complex numbers. Suppose that $b_k \neq b_j$ for all $k \neq j$, and that there exists $k_0 \in \{1, \ldots, m\}$ such that $c_{k_0} \neq 0$. Then, the number of zeros of $R(z)$ cannot exceed $m - 1$. For $c \in \mathbb{C}$, the number of zeros of $R(z) - c$ cannot exceed m.

Proof Note that $\sum_{k=1}^{m} \frac{c_k}{z-b_k} = \frac{r(z)}{\prod_{k=1}^{m}(z-b_k)}$, where

$$r(z) = \sum_{k=1}^{m} c_k \prod_{\substack{j=1 \\ j\neq k}}^{m} (z - b_j).$$

Obviously, $R(z) = 0$ implies that $r(z) = 0$. Since $c_{k_0} \neq 0$, we have

$$r(b_{k_0}) = c_{k_0} \prod_{\substack{j=1 \\ j\neq k_0}}^{m} (b_{k_0} - b_j) \neq 0.$$

This proves that $r(z)$ is a nonzero polynomial of degree at most $m-1$. Since every root of $R(z)$ is a root of $r(z)$, the number of zeros of $R(z)$ cannot exceed $m-1$. □

Recall the C^*-algebra $\mathfrak{C}_n$ and its pure states given in (3.4).

Lemma 4.4 *Let $v, w \in \mathbb{C}^n$ be unit vectors, and $x_0, x_1 \in [-\infty, \infty]$. Take $a_{N/2}(z) = e^{N\theta i}$ with N an odd integer, and $a_k(z) = e^{2N_k\theta i}$ with integers $N_{-n+1} < \cdots < N_{3n-2}$. Then $x_0 = x_1$ provided that*

$$f_{x_0,v}(\gamma^{n,a_{N/2}}) = f_{x_1,w}(\gamma^{n,a_{N/2}})$$

and

$$f_{x_0,v}(\gamma^{n,a_k}) = f_{x_1,w}(\gamma^{n,a_k}), \quad \forall\, k = -n+1, \ldots, 3n-2.$$

Proof Let $v, w \in \mathbb{C}^n$ be unit vectors, and $x_0, x_1 \in (-\infty, \infty)$. Introduce the vectors $\tilde{v} = [\Phi_n(x_0)]^t v$ and $\tilde{w} = [\Phi_n(x_1)]^t w$ in $\mathbb{C}^n$. We have

$$\begin{aligned} f_{x_0,v}(\gamma^{n,a_k}) &= \langle \gamma^{n,a_k}(x_0)v, v\rangle \\ &= \langle \overline{\Phi_n(x_0)} M^{n,a_k}(x_0)[\Phi_n(x_0)]^t v, v\rangle \\ &= \langle M^{n,a_k}(x_0)\tilde{v}, \tilde{v}\rangle. \end{aligned}$$

Analogously, $f_{x_1,w}(\gamma^{n,a_k}) = \langle M^{n,a_k}(x_1)\tilde{w}, \tilde{w}\rangle$. Suppose that $f_{x_0,v}(\gamma^{n,a_k}(x)) = f_{x_1,w}(\gamma^{n,a_k}(x))$ for all $k = -n+1, \ldots, 3n-2$. By formula (3.9),

$$\sum_{j=-n+1}^{n-1} A_j^{a_k}(x_0)\langle J_j\tilde{v}, \tilde{v}\rangle = \sum_{j=-n+1}^{n-1} A_j^{a_k}(x_1)\langle J_j\tilde{w}, \tilde{w}\rangle \tag{4.7}$$

for all $k = -n+1, \ldots, 3n-2$. Suppose that $x_0 \neq x_1$ and $x_0, x_1 \neq 0$. By formula (4.2) we have

$$A_j^{a_k}(x_l) = \frac{x_l i}{N_k - (j - x_l i)}, \quad l = 0, 1 \text{ and } j = -n+1, \ldots, n-1.$$

Define $b_j(x) = j - xi$ for $j = -n+1, \ldots, n-1$. Then, there exist scalars $c_j(x_0, \tilde{v})$ and $c_j(x_1, \tilde{w})$ such that (4.7) can be written as

$$R_0(N_k) - R_1(N_k) = 0, \quad \forall\, k = -n+1, \ldots, 3n-2,$$

where

$$R_0(z) = \sum_{j=-n+1}^{n-1} \frac{c_j(x_0, \tilde{v})}{z - b_j(x_0)} \text{ and } R_1(z) = \sum_{j=-n+1}^{n-1} \frac{c_j(x_1, \tilde{w})}{z - b_j(x_1)}.$$

It is easy to see that $c_0(x_0, \tilde{v}) = i x_0 \|\tilde{v}\|^2$. But $c_0(x_0, \tilde{v}) \neq 0$ since $[\Phi_n(x_0)]^t$ is invertible. Furthermore, $x_0 \neq x_1$ implies $b_j(x_0) \neq b_k(x_1)$ for all j, k. By Lemma 4.3, the function $R(z) := R_0(z) - R_1(z)$ cannot have more than $4n-3$ roots, contradicting that $R(z)$ has $4n-2$ roots. Therefore $x_0 = x_1$.

Suppose now that $x_0 \neq 0$ and $x_1 = 0$. Then $A_j^{a_k}(0) = \delta_{j,N_k}$, and $R_1(z)$ is a constant function. Thus, $R(z)$ has at most $2n-1$ roots. Therefore, the pure states $f_{x_0,v}$ and $f_{0,w}$ can be separated using $2n-2$ symbols.

According to (3.2), we have $\gamma^{n,a_{N/2}}(-\infty) = -I$ and $\gamma^{n,a_{N/2}}(\infty) = I$. Hence

$$f_{-\infty,v}(\gamma^{n,a_{N/2}}) \neq f_{+\infty,w}(\gamma^{n,a_{N/2}}).$$

Now take $x_0 \neq 0$, and suppose that $f_{x_0,v}(\gamma^{n,a_k}) = f_{\pm\infty,w}(\gamma^{n,a_k})$ for all $k = -n+1, \ldots, 3n-2$. Since $\gamma^{n,a_k}(\pm\infty) = I$, we have $f_{\pm\infty,w}(\gamma^{n,a_k}) = 1$. But the rational function $\tilde{R}(z) = R_0(z) - 1$ has at most $2n-1$ roots, thus, the pure states $f_{x_0,v}$ and $f_{\pm\infty,w}$ can be separated using $2n-2$ symbols.

Finally, take $x_0 = 0$, and choose $N_k \notin \{-n+1, \ldots, n-1\}$. Since $A_j^{a_k}(0) = \delta_{jN_k}$, we have $\gamma^{n,a_k}(x_0) = 0$. Therefore $f_{x_0,v}(\gamma^{n,a_k}) \neq f_{\pm\infty,w}(\gamma^{n,a_k})$. ☐

5 Toeplitz Operators Acting on the Polyharmonic Space

The C^*-algebra generated by all the Toeplitz operators (with homogeneous symbols) acting on the polyharmonic space $\mathcal{H}_n^2(\Pi) = \mathcal{A}_n^2(\Pi) \oplus \widetilde{\mathcal{A}}_n^2(\Pi)$ was described in [16]; fortunately, this algebra can be generated by a finite number of Toeplitz operators as shown in this section. The reflection map of Π with respect to the y-axis establishes a relationship between $\mathcal{A}_n^2(\Pi)$ and $\widetilde{\mathcal{A}}_n^2(\Pi)$, and the corresponding Toeplitz operators acting on these spaces. Indeed, define the self-adjoint unitary operator $\mathfrak{J} : L_2(\Pi) \longrightarrow L_2(\Pi)$ by $(\mathfrak{J} f)(z) = f(-\overline{z})$. In polar coordinates we have $(\mathfrak{J} f)(r,\theta) = f(r, \pi - \theta)$, where $f \in L_2(\Pi) = L_2(\mathbb{R}_+, rdr) \otimes L_2([0,\pi], d\theta)$. It is easy to see that $\left(\frac{\partial}{\partial z}\right)^n \mathfrak{J} = (-1)^n \mathfrak{J} \left(\frac{\partial}{\partial \overline{z}}\right)^n$. Thus $\widetilde{\mathcal{A}}_n^2(\Pi) = \mathfrak{J}(\mathcal{A}_n^2(\Pi))$ and $\widetilde{B}_{\Pi,n} = \mathfrak{J} B_{\Pi,n} \mathfrak{J}$, where $\widetilde{B}_{\Pi,n}$ is the orthogonal projection from $L_2(\Pi)$ onto $\widetilde{\mathcal{A}}_n^2(\Pi)$.

For $a \in L_\infty^{\{0,\pi\}}$, let $\widetilde{T}_{n,a}$ be the Toeplitz operator defined by

$$\widetilde{T}_{n,a} : \varphi \in \widetilde{\mathcal{A}}_n^2(\Pi) \longmapsto \widetilde{B}_{\Pi,n}(a\varphi) \in \widetilde{\mathcal{A}}_n^2(\Pi).$$

We have that $\mathfrak{J}\widetilde{T}_{n,a}\mathfrak{J} = T_{n,\widetilde{a}}$, where $\widetilde{a}(\theta) = a(\theta - \pi)$. Introduce the operator $\widetilde{R}_n = R_n \mathfrak{J} : L_2(\Pi) \to (L_2(\mathbb{R}))^n$, which is an isometric isomorphism from $\widetilde{\mathcal{A}}_n^2(\Pi)$ onto $(L_2(\mathbb{R}))^n$. Then, the Toeplitz operator $\widetilde{T}_{n,a}$ is unitarily equivalent to the multiplication operator $\gamma^{n,\widetilde{a}}(x) I = \widetilde{R}_n \widetilde{T}_{n,a} \widetilde{R}_n^*$ acting on $(L_2(\mathbb{R}))^n$. Since $\overline{b_{nk}(x)} = b_{nk}(-x)$, we have $p_n(x, \pi - \theta) = \overline{p_n(-x,\theta)}$. In addition, $e^{-\pi x}\psi(x) = \psi(-x)$. Hence $H_n(x, \pi - \theta) = -\overline{H_n(-x,\theta)}$. Thus

$$\gamma^{n,\widetilde{a}}(x) = \overline{\gamma^{n,\overline{a}}(-x)} = \int_0^\pi a(\theta) H_n(-x,\theta)[\overline{H_n(-x,\theta)}]^t \, d\theta. \tag{5.1}$$

On the other hand, $\Phi_n(-x) = \overline{\Phi_n(x)}$. Therefore, (3.6) and (5.1) imply that

$$\gamma^{n,\widetilde{a}}(x) = \overline{\Phi_n(x)}\;\overline{M^{n,\overline{a}}(-x)}[\Phi_n(x)]^t.$$

Certainly, $Q_{\Pi,n} = B_{\Pi,n} \oplus \widetilde{B}_{\Pi,n}$ is the orthogonal projection onto $\mathcal{H}_n^2(\Pi)$. Introduce the Toeplitz operator

$$\widehat{T}_{n,a} : \mathcal{H}_n^2(\Pi) \longrightarrow \mathcal{H}_n^2(\Pi)$$

by the rule $\widehat{T}_{n,a}(f) = Q_{\Pi,n}(af)$. Define $W_n = R_n \oplus \widetilde{R}_n$, which is an isometric isomorphism from $\mathcal{H}_n^2(\Pi)$ onto $(L_2(\mathbb{R}))^n \times (L_2(\mathbb{R}))^n$. Thus, $W_n W_n^* = I$ and $W_n^* W_n = B_{\Pi,n} \oplus \widetilde{B}_{\Pi,n}$.

Let $a(\theta) \in L_\infty^{\{0,\pi\}}$. A straightforward computation shows that the Toeplitz operator $\widehat{T}_{n,a}$ is unitary equivalent to the multiplication operator $\widehat{\gamma}^{n,a}(x) I = W_n \widehat{T}_{n,a} W_n^*$ acting on $(L_2(\mathbb{R}))^n \times (L_2(\mathbb{R}))^n$, where

$$\widehat{\gamma}^{n,a}(x) = \int_0^\pi a(\theta) \begin{pmatrix} \overline{H_n(x,\theta)}[H_n(x,\theta)]^t & -\overline{H_n(x,\theta)}\;\overline{H_n(-x,\theta)}^t \\ -H_n(-x,\theta) H_n(x,\theta)^t & H_n(-x,\theta)[\overline{H_n(-x,\theta)}]^t \end{pmatrix} d\theta. \tag{5.2}$$

Actually, $\widehat{\gamma}^{n,a}(x)$ belongs to the C^*-algebra $\widehat{\mathfrak{C}}_n$ consisting of all matrix-valued functions $f = (f_{ij}) \in M_{2n}(\mathbb{C}) \otimes C(\overline{\mathbb{R}})$ such that

$$f(-\infty) = \begin{pmatrix} \lambda_1 I & 0I \\ 0I & \lambda_2 I \end{pmatrix}, \quad \text{and} \quad f(+\infty) = \begin{pmatrix} \lambda_2 I & 0I \\ 0I & \lambda_1 I \end{pmatrix} \quad \lambda_1, \lambda_2 \in \mathbb{C},$$

where I denotes the $n \times n$ identity matrix. For $\widehat{\gamma}^{n,a}(x)$ we have $\widehat{\gamma}^{n,a}(-\infty) = \mathrm{diag}\,\{a(\pi)I, a(0)I\}$ [16].

Let $\widehat{\mathcal{T}}^n_{-\infty\infty}$ be the C^*-algebra generated by all the Toeplitz operators $\widehat{T}_{n,a}$ acting on $\mathcal{H}^2_n(\Pi)$, with $a(\theta) \in L_\infty^{\{0,\pi\}}$.

Theorem 5.1 *Consider $a_N(z) = e^{2N\theta i}$ and $a_k(\theta) = e^{N_k\theta i}$, where $N_{-2n+1},\ldots, N_{10n-6}$ are distinct odd integers and N is an integer not in $\{-2n+1,\ldots,2n-1\}$. Then $\widehat{\mathcal{T}}^n_{-\infty\infty}$ is generated by the Toeplitz operators $\widehat{T}_{n,a_N}$ and $\widehat{T}_{n,a_k}$ for $k = -2n+1,\ldots,10n-6$. Equivalently, the C^*-algebra $\widehat{\mathfrak{C}}_n$ is generated by all the matrix-valued functions*

$$\widehat{\gamma}^{n,a_N}(x), \widehat{\gamma}^{n,a_{-2n+1}}(x), \ldots, \widehat{\gamma}^{n,a_{10n-6}}(x). \tag{5.3}$$

Moreover, the map $\widehat{\mathcal{T}}^n_{-\infty\infty} \ni T \mapsto W_n T W_n^* \in \widehat{\mathfrak{C}}_n$ is an isometric isomorphism, where

$$\widehat{T}_{n,a} \longmapsto \widehat{\gamma}^{n,a}(x).$$

Proof Note that $\widehat{\mathfrak{C}}_n$ is a type I C^*-algebra. Let $\widehat{\mathfrak{B}}$ be the C^*-subalgebra of $\widehat{\mathfrak{C}}_n$ generated by all the matrix-valued functions (5.3). Then $\widehat{\mathfrak{B}}$ separates all the pure states of $\widehat{\mathfrak{C}}_n$ as shown in Lemmas 5.3, 5.4 and 5.5 below. By *the noncommutative Stone-Weierstrass conjecture* proved by I. Kaplansky [12] for type I or GCR C^*-algebras, we have that $\widehat{\mathfrak{C}}_n = \widehat{\mathfrak{B}}$. □

As for $\gamma^{n,a}(x)$, we need a convenient factorization of $\widehat{\gamma}^{n,a}(x)$.

Lemma 5.2 *For $a \in L_\infty^{\{0,\pi\}}$, the function $\widehat{\gamma}^{n,a}(x)$ can be written as*

$$\widehat{\gamma}^{n,a}(x) = \widehat{\Phi}(x)\mathcal{M}^{n,a}(x)[\widehat{\Phi}(x)]^*,$$

where $\widehat{\Phi}(x) = \begin{pmatrix} \overline{\Phi_n(x)} & 0 \\ 0 & \overline{\Phi_n(x)} \end{pmatrix}$, $\mathcal{M}^{n,a}(x) = \begin{pmatrix} M^{n,a}(x) & \overline{N^{n,\overline{a}}(-x)} \\ N^{n,a}(x) & \overline{M^{n,\overline{a}}(-x)} \end{pmatrix}$, *and*

$$N^{n,a}(x) = -(\psi(x))^2 e^{-\pi x} \int_0^\pi a(\theta) e^{-2\theta i} L_n(e^{-2\theta i})[L_n(e^{-2\theta i})]^t\, d\theta.$$

Proof It follows from (3.5), (5.2), and the equality $\Phi_n(-x) = \overline{\Phi_n(x)}$. □

Note that $N^{n,a}(x)$ is a Hankel matrix and it can be written as

$$N^{n,a}(x) = \sum_{j=-n+1}^{n-1} \eta^a_{n+j}(x) M_{n+j}, \tag{5.4}$$

where

$$\eta_m^a(x) = -\pi(\psi(x))^2 e^{-\pi x} \int_0^{2\pi} a(\theta/2) e^{-\theta m i} \, \frac{d\theta}{2\pi}, \quad m = 1, \dots, 2n-1, \tag{5.5}$$

and M_m is the Hankel matrix whose (j, k)-entry is equal to 1 for $j + k = m + 1$, and 0 elsewhere. For $a_k(\theta) = e^{N_k \theta i}$ with N_k an odd integer, we have

$$\eta_j^{a_k}(x) = -2i(\psi(x))^2 e^{-\pi x} \frac{1}{N_k - 2j}, \quad j = 1, \dots, 2n-1, \tag{5.6}$$

$$\overline{\eta_j^{\overline{a_k}}(-x)} = -2i(\psi(x))^2 e^{-\pi x} \frac{1}{N_k + 2j}, \quad j = 1, \dots, 2n-1. \tag{5.7}$$

We proceed now to separate all the pure states of the C^*-algebra $\widehat{\mathfrak{C}}_n$ using the elements of the C^*-algebra $\widehat{\mathfrak{B}}$ generated by all the matrix-valued functions given in (5.3). Each pure state [3, 12, 13] of $\widehat{\mathfrak{C}}_n$ has the form

$$f_{x,u}(M) = \langle M(x)u, u \rangle, \quad M \in \widehat{\mathfrak{C}}_n,$$

where $u \in \mathbb{C}^n \times \mathbb{C}^n$ is unimodular, and $x \in \overline{\mathbb{R}}$.

Lemma 5.3 *There are only two pure states of* $\widehat{\mathfrak{C}}_n$ *corresponding to* $x_0 = -\infty$*:* f_{x_0,v_1} *and* f_{x_0,v_2}*, where* $v_1 = (e_1^t, \mathbf{0}^t)$*,* $v_2 = (\mathbf{0}^t, e_1^t)$*,* $e_1 = (1, 0, \dots, 0)^t$*, and* $\mathbf{0} = (0, \dots, 0)^t \in \mathbb{C}^n$*. These pure states can be separated by any* $\widehat{\gamma}^{n,a_k}$*.*

Proof Since $\widehat{\gamma}^{n,a_k}(-\infty) = \operatorname{diag}\{-I, I\}$, we have that $f_{x_0,v_1}(\widehat{\gamma}^{n,a_k}) = -1$ and $f_{x_0,v_2}(\widehat{\gamma}^{n,a_k}) = 1$. □

Note that the pure states of $\widehat{\mathfrak{C}}_n$ corresponding to $x = \infty$ depend on the pure states of $\widehat{\mathfrak{C}}_n$ at $-\infty$.

For $k = -n+1, \dots, n-1$, introduce

$$\mathbf{J}_k = \begin{pmatrix} J_k & 0I \\ 0I & 0I \end{pmatrix}, \quad \widetilde{\mathbf{J}}_k = \begin{pmatrix} 0I & 0I \\ 0I & J_k \end{pmatrix},$$

$$\mathbf{M}_k = \begin{pmatrix} 0I & 0I \\ M_{n+k}I & 0I \end{pmatrix}, \quad \widetilde{\mathbf{M}}_k = \begin{pmatrix} 0I & M_{n+k}I \\ 0I & 0I \end{pmatrix}.$$

Lemma 5.4 *Take* $a_k(z) = e^{N_k \theta i}$*, where* $N_{-2n+1}, \dots, N_{6n-4}$ *are distinct odd integers. For* $x_0 \in (-\infty, \infty)$ *fixed, the* C^**-algebra generated by the matrices* $\widehat{\gamma}^{n,a_{-2n+1}}(x_0), \dots, \widehat{\gamma}^{n,a_{6n-4}}(x_0)$ *is equal to* $M_{2n}(\mathbb{C})$*.*

Proof Take $x_0 \in (-\infty, \infty)$. We will show that $\widehat{S} := \{\widehat{\gamma}^{n,a_k}(x_0)\}_{k=-2n-1}^{6n-4}$ is linearly independent for $x_0 \neq 0$. Suppose that

$$0 = \sum_{k=-2n+1}^{6n-4} c_k \widehat{\gamma}^{n,a_k}(x_0) = \widehat{\Phi}(x_0) \left(\sum_{k=-2n+1}^{6n-4} \mathcal{M}^{n,a_k}(x_0) \right) [\widehat{\Phi}(x_0)]^*.$$

According to formulas (3.9) and (5.4),

$$\begin{aligned} 0 &= \sum_{k=-2n+1}^{6n-4} c_k \mathcal{M}^{n,a_k}(x_0) \\ &= \sum_{j=-n+1}^{n-1} \sum_{k=-2n+1}^{6n-4} \left[c_k A_j^{a_k}(x_0) \mathbf{J}_j + c_k \overline{A_j^{\overline{a_k}}(-x_0)} \widetilde{\mathbf{J}}_j \right] \\ &+ \sum_{j=-n+1}^{n-1} \sum_{k=-2n+1}^{6n-4} \left[c_k \eta_{n+j}^{a_k}(x_0) \mathbf{M}_j + c_k \overline{\eta_{n+j}^{\overline{a}_k}(-x_0)} \widetilde{\mathbf{M}}_j \right]. \end{aligned}$$

Since $\widehat{\mathcal{J}} = \{\mathbf{J}_j,\ \widetilde{\mathbf{J}}_j,\ \mathbf{M}_j,\ \widetilde{\mathbf{M}}_j\}_{j=-n+1}^{n-1}$ is linearly independent, we have

$$\sum_{k=-2n+1}^{6n-4} c_k q_j^{a_k}(x_0) = 0, \quad j = -2n+1, \ldots, 6n-4,$$

where

$$q_j^{a_k}(x_0) = \begin{cases} A_{j+n}^{a_k}(x_0) & \text{if } j = -2n+1, \ldots, -1, \\ \overline{A_{j-n+1}^{\overline{a_k}}(-x_0)} & \text{if } j = 0, \ldots, 2n-2, \\ \eta_{j-2n+2}^{a_k}(x_0) & \text{if } j = 2n-1, \ldots, 4n-3, \\ \overline{\eta_{j-4n+3}^{\overline{a_k}}(-x_0)} & \text{if } j = 4n-2, .., 6n-4. \end{cases}$$

Now $\widehat{S}$ is linearly independent if and only if the determinant of $(q_j^{a_k}(x_0))$ is nonzero. By formulas (4.3), (4.4), (5.6) and (5.7),

$$\det(q_j^{a_k}(x_0))_{j,k=-2n+1}^{6n-4} = \nu(x_0) \det \left(\frac{1}{\alpha_j + \beta_k} \right)_{j,k=-2n+1}^{6n-4},$$

where $\nu(x) = (i\psi(x)^2)^{8n-4}(2e^{-\pi x}(1+e^{-2\pi x}))^{4n-2}$, $\beta_k = N_k$, and

$$\alpha_j = \begin{cases} -2(j+n) + 2x_0 i & \text{if } j = -2n+1, \dots, -1, \\ 2(j-n+1) - 2x_0 i & \text{if } j = 0, \dots, 2n-2, \\ -2(j-2n+2) & \text{if } j = 2n-1, \dots, 4n-3, \\ 2(j-4n+3) & \text{if } j = 4n-2, \dots, 6n-4. \end{cases}$$

The complex numbers α_j's are distinct for $x_0 \neq 0$, thus

$$\det(q_j^{a_k}(x_0))_{j,k=-2n+1}^{6n-4} = \nu(x_0) \frac{\prod_{-2n+1 \le j < k \le 6n-4} (\alpha_k - \alpha_j)(\beta_k - \beta_j)}{\prod_{j,k=-2n+1}^{6n-4} (\alpha_j + \beta_k)} \neq 0.$$

This proves that $\widehat{S}$ is linearly independent in the case $x_0 \neq 0$. Therefore, span $\widehat{S} = \widehat{\Phi}(x_0)(\text{span}\, \widehat{\mathcal{J}})\widehat{\Phi}(x_0)^*$. Let $\widehat{\mathcal{B}}$ be the C^*-algebra generated by $\widehat{S}$. As in the proof of Lemma 4.2, $\widehat{\Phi}(x_0)(\text{span}\, \widehat{\mathcal{J}})\widehat{\Phi}(x_0)^{-1} \subset \widehat{\mathcal{B}}$. It easy to see that $F_{n+j+1} := \mathbf{M}_j \widetilde{\mathbf{M}}_{n-1} = \text{diag}\,\{0I, E_{j+1,n}\}$ for $j = 0, \dots, n-1$. Besides

$$F_{j+1} := \widetilde{\mathbf{M}}_j (\mathbf{M}_{n-1} \widetilde{\mathbf{M}}_{n-1}) = \begin{pmatrix} 0I & E_{j+1,n} \\ 0I & 0I \end{pmatrix}, \quad j = 0, \dots, n-1.$$

Note that $\{F_j, F_j^t\}_{j=1}^{2n}$ generates $M_{2n}(\mathbb{C})$. That is, $\{\mathbf{M}_j, \widetilde{\mathbf{M}}_j\}_{j=0}^{n-1}$ generates $M_{2n}(\mathbb{C})$. Consequently, $\widehat{S}$ generates $M_{2n}(\mathbb{C})$.

Take now $x_0 = 0$. Then $\widehat{\Phi}(0) = \text{diag}\,\{I, I\}$, and

$$A_j^{a_k}(0) = \frac{2i}{\pi} \frac{1}{N_k - 2j}, \quad \overline{A_j^{\overline{a_k}}(0)} = \frac{2i}{\pi} \frac{1}{N_k + 2j}, \quad j = -n+1, \dots, n-1,$$

$$\eta_j^{a_k}(0) = -\frac{2i}{\pi} \frac{1}{N_k - 2j}, \quad \overline{\eta_j^{\overline{a_k}}(0)} = -\frac{2i}{\pi} \frac{1}{N_k + 2j}, \quad j = 1, \dots, 2n-1.$$

Thus

$$\begin{aligned} \widehat{\gamma}^{n,a_k}(0) = {} & A_0^{a_k}(0)(\mathbf{J}_0 + \widetilde{\mathbf{J}}_0) + \sum_{j=1}^{n-1} A_{-j}^{a_k}(0)(\mathbf{J}_{-j} + \widetilde{\mathbf{J}}_j - \widetilde{\mathbf{M}}_{-n+j}) \\ & + \sum_{j=1}^{n-1} A_j^{a_k}(0)(\mathbf{J}_j + \widetilde{\mathbf{J}}_{-j} - \mathbf{M}_{-n+j}) \\ & + \sum_{j=0}^{n-1} \left[\eta_{n+j}^{a_k}(0)\mathbf{M}_j + \overline{\eta_{n+j}^{\overline{a_k}}(0)} \widetilde{\mathbf{M}}_j \right]. \end{aligned}$$

It can be proved that $S = \{\widehat{\gamma}^{n,a_{-2n+1}}(0), \ldots, \widehat{\gamma}^{n,a_{2n-1}}(0)\}$ is linearly independent by using the Cauchy double alternant. Thus, the set of matrices $\{\mathbf{M}_j, \widetilde{\mathbf{M}}_j\}_{j=0}^{n-1}$ is contained in span S. We know that $\{\mathbf{M}_j, \widetilde{\mathbf{M}}_j\}_{j=0}^{n-1}$ generates $M_{2n}(\mathbb{C})$. □

Lemma 5.5 *Let $v, w \in \mathbb{C}^{2n}$ be unit vectors, and $x_0, x_1 \in [-\infty, \infty]$. Take N an integer not in $\{-2n+1, \ldots, 2n-1\}$, and $N_{-2n+1}, \ldots, N_{10n-6}$ distinct odd integers. Define $a_N(z) = e^{2N\theta i}$, and $a_k(z) = e^{N_k \theta i}$ for $k = -2n+1, \ldots, 10n-6$. Then $x_0 = x_1$ whenever*

$$f_{x_0,v}(\widehat{\gamma}^{n,a_N}) = f_{x_1,w}(\widehat{\gamma}^{n,a_N})$$

and

$$f_{x_0,v}(\widehat{\gamma}^{n,a_k}) = f_{x_1,w}(\widehat{\gamma}^{n,a_k}), \quad k = -2n+1, \ldots, 10n-6. \tag{5.8}$$

Proof Let $v, w \in \mathbb{C}^{2n}$ be unit vectors, and $x_0, x_1 \in (-\infty, \infty)$. Define $\tilde{v} = [\widehat{\Phi}(x_0)]^* v$ and $\tilde{w} = [\widehat{\Phi}(x_1)]^* w$. Then $f_{x_0,v}(\widehat{\gamma}^{n,a_k}) = \left\langle \mathcal{M}^{n,a_k}(x_0)\tilde{v}, \tilde{v}\right\rangle$ and $f_{x_1,w}(\widehat{\gamma}^{n,a_k}) = \langle \mathcal{M}^{n,a_k}(x_1)\tilde{w}, \tilde{w}\rangle$. Write $\tilde{v} = (v_1^t, v_2^t)^t$ and $\tilde{w} = (w_1^t, w_2^t)^t$, where $v_1, v_2, w_1, w_2 \in \mathbb{C}^n$. Then

$$\begin{aligned} f_{x_0,v}(\widehat{\gamma}^{n,a_k}) = {} & v_1^t M^{n,a_k}(x_0)\overline{v_1} + v_1^t \overline{N^{n,\overline{a_k}}(-x_0)}\overline{v_2} \\ & + v_2^t N^{n,a_k}(x_0)\overline{v_1} + v_2^t \overline{M^{n,\overline{a_k}}(-x_0)}\overline{v_2}. \end{aligned} \tag{5.9}$$

We have a similar representation for $f_{x_1,w}(\widehat{\gamma}^{n,a_k})$. Introduce the complex numbers $b_{1j}^{\pm}(x) = \pm(2j - 2xi)$ for $j = -n+1, \ldots, n-1$, and $b_{2j}^{\pm} = \pm 2j$ for $j = 1, \ldots, 2n-1$. Equations (5.8) can be written as $R_0(N_k) - R_1(N_k) = 0$, where $R_0(z)$ and $R_1(z)$ are rational functions of the form (4.6). In fact, $R_0(z)$ can be obtained using formulas (3.9), (4.3), (4.4), (5.4), (5.6) and (5.7) in the right-hand side of (5.9). Thus, the singularities of $R_0(z)$ are the complex numbers $b_{1j}^{\pm}(x_0)$ and $b_{2j}^{\pm} = \pm 2j$. Note that $R_0(z)$ and $R_1(z)$ share the singularities $b_{2j}^{\pm}$. If $x_0, x_1 \neq 0$ and $x_0 \neq x_1$, the rational function $R(z) := R_0(z) - R_1(z)$ has at most $12n - 5$ roots. If $x_0 = 0$ and $x_1 \neq 0$, then $R(z)$ has $8n - 4$ singularities, and thus it has at most $8n - 3$ roots. Thus, Eqs. (5.8) imply that $x_0 = x_1$.

Take $x_0 \neq 0$, and suppose that $f_{x_0,v}(\widehat{\gamma}^{n,a_k}) = f_{-\infty,v_m}(\widehat{\gamma}^{n,a_k})$ for all $k = -2n+1, \ldots, 10n-6$, where v_1 and v_2 are given in Lemma 5.3. We have that $f_{-\infty,v_1}(\widehat{\gamma}^{n,a_k}) = -1$ and $f_{-\infty,v_2}(\widehat{\gamma}^{n,a_k}) = 1$ for any k. Since $\tilde{R}(z) := R_0(z) \pm 1$ have at most $8n - 4$ roots, the pure states $f_{x_0,v}$ and $f_{-\infty,v_m}$ can be separated.

Finally, take $x_0 = 0$. Since $N \notin \{-2n+1, \ldots, 2n-1\}$ we have $A_j^{a_N}(0) = \overline{A_j^{\overline{a_N}}(0)} = 0$ for $j = -n+1, \ldots, n-1$, and $\eta_j^{a_N}(0) = \overline{\eta_j^{\overline{a_N}}(0)} = 0$ for $j = 1, \ldots, 2n-1$. That is, $\widehat{\gamma}^{n,a_N}(0) = 0$. Thus $f_{x_0,v}(\widehat{\gamma}^{n,a_N}) \neq f_{-\infty,v_m}(\widehat{\gamma}^{n,a_N})$. □

References

1. S. T. Ali, J. P. Antoine and J. P. Gazeau, *Coherent States, Wavelets and Their Generalizations.* Graduate Texts in Contemporary Physics, Springer-Verlag, New York, 2000.
2. W. Chu, *The Cauchy double alternant and divided differences.* Electron. J. Linear Algebra **15** (2006), 14–21.
3. J. M. G. Fell and R. S. Doran, *Representations of *-Algebras, Locally Compact Groups, and Banach *-Algebraic Bundles, Vol. 1.* Academic Press Inc., New York, London, 1988.
4. J. P. Gazeau, *Coherent States in Quantum Physics.* Wiley-VCH, Berlin, 2009.
5. A. Grossmann and J. Morlet, *Decomposition of Hardy functions into square integrable wavelets of constant shape.* SIAM J. Math. Anal. **15** (1984), no. 4, 723–736.
6. S. Grudsky, R. Quiroga-Barranco and N. L. Vasilevski, *Commutative C^*-algebras of Toeplitz operators and quantization on the unit disk.* J. Funct. Anal. **234** (2006), no. 1, 1–44.
7. O. Hutník, *A note on wavelet subspaces.* Monatsh. Math. **160** (2010), 59–72.
8. O. Hutník, *Wavelets from Laguerre polynomials and Toeplitz-type operators.* Integr. Equ. Oper. Theory **71** (2011), 357–388.
9. O. Hutník and M. Hutníková, *Toeplitz operators on poly-analytic spaces via time-scale analysis.* Oper. Matrices **8** (2014), no. 4, 1107–1129.
10. H. Jianxun and P. Lizhong, *Admissible wavelets associated with the transform group on $\mathbb{R}^n$.* Appl. Math. JCU **12B** (1997), 473–482.
11. W. P. Johnson and E. Lalov, *Some extensions of Brioschi's double alternant.* Linear Algebra Appl. **436** (2012), 1825–1836.
12. I. Kaplansky, *The structure of certain operator algebras.* Trans. Amer. Math. Soc. **70** (1951), no. 2, 219–255.
13. T. K. Lee, *Extreme points related to matrix algebras.* Kangweon-Kyungki Mathematical Journal **9** (2001), no. 1, 45–52.
14. M. Loaiza and C. Lozano, *On C^*-algebras of Toeplitz operators on the harmonic Bergman space.* Integr. Equ. Oper. Theory **76** (2013), no. 1, 105–130.
15. M. Loaiza and C. Lozano, *On Toeplitz operators on the weighted harmonic Bergman space on the upper half-plane.* Complex Anal. Oper. Theory **9** (2015), 139–165.
16. M. Loaiza, I. Morales-García and J. Ramírez-Ortega, *Toeplitz operators with homogeneous symbols on polyharmonic spaces.* Complex Anal. Oper. Theory. Accepted.
17. M. Loaiza and J. Ramírez-Ortega, *Toeplitz operators with homogeneous symbols acting on the poly-Bergman spaces of the upper half-plane.* Integr. Equ. Oper. Theory **87** (2017), 391–410.
18. J. Ramírez-Ortega, M. Ramírez-Mora and A. Sánchez-Nungaray, *Toeplitz operators with vertical symbols acting on the poly-Bergman spaces of the upper half-plane II.* Complex Anal. Oper. Theory. **13** (2019), no. 5, 2443–2462.
19. J. Ramírez-Ortega and A. Sánchez-Nungaray, *Toeplitz operators with vertical symbols acting on the poly-Bergman spaces of the upper half-plane.* Complex Anal. Oper. Theory **9** (2015), no. 8, 1801–1817.
20. N. L. Vasilevski, *On the structure of Bergman and poly-Bergman spaces.* Integr. Equ. Oper. Theory **33** (1999), 471–488.
21. N. L. Vasilevski, *Commutative Algebras of Toeplitz Operators on the Bergman Space.* Operator Theory: Advances and Applications 185, Birkhäuser, Boston, 2008.

Toeplitz Operators with Singular Symbols in Polyanalytic Bergman Spaces on the Half-Plane

Grigori Rozenblum and Nikolai Vasilevski

From the first named author, with best wishes to the second named one, on the occasion of his Jubilee

Abstract Using the approach based on sesquilinear forms, we introduce Toeplitz operators in the analytic Bergman space on the upper half-plane with strongly singular symbols, derivatives of measures. Conditions for boundedness and compactness of such operators are found. A procedure of reduction of Toeplitz operators in Bergman spaces of polyanalytic functions to operators with singular symbols in the analytic Bergman space by means of the creation-annihilation structure is elaborated, which leads to the description of the properties of the former operators.

Keywords Bergman space · Toeplitz operators

Mathematics Subject Classification (2000) Primary 47A75; Secondary 58J50

1 Introduction

In a series of papers [5–8] the authors developed the approach to defining the Toeplitz operators in Bergman type spaces by means of bounded sesquilinear forms, which permitted them to study Toeplitz operators with strongly singular symbols. The cases of the classical Bergman space of analytic functions on the unit disk

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$\mathbb{D}$ and the Fock space on the whole complex plane were considered. For further characterizations and sufficient conditions for boundedness (and compactness) of Toeplitz operators on Bergman spaces of the unit disk, see [9, 10, 13], and references therein.

The present paper is devoted to the study of such Toeplitz operators in one more classical Bergman space, the one of analytic functions on the upper half-plane Π, square integrable with respect to the Lebesgue measure. It is well known that for sufficiently regular, say, bounded symbols, the theories of Toeplitz operators in the Bergman spaces on the disk and on the upper half-plane are equivalent, in the sense that the Möbius transform

$$z \mapsto \zeta = M(z) := \frac{z-i}{1-iz} : \Pi \to \mathbb{D}, \tag{1.1}$$

generates the mapping

$$\mathbf{U} : f \mapsto g = \mathbf{U}f, \, (\mathbf{U}f)(z) = f(M(z))\frac{1}{(1-iz)^2}, \, z \in \Pi, \tag{1.2}$$

which is an isometry of the Bergman spaces $\mathcal{A}^2(\mathbb{D})$ and $\mathcal{A}_1 := \mathcal{A}^2(\Pi)$. However, when passing to more singular symbols that generate Toeplitz operators via sesquilinear forms, the boundedness conditions no longer carry over in such a simple way.

In the present paper, we introduce Carleson measures for derivatives of order k (k-C measures) for the Bergman space on the half-plane and find conditions for a given measure to be a k-C measure. As usual, these conditions are sufficient for complex measures but are also necessary for positive ones. An estimate for the properly defined norm of k-C measures, with explicit dependence on the derivative order k, is obtained. This makes it possible to consider strongly singular symbols, containing derivatives of unbounded order.

The above results are applied to the study of Toeplitz operators in polyanalytic Bergman spaces on the upper half-plane, using the creation-annihilation structure discovered in [2, 12].

2 The Bergman Space on the Half-Plane and Related Structures: Singular Symbols

Similar to the hyperbolic metric on the disk, the pseudo-hyperbolic metric on the half-plane is useful. As known, for $z, w \in \Pi$, the pseudo-hyperbolic distance between z, w is defined by

$$d(z, w) = \left| \frac{z-w}{z-\overline{w}} \right|.$$

We will denote by $D(z, R)$, $z = x + iy$, $R < 1$, the pseudo-hyperbolic disk in Π, centered at z with 'radius' R. It is easy to check that the disk $D(z, R)$ coincides with the Euclidean disk $B(w, r)$,

$$D(x + iy, R) = B(w, r), \quad \text{where } w = x + i\frac{1 + R^2}{1 - R^2}y, \quad r = \frac{2Ry}{1 - R^2}. \tag{2.1}$$

Thus, the area of the disk $D(z, R)$ equals $\frac{4\pi R^2 y^2}{(1-R^2)^2}$.

Conversely, the Euclidean disk $B(x + i\eta, r) \in \Pi$ is the pseudo-hyperbolic disk $D(x + iy, R)$ with

$$R = \frac{\eta}{r} - \sqrt{\frac{\eta^2}{r^2} - 1} \quad \text{and} \quad y = \frac{1 - R^2}{1 + R^2}\eta.$$

Note that, while the pseudo-hyperbolic radius is fixed, the y-coordinate of the center of the pseudo-hyperbolic disk is proportional to the corresponding coordinate of the Euclidean disk.

Recall that the Bergman space $\mathcal{A}_1 = \mathcal{A}^2(\Pi)$ is a subspace in $L^2(\Pi)$ which consists of functions analytic in Π. It is a reproducing kernel space, with the reproducing kernel $\kappa(z, w) = -(\pi(z - \overline{w})^2)^{-1}$. Thus, the integral operator $\mathbf{P} = \mathbf{P}_\Pi$ with kernel $\kappa(z, w)$ is the orthogonal (Bergman) projection of $L^2(\Pi)$ onto $\mathcal{A}^2(\Pi)$. This projection is connected with the Bergman projection $\mathbf{P}_\mathbb{D}$ for the Bergman space $\mathcal{A}^2(\mathbb{D})$ by means of the operator $\mathbf{U}$ in (1.2):

$$\mathbf{P}_\Pi = \mathbf{U}^*\mathbf{P}_\mathbb{D}\mathbf{U}. \tag{2.2}$$

Given a bounded function $a(z)$, $z \in \Pi$, the Toeplitz operator in $\mathcal{A}_1$ is defined in the usual way as

$$(\mathbf{T}_a f)(z) = (\mathbf{P}af)(z) = \int_\Pi \kappa(z, w)a(w)f(w)dA(w), \tag{2.3}$$

dA being the Lebesgue measure. This operator is bounded in $\mathcal{A}_1$, as a composition of two bounded operators. The function $a(z)$ is called the *symbol* of the Toeplitz operator. It was the object of many studies to extend this definition to symbols being objects, more singular than bounded functions, still producing bounded operators. This program was implemented, in particular, in [5, 6] for the Bergman space on the disk and for the Fock space. Now we follow the pattern of these papers for the upper half-plane case.

The first stage here is considering (complex) measures as symbols. Let μ be an absolutely continuous measure on Π, with a bounded density $a(z)$ with respect to the Lebesgue measure dA. Then the action of operator (2.3) can be written as

$$(\mathbf{T}_a f)(z) = (\mathbf{P}af)(z) = \int_\Pi \kappa(z,w)a(w)f(w)dA(w) = \int_\Pi \kappa(z,w)f(w)d\mu(w). \tag{2.4}$$

Such a definition of the Toeplitz operator by means of the expression on the right-hand side of (2.4) can be, at least formally, extended to measures μ which are not necessarily absolutely continuous with respect to A with bounded density, and even to those that are not absolutely continuous with respect to A at all; what is, actually, needed is just the boundedness of the operator defined by the right-hand side in (2.4). To find some effective analytical conditions for this boundedness is rather a quite hard task. At the same time the approach based upon sesquilinear forms turns out to be very efficient here. In fact, in case of $d\mu = a(z)dA(z)$, we consider the sesquilinear form $\mathbf{F}_\mu[f,g] = \langle \mathbf{T}_a f, g\rangle$, where $f, g \in \mathcal{A}_1$. By the above definition of the operator $\mathbf{T}_a$,

$$\mathbf{F}_\mu[f,g] = \langle \mathbf{P}af, g\rangle = \langle af, \mathbf{P}g\rangle = \langle af, g\rangle = \tag{2.5}$$
$$\int_\Pi a(z)f(z)\overline{g(z)}dA(z) = \int_\Pi f(z)\overline{g(z)}d\mu(z), \quad f, g \in \mathcal{A}_1.$$

The left-hand side in (2.5) is defined for a being a (sufficiently nice) function, however, the right-hand side makes sense for a measure μ and can be thus used for a definition of a Toeplitz operator. The sesquilinear form (2.5) defines a bounded operator in $\mathcal{A}_1$ in case it is bounded, i.e., $|\mathbf{F}_\mu[f,g]| \le C\|f\|\|g\|$. This boundedness follows, as soon as the inequality

$$\left|\int_\Pi |f(z)|^2 d\mu(z)\right| \le C\|f\|^2 \tag{2.6}$$

is satisfied for all $f \in \mathcal{B}$. This estimate is, surely, satisfied provided

$$\int_\Pi |f(z)|^2 d|\mu(z)| \le C\|f\|^2,$$

where $|\mu|$ denotes the variation of the measure μ. Note that the considerations involving sesquilinear forms are essentially more convenient in analysis since they evade using reproducing kernels and deal with inequalities containing only the functions f, g and the measure μ.

Measures μ subject to the estimate (2.6) are called Carleson measures for the space $\mathcal{A}_1$. A description for such Carleson measures was given for the case of the Bergman space on the disk, for the Fock space and some other Bergman and Hardy type spaces, see, e.g., [14–16]. The criterion for a measure to be a Carleson measure for $\mathcal{A}_1$ follows from the known one for the space $\mathcal{A}(\mathbb{D})$.

Proposition 2.1 *Let μ be a complex Borel measure on Π. For a fixed $R \in (0, 1)$, if*

$$|\mu|(D(z, R)) \le C_\mu A(D(z, R)), \quad z \in \Pi, \quad R < 1, \tag{2.7}$$

with constant C_μ not depending on z, then the sesquilinear form (2.5) is bounded in $\mathcal{A}_1$. That is, the inequality (2.6) is satisfied for all $f \in \mathcal{A}_1$, with constant $C = C(R)C_\mu$, where $C(R)$ depends only on R. If the measure μ is positive, the condition (2.7) is also necessary for the sesquilinear form (2.5) to be bounded.

Proof The result follows from a similar statement concerning Carleson measures for the Bergman space on the disk, see, e.g., [14], by means of the unitary equivalence (2.2). A reasoning establishing directly this property can be found, for example, in [1, Proposition 2.6]. □

We note here that the condition (2.7) can be (although just formally !) relaxed, when replaced by

$$|\mu|(D(z, R)) \le C_\mu A(D(z, R_1)), \quad z \in \Pi, \quad R < 1, \tag{2.8}$$

with any fixed $R_1 > R$. This remark will be used when considering Carleson measures for derivatives later on. A measure μ on Π, having compact support, can be considered as a distribution in $\mathcal{E}'(\Pi)$. At the same time, the function $f(z)\overline{g(z)}$ is infinitely differentiable in Π, $f(z)\overline{g(z)} \in \mathcal{E}(\Pi)$, moreover, it is real-analytic in Π. Thus, this function can be represented as

$$f(z)\overline{g(z)} = \text{Diag}^*(f \otimes \bar{g}) = \text{Diag}^*((f \otimes 1)(1 \otimes \bar{g})),$$

where Diag is the diagonal embedding of Π into $\Pi \times \Pi$, $\text{Diag}(z) = (z, z)$, and Diag^* is the induced mapping $\mathcal{A}_1 \otimes \overline{\mathcal{A}_1}$ to $\mathfrak{A}(\Pi)$ (the space of real-analytic functions), $\text{Diag}^*(f \otimes \overline{g}) = f(z)\overline{g(z)}$. We denote by $\mathcal{M}$ the image of $\mathcal{A}_1 \otimes \overline{\mathcal{A}_1}$ in $\mathfrak{A}(\Pi)$ under the mapping Diag^*. Therefore the expression (2.5) can be understood as

$$\mathbf{F}_\mu[f, g] = (\mu, f(z)\overline{g(z)}) = (\mu, h), \text{ with } h = \text{Diag}^*(f \otimes \bar{g}) \in \mathfrak{A}(\Pi), \tag{2.9}$$

where parentheses denote the intrinsic paring of $\mathcal{E}'(\Pi)$ and $\mathcal{E}(\Pi)$. Proposition 2.1 can be now understood in the sense that as soon as the condition (2.7) is satisfied, the sesquilinear form (2.9) can be extended to $\mathcal{M}$ for the measure μ not necessarily having compact support. Moreover, the estimate $|(\mu, h)| \le C_\mu \|f\| \|g\|$, $h \in \mathcal{M}$, holds. We recall here how such extension of the distribution is standardly performed; this will make our further considerations for distributions of more general nature more clear.

Let μ_j be a sequence of measures with compact support, each one in a (closed) quasi-hyperbolic disk $D_j \subsetneq \Pi$ of radius r, and let these disks form a covering of Π with finite multiplicity m: each point of Π is covered by not more than m disks D_j.

Suppose that for each measure μ_j, the estimate $|\mathbf{F}_{\mu_j}[f,g]| \le C\|f\|\|g\|$ is satisfied, with the same constant C. Then, we can sum up such estimates over j and, using the finite multiplicity property, arrive at the same estimate for the measure $\mu = \sum \mu_j$, automatically locally finite in Π (just with a controllably larger constant.) This line of reasoning, considering first the distributions in $\mathcal{E}'(\Pi)$, with compact support, obtaining proper estimates, and extending then these estimates to certain distributions without compact support condition, so that the estimates hold on $\mathcal{A}_1$, will be further implemented for more general distributions.

A few words to explain our philosophy. Let Ω be an open subset in $\mathbb{C}$ ($\Omega = \Pi$ in our case). If F is a distribution in $\mathcal{E}'(\Omega)$ i.e., a distribution with compact support in Ω, its derivative, say, ∂F is standardly defined as the distribution ∂F acting on functions $\phi \in \mathcal{E}(\Omega)$ by the rule $(\partial F, \phi) = -(F, \partial\phi)$. If, however, F is a distribution in a wider space, $F \in \mathcal{D}'(\Omega)$, the action $(\partial F, \phi)$ is not necessarily defined for all $\phi \in \mathcal{E}(\Omega)$. In particular, if ϕ is a nontrivial function in the Bergman space or a function in $\mathcal{M}$, it can never belong to $\mathcal{D}(\Omega)$, so the action of F on such functions and, further on, the definition of derivatives of F needs to be specified anew, however being consistent with the usual definition. In what follows, we consider a class of distributions for which such construction works, preserving the usual properties of distributions. The natural compensation for this frivolity is the narrowing of the set of functions on which such 'distributions' act.

3 Carleson Measures for Derivatives

Following the pattern in [5, 6], we introduce now a class of sesquilinear forms involving derivatives of functions f, g, corresponding thus to (formal) distributional derivatives of Carleson measures.

Definition 3.1 Let μ be a regular complex measure on Π and α, β be two nonnegative integers. We denote by $\mathbf{F}_{\alpha,\beta,\mu}$ the sesquilinear form

$$\mathbf{F}_{\alpha,\beta,\mu}[f,g] = (-1)^{\alpha+\beta} \int_\Pi \partial^\alpha f(z)\overline{\partial^\beta g(z)}d\mu(z), \quad f, g \in \mathcal{A}_1, \tag{3.1}$$

which we denote as well by

$$\mathbf{F}_{\alpha,\beta,\mu}[f,g] = (\partial^\alpha \bar{\partial}^\beta \mu, f\bar{g}). \tag{3.2}$$

This definition is consistent with our approach as explained above. In fact (3.1), (3.2) act as the *definition* of the action of the 'distribution' $\partial^\alpha \bar{\partial}^\beta \mu$ on elements in $\mathcal{A}_1$. Note that this is consistent with the standard distributional definition of the derivative of a measure in the case when μ is a compactly supported measure in Π.

The first set of properties of such forms and corresponding operators, similar to the ones for other Bergman spaces, is the following.

Theorem 3.2 *Let μ be a measure with compact support in Π and let α, β be nonnegative integers. Then*

1. *For any $f, g \in \mathcal{A}_1$, the integral in (3.1) converges, moreover,*

$$\mathbf{F}[f, g] = \mathbf{F}_{\alpha,\beta,\mu}[f, g] = (\partial^\alpha \bar{\partial}^\beta \mu, f\bar{g}),$$

   where the derivatives are understood in the sense of distributions in $\mathcal{E}'(\Pi)$ and the parentheses mean the intrinsic paring in $(\mathcal{E}'(\Pi), \mathcal{E}(\Pi))$.
2. *The sesquilinear form (3.1) is bounded, considered on $\mathcal{A}_1 \times \mathcal{A}_1$ and therefore defines a bounded Toeplitz operator by $\langle \mathbf{T_F} f, g\rangle = \mathbf{F}[f, g]$, or $(\mathbf{T_F} f)(z) = \mathbf{F}[f, \kappa_z(\cdot)]$, where, as usual, $\kappa_z(w) = \overline{k(z, w)} = k(w, z)$.*
3. *The sesquilinear form (3.1) determines a compact Toeplitz operator $\mathbf{T_F}$ in $\mathcal{A}_1$.*
4. *If $s_n(\mathbf{T_F})$ denote the singular numbers of the operator $\mathbf{T_F}$, then the following estimate holds*

$$s_n(\mathbf{T}) \leq C \exp(-n\sigma), \quad n \in \mathbb{N}, \tag{3.3}$$

   where $\sigma > 0$ is a constant determined by the measure μ and integers α, β.

Proof The property (4) absorbs the other ones, so we will prove only it. Due to Ky Fan's inequalities for singular numbers of compact operators, it is sufficient to establish (3.3) for a positive measure μ. Consider a closed Euclidean disk $B \subset \Pi$ with radius R such that for some $r > 0$, the support of μ lies strictly inside B, thus $\operatorname{dist}(z, \partial B) > r > 0$ for all $z \in \operatorname{supp}\mu$. The Cauchy integral formula implies that for any $z \in \operatorname{supp}\mu$ and any $\alpha \in \mathbb{Z}_+$,

$$|\partial^\alpha f(z)|^2 \leq C_\alpha \int_{\partial B} |f(\zeta)|^2 dl(\zeta)$$

for each function $f \in \mathcal{A}_1$, with constant C_α depending only on α, but not depending on f and z. By the same reason, for any $z \in \operatorname{supp}\mu$, the estimate

$$|\partial^\alpha f(z)|^2 \leq C_\alpha \int_{|\zeta - z| \in (R, R+r/2)} |f(\zeta)|^2 dA(\zeta) \tag{3.4}$$

holds. By the Cauchy-Schwartz inequality,

$$|\mathbf{F}_{\alpha,\beta,\mu}[f, g]| \leq \left(\int_{\operatorname{supp}\mu} |\partial^\alpha f(z)|^2 d|\mu|(z)\right)^{\frac{1}{2}} \left(\int_{\operatorname{supp}\mu} |\partial^\beta g(z)|^2 d|\mu|(z)\right)^{\frac{1}{2}},$$

for all $f, g \in \mathcal{A}_1(\Pi)$, and then, due to (3.4),

$$|\mathbf{F}_{\alpha,\beta,\mu}[f, g]| \leq C'_\alpha C'_\beta |\mu|(B) \|f\|_{L^2(B')} \|g\|_{L^2(B')}.$$

The last relation means that the sesquilinear form $\mathbf{F}_{\alpha,\beta,\mu}$ is bounded not only in $\mathcal{A}_1(\Pi)$, but in $\mathcal{A}_1(B')$ as well, where B' is the Euclidean disk $B' = B(z, R + r/2)$. Now we represent the Toeplitz operator $\mathbf{T_F}$ as the composition

$$\mathbf{T_F} = \mathbf{T}_{\mathbf{F}_{\mathcal{A}_1(B')}} \mathbf{I}_{B \Rrightarrow B'} \mathbf{I}_{B' \Rrightarrow \Pi}, \tag{3.5}$$

where $\mathbf{I}_{B \Rrightarrow B'} : \mathcal{A}_1(B') \to \mathcal{A}_1(B)$, $\mathbf{I}_{B' \Rrightarrow \Pi} : \mathcal{A}_1(\Pi) \to \mathcal{A}_1(B')$ are operators generated by restrictions of functions defined on a larger set to the corresponding smaller set. The equality (3.5) can be easily checked by writing the sesquilinear forms of operators on the left-hand and on the right-hand side. Finally, the first and the third operators on the right-hand side are bounded, while the middle one, the operator generated by the embedding of the disk B to B', is known to have the exponentially fast decaying sequence of singular numbers see, e.g., [3]. □

Remark 3.3 Using the results in [3], one can give an upper estimate for the constant σ in (3.3). In fact, let $Q \subset \mathbb{D}$ be the image of $\operatorname{supp}\mu$ in the unit disk, under the mapping (1.1). We denote by $\mathbf{C}(Q)$ the set of *connected* closed sets $V \subset \mathbb{D}$ containing Q. For each $V \in \mathbf{C}(Q)$, let $\operatorname{cap}(V)$ denote the logarithmic capacity of V (see the definition, e.g., in [3]) and we set $\mathbf{c}(\mu)$ as

$$\mathbf{c}(\mu) = \inf_{V \in \mathbf{C}(Q)} \operatorname{cap}(V).$$

Then estimate (3.3) holds for any $\sigma < \mathbf{c}(\mu)$. In fact, for a *positive* measure μ, $\alpha, \beta = 0$, and for a connected set Q, the asymptotics of the singular numbers of the Toeplitz operator was found in [3], and our estimate for the exponent in (3.3) follows from this result and the natural monotonicity of singular numbers under the extension of the set where the measure is supported. Our considerations give only the upper estimate for these singular numbers. It is remarkable that for a non-connected support of the measure, even in the setting of [3], the terms in which the singular numbers asymptotics or even sharp order estimates can be expressed are so far unknown.

Next, we present our main definition.

Definition 3.4 A measure on Π is called a Carleson measure for derivatives of order k (k-C measure) if for some constant $C_k(\mu) > 0$,

$$|\mathbf{F}_{k,\mu}[f, f]| \equiv \left| \int_\Pi |\partial^k f(z)|^2 d\mu(z) \right| \le C_k(\mu) \|f\|^2 \tag{3.6}$$

for all $f \in \mathcal{A}_1$.

Now, we find a sufficient condition for a measure to be a k-C measure. Here it is important to control the dependence of the constant $C_k(\mu)$ in (3.6) on the number k.

Theorem 3.5 *Let $\gamma \in (0,1)$ be fixed, and let the measure μ on Π satisfy the condition*

$$|\mu|(D(z_0,\gamma)) \leq \varpi_\kappa(\mu)|\operatorname{Im} z_0|^{2k} A(D(z_0,\gamma)) \tag{3.7}$$

with some $\varpi_\kappa(\mu)$ for all $z_0 \in \Pi$. Then inequality (3.6) is satisfied, with $C_k(\mu) = (k!)^2\varpi_\kappa(\mu)\gamma^{-2k}$, that is, μ is a k-C measure.

Proof Given a point $z_0 \in \Pi$, the pseudo-hyperbolic disk $D(z_0,\gamma)$ coincides with the Euclidean disk $B(w_0,s)$, where their centra and radii are connected by (2.1). We write then the standard representation of the derivative at a point w, $|w_0 - w| < s_1 < s$ of an analytic function $f(w)$:

$$f^{(k)}(w) = k!(2\pi\imath)^{-1}\int_{|w_0-\zeta|=\sigma}(\zeta - w)^{-k-1}f(\zeta)d\zeta, \quad s_1 < \sigma < s. \tag{3.8}$$

Now we fix $s_2 \in (s_1, s)$ and integrate (3.8) in σ variable from s_2 to s, which gives us the estimate

$$|f^{(k)}(w)| \leq (s - s_2)^{-1}k!(2\pi)^{-1}\int_{s_2\leq|w_0-\zeta|\leq s}|\zeta - w|^{-k-1}|f(\zeta)|dA(\zeta).$$

We choose then $s_1 = \frac{1}{4}s$, $s_2 = \frac{3}{4}s$. The inequality $|\zeta - w| \geq \frac{1}{2}s$, together with the Cauchy-Schwartz, yield

$$\begin{aligned} |f^{(k)}(w)| &\leq (s/2)^{-k-2}k!\pi^{-1}\int_{|w_0-\zeta|\leq s}|f(\zeta)|dA(\zeta) \\ &\leq \frac{2}{\sqrt{\pi}}(s/2)^{-(k+1)}k!\left(\int_{|w_0-\zeta|\leq s}|f(\zeta)|^2dA(\zeta)\right)^{\frac{1}{2}}, \end{aligned}$$

or

$$|f^{(k)}(w)|^2 \leq (4/\pi)(s/2)^{-2(k+1)}(k!)^2\int_{|w_0-\zeta|\leq s}|f(\zeta)|^2dA(\zeta). \tag{3.9}$$

The estimate (3.9) holds for all w, $|w - w_0| < s_1 = s/4$. Therefore we can integrate it over the Euclidean disk $B(w_0, s_1)$ with respect to the measure μ, which gives

$$\begin{aligned} &\left|\int_{B(w_0,s_1)}|f^{(k)}(w)|^2d\mu(w)\right| \\ &\leq |\mu|(B(w_0,s_1))(4/\pi)(s/2)^{-2(k+1)}(k!)^2\int_{|w_0-\zeta|\leq s}|f(\zeta)|^2dA(\zeta). \end{aligned} \tag{3.10}$$

We substitute now the estimate (3.7) into (3.10) and use (2.1) to arrive at

$$\left|\int_{B(w_0,s_1)} |f^{(k)}(w)|^2 d\mu(w)\right|$$
$$\leq \varpi_k(\mu)\, y_0^{2k}\pi(s/2)^2(4/\pi)(s/2)^{-2(k+1)}(k!)^2 \int_{|w_0-\zeta|\leq s} |f(\zeta)|^2 dA(\zeta)$$
$$\leq \varpi_k(\mu)\,\gamma^{-2k}(k!)^2 \int_{|w_0-\zeta|\leq s} |f(\zeta)|^2 dA(\zeta),$$

or, returning to the pseudo-hyperbolic disks,

$$\left|\int_{D(z_1,\gamma_1)} |f^{(k)}(w)|^2 d\mu(w)\right| \leq \varpi_k(\mu)\,\gamma^{-2k}(k!)^2 \int_{D(z_0,\gamma)} |f(\zeta)|^2 dA(\zeta), \tag{3.11}$$

where $D(z_1, \gamma_1) = B(w_0, s_1) \subset B(w_0, s) = D(z_0, \gamma)$.

Now we follow the reasoning in [14, Theorem 7.4], where estimates of the type (3.11) were summed to obtain the required k-Carleson property. One just should replace the hyperbolic disks with pseudo-hyperbolic ones. Like in [14], it is possible to find a locally finite covering Ξ of the half-plane Π by disks of the type $D(z_1, \gamma_1)$, with $z_1 \in \Pi$, so that the larger disks $D(z_0, \gamma)$ form a covering $\widetilde{\Xi}$ of Π of finite, moreover, controlled multiplicity. The latter means that the number $m(\widetilde{\Xi}) = \max_{z\in\Pi} \#\{D \in \widetilde{\Xi} : z \in D\}$ is finite. After adding up all inequalities of the form (3.11) over all disks $D = D(z_1, \gamma_1) \in \Xi$, we obtain the required estimate for $\int_\Pi |f^{(k)}|^2 d\mu$. □

Having Theorem 3.5 at our disposal, we introduce the classes of k-C measures.

Definition 3.6 Fix a number $\gamma \in (0, 1)$. The class $\mathfrak{M}_{k,\gamma}$ consists of measures μ on Π such that

$$\varpi_{k,\gamma}(\mu) := \sup_{z\in\Pi}\{|\mu|(D(z,\gamma))(\text{Im } z)^{-2(k+1)}(k!)^2\gamma^{-2k}\} < \infty.$$

Theorem 3.5 implies that the class $\mathfrak{M}_{k,\gamma}$ consists of k-C measures. This class, in fact, does not depend on the value of γ chosen, however the value $\varpi_{k,\gamma}(\mu)$ does.

It is convenient to extend the definition of $\mathfrak{M}_{k,\gamma}$ to half-integer values of k:

Definition 3.7 Let $k \in \mathbb{Z}_+ + \frac{1}{2}$ be a half-integer. The class $\mathfrak{M}_{k,\gamma}$, $\gamma \in (0, 1)$ consists of measures satisfying

$$\varpi_{k,\gamma}(\mu) := \sup_{z\in\Pi}\left\{|\mu|\left(D(z,\gamma)\right)(\text{Im } z)^{-2(k+1)}\Gamma(k+1)^2\gamma^{-2k}\right\} < \infty. \tag{3.12}$$

Further on, the parameter γ will be fixed and will be often omitted in our notations. The quantity $\varpi_k(\mu)$ will be called the k-norm of the measure μ. According to

Definition 3.7, the spaces $\mathfrak{M}_{k,\gamma}$ for different $k \in \mathbb{Z}_+/2$ are related by

$$\mathfrak{M}_k = (\text{Im } z)^{2(l-k)}\mathfrak{M}_l,$$

with

$$\varpi_k(\mu) = \gamma^{2(l-k)}(\Gamma(k+1)/\Gamma(l+1))^2\varpi_l(\mu).$$

Theorem 3.5 enables us to find conditions for boundedness of the operators defined by differential sesquilinear forms $\mathbf{F}_{\alpha,\beta,\mu}$ in (3.1). They look similar to the corresponding conditions for sesquilinear forms in the Bergman space on the unit disk and are derived from Theorem 3.5 in the same way as Proposition 6.8 is derived from Theorem 6.3 in [6], so we restrict ourselves to formulations only.

Theorem 3.8 *Let the measure μ satisfy (3.12) with some $k \in \mathbb{Z}_+/2$. Then for $\alpha, \beta \in \mathbb{Z}_+$, $\alpha+\beta = 2k$, the sesquilinear form*

$$\mathbf{F}_{\alpha,\beta,\mu}[f,g] = (-1)^{\alpha+\beta}\int_\Pi \partial^\alpha f\overline{\partial^\beta g}d\mu$$

is bounded in $\mathcal{A}_1$ and defines a bounded Toeplitz operator $\mathbf{T}_{\alpha,\beta,\mu}$ in the Bergman space $\mathcal{A}_1$, moreover its norm is majorated by $\varpi_k(\mu)$.

Taking into account our above agreement concerning distributional derivatives of measures without compact support condition, Theorem 3.8 can be reformulated as

Theorem 3.9 *Under the conditions of Theorem 3.8, the sesquilinear form*

$$\mathbf{F}_{\partial^\alpha\overline{\partial^\beta}\mu}[f,g] = (\partial^\alpha\overline{\partial^\beta}\mu, f\bar{g})$$

is bounded in $\mathcal{A}_1$ and defines a bounded Toeplitz operator $\mathbf{T}_{\partial^\alpha\overline{\partial^\beta}\mu}$ in $\mathcal{A}_1$.

As usual for Toeplitz type operators, boundedness conditions lead to compactness conditions, formulated in similar terms.

Theorem 3.10 *For $R > 0$, denote by Q_R the rectangle in Π:*

$$Q_R = \{z = x + y \in \Pi : x \in (-R, R), y \in (R^{-1}, R)\}.$$

Suppose that $\alpha + \beta = 2k$ and

$$\lim_{R\to\infty}\sup_{z\in\Pi\setminus Q_R}\left\{|\mu|\left(D(z,\gamma)\right)(\text{Im } z)^{-2(k+1)}\Gamma(k+1)^2\gamma^{-2k}\right\} = 0. \tag{3.13}$$

Then the operator $\mathbf{T}_{\partial^\alpha\overline{\partial^\beta}\mu}$ is compact in $\mathcal{A}_1$.

Proof It goes in a standard way. Split the measure μ into two parts, $\mu = \mu_R + \mu'_R$, where μ_R has support outside Q_R and μ'_R has compact support. Correspondingly, the operator $\mathbf{T}_{\partial^\alpha \overline{\partial^\beta} \mu}$ splits into two terms, $\mathbf{T}_R + \mathbf{T}'_R$. The first operator, by Theorem 3.9, has small norm, as soon as R is chosen sufficiently large. The operator $\mathbf{T}'_R$ is compact by Theorem 3.2. Therefore, $\mathbf{T}_{\partial^\alpha \overline{\partial^\beta} \mu}$ is compact. □

4 Examples

We give some examples of symbols-distributions and -hyperfunctions. More examples can be constructed, following the pattern seen in [5, 6].

Example 4.1 Let the measure μ be supported on the lattice $\mathbb{L} = \mathbb{Z} + i\mathbb{N}$. With an integer point $\mathbf{n} = (n_1 + in_2)$ we assign the weight $\mathbf{m_n} > 0$. Suppose that $\sup_{\mathbf{n}} \mathbf{m_n} < \infty$. Then the Toeplitz operator, with the measure $\mu = \sum_{\mathbf{n}} \mathbf{m_n} \delta(z - \mathbf{n})$ as symbol, is bounded.

Example 4.2 In the setting of Example 4.1, consider the Toeplitz operator with distributional symbol $\mu_{\alpha,\beta,W} = W(\mathbf{n}) \partial^\alpha \overline{\partial^\beta} \mu$, where $W(\mathbf{n})$ is a weight function, $W(n_1 + in_2) = |n_2|^{-\alpha-\beta}$. Then the conditions of Theorem 3.9 are satisfied for the measure $W(\mathbf{n})\mu$ and, therefore, the Toeplitz operator with symbol $\mu_{\alpha,\beta,W}$ is bounded. If $W(\mathbf{n})$ is a function on the lattice satisfying $W(\mathbf{n}) \to 0$ as $|\mathbf{n}| \to \infty$ then, by Theorem 3.10, this Toeplitz operator is compact.

Example 4.3 In the setting of Example 4.1, consider the, initially formal, sum

$$a = \sum_{\mathbf{n} \in \mathbb{L}} W(\mathbf{n}) \partial^{\alpha_{\mathbf{n}}} \overline{\partial^{\beta_{\mathbf{n}}}} \mu(\{\mathbf{n}\}),$$

where $(\alpha_{\mathbf{n}}, \beta_{\mathbf{n}})$ is a collection of orders of differentiation and $W(\mathbf{n})$ is a weight sequence. This is a sum of distributions supported at single points of the lattice $\mathbb{L}$. To each of them, we can apply Theorem 3.9 and obtain an estimate of the norm of the corresponding sesquilinear form $\mathbf{F_n}$, where the order $k_{\mathbf{n}} = \alpha_{\mathbf{n}} + \beta_{\mathbf{n}}$ is involved:

$$\begin{aligned} |\mathbf{F_n}[f, g]| &\leq C |W(\mathbf{n})| ((\alpha_{\mathbf{n}}! \beta_{\mathbf{n}}!)(\gamma/2)^{-(\alpha_{\mathbf{n}}+\beta_{\mathbf{n}})} n_2^{-\alpha_{\mathbf{n}}+\beta_{\mathbf{n}}} \|f\| \|g\| \\ &= C |W(\mathbf{n})| \tau(\mathbf{n}) \|f\| \|g\|. \end{aligned} \tag{4.1}$$

A rough way to estimate the sesquilinear form $\mathbf{F}_a$ would be to consider the sum of the terms in (4.1),

$$|\mathbf{F}_a[f, g]| \leq \sum (|W(\mathbf{n})| \tau(\mathbf{n})) \|f\| \|g\|, \tag{4.2}$$

so the sesquilinear form is bounded as soon as the series in (4.2) converges. A more exact treatment uses the finite multiplicity covering by disks containing no more than, say, 10 points of the lattice $\mathbb{L}$ similarly to how this was done in Sect. 2. In this way, the sesquilinear form is majorated by a smaller quantity,

$$|\mathbf{F}_a[f,g]| \le \sup\{(|W(\mathbf{n})|\tau(\mathbf{n}))\|f\|\|g\|\},$$

for which the finiteness condition requires a considerably milder decay requirement for $W(\mathbf{n})$ than the finiteness of the coefficient in (4.2).

Now we consider some symbols with support touching the boundary of the upper half-plane Π.

Example 4.4 Let $\mathcal{L}_j \subset \Pi$ be the straight line $\{z = x + iy : x \in \mathbb{R},\ y = 2^{-j}\}$, and μ_j be the measure $W(j)\delta(\mathcal{L}_j)$ with some weight sequence $W(j)$. In other words, it is the Lebesgue measure on the line $\mathcal{L}_j$ with the weight factor $W(j)$. The sum $\mu = \sum_j \mu_j$ is a measure on Π, which, however, does not have compact support in Π. By Definition 3.6, the measure μ belongs to the class $\mathfrak{M}_{0,\gamma}$ (say, for $\gamma = \frac{1}{2}$), as soon as $W(j) = O(2^{-2j})$.

Example 4.5 For the same system of straight lines $\mathcal{L}_j$ we consider

$$\varsigma = \sum_j W(j)\varsigma_j \equiv \sum_j W(j)\partial^j\delta(\mathcal{L}_j) = \sum_j (i/2)^j W(j)(1 \otimes \partial_y^j \delta(y - 2^{-j})). \tag{4.3}$$

In (4.3), ς is a formal sum of distributions $W(j)\varsigma_j$, each being a derivative of a measure, of unbounded orders, and this formal sum corresponds to the sesquilinear form

$$\mathbf{F}_\varsigma[f,g] = \sum_j \mathbf{F}_{\varsigma_j}[f,g] = \sum_j (-1)^j W(j) \int_{\mathcal{L}_j} \partial^j f \cdot \bar{g} dx. \tag{4.4}$$

The sequence of weights $W(j)$ should be chosen in such a way that the sum (4.4) converges for $f, g \in \mathcal{A}_1$ and, moreover, is a bounded sesquilinear form on $\mathcal{A}_1$. By Theorem 3.8, the measure $\mu_j = 1 \otimes \delta(y - 2^{-j})$ belongs to the class $\mathfrak{M}_{j,\gamma}$ with estimate

$$\varpi_{j,\gamma}(\mu_j) \le C(2/\gamma)^{2j}(j!)^2.$$

Thus, if the sum $\sum_j W(j)(2/\gamma)^{2j}(j!)^2$ is finite, the sesquilinear form (4.4) is bounded on $\mathcal{A}_1$.

5 The Structure of the Bergman Spaces

Along with the Bergman space $\mathcal{A}_1$ of analytic functions on Π, we consider spaces of polyanalytic functions. We denote by $\mathcal{A}_j$, $j = 1, 2, \ldots$ the space of square integrable functions on Π satisfying the iterated Cauchy-Riemann equation $\overline{\partial}^j f = 0$ (the reader was, probably, intrigued by the subscript in the notation $\mathcal{A}_1$—now its use is justified). Of course, $\mathcal{A}_j \subset \mathcal{A}_{j'}$ for $j < j'$, so, to get rid of these 'less polyanalytic' functions, the *true polyanalytic* Bergman spaces have been introduced (see [11]), by

$$\mathcal{A}_{(j)} = \mathcal{A}_j \ominus \mathcal{A}_{j-1} = \mathcal{A}_j \cap \mathcal{A}_{j-1}^{\perp}, \quad j = 2, \ldots; \quad \mathcal{A}_{(1)} = \mathcal{A}_1.$$

Worth mentioning is the following direct sum decomposition of $L_2(\Pi)$:

$$L_2(\Pi) = \bigoplus_{n \in \mathbb{N}} \mathcal{A}_{(j)} \oplus \bigoplus_{n \in \mathbb{N}} \widetilde{\mathcal{A}}_{(j)},$$

where $\widetilde{\mathcal{A}}_{(j)}$ are true poly-antianalytic Bergman spaces (see, for details, [11]).

For these poly-Bergman spaces on the upper half-plane, there exists a system of creation and annihilation operators, described in [2, 12]. These operators are two-dimensional singular integral operators,

$$(\mathbf{S}_\Pi u)(w) = -\frac{1}{\pi} \int_\Pi \frac{u(z) dA(z)}{(z - w)^2} \qquad \text{and} \qquad (\mathbf{S}_\Pi^* u)(w) = -\frac{1}{\pi} \int_\Pi \frac{u(z) dA(z)}{(\bar{z} - \bar{w})^2}.$$

Being understood in the principal value sense, they are bounded in $L^2(\Pi)$ and adjoint to each other. They are, in fact, the Beurling–Ahlfors operators compressed to the half-plane, and are surjective isometries,

$$\begin{aligned} &\mathbf{S}_\Pi : \mathcal{A}_{(j)} \to \mathcal{A}_{(j+1)}, \ \ \mathbf{S}_\Pi^* : \mathcal{A}_{(j)} \to \mathcal{A}_{(j-1)}, \quad j > 1, \\ &\mathbf{S}_\Pi^* : \widetilde{\mathcal{A}}_{(j)} \to \widetilde{\mathcal{A}}_{(j+1)}, \ \ \mathbf{S}_\Pi : \widetilde{\mathcal{A}}_{(j)} \to \widetilde{\mathcal{A}}_{(j-1)}, \quad j > 1, \end{aligned} \tag{5.1}$$

while

$$\mathbf{S}_\Pi^* : \mathcal{A}_1 \to \{0\}, \quad \mathbf{S}_\Pi : \widetilde{\mathcal{A}_1} \to \{0\}.$$

Thus, in particular, we have surjective isometries

$$(\mathbf{S}_\Pi)^j \mathcal{A}_{(1)} = \mathcal{A}_{(j+1)}(\Pi).$$

Formulas (5.1), possessing the structure similar to the ones of the Landau subspaces for the Schrödinger equation with uniform magnetic field, justify calling $\mathbf{S}_\Pi$, $\mathbf{S}_\Pi^*$ creation and annihilation operators.

Remark 5.1 Here one can notice a certain discrepancy in notations: $\mathbf{S}$ denotes usually the *annihilation* operator in the poly-Fock spaces while $\mathbf{S}_\Pi$ denotes here the *creation* operator in the poly-Bergman spaces—however, this is the tradition and we do not want to break it.

The operators $\mathbf{S}_{\Pi}^{j}$, restricted to $\mathcal{A}_1$, admit a representation, found in [4], which is much more convenient for using in further reductions.

Theorem 5.2 ([4, Theorem 3.3]) *For* $u \in \mathcal{A}_1$,

$$(\mathbf{S}_{\Pi}^{j}u)(z) = \frac{\partial^j[(z-\bar{z})^j u(z)]}{j!}, \; j \geq 0. \tag{5.2}$$

Note that the isometry $\mathbf{U}$ of the Bergman spaces on the upper half-plane Π and on the disk $\mathbb{D}$ is not carried over to the poly-analytic Bergman spaces.

6 Relations Among the Toeplitz Operators in the True Poly-Bergman Spaces

Let a be a distribution on Π, defined at least on $C^{\infty}(\bar{\Pi}) \cap L^1(\Pi)$. We consider the sesquilinear form

$$\mathbf{F}_a[u, v] = (au, \bar{v}) = (a, u\bar{v}), \quad u = \mathbf{S}_{\Pi}^{j} f \in \mathcal{A}_{(j+1)}, \quad v = \mathbf{S}_{\Pi}^{j} g \in \mathcal{A}_{(j+1)}, \tag{6.1}$$

with f, g being elements of the standard orthonormal basis in $\mathcal{A}_{(1)}$. The sesquilinear form (6.1) is defined for f, g in the basis in $\mathcal{A}_{(1)}$ and can be extended by sesquilinearity to the linear span of the basis. If it turns out that (6.1) is bounded on this span, it can be extended by continuity to the whole $\mathcal{A}_{(1)}$ and thus it would define a bounded operator in $\mathcal{A}_{(1)}$. On the other hand, since $\mathbf{S}_{\Pi}^{j}$ is a unitary operator, $\mathbf{F}_a[u, v]$ can be therefore extended to a bounded sesquilinear form defined for u, v being arbitrary elements in $\mathcal{A}_{(j+1)}$, thus defining a bounded operator in $\mathcal{A}_{(j+1)}$. The present section is devoted to finding an explicit relation between these two operators. Further on, in this section we will only use the Hilbert space $L^2(\Pi, dA)$ and therefore we will suppress the notation of the space in the scalar product $\langle ., . \rangle$; the parentheses $(., .)$ still denote the action of a distribution on a smooth function, without the complex conjugation. Since u, v are elements in the poly-Bergman space, the product $u\bar{v}$ is a smooth function in $L^1(\Pi)$.

The following result leads to establishing a relation between Toeplitz operators in the true poly-Bergman space $\mathcal{A}_{(j)}$ and the Bergman space $\mathcal{A}_1$.

Proposition 6.1 *Let a be a distribution in the half-plane* Π. *Then*

$$\mathbf{F}_a[u, v] = \mathbf{F}_a[\mathbf{S}_{\Pi}^{j} f, \mathbf{S}_{\Pi}^{j} g] = (a, \mathbf{S}_{\Pi}^{j} f \overline{\mathbf{S}_{\Pi}^{j} g}) = \langle (\mathbf{K}_{(j)} a) f, g \rangle, \tag{6.2}$$

with $\mathbf{S}_{\Pi}^{j}$ *defined in* (5.2)*, where* $\mathbf{K}_{(j)}$ *is a differential operator of order* $2j$ *having the form*

$$\mathbf{K}_{(j)} = \mathcal{K}_{(j)}(\Delta(y^2\cdot), \bar{\partial}(y\cdot), \partial(y\cdot)), \tag{6.3}$$

and $\mathcal{K}_{(j)}$ *being a polynomial of degree* j*. Moreover, if we assign the weight* -1 *to the differentiation and the weight* 1 *to the multiplication by* y*, with weights adding under the multiplication, then all monomials in* $\mathcal{K}_{(j)}$ *have weight* 0*.*

Proof We demonstrate the reasoning for the case $j = 1$. The general case uses the same machinery with some tedious bookkeeping. We set $u = \mathbf{S}_{\Pi} f$, $v = \mathbf{S}_{\Pi} g$ and consider the sesquilinear form $\mathbf{F}_a[f, g] = \langle a\partial(yf), \partial(yg)\rangle \equiv (a, \partial(yf)\overline{\partial(yg)})$ for f, g being some elements in the standard orthonormal basis in $\mathcal{A}(\Pi)$. Due to $\partial^* = -\bar{\partial}$:

$$\begin{aligned}(a, \partial(yf)\overline{\partial(yg)}) &= (a, (-if + y\partial f)\overline{(-ig + y\partial g)}) \\ &= (a, f\bar{g}) + (a, -ify\bar{\partial}\bar{g}) + (a, i\bar{g}y\partial f) + (a, y^2(\partial f)\bar{\partial}\bar{g})\end{aligned} \tag{6.4}$$

(the last transformation uses $\bar{\partial} g = 0$). For the second term on the right-hand side in (6.4), by the general rules of manipulation with distributions, we have

$$(a, -ify\bar{\partial}\bar{g}) = (-iya, f\overline{\partial g}) = (-iya, \overline{\partial(\bar{f}g)}) - (-iyF, (\bar{\partial} f)g) = (\bar{\partial}(iya), f\bar{g}),$$

because $\bar{\partial} f = 0$. The third term on the right in (6.4) is transformed in a similar way, and for the last one,

$$\begin{aligned}(a, y^2\partial f\bar{\partial}\bar{g}) &= (y^2 a, \bar{\partial}(\partial f\bar{g}) - \bar{\partial}(\partial f)\bar{g}) = -(\bar{\partial}(y^2 a), (\partial f)\bar{g}) \\ &= -(\bar{\partial}(y^2 a), \partial(f\bar{g}) - \{f\partial\bar{g}\}) = (\partial\bar{\partial}(y^2 a), f\bar{g}),\end{aligned}$$

again, the terms in curly bracket vanishing due to $\bar{\partial} g = 0$. Collecting the terms in (6.4), after simple transformations, we obtain the required relation.

For higher order, the procedure of transformation is similar, by means of formally commuting a and factors in the creation operators $\mathbf{S}_{\Pi}^{j}$ in the expression $\langle a\mathbf{S}_{\Pi}^{j} f, \mathbf{S}_{\Pi}^{j} g\rangle \equiv (a, \mathbf{S}_{\Pi}^{j} f\overline{\mathbf{S}_{\Pi}^{j} g})$, so that the Cauchy-Riemann operator falls on the functions f, g, while any commutation with a produces a derivative of a. It remains to notice that when commuting the terms in the expression on the left-hand side in (6.2), the weight of the terms does not change. Alternatively, one can make the calculations similar to the ones shown above, again by moving the Cauchy-Riemann operator to the functions f, g and on F.

To make the general reasoning more transparent, we present here our transformations for the case $j = 2$. So, we set $u = \mathbf{S}^2_\Pi f$, $v = \mathbf{S}^2_\Pi g$. We start with

$$\mathbf{F}_a[u, v] = \mathbf{F}_a[\mathbf{S}^2_\Pi f, \mathbf{S}^2_\Pi g] = ((\mathbf{S}^2_\Pi f)a, \overline{\mathbf{S}^2_\Pi g}) = -2((\partial^2(y^2 f) \times a), \bar\partial^2(y^2 \bar g)). \tag{6.5}$$

We expand in (6.5) the derivatives of the product by the Leibnitz formula, to obtain

$$\mathbf{F}_a[u, v] = (\partial^2((y^2 f)a) - 2\partial((y^2 f)\partial a) + y^2 f \partial^2 a, \bar\partial^2(y^2 \bar g). \tag{6.6}$$

Now we carry over the derivatives ∂, ∂^2 to the second factor in (6.6) (this is legal due to the definition of the derivatives of distributions):

$$\begin{aligned} \mathbf{F}_a[u, v] = ((y^2 f)a, \partial^2(\bar\partial^2(y^2 \bar g))) \\ +2((y^2 f)\partial a, \partial(\bar\partial^2(y^2 \bar g))) + (y^2 f \partial^2 a, \bar\partial^2(y^2 \bar g). \end{aligned} \tag{6.7}$$

We consider then the terms in (6.7) separately. In the first term, we commute ∂^2 and $\bar\partial^2$ in the second factor:

$$\partial^2(\bar\partial^2(y^2 \bar g)) = \bar\partial^2(\partial^2(y^2 \bar g)) = \tfrac{1}{2}\bar\partial^2(\bar g),$$

since $\partial \bar g = 0$. Therefore, the first term in (6.7) transforms to

$$\begin{aligned} \tfrac{1}{2}((y^2 f)a, \bar\partial^2 \bar g) = \tfrac{1}{2}(\bar\partial^2(y^2 f a), \bar g) = \tfrac{1}{2}(f \bar\partial^2(y^2 F), \bar g) \\ = (f\mathbf{K}_1 a, \bar g) = (\mathbf{K}_1 a, f \bar g), \end{aligned}$$

with $\mathbf{K}_1 a$ being the distribution

$$\mathbf{K}_1 a = \tfrac{1}{2}\bar\partial^2(y^2 a).$$

Next, for the second term in (6.7), we have

$$\begin{aligned} 2((y^2 f)\partial a, \partial(\bar\partial^2(y^2 \bar g))) = -2((y^2 f)\partial a, \bar\partial^2 \partial(y^2 \bar g))) = \\ 2((y^2 f)\partial a, \bar\partial^2(y \bar g)) = 2(\bar\partial^2(y^2 f \partial a), y \bar g). \end{aligned}$$

Now,

$$\bar\partial^2(y^2 f \partial a) = f \bar\partial^2(y^2 \partial a) = f(2\partial a + 2y\bar\partial\partial a + y^2 \bar\partial^2 \partial a).$$

Thus, the second term in (6.7) equals to

$$2((y^2 f)\partial a, \partial(\bar\partial^2(y^2 \bar g))) = (f\mathbf{K}_2 a, \bar g) = (\mathbf{K}_2 a, f \bar g),$$

where

$$\mathbf{K}_2 a = y(2\partial a + 2y\bar{\partial}\partial a + y^2\bar{\partial}^2\partial a).$$

Finally, the third term in (6.7) is transformed as

$$(y^2 f\partial^2 a, \bar{\partial}^2(y^2\bar{g}) = (\bar{\partial}^2 f y^2\partial^2 a, y^2\bar{g}) =$$
$$(f\bar{\partial}^2(y^2 a), y^2\bar{g}) = (fy^2\bar{\partial}^2(y^2 a), \bar{g}) = (f\mathbf{K}_3 a, \bar{g}) = (\mathbf{K}_3 a, f\bar{g}),$$

where $\mathbf{K}_3 a = y^2\bar{\partial}^2(y^2 a)$. A simple bookkeeping shows that the operators $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ have the structure claimed by the theorem, $\mathbf{K}_{(2)} = \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3$. Note again that although the distribution a does not necessarily have compact support in Π, our definition of derivatives of such distributions conserves the formal differentiation rules we used in these calculations. □

As explained above, the equality (6.2) extends to the whole of $\mathcal{A}_1$, as soon as we know that the right-hand side or on the left-hand side is a bounded sesquilinear form in $\mathcal{A}_1$. Thus, the statement of Proposition 6.1 can be formulated as the following theorem.

Theorem 6.2 *The operators $\mathbf{T}_a(\mathcal{A}_{(j+1)})$ and $\mathbf{T}_{\mathbf{K}a}(\mathcal{A}_1)$ are unitarily equivalent (up to a numerical factor) as soon as one of them is bounded. In this case, if one of these operators is compact, or belongs to a Schatten class, or is of finite rank, or zero, then the same holds for the other one.*

The terms in the differential operator $\mathbf{K}_{(j)}$ can be regrouped so that it takes the form

$$\mathbf{K}_{(j)} = \sum_{p+\bar{p}+2q\leq 2j} b_{p,\bar{p},q}(\partial)^p(\bar{\partial})^{\bar{p}}\Delta^q y^{p+\bar{p}+2q}.$$

Now we can apply the boundedness conditions obtained earlier, in Sect. 3, for differential sesquilinear forms to obtain boundedness conditions for Toeplitz operators in true poly-Bergman spaces.

Theorem 6.3 *Let μ be a measure on Π such that $y^k\mu$ are k-C measures for $\mathcal{A}_1$ for $k = 0, 1, \ldots, 2j$. Then the sesquilinear form $\int f\bar{g}d\mu$ is bounded in $\mathcal{A}_{(j+1)}$ and defines a bounded Toeplitz operator in $\mathcal{A}_{(j+1)}$. If, moreover, $y^k\mu$ are vanishing k-C measures for $\mathcal{A}_1$, then the corresponding operator in $\mathcal{A}_{(j+1)}$ is compact.*

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References

1. S.H.Kang, J.Y.Kim, *Toeplitz operators on Bergman spaces defined on upper planes.* Comm. Korean Math. Soc., **14** (1), 1999, 171–177.
2. Yu. Karlovich, L. Pessoa, *C^*-Algebras of Bergman type operators with piecewise continuous coefficients.* Int. Equat and Oper. Theory, 57 (4) (2007) 521–565.
3. O. Parfenov, *The asymptotic behavior of singular numbers of imbedding operators of some classes of analytic functions*. (Russian) Mat. Sb. (N.S.) 115(157) (1981), no. 4, 632–641; translation: Mathematics of the USSR-Sbornik, 1982, 43:4, 563–571.
4. L. Pessoa, A.M. Santos, *Theorems of Paley–Wiener type for spaces of polyanalytic functions*, Current trends in analysis and its applications, 605–613, Trends Math., Birkhäuser/Springer, Cham, 2015.
5. G. Rozenblum, N. Vasilevski,*Toeplitz operators defined by sesquilinear forms: Fock space case*. Journal of Functional Analysis, **267** (2014), 4399–4430.
6. G. Rozenblum, N. Vasilevski, *Toeplitz operators defined by sesquilinear forms: Bergman space case*, J. Math. Sci. (N.Y.) **213** (2016), no. 4, 582–609.
7. G. Rozenblum, N. Vasilevski, *Toeplitz operators in the Herglotz space.* Integral Equations Operator Theory **86** (2016), no. 3, 409–438
8. G. Rozenblum, N. Vasilevski, *Toeplitz operators in polyanalytic Bergman type spaces*, Functional Analysis and Geometry. Selim Grigorievich Krein Centennial. Contemporary Mathematics, **733**, AMS, (2019), 273–290.
9. J. Taskinen, J. Virtanen, *Toeplitz operators on Bergman spaces with locally integrable symbols.* Rev. Mat. Iberoam. **26** (2010), no. 2, 693–706.
10. J. Taskinen, J. Virtanen, *On generalized Toeplitz and little Hankel operators on Bergman spaces.* Arch. Math. (Basel) 110 (2018), no. 2, 155–166.
11. N. Vasilevski, *On the structure of Bergman and poly-Bergman spaces.* Integral Equations Operator Theory 33 (1999), no. 4, 471–488.
12. N. Vasilevski, *Poly-Bergman spaces and two-dimensional singular integral operators.* The Extended Field of Operator Theory, 349–359, Oper. Theory Adv. Appl., 171, Birkhauser, Basel, 2007.
13. F. Yan, D. Zheng, *Bounded Toeplitz operators on Bergman space.* Banach J. Math. Anal. 13 (2019), no. 2, 386–406.
14. K. Zhu, *Operator Theory in Function Spaces.* Second edition. Mathematical Surveys and Monographs, 138. American Mathematical Society, Providence, RI, 2007.
15. K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball.* Graduate Texts in Mathematics, 226. Springer-Verlag, New York, 2005.
16. K. Zhu, *Analysis on Fock Spaces,* Graduate Texts in Mathematics, 263. Springer-Verlag, New York, 2012.

Quantum Differentials and Function Spaces

Armen Sergeev

To Nikolai Vasilevskii on behalf of his 70th birthdate

Abstract Following the general scheme of Connes quantization we obtain interpretation of Schatten and interpolation ideals of compact operators in a Hilbert space in terms of function spaces. The main attention is paid to the case of Hilbert–Schmidt operators. In one-dimensional case the symmetry operator is given by the Hilbert transform. In the case of several variables the symmetry operator can be defined in terms of Riesz operators and Dirac matrices.

Keywords Connes quantization · Schatten ideals

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One of the goals of noncommutative geometry is the translation of basic notions of analysis into the language of Banach algebras. This translation is done using the quantization procedure which establishes a correspondence between function spaces and operator algebras in a Hilbert space H. The differential df of a function f (when it is correctly defined) corresponds under this procedure to the commutator of its operator image with some symmetry operator S which is a self-adjoint operator in H with square $S^2 = I$. The image of df under quantization is the quantum differential of f which is correctly defined even for non-smooth functions f. The arising operator calculus is called the quantum calculus.

In this paper we will give several assertions from this calculus concerning the interpretation of Schatten and interpolation ideals of compact operators in a Hilbert space in terms of function spaces on the circle. The main attention is paid to the case

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of Hilbert–Schmidt operators. The role of the symmetry operator S is played in this case by the Hilbert transform. In the case of function spaces of several variables the symmetry operator can be defined in terms of Riesz operators and Dirac matrices.

Briefly on the content of the paper. In the first Section basic definitions related to the ideals in the algebra of compact operators in a Hilbert space are recalled. In the second Section we introduce the quantum correspondence and formulate several assertions giving an interpretation of some operator algebras in terms of function theory. In the third Section we consider the ideal of Hilbert–Schmidt operators and present its interpretation in terms of function spaces. The fourth Section is devoted to the interpretation of Schatten and interpolation ideals. In the last fifth Section we consider the quantum differentials in spaces of functions of several variables.

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1 Ideals in the Algebra of Compact Operators

Let T be a compact operator in a Hilbert space H and $|T| = \sqrt{T^*T}$ is the non-negative square root of T^*T. Denote by $\{s_n(T)\}$ the sequence of *singular numbers* (*s-numbers*) of operator T given by the eigenvalues of the operator $|T|$ numerated in the decreasing order:

$$s_0(T) \geq s_1(T) \geq \dots$$

so that $s_n(T) \to 0$ for $n \to \infty$.

The singular numbers of T may be computed from the *minimax principle*, namely:

$$s_n(T) = \inf_E \{\|T|E^\perp\| : \dim E = n\}$$

so that $s_n(T)$ coincides with the infimum of the norms of restrictions of T to the orthogonal complements $E^\perp$ of different n-dimensional subspaces $E \subset H$. In fact, this infimum is attained at the subspace E_n generated by the first n eigenvectors of $|T|$ corresponding to the eigenvalues $s_0, \dots, s_{n-1}$.

In another way, we can define $s_n(T)$ as the distance from T to the subspace Fin_n of operators of rank$\leq n$. Namely,

$$s_n(T) = \inf_R \{\|T - R\| : R \in \mathrm{Fin}_n\}.$$

Definition 1.1 Let T be a compact operator in the Hilbert space H. We will say that T belongs to the space $\mathfrak{S}^p = \mathfrak{S}^p(H)$, $1 \leq p < \infty$, if

$$\sum_{n=0}^{\infty} s_n(T)^p < \infty.$$

The space $\mathfrak{S}^p$ is an ideal in the algebra $\mathcal{K} = \mathcal{K}(H)$ of compact operators and in the algebra $\mathcal{L} = \mathcal{L}(H)$ of bounded linear operators, acting in the Hilbert space H, and is called the *Schatten ideal*.

An important particular case is the class of Hilbert–Schmidt operators.

Definition 1.2 A compact operator T in the Hilbert space H is called the *Hilbert–Schmidt operator* if

$$\sum_{n=0}^{\infty} s_n(T)^2 < \infty.$$

The quantity

$$\|T\|_2 := \left(\sum_{n=0}^{\infty} s_n(T)^2\right)^{1/2}$$

is called the *Hilbert–Schmidt norm* of T.

If T is a Hilbert–Schmidt operator then for any orthonormal basis $\{e_k\}_{k=1}^{\infty}$ in H the series

$$\sum_{k=1}^{\infty} \|Te_k\|^2$$

converges. Its sum does not depend on the choice of the orthonormal basis $\{e_k\}$ and coincides with $\|T\|_2^2$.

The space $\mathfrak{S}_2(H)$ of Hilbert–Schmidt operators, acting in the Hilbert space H, is a Hilbert space with the norm $\|\cdot\|_2$ and an ideal in the algebra $\mathcal{K}(H)$ of compact operators closed with respect to the Hilbert–Schmidt norm.

We introduce now the function

$$\sigma_N(T) := \sum_{n=0}^{N-1} s_n(T).$$

In a different way it can be defined as

$$\sigma_N(T) = \sup_E \{\|TP_E\|_1 : \dim E = N\},$$

where P_E is the orthogonal projector to the subspace E and $\|TP_E\|_1$ is the nuclear norm of the operator TP_E:

$$\|TP_E\|_1 := \sum_{n=0}^{\infty} s_n(TP_E).$$

The supremum in the above formula is again attained on the subspace E_N generated by the first N eigenvectors of operator T.

Apart from ideals $\mathfrak{S}^p$ we introduce also the interpolation ideals $\mathfrak{S}^{p,q}$.

Definition 1.3 An operator $T \in \mathfrak{S}^{p,q} = \mathfrak{S}^{p,q}(H)$ if

$$\sum_{N=1}^{\infty} N^{(\alpha-1)q-1} \sigma_N(T)^q < \infty$$

where $\alpha = 1/p$. Extend this definition to $q = \infty$ by stating that $T \in \mathfrak{S}^{p,\infty}$ if the sequence of numbers $\{N^{\alpha-1}\sigma_N(T)\}_{N=1}^{\infty}$ is bounded.

Each of the introduced spaces $\mathfrak{S}^{p,q}$ is a two-sided ideal in the algebra $\mathcal{K}$ of compact operators. For $p_1 < p_2$ and for $p_1 = p_2, q_1 < q_2$ there are inclusions

$$\mathfrak{S}^{p_1,q_1} \subset \mathfrak{S}^{p_2,q_2}.$$

Here are some particular examples of the spaces $\mathfrak{S}^{p,q}$.

The space $\mathfrak{S}^{p,p}$, $1 \le p < \infty$, coincides with the space $\mathfrak{S}^p$, introduced above, with the norm given by the formula

$$\|T\|_p = \left(\mathrm{Tr}|T|^p\right)^{1/p} = \left[\sum_{n=0}^{\infty} s_n(T)^p\right]^{1/p}.$$

The space $\mathfrak{S}^{p,\infty}$, $1 < p < \infty$, consists of compact operators T for which $\sigma_N(T) = O(N^{1-\alpha})$, i.e. $s_n(T) = O(n^{-\alpha})$. There is a natural norm on this space given by

$$\|T\|_{p,\infty} = \sup_N \frac{1}{N^{1-\alpha}} \sigma_N(T).$$

The space $\mathfrak{S}^{p,1}$ consists of compact operators T for which the series

$$\sum_{N=1}^{\infty} N^{\alpha-2} \sigma_N(T)$$

converges which is equivalent to the convergence of the series $\sum_{n=1}^{\infty} n^{\alpha-1} s_{n-1}(T)$.

2 Quantum Correspondence

Using the quantization procedure, we will associate with function spaces the ideals in the algebra of bounded linear operators in a Hilbert space.

Let A be an *algebra of observables*, i.e. an associative algebra provided with involution. We suppose also that A has the exterior differential $d : A \to \Omega^1(A)$, i.e. a linear map from A to the space $\Omega^1(A)$ of 1-forms on this algebra satisfying Leibniz rule (cf. [2]).

The *quantization* of A is a linear representation π of observables from A by the densely defined closed linear operators, acting in a complex Hilbert space H called the *quantization space*. It is required that the involution in A transforms to Hermitian conjugation, and the action of the exterior derivative operator $d : A \to \Omega^1(A)$ corresponds to the commutator with some *symmetry operator* S, which is a selfadjoint operator on H with square $S^2 = I$. In other words,

$$\pi : df \longmapsto d^q f := [S, \pi(f)], \quad f \in A,$$

where $d^q f := [S, \pi(f)]$ is the *quantum differential* of f. We call the Lie algebra A^q, generated by quantum differentials $d^q f$, the *quantum algebra of observables* and the differential $d^q f$ the *quantum observable* associated with observable f.

Recall that the *differentiation* of the algebra A is a linear map $D : A \to A$ satisfying the Leibniz rule $D(ab) = (Da)b + a(Db)$. Denote by $\mathrm{Der}(A)$ the Lie algebra of all differentiations of the algebra A. In terms of $\mathrm{Der}(A)$ the quantization is an irreducible representation of the Lie algebra $\mathrm{Der}(A)$ in the Lie algebra of linear operators on H provided with commutator as the Lie bracket.

Consider a particular case of the above quantization problem in which the algebra of observables coincides with the algebra $A = L^\infty(\mathbb{R}, \mathbb{C})$ of bounded functions on the real line $\mathbb{R}$ provided with the natural involution given by complex conjugation.

A function $f \in A$ determines the bounded multiplication operator M_f in the Hilbert space $H = L^2(\mathbb{R})$ acting by the formula:

$$M_f : h \in H \longmapsto fh \in H.$$

The assignment $f \mapsto M_f$ defines a linear representation of the algebra A in the space H.

The differential of a general observable $f \in A$ is not defined in the classical sense so we cannot provide A with classical differential d. However, its quantum analogue d^q can be correctly defined as we will see below. We use the corresponding quantum algebra of observables A^q as a replacement of the (non-defined) classical algebra of observables A.

The symmetry operator S on H is given by the *Hilbert transform*

$$(Sf)(x) = \frac{i}{\pi}\text{P.V.}\int_{\mathbb{R}} K(x,t)f(t)dt, \quad f \in H, \tag{2.1}$$

where

$$K(x,t) = \frac{1}{t-x}$$

and the integral is taken in the principal value sense, i.e.

$$\text{P.V.} \int_{\mathbb{R}} K(x,t) f(t) dt := \lim_{\epsilon \to 0} \int_{|t| \geq \epsilon} K(x,t) f(t) dt.$$

It is well known (cf. [6, 7]) that S is a symmetry operator in H and has the following properties:

1. S commutes with translations;
2. S commutes with positive dilations and anti-commutes with reflections.

It turns out that the only bounded operator in H with such properties is a multiple of the Hilbert operator.

The quantum differential

$$d^q f := [S, M_f] = SM_f - M_f S$$

is equal to

$$(d^q f)(h) = \frac{i}{\pi} \int_{\mathbb{R}} K(x,t) f(t) h(t) dt - \frac{i}{\pi} \int_{\mathbb{R}} K(x,t) f(x) h(t) dt = \\ = \frac{i}{\pi} \int_{\mathbb{R}} K(x,t) \left[f(t) - f(x) \right] h(t) dt.$$

It is correctly defined as an operator in H for functions $f \in A$ (and even for functions from the space BMO($\mathbb{R}$)).

It is an integral operator given by the formula

$$(d^q f)(h)(x) = \frac{i}{\pi} \int_{\mathbb{R}} k_f(x,t) h(t) dt, \quad h \in H, \tag{2.2}$$

where

$$k_f(x,t) = \frac{f(t) - f(x)}{t - x}.$$

So in the considered example the quantization is essentially reduced to the replacement of the derivative by its finite-difference analogue. Such quantization, given by the correspondence $A \ni f \mapsto d^q f : H \to H$, Connes [2] calls the *"quantum calculus"* by analogy with the finite-difference calculus.

Here are several examples from the quantum calculus.

1. Quantum differential $d^q f$ is a finite rank operator if and only if the function f is rational (Kronecker theorem).
2. Quantum differential $d^q f$ is a compact operator if and only if the function f belongs to the class $\mathrm{VMO}(\mathbb{R})$;
3. Quantum differential $d^q f$ is a bounded operator if and only if the function f belongs to the class $\mathrm{BMO}(\mathbb{R})$.

These results are easily deduced from the corresponding assertions for Hankel operators (cf. [4, 5]) using the relation between such operators and quantum differentials pointed out in Sec. 4.

Recall for completeness the definitions of the space $\mathrm{BMO}(\mathbb{R})$ of functions with bounded mean oscillation and the space $\mathrm{VMO}(\mathbb{R})$ of functions with vanishing mean oscillation.

Denote by

$$f_I := \frac{1}{|I|}\int_I f(x)dx$$

the average of such function over the interval I of the real line of length $|I|$. If

$$M(f) := \sup_I \frac{1}{|I|}\int_I |f(x) - f_I|dx < \infty$$

then we will say that the function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ belongs to the space $\mathrm{BMO}(\mathbb{R})$.

Introduce one more notation

$$M_\delta(f) := \sup_{|I|<\delta} \frac{1}{|I|}\int_I |f(x) - f_I|dx$$

where $\delta > 0$. In terms of this function $f \in \mathrm{BMO}(\mathbb{R})$ if and only if the supremum $\sup_{\delta>0} M_\delta(f)$ is finite. We say that a function $f \in \mathrm{BMO}(\mathbb{R})$ belongs to the space $\mathrm{VMO}(\mathbb{R})$ if $M_\delta(f) \to 0$ $\delta \to 0$.

3 Hilbert–Schmidt Operators

In this case the role of quantization space is played by the Sobolev space of half-differentiable functions on the circle.

Definition 3.1 The *Sobolev space of half-differentiable functions* is the Hilbert space

$$V = H_0^{1/2}(S^1, \mathbb{R})$$

consisting of functions $f \in L_0^2(S^1, \mathbb{R})$ with zero average along the circle having the generalized derivative of order 1/2 in $L^2(S^1, \mathbb{R})$. In other words, it consists of functions $f \in L^2(S^1, \mathbb{R})$ having Fourier series of the form

$$f(z) = \sum_{n \neq 0} f_n z^n, \quad \bar{f}_n = f_{-n},\ z = e^{i\theta},$$

with finite Sobolev norm of order 1/2:

$$\|f\|_{1/2}^2 = \sum_{n \neq 0} |n||f_n|^2 = 2\sum_{n=1}^{\infty} n|f_n|^2 < \infty.$$

The inner product on V in terms of Fourier coefficients is given by the formula

$$(\xi, \eta) = \sum_{n \neq 0} |n|\xi_n \bar{\eta}_n = 2\,\mathrm{Re} \sum_{n=1}^{\infty} n\xi_n \bar{\eta}_n,$$

for vectors $\xi, \eta \in V$.

The complexification $V^{\mathbb{C}} = H_0^{1/2}(S^1, \mathbb{C})$ of the space V is a complex Hilbert space consisting of functions $f \in L^2(S^1, \mathbb{C})$ with Fourier decompositions of the form

$$f(z) = \sum_{n \neq 0} f_n z^n, \quad z = e^{i\theta},$$

and finite Sobolev norm $\|f\|_{1/2}^2 = \sum_{n \neq 0} |n||f_n|^2 < \infty$. Complexified Sobolev space $V^{\mathbb{C}}$ decomposes into the direct sum

$$V^{\mathbb{C}} = W_+ \oplus W_-$$

of subspaces $W_\pm$ consisting of functions

$$f(z) = \sum_{n \neq 0} f_n z^n, \quad z = e^{i\theta},$$

with Fourier coefficients f_n vanishing for $\mp n > 0$.

The space V admits a realization as the *Dirichlet space* $\mathcal{D}$ of functions in the unit disk $\mathbb{D}$ consisting of harmonic functions $h : \mathbb{D} \to \mathbb{R}$ normalized by the condition $h(0) = 0$ and having finite energy

$$E(h) = \frac{1}{2\pi} \int_{\mathbb{D}} |\mathrm{grad}\, h(z)|^2 dxdy = \frac{1}{2\pi} \int_{\mathbb{D}} \left(\left| \frac{\partial h}{\partial x} \right|^2 + \left| \frac{\partial h}{\partial y} \right|^2 \right) dxdy < \infty.$$

It is well known that the Poisson transform

$$Pf(z) = \frac{1}{2\pi}\int_0^{2\pi} P(\zeta, z) f(\zeta) d\theta, \quad \zeta = e^{i\theta},$$

where $P(\zeta, z)$ is the Poisson kernel in the disk D:

$$P(\zeta, z) = \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2},$$

establishes an isometric isomorphism

$$P : V \longrightarrow \mathcal{D}$$

between the Sobolev space V and the Dirichlet space $\mathcal{D}$ provided with the norm

$$\|h\|^2_{\mathcal{D}} := E(h).$$

In the case of upper halfplane $\mathbb{H}$ the above definition admits another useful interpretation. In this case the Sobolev space V coincides with the space $H^{1/2}(\mathbb{R})$ of half-differentiable functions on $\mathbb{R}$.

There is a *Douglas formula* expressing the energy of a map $f \in H^{1/2}(\mathbb{R})$ in terms of the finite-difference derivative of f:

$$E(Pf) = \|f\|^2_{1/2} = \frac{1}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \left[\frac{f(x) - f(y)}{x - y}\right]^2 dxdy. \tag{3.1}$$

It implies, in particular, that functions $f \in H^{1/2}(\mathbb{R})$ have L^2-bounded finite-difference derivatives.

We return now to the quantization problem formulated above and take for the algebra of observables the algebra $A = L^\infty(S^1, \mathbb{C})$ of bounded functions on the circle S^1 provided with the natural involution given by complex conjugation.

For a function $f \in A$ we denote again by M_f the bounded multiplication operator in the Hilbert space $V^{\mathbb{C}}$ acting by the formula $M_f : h \mapsto fh$. The symmetry operator S on $V^{\mathbb{C}}$ coincides again with the *Hilbert transform* given in this case by the formula

$$(Sh)(\phi) = \frac{1}{2\pi}\text{P.V.}\int_0^{2\pi} K(\phi, \psi) h(\psi) d\psi, \quad h \in H, \tag{3.2}$$

(Here and in the sequel we identify functions $h(z)$ on the circle S^1 with functions $h(\phi) := h(e^{i\phi})$ on the interval $[0, 2\pi]$.) The Hilbert kernel in the formula (3.2) is given by the expression

$$K(\phi, \psi) = 1 + i \cot \frac{\phi - \psi}{2} .$$

Note that for $\phi \to \psi$ it behaves like $1 + \frac{2i}{\phi - \psi}$.

The quantum differential

$$d^q f := [S, M_f]$$

is correctly defined as an operator on $V^{\mathbb{C}}$ for functions $f \in A$. It is an integral operator given by the formula

$$(d^q f)(h)(\phi) = \frac{1}{2\pi} \int_0^{2\pi} k_f(\phi, \psi) h(\psi) d\psi, \quad h \in H, \tag{3.3}$$

where

$$k_f(\phi, \psi) = K(\phi, \psi)(f(\phi) - f(\psi)),$$

and $K(\phi, \psi)$ is the Hilbert kernel. For $\phi \to \psi$ the kernel $k_f(\phi, \psi)$ behaves (up to a constant) like

$$\frac{f(\phi) - f(\psi)}{\phi - \psi} .$$

We supplement the above list of correspondences between the algebras of quantum differentials and function spaces by the following interpretation of Sobolev space of half-differentiable functions in terms of quantum correspondence.

Theorem 3.2 *A function f belongs to the Sobolev space $V^{\mathbb{C}}$ if and only if its quantum differential $d^q f$ is a Hilbert–Schmidt operator on $V^{\mathbb{C}}$. Moreover, the Hilbert–Schmidt norm of the operator $d^q f$ coincides with the Sobolev norm of the function f.*

To prove this Theorem recall that the commutator $d^q f := [S, M_f]$ is an integral operator on $V^{\mathbb{C}}$ with the kernel equal to

$$k_f(\phi, \psi) = K(\phi, \psi)(f(\phi) - f(\psi)).$$

This operator is Hilbert–Schmidt if and only if its kernel $k_f(\phi, \psi)$ is square integrable on $S^1 \times S^1$ which is equivalent to the condition

$$\int_0^{2\pi} \int_0^{2\pi} \frac{|f(\phi) - f(\psi)|^2}{\sin^2 \left(\frac{\phi - \psi}{2}\right)} d\phi \, d\psi < \infty. \tag{3.4}$$

Now the assertion of the Theorem follows from the Douglas formula (3.1) given above. In order to see that it is sufficient to switch in the formula (3.4) from the circle S^1 to the real line $\mathbb{R}$. Then the left hand side of the inequality (3.4) will be replaced by the expression

$$\frac{1}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\frac{f(x)-f(y)}{x-y}\right]^2 dxdy = \|f\|_{1/2}^2$$

which implies the assertion of the Theorem.

4 Interpretation of Schatten Classes and Interpolation Ideals

The introduced quantum differentials are closely related to Hankel operators which are defined in the following way.

Suppose that a function φ belongs to the space $V^{\mathbb{C}}$. Denote by $P_\pm$ the orthogonal projectors $P_\pm : V^{\mathbb{C}} \to W_\pm$.

The *Hankel operator* $H_\varphi : W_+ \to W_-$ is given by the formula

$$H_\varphi h := P_-(\varphi h).$$

It is well known (cf. [4]) that it is bounded in W_+ if $P_-\varphi \in \text{BMO}(S^1)$. In analogous way we can introduce the Hankel operators $\widetilde{H_\varphi} : W_- \to W_+$ given by the formula $\widetilde{H_\varphi} h := P_+(\varphi h)$.

The quantum differential $(d^q f)h = [S, M_f]h$, where $f \in A = L^\infty(S^1, \mathbb{C})$, $h \in V^{\mathbb{C}}$, may be rewritten, using the well-known relations: $S = P_+ - P_-$, $P_+ + P_- = I$ and $P_+P_- = P_-P_+ = 0$, as follows

$$[S, M_f] = SM_f - M_f S = (P_+ - P_-)f(P_+ + P_-) - f(P_+ - P_-)(P_+ + P_-) =$$
$$= P_+fP_+ + P_+fP_- - P_-P_-fP_+ - P_-fP_- - fP_+ + fP_- =$$
$$= -P_-(fP_+) + P_+(fP_-) + P_+fP_- - P_-fP_+ = -2P_-fP_+ + 2P_+fP_-.$$

This chain of equalities implies that

$$[S, M_f]h = -2P_-fP_+h + 2P_+fP_-h.$$

The last expression coincides with $-2P_-fh$ for $h \in W_+$ and with $2P_+fh$ for $h \in W_-$. In other words, the operator $d^q f$ with $f \in A$ is the direct orthogonal sum of two Hankel operators. So the description of various algebras of quantum differentials $d^q f$ is reduced to the description of the corresponding classes of Hankel operators. The latter was obtained by Peller in [4]. In order to formulate his result recall the definition of Besov classes B_p^s. Denote by Δ_ζ the difference operator

$$(\Delta_\zeta f)(z) := f(\zeta z) - f(z), \quad \zeta, z \in S^1,$$

and define the nth difference Δ_ζ^n as the nth power of operator Δ_ζ. Then the *Besov space* B_p^s, $s > 0$, $1 < p < \infty$, is defined as

$$B_p^s = \left\{ f \in L^p : \quad \int_{S^1} \frac{\|\Delta_\zeta^n f\|_p^p}{|1-\zeta|^{1+sp}} d\vartheta < \infty \right\}, \ \zeta = e^{i\vartheta},$$

where n is an arbitrary integer greater than s. In particular, for $s = 1/p$ we get

$$B_p^{1/p} = \left\{ f \in L^p : \quad \int_{S^1} \frac{\|f(\zeta z) - f(z)\|_p^p}{|1-\zeta|^2} d\vartheta < \infty \right\}.$$

Theorem 4.1 (Peller) *Let $f \in A$. Then the Hankel operator H_f belongs to the Schatten class $\mathfrak{S}^p$ with $1 < p < \infty$ if and only if $P_- f \in B_p^{1/p}$.*

Note that the analogous result holds for Hankel operators from the classes $\mathfrak{S}^p$ with $0 < p < \infty$ (cf. [4]).

The above theorem implies that the quantum differential $d^q f$ belongs to the Schatten class $\mathfrak{S}^p$ with $1 < p < \infty$ if and only if $P_\pm f \in B_p^{1/p}$, i.e. we have the following

Theorem 4.2 *The quantum differential $d^q f$ belongs to the Schatten class $\mathfrak{S}^p$ with $1 < p < \infty$ if and only if $f \in B_p^{1/p}$.*

Interpolation ideals $\mathfrak{S}^{p,q}$, as it is clear from their name, may be obtained from Schatten ideals $\mathfrak{S}^p$ by interpolation. Recall the general definition of interpolation spaces which may be found in [1]. Suppose that we have a pair (X_0, X_1) of subspaces of a Banach space X. Introduce a K-*functional* on the space

$$X_0 + X_1 = \{x_0 + x_1 : x_0 \in X_0, x_1 \in X_1\}$$

given by the formula

$$K(t, x, X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}$$

for $t > 0$.

For given $0 < \theta < 1$, $0 < q < \infty$ the interpolation space $(X_0, X_1)_{\theta,q}$ consists of elements $x \in X_0 + X_1$ having the finite norm

$$\|x\|_{\theta,q} = \left(\int_0^\infty \left(\frac{K(t, x, X_0, X_1)}{t^\theta} \right)^q \frac{dt}{t} \right)^{1/q}$$

with evident modification for $q = \infty$. Then the interpolation ideal $\mathfrak{S}^{p,q}$ coincides with interpolation space

$$\mathfrak{S}^{p,q} = (\mathfrak{S}_p, \mathfrak{S}_\infty)_{\theta,q}$$

where $0 < \theta < 1, 0 < q \le \infty$, $p = \frac{p_0}{1-\theta}$ with $0 < p_0 < \infty$.

It implies that the set of quantum differentials, belonging to the ideal $\mathfrak{S}^{p,q}$, coincides with the interpolation space

$$\left(B_{p_0}^{1/p_0}, B_{p_1}^{1/p_1}\right)_{\theta,q}$$

where $p_0 < p < p_1$, $1/p = (1-\theta)/p_0 + \theta/p_1$. An explicit description of this space may be found in [4].

5 Quantum Differentials in Spaces of Several Variables

The role of Hilbert transform in the case of function spaces of several real variables is played by the Riesz operators R_j, $1 \le j \le n$, which act on $L^2(\mathbb{R}^n)$ by the formula

$$R_j h(x) := c_n P.V. \int_{\mathbb{R}^n} \frac{(t_j - x_j)h(t)}{|t-x|^{n+1}} d^n t =: c_n P.V. \int_{\mathbb{R}^n} K^j(x,t) h(t) d^n t$$

where c_n is a coefficient (depending only on n) equal to

$$c_n = \frac{2i}{\Omega_{n+1}}$$

and Ω_{n+1} is the volume of the unit sphere S^n equal to

$$\Omega_{n+1} = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}.$$

The Riesz operators (cf. [6, 7]) also commute with translations and dilatations, moreover

$$\sum_{j=1}^{n} R_j^2 = 1.$$

Introduce the space of vector-functions

$$H = \left(L^2(\mathbb{R}^n)\right)^N.$$

The Riesz operators act on the vector-functions $\mathbf{h}(x) = (h_1(x), \dots, h_N(x)) \in H$ componentwise.

To define the symmetry operator, associated with Riesz operators, consider a collection of $(N \times N)$-matrices $\gamma_1, \dots, \gamma_n$ such that

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij}.$$

These matrices γ_j coincide with the Dirac matrices generating the spin representation of the Clifford algebra $\mathrm{Cl}^{\mathbb{C}}(\mathbb{R}^n)$ in the space $\mathbb{C}^N$ where $N = 2^{[n/2]}$ (cf. [3]). Having such collection of matrices γ_j, we can define the *symmetry operator* S acting on the space $H = \left(L^2(\mathbb{R}^n)\right)^N$ by the formula

$$S\mathbf{h} := \sum_{j=1}^{n} \gamma_j R_j \mathbf{h}.$$

The associated quantum differential $d^q f = [S, M_f]$ is equal to

$$(d^q f)(\mathbf{h}) = SM_f(\mathbf{h}) - M_f S(\mathbf{h}) = \sum_{j=1}^{n} \gamma_j R_j (f\mathbf{h}) - \sum_{j=1}^{n} f\gamma_j (R_j \mathbf{h}) \tag{5.1}$$

where

$$\gamma_j R_j (f\mathbf{h}) = c_n \gamma_j \int_{\mathbb{R}^n} K^j(x,t) f(t)\mathbf{h}(t) d^n t,$$

$$f\gamma_j R_j (\mathbf{h}) = c_n \gamma_j \int_{\mathbb{R}^n} K^j(x,t) f(x)\mathbf{h}(t) d^n t.$$

So

$$(d^q f)(\mathbf{h}) =$$

$$= \sum_{j=1}^{n} c_n \gamma_j \int_{\mathbb{R}^n} \left[f(t) - f(x)\right] K^j(x,t)\mathbf{h}(t) d^n t = \sum_{j=1}^{n} c_n \gamma_j \int_{\mathbb{R}^n} k_f^j(x,t)\mathbf{h}(t) d^n t$$

where

$$k_f^j(x,t) = \frac{\left[f(t) - f(x)\right](t_j - x_j)}{|t-x|^{n+1}}.$$

The introduced quantum differential can be considered as a quantum version of the *Dirac operator* $D = \sum_{j=1}^{n} \gamma_j \partial_{t_j}$ associated with the spin representation of the Clifford algebra $\mathrm{Cl}^{\mathbb{C}}(\mathbb{R}^n)$ determined by the γ-matrices $\gamma_1, \dots, \gamma_n$ (cf. [3]).

It would be interesting to study the properties of the quantum correspondence $f \mapsto d^q f$, defined by the formula (5.1), as in the case of one real variable.

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References

1. J.Bergh, J.Löfstrem, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin–Heidelberg–N.Y., 1976.
2. A.Connes, *Noncommutative Geometry*, Academic Press, London–San Diego, 1994.
3. H.D.Lawson,M.-L.Michelsohn, *Spin Geometry*, Princeton Univ. Press, Princeton, New Jersey, 1989.
4. V.V.Peller, *Hankel Operators and Their Applications*, Springer, Berlin-Heidelberg-New York, 2003.
5. S.Power, *Hankel operators on Hilbert space*, Research Notes in Math. 64(1982).
6. E.M.Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, New Jersey, 1970.
7. E.M.Stein, G.Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, New Jersey, 1971.

Toeplitz Quantization of a Free $*$-Algebra

Stephen Bruce Sontz

To Nikolai Vasilevski in celebration of his 70th birthday

Abstract In this note we quantize the free $*$-algebra generated by finitely many variables, which is a new example of the theory of Toeplitz quantization of $*$-algebras as developed previously by the author. This is achieved by defining Toeplitz operators with symbols in that non-commutative free $*$-algebra. These are densely defined operators acting in a Hilbert space. Then creation and annihilation operators are introduced as special cases of Toeplitz operators, and their properties are studied.

Keywords Toeplitz operators · Creation and annihilation operators

1 Introduction

The basic reference for this paper is [3] where a general theory of Toeplitz quantization of $*$-algebras is defined and studied. More details including motivation and references can be found in [3].

2 The Free $*$-Algebra

The example in this paper is the free algebra on $2n$ non-commuting variables $\mathcal{A} = \mathbb{C}\{\theta_1, \overline{\theta}_1, \ldots, \theta_n, \overline{\theta}_n\}$. In particular, the variables $\theta_j, \overline{\theta}_j$ do not commute for $1 \leq j \leq n$. The *holomorphic sub-algebra* is defined by $\mathcal{P} := \mathbb{C}\{\theta_1, \ldots, \theta_n\}$, the free

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algebra on n variables. The $*$-*operation* (or *conjugation*) on $\mathcal{A}$ is defined on the generators by

$$\theta_j^* := \overline{\theta}_j \quad \text{and} \quad \overline{\theta}_j{}^* := \theta_j,$$

where $j = 1, \ldots, n$. This is then extended to finite products of these $2n$ elements in the unique way that will make $\mathcal{A}$ into a $*$-algebra with $1^* = 1$. As explained in more detail in a moment these products form a vector space basis of $\mathcal{A}$, and so we extend the $*$-operation to finite linear combinations of them to make it an *anti-linear* map over the field $\mathbb{C}$ of complex numbers. Therefore, $\mathcal{P}$ is not a sub-$*$-algebra. Rather, we have $\mathcal{P} \cap \mathcal{P}^* = \mathbb{C}1$. Moreover, $\mathcal{P}$ is a non-commutative sub-algebra of $\mathcal{A}$ if $n \geq 2$. This set-up easily generalizes to infinitely many pairs of non-commuting variables $\theta_j, \overline{\theta}_j$.

The definition of $\mathcal{P}$ is motivated as a non-commutative analogy to the commutative algebra of holomorphic polynomials in the Segal-Bargmann space $L^2(\mathbb{C}^n, e^{-|z|^2}\mu_{Leb})$, where μ_{Leb} is Lebesgue measure on the Euclidean space $\mathbb{C}^n$. (See [1] and [2].) This is one motivation behind using the notation $\mathcal{P}$ for this sub-algebra.

We will later introduce a projection operator $P : \mathcal{A} \to \mathcal{P}$ using a sesqui-linear form defined on $\mathcal{A}$. This is an essential ingredient in the following definition.

Definition 2.1 Let $g \in \mathcal{A}$ be given. Then we define the *Toeplitz operator* T_g with symbol g as $T_g\phi := P(\phi g)$ for all $\phi \in \mathcal{P}$. It follows that $T_g : \mathcal{P} \to \mathcal{P}$ is linear. We let $\mathcal{L}(\mathcal{P}) := \{T : \mathcal{P} \to \mathcal{P} \mid T \text{ is linear}\}$. Then the linear map $\mathcal{A} \ni g \mapsto T_g \in \mathcal{L}(\mathcal{P})$ is called the *Toeplitz quantization*.

Multiplying the symbol g on the left of ϕ gives a similar theory, which we will not expound on in further detail.

The sesqui-linear form on $\mathcal{A}$ when restricted to $\mathcal{P}$ will turn out to be an inner product. So $\mathcal{P}$ will be a pre-Hilbert space that is dense in its completion, denoted as $\mathcal{H}$. This is another motivation for using the notation $\mathcal{P}$ for this sub-algebra. So every Toeplitz operator T_g will be a densely defined linear operator acting in the Hilbert space $\mathcal{H}$.

The definition of the sesqui-linear form on $\mathcal{A}$ is a more involved story. To start it we let $\mathcal{B}$ be the standard basis of $\mathcal{A}$ consisting of all finite *words* (or *monomials*) in the finite alphabet $\{\theta_1, \overline{\theta}_1, \ldots, \theta_n, \overline{\theta}_n\}$, which has $2n$ letters. The empty word (with zero letters) is the identity element $1 \in \mathcal{A}$. Let $f \in \mathcal{B}$ be a word in our alphabet. We let $l(f)$ denote the *length* of f, that is, the number of letters in f. Therefore $l(f) = 0$ if and only if $f = 1$.

Definition 2.2 Let $f \in \mathcal{B}$ with $l(f) > 0$. Then we say that f *begins with a* θ if the first letter of f (as read from the left) is an element of $\{\theta_1, \ldots, \theta_n\}$; otherwise, we say that f *begins with a* $\overline{\theta}$.

If $f = 1$, then we say that f *begins with a* θ and f *begins with a* $\overline{\theta}$.

Remark Suppose $l(f) > 0$ and that f begins with a θ. Then f has a *unique* representation as

$$f = \theta_{i_1} \cdots \theta_{i_r} \overline{\theta}_{j_1} \cdots \overline{\theta}_{j_s} f', \tag{2.1}$$

where $r \geq 1$, $s \geq 0$ and f' begins with a θ. That is to say, the word f begins with $r \geq 1$ occurrences of θ's followed by $s \geq 0$ occurrences of $\overline{\theta}$'s and finally another word f' that begins with a θ. Note that if $s = 0$, then $f' = 1$. We also have that $l(f') < l(f)$. As a simple example of this representation, note that each basis element $f = \theta_{i_1} \cdots \theta_{i_r}$ in $\mathcal{P}$ with $r \geq 1$ has this representation with $s = 0$ and $f' = 1$.

Dually, suppose that $l(f) > 0$ and that f begins with a $\overline{\theta}$. Then f has the obvious dual representation.

Now we are going to define a sesqui-linear form $\langle f, g \rangle$ for $f, g \in \mathcal{A}$ by first defining it on pairs of elements of the basis $\mathcal{B}$ and extending sesqui-linearly, which for us means anti-linear in the first entry and linear in the second. The definition on pairs will be by recursion on the length of the words. To start off the recursion for $l(f) = l(g) = 0$ (that is, $f = g = 1$) we define

$$\langle f, g \rangle = \langle 1, 1 \rangle := 1.$$

This choice is a convenient normalization convention.

The next case we consider is $l(f) > 0$, f begins with a θ and $g = 1$. In that case using (2.1) we define recursively

$$\begin{aligned} \langle f, 1 \rangle &= \langle \theta_{i_1} \cdots \theta_{i_r} \overline{\theta}_{j_1} \cdots \overline{\theta}_{j_s} f', 1 \rangle \\ &:= w(i_1, \ldots, i_r) \delta_{r,s} \delta_{i_1, j_r} \cdots \delta_{i_r, j_1} \langle f', 1 \rangle \\ &= w(i) \delta_{r,s} \delta_{i, j^T} \langle f', 1 \rangle \end{aligned}$$

where $r \geq 1$ and $w(i) \equiv w(i_1, \ldots, i_r) > 0$ is a positive weight. Here we also define the (variable length) multi-index $i = (i_1, \ldots, i_r)$ and $j^T := (j_s, \ldots, j_1)$ to be the reversed multi-index of the multi-index $j = (j_1, \ldots, j_s)$. It follows that $\langle f, 1 \rangle \neq 0$ in this case implies that we necessarily have $\langle f', 1 \rangle \neq 0$ and

$$f = \theta_{i_1} \cdots \theta_{i_r} \overline{\theta}_{i_r} \cdots \overline{\theta}_{i_1} f'.$$

Moreover, by recursion f' must also has this same form as f. Since the lengths are strictly decreasing ($l(f) > l(f') > \cdots$), this recursion terminates in a finite number of steps. Thus the previous equation can then be written using the obvious notations $\theta_i := \theta_{i_1} \cdots \theta_{i_r}$ and $\overline{\theta}_{i^T} := \overline{\theta}_{i_r} \cdots \overline{\theta}_{i_1}$ as

$$f = \theta_i \overline{\theta}_{i^T} f'.$$

Symmetrically, for $f = 1$, $l(g) > 0$ and g begins with a θ we write $g = \theta_{k_1} \cdots \theta_{k_t} \overline{\theta}_{l_1} \cdots \overline{\theta}_{l_u} g'$ uniquely so that $t \geq 1$ and g' begins with a θ and define recursively

$$\begin{aligned}\langle 1, g \rangle &= \langle 1, \theta_{k_1} \cdots \theta_{k_t} \overline{\theta}_{l_1} \cdots \overline{\theta}_{l_u} g' \rangle \\ &:= w(k_1, \ldots, k_t) \delta_{t,u} \delta_{k_1, l_u} \cdots \delta_{k_t, l_1} \langle 1, g' \rangle \\ &= w(k) \delta_{t,u} \delta_{k, l^T} \langle 1, g' \rangle.\end{aligned}$$

Next suppose that $l(f) > 0$ and $l(g) > 0$ and that both f and g begin with a θ and are written as above. In that case, we define

$$\begin{aligned}\langle f, g \rangle &= \langle \theta_{i_1} \cdots \theta_{i_r} \overline{\theta}_{j_1} \cdots \overline{\theta}_{j_s} f',\ \theta_{k_1} \cdots \theta_{k_t} \overline{\theta}_{l_1} \cdots \overline{\theta}_{l_u} g' \rangle \\ &:= w(i, l^T) \delta_{r+u, s+t} \delta_{(i, l^T), (k, j^T)} \langle f', g' \rangle,\end{aligned} \tag{2.2}$$

where $(i, l^T) := (i_1, \ldots, i_r, l_u, \ldots, l_1)$ is the concatenation of the two multi-indices i and $l^T = (l_u, \ldots, l_1)$. (Similarly for the notation (k, j^T).)

The definitions for two words that begin with a $\overline{\theta}$ are dual to these definitions. We use the same weight factors for this dual part, though new real weight factors could have been used.

There is still one remaining case for which we have yet to define the sesqui-linear form. That case is when f begins with a θ, g begins with a $\overline{\theta}$, (or *vice versa*), $l(f) > 0$ and $l(g) > 0$. In that case we define $\langle f, g \rangle := 0$.

Theorem 2.1 *The sesqui-linear form on $\mathcal{A}$ is* complex symmetric*, that is,*

$$\langle f, g \rangle^* = \langle g, f \rangle \quad \text{for all } f, g \in \mathcal{A}.$$

Proof The proof is by induction following the various cases of the recursive definition of the sesqui-linear form. First, for $l(f) = l(g) = 0$ we have $f = g = 1$ in which case

$$\langle 1, 1 \rangle^* = 1^* = 1 = \langle 1, 1 \rangle.$$

Next we take the case $l(f) > 0$, f begins with a θ and $l(g) = 0$. Then we write $f = \theta_i \overline{\theta}_j f'$ for multi-indices i, j of lengths r, s respectively and f' begins with a θ. So we calculate

$$\langle f, 1 \rangle^* = \big(w(i) \delta_{r,s} \delta_{i, j^T} \langle f', 1 \rangle \big)^* = w(i) \delta_{r,s} \delta_{i, j^T} \langle 1, f' \rangle,$$

where we used the induction hypothesis and the reality of the weight $w(i)$ for the last step. On the other hand, we have by definition that

$$\langle 1, f \rangle = \langle 1, \theta_i \overline{\theta}_j f' \rangle = w(i) \delta_{r,s} \delta_{i, j^T} \langle 1, f' \rangle.$$

This proves that $\langle f, 1\rangle^* = \langle 1, f\rangle$. Similarly, one shows $\langle 1, g\rangle^* = \langle g, 1\rangle$, where $l(g) > 0$ and g begins with a θ.

For the case where $l(f) > 0$ and $l(g) > 0$ and both f and g begin with a θ, we write $f = \theta_i \overline{\theta}_j f'$ and $g = \theta_k \overline{\theta}_l g'$, where i, j, k, l are multi-indices of lengths r, s, t, u respectively and f', g' begin with a θ. Then we see by induction that

$$\langle g, f\rangle^* = \langle \theta_k \overline{\theta}_l g', \theta_i \overline{\theta}_j f'\rangle^* = \big(w(k, j^T)\, \delta_{t+s,u+r}\, \delta_{(k,j^T),(i,l^T)}\, \langle g', f'\rangle\big)^*$$
$$= w(i, l^T)\, \delta_{r+u,s+t}\, \delta_{(i,l^T),(k,j^T)}\, \langle f', g'\rangle = \langle f, g\rangle.$$

The proofs for words that begin with $\overline{\theta}$ are similar. The final case is if one of the pair f, g begins with a θ and the other begins with a $\overline{\theta}$. But then $\langle f, g\rangle = 0$ as well as $\langle g, f\rangle = 0$. So in this final case the identity is trivially true. ∎

While the sesqui-linear form is complex symmetric according to this proposition, when $n \geq 2$ it does not satisfy the nice properties with respect to the ∗-operation as were given in [3]. We recall that those properties are

$$\langle f_1, f_2 g\rangle = \langle f_1 g^*, f_2\rangle, \tag{2.3}$$
$$\langle f_1, f_2 g\rangle = \langle f_1 f_2^*, g\rangle, \tag{2.4}$$

where $f_1, f_2 \in \mathcal{P}$ and $g \in \mathcal{A}$. It seems reasonable to conjecture that these identities do hold for $n = 1$. This detail is left to the reader's further consideration.

For the first property (2.3) the counterexample is provided by taking $f_1 = \theta_1$, $f_2 = \theta_1\theta_2$ and $g = \overline{\theta}_2\theta_1\overline{\theta}_1$. Then we have on the one hand that

$$\langle f_1, f_2 g\rangle = \langle \theta_1, \theta_1\theta_2\overline{\theta}_2\theta_1\overline{\theta}_1\rangle = w(1,2)\delta_{(1,2)(1,2)}\langle 1, \theta_1\overline{\theta}_1\rangle = w(1,2)w(1) \neq 0.$$

On the other hand

$$\langle f_1 g^*, f_2\rangle = \langle \theta_1\theta_1\overline{\theta}_1\theta_2, \theta_1\theta_2\rangle = 0.$$

For the second property (2.4), we take $f_1 = f_2 = \theta_1$ and $g = \theta_2\overline{\theta}_2$. Then we have for the left side that

$$\langle f_1, f_2 g\rangle = \langle \theta_1, \theta_1\theta_2\overline{\theta}_2\rangle = w(1,2)\delta_{(1,2),(1,2)}\langle 1, 1\rangle = w(1,2) \neq 0.$$

But for the right side we get

$$\langle f_1 f_2^*, g\rangle = \langle \theta_1\overline{\theta}_1, \theta_2\overline{\theta}_2\rangle = w(1,2)\delta_{(1,2),(2,1)}\langle 1, 1\rangle = 0.$$

Thus this example is not compatible with all of the general theory presented in [3] when $n \geq 2$. But it still is an illuminating example as we shall discuss in more detail a bit later on. However, we do have a particular case of (2.3) in this example.

The only change from (2.3) in the following is that now $g \in \mathcal{P} \cup \mathcal{P}^*$ is required instead of $g \in \mathcal{A}$.

Theorem 2.2 *Suppose that* $f_1, f_2, \in \mathcal{P}$ *and* $g \in \mathcal{P} \cup \mathcal{P}^*$. *Then*

$$\langle f_1, f_2 g\rangle = \langle f_1 g^*, f_2\rangle$$

Proof We first prove the result for $g \in \mathcal{P}$. It suffices to consider $f_1 = \theta_i$, $f_2 = \theta_j$ and $g = \theta_k$ for multi-indices i, j, k. Then we get

$$\langle f_1, f_2 g\rangle = \langle \theta_i, \theta_j \theta_k\rangle = \langle \theta_i, \theta_{(j,k)}\rangle = w(i)\delta_{i,(j,k)}.$$

Next the other side evaluates to

$$\langle f_1 g^*, f_2\rangle = \langle \theta_i (\theta_k)^*, \theta_j\rangle = \langle \theta_i \overline{\theta}_{k^T}, \theta_j\rangle = w(i)\delta_{i,(j,k)},$$

using $(k^T)^T = k$. And so the identity holds in this case.

Next suppose that $g \in \mathcal{P}^*$. Then we apply the result of the first case to the element $g^* \in \mathcal{P}$. And that will prove this second case as the reader can check by using Theorem 2.1. ∎

Continuing our comments about why this is an illuminating example, let us note that it satisfies the first seven of the eight properties used for the more general theory presented in [3]. While it does not satisfy in general the eighth property (that T_g and T_{g^*} are adjoints on the domain $\mathcal{P}$ for all $g \in \mathcal{A}$), it satisfies the weaker version of this property given in Theorem 2.5 below.

According to the general theory we have to find a set Φ which must be a Hamel basis of $\mathcal{P}$ as well as being an orthonormal set. Clearly, the candidate is

$$\{\theta_{i_1} \cdots \theta_{i_r} \mid 1 \le i_i \le n, \dots, 1 \le i_r \le n\},$$

the words in the sub-alphabet $\{\theta_1, \dots, \theta_n\}$. And this almost works. We need only to normalize these words. Taking $f = \theta_{i_1} \cdots \theta_{i_r}$ and $g = \theta_{k_1} \cdots \theta_{k_t}$ in (2.2) (so we have $s = u = 0$, $f' = 1$ and $g' = 1$) we get

$$\langle f, g\rangle = w(i)\delta_{r,t}\delta_{i,k},$$

where $i = (i_1, \dots, i_r)$ and $k = (k_1, \dots, k_t)$ are multi-indices of lengths r, t respectively. In particular, $\langle f, g\rangle = 0$ if $f \ne g$. On the other hand, $\langle f, f\rangle = w(i) > 0$. So we define $\varphi_i := w(i)^{-1/2}\theta_{i_1} \cdots \theta_{i_r} = w(i)^{-1/2}\theta_i$, and the orthonormal Hamel basis is defined by

$$\Phi := \{\varphi_i \mid i = (i_1, \dots, i_r) \text{ with } r \ge 0,\ 1 \le i_i \le n, \dots, 1 \le i_r \le n\}.$$

This argument shows that the complex symmetric sesqui-linear form restricted to $\mathcal{P}$ is positive definite, that is, it is an inner product. We let $\mathcal{H}$ denote the completion of $\mathcal{P}$ with respect to this inner product. Then Φ is an orthonormal basis of $\mathcal{H}$. However, it is sometimes more convenient to work with the orthogonal (but perhaps not orthonormal) set $\{\theta_i = \theta_{i_1} \cdots \theta_{i_r}\}$ of $\mathcal{H}$.

We now have enough information about the sesqui-linear form in order to define the projection $P : \mathcal{A} \to \mathcal{P}$.

Definition 2.3 Let $g \in \mathcal{A}$ be given. Then define

$$Pg := \sum_{\varphi_i \in \Phi} \langle \varphi_i, g \rangle \, \varphi_i. \tag{2.5}$$

This will be well defined when we show in a moment that the sum on the right side of (2.5) is finite. If this were a Hilbert space setting, we could write $P = \sum_i |\varphi_i\rangle\langle\varphi_i|$ in Dirac notation, and P would be the orthogonal projection onto the closed subspace $\mathcal{P}$. Anyway, this abuse of notation motivates the definition of P. Given that P is well-defined, it is clear that P is linear, that it acts as the identity on $\mathcal{P}$ (since Φ is an orthonormal basis of $\mathcal{P}$) and that its range is $\mathcal{P}$.

Theorem 2.3 *The sum on the right side of* (2.5) *has only finitely many non-zero terms. Consequently, P is well-defined.*

Proof Take $g \in \mathcal{A}$. It suffices to show that $\langle \theta_i, g \rangle = 0$ except for finitely many multi-indices i, since φ_i is proportional to θ_i. So it suffices to calculate $\langle \theta_i, g \rangle$ for all possible multi-indices i. We do this by cases.

If g begins with a $\overline{\theta}$, then $\langle \theta_i, g \rangle = 0$ for all multi-indices $i \neq \emptyset$, the empty multi-index. (Note that $\theta_\emptyset = 1$.) It follows that all of the terms, except possibly one term, on the right side of (2.5) are 0 and so $P(g) = \langle 1, g \rangle \, 1$ in this case. For the particular case $g = 1$ (i.e., $l(g) = 0$) we have that $P(1) = 1$, using $\langle 1, 1 \rangle = 1$.

So the only remaining the case is when g begins with a θ and $l(g) > 0$. Then, using $g = \theta_k \overline{\theta}_l g'$ for multi-indices k, l of lengths $t \geq 1$ and $u \geq 0$ respectively and g' begins with a θ, we have

$$\begin{aligned} \langle \theta_i, g \rangle &= \langle \theta_{i_1} \cdots \theta_{i_r}, \theta_{k_1} \cdots \theta_{k_t} \overline{\theta}_{l_1} \cdots \overline{\theta}_{l_u} g' \rangle \\ &= w(i, l^T) \, \delta_{r+u,t} \, \delta((i, l^T), k) \, \langle 1, g' \rangle \\ &= w(k) \, \delta_{r+u,t} \, \delta((i, l^T), k) \, \langle 1, g' \rangle. \end{aligned} \tag{2.6}$$

Whether this is non-zero now is the question. More explicitly, for a given g of this form how many θ_i's are there such that this expression could be non-zero? However, if the factor $\delta((i, l^T), k)$ is non-zero, then we have necessarily that the multi-index $i = (i_1, \ldots, i_r)$ of variable length $r \geq 0$ forms the initial r entries in the given multi-index k of length $t \geq 1$. Thus for a given g there are at most finitely many θ_i for which (2.6) could be non-zero. So the sum on the right side of (2.5) has only finitely many non-zero terms, and P is well-defined. ∎

Theorem 2.4 *P is symmetric with respect to the sesqui-linear form, that is, $\langle Pf, g\rangle = \langle f, Pg\rangle$ for all $f, g \in \mathcal{A}$.*

Proof Using Theorem 2.1 to justify the third equality, we calculate

$$\begin{aligned}\langle Pf, g\rangle &= \Big\langle \sum_i \langle \varphi_i, f\rangle \varphi_i, g\Big\rangle = \sum_i \Big\langle \langle \varphi_i, f\rangle \varphi_i, g\Big\rangle = \sum_i \langle f, \varphi_i\rangle \langle \varphi_i, g\rangle \\ &= \sum_i \Big\langle f, \langle \varphi_i, g\rangle \varphi_i\Big\rangle = \Big\langle f, \sum_i \langle \varphi_i, g\rangle \varphi_i\Big\rangle = \langle f, Pg\rangle.\end{aligned}$$

We also used the finite additivity of the sesqui-linear form in each entry, since the sums have only finitely many non-zero terms. ∎

This result says that P has *an* adjoint on $\mathcal{A}$, namely P itself. Since the sesqui-linear form may be degenerate, adjoints need not be unique.

Theorem 2.5 *Suppose that $g \in \mathcal{P} \cup \mathcal{P}^*$. Then for all $f_1, f_2 \in \mathcal{P}$ we have $\langle f_1, T_g f_2\rangle = \langle T_{g^*} f_1, f_2\rangle$.*

Proof Using the previous result and Theorem 2.2 we calculate

$$\begin{aligned}\langle f_1, T_g f_2\rangle &= \langle f_1, P(f_2 g)\rangle = \langle Pf_1, f_2 g\rangle = \langle f_1, f_2 g\rangle = \langle f_1 g^*, f_2\rangle \\ &= \langle f_1 g^*, Pf_2\rangle = \langle P(f_1 g^*), f_2\rangle = \langle T_{g^*} f_1, f_2\rangle.\end{aligned}$$ ∎

Since the sesqui-linear form is an inner product when restricted to $\mathcal{P}$, T_{g^*} is the *unique* adjoint of T_g on $\mathcal{P}$. Symmetrically, T_g is the *unique* adjoint of T_{g^*} on $\mathcal{P}$.

Next, for all $\phi \in \mathcal{P}$ we define the *creation* and *annihilation operators* associated to the variables $\theta_j, \overline{\theta}_j$ for $1 \le j \le n$ by

$$A_j^\dagger(\phi) := T_{\theta_j}(\phi) = P(\phi\theta_j) = \phi\theta_j \quad \text{and} \quad A_j(\phi) := T_{\overline{\theta}_j}(\phi) = P(\phi\overline{\theta}_j),$$

respectively. These are operators densely defined in $\mathcal{H}$ sending $\mathcal{P}$ to itself. By Theorem 2.5 the operators $A_j^\dagger$ and A_j are adjoints of each other on the domain $\mathcal{P}$.

We now evaluate these operators on the basis elements φ_i of $\mathcal{P}$, where i is a multi-index. First, for the creation operator we have

$$\begin{aligned}A_j^\dagger(\varphi_i) &= P(\varphi_i\theta_j) = \varphi_i\theta_j = w(i)^{-1/2}\,\theta_i\,\theta_j = w(i)^{-1/2}\,\theta_{(i,j)} \\ &= \left(\frac{w(i,j)}{w(i)}\right)^{1/2} \varphi_{(i,j)}.\end{aligned}$$

Here $j = 1, \ldots, n$ and also j denotes the multi-index with exactly one entry, namely the integer j. Also we are using the notation (i, j) for the multi-index with the integer j concatenated to the right of the multi-index i. It follows that the kernel of $A_j^\dagger$ is zero as the reader can check. Also, the weight of the 'higher' state $\varphi_{(i,j)}$ appears in the numerator while the weight of the 'lower' state φ_i is in the denominator. This turns out to be consistent with the way the weights (which are products of factorials) work in the case of standard quantum mechanics.

Next, for the annihilation operator A_j for $1 \le j \le n$ we have to evaluate $A_j(\varphi_k) = P(\varphi_k \overline{\theta}_j) = \sum_i \langle \varphi_i, \varphi_k \overline{\theta}_j \rangle \varphi_i$ for every multi-index k. To do this, consider

$$\begin{aligned}\langle \varphi_i, \varphi_k \overline{\theta}_j \rangle &= (w(i)w(k))^{-1/2} \langle \theta_i, \theta_k \overline{\theta}_j \rangle \\ &= (w(i)w(k))^{-1/2}\, w(k)\, \delta_{r+1,t}\, \delta((i, j^T), k)\, \langle 1, 1 \rangle \\ &= \left(\frac{w(k)}{w(i)} \right)^{1/2} \delta_{r+1,t}\, \delta((i, j), k),\end{aligned}$$

where the multi-indices $i = (i_1, \dots, i_r)$ and $k = (k_1, \dots, k_t)$ have lengths r and t respectively. We also used $j^T = j$, since j is a multi-index with exactly one entry in it. The only possible non-zero value occurs when the concatenated multi-index (i, j) is equal to the multi-index k. So for $k \neq (i, j)$ we have that $\langle \varphi_i, \varphi_k \overline{\theta}_j \rangle = 0$. If the last entry in the multi-index k is not j (i.e., $k_t \neq j$), then $k \neq (i, j)$ for all multi-indices i. Consequently, in this case we calculate

$$A_j(\varphi_k) = P(\varphi_k \overline{\theta}_j) = \sum_i \langle \varphi_i, \varphi_k \overline{\theta}_j \rangle \varphi_i = 0.$$

Therefore, in this example the annihilation operator A_j has infinite dimensional kernel. As a very particular case, we take $k = \emptyset$, the empty multi-index, and get that $A_j(\varphi_\emptyset) = A_j(1) = 0$ for all $1 \le j \le n$, that is, $1 \in \cap_{j=1}^n \ker A_j$. So 1 is a normalized vacuum state in $\mathcal{H}$.

On the other hand if $k = (i, j)$ for some clearly unique multi-index i (and in particular $r + 1 = t$), then we find that

$$\langle \varphi_i, \varphi_{(i,j)} \overline{\theta}_j \rangle = \left(\frac{w(i, j)}{w(i)} \right)^{1/2} > 0,$$

and consequently in this case

$$A_j(\varphi_{(i,j)}) = P(\varphi_{(i,j)} \overline{\theta}_j) = \left(\frac{w(i, j)}{w(i)} \right)^{1/2} \varphi_i.$$

Again, the weight of the 'higher' state $\varphi_{(i,j)}$ appears in the numerator while the weight of the 'lower' state φ_i is in the denominator. And again this is consistent with standard quantum mechanics.

It is now an extended exercise to compute the commutation relations of these operators. For example, $[A_j^\dagger, A_k^\dagger] \neq 0$ if $j \neq k$, since $\theta_j \theta_k \neq \theta_k \theta_j$. The formulas for these relations are simpler if we take the weights to be $w_i = w_{i_1,\dots,i_r} := \mu_{i_1} \cdots \mu_{i_r}$ for positive real numbers $\mu_1, \dots, \mu_n$.

3 Concluding Remarks

I conclude with possibilities for future related research concerning algorithms that manipulate the words in the basis of the algebra $\mathcal{A}$.

The sesqui-linear form on $\mathcal{A}$ serves mainly to define the projection operator P, which is crucial in this quantization theory. Using this, creation operators tack on a holomorphic variable on the right (up to a weight factor), while annihilation operator chop off the appropriate holomorphic variable on the right, if present (again up to a weight), and otherwise map the word to zero. One can define other projection operators that see more deeply into the word, rather than looking at only the rightmost part of the word. In general each occurrence of $\overline{\theta}_j$ is erased while at the same time some corresponding occurrence of θ_j is also erased. The end result is a word with no $\overline{\theta}$'s at all. Moreover, if the original word had no $\overline{\theta}$'s to begin with, then it will remain unchanged. Basically, the projection map is an algorithm that scans a word from one end to the other, eliminating all $\overline{\theta}$'s and some θ's.

There are many such algorithms. To give the reader an idea of this, let us consider scanning a word from left to right until we hit the first occurrence of $\overline{\theta}_j$ for some j. We change the word by eliminating this $\overline{\theta}_j$ and the rightmost occurrence of θ_j to the left of this $\overline{\theta}_j$, if there is such an occurrence. If there is no occurrence of θ_j to the left, we define P on this word to be zero. Otherwise, we continue scanning from our current position in the word looking for the next $\overline{\theta}_k$ for some k. We repeat the same procedure. Since the word is finite in length, this algorithm will terminate. At such time there will be no occurrences of $\overline{\theta}$'s left. The resulting word (or zero) will be P evaluated on the original word.

The reader is invited to produce other algorithms for finding one (or various) occurrences of θ_j to pair with an identified occurrence of $\overline{\theta}_j$. There are other deterministic algorithms for sure, but there are even stochastic algorithms as well. These stochastic algorithms could pair a random number of occurrences of θ_j, including zero occurrences with non-zero probability, with a given occurrence of $\overline{\theta}_j$. Also, the locations of these occurrences could be random. Then all Toeplitz operators, including those of creation and annihilation, would become random operators.

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References

1. V. Bargmann, *On a Hilbert space of analytic functions and its associated integral transform. I*, Commun. Pure Appl. Math. **14** (1961) 187–214.
2. B.C. Hall, *Holomorphic methods in analysis and mathematical physics*, in: “First Summer School in Analysis and Mathematical Physics” (S. Pérez-Esteva and C. Villegas-Blas, Eds.), 1–59, Contemp. Math., **260**, Am. Math. Soc., 2000.
3. S.B. Sontz, *Toeplitz Quantization without Measure or Inner Product*, Proceedings of: Geometric Methods in Physics. XXXII Workshop 2013. Trends in Mathematics, (2014) 57–66. arXiv:1312.0588

Making the Case for Pseudodifferential Arithmetic

André Unterberger

To our longtime friend Nikolai Vasilevski

Abstract Let $\Gamma = SL(2,\mathbb{Z})$ act in the plane by linear changes of coordinates. The resulting spectral theory of the automorphic Euler operator, which refines the theory of modular forms of non-holomorphic type, has definite advantages. One of these lies in the enrichment gained by interpreting automorphic distributions as symbols in the Weyl pseudodifferential calculus: the formula making the sharp composition of modular distributions explicit is given in terms of L-function theory. On the other hand, starting from distributions of arithmetic interest for symbols, we obtain operators the structure of which expresses itself nicely in terms of congruence arithmetic, providing a possible new approach to the Riemann hypothesis.

1 Introduction

This paper is based on the following ideas. First, that pseudodifferential analysis and arithmetic can cooperate in an interesting way. Next, that in order to make such an association possible, one should start with refurbishing classical modular form theory, here relative to the full modular group for simplicity, by letting the plane $\mathbb{R}^2$ take the place usually ascribed to the hyperbolic half-plane: we believe that other advantages originate also from this substitution. The developments towards this program have been obtained in a series of books going back a number of years: they will get a better focus from the concentrated exposition that follows, leaving aside most (usually lengthy) proofs.

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The first section deals with a new way to realize pseudodifferential analysis, which in an unexpected way lets the main objects of modular form theory appear in a fully natural way: we consider this approach to modular form theory as being especially suitable for analysts. Recall that pseudodifferential analysis starts with a defining rule Op associating linear operators from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$ to distributions in $\mathcal{S}'(\mathbb{R}^2)$: the (unique) distribution a given operator originates from is called the symbol of this operator. The point of view developed in Sect. 2 consists in letting symbols appear as (continuous and discrete) superpositions of elements of the kind $\mathfrak{s}_b^a$, with $\mathfrak{s}_b^a(x,\xi) = e^{2i\pi ax}\delta(\xi - b)$.

There are definite advantages originating from this choice in pseudodifferential analysis. Indeed, denoting as # the (far from everywhere defined) bilinear operation on symbols that corresponds to the composition of the associated operators, one has the equation, not to be understood in a pointwise sense,

$$\mathfrak{s}_{b_1}^{a_1} \,\#\, \mathfrak{s}_{b_2}^{a_2} = \delta(a_1 + a_2 - b_1 + b_2)\; \mathfrak{s}_{a_1+b_2}^{a_1+a_2}, \tag{1.1}$$

which has interesting features. First, the presence of delta factors implies that a general #-product formula will necessitate a 1-dimensional integral only, while a 4-dimensional one is necessary in the traditional approach.

The main application of this formula is obtained when coupling it with the process of decomposing all symbols (the two ones one starts with as well as their #-product) into their homogeneous components. One is left with the problem of analyzing the sharp product of two factors of the kind $|\xi|^{-1-\nu} \exp\left(2i\pi \frac{kx}{\xi}\right)$. In the case when either $k_1 + k_2 \neq 0$ or $k_1 > 0$, and assuming that $|\mathrm{Re}\,(\nu_1 \pm \nu_2)| < 1$, the composition of the two associated operators acts from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$, thus has a symbol in the usual sense, the homogeneous parts of which were computed in [5, Section 4.5]. The situation can be saved in the remaining case by applying the Euler operator $2i\pi\mathcal{E} = 1 + x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi}$ to the result. Applying the differential operator $\mathcal{E}$ to a symbol amounts to replacing the associated operator A by $PAQ - QAP$, where $Q = x$ and $P = \frac{1}{2i\pi}\frac{d}{dx}$ are the infinitesimal operators of Heisenberg's representation: this gives an operator-theoretic meaning to the trick above, which originated from geometrical considerations.

Let us introduce now (Sect. 3) automorphic distributions, by definition tempered distributions invariant under the action, by linear changes of coordinates, of the arithmetic group $\Gamma = SL(2,\mathbb{Z})$, and modular distributions, to wit automorphic distributions homogeneous of some degree $-1-\nu$. A linear operator Θ_0, with a nice pseudodifferential interpretation—remindful of the link between pseudodifferential analysis and Toeplitz operator theory—sends automorphic and modular distributions in $\mathbb{R}^2$ to automorphic functions and modular forms of the non-holomorphic type, in the hyperbolic half-plane Π. But, in order to characterize an automorphic distribution $\mathfrak{S}$, we need to know, besides $\Theta_0\mathfrak{S}$, the image under Θ_0 of the transform of $\mathfrak{S}$ under the symplectic Fourier transformation. Automorphic distributions (in $\mathbb{R}^2$) carry slightly more information than automorphic functions (in Π).

In Sect. 4, we shall show that the measures $\mathfrak{s}_b^a$ used (for pseudodifferential purposes) in Sect. 2 lead immediately to the construction of a full set of modular distributions. It suffices, introducing a character χ of $\mathbb{Q}^\times$, to consider the series

$$\mathfrak{T}_\chi = \pi \sum_{m,n \in \mathbb{Z}^\times} \chi\left(\frac{m}{n}\right) \mathfrak{s}_n^m, \tag{1.2}$$

replacing when χ is the trivial character the subscript $mn \neq 0$ by $|m| + |n| \neq 0$. In this case, decomposing $\mathfrak{T}_\chi$ into homogeneous components defines the Eisenstein distributions $\mathfrak{E}_{i\lambda}$, where $\mathfrak{E}_\nu$ makes sense for $\nu \in \mathbb{C}$, $\nu \neq \pm 1$: it is a modular distribution homogeneous of degree $-1-\nu$. For a special, discrete set of real numbers λ, the component of degree $-1-i\lambda$ of $\mathfrak{T}_\chi$ will also be automorphic for some choices of a non-trivial character χ, and we shall call it a Hecke distribution.

Needless to say, the terminology has been chosen to correspond, under the two-to-one map Θ_0, to the one used in non-holomorphic modular form theory. Eisenstein distributions are transformed to Eisenstein series and Hecke distributions are transformed to Hecke eigenforms. A more difficult question, treated in Sect. 6, consists in defining a Hilbert space $L^2(\Gamma\backslash\mathbb{R}^2)$, to substitute for the classical space $L^2(\Gamma\backslash\Pi)$. There is indeed an independent construction, a rather hard one because there is no fundamental domain for the action of Γ in $\mathbb{R}^2$ (by linear changes of coordinates). The Euler operator has a natural self-adjoint realization in $L^2(\Gamma\backslash\mathbb{R}^2)$, and general elements in this space decompose as integrals of Eisenstein distributions, on the spectral line $\mathrm{Re}\,\nu = 0$, and series of Hecke distributions: this is an analogue of the Roelcke-Selberg expansion theorem from automorphic function theory.

Given two modular distributions $\mathfrak{N}_1$ and $\mathfrak{N}_2$ (either can be an Eisenstein or a Hecke distribution), one can by the methods indicated in Sect. 2 give a meaning (Sect. 5) to the classically undefined sharp composition $\mathfrak{N}_1 \# \mathfrak{N}_2$. A full decomposition of the result, as an integral and series of Eisenstein and Hecke distributions, can be obtained. The coefficients of the decomposition are quite interesting: they involve the L-functions of the modular distributions concerned by each term of the expansion under consideration (two in input, one in output), as well as the results of bi- or trilinear operations on such L-functions. However, the results will be very briefly hinted at, and fully referenced, since the whole book [5] was needed to answer this question.

In the spectral theory of the automorphic Euler operator, the non-trivial zeros of the Riemann zeta function show up not as eigenvalues, but as resonances. The consideration of special automorphic symbols, integrals of Eisenstein distributions (no Hecke distributions appear in this context) lead to equivalent formulations of the Riemann hypothesis (Sect. 8). The symbol

$$\mathfrak{T}_\infty^1(x,\xi) = \sum_{|j|+|k| \neq 0} a_1((j,k))\,\delta(x-j)\,\delta(\xi-k), \tag{1.3}$$

with $a_1(r) = \prod_{p|r}(1-p)$, decomposes exactly (this is not a spectral decomposition) as a series of Eisenstein distributions $\mathfrak{E}_{-\nu}$, where ν runs through the set of zeros, both trivial and non-trivial, of zeta: in the usual way, (j, k) denotes the g.c.d. of the integers j and k, when not both zero. It is therefore not surprising that one should be able to express the Riemann hypothesis in terms involving the operator with symbol $\mathfrak{T}^1_\infty$ or, which is just as well when tested on pairs of functions compactly supported in $[0, \infty[$, $\mathfrak{T}^1_M$, where M is a squarefree integer and one replaces $a_1((j, k))$ by $a_1((j, k, M))$. Now, the structure of this operator expresses itself nicely in terms of congruence arithmetic, and it transfers under an appropriate map to a linear endomorphism of the space $\mathbb{C}^{4M^2}$ with a Eulerian structure. Whether, in possession of rather sharp methods of pseudodifferential analysis, one could use this information towards a better understanding of the zeros of zeta, is far from sure. At least, it takes one to the question of up to which point congruence arithmetic and real analysis can be made to work together, which may well be the heart of the Riemann hypothesis.

In a very short last section, we show on an example of historical importance (that of the Ramanujan Delta function) that modular forms of the holomorphic type, just as those of non-holomorphic type (as introduced in Sects. 3 and 4) are in a very natural way associated to pseudodifferential analysis.

2 A Composition Formula

Given a tempered distribution $\mathfrak{S} = \mathfrak{S}(x, \xi)$ in $\mathbb{R}^d \times \mathbb{R}^d$, one considers the "pseudodifferential" operator with symbol $\mathfrak{S}$, to wit the linear operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ weakly defined by the equation

$$\left(\mathrm{Op}^{[2]}(\mathfrak{S})\, u\right)(x) = 2^{-d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathfrak{S}\left(\frac{x+y}{2},\, \xi\right) e^{i\pi \langle x-y, \xi\rangle}\, u(y)\, dy\, d\xi. \qquad (2.1)$$

The superscript [2] expresses the fact that this is really the definition corresponding to having chosen 2 for Planck's constant. There is no doubt that this is the best normalization in "pseudodifferential arithmetic", by which we shall essentially mean pseudodifferential analysis with symbols of arithmetic interest. But choosing 1 instead (to wit, replacing $2^{-d} e^{i\pi\langle x-y,\xi\rangle}$ by $e^{2i\pi\langle x-y,\xi\rangle}$) yields the benefit that the trace and Hilbert-Schmidt norms of an operator, when defined—which is never the case in pseudodifferential arithmetic—correspond to the integral and L^2-norm of its symbol. We shall only use the version (2.1), but we cannot yet dispense with the superscript [2] because quotations from previous work will sometimes be needed. One has $\mathrm{Op}^{[1]}\left(\mathrm{Resc}\, \mathfrak{S}\right) = U[2]\, \mathrm{Op}^{[2]}(\mathfrak{S})\, U[2]^{-1}$ with $(\mathrm{Resc}\, \mathfrak{S})(x, \xi) = \mathfrak{S}(x\sqrt{2}, \xi\sqrt{2})$ and $(U[2]\, u)(x) = 2^{\frac{1}{4}} u(x\sqrt{2})$, which makes translations easy.

There is a well-known "composition formula", which expresses the symbol $\mathfrak{S}_1 \overset{[2]}{\#} \mathfrak{S}_2$ of the composition of two operators with symbols $\mathfrak{S}_1$ and $\mathfrak{S}_2$, assuming of course that this composition makes sense, which is the case if the image of $\mathrm{Op}^{[2]}(\mathfrak{S}_2)$ is contained in a domain of sorts of $\mathrm{Op}^{[2]}(\mathfrak{S}_1)$. The formula (there is nothing in it but Weyl's exponential version of Heisenberg's commutation relation, after one has written $\mathfrak{S}(x,\xi)$, in both cases, as a superposition of exponentials $e^{i\pi(\langle a,x\rangle+\langle b,\xi\rangle)}$), is expressed by means of a $(4d)$-dimensional integral. We wish, in this section, to put into evidence the benefits obtained in some cases of replacing the realization of symbols as superpositions of exponentials by the formula

$$\mathfrak{S}_f = \int_{\mathbb{R}^d\times\mathbb{R}^d} f(a,b)\,\mathfrak{s}_b^a\,da\,db \qquad \text{with} \qquad \mathfrak{s}_b^a(x,\xi) = e^{2i\pi\langle a,x\rangle}\delta(\xi-b). \tag{2.2}$$

A first one is that the above-mentioned $(4d)$-dimensional integral will become a d-dimensional one; at the same time, one can give at one stroke the formula for the composition of an arbitrary number of operators. Indeed [6, p. 107], one has the equation

$$\mathfrak{s}_{b_1}^{a_1} \overset{[2]}{\#} \dots \overset{[2]}{\#} \mathfrak{s}_{b_k}^{a_k} = \prod_{j=1}^{k-1} \delta(a_j + a_{j+1} - b_j + b_{j+1})\,\mathfrak{s}\left[a_1+\cdots+a_k,\ a_1+\cdots+a_{k-1}+b_k\right] \tag{2.3}$$

or, in the case when $d=1$, assuming that $a_1+\cdots+a_k \neq 0$,

$$\mathfrak{s}_{b_1}^{a_1} \overset{[2]}{\#} \dots \overset{[2]}{\#} \mathfrak{s}_{b_k}^{a_k} = \prod_{j=1}^{k-1} \delta(a_j + a_{j+1} - b_j + b_{j+1})\,\mathfrak{s}\left[a_1+\cdots+a_k,\ \frac{a_1b_1+\cdots+a_kb_k}{a_1+\cdots+a_k}\right]. \tag{2.4}$$

Taking two factors, one obtains from (2.2) and (2.3) the formula $\mathfrak{S}_{f_1} \overset{[2]}{\#} \mathfrak{S}_{f_2} = \mathfrak{S}_f$, with

$$f(a,b) = \int_{\mathbb{R}^d} f_1(x, a+b-x)\, f_2(a-x, b-x)\,dx. \tag{2.5}$$

Specializing from now on in the case when $d=1$, we systematically decompose symbols into homogeneous components, extending to appropriate distributions the rule, valid if $h \in \mathcal{S}_{\mathrm{even}}(\mathbb{R}^2)$ (one may, at the price of introducing an extra parameter ± 1, dispense with the parity condition, but this is not necessary for our present purpose),

$$h(x,\xi) = \frac{1}{i}\int_{\mathrm{Re}\,\nu=-a} h_\nu(x,\xi)\,d\nu, \qquad a<1,\ |x|+|\xi|\neq 0, \tag{2.6}$$

with

$$h_\nu(x,\xi) = \frac{1}{4\pi}\int_{-\infty}^{\infty} |t|^\nu h(tx,t\xi)\,dt, \qquad \mathrm{Re}\,\nu > -1, \tag{2.7}$$

a consequence of the Mellin (or Fourier) inversion formula. The function h_ν, undefined at the origin, is homogeneous of degree $-1-\nu$. For instance, with $(\mathfrak{s}_b^a)_{\mathrm{even}} = \frac{1}{2}\left(\mathfrak{s}_b^a + \mathfrak{s}_{-b}^{-a}\right)$, one has [6, p. 116]

$$\left[\left(\mathfrak{s}_b^a\right)_{\mathrm{even}}\right]_{i\lambda}(x,\xi) = \frac{1}{4\pi}|b|^{i\lambda}|\xi|^{-1-i\lambda}\exp\left(2i\pi\frac{abx}{\xi}\right), \qquad \lambda \in \mathbb{R}^\times,\ b \neq 0. \tag{2.8}$$

A pointwise application of (2.4) (in the case of two factors) is not possible: the result would be meaningless when $a_1 + a_2 - b_1 + b_2 \neq 0$, and would be zero in the remaining case. However, decomposing the right-hand side of (2.4) into homogeneous components will involve a factor of the kind $\left|\frac{a_1b_1+a_2b_2}{a_1+a_2}\right|^{-1-i\lambda}$ against which, in general, the troublesome delta factor can be tested (one delta factor will remain in the result). This will not be the case if $a_1b_1 + a_2b_2 = 0$ or $a_1+a_2 = 0$: under these conditions, an examination of whether the arguments of two delta factors present are transversal or not leaves only the case when $a_1b_1+a_2b_2 = 0$ and $a_1b_1 < 0$ to worry about. Even so [6, section 6.2], one recovers a meaningful result if one first replaces the right-hand side of (2.4) by its image under the Euler operator

$$2i\pi\mathcal{E} = 1 + x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi}. \tag{2.9}$$

This repair will be fully operational after we have made it explicit, in (5.6) below, which operation on an operator corresponds to applying its symbol the differential operator $\mathcal{E}$. To sum up the (quite lengthy) developments in [5, Chapter 4]: one can in interesting cases, especially in pseudodifferential arithmetic, give a meaning to all homogeneous components of the #-composition of two symbols, except the one homogeneous of degree -1. This does not require that the composition of the two associated operators be fully meaningful in any classical sense.

3 Automorphic Distributions and Modular Distributions

An automorphic distribution is a tempered distribution $\mathfrak{S}$ in the plane, invariant under the action by linear changes of coordinates of the group $\Gamma = SL(2,\mathbb{Z})$. One can characterize $\mathfrak{S}$ by a pair of (dependent) functions $\Theta_0^{[2]}\mathfrak{S}$ and $\Theta_1^{[2]}\mathfrak{S}$ in the hyperbolic half-plane $\{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$, setting

$$\left(\Theta_0^{[2]}\mathfrak{S}\right)(z) = \langle \mathfrak{S},\ (x,\xi) \mapsto \exp\left(-\frac{\pi}{\operatorname{Im} z}|x - z\xi|^2\right)\rangle \tag{3.1}$$

and $\Theta_1^{[2]}\mathfrak{S} = \Theta_0^{[2]}\left((2i\pi\mathcal{E})\,\mathfrak{S}\right)$: the superscript [2] makes this definition coherent with the choice of 2 for Planck's constant. The two functions so obtained are automorphic in the hyperbolic half-plane Π, with a reference to the action of Γ there by fractional-linear transformations of the complex variable z. Besides, under $\Theta_0^{[2]}$ or $\Theta_1^{[2]}$, the operator $\pi^2\mathcal{E}^2$, where $2i\pi\mathcal{E}$ is the Euler operator (2.9), transfers to the operator $\Delta - \frac{1}{4}$, where $\Delta = (z - \overline{z})^2 \frac{\partial^2}{\partial z \partial \overline{z}}$ is the hyperbolic Laplacian. It follows that if an automorphic distribution $\mathfrak{S}$ is homogeneous of some degree $-1-\nu$, in which case we shall call it a modular distribution, its image under $\Theta_0^{[2]}$ or $\Theta_1^{[2]}$ is a generalized eigenfunction of Δ for the (generalized) eigenvalue $\frac{1-\nu^2}{4}$, in other words a modular form of the non-holomorphic type. But the notion of automorphic or modular distribution (in $\mathbb{R}^2$) is more precise than that of automorphic function or modular form (in Π): if two automorphic distributions are related under the symplectic Fourier transformation $\mathcal{F}^{\text{symp}}$ in $\mathbb{R}^2$ (the one with the combination $x\eta - y\xi$ in the exponent, which commutes with the action of $SL(2,\mathbb{R})$ by linear changes of coordinates with determinant 1), they have the same image under $\Theta_0^{[2]}$, and images the negative of each other under $\Theta_1^{[2]}$, so that a pair of automorphic functions is needed to characterize just one automorphic distribution.

The connection between the two environments expresses itself nicely with the help of the symbolic calculus (2.1). Let us introduce the (even and odd) functions $\phi_z^{0,[2]}$ and $\phi_z^{1,[2]}$ in $\mathcal{S}(\mathbb{R})$, depending on a point z of the hyperbolic half-plane, such that

$$\begin{aligned}\phi_z^{0,[2]}(x) &= \left(\operatorname{Im}\,(-z^{-1})\right)^{\frac{1}{4}} \exp\frac{i\pi x^2}{2\overline{z}},\\ \phi_z^{1,[2]}(x) &= 2\pi^{\frac{1}{2}} \left(\operatorname{Im}\,(-z^{-1})\right)^{\frac{3}{4}}\ x\ \exp\frac{i\pi x^2}{2\overline{z}}.\end{aligned} \tag{3.2}$$

They make up [5, p. 56] total sets in the even and odd parts of $L^2(\mathbb{R})$; they are linked to the metaplectic representation (here, a version $\text{Met}^{[2]}$) since, under any element of the metaplectic group (a twofold cover of $SL(2,\mathbb{R})$) lying above a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the way $\phi_z^{0,[2]}$ and $\phi_z^{1,[2]}$ transform can be caught, up to a scalar factor, by the transformation $z \mapsto \frac{az+b}{cz+d}$. For our present purpose, we note that, with $\kappa = 0$ or 1, one has the identity

$$\left(\Theta_\kappa^{[2]}\mathfrak{S}\right)(z) = \left(\phi_z^{\kappa,[2]}\,\middle|\,\text{Op}^{[2]}(\mathfrak{S})\,\phi_z^{\kappa,[2]}\right) \tag{3.3}$$

(we use in $L^2(\mathbb{R})$ the notation $(u\,|\,v) = \int_{-\infty}^{\infty} \overline{u}(x)\,v(x)\,dx$).

4 Eisenstein Distributions and Hecke Distributions

Considering a non-constant character χ on $\mathbb{Q}^\times$, tempered in the sense that, for some $C > 0$, one has $|\chi\left(\frac{m}{n}\right)| \leq |mn|^C$ for every pair m, n of non-zero integers, one introduces the even distribution

$$\mathfrak{T}_\chi = \pi \sum_{m,n\neq 0} \chi\left(\frac{m}{n}\right) \mathfrak{s}_n^m. \tag{4.1}$$

When χ coincides with the trivial character $\chi_0 = 1$, we extend the definition, with the difference that the domain of summation will then be $\{m, n \colon |m| + |n| \neq 0\}$ in place of $\{(m, n) \in \mathbb{Z}^2 \colon mn \neq 0\}$. The distribution $\mathfrak{T}_{\chi_0}$ is invariant under the action by linear transformations in the full group $SL(2, \mathbb{Z})$, i.e., is an automorphic distribution: this is not the case for $\mathfrak{T}_\chi$ in general.

With the help of (2.8), one obtains if χ is distinct from the trivial character the decomposition into homogeneous components of $\mathfrak{T}_\chi$: it is the integral $\int_{-\infty}^{\infty} \mathfrak{N}[\chi, i\lambda]\, d\lambda$, with

$$\langle \mathfrak{N}[\chi, i\lambda],\, h\rangle = \frac{1}{4} \sum_{m,n\neq 0} \chi\left(\frac{m}{n}\right) \int_{-\infty}^{\infty} |t|^{-1-i\lambda} \left(\mathcal{F}_1^{-1} h\right)\left(\frac{m}{t},\, nt\right) dt, \tag{4.2}$$

where $\mathcal{F}_1^{-1}$ denotes the inverse Fourier transformation with respect to the first variable of a pair.

When χ is the constant character χ_0, one obtains a decomposition of $\mathfrak{T}_{\chi_0}$, the terms of which on the line $\operatorname{Re}\nu = 0$ are denoted as $\mathfrak{E}_{i\lambda}$, where the distribution $\mathfrak{E}_\nu$ so defined when $\nu \in i\mathbb{R}$, as well as its analytic continuation with respect to ν, will be called an Eisenstein distribution: there are two extra terms, to the left and right of the above line. The distribution $\mathfrak{E}_\nu$ can be continued to the plane, with the exception of two simple poles at $\nu = \pm 1$. When $|\operatorname{Re}\nu| > 1$, it is given by the equations

$$\langle \mathfrak{E}_\nu,\, h\rangle = \frac{1}{2} \sum_{|m|+|n|\neq 0} \int_{-\infty}^{\infty} |t|^{-\nu} h(mt, nt)\, dt, \qquad \operatorname{Re}\nu < -1,$$

$$\mathfrak{E}_\nu(x, \xi) = \frac{1}{2}\, \zeta(-\nu) \sum_{(j,k)=1} |-kx + j\xi|^{-\nu-1}, \qquad \operatorname{Re}\nu > 1. \tag{4.3}$$

One has [5, p. 13]

$$\operatorname{Res}_{\nu=-1} \mathfrak{E}_\nu = -1 \qquad \text{and} \qquad \operatorname{Res}_{\nu=1} \mathfrak{E}_\nu = \delta_0, \tag{4.4}$$

where δ_0 is the unit mass at the origin of $\mathbb{R}^2$. Also, one has $\mathcal{F}^{\text{symp}} \mathfrak{E}_\nu = \mathfrak{E}_{-\nu}$ and $\mathcal{F}^{\text{symp}} \mathfrak{N}[\chi, i\lambda] = \mathfrak{N}[\chi^{-1}, -i\lambda]$ if χ is a non-trivial character.

The Eisenstein distribution $\mathfrak{E}_\nu$, automorphic and homogeneous of degree $-1-\nu$, is a modular distribution. The distribution $\mathfrak{N}[\chi, i\lambda]$ is not automorphic in general, but it satisfies the invariance property $\langle \mathfrak{N}[\chi, i\lambda]\,,\; h \circ \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)\rangle = \langle \mathfrak{N}[\chi, i\lambda]\,,\; h\rangle$ for every function $h \in \mathcal{S}(\mathbb{R}^2)$. For a special, discrete set of real numbers λ, it will also be invariant for some choices of χ under the transformation associated to the matrix $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$: in this case, it will be a modular form of degree $-1 - i\lambda$, to be called a Hecke distribution. Note that, given λ, a character χ making it possible to write a given modular distribution as $\mathfrak{N}[\chi, i\lambda]$ is far from unique. Indeed, a character of $\mathbb{Q}^\times$ is determined by the values it takes on primes, and it is always possible, for any finite set of primes, to change $\chi(p)$ to $(\chi(p))^{-1} p^{i\lambda}$ without changing $\mathfrak{N}[\chi, i\lambda]$: it is only the square of the character $\widetilde{\chi}$ such that $\widetilde{\chi}(p) = \chi(p)\, p^{-\frac{i\lambda}{2}}$ that is determined by $\mathfrak{N}[\chi, i\lambda]$.

The terminology regarding modular distributions has been chosen to fit the one used in modular form theory of the non-holomorphic type as closely as possible. If $\mathrm{Re}\,\nu < -1$, the non-holomorphic Eisenstein series $E_{\frac{1-\nu}{2}}$ is the function in the hyperbolic half-plane defined as the series

$$E_{\frac{1-\nu}{2}}(z) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z}\\ (m,n)=1}} \left(\frac{|mz-n|^2}{\mathrm{Im}\, z} \right)^{\frac{\nu-1}{2}}, \tag{4.5}$$

where (m, n) denotes the g.c.d. of the pair m, n. If one sets $\zeta^*(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\,\zeta(s)$ (so as to obtain a function invariant under the symmetry $s \mapsto 1-s$), and $E^*_{\frac{1-\nu}{2}}(z) = \zeta^*(\nu)\, E_{\frac{1-\nu}{2}}(z)$, one has the transferring identity $\Theta_0^{[2]}\mathfrak{E}_\nu = E^*_{\frac{1-\nu}{2}} = E^*_{\frac{1+\nu}{2}}$: the Eisenstein distributions $\mathfrak{E}_\nu$ and $\mathfrak{E}_{-\nu}$, while distinct (they are related under $\mathcal{F}^{\mathrm{symp}}$), have the same image under $\Theta_0^{[2]}$ [6, p. 119].

All these matters are discussed in detail in [5, Chapter 1]. For people with some experience in modular form theory, let us indicate also the following three facts. First, the images under $\Theta_0^{[2]}$ or $\Theta_1^{[2]}$ of Hecke distributions are automatically Hecke eigenforms: note that Hecke operators did not enter the definition of $\mathfrak{N}[\chi, i\lambda]$. Next, the L-function associated to a Hecke eigenform (or the one, with a more precise coefficient, associated to a Hecke distribution), has a natural spectral interpretation, as the coefficient of the decomposition of the Hecke distribution into generalized eigenfunctions of the operator $2i\pi\,\mathcal{E}^\natural = x\frac{\partial}{\partial x} - \xi\frac{\partial}{\partial \xi}$ (observe the sign change). Finally, there is a "converse theorem" identifying the fact that $\mathfrak{N}[\chi,\; i\lambda]$ is a modular distribution with a functional equation satisfied by its L-function.

To make our list of modular distributions complete, we must not forget the constant 1 and the unit mass δ_0 at the origin of $\mathbb{R}^2$, which are modular distributions of degrees of homogeneity 0 and -2, albeit uninteresting ones. The set of all real numbers $\lambda > 0$ such that $\mathfrak{N}[\chi, i\lambda]$ is a Hecke distribution for some choice of χ can be written as a sequence $(\lambda_r)_{r\geq 1}$ going to infinity: for each r, there is a finite set (possibly always reduced to one element, but whether this is the case is not known) of possible choices of classes of characters, two characters being in the same class for a given λ_r if they lead to the same Hecke distribution (cf. harmless changes of $\chi(p)$ as indicated above). We thus label the Hecke distributions $\mathfrak{N}[\chi, i\lambda_r]$ with $r \geq 1$ as $\mathfrak{N}_{r,\ell}$, the index ℓ (the finite domain of which may depend on r) standing for the set of classes of characters just defined. The degree of homogeneity of $\mathfrak{N}_{r,\ell}$ is $-1-i\lambda_r$. We also set, for $r = -1, -2, \dots, \lambda_r = -\lambda_{-r}$ and $\mathfrak{N}_{r,\ell} = \mathcal{F}^{\text{symp}}\mathfrak{N}_{-r,m}$, where m is the class of χ^{-1} if ℓ is the class of χ, obtaining in this way a complete set of Hecke distributions: note that the degree of homogeneity of $\mathfrak{N}_{r,\ell}$ is still $-1-i\lambda_r$ whatever the sign of r. It is convenient to set $\Theta_0^{[2]}\mathfrak{N}_{r,\ell} = \mathcal{N}_{|r|,\ell}$, observing that r has been changed to $|r|$ in the process.

5 The #-Product of Modular Distributions

The whole book [5] was devoted to this question, and some progress, which led to the point of view developed in Sect. 2, was made in [6, Chapter 6].

In the hyperbolic half-plane Π, the action of Γ has a fundamental domain D, which makes it possible to define a Hilbert space $L^2(\Gamma\backslash\Pi) = L^2(D)$ and a self-adjoint realization there of the Laplacian Δ. But the modular form $\mathcal{N}_{|r|,\ell}$, while normalized in the so-called Hecke sense (this observation for people familiar with modular form theory of the non-holomorphic type), is not normalized in the $L^2(D)$-sense, and we denote as $\parallel \mathcal{N}_{|r|,\ell} \parallel$ its norm there. The Roelcke-Selberg expansion theorem is the fact that any element of $L^2(\Gamma\backslash\Pi)$ can be written as the sum of an integral over the line $\operatorname{Re}\nu = 0$, with suitable coefficients, of the Eisenstein series $E_{\frac{1-\nu}{2}}$, and of a series of Hecke eigenforms $\mathcal{N}_{r,\ell}$ with $r \geq 1$. The set of Hecke eigenforms $\parallel \mathcal{N}_{r,\ell} \parallel^{-1}\mathcal{N}_{r,\ell}$ with $r \geq 1$, together with the constant $(\frac{3}{\pi})^{\frac{1}{2}}$, constitutes a complete orthonormal basis of the part of $L^2(\Gamma\backslash\Pi)$ corresponding to the discrete part of the spectrum of Δ.

It is often helpful to make use of an automorphic distribution the decomposition of which involves all Eisenstein distributions homogeneous of degrees $-1-i\lambda$ with $\lambda \in \mathbb{R}$, as well as all Hecke distributions: one such object is constructed as follows. Starting from the distribution $\mathfrak{s}_1^1(x,\xi) = e^{2i\pi x}\delta(\xi - 1)$, which is invariant under the linear transformation with matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and taking $m = 1, 2, \dots$, define ([3, p. 23] or [6, p. 120]) the distribution

$$\mathfrak{b}_m = \pi^2\mathcal{E}^2 \left(\pi^2\mathcal{E}^2 + 1\right) \dots \left(\pi^2\mathcal{E}^2 + (m-1)^2\right) \mathfrak{s}_1^1. \tag{5.1}$$

and, with $\Gamma_\infty^o = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z}\}$, consider the series, convergent in $\mathcal{S}'(\mathbb{R}^2)$,

$$\mathfrak{B}_m = \frac{1}{2} \sum_{g \in \Gamma/\Gamma_\infty^o} \mathfrak{b}_m \circ g^{-1}, \tag{5.2}$$

The image under $\Theta_0^{[2]}$ of this automorphic distribution is a special case of a collection of automorphic functions introduced by Selberg [2]. It decomposes into homogeneous components as

$$\mathfrak{B}_m = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\Gamma(m-\frac{i\lambda}{2})\,\Gamma(m+\frac{i\lambda}{2})}{\zeta^*(i\lambda)\,\zeta^*(-i\lambda)}\,\mathfrak{E}_{i\lambda}\,d\lambda + \frac{1}{2}\sum_{\substack{r,\,\ell\\ r\in\mathbb{Z}^\times}} \frac{\Gamma(m-\frac{i\lambda_r}{2})\,\Gamma(m+\frac{i\lambda_r}{2})}{\|\mathcal{N}_{|r|,\ell}\|^2}\,\mathfrak{N}_{r,\ell}. \tag{5.3}$$

When $m=0$, the right-hand side is still meaningful, the coefficient of $\mathfrak{E}_{i\lambda}$ in the integral reducing then to $\left(\zeta(i\lambda)\zeta(-i\lambda)\right)^{-1}$: but $\mathfrak{B}_0$, so defined, could not be defined by the then divergent series (5.2).

We come to the question of defining properly and computing the sharp product of any two modular distributions $\mathfrak{N}_1$ and $\mathfrak{N}_2$ (either can be an Eisenstein distribution or a Hecke distribution). Splitting the rule of composition into its commutative and anti-commutative part, i.e., writing for $j=0$ or 1

$$\left[\mathfrak{N}_1 \overset{[2]}{\#} \mathfrak{N}_2\right]^{(j)} = \frac{1}{2}\left[\mathfrak{N}_1 \overset{[2]}{\#} \mathfrak{N}_2 + (-1)^j\,\mathfrak{N}_2 \overset{[2]}{\#} \mathfrak{N}_1\right], \tag{5.4}$$

the aim is to find (in imitation of the expansion (5.3) of $\mathfrak{B}_0$) coefficients making the identity

$$\left[\mathfrak{N}_1 \overset{[2]}{\#} \mathfrak{N}_2\right]^{(j)} = \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathcal{C}^j_{i\lambda}(\mathfrak{N}_1,\,\mathfrak{N}_2)\,\frac{\mathfrak{E}_{i\lambda}}{\zeta(i\lambda)\zeta(-i\lambda)}\,d\lambda + \frac{1}{2}\sum_{r\in\mathbb{Z}^\times}\sum_{\ell} \mathcal{C}^j_{r,\ell}(\mathfrak{N}_1,\,\mathfrak{N}_2)\,\frac{\Gamma(\frac{i\lambda_r}{2})\Gamma(-\frac{i\lambda_r}{2})}{\|\mathcal{N}_{|r|,\ell}\|^2}\,\mathfrak{N}_{r,\ell} \tag{5.5}$$

valid. An almost complete answer is to be found in [6, section 6.4]. The coefficients obtained are quite interesting: besides products of zeta factors, they involve L-functions of modular forms (of the non-holomorphic type), product L-functions and even triple products of L-functions. We thus consider pseudodifferential analysis, in this arithmetic environment, as being a good approach to the theory of bi- or trilinear operations on L-functions.

We shall not reproduce the formulas here. One point needs be stressed, however. The composition of two operators A_1 and A_2 the symbols of which are modular distributions is meaningless in the usual sense, because the image of A_2 is not a subset of the domain of A_1. However, denoting as $P=\frac{1}{2i\pi}\frac{\partial}{\partial x}$ and $Q=(x)$ the infinitesimal operators of the Heisenberg representation, one has for any symbol

$\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^2)$ the identity

$$P\,\mathrm{Op}^{[2]}(\mathfrak{S})\,Q - Q\,\mathrm{Op}^{[2]}(\mathfrak{S})\,P = \mathrm{Op}^{[2]}(\mathcal{E}\,\mathfrak{S}). \tag{5.6}$$

Extending this identity makes it sometimes possible to define the image under $\mathcal{E}$ of the symbol of an operator without it being possible to give this symbol any meaning ! This is precisely what occurs in the situation under discussion, since each of the two terms of the difference $P\,(A_1A_2)\,Q - Q\,(A_1A_2)\,P$, redefined as $(PA_1)(A_2Q) - (QA_1)(A_2P)$, makes sense. Note that the knowledge of $\mathcal{E}\mathfrak{S}$, if $\mathfrak{S}$ is automorphic, is equivalent to that of $\mathfrak{S}$ up to the addition of a multiple of $\mathfrak{E}_0$.

At the end of Sect. 2, we explained in which way applying a meaningless sharp product the operator $2i\pi\mathcal{E}$, and coupling this with decompositions of symbols (the input as well as the output ones) into homogeneous components, did in some cases save the situation. The formula (5.6) completes the trick, giving it significance on the operator-theoretic level. One should however be careful with the notation # in this setting: in particular, it denotes an operation no longer quite associative.

6 The Hilbert Space $L^2(\Gamma\backslash\mathbb{R}^2)$ and the Automorphic Euler Operator

Defining in an independent way a Hilbert space $L^2(\Gamma\backslash\mathbb{R}^2)$ is more difficult than defining the space $L^2(\Gamma\backslash\Pi)$, since there is no fundamental domain for the action of $\Gamma = SL(2,\mathbb{Z})$ in $\mathbb{R}^2$ (by linear changes of coordinates): how to do this will be recalled here. Despite the fact that the pair of operators $\Theta_0^{[2]}, \Theta_1^{[2]}$ transfers in a one-to-one way automorphic distribution theory to a theory of pairs of (related) automorphic functions, the two are not equivalent in a topological sense, as indicated by (6.4) below. Indeed, as the Gamma function is rapidly decreasing on vertical lines, the transferring operator is far from having a continuous inverse: this, again, indicates that automorphic distribution theory carries more information.

Given $f \in \mathcal{S}(\mathbb{R}^2)$, we wish to define as a distribution the sum of the series $\sum_{g\in\Gamma} f \circ g^{-1}$: note that f cannot be invariant under any infinite subgroup of Γ so that, unlike the situation that occurred in (5.2), the summation over Γ needs not (and, generally, cannot) be replaced by the summation over a quotient of Γ. Consider the series, depending on a pair h, f of functions in $\mathcal{S}(\mathbb{R}^2)$,

$$\langle \mathfrak{P}, h \otimes f\rangle = \sum_{g\in\Gamma} \int_{\mathbb{R}^2} (h \circ g)(x,\xi)\, f(x,\xi)\, dx\, d\xi. \tag{6.1}$$

This series is not convergent in general, but it is (an already delicate question) if one assumes that f and h both lie in the image of $\mathcal{S}_{\text{even}}(\mathbb{R}^2)$ under the operator $2i\pi\mathcal{E}(2i\pi\mathcal{E}+1)$ [4, p. 191]. The object $\mathfrak{P}$, undistinguishable from the Poincaré summation process, is doubly automorphic in the sense that the right-hand side of (6.1) remains invariant if h and f undergo transformations by two independent elements of Γ. One may thus ask for a full decomposition of $\mathfrak{P}$ as an integral and series of tensor products of modular distributions: this is a quite lengthy, and rather difficult task, developed in [4, Chapter 5]. Introducing the Hecke distributions $\mathfrak{N}_{r,\ell}$ and their images $\mathcal{N}_{|r|,\ell}$ under $\Theta_0^{[2]}$, and defining $\epsilon_{|r|,\ell} = \pm 1$ as the parity of the Hecke eigenform $\mathcal{N}_{|r|,\ell}$ under the symmetry $z \mapsto -\bar{z}$, one obtains the formula

$$\langle \mathfrak{P},\, h \otimes f \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle \mathfrak{E}_{i\lambda},\, h \rangle \, \langle \mathfrak{E}_{-i\lambda},\, f \rangle \, \frac{d\lambda}{\zeta(i\lambda)\, \zeta(-i\lambda)} + 2 \sum_{r \in \mathbb{Z}^\times, \ell} \Gamma\left(\frac{i\lambda_r}{2}\right) \Gamma\left(-\frac{i\lambda_r}{2}\right) \epsilon_{|r|,\ell} \, \frac{\langle \mathfrak{N}_{r,\ell}\,,\, h \rangle \, \langle \mathfrak{N}_{-r,\ell}\,,\, f \rangle}{\| \mathcal{N}_{|r|,\ell} \|^2}. \tag{6.2}$$

Then, one shows the identity $\langle \mathfrak{N}_{r,\ell},\, \overline{h} \rangle = \epsilon_{|r|,\ell} \, \overline{\langle \mathfrak{N}_{-r,\ell},\, h \rangle}$, from which it follows that if $h \in (2i\pi\mathcal{E})(1 + 2i\pi\mathcal{E})\, \mathcal{S}_{\text{even}}(\mathbb{R}^2)$, one has $\langle \mathfrak{P},\, \overline{h} \otimes h \rangle \geq 0$. If $h \in (2i\pi\mathcal{E})^2(1 + 2i\pi\mathcal{E})(1 - 2i\pi\mathcal{E})\, \mathcal{S}_{\text{even}}(\mathbb{R}^2)$, the series $\mathfrak{S} = \sum_{g \in \Gamma} h \circ g$ defines a tempered distribution and the number $\langle \mathfrak{P},\, \overline{h} \otimes h \rangle$ depends only on $\mathfrak{S}$: observe, however, that distinct functions h can lead to the same distribution $\mathfrak{S}$, since the latter depends only, as can be seen, on the restriction of $(\mathcal{F}_1^{-1} h)(\eta, \xi)$ to the set where $\eta\xi \in \mathbb{Z}$. One may thus set

$$\| \mathfrak{S} \|^2_{L^2(\Gamma \backslash \mathbb{R}^2)} : = \langle \mathfrak{P},\, \bar{h} \otimes h \rangle. \tag{6.3}$$

One shows that this defines a Hilbert norm on the (incomplete) space of distributions $\mathfrak{S}$.

Finally, one cannot fail to ask how the Hilbert space $L^2(\Gamma \backslash \mathbb{R}^2)$ relates to the space $L^2(\Gamma \backslash \Pi)$. The answer is given by the equations

$$\begin{aligned} \| \Theta_0^{[2]} \mathfrak{S} \|_{L^2(\Gamma \backslash \Pi)} &= \| \Gamma(i\pi\mathcal{E})\, \mathfrak{S} \|_{L^2(\Gamma \backslash \mathbb{R}^2)} && \text{if } \mathcal{F}^{\text{symp}} \mathfrak{S} = \mathfrak{S}, \\ \| \Theta_0^{[2]} \mathfrak{S} \|_{L^2(\Gamma \backslash \Pi)} &= 2 \, \| \Gamma(1 + i\pi\mathcal{E})\, \mathfrak{S} \|_{L^2(\Gamma \backslash \mathbb{R}^2)} && \text{if } \mathcal{F}^{\text{symp}} \mathfrak{S} = -\mathfrak{S}. \end{aligned} \tag{6.4}$$

The same pair of identities also holds in a non-automorphic environment [4, p. 24], in which its proof is considerably easier.

In the Hilbert space $L^2(\Gamma \backslash \mathbb{R}^2)$, the operator $\mathcal{E}$ is self-adjoint and the Hecke distributions make up a complete set of orthogonal eigenvectors of $\mathcal{E}$ (for their normalization, use (6.4) and the equation $\Theta_0^{[2]} \mathfrak{N}_{r,\ell} = \mathcal{N}_{|r|,\ell}$); the Eisenstein distributions $\mathfrak{E}_{i\lambda}$ contribute the continuous part of the spectrum of $\mathcal{E}$. Equation (6.2) yields a "resolution of the identity", to wit a decomposition of h, or f, as an integral and series of modular distributions: simply drop the other element of the pair h, f. It is thus immediate to obtain a formula for the resolvent $(2i\pi\mathcal{E} - \mu)^{-1}$ ($\mu \notin i\mathbb{R}$) of the automorphic Euler operator. Zeros of the Riemann zeta function then show up as poles of the resolvent, i.e., as resonances.

7 An Automorphic Distribution Decomposing over the Zeros of Zeta

Define the function $a_1(r) = \prod_{p|r}(1-r)$ (as an index in a product, p is always assumed to run through primes only). Consider the pair of tempered distributions

$$\mathfrak{T}^1_\infty(x,\xi) = \sum_{|j|+|k|\neq 0} a_1((j,k))\,\delta(x-j)\,\delta(\xi-k),$$
$$\mathfrak{R}_\infty(x,\xi) = \sum_{(j,k)=1} [\delta(-kx+j\xi+1) - \delta(-kx+j\xi)]. \qquad (7.1)$$

In the second sum (a series of line measures), pairs j, k and $-j, -k$ are grouped, for convergence, before summation. Both distributions are automorphic, and they decompose into Eisenstein distributions (no Hecke distributions are needed here), as given by the equations [6, section 3.2]

$$\langle \mathfrak{R}_\infty,\, h\rangle = 2\sum_{n\geq 0} \frac{(-1)^{n+1}}{(n+1)\,!}\, \frac{\pi^{\frac{5}{2}+2n}}{\Gamma(\frac{3}{2}+n)\zeta(3+2n)}\, \mathfrak{E}_{2n+2},$$
$$\mathfrak{T}^1_\infty = 12\,\delta_0 + \mathfrak{R}_\infty + \sum^{\text{reg}}_{\zeta^*(\rho)=0} \text{Res}_{\nu=\rho}\left(\frac{\mathfrak{E}_{-\nu}}{\zeta(\nu)}\right): \qquad (7.2)$$

with $\zeta^*(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\,\zeta(s)$, the equation $\zeta^*(\rho) = 0$ singles out the non-trivial zeros of zeta.

The distribution $\mathfrak{R}_\infty$ decomposes over the set of trivial zeros of zeta only. This pair of equations makes it tempting to believe that either distribution could be of some use in a possible approach to the zeros of zeta. This is indeed the case, but pseudodifferential analysis is required to make the most of this idea.

8 A Possible Approach to the Zeros of Zeta

A popular way [1] of approaching the Riemann hypothesis was suggested by Hilbert and Polya, independently: search for an interpretation of the numbers $i\,(\rho - \frac{1}{2})$, with ρ in the set of non-trivial zeros of zeta, as the eigenvalues of a self-adjoint operator A in some Hilbert space. The self-adjointness of A implies that all operators q^{iA}, where q runs through the set of prime numbers, are unitary. We start here with the construction of an operator A in some space of tempered distributions in the plane, with the property that the Riemann hypothesis is equivalent to the fact that, for every $\varepsilon > 0$, the operator q^{iA} is in a very weak sense a $\mathrm{O}(q^\varepsilon)$. There is no Hilbert space here and, while the operator A decomposes over the set of zeros, trivial or not, of zeta, there is no need for separating the two sets of zeros.

The operator A is just the operator $2\pi\mathcal{E}+\frac{i}{2}$, acting on automorphic distributions, but it will suffice in view of obtaining a criterion for the validity of R.H. to test it on the distribution $\mathfrak{T}_\infty^1$ only, asking for a very weak estimate at that. Combining the decomposition (7.2) with the definition of the Weyl calculus $\mathrm{Op} = \mathrm{Op}^{(1)}$ (choosing 2 for Planck's constant would do with insignificant changes) and with the Fourier series decomposition which is the analogue of (4.2) for Eisenstein distributions, one obtains the following necessary and sufficient condition for the Riemann hypothesis to hold: that, given $\varepsilon > 0$ and any pair c, d with $0 \le c < d$ and $d^2 - c^2 > 2$, one should have for every function $w \in C^\infty(\mathbb{R})$ supported in $[c, d]$ the estimate

$$\left(w \,\middle|\, \mathrm{Op}\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty^1\right) w\right) = \mathrm{O}\left(Q^{\frac{1}{2}+\varepsilon}\right), \qquad Q \text{ squarefree } \to \infty. \tag{8.1}$$

The criterion remains valid if one subjects Q to the constraint of being prime. Using the definition (7.1) of $\mathfrak{T}_\infty^1$, it is immediate that, if M is a squarefree integer divisible by Q and by all primes $< dQ$, one can in (8.1) replace $\mathfrak{T}_\infty^1$ by the symbol

$$\mathfrak{T}_M^1(x,\xi) = \sum_{|j|+|k|\neq 0} a_1((j,k,M))\,\delta(x-j)\,\delta(\xi-k). \tag{8.2}$$

Now, set $N = 2M = QR$, and associate to any function $w \in C^\infty(\mathbb{R})$ the function $\theta_N w$ on $\mathbb{Z}$, periodic of period S^2, defined as

$$(\theta_N w)(n) = \sum_{\substack{n_1 \in \mathbb{Z} \\ n_1 \equiv n \bmod N^2}} w\left(\frac{n_1}{N}\right). \tag{8.3}$$

In [6, Chapter 4], it has been shown that there is an explicit set $h_{R,Q}(m,n)$ of coefficients ($m, n \in \mathbb{Z}/N^2\mathbb{Z}$) making the identity

$$\left(w \,\middle|\, \mathrm{Op}\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_M^1\right) w\right) = \sum_{m,n \bmod N^2} h_{R,Q}(m,n)\,\overline{(\theta_N w)(m)}\,(\theta_N w)(n) \tag{8.4}$$

valid. The matrix $\left(h_{R,Q}(m,n)\right)$ has a Eulerian structure with rather simple individual factors [6, Prop.4.3.5]. To understand what (8.1) really means, we must make the role of Q in this Hermitian form totally explicit.

Denote as $n \mapsto \overset{\vee}{n}$ the automorphism of $\mathbb{Z}/N^2 = \mathbb{Z}/R^2Q^2$ defined by the pair of equations

$$\overset{\vee}{n} \equiv n \bmod R^2, \quad \overset{\vee}{n} \equiv -n \bmod Q^2, \tag{8.5}$$

and define the reflection $\Lambda_{R,Q}$ of the linear space of complex-valued functions on $\mathbb{Z}/N^2\mathbb{Z}$ such that $\left(\Lambda_{R,Q}\psi\right)(n) = \psi(\overset{\vee}{n})$. Assuming that Q is odd, one has for every function $w \in C^\infty(\mathbb{R})$ the identity

$$\left(w \,\middle|\, \mathrm{Op}\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}^1_M\right) w\right) = \mu(M) \sum_{m,n \bmod N^2} h_{N,1}(m,n)\, \overline{(\theta_N w)(m)}\, (\Lambda_{R,Q}\theta_N w)(n) \tag{8.6}$$

(the Möbius factor $\mu(M) = \pm 1$ indicates the parity of the number of primes dividing M). Now, while it is possible, using the Heisenberg representation, to define a linear automorphism $\Lambda^\sharp_{R,Q}$ of $\mathcal{S}(\mathbb{R})$ such that $\Lambda_{R,Q}\theta_N w = \theta_N \Lambda^\sharp_{R,Q} w$ for every w, it is only the "arithmetic side" $\Lambda_{R,Q}$ that is truly simple.

The difficulty of the present approach lies in the necessity to combine information from both sides (an arithmetic and an analytic one) of the same operator to analyze (8.1). There is work in progress on this question.

9 Pseudodifferential Analysis and Modular Form Theory

In Sect. 4, we have built all modular distributions relative to the group $SL(2,\mathbb{Z})$, and we have shown in Sect. 3 how this leads to a construction of modular forms (in the hyperbolic half-plane) of the non-holomorphic type. Making an example of a pseudodifferential operator, of the kind considered in the last section, totally explicit, leads to an approach to modular form theory of the holomorphic type.

Consider the measure $\mathfrak{d}_{12}$ on the line defined as

$$\mathfrak{d}_{12}(x) = \sum_{m\in\mathbb{Z}} \chi^{(12)}(m)\, \delta\left(x - \frac{m}{\sqrt{12}}\right), \tag{9.1}$$

where $\chi^{(12)}$ is the (Dirichlet) character mod 12 such that $\mathfrak{d}_{12}(m) = 0$ if $(m, 12) > 1$, while $\mathfrak{d}_{12}(\pm 5) = -1$ and $\mathfrak{d}_{12}(\pm 1) = 1$. One has then if $w \in \mathcal{S}(\mathbb{R})$ the identity

$$\left(w \,\middle|\, \mathrm{Op}\left(12^{i\pi\mathcal{E}}\mathfrak{T}^1_{12}\right) w\right) = \frac{1}{4}\left|\langle \mathfrak{d}_{12},\, w\rangle\right|^2. \tag{9.2}$$

Consider now the Gaussian transform of $\mathfrak{d}_{12}$, to wit the function η, holomorphic in the upper half-plane, defined by the equation

$$\eta(z) = \langle \mathfrak{d}_{12},\, x \mapsto e^{i\pi z x^2}\rangle. \tag{9.3}$$

This is the so-called Dedekind eta function, of historical importance, the 24th power of which is the celebrated Ramanujan Delta function Δ, from which modular form theory originated. It is a modular form (of holomorphic type) of weight 12, which means that one has the identity

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z), \qquad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}). \tag{9.4}$$

This is a special case of [6, theorem 4.3.4], in which a similar analysis is performed for the operator with symbol $Q^{i\pi\mathcal{E}}\mathfrak{T}^1_Q$, where $\frac{Q}{2}$ is assumed to be an even squarefree integer: a more detailed analysis is performed in [6, Chapter 5]. Most readers of the present volume are experts in Toeplitz operator theory: note the analogy (and the essential difference) between the Gaussian transform which occurs in (9.3) and the one, so basic in the Toeplitz theory, which connects the real-type and holomorphic-type realizations of the same function.

References

1. A.Connes, *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function*, Selecta Math. (N.S.) **5**,1, 1999, 29–106.
2. A.Selberg, *On the Estimation of Fourier Coefficients of Modular Forms*, Proc.Symp.Pure Math. **8** (1963), 1–15.
3. A.Unterberger, *Automorphic pseudodifferential analysis and higher–level Weyl calculi*, Progress in Math. **209**, Birkhäuser, Basel–Boston–Berlin, 2002.
4. A.Unterberger, *Pseudodifferential analysis, automorphic distributions in the plane and modular forms*, Birkhäuser, Basel-Boston-Berlin, 2011.
5. A.Unterberger, *Pseudodifferential operators with automorphic symbols*, Pseudodifferential Operators **11**, Birkhäuser, Basel–Boston–Berlin, 2015.
6. A.Unterberger, *Pseudodifferential methods in number theory*, Pseudodifferential Operators **13**, Birkhäuser, Basel-Boston-Berlin, 2018.