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p-adic Hodge Theory

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Preface

The basic theme of p-adic Hodge theory is to understand the relationship between various p-adic cohomology theories associated to algebraic varieties over p-adic fields. In the standard formulation, it is concerned with comparisons between algebraic de Rham cohomology, p-adic étale cohomology, and crystalline cohomology. Each of these cohomology theories carry additional structure: de Rham cohomology comes equipped with a filtration, étale cohomology with a Galois action, and crystalline cohomology with a semi-linear Frobenius operator. Comparisons between these theories shed light on each of these individual structures, and the package of all of these cohomology theories and the comparison isomorphisms between them is a very rich structure associated to algebraic varieties over p-adic fields.

In recent years, there has been a surge of activity in the area related to integral p-adic Hodge, non-Abelian phenomena, and connections to notions in algebraic topology. The basic comparison isomorphisms of p-adic Hodge theory are defined rationally and don't directly provide information about the integral structures present in the cohomology theories, and there have been recent developments in the area to understand integral and torsion phenomena. Non-abelian phenomena can be understood on several levels, but the most basic one is the development of theories with coefficients. The connections with algebraic topology arise from the strong relationship between crystalline cohomology and topological Hochschild homology. This is also closely tied to the theory of the de Rham–Witt complex.

This proceedings volume contains chapters related to the research presented at the 2017 Simons Symposium on p-adic Hodge theory. This symposium was focused on recent developments in p-adic Hodge theory, especially those concerning integral questions and their connections to notions in algebraic topology.

The volume begins with the chapter of Morrow on the $\mathbf{A}_{\text{inf}}$-cohomology theory which was introduced in the earlier fundamental paper of Bhatt, Morrow, and Scholze on integral p-adic Hodge theory. The present chapter contains a detailed presentation of the $\mathbf{A}_{\text{inf}}$-cohomology theory, largely self-contained. The author focuses, in particular, on de Rham–Witt theory and the p-adic analogue of the Cartier isomorphism.

The chapter of Colmez and Niziol is concerned with a fundamental computation of the pro-étale cohomology of the rigid analytic affine space in any dimension. Contrary to the standard results for étale cohomology of algebraic varieties, these pro-étale cohomology groups are nonzero and the authors describe them using differential forms.

The third chapter by Chung, Kim, Kim, Park, and Yoo is concerned with a certain invariant attached to representations of the fundamental group of the ring of S-integers $\mathcal{O}_F[1/S]$ of a number field F, for some finite set of primes S. The authors describe a theory of the "arithmetic Chern-Simons action", inspired by the topological theory. The main result is a formula relating an invariant of a torsor over $\mathcal{O}_F[1/S]$ to locally defined data. The authors also give several interesting applications of this formula.

Throughout the subject of p-adic Hodge theory various large rings play a central role. The chapter of Kedlaya discusses various key basic algebraic properties of the ring $\mathbf{A}_{\mathrm{inf}}$, which is the ring of Witt vectors of a perfect valuation ring in characteristic p. This ring is, in particular, fundamental for the $\mathbf{A}_{\mathrm{inf}}$-cohomology developed by Bhatt, Morrow, and Scholze, and in integral p-adic Hodge theory. This ring is quite different from the ones occurring in classical algebraic geometry: for example, it is not Noetherian. Nevertheless, the author discusses several favorable properties, e.g., those related to vector bundles.

A fundamental result in complex Hodge theory is the Simpson correspondence relating local systems and Higgs bundles. An analogue of this theory was developed in characteristic p by Ogus and Vologodsky. The chapter of Gros is concerned with the problem of lifting this characteristic p correspondence to a mixed characteristic correspondence via a q-deformation.

The final chapter of Tsuji concerns the study of integral p-adic Hodge theory with coefficients. Early in the development of p-adic Hodge theory, Faltings constructed a theory of coefficients for integral p-adic Hodge theory. The present chapter refines this theory and generalizes the work of Bhatt, Morrow, and Scholze to this context. The chapter contains a detailed exposition of the many technical aspects of the theory and contains many improvements in this regard to the existing literature as well.

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Contents

Notes on the $\mathbb{A}_{\mathrm{inf}}$-Cohomology of *Integral p-Adic Hodge Theory*

Matthew Morrow

Abstract We present a detailed overview of the construction of the $\mathbb{A}_{\mathrm{inf}}$-cohomology theory from the preprint *Integral p-adic Hodge theory*, joint with Bhatt and Scholze. We focus particularly on the p-adic analogue of the Cartier isomorphism via relative de Rham–Witt complexes.

Keywords p-adic Hodge theory · Prismatic cohomology · Perfectoid · de Rham–Witt complex

Extended abstract

These are expanded notes of a mini-course, given at l'Institut de Mathématiques de Jussieu–Paris Rive Gauche, 15 Jan.–1 Feb. 2016, detailing some of the main results of the article

> [5] B. Bhatt, M. Morrow, P. Scholze, *Integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci. 128 (2018), 219–397.

More precisely, the goal of these notes is to give a detailed, and largely self-contained, presentation of the construction of the $\mathbb{A}_{\mathrm{inf}}$-cohomology theory from [5], focussing on the p-adic analogue of the Cartier isomorphism via relative de Rham–Witt complexes. By restricting attention to this particular aspect of [5], we hope to have made the construction more accessible. However, the reader should only read these notes in conjunction with [5] itself and is strongly advised also to consult the surveys [2, 26] by the other authors, which cover complementary aspects of the theory. In particular, in these notes we do not discuss q-de Rham complexes, cotangent complex

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calculations, Breuil–Kisin(–Fargues) modules, or the crystalline and de Rham comparison theorems of [5, Sect. 12–14], as these topics are not strictly required for the construction of the $\mathbb{A}_{\mathrm{inf}}$-cohomology theory.[1] Moreover, we refer to [5] for several self-contained proofs to avoid verbatim repetition.

Section 1 is an introduction which begins by recalling some classical problems and results of p-adic Hodge theory before stating the main theorem of the course, namely the existence of a new cohomology theory for p-adic schemes which integrally interpolates étale, crystalline and de Rham cohomologies.

Section 2 introduces the décalage functor, which modifies a given complex by a small amount of torsion. This functor is absolutely essential to our constructions, as it kills the "junk torsion" which so often appears in p-adic Hodge theory and thus allows us to establish results integrally. An example of this annihilation of torsion, in the context of Faltings' almost purity theorem, is given in Sect. 2.2.

Section 3 develops the necessary elementary theory of perfectoid rings, emphasising the importance of certain maps $\theta_r, \widetilde{\theta}_r$ which generalise Fontaine's usual map θ of p-adic Hodge theory and are central to the later constructions.

Section 4 is a minimal summary of Scholze's theory of pro-étale cohomology for rigid analytic varieties. In particular, in Sect. 4.3 we explain the usual technique by which the pro-étale manifestation of the almost purity theorem allows the pro-étale cohomology of "small" rigid affinoids to be (almost) calculated in terms of group cohomology related to perfectoid rings.

Section 5 revisits the main theorem and defines the new cohomology theory as the hypercohomology of a certain complex $\mathbb{A}\Omega_{\mathfrak{X}}$. In Theorem 4 we state a *p-adic Cartier isomorphism*, which identifies the cohomology sheaves of the base change of $\mathbb{A}\Omega_{\mathfrak{X}}$ along $\widetilde{\theta}_r$ with Langer–Zink's relative de Rham–Witt complex of the p-adic scheme $\mathfrak{X}$. We then deduce all main properties of the new cohomology theory from this p-adic Cartier isomorphism.

Section 6 reviews Langer–Zink's theory of the relative de Rham–Witt complex, which may be seen as the initial object in the category of Witt complexes, i.e., families of differential graded algebras over the Witt vectors which are equipped with compatible Restriction, Frobenius, and Verschiebung maps. In Sect. 6.2 we present one of our main constructions, namely building Witt complexes from the data of a commutative algebra (in a derived sense), equipped with a Frobenius, over the infinitesimal period ring $\mathbb{A}_{\mathrm{inf}}$. In Sect. 6.3 we apply this construction to the group cohomology of a Laurent polynomial algebra and prove that the result is precisely the relative de Rham–Witt complex itself; this is the local calculation which underlies the p-adic Cartier isomorphism.

Finally, Sect. 7 sketches the proof of the p-adic Cartier isomorphism by reducing to the final calculation of the previous paragraph. This reduction is based on various technical lemmas that the décalage functor behaves well under base change and

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[1] To be precise, there is one step in the construction, namely the equality $(\dim_{\mathfrak{X}})$ in the proof of Theorem 7, where we will have to assume that the p-adic scheme $\mathfrak{X}$ is defined over a discretely valued field; this assumption can be overcome using the crystalline comparison theorems of [5].

taking cohomology, and that it transforms certain almost quasi-isomorphisms into quasi-isomorphisms.

The appendices provide an introduction to Fontaine's infinitesimal period ring $\mathbb{A}_{\inf}$ and state a couple of lemmas about Koszul complexes which are used repeatedly in calculations.

1 Introduction

1.1 Mysterious Functor and Crystalline Comparison

Here in Sect. 1.1 we consider the following common situation:

- K a complete discrete valuation field of mixed characteristic; ring of integers $\mathcal{O}_K$; perfect residue field k.
- $\mathfrak{X}$ a proper, smooth scheme over $\mathcal{O}_K$.

For $\ell \neq p$, proper base change in étale cohomology gives a canonical isomorphism

$$H^i_{\text{ét}}(\mathfrak{X}_{\overline{k}}, \mathbb{Z}_\ell) \cong H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}, \mathbb{Z}_\ell)$$

which is compatible with Galois actions.[2] Grothendieck's question of the mysterious functor is often now interpreted as asking what happens in the case $\ell = p$. More precisely, how are $H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}) := H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}, \mathbb{Z}_p)$ and $H^i_{\text{crys}}(\mathfrak{X}_k) := H^i_{\text{crys}}(\mathfrak{X}_k/W(k))$ related? In other words, how does p-adic cohomology of $\mathfrak{X}$ degenerate from the generic to the special fibre?

Grothendieck's question is answered after inverting p by the Crystalline Comparison Theorem (Fontaine–Messing [15], Bloch–Kato [7], Faltings [12], Tsuji [28] Nizioł [23], ...), stating that there are natural isomorphisms

$$H^i_{\text{crys}}(\mathfrak{X}_k) \otimes_{W(k)} \mathbb{B}_{\text{crys}} \cong H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{crys}}$$

which are compatible with Galois and Frobenius actions (and filtrations after base changing to $\mathbb{B}_{\text{dR}}$), where $\mathbb{B}_{\text{crys}}$ and $\mathbb{B}_{\text{dR}}$ are Fontaine's period rings (which we emphasise contain $1/p$; they will not appear again in these notes, so we do not define them). Hence general theory of period rings implies that

$$H^i_{\text{crys}}(\mathfrak{X}_k)\left[\tfrac{1}{p}\right] = (H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{crys}})^{G_K}$$

[2] To be precise, the isomorphism depends only on a choice of specialisation of geometric points of Spec $\mathcal{O}_K$. A consequence of the compatibility with Galois actions is that the action of G_K on $H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}, \mathbb{Z}_\ell)$ is unramified.

(i.e., the crystalline Dieudonné module of $H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}})[\frac{1}{p}]$, by definition) with φ on the left induced by $1 \otimes \varphi$ on the right. In summary, $(H^n_{\text{ét}}(\mathfrak{X}_{\overline{K}})[\frac{1}{p}], G_K)$ determines $(H^n_{\text{crys}}(\mathfrak{X}_k)[\frac{1}{p}], \varphi)$. Similarly, in the other direction, $(H^n_{\text{ét}}(\mathfrak{X}_{\overline{K}})[\frac{1}{p}], G_K)$ is determined by $(H^n_{\text{crys}}(\mathfrak{X}_k)[\frac{1}{p}], \varphi, \text{Hodge fil.})$.

But what if we do not invert p? There are various partial results in the literature, including [8, 13], and a simplifying summary would be to claim that "everything seems to work integrally if $ie < p - 1$",[3] where e is the absolute ramification degree of K. With no assumptions on ramification degree, dimension, value of p, etc., we prove in [5] results of the following form:

(i) The torsion in $H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}})$ is "less than" that of $H^i_{\text{crys}}(\mathfrak{X}_k)$. To be precise,

$$\text{length}_{\mathbb{Z}_p} H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}})/p^r \leq \text{length}_{W(k)} H^i_{\text{crys}}(\mathfrak{X}_k)/p^r$$

for all $r \geq 1$, as one would expect for a degenerating family of cohomologies. In particular, if $H^i_{\text{crys}}(\mathfrak{X}_k)$ is torsion-free then so is $H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}})$.

(ii) If $H^*_{\text{crys}}(\mathfrak{X}_k)$ is torsion-free for $* = i, i+1$, then $(H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}), G_K)$ determines $(H^i_{\text{crys}}(\mathfrak{X}_k), \varphi)$.

It really is possible that additional torsion appears when degenerating the p-adic cohomology from the generic fibre to the special fibre, as the following example indicates (which is labeled a theorem as there seems to be no case of an $\mathfrak{X}$ as above for which $H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}) \otimes_{\mathbb{Z}_p} W(k)$ and $H^i_{\text{crys}}(\mathfrak{X})$ were previously known to have non-isomorphic torsion submodules):

Theorem 0 *There exists a smooth projective relative surface $\mathfrak{X}$ over $\mathbb{Z}_2$ such that $H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}})$ is torsion-free for all $i \geq 0$ but such that $H^2_{\text{crys}}(\mathfrak{X}_k)$ contains non-trivial 2-torsion.*[4]

Proof We do not reproduce the construction here; see [5, Proposition 2.2].

1.2 Statement of Main Theorem and Outline of Notes

The following notation will be used repeatedly in these notes:

- $\mathbb{C}$ is a complete, non-archimedean, algebraically closed field of mixed characteristic[5]; ring of integers $\mathcal{O}$; residue field k.

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[3] Our results can presumably make this more precise.

[4] In [5, Theorem 2.10] we also give an example for which $H^2_{\text{ét}}(\mathfrak{X}_{\overline{K}})_{\text{tors}} = \mathbb{Z}/p^2\mathbb{Z}$ and $H^2_{\text{crys}}(\mathfrak{X}_k)_{\text{tors}} = k \oplus k$.

[5] More general, most of the theory which we will present works for any perfectoid field of mixed characteristic which contains all p-power roots of unity.

- $\mathcal{O}^\flat := \varprojlim_\varphi \mathcal{O}/p\mathcal{O}$ is the *tilt* (using Scholze's language [24]—or $\mathcal{R}_\mathcal{O}$ in Fontaine's original notation [14]) of $\mathcal{O}$; so $\mathcal{O}^\flat$ is a perfect ring of characteristic p which is the ring of integers of $\mathbb{C}^\flat := \operatorname{Frac}\mathcal{O}^\flat$, which is a complete, non-archimedean, algebraically closed field with residue field k.
- $\mathbb{A}_{\inf} := W(\mathcal{O}^\flat)$ is the first period ring of Fontaine[6]; it is equipped with the usual Witt vector Frobenius φ. There are three key specialisation maps:

$$\begin{array}{ccccc} & & W(\mathbb{C}^\flat) & & \\ & & \uparrow{\scriptstyle \text{étale}} & & \\ \mathcal{O} & \xleftarrow[\text{Fontaine's map }\theta]{\text{de Rham}} & \mathbb{A}_{\inf} & \xrightarrow{\text{crystalline}} & W(k) \end{array}$$

where Fontaine's map θ will be discussed in detail, and in greater generality, in Sect. 3.

The goal of these notes is to provide a relatively detailed overview of the proof of the following theorem, establishing the existence of a cohomology theory, taking values in $\mathbb{A}_{\inf}$-modules, which integrally interpolates the étale, crystalline, and de Rham cohomologies of a smooth p-adic scheme:

Theorem 1 *For any proper, smooth (possibly p-adic formal) scheme $\mathfrak{X}$ over $\mathcal{O}$, there is a perfect complex $R\Gamma_{\mathbb{A}}(\mathfrak{X})$ of $\mathbb{A}_{\inf}$-modules, functorial in $\mathfrak{X}$ and equipped with a φ-semi-linear endomorphism φ, with the following specialisations (which are compatible with Frobenius actions where they exist):*

(i) Étale: $R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\inf}} W(\mathbb{C}^\flat) \simeq R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} W(\mathbb{C}^\flat)$, where $X := \mathfrak{X}_\mathbb{C}$ is the generic fibre of $\mathfrak{X}$ (viewed as a rigid analytic variety over $\mathbb{C}$ in the case that $\mathfrak{X}$ is a formal scheme)

(ii) Crystalline: $R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\inf}} W(k) \simeq R\Gamma_{\text{crys}}(\mathfrak{X}_k/W(k))$.

(iii) de Rham: $R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\inf}} \mathcal{O} \simeq R\Gamma_{\text{dR}}(\mathfrak{X}/\mathcal{O})$.

The individual cohomology groups

$$H^i_{\mathbb{A}}(\mathfrak{X}) := H^i(R\Gamma_{\mathbb{A}}(\mathfrak{X}))$$

have the following properties:

(iv) $H^i_{\mathbb{A}}(\mathfrak{X})$ is a finitely presented $\mathbb{A}_{\inf}$-module;

(v) $H^i_{\mathbb{A}}(\mathfrak{X})[\frac{1}{p}]$ is finite free over $\mathbb{A}_{\inf}[\frac{1}{p}]$;

(vi) $H^i_{\mathbb{A}}(\mathfrak{X})$ is equipped with a Frobenius-semi-linear endomorphism φ which becomes an isomorphism after inverting any generator $\xi \in \mathbb{A}_{\inf}$ of Ker θ*, i.e., $\varphi : H^i_{\mathbb{A}}(\mathfrak{X})[\frac{1}{\xi}] \xrightarrow{\simeq} H^i_{\mathbb{A}}(\mathfrak{X})[\frac{1}{\varphi(\xi)}]$;*

[6] A brief introduction to $\mathcal{O}^\flat$ and $\mathbb{A}_{\inf}$ may be found at the beginning of Appendix 1.

(vii) *Étale:* $H^i_{\mathbb{A}}(\mathfrak{X}) \otimes_{\mathbb{A}_{\inf}} W(\mathbb{C}^\flat) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(\mathbb{C}^\flat)$, *whence*

$$(H^i_{\mathbb{A}}(\mathfrak{X}) \otimes_{\mathbb{A}_{\inf}} W(\mathbb{C}^\flat))^{\varphi=1} \cong H^i_{\text{ét}}(X, \mathbb{Z}_p).$$

(viii) *Crystalline: there is a short exact sequence*

$$0 \longrightarrow H^i_{\mathbb{A}}(\mathfrak{X}) \otimes_{\mathbb{A}_{\inf}} W(k) \to H^i_{\text{crys}}(\mathfrak{X}_k/W(k)) \longrightarrow \mathrm{Tor}_1^{\mathbb{A}_{\inf}}(H^{i+1}_{\mathbb{A}}(\mathfrak{X}), W(k)) \longrightarrow 0,$$

where the Tor_1 *term is killed by a power of* p.

(ix) *de Rham: there is a short exact sequence*

$$0 \longrightarrow H^i_{\mathbb{A}}(\mathfrak{X}) \otimes_{\mathbb{A}_{\inf}} \mathcal{O} \to H^i_{\text{dR}}(\mathfrak{X}/\mathcal{O}) \longrightarrow H^{i+1}_{\mathbb{A}}(\mathfrak{X})[\xi] \longrightarrow 0,$$

where the third term is again killed by a power of p.

(x) *If* $H^i_{\text{crys}}(\mathfrak{X}_k/W(k))$ *or* $H^i_{\text{dR}}(\mathfrak{X}/\mathcal{O})$ *is torsion-free, then* $H^i_{\mathbb{A}}(\mathfrak{X})$ *is a finite free* $\mathbb{A}_{\inf}$*-module.*

Corollary 1 *Let* $\mathfrak{X}$ *be as in the previous theorem, fix* $i \geq 0$, *and assume* $H^i_{\text{crys}}(\mathfrak{X}_k/W(k))$ *is torsion-free. Then* $H^i_{\text{ét}}(X, \mathbb{Z}_p)$ *is also torsion-free. If we assume further that* $H^{i+1}_{\text{crys}}(\mathfrak{X}_k/W(k))$ *is torsion-free, then*

$$H^i_{\mathbb{A}}(\mathfrak{X}) \otimes_{\mathbb{A}_{\inf}} W(k) = H^i_{\text{crys}}(\mathfrak{X}_k/W(k)) \quad \text{and} \quad H^i_{\mathbb{A}}(\mathfrak{X}) \otimes_{\mathbb{A}_{\inf}} \mathcal{O} = H^i_{\text{dR}}(\mathfrak{X}/\mathcal{O}).$$

Proof We first mention that the "whence" assertion of part (vii) of the previous theorem is the following general, well-known assertion: if M is a finitely generated $\mathbb{Z}_p$-module and F is any field of characteristic p, then $(M \otimes_{\mathbb{Z}_p} W(F))^{\varphi=1} = M$ (where φ really means $1 \otimes \varphi$).

Now assume $H^i_{\text{crys}}(\mathfrak{X}_k/W(k))$ is torsion-free. Then part (x) of the previous theorem implies that $H^i_{\mathbb{A}}(\mathfrak{X})$ is finite free; so from part (vii) we see that $H^i_{\text{ét}}(X, \mathbb{Z}_p)$ cannot have torsion. If we also assume $H^{i+1}_{\text{crys}}(\mathfrak{X}_k/W(k))$ is torsion-free, then $H^{i+1}_{\mathbb{A}}(\mathfrak{X})$ is again finite free by (x), and so no torsion terms appear in the short exact sequences in parts (viii) and (ix) of the previous theorem.

Having stated the main theorem, we now give a very brief outline of the ideas which will be used to construct the $\mathbb{A}_{\inf}$-cohomology theory.

(i) We will define $R\Gamma_{\mathbb{A}}(\mathfrak{X})$ to be the Zariski hypercohomology of the following complex of sheaves of $\mathbb{A}_{\inf}$-modules on the formal scheme $\mathfrak{X}$:

$$\mathbb{A}\Omega_{\mathfrak{X}} := L\eta_\mu\big(R\nu_*\widehat{(\mathbb{A}_{\inf,X})}\big)$$

where:

- $\mathbb{A}_{\inf,X}$ is a certain period sheaf of $\mathbb{A}_{\inf}$-modules on the pro-étale site $X_{\text{proét}}$ of the rigid analytic variety X (note that even if $\mathfrak{X}$ is an honest scheme over $\mathcal{O}$, we must view its generic fibre as a rigid analytic variety);

- $\nu : X_{\text{proét}} \to \mathfrak{X}_{\text{Zar}}$ is the projection map to the Zariski site of $\mathfrak{X}$;
- the hat indicates the derived p-adic completion of $R\nu_*(\mathbb{A}_{\mathrm{inf},X})$ (see also the end of item (iv));
- $L\eta$ is the décalage functor which modifies a given complex by a small amount of torsion (in this case with respect to a prescribed element $\mu \in \mathbb{A}_{\mathrm{inf}}$).

(ii) Parts (ii) and (iii) of Theorem 1 are proved simultaneously by relating $\mathbb{A}\Omega_{\mathfrak{X}}$ to Langer–Zink's relative de Rham–Witt complex $W_r\Omega^\bullet_{\mathfrak{X}/\mathcal{O}}$; indeed, this equals $\Omega^\bullet_{\mathfrak{X}/\mathcal{O}}$ if $r = 1$ (which computes de Rham cohomology of $\mathfrak{X}$) and satisfies $W_r\Omega^\bullet_{\mathfrak{X}/\mathcal{O}} \otimes_{W_r(\mathcal{O})} W_r(k) = W_r\Omega^\bullet_{\mathfrak{X}_k/k}$ (where $W_r\Omega^\bullet_{\mathfrak{X}_k/k}$ is the classical de Rham–Witt complex of Bloch–Deligne–Illusie computing crystalline cohomology of $\mathfrak{X}_k$).

(iii) If Spf R is an affine open of $\mathfrak{X}$ (so R is a p-adically complete, formally smooth $\mathcal{O}$-algebra[7]) which is *small*, i.e., formally étale over $\mathcal{O}\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle$ (:= the p-adic completion of $\mathcal{O}[T_1^{\pm 1}, \dots, T_d^{\pm 1}]$), then we will use the almost purity theorem to explicitly calculate $R\Gamma_{\text{Zar}}(\text{Spf } R, \mathbb{A}\Omega_{\mathfrak{X}})$ in terms of group cohomology and Koszul complexes. These calculations can be rephrased using "q-de Rham complexes" over $\mathbb{A}_{\mathrm{inf}}$ (=deformations of the de Rham complex), but we do not do so in these notes.

(iv) Some remarks on the history and development of the results:

- Early motivation for the existence of $R\Gamma_{\mathbb{A}}(\mathfrak{X})$ (e.g., as discussed by Scholze at Harris' 2014 MSRI birthday conference) came from topological cyclic homology. These notes say nothing about that point of view, which may now be found in [6].
- At the time of writing the announcement of our results [4], we only knew that the definition of $R\Gamma_{\mathbb{A}}(\mathfrak{X})$ in part (i) of the remark almost (in the precise sense of Faltings' almost mathematics) had the desired properties of Theorem 1, so it was necessary to modify the definition slightly; this modification is no longer necessary.
- Further simplifications of some of the proofs were explained in [2], some of which are also taken into account in these notes.
- The definition of $\mathbb{A}\Omega_{\mathfrak{X}}$ continues to make sense for any p-adic formal $\mathcal{O}$-scheme $\mathfrak{X}$, not necessarily smooth, and in particular the comparison isomorphisms of Theorem 1 have been extended to case of semi-stable reduction by Česnavičus and Koshikawa [9].
- In late 2018 the authors of [5] realised that the period sheaf $\mathbb{A}_{\mathrm{inf},X}$ on $X_{\text{proét}}$ might not be derived p-adically complete, though this had been implicitly used in the construction. This is easily fixed, without changing any of the

[7] Throughout these notes we follow the convention that *formally smooth/étale* includes the condition of being topologically finitely presented, i.e., a quotient of $\mathcal{O}\langle T_1, \dots, T_N\rangle$ by a finitely generated ideal. Under this convention formal smoothness implies flatness. In fact, according to a result of Elkik [11, Theorem 7] (see Rmq. 2 on p. 587 for elimination of the Noetherian hypothesis), a p-adically complete $\mathcal{O}$-algebra is formally smooth if and only if it is the p-adic completion of a smooth $\mathcal{O}$-algebra.

ensuing arguments, either by replacing $\mathbb{A}_{\mathrm{inf},X}$ by its derived p-adic completion (which is then a complex of sheaves) or else by derived p-adically completing all occurrences of $R\nu_*(\mathbb{A}_{\mathrm{inf},X})$ and $R\Gamma_{\mathrm{pro\acute{e}t}}(-, \mathbb{A}_{\mathrm{inf},X})$ in the theory. In the published version of [5] the former approach is adopted, but in these notes we will follow the latter route which has the conceptual advantage that $\mathbb{A}_{\mathrm{inf},X}$ remains an honest sheaf of rings. Unfortunately this leads to a notation inconsistency: the $\mathbb{A}_{\mathrm{inf},X}$ of these notes is $\mathcal{H}^0(-)$ of the complex of sheaves $\mathbb{A}_{\mathrm{inf},X}$ of [5].

- Most recently, a site theoretic definition of the $\mathbb{A}_{\mathrm{inf}}$-cohomology is now available through the prismatic theory of Bhatt–Scholze [3].

2 The décalage Functor $L\eta$: Modifying Torsion

For a ring A and non-zero divisor $f \in A$, we define the *décalage functor* which was introduced first by Berthelot–Ogus [1, Chap. 8] following a suggestion of Deligne. It will play a fundamental role in our constructions.

Definition 1 Suppose that C is a cochain complex of f-torsion-free A-modules. Then we denote by $\eta_f C$ the subcomplex of $C[\frac{1}{f}]$ defined as

$$(\eta_f C)^i := \{x \in f^i C^i : dx \in f^{i+1} C^{i+1}\}$$

i.e., $\eta_f C$ is the largest subcomplex of $C[\frac{1}{f}]$ which in degree i is contained in $f^i C^i$ for all $i \in \mathbb{Z}$.

It is easy to compute the cohomology of $\eta_f C$:

Lemma 1 *The map on cocycles $Z^i C \to Z^i(\eta_f C)$ given by $m \to f^i m$ induces a natural isomorphism*

$$H^i(C)/H^i(C)[f] \xrightarrow{\simeq} H^i(\eta_f C).$$

Proof It is easy to see that the map induces $H^i(C) \to H^i(\eta_f C)$, and the kernel corresponds to those $x \in C^i$ such that $dx = 0$ and $fx \in d(C^{i-1})$, i.e., $H^i(C)[f]$.

Corollary 2 *If $C \xrightarrow{\sim} C'$ is a quasi-isomorphism of complexes of f-torsion-free A-modules, then the induced map $\eta_f C \to \eta_f C'$ is also a quasi-isomorphism.*

Proof Immediate from the previous lemma.

We may now derive η_f. There is a well-defined endofunctor $L\eta_f$ of the derived category $D(A)$ defined as follows: if $D \in D(A)$ then pick a quasi-isomorphism $C \xrightarrow{\sim} D$ where C is a cochain complex of f-torsion-free A-modules (e.g., pick a projective resolution, at least if D is bounded above) and set

$$L\eta_f D := \eta_f C.$$

This is well-defined by the previous corollary and standard formalism of derived categories.

> **Warning:** $L\eta_f$ does not preserve distinguished triangles! For example, if $A = \mathbb{Z}$ then $L\eta_p(\mathbb{Z}/p\mathbb{Z}) = 0$ but $L\eta_p(\mathbb{Z}/p^2\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.

The general theory of the functor $L\eta_f$ will be spread out through the notes (see especially Remarks 7 and 9); now we proceed to two important examples.

2.1 Example 1: Crystalline Cohomology

The following proposition is the origin of the décalage functor, in which $A = W(k)$ and $f = p$; it is closely related to the Cartier isomorphism for the de Rham–Witt complex.

Proposition 1 *Let k be a perfect field of characteristic p and R a smooth k-algebra. Then*

(i) (Illusie 1979) The absolute Frobenius $\varphi : W\Omega^\bullet_{R/k} \to W\Omega^\bullet_{R/k}$ is injective and has image $\eta_p W\Omega^\bullet_{R/k}$, thus inducing a Frobenius-semi-linear isomorphism

$$\Phi : W\Omega^\bullet_{R/k} \xrightarrow{\simeq} \eta_p W\Omega^\bullet_{R/k}.$$

(ii) (Berthelot–Ogus 1978) There exists a Frobenius-semi-linear quasi-isomorphism

$$\Phi : R\Gamma_{\text{crys}}(R/W(k)) \xrightarrow{\sim} L\eta_p R\Gamma_{\text{crys}}(R/W(k)).$$

Proof Obviously (i)⇒(ii), but (ii) was proved earlier and is the historical origin of $L\eta$: see [1, Theorem 8.20] (with the zero gauge). Berthelot–Ogus applied it to study the relation between the Newton and Hodge polygons associated to a proper, smooth variety over k.

(i) is a consequence of the following standard de Rham–Witt identities:

- φ has image in $\eta_p W\Omega^\bullet_{R/k}$ since $\varphi = p^i F$ on $W\Omega^i_{R/k}$ and $d\varphi = \varphi d$.
- φ is injective since $FV = VF = p$.
- the image of φ is exactly $\eta_p W\Omega^\bullet_{R/k}$ since $d^{-1}(pW\Omega^{i+1}_{R/k}) = F(W\Omega^i_{R/k})$ [18, Equation I.3.21.1.5].

2.2 "Example 2": An Integral Form of Faltings' Almost Purity Theorem

We now present an integral form of (the main consequence of) Faltings' almost purity theorem; we do not need this precise result, but we will make use of Lemma 2 and the "goodness" of the group cohomology established in the course of the proof of Theorem 2. Moreover, readers familiar with Faltings' approach to p-adic Hodge theory may find this result motivating. To recall Faltings' almost purity theorem we consider the following situation:

- $\mathbb{C}$ is a complete, non-archimedean, algebraically closed field of mixed characteristic; ring of integers $\mathcal{O}$.
- R is a p-adically complete, formally smooth $\mathcal{O}$-algebra, which we further assume is connected and *small*, i.e., formally étale over $\mathcal{O}\langle \underline{T}^{\pm 1}\rangle := \mathcal{O}\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle$. As usual in Faltings' theory, we associate to this the following two rings:
- $R_\infty := R\widehat{\otimes}_{\mathcal{O}\langle \underline{T}^{\pm 1}\rangle}\mathcal{O}\langle \underline{T}^{\pm 1/p^\infty}\rangle$—this is acted on by $\Gamma := \mathbb{Z}_p(1)^d$ via R-algebra automorphisms in the usual way: given $\gamma \in \Gamma = \mathrm{Hom}_{\mathbb{Z}_p}((\mathbb{Q}_p/\mathbb{Z}_p)^d, \mu_{p^\infty})$ and $k_1, \dots, k_d \in \mathbb{Z}[\frac{1}{p}]$, the action is $\gamma \cdot T_1^{k_1} \dots T_d^{k_d} := \gamma(k_1, \dots, k_d) T_1^{k_1} \dots T_d^{k_d}$.
- $\overline{R}$:= the p-adic completion of the normalisation of R in the maximal (ind)étale extension of $R[\frac{1}{p}]$—this is acted on by $\Delta := \mathrm{Gal}(R[\frac{1}{p}])$ via R-algebra automorphisms, and its restriction to R_∞ gives the Γ-action there.

Faltings' almost purity theorem states $\overline{R}$ is an "almost étale" R_∞-algebra, and the main consequence of this is that the resulting map on continuous group cohomology

$$R\Gamma_{\mathrm{cont}}(\Gamma, R_\infty) \longrightarrow R\Gamma_{\mathrm{cont}}(\Delta, \overline{R})$$

is an almost quasi-isomorphism (i.e., all cohomology groups of the cone are killed by the maximal ideal $\mathfrak{m} \subset \mathcal{O}$). This is his key to calculating étale cohomology in terms of de Rham cohomology; indeed, $R\Gamma_{\mathrm{cont}}(\Delta, \overline{R})$ is a priori hard to calculate and encodes Galois/étale cohomology, while $R\Gamma_{\mathrm{cont}}(\Gamma, R_\infty)$ is easy to calculate using Koszul complexes (as we will see in the proof of Theorem 2) and differential forms.

The following is our integral form of this result, in which we apply $L\eta$ with respect to any element $f \in \mathfrak{m} \subset \mathcal{O}$:

Theorem 2 *Under the above set-up, the induced map*

$$L\eta_f R\Gamma_{\mathrm{cont}}(\Gamma, R_\infty) \longrightarrow L\eta_f R\Gamma_{\mathrm{cont}}(\Delta, \overline{R})$$

is a quasi-isomorphism (not just an almost quasi-isomorphism!) for any non-zero $f \in \mathfrak{m}$.

Remark 1 (i) The proof of Theorem 2 requires knowing nothing new about $R\Gamma_{\mathrm{cont}}(\Delta, \overline{R})$: a key remarkable property of $L\eta$ is that it can transform almost quasi-isomorphisms into actual quasi-isomorphisms, having only imposed hypotheses on the domain, not the codomain, of the morphism; this will be explained in the next lemma.

(ii) The theorem implies that the kernel and cokernel of $H^i_{\mathrm{cont}}(\Gamma, R_\infty) \to H^i_{\mathrm{cont}}(\Delta, \overline{R})$ are killed by f; since f is any element of $\mathfrak{m}$, the kernel and cokernel are killed by $\mathfrak{m}$. Thus Theorem 2 is a family of on-the-nose integral results which recovers Faltings' almost quasi-isomorphism $R\Gamma_{\mathrm{cont}}(\Gamma, R_\infty) \to R\Gamma_{\mathrm{cont}}(\Delta, \overline{R})$.

Lemma 2 *Let $\mathfrak{M} \subseteq A$ be an ideal of a ring and $f \in \mathfrak{M}$ a non-zero-divisor. Say that an A-module M is "good" if and only if both M and M/fM contain no non-zero elements killed by $\mathfrak{M}$. Then the following statements hold:*

(i) If $M \to N$ is a homomorphism of A-modules with kernel and cokernel killed by $\mathfrak{M}$, and M is good, then $M/M[f] \to N/N[f]$ is an isomorphism.
(ii) If $C \to D$ is a morphism of complexes of A-modules whose cone is killed by $\mathfrak{M}$, and all cohomology groups of C are good, then $L\eta_f C \to L\eta_f D$ is a quasi-isomorphism.

Proof Clearly (ii) is a consequence of (i) and Lemma 1. So we must prove (i).

Since the kernel of M is killed by $\mathfrak{M}$, but M contains no non-zero elements killed by $\mathfrak{M}$, we see that $M \to N$ is injective, and we will henceforth identify M with a submodule of N. Then $M[f] = M \cap N[f]$ and so $M/M[f] \to N/N[f]$ is also injective.

Since the quotient N/M is killed by $\mathfrak{M}$, there is a chain of inclusions $\mathfrak{M}fN \subseteq fM \subseteq fN \subseteq M$. But M/fM contains no non-zero elements killed by $\mathfrak{M}$, so $fM = fN$, and this completes the proof: any $n \in N$ satisfies $fn = fm$ for some $m \in M$, whence $n \equiv m \bmod N[f]$.

Proof (*Proof of Theorem* 2). To prove Theorem 2 we use Faltings' almost purity theorem and Lemma 2 (in the context $A = \mathcal{O}$, $f \in \mathfrak{M} = \mathfrak{m}$): so it is enough to show that $H^i_{\mathrm{cont}}(\Gamma, R_\infty)$ is "good" for all $i \geq 0$. This is a standard type of explicit calculation of $H^i_{\mathrm{cont}}(\Gamma, R_\infty)$ in terms of Koszul complexes. For the sake of the reader unfamiliar with this type of calculation, the special case that $R = \mathcal{O}\langle T^{\pm 1}\rangle$ is presented in a footnote[8]; here in the main text we will prove the general case. Both there and

[8]In this footnote we carry out the calculation of the proof of Theorem 2 when $R = \mathcal{O}\langle T^{\pm 1}\rangle$, in which case $R_\infty = \mathcal{O}\langle T^{\pm 1/p^\infty}\rangle$. To reiterate, we must show that $H^i_{\mathrm{cont}}(\Gamma, R_\infty)$ is good for all $i \geq 0$.

First note that R_∞ admits a Γ-equivariant decomposition into $\mathcal{O}$-submodules

$$R_\infty = \widehat{\bigoplus}_{k \in \mathbb{Z}\left[\frac{1}{p}\right]} \mathcal{O}T^k$$

(where the hat denotes p-adic completion of the sum), with the generator $\gamma \in \Gamma$ acting on the rank-one free $\mathcal{O}$-module $\mathcal{O}T^k$ as multiplication by ζ^k. Thus $R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p, \mathcal{O}T^k) \simeq [\mathcal{O} \xrightarrow{\zeta^k - 1} \mathcal{O}]$ (since the group cohomology of an infinite cyclic group with generator γ is computed by the invariants and coinvariants of γ, and similarly in the case of continuous group cohomology), and so

$$R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p, R_\infty) \simeq \widehat{\bigoplus}_{k \in \mathbb{Z}\left[\frac{1}{p}\right]} [\mathcal{O} \xrightarrow{\zeta^k - 1} \mathcal{O}]$$

here we pick a compatible sequence $\zeta_p, \zeta_{p^2}, \ldots, \in \mathcal{O}$ of p-power roots of unity to get a generator $\gamma \in \mathbb{Z}_p(1)$ and an identification $\Gamma \cong \mathbb{Z}_p^d$; as a convenient abuse of notation, we write $\zeta^k := \zeta_{p^j}^a$ when $k = a/p^j \in \mathbb{Z}[\frac{1}{p}]$.

First note that $\mathcal{O}\langle \underline{T}^{\pm 1/p^\infty}\rangle$ admits a Γ-equivariant decomposition into $\mathcal{O}\langle \underline{T}^{\pm 1}\rangle$-modules:

$$\mathcal{O}\langle \underline{T}^{\pm 1/p^\infty}\rangle = \mathcal{O}\langle \underline{T}^{\pm 1}\rangle \oplus \mathcal{O}\langle \underline{T}^{\pm 1}\rangle^{\text{non-int}},$$

where

$$\mathcal{O}\langle \underline{T}^{\pm 1}\rangle^{\text{non-int}} := \widehat{\bigoplus_{\substack{k_1,\ldots,k_d \in \mathbb{Z}\left[\frac{1}{p}\right]\cap[0,1) \\ \text{not all zero}}}} \mathcal{O}\langle \underline{T}^{\pm 1}\rangle T_1^{k_1} \ldots T_d^{k_d}$$

(where the hat denotes p-adic completion of the sum), with the generators $\gamma_1, \ldots, \gamma_d \in \Gamma$ acting on the rank-one free $\mathcal{O}$-module $\mathcal{O}T_1^{k_1} \ldots T_d^{k_d}$ respectively as multiplication by $\zeta^{k_1}, \ldots, \zeta^{k_d}$.

Base changing to R we obtain a similar Γ-equivariant decomposition of R_∞ into R-modules

$$R_\infty = R \oplus R_\infty^{\text{non-int}}, \qquad R_\infty^{\text{non-int}} := \widehat{\bigoplus_{\substack{k_1,\ldots,k_d \in \mathbb{Z}\left[\frac{1}{p}\right]\cap[0,1) \\ \text{not all zero}}}} RT_1^{k_1} \ldots T_d^{k_d},$$

and so $R\Gamma_{\text{cont}}(\mathbb{Z}_p^d, R_\infty) \simeq R\Gamma_{\text{cont}}(\mathbb{Z}_p^d, R) \oplus R\Gamma_{\text{cont}}(\mathbb{Z}_p^d, R_\infty^{\text{non-int}})$, where

$$R\Gamma_{\text{cont}}(\mathbb{Z}_p^d, R_\infty^{\text{non-int}}) \simeq \widehat{\bigoplus_{\substack{k_1,\ldots,k_d \in \mathbb{Z}\left[\frac{1}{p}\right]\cap[0,1) \\ \text{not all zero}}}} R\Gamma_{\text{cont}}(\mathbb{Z}_p^d, RT_1^{k_1} \ldots T_d^{k_d})$$

(where the hat now denotes the derived p-adic completion of the sum of complexes). Now we must calculate $H^i_{\text{cont}}(\mathbb{Z}_p, ?)$ for $? = R$ or $RT_1^{k_1} \ldots T_d^{k_d}$.

In the first case, the action of $\mathbb{Z}_p^d$ on R is trivial and so a standard group cohomology fact says that $H^i_{\text{cont}}(\mathbb{Z}_p^d, R) \cong \bigwedge_R^i R^d$. In the second case, another standard group

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(where the hat now denotes the derived p-adic completion of the sum of complexes), which has cohomology groups

$$H^0_{\text{cont}}(\mathbb{Z}_p, R_\infty) \cong \widehat{\bigoplus}_{k\in\mathbb{Z}} \mathcal{O} \oplus 0, \qquad H^1_{\text{cont}}(\mathbb{Z}_p, R_\infty) \cong \widehat{\bigoplus}_{k\in\mathbb{Z}} \mathcal{O} \oplus \bigoplus_{k\in\mathbb{Z}\left[\frac{1}{p}\right]\setminus\mathbb{Z}} \mathcal{O}/(\zeta^k - 1)\mathcal{O}$$

(once some care is taken regarding the p-adic completions: see footnote 9).

We claim that both cohomology groups are good. Since $\mathcal{O}$ has no non-zero elements killed by $\mathfrak{m}$, it remains only to prove that the same is true of $\mathcal{O}/a\mathcal{O}$, where $a = f$ or $\zeta^k - 1$ for some $k \in \mathbb{Z}[\frac{1}{p}] \setminus \mathbb{Z}$. But this is an easy argument with valuations: if $x \in \mathcal{O}$ is almost a multiple of a, then $\nu_p(x) + \varepsilon \geq \nu_p(a)$ for all $\varepsilon > 0$, whence $\nu_p(x) \geq \nu_p(a)$ and so x is actually a multiple of a.

cohomology fact says that $R\Gamma_{\text{cont}}(\mathbb{Z}_p^d, RT_1^{k_1}\dots T_d^{k_d})$ can be calculated by the Koszul complex $K_R(\zeta^{k_1}-1,\dots,\zeta^{k_d}-1)$; then Lemma 23 reveals (crucially using that not all k_i are zero) that

$$H^i_{\text{cont}}(\mathbb{Z}_p^d, RT_1^{k_1}\dots T_d^{k_d}) \cong R/(\zeta_{p^r}-1)R^{\binom{d-1}{i-1}}$$

where $r := -\min_{1\le i\le d}\nu_p(k_i) \ge 1$ is the smallest integer such that $\zeta_{p^r}-1|\zeta^{k_i}-1$ for all $i=1,\dots,d$.

Assembling[9] these calculations yields isomorphisms

$$H^i_{\text{cont}}(\Gamma, R_\infty) \cong \bigwedge_R^i R^d \oplus \bigoplus_{\substack{k_1,\dots,k_d\in\mathbb{Z}\left[\frac{1}{p}\right]\cap[0,1)\\ \text{not all zero}}} R/(\zeta_{p^{-\min_{1\le i\le d}\nu_p(k_i)}}-1)R^{\binom{d-1}{i-1}},$$

which we claim is good for each $i\ge 0$. That is, we must show that R, R/fR, and $R/(\zeta_{p^r}-1)R$, for $r\ge 1$, contain no non-zero elements killed by $\mathfrak{m}$. This is trivial for R itself since it is a torsion-free $\mathcal{O}$-algebra, so it remains to show, for each non-zero $a\in\mathfrak{m}$, that R/aR contains no non-zero elements killed by $\mathfrak{m}$; but R is a topologically free $\mathcal{O}$-module [5, Lemma 8.10] and so R/aR is a free $\mathcal{O}/a\mathcal{O}$-module, thereby reducing the problem to the analogous assertion for $\mathcal{O}/a\mathcal{O}$, which was proved in the final paragraph of footnote 8.

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[9]This step requires some care about p-adic completions: the following straightforward result is sufficient. Suppose $(C_\lambda)_\lambda$ is a family of complexes satisfying the following for all $i\in\mathbb{Z}$: the group $H^i(C_\lambda)$ is p-adically complete and separated for all λ, with a bound on its p-power-torsion which is independent of λ. Then $H^i(\widehat{\bigoplus}_\lambda C_\lambda) = \widehat{\bigoplus}_\lambda H^i(C_\lambda)$, where the left hat is the derived p-adic completion of the sum of complexes, and the right hat is the usual p-adic completion of the sum of cohomology groups. *Proof*. Set $C_{\text{disc}} := \bigoplus_\lambda C_\lambda$ and $C = \widehat{C_{\text{disc}}}$ (derived p-adic completion); then the usual short exact sequences associated to a derived p-adic completion are

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & \varprojlim^1_r H^i(C_{\text{disc}})[p^r] & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & \operatorname{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, H^i(C_{\text{disc}})) & \longrightarrow & H^i(C) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, H^{i+1}(C_{\text{disc}})) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & \widehat{H^i(C_{\text{disc}})} = \widehat{\bigoplus}_\lambda H^i(C_\lambda) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Our assumption that $\bigoplus_\lambda H^i(C_\lambda)$ has bounded p-power-torsion implies that the right and top terms vanish.

3 Algebraic Preliminaries on Perfectoid Rings

Fix a prime number p, and let A be a commutative ring which is π-adically complete (and separated) for some element $\pi \in A$ dividing p. Denoting by $\varphi : A/pA \to A/pA$ the absolute Frobenius, we have:

- the *tilt* $A^\flat := \varprojlim_\varphi A/pA$ of A, which is a perfect $\mathbb{F}_p$-algebra, on which we also denote the absolute Frobenius by φ. We sometimes write elements of $A^\flat$ as $x = (x_0, x_1, \dots)$, where $x_i \in A/pA$ and $x_i^p = x_{i-1}$ for all $i \geq 1$, and unless indicated otherwise the "projection $A^\flat \to A/pA$" refers to the map $x \mapsto x_0$.
- the associated "infinitesimal period ring" $W(A^\flat)$ of Fontaine, which is denoted by $\mathbb{A}_{\inf}(A)$ in [5]. Note that, since $A^\flat$ is a perfect ring, $W(A^\flat)$ behaves just like the ring of Witt vectors of a perfect field of characteristic p: in particular p is a non-zero divisor of $W(A^\flat)$, each element has a unique expansion of the form $[x] + p[y] + p^2[z] + \cdots$, and $W(A^\flat)/p^r = W_r(A^\flat)$ for any $r \geq 1$.

The goal of this section is to study these constructions in more detail, in particular to introduce ring homomorphisms

$$\widetilde{\theta}_r, \theta_r : W(A^\flat) \longrightarrow W_r(A)$$

which play a fundamental role in the paper, and to define perfectoid rings.

3.1 The Maps θ_r, $\widetilde{\theta}_r$

The following lemma is helpful in understanding $A^\flat$ and will be used several times; we omit the proof since it is relatively well-known and based on standard p-adic or π-adic approximations:

Lemma 3 *The canonical maps*

$$\varprojlim_{x \mapsto x^p} A \longrightarrow A^\flat = \varprojlim_{\varphi} A/pA \longrightarrow \varprojlim_{\varphi} A/\pi A$$

are isomorphisms of monoids (resp. rings).

Before stating the main lemma which permits us to define the maps θ_r, we recall that if B is any ring, then the associated rings of Witt vectors $W_r(B)$ are equipped with three operators:

$$R, F : W_{r+1}(B) \to W_r(B) \qquad V : W_r(B) \to W_{r+1}(B),$$

where R, F are ring homomorphisms, and V is merely additive. Therefore we can take the limit over r in two ways (of which the second is probably more familiar):

$$\varprojlim_{r \text{ wrt } F} W_r(B) \qquad \text{or} \qquad W(B) = \varprojlim_{r \text{ wrt } R} W_r(B).$$

Lemma 4 *Let A be as above, i.e., a ring which is π-adically complete with respect to some element $\pi \in A$ dividing p. Then the following three ring homomorphisms are isomorphisms:*

$$\begin{array}{ccc} W(A^\flat) = \varprojlim_{r \text{ wrt } R} W_r(A^\flat) & \xleftarrow[(i)]{\varphi^\infty} & \varprojlim_{r \text{ wrt } F} W_r(A^\flat) \\ & & \downarrow (ii) \\ \varprojlim_{r \text{ wrt } F} W_r(A) & \xrightarrow[(iii)]{} & \varprojlim_{r \text{ wrt } F} W_r(A/\pi A) \end{array}$$

where

(i) φ^∞ is induced by the homomorphisms $\varphi^r : W_r(A^\flat) \to W_r(A^\flat)$ for $r \geq 1$;
(ii) the right vertical arrow is induced by the projection $A^\flat \to A/pA \to A/\pi A$;
(iii) the bottom horizontal arrow is induced by the projection $A \to A/\pi A$.

There is therefore an induced isomorphism

$$W(A^\flat) \xrightarrow{\simeq} \varprojlim_{r \text{ wrt } F} W_r(A)$$

making the diagram commute.

Proof We refer the reader to [5, Lemma 3.2] for the elementary proofs of the isomorphisms.

Definition 2 Continue to let A be as in the previous lemma, and $r \geq 1$. Define $\widetilde{\theta}_r : W(A^\flat) \to W_r(A)$ to be the composition

$$\boxed{\widetilde{\theta}_r : W(A^\flat) \xrightarrow{\simeq} \varprojlim_{r \text{ wrt } F} W_r(A) \longrightarrow W_r(A),}$$

where the first map is the isomorphism of the previous lemma, and the second map is the canonical projection. Also define

$$\boxed{\theta_r := \widetilde{\theta}_r \circ \varphi^r : W(A^\flat) \longrightarrow W_r(A).}$$

We stress that the Frobenius maps $F : W_{r+1}(A) \to W_r(A)$ need not be surjective, and thus $\theta_r, \widetilde{\theta}_r$ need not be surjective; indeed, such surjectivity will be part of the definition of a perfectoid ring (see Lemma 7).

To explicitly describe the maps θ_r and $\widetilde{\theta}_r$, we follow the usual convention of exploiting the isomorphism of monoids of Lemma 3 to denote an element $x \in A^\flat$ either as $x = (x_0, x_1, \dots) \in \varprojlim_{\varphi} A/pA$ or $x = (x^{(0)}, x^{(1)}, \dots) \in \varprojlim_{x \mapsto x^p} A$:

Lemma 5 *For any $x \in A^\flat$ we have $\theta_r([x]) = [x^{(0)}] \in W_r(A)$ and $\widetilde{\theta}_r([x]) = [x^{(r)}]$ for $r \geq 1$.*

Proof The formula for $\widetilde{\theta}_r$ follows from a straightforward chase through the above isomorphisms, and the corresponding formula for θ_r is an immediate consequence.

In particular, Lemma 5 implies that $\theta := \theta_1 : W(A^\flat) \to A$ is the usual map of p-adic Hodge theory as defined by Fontaine [14, Sect. 1.2], and also shows that the diagram

$$\begin{array}{ccc} W(A^\flat) & \xrightarrow{\theta_r} & W_r(A) \\ \downarrow & & \downarrow \\ W_r(A^\flat) & \longrightarrow & W_r(A/pA) \end{array}$$

commutes, where the left arrow is the canonical restriction map and the bottom arrow is induced by the projection $A^\flat \to A/pA$.

The following records the compatibility of the maps θ_r and $\widetilde{\theta}_r$ with the usual operators on the Witt groups; though it is probably the first set of diagrams which initially appears more natural, it is the second set which we we will use when constructing Witt complexes:

Lemma 6 *Continue to let A be as in the previous two lemmas. Then the following diagrams commute:*

$$\begin{array}{ccc} W(A^\flat) & \xrightarrow{\theta_{r+1}} & W_{r+1}(A) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle R} \\ W(A^\flat) & \xrightarrow{\theta_r} & W_r(A) \end{array} \qquad \begin{array}{ccc} W(A^\flat) & \xrightarrow{\theta_{r+1}} & W_{r+1}(A) \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle F} \\ W(A^\flat) & \xrightarrow{\theta_r} & W_r(A) \end{array} \qquad \begin{array}{ccc} W(A^\flat) & \xrightarrow{\theta_{r+1}} & W_{r+1}(A) \\ {\scriptstyle \lambda_{r+1}\varphi^{-1}}\uparrow & & \uparrow{\scriptstyle V} \\ W(A^\flat) & \xrightarrow{\theta_r} & W_r(A) \end{array}$$

where the third diagram requires an element $\lambda_{r+1} \in W(A^\flat)$ satisfying $\theta_{r+1}(\lambda_{r+1}) = V(1)$ in $W_{r+1}(A)$. Equivalently, the following diagrams commute:

$$\begin{array}{ccc} W(A^\flat) & \xrightarrow{\widetilde{\theta}_{r+1}} & W_{r+1}(A) \\ {\scriptstyle \varphi^{-1}}\downarrow & & \downarrow{\scriptstyle R} \\ W(A^\flat) & \xrightarrow{\widetilde{\theta}_r} & W_r(A) \end{array} \qquad \begin{array}{ccc} W(A^\flat) & \xrightarrow{\widetilde{\theta}_{r+1}} & W_{r+1}(A) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle F} \\ W(A^\flat) & \xrightarrow{\widetilde{\theta}_r} & W_r(A) \end{array} \qquad \begin{array}{ccc} W(A^\flat) & \xrightarrow{\widetilde{\theta}_{r+1}} & W_{r+1}(A) \\ {\scriptstyle \times\varphi^{r+1}(\lambda_{r+1})}\uparrow & & \uparrow{\scriptstyle V} \\ W(A^\flat) & \xrightarrow{\widetilde{\theta}_r} & W_r(A) \end{array}$$

Proof See [5, Lemma 3.4] for the short verification.

3.2 Perfectoid Rings

The next goal is to define what it means for A to be perfectoid, which requires discussing surjectivity and injectivity of the Frobenius on A/pA. We do this in greater generality than we require, but this greater generality reveals the intimate relation to the map θ and its generalisations $\theta_r, \widetilde{\theta}_r$.

Lemma 7 *Let A be a ring which is π-adically complete with respect to some element $\pi \in A$ such that π^p divides p. Then the following are equivalent:*

(i) Every element of $A/\pi pA$ is a p^{th}-power.
(ii) Every element of A/pA is a p^{th}-power.
(iii) Every element of $A/\pi^p A$ is a p^{th}-power.
(iv) The Witt vector Frobenius $F : W_{r+1}(A) \to W_r(A)$ is surjective for all $r \geq 1$.
(v) $\theta_r : W(A^\flat) \to W_r(A)$ is surjective for all $r \geq 1$.
(vi) $\theta : W(A^\flat) \to A$ is surjective.

Moreover, if these equivalent conditions hold then there exist $u, v \in A^\times$ such that $u\pi$ and vp admit systems of p-power roots in A.

Proof The implications (i)⇒(ii)⇒(iii) are trivial since $\pi pA \subseteq pA \subseteq \pi^p A$. (v)⇒ (vi) is also trivial since $\theta = \theta_1$.

(iii)⇒(i): a simple inductive argument allows us to write any given element $x \in A$ as an infinite sum $x = \sum_{i=0}^{\infty} x_i^p \pi^{pi}$ for some $x_i \in A$; but then $x \equiv (\sum_{i=0}^{\infty} x_i \pi^i)^p$ mod $p\pi A$.

(iv)⇒(ii): Clear from the fact that the Frobenius $F : W_2(A) \to W_1(A) = A$ is explicitly given by $(\alpha_0, \alpha_1) \mapsto \alpha_0^p + p\alpha_1$.

(iv)⇒(v): The hypothesis states that the transition maps in the inverse system $\varprojlim_{r \text{ wrt } F} W_r(A)$ are surjective, which implies that each map $\widetilde{\theta}_r$ is surjective, and hence that each map θ_r is surjective.

(vi)⇒(ii): Clear since any element of A in the image of θ is a p^{th}-power mod p.

It remains to show that (ii)⇒(iv), but we will first prove the "moreover" assertion using only (i) (which we have shown is equivalent to (ii)). Applying Lemma 3 to both A and $A/\pi pA$ implies that the canonical map $\varprojlim_{x \mapsto x^p} A \to \varprojlim_{x \mapsto x^p} A/\pi pA$ is an isomorphism. Applying (i) repeatedly, there therefore exists $\omega \in \varprojlim_{x \mapsto x^p} A$ such that $\omega^{(0)} \equiv \pi$ mod πpA (resp. $\equiv p$ mod πpA). Writing $\omega^{(0)} = \pi + \pi px$ (resp. $\omega^{(0)} = p + \pi px$) for some $x \in A$, the proof of the "moreover" assertion is completed by noting that $1 + px \in A^\times$ (resp. $1 + \pi x \in A^\times$).

(ii)⇒(iv): By the "moreover" assertion, there exist $\pi' \in A$ and $v \in A^\times$ satisfying $\pi'^p = vp$. Note that A is π'-adically complete, and so we may apply the implication (ii)⇒(i) for the element π' to deduce that every element of $A/\pi' pA$ is a p^{th}-power; it follows that every element of A/Ip is a p^{th}-power, where I is the ideal $\{a \in A : a^p \in pA\}$. Now apply implication "(xiv)$'$ ⇒(ii)" of Davis–Kedlaya [10].

Lemma 8 *Let A be a ring which is π-adically complete with respect to some element $\pi \in A$ such that π^p divides p, and assume that the equivalent conditions of the previous lemma are true.*

(i) If Ker θ *is a principal ideal of* $W(A^\flat)$*, then*

(a) $\Phi : A/\pi A \to A/\pi^p A$*,* $a \mapsto a^p$*, is an isomorphism;*
(b) any generator of Ker θ *is a non-zero-divisor*[10]*;*
(c) an element $\xi \in$ Ker θ *is a generator if and only if it is "distinguished", i.e., its Witt vector expansion* $\xi = (\xi_0, \xi_1, \ldots)$ *has the property that* ξ_1 *is a unit of* $A^\flat$.
(d) any element $\xi \in$ Ker θ *satisfying* $\theta_r(\xi) = V(1) \in W_r(A)$ *for some* $r > 1$ *is distinguished (and such an element exists for any given* $r > 1$*).*

(ii) Conversely, if π *is a non-zero-divisor and* $\Phi : A/\pi A \to A/\pi^p A$ *is an isomorphism (which is automatic if* A *is integrally closed in* $A[\frac{1}{\pi}]$*), then* Ker θ *is a principal ideal.*

Proof Rather than copying the proof here, we refer the reader to Lemma 3.10 and Remark 3.11 of [5]. The only assertion which is not proved there is the parenthetical assertion in (ii), for which we just note that if A is integrally closed in $A[\frac{1}{\pi}]$, then Φ is automatically injective: indeed, if a^p divides π^p, then $(a/\pi)^p \in A$ and so $a/\pi \in A$.

We can now define a perfectoid ring[11]:

Definition 3 A ring A is *perfectoid* if and only if the following three conditions hold:

- A is π-adically complete for some element $\pi \in A$ such that π^p divides p;
- the Frobenius map $\varphi : A/pA \to A/pA$ is surjective (equivalently, $\theta : W(A^\flat) \to A$ is surjective);
- the kernel of $\theta : W(A^\flat) \to A$ is principal.

Remark 2 The first condition of the definition could be replaced by the seemingly stronger, but actually equivalent and perhaps more natural, condition that "A is p-adically complete and there exists a unit $u \in A^\times$ such that pu is a p^{th}-power." Indeed, this follows from the final assertion of Lemma 7.

We return to the maps θ_r, describing their kernels in the case of a perfectoid ring:

Lemma 9 *Suppose that* A *is a perfectoid ring, and let* $\xi \in W(A)$ *be any element generating* Ker θ *(this exists by Lemma 7). Then* Ker θ_r *is generated by the non-zero-divisor*

[10] In all our cases of interest the ring A will be an integral domain, in which case it may be psychologically comforting to note that $A^\flat$ and $W(A^\flat)$ are also integral domains. *Proof.* The ring $W(A^\flat)$ is p-adically separated, satisfies $W(A^\flat)/p = A^\flat$, and p is a non-zero-divisor in it (these properties all follow simply from $A^\flat$ being perfect). So, once we show that $A^\flat$ is an integral domain, it will easily follow that $W(A^\flat)$ is also an integral domain. But the fact that $A^\flat$ is an integral domain follows at once from the same property of A using the isomorphism of monoids $\varprojlim_{x \mapsto x^p} A \xrightarrow{\simeq} A^\flat$ which already appeared in Lemma 4.

[11] Perhaps "*integral* perfectoid ring" would be better terminology to avoid conflict with the more common notion of perfectoid algebras in which p is invertible.

$$\xi_r := \xi\varphi^{-1}(\xi)\dots\varphi^{-(r-1)}(\xi)$$

for any $r \geq 1$*, and so* Ker $\widetilde{\theta}_r$ *is generated by the non-zero-divisor*

$$\widetilde{\xi}_r := \varphi^r(\xi_r) = \varphi(\xi)\dots\varphi^r(\xi).$$

Proof It is enough to prove the claim about ξ_r, since the claim about $\widetilde{\xi}_r$ then follows by applying φ^r. The proof is by induction on $r \geq 1$, using the diagrams of Lemma 6 for the inductive step; we refer to [5, Lemma 3.12] for the details.

We finish this introduction to perfectoid rings with some examples:

Example 1 (*Perfect rings of characteristic p*) Suppose that A is a ring of characteristic p. Then A is perfectoid if and only if it is perfect. Indeed, if A is perfect, then it is 0-adically complete, the Frobenius is surjective, and the kernel of $\theta : W(A) \to A$ is generated by p. Conversely, if A is perfectoid, then Lemma 8(i)(c) implies that the distinguished element $p \in \mathrm{Ker}(\theta : W(A^\flat) \to A)$ must be a generator, whence $W(A^\flat)/p \cong A$; but $W(A^\flat)/p = A^\flat$ is perfect.

In particular, in this case $A^\flat = A$ and the maps $\theta_r : W(A^\flat) \to W_r(A)$ are the canonical Witt vector restriction maps.

Example 2 If $\mathbb{C}$ is a complete, non-archimedean algebraic closed field of residue characteristic $p > 0$, then its ring of integers $\mathcal{O}$ is a perfectoid ring. Indeed, if $\mathbb{C}$ has equal characteristic p then $\mathcal{O}$ is perfect and we may appeal to the previous lemma. If $\mathbb{C}$ has mixed characteristic (our main case of interest), then $\mathcal{O}$ is $p^{1/p}$-adically complete, integrally closed in $\mathcal{O}[\frac{1}{p^{1/p}}] = \mathbb{C}$, and every element of $\mathcal{O}/p\mathcal{O}$ is a p^{th}-power since $\mathbb{C}$ is algebraically closed, so we may appeal to Lemma 8(ii); in this situation the ring $W(\mathcal{O}^\flat)$ will always be denoted by $\mathbb{A}_{\mathrm{inf}}$.

Example 3 Let A be a perfectoid ring which is π-adically complete with respect to some non-zero-divisor $\pi \in A$ such that π^p divides p. Here we offer some constructions of new perfectoid rings from A:

(i) The rings $A\langle T_1^{1/p^\infty}, \dots, T_d^{1/p^\infty}\rangle$ and $A\langle T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\rangle$, which are by definition the π-adic completions of $A[T_1^{1/p^\infty}, \dots, T_d^{1/p^\infty}]$ and $A[T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}]$ respectively, are also perfectoid.
(ii) Any π-adically complete, formally étale A-algebra is also perfectoid.

Proof Since the π-adic completeness of the given ring is tautological in each case, we only need to check that $\Phi : B/\pi B \to B/\pi^p B, b \mapsto b^p$ is an isomorphism in each case. This is clear for $B = A\langle \underline{T}^{\pm 1/p^\infty}\rangle$ and $A\langle \underline{T}^{1/p^\infty}\rangle$, and it hold for and A-algebra B as in (ii) since the square

$$\begin{array}{ccc} B/\pi & \xrightarrow{\varphi} & B/\pi \\ \uparrow & & \uparrow \\ A/\pi & \xrightarrow[\varphi]{} & A/\pi \end{array}$$

is a pushout diagram (the base change of the Frobenius along an étale morphism in characteristic p is again the Frobenius).

3.3 Main Example: Perfectoid Rings Containing Enough Roots of Unity

Here in Sect. 3.3 we fix a perfectoid ring A which has no p-torsion and which contains a compatible system $\zeta_p, \zeta_{p^2}, \dots$ of primitive p-power roots of unity (to be precise, since A is not necessarily an integral domain, this means that ζ_{p^r} is a root of the p^{rth} cyclotomic polynomial), which we fix. The simplest example is $\mathcal{O}$ itself, but we also need the theory for perfectoid algebras containing $\mathcal{O}$ such as $\mathcal{O}\langle T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\rangle$.

In particular we define particular elements $\varepsilon, \xi, \mu, \dots$, which will be used repeatedly in our main constructions, and so we highlight (or rather box) the primary definitions and relations. Firstly, set

$$\boxed{\varepsilon := (1, \zeta_p, \zeta_{p^2}, \dots) \in A^\flat, \qquad \mu := [\varepsilon] - 1 \in W(A^\flat),}$$

and

$$\boxed{\xi := 1 + [\varepsilon^{1/p}] + [\varepsilon^{1/p}]^2 + \dots + [\varepsilon^{1/p}]^{p-1} \in W(A^\flat).}$$

Lemma 10 *ξ is a generator of* Ker θ *satisfying $\theta_r(\xi) = V(1)$ for all $r \geq 1$.*

Proof By Lemma 8(i)(d) it is sufficient to show that $\theta_r(\xi) = V(1)$ for all $r \geq 1$. The ghost map gh : $W_r(A) \to A^r$ is injective since A is p-torsion-free, and so it is sufficient to prove that $\text{gh}(\theta_r(\xi)) = \text{gh}(V(1))$. But it follows easily from Lemma 5 that the composition $\text{gh} \circ \theta_r : W(A^\flat) \to A^r$ is given by $(\theta, \theta\varphi, \dots, \theta\varphi^{r-1})$, and so in particular that

$$\text{gh}(\theta_r(\xi)) = (\theta(\xi), \theta\varphi(\xi), \dots, \theta\varphi^{r-1}(\xi)).$$

Since $\theta(\xi) = 0$ and $\text{gh}(V(1)) = (0, p, p, p, \dots)$, it remains only to check that $\theta\varphi^i(\xi) = p$ for all $i \geq 1$, which is straightforward:

$$\theta\varphi^i(\xi) = \theta(1 + [\varepsilon^{p^{i-1}}] + [\varepsilon^{p^{i-1}}]^2 + \dots + [\varepsilon^{p^{i-1}}]^{p-1}) = 1 + 1 + \dots + 1 = p.$$

It now follows from Lemma 9 that Ker θ_r is generated by

$$\boxed{\xi_r := \xi\varphi^{-1}(\xi) \dots \varphi^{-(r-1)}(\xi) = \sum_{i=0}^{p^r-1} [\varepsilon^{1/p^r}]^i,}$$

and that Ker $\widetilde{\theta}_r$ is generated by

$$\boxed{\widetilde{\xi}_r := \varphi^r(\xi_r) = \varphi(\xi)\dots\varphi^r(\xi)}.$$

Proposition 2 *μ is a non-zero divisor of $W(A^\flat)$ which satisfies*

$$\boxed{\mu = \xi_r\varphi^{-r}(\mu), \qquad \varphi^r(\mu) = \widetilde{\xi}_r\mu, \qquad \widetilde{\theta}_r(\mu) = [\zeta_{p^r}] - 1 \in W_r(A)}$$

for all $r \geq 1$.

Proof The final identity is immediate from Lemma 5. It is clear that $\mu = \xi\varphi^{-1}(\mu)$, whence the identity $\mu = \xi_r\varphi^{-r}(\mu)$ follows by a trivial induction on r, and the central identity then follows by applying φ^r. To prove that μ is a non-zero-divisor, it suffices to show that $\widetilde{\theta}_r(\mu) = [\zeta_{p^r}] - 1$ is a non-zero-divisor of $W_r(A)$ for all $r \geq 1$ (since $W(A^\flat) = \varprojlim_{r \text{ wrt } F} W_r(A)$). Since A is p-torsion-free the ghost map is injective and so we may check this by proving that

$$\text{gh}([\zeta_{p^r}] - 1) = (\zeta_{p^r} - 1, \zeta_{p^{r-1}} - 1, \dots, \zeta_p - 1)$$

is a non-zero-divisor of A^r; i.e., we must show that $\zeta_{p^r} - 1$ is a non-zero-divisor in A for all $r \geq 1$. But $\zeta_{p^r} - 1$ divides p, and A is assumed to be p-torsion-free.

Remark 3 The reader may wish to note that the Teichmüller lifts $[\zeta_p], [\zeta_{p^2}], \dots$ are not primitive p-power roots unity in $W_r(A)$ in any reasonable sense. Indeed, it follows from its ghost components $\text{gh}([\zeta_p]) = (\zeta_p, 1, 1, \dots, 1)$ that $[\zeta_p]$ is not a root of $X^{p-1} + \cdots + X + 1$ when $r > 1$.

However, the element $[\zeta_{p^r}] - 1 \in W_r(A)$ will play a distinguished role in our constructions and so we point out that it is a non-zero-divisor whose powers define the p-adic topology. Indeed, it follows from the ghost component calculation of the previous proposition that $[\zeta_{p^r}] - 1$ is a root of the polynomial

$$((X+1)^{p^r} - 1)/X = X^{p^r-1} + pX(\cdots) + p^r,$$

whence p divides $([\zeta_{p^r}] - 1)^{p^r-1}$, and $[\zeta_{p^r}] - 1$ divides p^r. A particularly important consequence of this is that $L\eta_{[\zeta_{p^r}]-1}$ commutes with derived p-adic completion, by [5, Lemma 6.20].

4 The Pro-étale Site and Its Sheaves

In this section we review aspects of pro-étale cohomology following [25, Sects. 3–4], working under the following set-up:

- $\mathbb{C}$ is a complete, non-archimedean, algebraically closed field of mixed characteristic; ring of integers $\mathcal{O}$ with maximal ideal $\mathfrak{m}$; residue field k.
- X is a quasi-separated rigid analytic variety over $\mathbb{C}$.

In particular, we will introduce various pro-étale sheaves on X which will play an essential role in our constructions, and explain how to calculate their cohomology via affinoid perfectoids and almost purity theorems.

4.1 The Pro-étale Site $X_{\mathbf{pro\acute{e}t}}$

We will take for granted that the reader is either familiar with, or can reasonably imagine, étale morphisms and coverings of rigid analytic varieties, and we let $X_{\text{ét}}$ denote the associated étale site of X. To define coverings in $X_{\text{ét}}$ (and soon in $X_{\text{proét}}$) it is useful to view X as an adic space,[12] and we therefore denote by $|X|$ the underlying topological space of its associated adic space X^{ad}: for example, if T is an affinoid $\mathbb{C}$-algebra, then $|\operatorname{Sp} T|$ denotes the topological space of (equivalences classes of) all continuous valuations on T, not merely those factoring through a maximal ideal (which correspond to the closed points of the adic space).

We now define (a countable version of) Scholze's pro-étale site $X_{\text{proét}}$ in several steps:

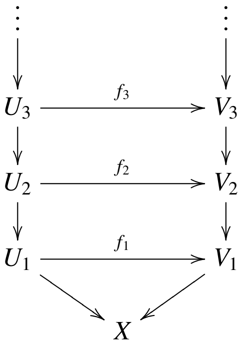
- An object of $X_{\text{proét}}$ is simply a formal inverse system $\mathcal{U} = \text{“}\varprojlim\text{”}_i\, U_i$ in $X_{\text{ét}}$ of the form

$$\begin{array}{c} \vdots \\ \downarrow \\ U_3 \\ \downarrow{\scriptstyle \text{fin. ét. surj.}} \\ U_2 \\ \downarrow{\scriptstyle \text{fin. ét. surj.}} \\ U_1 \\ \downarrow{\scriptstyle \text{ét.}} \\ X \end{array}$$

  In other words, $\mathcal{U}$ is the data of a tower of finite étale covers of U_1, which is étale over X. The underlying topological space of $\mathcal{U}$ is by definition $|\mathcal{U}| := \varprojlim_i |U_i|$.

[12] There is an equivalence of categories between quasi-separated rigid analytic varieties over $\mathbb{C}$ and those adic spaces over $\operatorname{Spa}(\mathbb{C}, \mathcal{O})$ whose structure map is quasi-separated and locally of finite-type [16, Proposition 4.5]. A collection of étale maps $\{f_\lambda : U_\lambda \to U\}$ in $X_{\text{ét}}$ is a cover if and only if it is jointly “strongly surjective”, which is equivalent to being jointly surjective at the level of adic points [17, Sect. 2.1].

- Up to isomorphism,[13] a morphism $f : \mathcal{U} \to \mathcal{V}$ in $X_{\text{proét}}$ is simply a compatible family of morphisms between the towers



- A morphism f as immediately above is called *pro-étale* if and only if it satisfies the following additional condition: the induced finite étale map

$$U_{i+1} \longrightarrow U_i \times_{V_i} V_{i+1}$$

  is surjective for each $i \geq 1$. It can be shown that this implies that the induced continuous map of topological spaces $|f| : |\mathcal{U}| \to |\mathcal{V}|$ is an open mapping [25, Lemma 3.10(iv)].
  Then a collection of morphisms $\{f_\lambda : \mathcal{U}_\lambda \to \mathcal{U}\}$ in $X_{\text{proét}}$ is defined to be a cover if and only if each morphism f_λ is pro-étale and the collection $\{|f_\lambda| : |\mathcal{U}_\lambda| \to |\mathcal{U}|\}$ is a pointwise covering of the topological space $|\mathcal{U}|$. For the proof that this indeed defines a Grothendieck topology we refer the reader to [25, Lemma 3.10].

This completes the definition of the pro-étale site $X_{\text{proét}}$.[14]

[13]This means that we are permitted to replace the towers "$\varprojlim$"$_i\, U_i$ and "$\varprojlim$"$_i\, V_i$ by "obviously isomorphic" towers, e.g., by inserting or removing some stages of the tower. To be precise, first let pro-$X_{\text{ét}}$ denote the usual category of countable inverse systems in $X_{\text{ét}}$: its objects are inverse systems "$\varprojlim$"$_i\, U_i$ in $X_{\text{ét}}$, and its morphisms are defined by $\mathrm{Hom}(\text{“}\varprojlim\text{”}_i\, U_i, \text{“}\varprojlim\text{”}_j\, V_j) := \varprojlim_j \varinjlim_i \mathrm{Hom}_{X_{\text{ét}}}(U_i, V_j)$. Then call an object $\mathcal{U}$ of pro-$X_{\text{ét}}$ *pro-étale* if and only if it is isomorphic in pro-$X_{\text{ét}}$ to an inverse system "$\varprojlim$"$_i\, U_i$ whose transition maps are finite étale surjective; and call a morphism $f : \mathcal{U} \to \mathcal{V}$ *pro-étale* if and only if there exist isomorphisms $\mathcal{U} \cong$ "$\varprojlim$"$_i\, U_i$ and $\mathcal{V} \cong$ "$\varprojlim$"$_i\, V_i$ in pro-$X_{\text{ét}}$ such that "$\varprojlim$"$_i\, U_i$ and "$\varprojlim$"$_i\, V_i$ have finite étale surjective transition maps and such that the resulting morphism "$\varprojlim$"$_i\, U_i \to$ "$\varprojlim$"$_i\, V_i$ has the shape described in the main text. Then the category $X_{\text{proét}}$ is more correctly defined as the full subcategory of pro-$X_{\text{ét}}$ consisting of pro-étale objects, and covers are defined as in the main text using the more correct definition of a pro-étale morphism.

[14]The topos of abelian sheaves on $X_{\text{proét}}$ is "algebraic" in the sense of [27, Definition VI.2.3]; see [25, Proposition 3.12] for this and further properties of the site. In particular, it then follows from [27, Corollary VI.5.3] that if $\mathcal{U} \in X_{\text{proét}}$ is such that $|\mathcal{U}|$ is quasi-compact and quasi-separated, then $H^*_{\text{proét}}(\mathcal{U}, -)$ commutes with filtered inductive colimits of sheaves.

There is an obvious projection functor of sites

$$\nu : X_{\text{proét}} \longrightarrow X_{\text{ét}}$$

obtained by pulling back any $U \in X_{\text{ét}}$ to the constant tower $\cdots \to U \to U \to U \to X$ in $X_{\text{proét}}$; this satisfies the unsurprising[15] property that if $\mathcal{F}$ is a sheaf on $X_{\text{ét}}$, and $\mathcal{U} = \text{“}\varprojlim\text{”}_i U_i \in X_{\text{proét}}$, then

$$\nu^*\mathcal{F}(\mathcal{U}) = \varinjlim_i \mathcal{F}(U_i)$$

and more generally

$$H^i_{\text{proét}}(\mathcal{U}, \nu^*\mathcal{F}) = \varinjlim H^i_{\text{ét}}(U_i, \mathcal{F})$$

for all $i \geq 0$ [25, Lemma 3.16]. For this reason the most interesting sheaves on $X_{\text{proét}}$ are not obtained via pullback from $X_{\text{ét}}$, although our first examples of sheaves on $X_{\text{proét}}$ are of this form.

The *integral* and *rational structure sheaves* $\mathcal{O}^+_{X_{\text{ét}}}$ and $\mathcal{O}_{X_{\text{ét}}}$ on $X_{\text{ét}}$ are defined by

$$\mathcal{O}^+_{X_{\text{ét}}}(\operatorname{Sp} T) := T^\circ \subset T =: \mathcal{O}_{X_{\text{ét}}}(\operatorname{Sp} T)$$

where $\operatorname{Sp} T \in X_{\text{ét}}$ is any rigid affinoid, and T° denotes the subring of power bounded elements inside T. The integral structure sheaf was not substantially studied in the classical theory.[16] Pulling back then defines the *integral* and *rational structure sheaves* $\mathcal{O}^+_X$ and $\mathcal{O}_X$ on $X_{\text{proét}}$

$$\mathcal{O}^+_X := \nu^*\mathcal{O}^+_{X_{\text{ét}}} \subset \mathcal{O}_X := \nu^*\mathcal{O}_{X_{\text{ét}}},$$

which are our first examples of sheaves on $X_{\text{proét}}$.

We now describe the finer, local nature of the pro-étale site by introducing affinoid perfectoids and stating the fundamental role which they play in the theory.

Definition 4 An object $\mathcal{U} = \text{“}\varprojlim\text{”}_i U_i$ in $X_{\text{proét}}$ is called *affinoid perfectoid* if and only if it satisfies the following two conditions:

- U_i is a rigid affinoid, i.e., $U_i = \operatorname{Sp} T_i$ for some affinoid $\mathbb{C}$-algebra T_i, for each $i \geq 1$;
- and the p-adic completion of the ring $\mathcal{O}^+_X(\mathcal{U}) = \varinjlim_i T_i^\circ$ is a perfectoid ring.[17]

[15] Nonetheless, a condition is required: we must assume that the topological space $|\mathcal{U}|$ is quasi-compact and quasi-separated; this is satisfied in particular when $\mathcal{U}$ is a tower of rigid affinoids.

[16] Unlike the rational structure sheaf, the integral structure sheaf can have non-zero higher cohomology on rigid affinoids.

[17] We emphasise that, in our current set-up, this perfectoid ring will always be the type considered in Sect. 3.3: indeed, it is p-torsion-free since each T_i is p-torsion-free, and it contains a compatible sequence of primitive p-power roots of unity since it contains $\mathcal{O}$.

The following key result makes precise the idea that X looks locally perfectoid in the pro-étale topology, and that affinoid perfectoids are small enough for their cohomology to almost vanish, thereby allowing them to be used for almost calculations à la Čech, as we will see further in Sect. 4.3.

Proposition 3 (Scholze)

(i) The affinoid perfectoid objects of $X_{\text{proét}}$ form a basis for the site.
(ii) If $\mathcal{U} \in X_{\text{proét}}$ is affinoid perfectoid, then $H^*_{\text{proét}}(\mathcal{U}, \mathcal{O}^+_X/p)$ is almost zero (i.e., killed by $\mathfrak{m}$) for $* > 0$.

Proof These are consequences of the tilting formalism and almost purity theorems developed in [24]. See Corollary 4.7 and Lemma 4.10 of [25].

To complement the previous local result we recall also the key global result about pro-étale cohomology, which we will need:

Theorem 3 (Scholze) *If the rigid analytic variety X is moreover proper and smooth over $\mathbb{C}$, then the canonical map of $\mathcal{O}/p\mathcal{O}$-modules*

$$H^i_{\text{ét}}(X, \mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{Z}/p\mathbb{Z}} \mathcal{O}/p\mathcal{O} \longrightarrow H^i_{\text{proét}}(X, \mathcal{O}^+_X/p)$$

is an almost isomorphism (i.e., the kernel and cokernel are killed by $\mathfrak{m}$) for all $i \geq 0$.

Proof See [25, Sect. 5].

4.2 More Sheaves on $X_{\text{proét}}$

As indicated by Proposition 3(ii) and Theorem 3, the pro-étale sheaf $\mathcal{O}^+_X/p$ on X enjoys some special properties, and this richness passes to the *completed integral structure sheaf*

$$\widehat{\mathcal{O}}^+_X := \varprojlim_s \mathcal{O}^+_X/p^s,$$

which is probably the most important sheaf on $X_{\text{proét}}$. We stress that it is not known whether $\mathcal{O}^+_X(\mathcal{U})$ coincides with the p-adic completion of $\mathcal{O}^+_X(\mathcal{U})$ for arbitrary objects $\mathcal{U} \in X_{\text{proét}}$.

Further sheaves of interest on $X_{\text{proét}}$ are collected in the following definition:

Definition 5 The *tilted integral structure sheaf* [18] is

$$\mathcal{O}^{+\flat}_X := \varprojlim_{\varphi} \mathcal{O}^+_X/p,$$

[18] Usually denoted by $\widehat{\mathcal{O}}^+_{X^\flat}$ to evoke the idea of it being the completed integral structure sheaf on the tilt $X^\flat$ of X.

where the limit is taken over iterations of the Frobenius map φ on the sheaf of $\mathbb{F}_p$-algebras $\mathcal{O}_X^+/p$. We will also need Witt vector forms[19] of the completed and tilted integral structure sheaves

$$W_r(\widehat{\mathcal{O}}_X^+) \qquad \text{and} \qquad W_r(\mathcal{O}_X^{+\flat}),$$

and the *infinitesimal period sheaf*

$$\mathbb{A}_{\inf,X} := W(\mathcal{O}_X^{+\flat}).$$

By repeating Lemma 4 in terms of presheaves on $X_{\text{proét}}$ and then sheafifying, we obtain a canonical isomorphism of pro-étale sheaves

$$\mathbb{A}_{\inf,X} \xrightarrow{\simeq} \varprojlim_{r \text{ wrt } F} W_r(\widehat{\mathcal{O}}_X^+).$$

As in the affine case in Sect. 3.1 we then denote the resulting projection maps and their Frobenius twists by

$$\widetilde{\theta}_r : \mathbb{A}_{\inf,X} \longrightarrow W_r(\widehat{\mathcal{O}}_X^+) \quad \text{and} \quad \theta_r = \widetilde{\theta}_r \circ \varphi^r : \mathbb{A}_{\inf,X} \longrightarrow W_r(\widehat{\mathcal{O}}_X^+).$$

To reduce further analysis of all these sheaves to the affine case of Sect. 3, we combine the fact that X is locally perfectoid in the pro-étale topology (Proposition 3(i)) with the fact that the sections of these sheaves on affinoid perfectoids are "as expected":

Lemma 11 (Scholze) *Let* $\mathcal{U} = \text{“}\varprojlim_i\text{”}\, U_i$ *be an affinoid perfectoid in* $X_{\text{proét}}$*, with associated perfectoid ring* $A := \mathcal{O}_X^+(\mathcal{U})\widehat{\ }_p$*. Then*

$$\widehat{\mathcal{O}}_X^+(\mathcal{U}) = A, \quad W_r(\widehat{\mathcal{O}}_X^+)(\mathcal{U}) = W_r(A), \quad \mathcal{O}_X^{+\flat}(\mathcal{U}) = A^\flat,$$
$$W_r(\mathcal{O}_X^{+\flat})(\mathcal{U}) = W_r(A^\flat), \quad \mathbb{A}_{\inf,X}(\mathcal{U}) = W(A^\flat).$$

On the other hand, for $* > 0$ *the pro-étale cohomology groups*

$$H^*_{\text{proét}}(\mathcal{U}, \widehat{\mathcal{O}}_X^+), \quad H^*_{\text{proét}}(\mathcal{U}, W_r(\widehat{\mathcal{O}}_X^+)), \quad H^*_{\text{proét}}(\mathcal{U}, \mathcal{O}_X^{+\flat}),$$
$$H^*_{\text{proét}}(\mathcal{U}, W_r(\mathcal{O}_X^{+\flat})), \quad H^*_{\text{proét}}(\mathcal{U}, \mathbb{A}_{\inf,X})$$

are almost zero, i.e., killed respectively by $\mathfrak{m}$, $W_r(\mathfrak{m})$, $\mathfrak{m}^\flat$, $W_r(\mathfrak{m}^\flat)$, $[\mathfrak{m}^\flat]$.[20]

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[19] If $\mathcal{R}$ is a sheaf of rings on a site $\mathcal{T}$, then $W_r(\mathcal{R})$ and $W(\mathcal{R})$ are the sheaves of rings obtained by applying the Witt vector construction section-wise, i.e., $W_r(\mathcal{R})(U) := W_r(\mathcal{R}(U))$ and $W(\mathcal{R})(U) := W(\mathcal{R}(U))$ for all $U \in \mathcal{T}$.

[20] Now seems to be an appropriate moment for mentioning some formalism of almost mathematics over Witt rings. By a "setting for almost mathematics" we mean a pair (V, I), where V is a ring and $I = I^2 \subseteq V$ is an ideal which is an increasing union of principal ideals $\bigcup_\lambda t_\lambda V$ generated by non-zero-divisors t_λ. Elementary manipulations of Witt vectors [5, Lemma 10.1 and

Proof See Lemmas 4.10, 5.11 and Theorem 6.5 of [25] for the description of the sections. The almost vanishings follow by taking suitable limits of Proposition 3(ii).

Corollary 3 *The maps of pro-étale sheaves $\theta_r, \widetilde{\theta}_r : \mathbb{A}_{\text{inf},X} \to W_r(\widehat{\mathcal{O}}_X^+)$ are surjective, with kernels generated respectively by the elements $\xi_r, \widetilde{\xi}_r \in \mathbb{A}_{\text{inf}} = W(\mathcal{O}^\flat)$ defined in Sect. 3.3; moreover, these elements (as well as $\mu \in \mathbb{A}_{\text{inf}}$, also defined in Sect. 3.3) are non-zero-divisors of the sheaf of rings $\mathbb{A}_{\text{inf},X}$.*

Proof All assertions are local, so by Proposition 3(i) it is sufficient to prove the analogous affine assertions after taking sections in any affinoid perfectoid $\mathcal{U} \in X_{\text{proét}}$; but using the descriptions of the sections given by the previous lemma, these affine assertions were covered in Sects. 3.2–3.3.

4.3 Calculating Pro-étale Cohomology

This section is devoted to an explanation of how Proposition 3(ii) is used in practice to (almost) calculate the pro-étale cohomology of our sheaves of interest; this is of course the pro-étale analogue of Faltings' purity theorem and techniques which we saw in Sect. 2.2. We assume in this section that our rigid analytic variety X is the generic fibre $\mathfrak{X}_{\mathbb{C}}$ of a smooth p-adic formal scheme $\mathfrak{X}$ over $\mathcal{O}$; this will be the set-up of our main results later.

Relatively elementary arguments show that $\mathfrak{X}$ admits a basis of affine opens $\{\operatorname{Spf} R\}$ where each R is a p-adically complete, formally smooth $\mathcal{O}$-algebra which is moreover small, i.e., formally étale over $\mathcal{O}\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle$. Fix such an open $\operatorname{Spf} R \subseteq \mathfrak{X}$ (as well as a formally étale map $\mathcal{O}\langle \underline{T}^{\pm 1}\rangle \to R$, sometimes called a "framing"); the associated generic fibre is the rigid affinoid space $U := \operatorname{Sp} R[\frac{1}{p}] \subseteq X$, which is equipped with an étale morphism to $\operatorname{Sp} \mathbb{C}\langle \underline{T}^{\pm 1}\rangle$. We will explain how to almost calculate the pro-étale cohomology groups $H^*_{\text{proét}}(\operatorname{Sp} R[\frac{1}{p}], ?)$ where ? is any of the sheaves from Lemma 11.

For each $i \geq 1$, let

$$R_i := R \otimes_{\mathcal{O}\langle \underline{T}^{\pm 1}\rangle} \mathcal{O}\langle \underline{T}^{\pm 1/p^i}\rangle$$

Corollary 10.2] then show that each Teichmüller lift $[t_\lambda] \in W_r(V)$ is a non-zero-divisor and that $W_r(I) := \operatorname{Ker}(W_r(V) \to W_r(V/I))$ equals the increasing union $\bigcup_\lambda [t_\lambda] W_r(V)$, which moreover coincides with its square; in conclusion, the pair $(W_r(V), W_r(I))$ is also a setting for almost mathematics. We apply this above, and elsewhere, in the cases $(V, I) = (\mathcal{O}, \mathfrak{m})$ and $(\mathcal{O}^\flat, \mathfrak{m}^\flat)$.

Upon taking the limit as $r \to \infty$, the inclusion $W(I) := \operatorname{Ker}(W(V) \to W(V/I)) \supset [I] := \bigcup_\lambda [t_\lambda] W(V)$ is strict; the pair $(W(V), [I])$ is a setting for almost mathematics, but $(W(V), W(I))$ typically is not. So, strictly, speaking, the almost language should be avoided for the ideal $W(\mathfrak{m}^\flat)$, though we will sometimes abuse this. However, if V is a perfect ring of characteristic p (e.g., $V = \mathcal{O}^\flat$), then $[I]$ and $W(I)$ coincide after p-adic completion (and derived p-adic completion) by the argument of the proof of Lemma 22; so a map between p-adically complete $W(V)$-modules (resp. derived p-adically complete complexes of $W(V)$-modules) has kernel and cokernel (resp. all cohomology groups of the cone) killed by $W(I)$ if and only if they are killed by $[I]$.

be the finite étale R-algebra obtained by adjoining p^i-roots of $T_1, \dots, T_d$. Then $\mathrm{Sp}\, R_{i+1}[\frac{1}{p}] \to \mathrm{Sp}\, R_i[\frac{1}{p}]$ is a finite étale cover of rigid affinoids for each $i \geq 0$, whence it easily follows that

$$\mathcal{U} := \text{“}\varprojlim_i\text{”}\, \mathrm{Sp}\, R_i\left[\tfrac{1}{p}\right] \longrightarrow U$$

is a cover in $X_{\text{proét}}$.

In fact, $\mathrm{Sp}\, R_i[\frac{1}{p}] \to U$ is a finite Galois cover with Galois group $\mu_{p^i}^d$, where $\underline{\zeta} = (\zeta_1, \dots, \zeta_d) \in \mu_{p^i}^d$ acts on R_i in the usual way via $\underline{\zeta} \cdot T_1^{j_1/p^i} \dots T_d^{j_d/p^i} := \zeta_1^{j_1} \dots \zeta_d^{j_d} T_1^{j_1/p^i} \dots T_d^{j_d/p^i}$, and so for each $s \geq 1$ there is an associated Cartan–Leray[21] spectral sequence

$$H^a_{\text{grp}}(\mu_{p^i}^d, H^b_{\text{proét}}(U_i, \mathcal{O}_X^+/p^s)) \implies H^{a+b}_{\text{proét}}(U, \mathcal{O}_X^+/p^s),$$

or writing in a more derived fashion

$$R\Gamma_{\text{grp}}(\mu_{p^i}^d, R\Gamma_{\text{proét}}(U_i, \mathcal{O}_X^+/p^s)) \xrightarrow{\sim} R\Gamma_{\text{proét}}(U, \mathcal{O}_X^+/p^s).$$

Taking the colimit over i yields an analogous quasi-isomorphism (and spectral sequence) for the “$\mathbb{Z}_p(1)^d$-Galois cover” $\mathcal{U} \to U$:

$$R\Gamma_{\text{grp}}(\mathbb{Z}_p(1)^d, R\Gamma_{\text{proét}}(\mathcal{U}, \mathcal{O}_X^+/p^s)) \xrightarrow{\sim} R\Gamma_{\text{proét}}(U, \mathcal{O}_X^+/p^s).$$

However, $\mathcal{U}$ is affinoid perfectoid: indeed, since the power bounded elements in the affinoid $\mathbb{C}$-algebra $R_i[\frac{1}{p}]$ are exactly R_i, we must show that $(\varinjlim_i R_i)\widehat{\ }_p = R\widehat{\otimes}_{\mathcal{O}\langle \underline{T}^{\pm 1}\rangle} \mathcal{O}\langle \underline{T}^{\pm 1/p^\infty}\rangle =: R_\infty$ is a perfectoid ring; but R_∞ is a p-adically complete, formally étale $\mathcal{O}\langle \underline{T}^{\pm 1/p^\infty}\rangle$-algebra, whence it is perfectoid by Example 3. Therefore the pro-étale cohomology $H^*_{\text{proét}}(\mathcal{U}, \mathcal{O}_X^+/p^s)$ almost vanishes for $* > 0$ (by Proposition 3(ii)) and almost equals $R_\infty/p^s R_\infty$ for $* = 0$ (by Lemma 11, using that $\mathcal{O}_X^+/p^s = \widehat{\mathcal{O}}_X^+/p^s$); so the edge map associated to the previous line is an almost quasi-isomorphism

$$R\Gamma_{\text{grp}}(\mathbb{Z}_p(1)^d, R_\infty/p^s R_\infty) \overset{\text{al. qu.-iso.}}{\longrightarrow} R\Gamma_{\text{proét}}(U, \mathcal{O}_X^+/p^s)$$

(i.e., all cohomology groups of the cone are killed by $\mathfrak{m}$), where we mention explicitly that $\mathbb{Z}_p(1)^d$ is acting on R_∞ as in Sect. 2.2. Finally, taking the derived inverse limit[22]

[21] Often called Hochschild–Serre in this setting. Here H^*_{grp} and $R\Gamma_{\text{grp}}$ refer to group cohomology for a topological group acting on discrete modules.

[22] This process of taking the inverse limit deserves further explanation. By definition, when G is a topological group and M is a complete topological G-module whose topology is defined by a system $\{N\}$ of open sub-G-submodules, we define its continuous group cohomology as $R\Gamma_{\text{cont}}(G, M) := R\lim_N R\Gamma_{\text{grp}}(G, M/N)$ and $H^*_{\text{cont}}(G, M) := H^*(R\Gamma_{\text{cont}}(G, M))$; of course, we

over s yields an almost quasi-isomorphism

$$\boxed{R\Gamma_{\text{cont}}(\mathbb{Z}_p(1)^d, R_\infty) \stackrel{\text{al. qu.-iso.}}{\longrightarrow} R\Gamma_{\text{proét}}(U, \widehat{\mathcal{O}}_X^+).}$$

Arguing by induction and taking inverse limits, these almost descriptions may be extended to the other sheaves in Lemma 11, giving in particular almost (wrt. $W_r(\mathfrak{m})$ and $W(\mathfrak{m}^\flat)$ respectively) quasi-isomorphisms

$$\boxed{R\Gamma_{\text{cont}}(\mathbb{Z}_p(1)^d, W_r(R_\infty)) \stackrel{\text{al. qu.-iso.}}{\longrightarrow} R\Gamma_{\text{proét}}(U, W_r(\widehat{\mathcal{O}}_X^+))}$$

and

$$\boxed{R\Gamma_{\text{cont}}(\mathbb{Z}_p(1)^d, W(R_\infty^\flat)) \stackrel{\text{al. qu.-iso.}}{\longrightarrow} R\Gamma_{\text{proét}}\widehat{(U, \mathbb{A}_{\text{inf},X})},}$$

where the hat indicates derived p-adic completion. These "Cartan–Leray almost quasi-isomorphisms" are crucial to all our calculations of pro-étale cohomology.

5 The Main Construction and Theorems

In this section we present the main construction and define the new cohomology theory introduced in [5], before proving that its main properties, as stated in Theorem 1, can be reduced to a certain p-adic analogue of the Cartier isomorphism. We work in the set-up of Sect. 1.2 throughout:

- $\mathbb{C}$ is a complete, non-archimedean, algebraically closed field of mixed characteristic; ring of integers $\mathcal{O}$ with maximal ideal $\mathfrak{m}$; residue field k.

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may always restrict the limit to any preferred system of open neighbourhoods of 0 by sub-G-modules. In particular, $R\Gamma_{\text{cont}}(\mathbb{Z}_p(1)^d, R_\infty) = R\lim_s R\Gamma_{\text{grp}}(\mathbb{Z}_p(1)^d, R_\infty/p^s R_\infty)$.

To take the inverse limit of the right, we show that the canonical map $R\Gamma_{\text{proét}}(U, \widehat{\mathcal{O}}_X^+) \to R\lim_s R\Gamma_{\text{proét}}(U, \mathcal{O}_X^+/p^s)$ is a quasi-isomorphism. Since the codomain may be rewritten as $R\Gamma_{\text{proét}}(U, R\lim_s \mathcal{O}_X^+/p^s)$ by general formalism of derived functors, it is enough to show that the canonical map $\widehat{\mathcal{O}}_X^+ \to R\lim_s \mathcal{O}_X^+/p^s$ is a quasi-isomorphism (note that the topos of pro-étale sheaves does not satisfy the necessary Grothendieck axioms to automatically imply that higher derived inverse limits vanish in the case of surjective transition maps!), for which it is enough to show that $R\Gamma_{\text{proét}}(\mathcal{V}, \widehat{\mathcal{O}}_X^+) \to R\lim_s R\Gamma_{\text{proét}}(\mathcal{V}, \mathcal{O}_X^+/p^s)$ is a quasi-isomorphism for every affinoid perfectoid $\mathcal{V} \in X_{\text{proét}}$; this is what we shall now do. Firstly, it is easily seen to be an almost quasi-isomorphism by Lemma 11, and so in particular the cone is derived p-adically complete; since the codomain is derived p-adically complete, therefore the domain is also; but the codomain is precisely the derived p-adic completion of the domain, and hence the map is a quasi-isomorphism.

Unfortunately the same argument does not work for the sheaf $\mathbb{A}_{\text{inf},X}$, which seemingly fails to be derived p-adically complete on $X_{\text{proét}}$; in particular, the canonical map $R\Gamma_{\text{proét}}(U, \mathbb{A}_{\text{inf},X}) \to R\lim_s R\Gamma_{\text{proét}}(U, \mathbb{A}_{\text{inf},X}/p^s)$ is only a quasi-isomorphism after replacing the domain by its derived p-adic completion.

- We pick a compatible sequence $\zeta_p, \zeta_{p^2}, \ldots \in \mathcal{O}$ of p-power roots of unity, and define $\mu, \xi, \xi_r, \widetilde{\xi}, \widetilde{\xi}_r \in \mathbb{A}_{\inf} = W(\mathcal{O}^\flat)$ as in Sect. 3.3.
- $\mathfrak{X}$ is a smooth p-adic formal scheme over $\mathcal{O}$, which we do not yet assume is proper; its generic fibre, as a rigid analytic variety over $\mathbb{C}$, is denoted by $X = \mathfrak{X}_{\mathbb{C}}$.
- $\nu : X_{\text{proét}} \to \mathfrak{X}_{\text{Zar}}$ is the projection map of sites obtained by pulling back any Zariski open in $\mathfrak{X}_{\text{Zar}}$ to the constant tower in $X_{\text{proét}}$ consisting of its generic fibre. That is, ν is the composition of maps of sites $X_{\text{proét}} \to X_{\text{ét}} \to \mathfrak{X}_{\text{ét}} \to \mathfrak{X}_{\text{Zar}}$, where the first projection map is what was previously denoted by ν in Sect. 4.1.[23]

The following is the fundamental new object at the heart of our cohomology theory:

Definition 6 Applying $\nu : X_{\text{proét}} \to \mathfrak{X}_{\text{Zar}}$ to the period sheaf $\mathbb{A}_{\inf,X}$ gives a "nearby cycles period sheaf" $R\nu_*\mathbb{A}_{\inf,X}$, which is a complex of sheaves of $\mathbb{A}_{\inf}$-modules on $\mathfrak{X}_{\text{Zar}}$; we p-adically complete this in the derived sense and then apply $L\eta_\mu$ to obtain a complex of sheaves of $\mathbb{A}_{\inf}$-modules on $\mathfrak{X}_{\text{Zar}}$:

$$\mathbb{A}\Omega_{\mathfrak{X}} := L\eta_\mu\big(\widehat{R\nu_*\mathbb{A}_{\inf,X}}\big).$$

We will soon equip $\mathbb{A}\Omega_{\mathfrak{X}}$ with a Frobenius-semi-linear endomorphism φ.

Remark 4 The previous definition used the décalage functor for a complex of sheaves, whereas we only defined it in Definition 1 for complexes of modules; here we explain the necessary minor modifications.

Let $\mathcal{T}$ be a site, A a ring, and $f \in A$ a non-zero-divisor. Call a complex C of sheaves of A-modules *strongly K-flat* if and only if

- C^i is a sheaf of flat A-modules for all $i \in \mathbb{Z}$,
- and the direct sum totalisation of the bicomplex $C \otimes_A D$ is acyclic for every acyclic complex D of sheaves of A-modules.[24]

For any such C we define a new complex of sheaves $\eta_f C$ by

$$\mathcal{T} \ni U \mapsto (\eta_f C)^i(U) := \{x \in f^i C^i(U) : dx \in f^{i+1}C^{i+1}(U)\}.$$

Any complex D of sheaves of A-modules may be resolved by a strongly K-flat complex $C \overset{\sim}{\to} D$ (e.g., see the proof of *The Stacks Project*, Tag 077J), and we define $L\eta_f D := \eta_f C$. This is a well-defined endofunctor of the derived category of sheaves of A-modules on $\mathcal{T}$. For further details, we refer the reader to [5, Sect. 6], the majority of which is established in the generality of ringed topoi.

Warning: The décalage functor does not commute with global sections: there is a natural "global-to-local" morphism

$$L\eta_f R\Gamma(\mathcal{T}, C) \longrightarrow R\Gamma(\mathcal{T}, L\eta_f C),$$

[23] We hope that this rechristening of ν does not lead to confusion, but we are following the (incompatible) notations of [25] and [5].

[24] This is not automatic from the first condition since C may be unbounded, and is a standard condition to impose when requiring flatness conditions on unbounded complexes of sheaves.

but this is not in general a quasi-isomorphism.[25]

Remark 5 Before saying anything precise, we offer some vague descriptions of how $\mathbb{A}\Omega_{\mathfrak{X}}$ looks and how it can be studied. Ignoring the décalage functor for the moment, $\widehat{R\nu_*\mathbb{A}_{\mathrm{inf},X}}$ is obtained by sheafifying $\mathfrak{X} \supseteq \mathrm{Spf}\, R \mapsto R\Gamma_{\mathrm{pro\acute{e}t}}(\widehat{\mathrm{Sp}\, R[\frac{1}{p}]}, \mathbb{A}_{\mathrm{inf},X})$, as Spf R runs over affine opens of $\mathfrak{X}$. We may suppose here that R is small and so put ourselves in the situation of Sect. 4.3: R is a small, formally smooth $\mathcal{O}$-algebra corresponding to an affine open Spf $R \subseteq \mathfrak{X}$, with associated pro-étale cover $\mathcal{U} = \text{“}\varprojlim_i\text{”}\, \mathrm{Sp}\, R_i[\frac{1}{p}] \to \mathrm{Sp}\, R[\frac{1}{p}]$, where $\mathcal{U}$ is affinoid perfectoid with associated perfectoid ring R_∞. As we saw in Sect. 4.3 there is an associated Cartan–Leray almost (wrt. $W(\mathfrak{m}^\flat)$) quasi-isomorphism

$$R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p(1)^d, W(R_\infty^\flat)) \longrightarrow R\Gamma_{\mathrm{pro\acute{e}t}}(\widehat{\mathrm{Sp}\, R\left[\tfrac{1}{p}\right]}, \mathbb{A}_{\mathrm{inf},X}).$$

Recalling from Sect. 2.2 that the décalage functor sometimes transforms almost quasi-isomorphisms into actual quasi-isomorphisms, $\mathbb{A}\Omega_{\mathfrak{X}}$ can therefore be analysed locally through the complexes

$$L\eta_\mu R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p(1)^d, W(R_\infty^\flat)),$$

as Spf R various over small affine opens of $\mathfrak{X}$.[26] These complexes will turn out to be relatively explicit and related to de Rham–Witt complexes, Koszul complexes, and q-de Rham complexes.

Remark 6 (*de Rham–Witt complexes*) Before continuing any further with Sect. 5 the reader should probably first read Sect. 6.1, where the relative de Rham–Witt complex $W_r\Omega^\bullet_{\mathfrak{X}/\mathcal{O}}$ on $\mathfrak{X}$ is defined; it provides an explicit complex computing both de Rham and crystalline cohomology.

In the subsequent Sect. 6.2, which the reader can ignore for the moment, we will explain methods of constructing “Witt complexes” over perfectoid rings. In particular, given a commutative algebra object $D \in D(\mathbb{A}_{\mathrm{inf}})$ equipped with a Frobenius-semi-linear automorphism φ_D and satisfying certain hypotheses, we will equip the cohomology groups

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[25]For example, given a proper smooth variety Y over k, Proposition 1 provides a quasi-isomorphism $W\Omega^\bullet_{Y/k} \xrightarrow{\sim} L\eta_p W\Omega^\bullet_{Y/k}$ of sheaves; if the global-to-local comparison morphism were an isomorphism, we would deduce that $R\Gamma_{\mathrm{crys}}(Y/W(k)) \xrightarrow{\sim} L\eta_p R\Gamma_{\mathrm{crys}}(Y/W(k))$, whence $H^n_{\mathrm{crys}}(Y/W(k)) \xrightarrow{\simeq} H^i_{\mathrm{crys}}(Y/W(k))/H^i_{\mathrm{crys}}(Y/W(k))[p]$ for all $i \geq 0$ by Lemma 1. But the p-power torsion in $H^i_{\mathrm{crys}}(Y/W(k))$ is bounded since it is a finitely generated $W(k)$-module, so the previous isomorphism would in fact force $H^i_{\mathrm{crys}}(Y/W(k))$ to be p-torsion-free for all $i \geq 0$; but this is well-known to be false, e.g., when Y is an Enriques surface in characteristic two and $i = 2$ or 3 [18, Proposition II.7.3.5].

[26]The astute reader may notice that in this argument we have just implicitly identified $L\eta_\mu\big(R\Gamma_{\mathrm{pro\acute{e}t}}(\widehat{\mathrm{Sp}\, R[\frac{1}{p}]}, \mathbb{A}_{\mathrm{inf},X})\big)$ and $R\Gamma_{\mathrm{Zar}}(\mathrm{Spf}\, R, \mathbb{A}\Omega_{\mathfrak{X}})$, contrary to the warning of Remark 4; this is precisely the type of technical obstacle which will need to be overcome in Sect. 7.1.

$$\mathcal{W}_r^\bullet(D) := H^\bullet((L\eta_\mu D)/\widetilde{\xi}_r), \quad \text{where} \quad (L\eta_\mu D)/\widetilde{\xi}_r = (L\eta_\mu D) \otimes^{\mathbb{L}}_{\mathbb{A}_{\mathrm{inf}},\widetilde{\theta}_r} W_r(\mathcal{O}),$$

with the structure of a Witt complex for $\mathcal{O} \to R$ (where R is an $\mathcal{O}$-algebra depending on D); the differential $d : \mathcal{W}_r^\bullet(D) \to \mathcal{W}_r^{\bullet+1}(D)$ will be given by the Bockstein $\mathrm{Bock}_{\widetilde{\xi}_r}$.

To explain the main theorems we recall from Sect. 3.1 that there are two ways of specialising from $\mathbb{A}_{\mathrm{inf}}$ to $W_r(\mathcal{O})$

$$\begin{array}{ccc} & \mathbb{A}_{\mathrm{inf}} & \\ {}^{\theta_r}\swarrow & & \searrow^{\widetilde{\theta}_r=\theta_r\circ\varphi^{-r}} \\ W_r(\mathcal{O}) = \mathbb{A}_{\mathrm{inf}}/\xi_r\mathbb{A}_{\mathrm{inf}} & & W_r(\mathcal{O}) = \mathbb{A}_{\mathrm{inf}}/\widetilde{\xi}_r\mathbb{A}_{\mathrm{inf}}, \end{array}$$

so we use these to form corresponding specialisations of the complex of sheaves of $\mathbb{A}_{\mathrm{inf}}$-modules $\mathbb{A}\Omega_{\mathfrak{X}}$:

$$\begin{array}{ccc} & \mathbb{A}\Omega_{\mathfrak{X}} & \\ {}^{\theta_r}\swarrow & & \searrow^{\widetilde{\theta}_r=\theta_r\circ\varphi^{-r}} \\ \mathbb{A}\Omega_{\mathfrak{X}}/\xi_r = \mathbb{A}\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{\mathbb{A}_{\mathrm{inf}},\theta_r} W_r(\mathcal{O}) & & \mathbb{A}\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{\mathbb{A}_{\mathrm{inf}},\widetilde{\theta}_r} W_r(\mathcal{O}) = \mathbb{A}\Omega_{\mathfrak{X}}/\widetilde{\xi}_r =: \widetilde{W_r\Omega}_{\mathfrak{X}/\mathcal{O}} \end{array}$$

The next theorem is the main new calculation at the heart of our results (and is the reason for the chosen notation $\widetilde{W_r\Omega}_{\mathfrak{X}/\mathcal{O}}$ on the right of the previous line), from which we will deduce all further results, in which $W_r\Omega^\bullet_{\mathfrak{X}/\mathcal{O}}$ is the relative de Rham–Witt complex of $\mathfrak{X}$ over $\mathcal{O}$:

Theorem 4 ("p-adic Cartier isomorphism"[27]) *There are natural*[28] *isomorphisms of Zariski sheaves of $W_r(\mathcal{O}) = \mathbb{A}_{\mathrm{inf}}/\widetilde{\xi}_r\mathbb{A}_{\mathrm{inf}}$-modules*

$$C_{\mathfrak{X}}^{-r} : W_r\Omega^i_{\mathfrak{X}/\mathcal{O}} \xrightarrow{\simeq} \mathcal{H}^i(\widetilde{W_r\Omega}_{\mathfrak{X}/\mathcal{O}})$$

for all $i \geq 0$, $r \geq 1$, which satisfy the following compatibilities:

(i) the restriction map $R : W_{r+1}\Omega^i_{\mathfrak{X}/\mathcal{O}} \to W_r\Omega^i_{\mathfrak{X}/\mathcal{O}}$ is compatible with the map $\widetilde{W_{r+1}\Omega}_{\mathfrak{X}/\mathcal{O}} \to \widetilde{W_r\Omega}_{\mathfrak{X}/\mathcal{O}}$ induced by the inverse Frobenius on $\mathbb{A}_{\mathrm{inf},X}$.

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[27] In the case $r = 1$, the accepted terminology is now "Hodge–Tate comparison".

[28] As written, this isomorphism is natural but not canonical: it depends on the chosen sequence of p-power roots of unity. To make it independent of any choices, the right side should be replaced by $\mathcal{H}^i(\widetilde{W_r\Omega}_{\mathfrak{X}/\mathcal{O}}) \otimes_{W_r(\mathcal{O})} (\mathrm{Ker}\,\widetilde{\theta}_r/(\mathrm{Ker}\,\widetilde{\theta}_r)^2)^{\otimes i}$. Here $\mathrm{Ker}\,\widetilde{\theta}_r/(\mathrm{Ker}\,\widetilde{\theta}_r)^2 = \widetilde{\xi}_r\mathbb{A}_{\mathrm{inf}}/\widetilde{\xi}_r^2\mathbb{A}_{\mathrm{inf}}$ is a certain canonical rank-one free $W_r(\mathcal{O})$-module, and so we are replacing the right side by a type of Tate twist $\mathcal{H}^i(\widetilde{W_r\Omega}_{\mathfrak{X}/\mathcal{O}})\{i\}$. This dependence arrises as follows: changing the chosen sequence of p-power roots of unity changes μ up to a unit in $\mathbb{A}_{\mathrm{inf}}$: this does not affect $L\eta_\mu$ (which depends only on the ideal generated by μ), but does affect the forthcoming isomorphism in Remark 7(a) (see footnote 29).

(ii) the de Rham–Witt differential $d : W_r\Omega^i_{\mathfrak{X}/\mathcal{O}} \to W_r\Omega^{i+1}_{\mathfrak{X}/\mathcal{O}}$ *is compatible with the Bockstein homomorphism* $\mathrm{Bock}_{\tilde{\xi}_r} : \mathcal{H}^i(\widetilde{W_r\Omega}_{\mathfrak{X}/\mathcal{O}}) \to \mathcal{H}^{i+1}(\widetilde{W_r\Omega}_{\mathfrak{X}/\mathcal{O}})$.

Proof (*Idea of forthcoming proof*) Using the construction of Sect. 6.2 (summarised in the previous remark), we will equip the sections $\mathcal{H}^\bullet(\widetilde{W_r\Omega}_{\mathfrak{X}/\mathcal{O}})(\mathrm{Spf}\,R)$ with the structure of a Witt complex for $\mathcal{O} \to R$, naturally as Spf R varies over all small affine opens of $\mathfrak{X}$, in Sect. 7.2. This will give rise to universal (hence natural) morphisms of Witt complexes $W_r\Omega^\bullet_{R/\mathcal{O}} \to \mathcal{H}^\bullet(\widetilde{W_r\Omega}_{\mathfrak{X}/\mathcal{O}})(\mathrm{Spf}\,R)$ which satisfy (i) and (ii) and which will be explicitly checked to be isomorphisms (after p-adically completing $W_r\Omega^\bullet_{R/\mathcal{O}}$) by reducing, via the type of argument sketched in Remark 5, to group cohomology calculations given in Sect. 6.3.

Theorem 5 (Relative de Rham–Witt comparison) *There are natural quasi-isomorphisms in the derived category of Zariski sheaves of* $W_r(\mathcal{O}) = \mathbb{A}_{\rm inf}/\xi_r\mathbb{A}_{\rm inf}$*-modules*

$$W_r\Omega^\bullet_{\mathfrak{X}/\mathcal{O}} \simeq \mathbb{A}\Omega_{\mathfrak{X}}/\xi_r,$$

for all $r \geq 1$, *such that the restriction map* $R : W_{r+1}\Omega^\bullet_{\mathfrak{X}/\mathcal{O}} \to W_r\Omega^\bullet_{\mathfrak{X}/\mathcal{O}}$ *is compatible with the canonical quotient map* $\mathbb{A}_{\rm inf}/\xi_{r+1}\mathbb{A}_{\rm inf} \to \mathbb{A}_{\rm inf}/\xi_r\mathbb{A}_{\rm inf}$.

In a moment we will equip $\mathbb{A}\Omega_{\mathfrak{X}}$ with a Frobenius and check that Theorem 4 implies Theorem 5, from which we will then deduce Theorem 1; first we require some additional properties of the décalage functor:

Remark 7 (*Elementary properties of the décalage functor, I*) Let A be a ring and $f \in A$ a non-zero-divisor.

(a) (Bockstein construction) One of the most important properties of the décalage functor is its relation to the Bockstein boundary map. Let C be a complex of f-torsion-free A-modules. From the definition of $\eta_f C$ it is easy to see that if $f^i x \in (\eta_f C)^i$ is a arbitrary element, then x mod fC^i is a cocycle of the complex C/fC (since $d(f^i x)$ is divisible by f^{i+1}), and so defines a class $\overline{x} \in H^i(C/fC)$; this defines a map of A-modules

$$(\eta_f C)^i \longrightarrow H^i(C/fC), \qquad f^i x \mapsto \overline{x}.$$

Next, the Bockstein $\mathrm{Bock}_f : H^\bullet(C/fC) \to H^{\bullet+1}(C/fC)$ gives the cohomology groups $H^\bullet(C/fC)$ the structure of a complex of A/fA-modules, and we leave it to the reader as an important exercise to check that the map

$$\eta_f C \longrightarrow [H^\bullet(C/fC), \mathrm{Bock}_f],$$

given in degree i by the previous line, is actually one of complexes, i.e., that the differential on $\eta_f C$ is compatible with Bock_f. Even more, the reader should check that the induced map

$$(\eta_f C) \otimes_A A/fA \longrightarrow [H^\bullet(C/fC), \mathrm{Bock}_f]$$

is a quasi-isomorphism. (The proof may be found as [5, Proposition 6.12].) More generally, if D is an arbitrary complex of A-modules, then this can be rewritten as a natural[29] quasi-isomorphism

$$(L\eta_f D) \otimes^{\mathbb{L}}_A A/fA \xrightarrow{\sim} [H^*(D \otimes^{\mathbb{L}}_A A/fA), \mathrm{Bock}_f]$$

of complexes of A/fA-modules.[30]

(b) (Multiplicativity) If $g \in A$ is another non-zero-divisor, and C is a complex of fg-torsion-free A-modules, then

$$\eta_g \eta_f C = \eta_{fg} C \subseteq C\left[\tfrac{1}{gf}\right].$$

Noting that η_f preserves the property the g-torsion-freeness, there is no difficulty deriving to obtain a natural equivalence of endofunctors of $D(A)$

$$L\eta_g \circ L\eta_f \simeq L\eta_{gf}.$$

(c) (Coconnective complexes) Let $D^{\geq 0}_{f\text{-tf}}(A)$ be the full subcategory of $D(A)$ consisting of those complexes D which have $H^i(D) = 0$ for $i < 0$ and $H^0(D)$ is f-torsion-free. Any such D admits a quasi-isomorphic replacement $C \xrightarrow{\sim} D$, where C is a cochain complex of f-torsion-free A-modules supported in positive degree (e.g., if D is bounded then pick a projective resolution $P \xrightarrow{\sim} D$ and set $C := \tau^{\geq 0} P$). Then

$$L\eta_f D = \eta_f C \subseteq C \xrightarrow{\sim} D,$$

whence $L\eta_f$ restricts to an endofunctor of $D^{\geq 0}_{f\text{-tf}}(A)$, and on this subcategory there is a natural transformation $j : L\eta_f \to \mathrm{id}$. In fact, all our applications of the décalage functor take place in this subcategory.

(d) (Functorial bound on torsion) We maintain the hypotheses of (c). Then the morphism $j : L\eta_f D \to D$ induces an isomorphism on H^0: indeed,

$$H^0(L\eta_f D) = \mathrm{Ker}((\eta_f C)^0 \xrightarrow{d} (\eta_f C)^1) = \mathrm{Ker}(C^0 \xrightarrow{d} C^1) = H^0(D).$$

More generally, for any $i \geq 0$, the map $j : H^i(L\eta_f D) \to H^i(D)$ has kernel $H^i(L\eta_f D)[f^i]$ and image $f^i H^i(D)$: indeed, the composition

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[29] Continuing the theme of the previous footnote, the left side depends only on the ideal fA while the right side currently depends on the chosen generator f; to make the construction and morphism independent of this choice, each cohomology group on the right should be replaced by the twist $H^*(D \otimes^{\mathbb{L}}_A A/fA) \otimes_{A/fA} (f^*A/f^{*+1}A)$.

[30] Curiously, this shows that the complex $(L\eta_f D) \otimes^{\mathbb{L}}_A A/fA$, which a priori lives only in the derived category of A/fA-modules, has a natural representative by an actual complex.

$$H^i(D)/H^i(D)[f] \overset{\simeq}{\to} H^i(L\eta_f D) \overset{j}{\to} H^i(D),$$

where the first isomorphism is Lemma 1, is easily seen to be multiplication by f^i, whence the assertion follows.
It may be useful to note that this f-power-torsion difference between D and its décalage $L\eta_f D$ can be functorially captured in the derived category, at least after truncation. More precisely, multiplication by f^i defines a map $\tau^{\le i} C \to \tau^{\le i}\eta_f C$, which induces a natural transformation of functors "f^i" : $\tau^{\le i} \to \tau^{\le i} L\eta_f$ on $D^{\ge 0}_{f\text{-tf}}(A)$ such that the compositions

$$\tau^{\le i} \xrightarrow{\text{“}f^i\text{”}} \tau^{\le i} L\eta_f \overset{j}{\to} \tau^{\le i}, \quad \tau^{\le i} L\eta_f \overset{j}{\to} \tau^{\le i} \xrightarrow{\text{“}f^i\text{”}} \tau^{\le i} L\eta_f$$

are both multiplication by f^i.

(e) (a)–(d) have obvious modifications for complexes of sheaves of A-modules on a site.

As promised, we will now equip $\mathbb{A}\Omega_{\mathfrak{X}}$ with a Frobenius:

Lemma 12 *The complex of sheaves of $\mathbb{A}_{\mathrm{inf}}$-modules $\mathbb{A}\Omega_{\mathfrak{X}}$ is equipped with a Frobenius-semi-linear endomorphism φ which becomes an isomorphism after inverting ξ, i.e.,*

$$\varphi : \mathbb{A}\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{\mathbb{A}_{\mathrm{inf}}} \mathbb{A}_{\mathrm{inf}}\left[\tfrac{1}{\xi}\right] \overset{\sim}{\to} \mathbb{A}\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{\mathbb{A}_{\mathrm{inf}}} \mathbb{A}_{\mathrm{inf}}\left[\tfrac{1}{\tilde{\xi}}\right]$$

(recall that $\tilde{\xi} = \varphi(\xi)$).

Proof The Frobenius automorphism φ on the period sheaf $\mathbb{A}_{\mathrm{inf},X}$ induces a Frobenius automorphism φ on the completion of its derived image $C := \widehat{R\nu_*\mathbb{A}_{\mathrm{inf},X}}$, which by functoriality then induces a quasi-isomorphism of complexes of Zariski sheaves

$$\varphi : L\eta_\mu C \overset{\sim}{\to} L\eta_{\varphi(\mu)} C.$$

We follow this map by

$$L\eta_{\varphi(\mu)} C = L\eta_{\tilde{\xi}} L\eta_\mu C \longrightarrow L\eta_\mu C$$

to ultimately define the desired Frobenius $\varphi : L\eta_\mu C \to L\eta_\mu C$, where it remains to explain the previous line. The equality is a consequence of Remark 7(b) of the previous remark since $\varphi(\mu) = \tilde{\xi}\mu$; the arrow is a consequence of Remark 7(c) since $\mathcal{H}^0(L\eta_\mu C)$ has no $\tilde{\xi}$-torsion.[31] Since the arrow becomes a quasi-isomorphism after inverting $\tilde{\xi}$, we see that the final Frobenius $\varphi : L\eta_\mu C \to L\eta_\mu C$ becomes a quasi-isomorphism after inverting ξ.

[31]*Proof.* $\mathcal{H}^0(C) = \nu_*\mathbb{A}_{\mathrm{inf},X}$ has no μ-torsion since $\mathbb{A}_{\mathrm{inf},X}$ has no μ-torsion by Corollary 3; thus $\mathcal{H}^0(L\eta_\mu C) \overset{\simeq}{\to} \mathcal{H}^0(C)$ by Remark 7(d). But since $\mathcal{H}^0(C)$ has no μ-torsion, it also has no $\varphi(\mu) = \tilde{\xi}\mu$-torsion, thus has no $\tilde{\xi}$-torsion.

Proof (*Proof that Theorem 4 implies Theorem 5*) As in the proof of the previous lemma we write $C := \widehat{R\nu_*\mathbb{A}_{\mathrm{inf},X}}$, which we equipped with a Frobenius-semi-linear automorphism φ. Thus we have

$$\begin{aligned}
W_r\Omega^\bullet_{\mathfrak{X}/\mathcal{O}} &\overset{C_{\mathfrak{X}}^{-r}}{\cong} [\mathcal{H}^\bullet(\widetilde{W_r\Omega}_{\mathfrak{X}/\mathcal{O}}), \mathrm{Bock}_{\widetilde{\xi}_r}] && \text{by Theorem 4}\\
&= [\mathcal{H}^\bullet((L\eta_\mu C)/\widetilde{\xi}_r), \mathrm{Bock}_{\widetilde{\xi}_r}] && \text{rewriting for clarify}\\
&\simeq (L\eta_{\widetilde{\xi}_r} L\eta_\mu C)/\widetilde{\xi}_r && \text{by the Bockstein} - L\eta \text{ relation, i.e., Remark 7(a)}\\
&= (L\eta_{\widetilde{\xi}_r\mu} C)/\widetilde{\xi}_r && \text{by Remark 7(b)}\\
&\overset{\varphi^{-r}}{\overset{\sim}{\to}} (L\eta_\mu C)/\xi_r && \text{functoriality and } \varphi^{-r}(\widetilde{\xi}_r\mu) = \mu,
\end{aligned}$$

which proves Theorem 5.

Now we deduce the beginning of Theorem 1 from Theorem 5:

Theorem 6 *If $\mathfrak{X}$ is moreover proper over $\mathcal{O}$, then $R\Gamma_{\mathbb{A}}(\mathfrak{X}) := R\Gamma_{\mathrm{Zar}}(\mathfrak{X}, \mathbb{A}\Omega_{\mathfrak{X}})$ is a perfect complex of $\mathbb{A}_{\mathrm{inf}}$-modules with the following specialisations, in which (i) and (ii) are compatible with the Frobenius actions:*

(i) Étale specialization: $R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\mathrm{inf}}} W(\mathbb{C}^\flat) \simeq R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} W(\mathbb{C}^\flat)$.
(ii) Crystalline specialization: $R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\mathrm{inf}}} W(k) \simeq R\Gamma_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$.
(iii) de Rham specialization: $R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\mathrm{inf}},\theta} \mathcal{O} \simeq R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O})$.

Proof We prove the specialisations in reverse order. Firstly, since $R\Gamma_{\mathbb{A}}(\mathfrak{X})$ is derived ξ-adically complete,[32] general formalism implies that $R\Gamma_{\mathbb{A}}(\mathfrak{X})$ is a perfect complex of $\mathbb{A}_{\mathrm{inf}}$-modules if and only if $R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\mathrm{inf}}} \mathbb{A}_{\mathrm{inf}}/\xi\mathbb{A}_{\mathrm{inf}}$ is a perfect complex of $\mathbb{A}_{\mathrm{inf}}/\xi\mathbb{A}_{\mathrm{inf}} = \mathcal{O}$-modules. But Theorem 5 in the case $r = 1$ implies that

$$R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\mathrm{inf}}} \mathbb{A}_{\mathrm{inf}}/\xi\mathbb{A}_{\mathrm{inf}} \simeq R\Gamma_{\mathrm{Zar}}(\mathfrak{X}, \Omega^\bullet_{\mathfrak{X}/\mathcal{O}}) = R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}),$$

[32] *"Proof"*. If f, g are non-zero-divisors of a ring A, and D is complex of A-modules which is derived g-adically complete, then we claim that $L\eta_f D$ is still derived g-adically complete: indeed, this follows from the fact that a complex is derived g-adically complete if and only if all of its cohomology groups are derived g-adically complete, that $H^i(L\eta_f D) \cong H^i(D)/H^i(D)[f]$ for all $i \in \mathbb{Z}$ by Lemma 1, and that kernels and cokernels of maps between derived g-adically complete modules are again derived g-adically complete. For a reference on such matters, see *The Stacks Project*, Tag 091N.

It is tempting to claim that the previous paragraph remains valid for the complex of sheaves $\widehat{R\nu_*\mathbb{A}_{\mathrm{inf},X}}$ (which is indeed derived ξ-adically complete, since $R\nu_*$ and derived p-adic completion preserve the derived ξ-adic completeness of the pro-étale sheaf $\mathbb{A}_{\mathrm{inf},X}$), which would complete the proof since $R\Gamma_{\mathrm{Zar}}(\mathfrak{X}, -)$ also preserves derived ξ-adic completeness, but unfortunately the previous paragraph does not remain valid for complexes of sheaves on a "non-replete" site (e.g., the Zariski site). In fact, it seems that the derived ξ-adic completeness of $R\Gamma_{\mathbb{A}}(\mathfrak{X})$ is not purely formal, and requires the technical lemmas established in Sect. 7.1; therefore we have postponed a proof of the completeness to Corollary 4.

which is indeed a perfect complex.[33]

It follows that $R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\inf}} W(k)$ is a perfect complex of $W(k)$-modules; since $W(k)$ is p-adically complete, any perfect complex over it is derived p-adically complete and so

$$\begin{aligned} R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\inf}} W(k) &\overset{\sim}{\to} \mathrm{Rlim}_r (R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\inf}} W_r(k)) \\ &= \mathrm{Rlim}_r (R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\inf}} W_r(\mathcal{O}) \otimes^{\mathbb{L}}_{W_r(\mathcal{O})} W_r(k)) \\ &\overset{\sim}{\to} \mathrm{Rlim}_r (R\Gamma_{\mathrm{Zar}}(\mathfrak{X}, W_r\Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}} \otimes^{\mathbb{L}}_{W_r(\mathcal{O})} W_r(k))) \end{aligned}$$

where the final line uses Theorem 5. But the canonical base change map $W_r\Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}} \otimes^{\mathbb{L}}_{W_r(\mathcal{O})} W_r(k) \overset{\sim}{\to} W_r\Omega^{\bullet}_{\mathfrak{X}_k/k}$ is a quasi-isomorphism for each $r \geq 1$ by Remark 8(vii), and so we deduce that

$$R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\inf}} W(k) \overset{\sim}{\to} \mathrm{Rlim}_r\, R\Gamma_{\mathrm{Zar}}(\mathfrak{X}_k, W_r\Omega^{\bullet}_{\mathfrak{X}_k/k}) = R\Gamma_{\mathrm{crys}}(\mathfrak{X}_k/W(k)).$$

It remains to prove the étale specialisation; we prove the stronger (since μ becomes invertible in $W(\mathbb{C}^{\flat})$) result that $R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\inf}} \mathbb{A}_{\inf}[\frac{1}{\mu}] \simeq R\Gamma_{\acute{e}t}(X, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathbb{A}_{\inf}[\frac{1}{\mu}]$. Since $L\eta_{\mu}$ only effects complexes up to μ^i-torsion in degree i (to be precise, use the morphisms "μ^i" on the truncations of $\mathbb{A}\Omega_{\mathfrak{X}} \to \widehat{R\nu_*\mathbb{A}_{\inf,X}}$, as in Remark 7), the kernel and cokernel of $H^i_{\mathbb{A}}(\mathfrak{X}) \to H^i_{\mathrm{Zar}}\big(\mathfrak{X}, \widehat{R\nu_*\mathbb{A}_{\inf,X}}\big)$ are killed by μ^i. The key to the étale specialisation is now the fact that the canonical map

$$R\Gamma_{\acute{e}t}(X, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathbb{A}_{\inf} \longrightarrow R\Gamma_{\mathrm{pro\acute{e}t}}\widehat{(X, \mathbb{A}_{\inf,X})}$$

(where the hat continues to denote derived p-adic completion) has cone killed by $W(\mathfrak{m}^{\flat}) \ni \mu$; this is deduced from Theorem 3 by taking a suitable limit (see [25, Proof of Theorem 8.4]); inverting μ completes the proof.

We remark that there is an alternative proof of the étale specialisation, due to Bhatt [2, Remark 8.4], which is simpler in that it avoids Theorem 3.

We next discuss the rest of Theorem 1 (continuing the same enumeration):

Theorem 7 *Continuing to assume that $\mathfrak{X}$ is a proper, smooth, p-adic formal scheme over $\mathcal{O}$, then the individual $\mathbb{A}_{\inf}$-modules $H^i_{\mathbb{A}}(\mathfrak{X}) := H^i_{\mathrm{Zar}}(\mathfrak{X}, \mathbb{A}\Omega_{\mathfrak{X}})$ vanish for $i > 2\dim \mathfrak{X}$ and enjoy the following properties:*

(iv) $H^i_{\mathbb{A}}(\mathfrak{X})$ *is a finitely presented $\mathbb{A}_{\inf}$-module;*
(v) $H^i_{\mathbb{A}}(\mathfrak{X})[\frac{1}{p}]$ *is finite free over $\mathbb{A}_{\inf}[\frac{1}{p}]$;*

[33] *Proof.* By derived p-adically completeness, it is enough to check that $R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}) \otimes^{\mathbb{L}}_{\mathcal{O}} \mathcal{O}/p\mathcal{O} = R\Gamma_{\mathrm{dR}}(\mathfrak{X} \otimes_{\mathcal{O}} \mathcal{O}/p\mathcal{O}/(\mathcal{O}/p\mathcal{O}))$ is a perfect complex of $\mathcal{O}/p\mathcal{O}$-modules; this follows from the facts that $\Omega^{\bullet}_{\mathfrak{X}\otimes_{\mathcal{O}}\mathcal{O}/p\mathcal{O}/(\mathcal{O}/p\mathcal{O})}$ is a perfect complex of $\mathcal{O}_{\mathfrak{X}\otimes_{\mathcal{O}}\mathcal{O}/p\mathcal{O}}$-modules by smoothness, and that the structure map $\mathfrak{X} \otimes_{\mathcal{O}} \mathcal{O}/p\mathcal{O} \to \mathrm{Spec}\, \mathcal{O}/p\mathcal{O}$ is proper, flat, and of finite presentation.

(vi) *$H^i_{\mathbb{A}}(\mathfrak{X})$ is equipped with a Frobenius-semi-linear endomorphism φ which becomes an isomorphism after inverting ξ (or any other preferred generator of* Ker θ*), i.e., $\varphi : H^i_{\mathbb{A}}(\mathfrak{X})[\frac{1}{\xi}] \stackrel{\simeq}{\to} H^i_{\mathbb{A}}(\mathfrak{X})[\frac{1}{\xi}]$.*

(vii) *Étale: $H^i_{\mathbb{A}}(\mathfrak{X})[\frac{1}{\mu}] \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{A}_{\text{inf}}[\frac{1}{\mu}]$.*

(viii) *Crystalline: there is a short exact sequence*

$$0 \longrightarrow H^i_{\mathbb{A}}(\mathfrak{X}) \otimes_{\mathbb{A}_{\text{inf}}} W(k) \to H^i_{\text{crys}}(\mathfrak{X}_k/W(k)) \longrightarrow \text{Tor}_1^{\mathbb{A}_{\text{inf}}}(H^{i+1}_{\mathbb{A}}(\mathfrak{X}), W(k)) \longrightarrow 0$$

(ix) *de Rham: there is a short exact sequence*

$$0 \longrightarrow H^i_{\mathbb{A}}(\mathfrak{X}) \otimes_{\mathbb{A}_{\text{inf}},\theta} \mathcal{O} \to H^i_{\text{dR}}(\mathfrak{X}/\mathcal{O}) \longrightarrow H^{i+1}_{\mathbb{A}}(\mathfrak{X})[\xi] \longrightarrow 0$$

(x) *If $H^i_{\text{crys}}(\mathfrak{X}_k/W(k))$ or $H^i_{\text{crys}}(\mathfrak{X}/\mathcal{O})$ is torsion-free, then $H^i_{\mathbb{A}}(\mathfrak{X})$ is a finite free $\mathbb{A}_{\text{inf}}$-module.*

Proof The étale and de Rham specialisations, i.e., (vii) and (ix), are immediate from the derived specialisations proved in the previous theorem.

As mentioned at the start of the previous proof, the complex $R\Gamma_{\mathbb{A}}(\mathfrak{X})$ is derived ξ-adically complete; so to prove that its cohomology vanishes in degree $> 2\dim\mathfrak{X}$, it is enough to note that the same is true of $R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\text{inf}}} \mathbb{A}_{\text{inf}}/\xi\mathbb{A}_{\text{inf}} \simeq R\Gamma_{\text{dR}}(\mathfrak{X}/\mathcal{O})$ (where we have applied the de Rham comparison of Theorem 6).

(vi) follows from Lemma 12 and, similarly to the étale specialisation in Theorem 6, one can give more precise bounds by observing that $\varphi : \mathbb{A}\Omega_{\mathfrak{X}} \to \mathbb{A}\Omega_{\mathfrak{X}}$ is invertible on any truncation up to an application of the morphism "ξ^i".

We now prove (iv) and (v) by a descending induction on i, noting that they are trivial when $i > 2\dim\mathfrak{X}$. By the inductive hypothesis we may suppose that all cohomology groups of $\tau^{>i} R\Gamma_{\mathbb{A}}(\mathfrak{X})$ are finitely presented and become free after inverting p, whence they are perfect $\mathbb{A}_{\text{inf}}$-modules by Theorem 11(ii). It follows that the complex of $\mathbb{A}_{\text{inf}}$-modules $\tau^{>i} R\Gamma_{\mathbb{A}}(\mathfrak{X})$ is perfect, which combined with the perfectness of $R\Gamma_{\mathbb{A}}(\mathfrak{X})$ implies that $\tau^{\le i} R\Gamma_{\mathbb{A}}(\mathfrak{X})$ is also perfect. Thus its top degree cohomology group $H^i(\tau^{\le i} R\Gamma_{\mathbb{A}}(\mathfrak{X})) = H^i_{\mathbb{A}}(\mathfrak{X})$ is the cokernel of a map between projective $\mathbb{A}_{\text{inf}}$-modules, and so is finitely presented.

To prove (v) we wish to apply Corollary 6, and must therefore check that $H^i_{\mathbb{A}}(\mathfrak{X})[\frac{1}{p\mu}]$ is a finite free $\mathbb{A}_{\text{inf}}[\frac{1}{p\mu}]$-module of the same rank as the $W(k)$-module $M \otimes_{\mathbb{A}_{\text{inf}}} W(k)$. Part (vii) implies that

$$H^i_{\mathbb{A}}(\mathfrak{X})\left[\frac{1}{p\mu}\right] \cong H^i_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{A}_{\text{inf}}\left[\frac{1}{p\mu}\right],$$

which is finite free over $\mathbb{A}_{\text{inf}}[\frac{1}{p\mu}]$, while the derived crystalline specialisation of Theorem 6 implies that

$$H^i_{\mathbb{A}}(\mathfrak{X}) \otimes_{\mathbb{A}_{\text{inf}}} W(k)\left[\frac{1}{p}\right] \stackrel{\simeq}{\to} H^i_{\text{crys}}(\mathfrak{X}_k/W(k))\left[\frac{1}{p}\right].$$

(There are no higher Tor obstructions since $H^*_{\mathbb{A}}(\mathfrak{X})[\frac{1}{p}]$ is finite free over $\mathbb{A}_{\mathrm{inf}}[\frac{1}{p}]$ by the inductive hypothesis for $* > i$.) Therefore we must check that the following equality of dimensions holds:

$$\dim_{\mathbb{Q}_p} H^i_{\text{ét}}(X, \mathbb{Q}_p) = \dim_{W(k)\left[\frac{1}{p}\right]} H^i_{\text{crys}}(\mathfrak{X}_k / W(k)) \left[\frac{1}{p}\right]. \qquad (\dim_{\mathfrak{X}})$$

This can be proved in varying degrees of generality as follows:

- In the special case that $\mathfrak{X}$ is obtained by base change from a smooth, proper scheme over the ring of integers of a discretely valued subfield of $\mathbb{C}$ (which is perhaps the main case of interest for most readers), then the equality ($\dim_{\mathfrak{X}}$) is classical (or a consequence of the known Crystalline Comparison Theorem): the crystalline cohomology (with p inverted) of the special fibre identifies with the de Rham cohomology of the generic fibre, which has the same dimension as the $\mathbb{Q}_p$-étale cohomology by non-canonically embedding into the complex numbers and identifying de Rham cohomology with singular cohomology.
- Slightly more generally, if $\mathfrak{X}$ is obtained by base change from a smooth, proper, p-adic *formal* scheme over the ring of integers of a discretely valued subfield of $\mathbb{C}$, then ($\dim_{\mathfrak{X}}$) follows from the rational Hodge–Tate decomposition [25, Corollary 1.8] (which is an easy consequence of the results in the remainder of these notes) and the same identification of crystalline and de Rham cohomology as in the previous case.
- In the full generality in which we are working (i.e., $\mathfrak{X}$ is an arbitrary proper, smooth, p-adic formal scheme over $\mathcal{O}$), then the equality ($\dim_{\mathfrak{X}}$) follows from our general Crystalline Comparison Theorem

  $$H^i_{\text{crys}}(\mathfrak{X}_k / W(k)) \otimes_{W(k)} \mathbb{B}_{\text{crys}} \cong H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{crys}}$$

  (Proposition 13.9 and Theorem 14.5(i) of [5]), whose proof we do not cover in these notes.[34]

Finally we must prove (viii) and (x): but (viii) follows from the derived form of the crystalline specialisation in Theorem 6, part (v), and Lemma 22, while (x) follows by combining (viii) or (ix) with Corollary 5.

This completes the proof of Theorem 6, or rather reduces it to the p-adic Cartier isomorphism of Theorem 4. The remainder of these notes is devoted to sketching a proof of this p-adic Cartier isomorphism.

[34]Possibly ($\dim_{\mathfrak{X}}$) can be proved in this case by combining spreading-out arguments of Conrad–Gabber with the relative p-adic Hodge theory of [25, Sect. 8], but the author has not seriously considered the problem.

6 Witt Complexes

This section is devoted to the theory of Witt complexes. We begin by defining Witt complexes and Langer–Zink's relative de Rham–Witt complex, and then in Sect. 6.2 present one of our main constructions: namely equipping certain cohomology groups with the structure of a Witt complex over a perfectoid ring. We apply this construction in Sect. 6.3 to the group cohomology of a Laurent polynomial algebra and prove that the result is precisely the relative de Rham–Witt complex itself; this is the key local result from which the p-adic Cartier isomorphism will then be deduced in Sect. 7.

6.1 Langer–Zink's Relative de Rham–Witt Complex

We recall the notion of a Witt complex, or F-V-procomplex, from the work of Langer–Zink [22].

Definition 7 Let $A \to B$ be a morphism of $\mathbb{Z}_{(p)}$-algebras. An associated relative *Witt complex*, or *F-V-procomplex*, consists of the following data $(\mathcal{W}_r^\bullet, R, F, V, \lambda_r)$:

(i) a commutative differential graded $W_r(A)$-algebra $\mathcal{W}_r^\bullet = \bigoplus_{n\geq 0} \mathcal{W}_r^n$ for each integer $r \geq 1$;
(ii) morphisms $R : \mathcal{W}_{r+1}^\bullet \to R_*\mathcal{W}_r^\bullet$ of differential graded $W_{r+1}(A)$-algebras for $r \geq 1$;
(iii) morphisms $F : \mathcal{W}_{r+1}^\bullet \to F_*\mathcal{W}_r^\bullet$ of graded $W_{r+1}(A)$-algebras for $r \geq 1$;
(iv) morphisms $V : F_*\mathcal{W}_r^\bullet \to \mathcal{W}_{r+1}^\bullet$ of graded $W_{r+1}(A)$-modules for $r \geq 1$;
(v) morphisms of $W_r(A)$-algebras $\lambda_r : W_r(B) \to \mathcal{W}_r^0$ for each $r \geq 1$ which commute with R, F, V.

such that the following identities hold:

- R commutes with both F and V;
- $FV = p$;
- $FdV = d$;
- the Teichmüller identity[35]: $Fd\lambda_{r+1}([b]) = \lambda_r([b])^{p-1}d\lambda_r([b])$ for $b \in B$, $r \geq 1$.

Example 4 If k is a perfect field of characteristic p and R is a smooth k-algebra (or, in fact, any k-algebra, but it is the smooth case that was studied most classically), then the classical de Rham–Witt complex $W_r\Omega_{R/k}^\bullet$ of Bloch–Deligne–Illusie, together with its operators R, F, V and the identification $\lambda_r : W_r(R) = W_r\Omega_{R/k}^0$, is a Witt complex for $k \to R$.

[35]The Teichmüller identity follows from the other axioms if $\mathcal{W}_r^1$ is p-torsion-free:

$$p\lambda_r([b])^{p-1}d\lambda_r([b]) = d\lambda_r([b]^p) = dF\lambda_r([b]) = FdVF\lambda_r([b])$$
$$= Fd(\lambda_r([b])V(1)) = F(V(1))d\lambda_r([b]) = pFd\lambda_r([b]).$$

There is an obvious definition of morphism between Witt complexes. In particular, it makes sense to ask for an initial object in the category of all Witt complexes for $A \to B$:

Theorem 8 (Langer–Zink 2004) *There is an initial object* $(W_r\Omega^\bullet_{B/A}, R, F, V, \lambda_r)$ *in the category of Witt complexes for* $A \to B$*, called the* relative de Rham–Witt complex. *(And this agrees with* $W_r\Omega^\bullet_{R/k}$ *of the previous example when* $A = k$ *and* $B = R$*).*

Remark 8 (i) The reason for the "relative" in the definition is that there has been considerable work recently, mostly by Hesselholt, on the *absolute de Rham–Witt complex* $W_r\Omega^\bullet_B$"$= W_r\Omega^\bullet_{B/\mathbb{F}_1}$".

(ii) Given a Witt complex for $A \to B$, each $\mathcal{W}^\bullet_r$ is in particular a commutative differential graded $W_r(A)$-algebra whose degree zero summand is a $W_r(B)$-algebra (via the structure maps λ_r). There are therefore natural maps of differential graded $W_r(A)$-algebras $\Omega^\bullet_{W_r(B)/W_r(A)} \to \mathcal{W}^\bullet_r$ for all $r \geq 1$ (which are compatible with the restriction maps on each side).

In the case of the relative de Rham–Witt complex itself, each map $\Omega^\bullet_{W_r(B)/W_r(A)} \to W_r\Omega^\bullet_{B/A}$ is surjective (indeed, the elementary construction of $W_r\Omega^\bullet_{B/A}$ is to mod out $\Omega^\bullet_{W_r(B)/W_r(A)}$ by the required relations so that the axioms of a Witt complex are satisfied) and is even an isomorphism when $r = 1$, i.e., $\Omega^\bullet_{B/A} \overset{\simeq}{\to} W_1\Omega^\bullet_{B/A}$.

(iii) If B is smooth over A, and p is nilpotent in A, then Langer–Zink construct natural comparison quasi-isomorphisms $R\Gamma_{\mathrm{crys}}(B/W_r(A)) \overset{\sim}{\to} W_r\Omega^\bullet_{B/A}$, where the left side is crystalline cohomology with respect to the usual pd-structure on the ideal $VW_{r-1}(A) \subseteq W_r(A)$ (note that the quotient $W_r(A)/VW_{r-1}(A)$ is A) defined by the rule $\gamma_n(V(\alpha)) := \frac{p^{n-1}}{n!}V(\alpha^n)$. This is a generalisation of Illusie's classical comparison quasi-isomorphism $R\Gamma_{\mathrm{crys}}(R/W_r(k)) \overset{\sim}{\to} W_r\Omega^\bullet_{R/k}$.

(iv) Langer–Zink's proof of the comparison quasi-isomorphism in (iii) uses an explicit description of $W_r\Omega^\bullet_{B/A}$ in the case that $B = A[T_1, \dots, T_d]$; in [5, Sect. 10.4] we extend their description to $B = A[T_1^{\pm 1}, \dots, T_d^{\pm 1}]$.

(v) If $B \to B'$ is an étale morphism of A-algebras, then $W_r(B) \to W_r(B')$ is known to be étale and it can be shown that $W_r\Omega^n_{B/A} \otimes_{W_r(B)} W_r(B') \overset{\simeq}{\to} W_r\Omega^n_{B'/A}$ [5, Lemma 10.8]. From these and similar base change results one sees that if Y is any A-scheme, then there is a well-defined Zariski (or even étale) sheaf $W_r\Omega^n_{Y/A}$ on Y whose sections on any Spec B are $W_r\Omega^n_{B/A}$.

(vi) If now $\mathfrak{X}$ is a p-adic formal scheme over A, then there is similarly a well-defined Zariski (or étale) sheaf $W_r\Omega^n_{\mathfrak{X}/A}$ whose sections on any Spf B are the following (identical[36]) p-adically complete $W_r(B)$-modules

$$(W_r\Omega^n_{B/A})^\wedge_p \qquad (W_r\Omega^n_{B/A})^\wedge_{[p]} \qquad \varprojlim_s W_r\Omega^n_{(B/p^sB)/(A/p^sA)}$$

[36]For the elementary proof that the three completions are the same, see Lemma 10.3 and Corollary 10.10 of [5].

(vii) (Base change) In [5, Proposition 10.14] we establish the following important base change property: if $A \to A'$ is a homomorphism between perfectoid rings, and R is a smooth A-algebra, then the canonical base change map $W_r\Omega^n_{R/A} \otimes_{W_r(A)} W_r(A') \to W_r\Omega^n_{R\otimes_A A'/A'}$ is an isomorphism; moreover, the $W_r(A)$-modules $W_r\Omega^n_{R/A}$ and $W_r(A')$ are Tor-independent, whence $W_r\Omega^\bullet_{R/A} \otimes^{\mathbb{L}}_{W_r(A)} W_r(A') \xrightarrow{\sim} W_r\Omega^\bullet_{R\otimes_A A'/A'}$.

In conclusion, in the set-up of Sect. 5, the relative de Rham–Witt complex $W_r\Omega^\bullet_{\mathfrak{X}/\mathcal{O}}$ is an explicit complex computing both de Rham and crystalline cohomologies.

6.2 Constructing Witt Complexes

From now until the end of Sect. 6 we fix the following:

- A is a perfectoid ring of the type discussed in Sect. 3.3, i.e., p-torsion-free and containing a compatible system $\zeta_p, \zeta_{p^2}, \ldots$ of primitive p-power roots of unity (which we fix); let $\varepsilon \in A^\flat$ and $\mu, \xi, \xi_r, \widetilde{\xi}_r \in W(A^\flat)$ be the elements constructed there.
- D is a coconnective (i.e., $H^*(D) = 0$ for $* < 0$), commutative algebra object[37] in $D(W(A^\flat))$ which is equipped with a φ-semi-linear quasi-isomorphism $\varphi_D : D \xrightarrow{\sim} D$ (of algebra objects), and is assumed to satisfy the following hypothesis:

  ($\mathcal{W}$1) $H^0(D)$ is μ-torsion-free.

Here we will explain how to functorially construct, from the data D, φ_D, certain Witt complexes over A: this will lead to universal maps from de Rham–Witt complexes to cohomology groups of D, which will eventually provide the maps in the p-adic Cartier isomorphism.

Example 5 The main examples are $A = \mathcal{O}$ with the following coconnective, commutative algebra objects over $\mathbb{A}_{\inf} = W(\mathcal{O}^\flat)$, which will be studied in Sects. 6.3 and 7.2 respectively:

(i) $R\Gamma_{\mathrm{grp}}(\mathbb{Z}^d, W(A^\flat)[U_1^{\pm 1/p^\infty}, \ldots, U_d^{\pm 1/p^\infty}])$, or its derived p-adic completion.
(ii) The derived p-adic completion of $R\Gamma_{\text{proét}}(\mathrm{Sp}\, R[\frac{1}{p}], \mathbb{A}_{\inf,X})$, where $\mathrm{Spf}\, R$ is a small affine open of a smooth p-adic formal $\mathcal{O}$-scheme with generic fibre X.

We first explain our preliminary construction of a Witt complex from the data D, φ_D, which will then be refined. In this construction, indeed throughout the rest of the section, it is important to recall from Sect. 3 the isomorphisms $\widetilde{\theta}_r : W(A^\flat)/\widetilde{\xi}_r \xrightarrow{\simeq}$

[37] By this we mean that D is a commutative algebra object in the category $D(W(A^\flat))$ in the most naive way: the constructions can be upgraded to the level of $\mathbb{E}_\infty$-algebras, but again this is not necessary for our existing results.

$W_r(A)$, which we often implicitly view as an identification. In particular, for each $r \geq 1$, we may form the coconnective[38] derived algebra object

$$D/\widetilde{\xi}_r := D \otimes^{\mathbb{L}}_{W(A^\flat)} W_r(A^\flat)/\widetilde{\xi}_r = D \otimes^{\mathbb{L}}_{W(A^\flat),\widetilde{\theta}_r} W_r(A)$$

over $W(A^\flat)/\widetilde{\xi}_r = W_r(A)$, and take its cohomology

$$\mathcal{W}_r^\bullet(D)_{\text{pre}} := H^\bullet(D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r)$$

to form a graded $W_r(A)$-algebra. Equipping these cohomology groups with the Bockstein differential $\text{Bock}_{\widetilde{\xi}_r} : \mathcal{W}_r^n(D)_{\text{pre}} \to \mathcal{W}_r^{n+1}(D)_{\text{pre}}$ associated to the distinguished triangle

$$D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r \xrightarrow{\widetilde{\xi}_r} D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r^2 \longrightarrow D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r$$

makes $\mathcal{W}_r^\bullet(D)_{\text{pre}}$ into a differential graded $W_r(A)$-algebra.

Next let

$$R' : \mathcal{W}_{r+1}^\bullet(D)_{\text{pre}} \to \mathcal{W}_r^\bullet(D)_{\text{pre}}$$
$$F : \mathcal{W}_{r+1}^\bullet(D)_{\text{pre}} \to \mathcal{W}_r^\bullet(D)_{\text{pre}}$$
$$V : \mathcal{W}_r^\bullet(D)_{\text{pre}} \to \mathcal{W}_{r+1}^\bullet(D)_{\text{pre}}$$

be the maps on cohomology induced respectively by

$$D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_{r+1} \xrightarrow{\varphi_D^{-1}\otimes\varphi^{-1}} D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r$$
$$D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_{r+1} \xrightarrow{\text{id}\,\otimes \text{can. proj.}} D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r$$
$$D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r \xrightarrow{\text{id}\,\otimes\varphi^{r+1}(\xi)} D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_{r+1},$$

which are compatible with the usual Witt vector maps R, F, V on $W_r(A) = W(A^\flat)/\widetilde{\xi}_r$ thanks to the second set of diagrams in Lemma 6.

As we will see in the proof of part (ii) of the next result, R' must be replaced by[39]

$$R := \widetilde{\theta}_r(\xi)^n R' : \mathcal{W}_{r+1}^n(D)_{\text{pre}} \to \mathcal{W}_r^n(D)_{\text{pre}}$$

if we are to satisfy the axioms of a Witt complex.

Proposition 4 *The data $(\mathcal{W}_r^\bullet(D)_{\text{pre}}, R, F, V)$ satisfies all those axioms appearing in the definition of a Witt complex (Definition 7) which only refer to R, F, V (i.e., which do not involve the additional ring B or the structure maps λ_r). More precisely:*

[38]From assumption ($\mathcal{W}$1) and the existence of φ_D, it follows that $H^0(D)$ has no $\varphi^r(\mu) = \widetilde{\xi}_r\mu$-torsion, hence no $\widetilde{\xi}_r$-torsion; so $D/\widetilde{\xi}_r$ is still coconnective.

[39]The reader should use the identities of Sect. 3.3 to calculate that $\widetilde{\theta}_r(\xi) = \frac{[\zeta_{p^r}]-1}{[\zeta_{p^{r+1}}]-1} \in W_r(A)$.

(i) $\mathcal{W}_r^\bullet(D)_{\mathrm{pre}}$ *is a commutative*[40] *differential graded* $W_r(A)$*-algebra for each* $r \geq 1$.
(ii) R' *is a homomorphism of graded rings, and* R *is a homomorphism of differential graded rings;*
(iii) V *is additive, commutes with* R' *and* R*, and is* F*-inverse-semi-linear (i.e.,* $V(F(x)y) = xV(y)$*);*
(iv) F *is a homomorphism of graded rings and commutes with both* R' *and* R*;*
(v) $FdV = d$*;*
(vi) FV *is multiplication by* p*.*

Proof Part (i) is a formal consequence of D being a commutative algebra object of $D(W(A^\flat))$.

(ii): R' is a homomorphism of graded rings by functoriality; the same is true of R since it is twisted by increasing powers of an element. Moreover, the commutativity of

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W(A^\flat)/\widetilde{\xi}_{r+1} & \xrightarrow{\widetilde{\xi}_{r+1}} & W(A^\flat)/\widetilde{\xi}_{r+1}^2 & \longrightarrow & W(A^\flat)/\widetilde{\xi}_{r+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \xi\varphi^{-1}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W(A^\flat)/\widetilde{\xi}_r & \xrightarrow{\widetilde{\xi}_r} & W(A^\flat)/\widetilde{\xi}_r^2 & \longrightarrow & W(A^\flat)/\widetilde{\xi}_r & \longrightarrow & 0
\end{array}$$

and functoriality of the resulting Bocksteins implies that

$$\begin{array}{ccc}
\mathcal{W}_{r+1}^n(D)_{\mathrm{pre}} & \xrightarrow{d} & \mathcal{W}_{r+1}^{n+1}(D)_{\mathrm{pre}} \\
\downarrow{\scriptstyle R'} & & \downarrow{\scriptstyle \widetilde{\theta}_r(\xi)R'} \\
\mathcal{W}_r^n(D)_{\mathrm{pre}} & \xrightarrow[d]{} & \mathcal{W}_r^{n+1}(D)_{\mathrm{pre}}
\end{array}$$

commutes; hence the definition of R was exactly designed to arrange that it commute with d.

(iii): V is clearly additive, and it commutes with R' since it already did so before taking cohomology. Secondly, the F-inverse-semi-linearity of V follows by passing to cohomology in the following commutative diagram:

[40] Unfortunately this is not strictly true: if $p = 2$ then the condition that $x^2 = 0$ for $x \in \mathcal{W}_r^{\mathrm{odd}}(D)_{\mathrm{pre}}$ need not be true; but this will be fixed when we improve the construction.

$$\begin{array}{ccc}
D/\widetilde{\xi}_{r+1} \otimes^{\mathbb{L}} D/\widetilde{\xi}_{r+1} & \xrightarrow{\text{mult}} & D/\widetilde{\xi}_{r+1} \\
\uparrow {\scriptstyle \mathrm{id}\otimes\varphi^{r+1}(\xi)} & & \\
D/\widetilde{\xi}_{r+1} \otimes^{\mathbb{L}} D/\widetilde{\xi}_{r} & & \uparrow {\scriptstyle \varphi^{r+1}(\xi)} \\
\downarrow {\scriptstyle \text{can. proj.}\otimes\mathrm{id}} & & \\
D/\widetilde{\xi}_{r} \otimes^{\mathbb{L}} D/\widetilde{\xi}_{r} & \xrightarrow{\text{mult}} & D/\widetilde{\xi}_{r}
\end{array}$$

It now easily follows that V also commutes with R'.

(iv): F is a graded ring homomorphism, and it commutes with R' by definition, and then easily also with R.

(v): This follows by tensoring the commutative diagram below with D over $W(A^\flat)$, and looking at the associated boundary maps on cohomology:

$$\begin{array}{ccccccccc}
0 & \to & W(A^\flat)/\widetilde{\xi}_r & \to & W(A^\flat)/\widetilde{\xi}_r^2 & \to & W(A^\flat)/\widetilde{\xi}_r & \to & 0 \\
 & & \| & & \downarrow {\scriptstyle \varphi^{r+1}(\xi)} & & \downarrow {\scriptstyle \varphi^{r+1}(\xi)} & & \\
0 & \to & W(A^\flat)/\widetilde{\xi}_r & \to & W(A^\flat)/\widetilde{\xi}_r\widetilde{\xi}_{r+1} & \to & W(A^\flat)/\widetilde{\xi}_{r+1} & \to & 0 \\
 & & \uparrow {\scriptstyle \text{can. proj.}} & & \uparrow & & \| & & \\
0 & \to & W(A^\flat)/\widetilde{\xi}_{r+1} & \to & W(A^\flat)/\widetilde{\xi}_{r+1}^2 & \to & W(A^\flat)/\widetilde{\xi}_{r+1} & \to & 0
\end{array}$$

(vi): This follows from the fact that $\widetilde{\theta}_r(\varphi^{r+1}(\xi)) = p$ for all $r \geq 1$ (which is true since $\theta_r(\varphi(\xi)) = \theta_r(\varphi(\xi)) = F(\theta_{r+1}(\xi)) = FV(1) = p$, where the third equality uses the second diagram of Lemma 6).

Unfortunately, there are various heuristic and precise reasons[41] that $\mathcal{W}_r^\bullet(D)_{\text{pre}}$ is "too large" to underlie an interesting Witt complex over A, and so we replace it by

$$\mathcal{W}_r^n(D) := ([\zeta_{p^r}] - 1)^n \mathcal{W}_r^n(D)_{\text{pre}} \subseteq \mathcal{W}_r^n(D)_{\text{pre}}.$$

Lemma 13 *The $W_r(A)$-submodules $\mathcal{W}_r^n(D) \subseteq \mathcal{W}_r^n(D)_{\text{pre}}$ define sub differential graded algebras of $\mathcal{W}_r^\bullet(D)_{\text{pre}}$, for each $r \geq 1$, which are closed under the maps*

[41]For example, suppose that B is an A-algebra and that we are given structure maps $\lambda_r : W_r(B) \to \mathcal{W}_r^0(D)$ under which $(\mathcal{W}_r^\bullet(D), R, F, V, \lambda_r)$ becomes a Witt complex for $A \to B$, thereby resulting in a universal map of Witt complexes $\lambda_r^\bullet : W_r\Omega^\bullet_{B/A} \to W_r^\bullet(D)$; then from the surjectivity of the restriction maps for $W_r\Omega^\bullet_{B/A}$ and the definition of the restriction map R for $\mathcal{W}_r^\bullet(D)_{\text{pre}}$, we see that

$$\operatorname{Im} \lambda_r^n \subseteq \bigcap_{s\geq 1} \operatorname{Im}(\mathcal{W}_{r+s}^n(D)_{\text{pre}} \xrightarrow{R^s} \mathcal{W}_r^n(D)_{\text{pre}}) \subseteq \bigcap_{s\geq 1} \widetilde{\theta}_r(\xi_s)^n \mathcal{W}_r^n(D)_{\text{pre}} = \bigcap_{s\geq 1} \left(\frac{[\zeta_{p^r}]-1}{[\zeta_{p^s}]-1}\right)^n \mathcal{W}_r^n(D)_{\text{pre}}.$$

The far right side contains, and often equals in realistic situations, $([\zeta_{p^r}] - 1)^n \mathcal{W}_r^n(D)_{\text{pre}}$, which motivates our replacement.

R, F, V (and hence Proposition 4 clearly remains valid for the data $(\mathcal{W}_r^\bullet(D), R, F, V)$).

Proof This is a consequence of the following simple identities, where $x \in \mathcal{W}_{r+1}^n(D)_{\text{pre}}$ and $y \in \mathcal{W}_r^n(D)_{\text{pre}}$:

$$R(([\zeta_{p^{r+1}}]-1)^n x) = \left(\tfrac{[\zeta_{p^{r+1}}]-1}{[\zeta_{p^r}]-1}\right)^n ([\zeta_{p^{r+1}}]-1)^n R'(x) = ([\zeta_{p^r}]-1)^n R'(x)$$

$$F(([\zeta_{p^{r+1}}]-1)^n x) = (F[\zeta_{p^{r+1}}]-1)^n F(x) = ([\zeta_{p^r}]-1)^n F(x)$$

$$V(([\zeta_{p^r}]-1)^n y) = V(F([\zeta_{p^{r+1}}]-1)^n y) = ([\zeta_{p^{r+1}}]-1)^n V(y)$$

Note that the first identity crucially used the definition of the restriction map R as a multiple of R'.

Next we relate the groups $\mathcal{W}_r^n(D)$ to the cohomology of the décalage $L\eta_\mu D$ of D. From the earlier assumption ($\mathcal{W}1$) and Remark 7(c) there is a canonical map $L\eta_\mu D \to D$, and by imposing the following two additional assumptions on D we will show in Lemma 14 that the resulting map on cohomology

$$H^n(L\eta_\mu D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r) \longrightarrow H^n(D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r) = \mathcal{W}_r^n(D)_{\text{pre}}$$

is injective and has image exactly $\mathcal{W}_r^n(D)$.

From now on we assume that D satisfies the following assumptions (in addition to ($\mathcal{W}1$)):

($\mathcal{W}2$) The cohomology groups $H^*(L\eta_\mu D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r)$ are p-torsion-free for all $r \geq 0$.

($\mathcal{W}3$) The canonical base change map $L\eta_\mu D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r \to L\eta_{[\zeta_{p^r}]-1}(D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r)$ is a quasi-isomorphism for all $r \geq 1$.

Remark 9 (*Elementary properties of the décalage functor, II—base change*) We explain the base change map of assumption ($\mathcal{W}3$). If $\alpha : R \to S$ is a ring homomorphism, $f \in R$ is a non-zero-divisor whose image $\alpha(f) \in S$ is still a non-zero-divisor, and $C \in D(R)$, then there is a canonical base change map

$$(L\eta_f C) \otimes^{\mathbb{L}}_R S \longrightarrow L\eta_{\alpha(f)}(C \otimes^{\mathbb{L}}_R S)$$

in $D(S)$ which the reader will construct without difficulty. This base change map is not a quasi-isomorphism in general,[42] but it is in the following cases:

(i) When $R \to S$ is flat. *Proof*: Easy.

[42]On the other hand, if $C \in D(S)$ then the canonical restriction map $L\eta_f(C|_A) \to L\eta_{\alpha(f)}(C)|_D$ in $D(A)$, which the reader will also easily construct, is always a quasi-isomorphism.

(ii) When $S = R/gR$ for some non-zero-divisor $g \in R$ (i.e., f, g is a regular sequence in R) and the cohomology groups of $C \otimes^{\mathbb{L}}_R R/fR$ are assumed to be g-torsion-free.[43] *Proof*: Since the base change map is always a quasi-isomorphism after inverting f, it is equivalent to establish the quasi-isomorphism after applying $- \otimes^{\mathbb{L}}_{R/gR} R/(f, g)$, after which the base change map becomes the canonical map

$$\begin{aligned}&[H^\bullet(C \otimes^{\mathbb{L}}_R R/fR), \mathrm{Bock}_f] \otimes^{\mathbb{L}}_{R/fR} R/(f, g)\\ &\quad\longrightarrow [H^\bullet(C \otimes^{\mathbb{L}}_R R/gR \otimes^{\mathbb{L}}_{R/gR} R/(f, g)), \mathrm{Bock}_{f \bmod gR}]\end{aligned}$$

by Remark 7(a). But our assumption implies that the left tensor product $\otimes^{\mathbb{L}}_{R/fR}$ is equivalently underived, and that hence it is enough to check that the canonical map $H^n(C \otimes^{\mathbb{L}}_R R/fR) \otimes_{R/fR} R/(f, g) \to H^n(C \otimes^{\mathbb{L}}_R R/(f, g))$ is an isomorphism for all $n \in \mathbb{Z}$; but this is again true because of the g-torsion-freeness assumption.

In the particular case of ($\mathcal{W}3$), we are base changing along $\widetilde{\theta}_r : W(A^\flat) \to W(A^\flat)/\widetilde{\xi}_r = W_r(A)$, noting that $\widetilde{\theta}_r(\mu) = [\zeta_{p^r}] - 1 \in W_r(A)$ is a non-zero-divisor by Remark 3. There is no a priori reason to expect hypothesis ($\mathcal{W}3$) to be satisfied in practice, but it will be in our cases of interest.[44]

Lemma 14 *The aforementioned map on cohomology*

$$H^n(L\eta_\mu D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r) \longrightarrow H^n(D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r) = \mathcal{W}^n_r(D)_{\text{pre}} \qquad (\dagger)$$

is injective with image $\mathcal{W}^n_r(D) = ([\zeta_{p^r}] - 1)^n \mathcal{W}^n_r(D)_{\text{pre}}$, *for all* $r \geq 1$ *and* $n \geq 0$.

Proof The canonical map $L\eta_\mu D \to D$ induces maps on cohomology whose kernels and cokernels are killed by powers of μ, by Remark 7(d); hence the map (†) of $W_r(A)$-modules has kernel and cokernel killed by a power of $\widetilde{\theta}_r(\mu) = [\zeta_{p^r}] - 1$. But $[\zeta_{p^r}] - 1$ divides p^r by Remark 3, so from assumption ($\mathcal{W}2$) we deduce that map (†) is injective for every $r \geq 1$ and $n \geq 0$.

Regarding its image, simply note that (†) factors as

$$H^n(L\eta_\mu D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r) \xrightarrow{\simeq} H^n(L\eta_{[\zeta_{p^r}]-1}(D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r)) \longrightarrow \mathcal{W}^n_r(D)_{\text{pre}},$$

where the first map is the base change isomorphism of assumption ($\mathcal{W}3$), and the second map has image $([\zeta_{p^r}] - 1)^n W^n_r(D)_{\text{pre}}$ by Remark 7(d).

We summarise our construction of Witt complexes by stating the following theorem:

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[43]This was erroneously asserted to be true in the official announcement without the g-torsion-freeness assumption.

[44]Note in particular that ($\mathcal{W}3$) is satisfied if the cohomology groups of $D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\mu$ are p-torsion-free; this follows from Remark 9(ii) since $\widetilde{\xi}_r \equiv p^r \bmod \mu W(A^\flat)$.

Theorem 9 *Let A and D, φ_D be as at the start of Sect. 6.2, and assume that D satisfies assumptions ($\mathcal{W}1$)–($\mathcal{W}3$). Suppose moreover that B is an A-algebra equipped with $W_r(A)$-algebra homomorphisms $\lambda_r : W_r(B) \to H^0(D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r)$ making the following diagrams commute for all $r \geq 1$:*

$$\begin{array}{ccc} W_{r+1}(B) & \xrightarrow{\lambda_{r+1}} & H^0(D/\widetilde{\xi}_{r+1}) \\ \downarrow R & & \downarrow \varphi_D^{-1} \\ W_r(B) & \xrightarrow{\lambda_r} & H^0(D/\widetilde{\xi}_r) \end{array} \qquad \begin{array}{ccc} W_{r+1}(B) & \xrightarrow{\lambda_{r+1}} & H^0(D/\widetilde{\xi}_{r+1}) \\ \downarrow F & & \downarrow \text{can. proj.} \\ W_r(B) & \xrightarrow{\lambda_r} & H^0(D/\widetilde{\xi}_r) \end{array} \qquad \begin{array}{ccc} W_{r+1}(B) & \xrightarrow{\lambda_{r+1}} & H^0(D/\widetilde{\xi}_{r+1}) \\ \uparrow V & & \uparrow \times\varphi^{r+1}(\xi) \\ W_r(B) & \xrightarrow{\lambda_r} & H^0(D/\widetilde{\xi}_r) \end{array}$$

Then the cohomology groups $\mathcal{W}_r^*(D) = H^*(L\eta_\mu D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r)$ may be equipped with the structure of a Witt complex for $A \to B$, and consequently there are associated universal maps of Witt complexes

$$\lambda_r^\bullet : W_r\Omega^\bullet_{B/A} \longrightarrow \mathcal{W}_r^\bullet(D)$$

(which are functorial with respect to D, φ_D and B, λ_r in the obvious sense).

Proof Combining the hypotheses of the theorem with Lemma 13, we see that $\mathcal{W}_r^*(D)$ satisfies all axioms for a Witt complex for $A \to B$, except perhaps for the following two: that $x^2 = 0$ for $x \in \mathcal{W}_r^{\text{odd}}(D)_{\text{pre}}$ when $p = 2$; and the Teichmüller identity. But these follow from the other axioms since $\mathcal{W}_r^*(D)$ is assumed to be p-torsion-free.[45]

Remark 10 (*p-completions*) In our cases of interest the complex D will sometimes be derived p-adically complete, whence the complexes $L\eta_\mu D$ and $L\eta_\mu \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r$ are also derived p-adically complete (by footnote 32); then each cohomology group $\mathcal{W}_r^n(D)$ is both p-torsion-free (by assumption ($\mathcal{W}2$)) and derived p-adically complete, hence p-adically complete in the underived sense. So, in this case, the associated universal maps $W_r\Omega^n_{B/A} \to \mathcal{W}_r^n(D)$ of the previous theorem factor through the p-adic completion $\widehat{(W_r\Omega^n_{B/A})}_p$ which was discussed in Remark 8(vi).

6.3 The de Rham–Witt Complex of a Torus as Group Cohomology

We continue to let A be a fixed perfectoid ring as at the start of Sect. 6.2, and we fix $d \geq 0$ and set

$$D^{\text{grp}} = D^{\text{grp}}_{A,d} := R\Gamma_{\text{grp}}(\mathbb{Z}^d, W(A^\flat)[U_1^{\pm 1/p^\infty}, \ldots, U_d^{\pm 1/p^\infty}]),$$

[45] $2x^2 = 0$ so $x^2 = 0$, c.f., footnote 40. For the Teichmüller identity see footnote 35.

where the i^{th}-generator $\gamma_i \in \mathbb{Z}^d$ acts on $W(A^\flat)[\underline{U}^{\pm 1/p^\infty}]$ via the $W(A^\flat)$-algebra homomorphism

$$\gamma_i U_j^k := \begin{cases} [\varepsilon^k] U_j^k & i = j \\ U_j^k & i \neq j \end{cases}$$

(here $k \in \mathbb{Z}[\frac{1}{p}]$, and $\varepsilon^k \in A^\flat$ is well-defined since $A^\flat$ is a perfect ring). Here we will apply the construction of Sect. 6.2 to D^{grp} to build a Witt complex $\mathcal{W}_r^\bullet(D^{\text{grp}})$ for $A \to A[\underline{T}^{\pm 1}]$, and show that the resulting universal maps $\lambda_r^\bullet : W_r\Omega^\bullet_{A[\underline{T}^{\pm 1}]/A} \to \mathcal{W}_r^\bullet(D^{\text{grp}})$ are in fact isomorphisms. This is the key local result from which the p-adic Cartier isomorphism will be deduced in Sect. 7.

In order to apply Theorem 9 to D^{grp} we must first check that all necessary hypotheses are fulfilled; we begin with the basic assumptions:

Lemma 15 *D^{grp} is a coconnective algebra object in $D(W(A^\flat))$ which is equipped with a φ-semi-linear quasi-isomorphism $\varphi_{\text{grp}} : D^{\text{grp}} \xrightarrow{\sim} D^{\text{grp}}$ and satisfies assumptions ($\mathcal{W}1$)–($\mathcal{W}3$).*

Proof Certainly D^{grp} is a coconnective, commutative algebra object in $D(W(A^\flat))$, and it is equipped with a φ-semi-linear quasi-isomorphism $\varphi_{\text{grp}} : D^{\text{grp}} \xrightarrow{\sim} D^{\text{grp}}$ induced by the obvious Frobenius automorphism on $W(A^\flat)[\underline{U}^{\pm 1/p^\infty}]$ (acting on the coefficients as the Witt vector Frobenius φ and sending U_i^k to U_i^{pk} for all $k \in \mathbb{Z}[\frac{1}{p}]$ and $i = 1, \dots, d$). Also, $H^0(D^{\text{grp}})$ is μ-torsion-free since μ is a non-zero-divisor of $W(A^\flat)$ by Proposition 2. Therefore D^{grp} satisfies the hypotheses from the start of Sect. 6.2, including ($\mathcal{W}1$).

Next we show that the cohomology groups of $L\eta_\mu D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r$ and $D \otimes^{\mathbb{L}}_{W(A^\flat)} W(A^\flat)/\mu$ are p-torsion-free, i.e., that hypotheses ($\mathcal{W}2$) and ($\mathcal{W}3$) (by footnote 44) are satisfied. This is a straightforward calculation of group cohomology in terms of Koszul complexes, in the same style as the proof of Theorem 2. Indeed, there is a $\mathbb{Z}^d$-equivariant decomposition of $W(A^\flat)$-modules

$$W(A^\flat)[\underline{U}^{\pm 1/p^\infty}] = \bigoplus_{k_1, \dots, k_d \in \mathbb{Z}\left[\frac{1}{p}\right]} W(A^\flat) U_1^{k_1} \dots U_d^{k_d},$$

where the generator $\gamma_i \in \mathbb{Z}^d$ acts on the rank-one free $W(A^\flat)$-module $W(A^\flat)U_1^{k_1} \dots U_d^{k_d}$ as multiplication by $[\varepsilon^{k_i}]$. By the standard group cohomology calculation of $R\Gamma_{\text{grp}}(\mathbb{Z}^d, W(A^\flat)U_1^{k_1} \cdots U_d^{k_d})$ as a Koszul complex, this shows that

$$R\Gamma_{\text{grp}}(\mathbb{Z}^d, W(A^\flat)[\underline{U}^{\pm 1/p^\infty}]) \simeq \bigoplus_{k_1, \dots, k_d \in \mathbb{Z}\left[\frac{1}{p}\right]} K_{W(A^\flat)}([\varepsilon^{k_1}] - 1, \dots, [\varepsilon^{k_d}] - 1).$$

It is now sufficient to show that the cohomology groups of $\eta_\mu K \otimes_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r$ and $K \otimes_{W(A^\flat)} W(A^\flat)/\mu$ are p-torsion-free, where K runs over the Koszul complexes appearing in the sum.

Since it is important for the forthcoming cohomology calculations, we explicitly point out now that, if $k, k' \in \mathbb{Z}[\frac{1}{p}]$, then $[\varepsilon^k] - 1$ divides $[\varepsilon^{k'}] - 1$ if and only if $\nu_p(k) \leq \nu_p(k')$.

We will first prove that the cohomology of $K \otimes_{W(A^\flat)} W(A^\flat)/\mu$ is p-torsion-free. Lemma 23 implies that there is an isomorphism of $W(A^\flat)$-modules $H^n(K) \cong W(A^\flat)/([\varepsilon^k] - 1)^{\binom{d-1}{n-1}}$, where $k = p^{-\min_{1\leq i\leq d} \nu_p(k_i)}$ and we have used that $[\varepsilon^k] - 1$ is a non-zero-divisor of $W(A^\flat)$ (so that the torsion term of that lemma vanishes). But $W(A^\flat)/([\varepsilon^k] - 1)$ is p-torsion-free since $p, [\varepsilon^k] - 1$ is a regular sequence[46] of $W(A^\flat)$, and so both $H^{n+1}(K)[\mu]$ and $H^n(K)/\mu \cong W(A^\flat)/([\varepsilon^{\min\{k,0\}}] - 1)^{\binom{d-1}{n-1}}$ are p-torsion-free; therefore $H^n(K \otimes_{W(A^\flat)} W(A^\flat)/\mu)$ is p-torsion-free.

Next we prove that the cohomology of $\eta_\mu K \otimes_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r$ is p-torsion-free. Lemma 24 implies that $\eta_\mu K \cong K_{W(A^\flat)}(([\varepsilon^{k_1}] - 1)/\mu, \ldots, ([\varepsilon^{k_d}] - 1)/\mu)$ if $k_i \in \mathbb{Z}$ for all i, and that $\eta_\mu K$ is acyclic otherwise. Evidently we may henceforth assume we are in the first case; then $\widetilde{\theta}_r$ induces an identification of complexes of $W(A^\flat)/\widetilde{\xi}_r = W_r(A)$-modules

$$\eta_\mu K \otimes_{W(A^\flat)} W(A^\flat)/\widetilde{\xi}_r \cong K_{W_r(A)}\left(\frac{[\zeta^{k_1/p^r}] - 1}{[\zeta_{p^r}] - 1}, \ldots, \frac{[\zeta^{k_d/p^r}] - 1}{[\zeta_{p^r}] - 1}\right),$$

and it remains to prove that the Koszul complex on the right has p-torsion-free cohomology. But Lemma 23 implies that each cohomology group of this Koszul complex is isomorphic to a direct sum of copies of

$$W_r(A)\left[\tfrac{[\zeta_{p^j}]-1}{[\zeta_{p^r}]-1}\right] \qquad \text{and} \qquad W_r(A)/\tfrac{[\zeta_{p^j}]-1}{[\zeta_{p^r}]-1},$$

where $j := -\min_{1\leq i\leq d} \nu_p(k_i/p^r) \leq r$. The left module is p-torsion-free since $W_r(A)$ is p-torsion-free, while the right module (which $= W_r(A)$ if $j \leq 0$, so we suppose $1 \leq j \leq r$) can be easily shown to be isomorphic to $W_{r-j}(A)$ via $F^j : W_r(A) \to W_{r-j}(A)$ [5, Corollary 3.18], which is again p-torsion-free.

Next we prove the existence of suitable structure maps:

Lemma 16 *There exists a unique collection of $W_r(A)$-algebra homomorphisms $\lambda_{r,\mathrm{grp}} : W_r(A[\underline{T}^{\pm 1}]) \to H^0(D^{\mathrm{grp}}/\widetilde{\xi}_r)$, for $r \geq 1$, making the diagrams of Theorem 9 commute and satisfying $\lambda_{r,\mathrm{grp}}([T_i]) = U_i$ for $i = 1, \ldots, d$.*

Proof The maps $\widetilde{\theta}_r$ induce identifications $W(A^\flat)[\underline{U}^{\pm 1/p^\infty}]/\widetilde{\xi}_r = W_r(A)[\underline{U}^{\pm 1/p^\infty}]$ and thus $H^0(D^{\mathrm{grp}}/\widetilde{\xi}_r) = W_r(A)[\underline{U}^{\pm 1}]^{\mathbb{Z}^d}$, where the latter term is the fixed points for $\mathbb{Z}^d$ acting on $W_r(A)[\underline{U}^{\pm 1}]$ via

$$\gamma_i U_j^k := \begin{cases} [\zeta^{k/p^r}] U_j^k & i = j \\ U_j^k & i \neq j \end{cases}$$

[46] *Proof.* We will show that $\varepsilon^k - 1$ is a non-zero-divisor of $A^\flat$. If $x \in A^\flat = \varprojlim_{x\mapsto x^p} A$ satisfies $\varepsilon^k x = x$, then $\zeta^{k/p^i} x^{(i)} = x^{(i)}$ for all $i \geq 0$, and so $x^{(i)} = 0$ for $i \gg 0$ since then $\zeta^{k/p^i} - 1$ is a non-zero-divisor of A, just as at the end of the proof of Proposition 2.

(where the notation ζ^{k/p^r} was explained at the start of the proof of Theorem 2). Under this identification of $H^0(D^{\mathrm{grp}}/\tilde{\xi}_r)$, it is easy to see that the maps $\varphi_{\mathrm{grp}}^{-1}$, "canonical projection", and $\times\varphi_{\mathrm{grp}}^{r+1}(\xi)$ in the diagrams of Theorem 9 are given respectively by:

- the ring homomorphism $R : W_{r+1}(A)[\underline{U}^{\pm 1/p^\infty}] \to W_r(A)[\underline{U}^{\pm 1/p^\infty}]$ which acts as the Witt vector Restriction map on the coefficients and satisfies $R(U_i^k) = U_i^{k/p}$ for all $k \in \mathbb{Z}[\frac{1}{p}]$ and $i = 1, \dots, d$;
- the ring homomorphism $F : W_{r+1}(A)[\underline{U}^{\pm 1/p^\infty}] \to W_r(A)[\underline{U}^{\pm 1/p^\infty}]$ which acts as the Witt vector Frobenius on the coefficients and fixes the variables;
- the additive map $V : W_r(A)[\underline{U}^{\pm 1/p^\infty}] \to W_{r+1}(A)[\underline{U}^{\pm 1/p^\infty}]$ which is defined by $V(\alpha U_1^{k_1} \dots U_d^{k_d}) := V(\alpha)U_1^{k_1} \dots U_d^{k_d}$ for all $\alpha \in W_r(A)$ and $k_1, \dots, k_d \in \mathbb{Z}[\frac{1}{p}]$.

Therefore the proof will be complete if we show that there is a unique collection of $W_r(A)$-algebra homomorphisms $\lambda_{r,\mathrm{grp}} : W_r(A[\underline{T}^{\pm 1}]) \to W_r(A)[\underline{U}^{\pm 1/p^\infty}]$ commuting with R, F, V on each side and satisfying $\lambda_{r,\mathrm{grp}}([T_i]) = U_i$ for $i = 1, \dots, d$.

To prove this, we first use the standard isomorphism of $W_r(A)$-algebras[47]

$$W_r(A)[\underline{U}^{\pm 1/p^\infty}] \overset{\simeq}{\to} W_r(A[\underline{T}^{\pm 1/p^\infty}]), \qquad U_i^k \mapsto [T_i^k] \qquad \left(k \in \mathbb{Z}\left[\tfrac{1}{p}\right]\right)$$

to define a modified isomorphism

$$\tau_r : W_r(A)[\underline{U}^{\pm 1/p^\infty}] \overset{\simeq}{\to} W_r(A[\underline{T}^{\pm 1/p^\infty}]), \qquad U_i^k \mapsto [T_i^{k/p^r}] \qquad \left(k \in \mathbb{Z}\left[\tfrac{1}{p}\right]\right),$$

noting that the new maps τ_r respect R, F, V on each side (the reader should check this by explicit calculation). Therefore the collection of maps

$$\lambda_{r,\mathrm{grp}} : W_r(A[\underline{T}^{\pm 1}]) \hookrightarrow W_r(A[\underline{T}^{\pm 1/p^\infty}]) \xrightarrow[\simeq]{\tau_r^{-1}} W_r(A)[\underline{U}^{\pm 1/p^\infty}]$$

satisfies the desired conditions (and their uniqueness was explained in the previous footnote).

The previous two lemmas show that all hypotheses of Theorem 9 are satisfied, and so there are associated universal maps of Witt complexes

$$\lambda_{r,\mathrm{grp}}^{\bullet} : W_r\Omega^{\bullet}_{A[\underline{T}^{\pm 1}]/A} \longrightarrow \mathcal{W}_r^{\bullet}(D^{\mathrm{grp}}).$$

As already explained, the key local result underlying the forthcoming proof of the p-adic Cartier isomorphism will be the fact that these are isomorphisms:

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[47] This isomorphism is proved by localising the analogous assertion for $A[\underline{T}^{1/p^\infty}]$, which is an easy consequence of [22, Corollary 2.4]. The cited result also implies that $W_r(A[\underline{T}^{\pm 1}])$ is generated as a $W_r(A)$-module by the elements $V^j([T_i^k])$, for $k \in \mathbb{Z}$, $j \geq 0$, $i = 1, \dots, d$, which proves the uniqueness of the maps $\lambda_{r,\mathrm{grp}}$.

Theorem 10 *The map* $\lambda^n_{r,\mathrm{grp}} : W_r\Omega^n_{A[\underline{T}^{\pm 1}]/A} \to \mathcal{W}^n_r(D^{\mathrm{grp}})$ *is an isomorphism for each* $r \geq 1$, $n \geq 0$.

Proof We will content ourselves here with proving that $\lambda^n_{r,\mathrm{grp},\kappa} := \lambda^n_{r,\mathrm{grp}} \otimes_{W_r(A)} W_r(\kappa)$ is an isomorphism,[48] where $\kappa := A/\sqrt{pA}$ is the perfect ring obtained by modding out A by its ideal of p-adically topologically nilpotent elements. Recalling from Remark 8(vii) that the canonical base change map $W_r\Omega^n_{A[\underline{T}^{\pm 1}]/A} \otimes_{W_r(A)} W_r(\kappa) \to W_r\Omega^n_{\kappa[\underline{T}^{\pm 1}]/\kappa}$ is an isomorphism, this means showing that $\lambda^n_{r,\mathrm{grp},\kappa}$ induces an isomorphism $W_r\Omega^n_{\kappa[\underline{T}^{\pm 1}]/\kappa} \xrightarrow{\simeq} \mathcal{W}^n_r(D^{\mathrm{grp}}) \otimes_{W_r(A)} W_r(\kappa)$; this will turn out to be exactly Illusie–Raynaud's Cartier isomorphism for the classical de Rham–Witt complex.

We now begin the proof that $\lambda^n_{r,\mathrm{grp},\kappa}$ is an isomorphism. By the Künneth formula and the standard calculation of group cohomology of an infinite cyclic group, we may represent D^{grp} by the particular complex of $W(A^\flat)$-modules

$$D^{\mathrm{grp}} = \bigotimes_{i=1}^{d} \left[W(A^\flat)[U_i^{\pm 1/p^\infty}] \xrightarrow{\gamma_i - 1} W(A^\flat)[U_i^{\pm 1/p^\infty}] \right],$$

where each length two complex is

$$W(A^\flat)[U_i^{\pm 1/p^\infty}] \xrightarrow{\gamma_i - 1} W(A^\flat)[U_i^{\pm 1/p^\infty}], \quad U_i^k \mapsto ([\varepsilon^k] - 1)U_i^k \qquad \left(k \in \mathbb{Z}\left[\tfrac{1}{p}\right]\right).$$

(Note: although we previously used D^{grp} to denote $R\Gamma(\mathbb{Z}^d, W(A^\flat)[\underline{U}^{\pm 1/p^\infty}])$ in a derived sense, in the rest of this proof we have this particular honest complex of flat $W(A^\flat)$-modules in mind when writing D^{grp}.) This length two complex obviously receives a injective map, given by the identity in degree 0 and by multiplication by μ in degree 1, from

$$D^{\mathrm{grp}}_{\mathrm{int},i} := \left[W(A^\flat)[U_i^{\pm 1}] \to W(A^\flat)[U_i^{\pm 1}] \right], \quad U_i^k \mapsto \tfrac{[\varepsilon^k]-1}{\mu} U_i^k \quad (k \in \mathbb{Z}),$$

and tensoring over $i = 1, \ldots, d$ defines a split injection of complexes of $W(A^\flat)$-modules[49]

$$D^{\mathrm{grp}}_{\mathrm{int}} := \bigotimes_{i=1}^{n} D^{\mathrm{grp}}_{\mathrm{int},i} \longrightarrow D^{\mathrm{grp}}.$$

[48]To then deduce that $\lambda^n_{r,\mathrm{grp}}$ itself is an isomorphism, one applies a form of Nakayama's lemma exploiting the fact that (the non-finitely generated $W_r(A)$-modules) $W_r\Omega^n_{A[\underline{T}^{\pm 1}]/A}$ and $\mathcal{W}^n_r(D^{\mathrm{grp}})$ admit compatible direct sum decompositions into certain finitely generated $W_r(A)$-modules for which Nakayama's lemma is valid; see [5, Lemma 11.14] for the details.

[49]The complex $D^{\mathrm{grp}}_{\mathrm{int}}$ (resp. $D^{\mathrm{grp}}_{\mathrm{int},i}$) is in fact the "$q$-de Rham complex" $[\varepsilon]\text{-}\Omega^\bullet_{W(A^\flat)[\underline{U}^{\pm 1}]/W(A^\flat)}$ (resp. $[\varepsilon]\text{-}\Omega^\bullet_{W(A^\flat)[U_i^{\pm 1}]/W(A^\flat)}$) of $W(A^\flat)[\underline{U}^{\pm 1}]$ (resp. $W(A^\flat)[U_i^{\pm 1}]$) associated to the element $q = [\varepsilon] \in W(A^\flat)$.

The content of the second sentence of the final paragraph of the proof of Lemma 15 was exactly that this inclusion has image in $\eta_\mu D^{\mathrm{grp}}$ and that the induced map $\mathfrak{q} : D^{\mathrm{grp}}_{\mathrm{int}} \hookrightarrow \eta_\mu D^{\mathrm{grp}}$ is a quasi-isomorphism.

The next important observation (which is most natural from the point of view of q-de Rham complexes) is that there is an identification $D^{\mathrm{grp}}_{\mathrm{int}} \otimes_{W(A^\flat)} W(\kappa) = \Omega^\bullet_{W(k)[\underline{U}^{\pm 1}]/W(\kappa)}$: indeed, the canonical projection $A^\flat \to A/pA \to \kappa$ sends ε to 1, and so the projection $W(A^\flat) \to W(\kappa)$ sends $([\varepsilon^k]-1)/\mu = 1 + [\varepsilon] + \cdots + [\varepsilon]^{k-1}$ to k, whence

$$D^{\mathrm{grp}}_{\mathrm{int}} \otimes_{W(A^\flat)} W(\kappa) = \bigotimes_{i=1}^n \left[W(\kappa)[U_i^{\pm 1}] \xrightarrow{U_i^k \mapsto k U_i^k} W(\kappa)[U_i^{\pm 1}] \right] = \Omega^\bullet_{W(\kappa)[\underline{U}^{\pm 1}]/W(\kappa)}.$$

The final identification here is most natural after inserting a dummy basis element $\operatorname{dlog} U_i$ in degree one of each two term complex.

Base changing the Bockstein construction[50] along $W(A^\flat) \to W(\kappa)$ therefore yields isomorphisms of complexes of $W_r(\kappa)$-modules

$$\begin{aligned} \mathcal{W}^\bullet_r(D^{\mathrm{grp}}) \otimes_{W_r(A)} W_r(\kappa) &\xrightarrow{\simeq} [H^\bullet(\eta_\mu D^{\mathrm{grp}} \otimes^{\mathbb{L}}_{W(A^\flat)} W(\kappa)/p^r), \mathrm{Bock}_{p^r}] \\ &\xleftarrow{\mathfrak{q}\simeq} [H^\bullet(\Omega^\bullet_{W(\kappa)[\underline{U}^{\pm 1}]/W(\kappa)} \otimes_{W(\kappa),\widetilde{\theta}_r} W_r(\kappa)), \mathrm{Bock}_{p^r}] \end{aligned}$$

But the complex on the right (hence on the left) identifies with $W_r\Omega^\bullet_{\kappa[\underline{T}^{\pm 1}]/\kappa}$ by the de Rham–Witt Cartier isomorphism of Illusie–Raynaud [19, Sect. III.1], and the resulting map

$$W_r\Omega^\bullet_{A[\underline{T}^{\pm 1}]/A} \otimes_{W_r(A)} W_r(\kappa) \xrightarrow{can.\ map} W_r\Omega^\bullet_{\kappa[\underline{T}^{\pm 1}]/\kappa} \cong \mathcal{W}^\bullet_r(D^{\mathrm{grp}}) \otimes_{W_r(A)} W_r(\kappa)$$

is precisely $\lambda^\bullet_{r,\mathrm{grp},\kappa}$: this is proved by observing that the above isomorphisms (including the de Rham–Witt Cartier isomorphism) are all compatible with multiplicative structure, whence it suffices to check in degree 0, which is not hard (see [5, Theorem 11.13] for a few more details). As we commented at the beginning of the proof,

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[50]If $\alpha : R \to S$ is a ring homomorphism, $f \in R$ is a non-zero-divisor whose image $\alpha(f) \in S$ is still a non-zero-divisor, and $C \in D(R)$, then there is a base change map $[H^\bullet(C \otimes^{\mathbb{L}}_R R/fR), \mathrm{Bock}_f] \otimes_{R/fR} S/\alpha(f)S \to [H^\bullet(C \otimes^{\mathbb{L}}_R S/\alpha(f)S), \mathrm{Bock}_{\alpha(f)}]$ of complexes of $S/\alpha(f)S$-modules; it is an isomorphism if the R/fR-modules $H^*(D \otimes^{\mathbb{L}}_R R/fR)$ are Tor-independent from $S/\alpha(f)S$, as the reader will easily prove (c.f., Remark 9(ii)).

Here we are applying this base change along the canonical map $W(A^\flat) \to W(\kappa)$, which sends $\widetilde{\xi}_r$ to p^r, and the complex $\eta_f D_{\mathrm{grp}}$. The Tor-independence condition is satisfied in this case since the $W_r(A)$-modules $\mathcal{W}^*_r(D_{\mathrm{grp}})$ are Tor-independent from $W_r(k)$: indeed, the proof of Lemma 15 shows that the cohomology groups of $\eta_\mu D_{\mathrm{grp}}/\widetilde{\xi}_r$ are direct sums of $W_r(A)$-modules of the form

$$W_r(A), \qquad W_r(A)\left[\tfrac{[\zeta_{p^j}]-1}{[\zeta_{p^r}]-1}\right], \qquad \text{and} \qquad W_r(A)/\tfrac{[\zeta_{p^j}]-1}{[\zeta_{p^r}]-1}, \qquad 1 \le j < r,$$

which are Tor-independent from $W_r(\kappa)$ by Lemmas 3.13 and 3.18(iii) and Remark 3.19 of [5].

the canonical base change map of relative de Rham–Witt complexes in the previous line is an isomorphism, and so in conclusion $\lambda^{\bullet}_{r,\mathrm{grp},\kappa}$ is an isomorphism.

7 The Proof of the *p*-Adic Cartier Isomorphism

This section is devoted to a detailed sketch of the p-adic Cartier isomorphism stated in Theorem 4. We adopt the set-up from the start of Sect. 5, namely

- $\mathbb{C}$ is a complete, non-archimedean, algebraically closed field of mixed characteristic; ring of integers $\mathcal{O}$ with maximal ideal $\mathfrak{m}$; residue field k.
- We pick a compatible sequence $\zeta_p, \zeta_{p^2}, \ldots \in \mathcal{O}$ of p-power roots of unity, and define $\mu, \xi, \xi_r, \tilde{\xi}, \tilde{\xi}_r \in \mathbb{A}_{\inf} = W(\mathcal{O}^\flat)$ as in Sect. 3.3.
- $\mathfrak{X}$ will denote various smooth formal schemes over $\mathcal{O}$.

7.1 Technical Lemmas: Base Change and Global-to-Local Isomorphisms

Here in Sect. 7.1 we state, and sketch the proofs of, certain technical lemmas which need to be established as part of the proof of the p-adic Cartier isomorphism. We adopt the following local set-up: let R be a p-adically complete, formally smooth $\mathcal{O}$-algebra and $\mathfrak{X} := \operatorname{Spf} R$, with associated generic fibre being the rigid affinoid $X = \operatorname{Sp} R[\frac{1}{p}]$. We will often impose the extra condition that R is *small*, i.e., that there exists a formally étale map (a "framing") $\mathcal{O}\langle \underline{T}^{\pm 1}\rangle = \mathcal{O}\langle T_1^{\pm 1}, \ldots, T_d^{\pm 1}\rangle \to R$; we stress however that we are careful to formulate certain results (e.g., Lemma 17) without reference to any such framing (its existence will simply be required in the course of the proof).

Firstly, as explained at the end of Remark 4 (taking $\mathcal{T} = \mathfrak{X}_{\mathrm{Zar}}$ and $C = R\widehat{\nu_* \mathbb{A}_{\inf,X}}$), there is a natural global-to-local morphism $L\eta_\mu R\Gamma_{\mathrm{Zar}}(\mathfrak{X}, R\widehat{\nu_* \mathbb{A}_{\inf,X}}) \to R\Gamma_{\mathrm{Zar}}(\mathfrak{X}, \mathbb{A}\Omega_{\mathfrak{X}})$ of complexes of $\mathbb{A}_{\inf}$-modules; this may be rewritten as

$$\mathbb{A}\Omega^{\text{proét}}_{R/\mathcal{O}} := L\eta_\mu\big(R\Gamma_{\text{proét}}\widehat{(X, \mathbb{A}_{\inf,X})}\big) \longrightarrow R\Gamma_{\mathrm{Zar}}(\mathfrak{X}, \mathbb{A}\Omega_{\mathfrak{X}}). \tag{t1}$$

There is an analogous global-to-local morphism of complexes of $W_r(\mathcal{O})$-modules

$$\widetilde{W_r\Omega}^{\text{proét}}_{R/\mathcal{O}} := L\eta_{[\zeta_{p^r}]-1} R\Gamma_{\text{proét}}(X, W_r(\widehat{\mathcal{O}}_X^+)) \longrightarrow R\Gamma_{\mathrm{Zar}}(\mathfrak{X}, L\eta_{[\zeta_{p^r}]-1} R\nu_* W_r(\widehat{\mathcal{O}}_X^+)). \tag{t2}$$

Thirdly, recalling Corollary 3 that $\tilde{\theta}_r : \mathbb{A}_{\inf,X}/\tilde{\xi}_r \xrightarrow{\simeq} W_r(\widehat{\mathcal{O}}_X^+)$ (which we continue to often implicitly view as an identification), there is a base change morphism (see Remark 9) of complexes of $W_r(\mathcal{O})$-modules

$$\begin{aligned}\mathbb{A}\Omega^{\text{proét}}_{R/\mathcal{O}}/\widetilde{\xi}_r &= L\eta_\mu\big(R\Gamma_{\text{proét}}\widehat{(X, \mathbb{A}_{\text{inf},X})}\big) \otimes^{\mathbb{L}}_{\mathbb{A}_{\text{inf}}} \mathbb{A}_{\text{inf}}/\widetilde{\xi}_r\mathbb{A}_{\text{inf}} \longrightarrow L\eta_{[\zeta_{p^r}]-1} R\Gamma_{\text{proét}}(X, W_r(\widehat{\mathcal{O}}^+_X)) \\ &= \widetilde{W_r\Omega}^{\text{proét}}_{R/\mathcal{O}}.\end{aligned} \tag{t3}$$

Here we implicitly use the facts that $R\Gamma_{\text{proét}}(X, W_r(\widehat{\mathcal{O}}^+_X))$ is already derived p-adic complete by footnote 22 and that $L\eta_{[\zeta_{p^r}]-1}$ preserves derived p-adic completeness by Remark 3, so that there is no need to complete the codomain. As we have commented earlier, global-to-local and base change morphisms associated to the décalage functor are not in general quasi-isomorphisms; remarkably, they are in our setting:

Lemma 17 *If R is small then maps (t1), (t2), and (t3) are quasi-isomorphisms and, moreover:*

(i) the cohomology groups of $\widetilde{W_r\Omega}^{\text{proét}}_{R/\mathcal{O}}$ are p-torsion-free;
(ii) if R' is a p-adically complete, formally étale R-algebra, then the canonical base change map

$$\widetilde{W_r\Omega}^{\text{proét}}_{R/\mathcal{O}} \widehat{\otimes}^{\mathbb{L}}{}_{W_r(R)} W_r(R') \to \widetilde{W_r\Omega}^{\text{proét}}_{R'/\mathcal{O}}$$

is a quasi-isomorphism.

The key to proving Lemma 17, and to performing necessary auxiliary calculations, is the Cartan–Leray almost quasi-isomorphisms of Sect. 4.3, for which we must assume that R is small and fix a framing $\mathcal{O}\langle \underline{T}^{\pm 1}\rangle \to R$; set $R_\infty := R\widehat{\otimes}_{\mathcal{O}\langle \underline{T}^{\pm 1}\rangle}\mathcal{O}\langle \underline{T}^{\pm 1/p^\infty}\rangle$ as in Sect. 4.3. Then, as explained in Sect. 4.3 and repeated in Remark 5, there are Cartan–Leray almost (wrt. $W(\mathfrak{m}^\flat)$ and $W_r(\mathfrak{m})$ respectively) quasi-isomorphisms of complexes of $\mathbb{A}_{\text{inf}}$- and $W_r(\mathcal{O})$-modules respectively

$$R\Gamma_{\text{cont}}(\mathbb{Z}_p(1)^d, W(R^\flat_\infty)) \longrightarrow R\Gamma_{\text{proét}}\widehat{(X, \mathbb{A}_{\text{inf},X})}$$

and

$$R\Gamma_{\text{cont}}(\mathbb{Z}_p(1)^d, W_r(R_\infty)) \longrightarrow R\Gamma_{\text{proét}}(X, W_r(\widehat{\mathcal{O}}^+_X)).$$

Applying $L\eta_\mu$ (resp. $L\eta_{[\zeta_{p^r}]-1}$) obtains

$$\mathbb{A}\Omega^{\square}_{R/\mathcal{O}} := L\eta_\mu R\Gamma_{\text{cont}}(\mathbb{Z}_p(1)^d, W(R^\flat_\infty)) \longrightarrow L\eta_\mu\big(R\Gamma_{\text{proét}}\widehat{(X, \mathbb{A}_{\text{inf},X})}\big) = \mathbb{A}\Omega^{\text{proét}}_{R/\mathcal{O}} \tag{t4}$$

and

$$\begin{aligned}\widetilde{W_r\Omega}^{\square}_{R/\mathcal{O}} &:= L\eta_{[\zeta_{p^r}]-1} R\Gamma_{\text{cont}}(\mathbb{Z}_p(1)^d, W_r(R_\infty)) \longrightarrow L\eta_{[\zeta_{p^r}]-1} R\Gamma_{\text{proét}}(X, W_r(\widehat{\mathcal{O}}^+_X)) \\ &= \widetilde{W_r\Omega}^{\text{proét}}_{R/\mathcal{O}}\end{aligned} \tag{t5}$$

(The squares $\square$ remind us that the objects depend on the chosen framing.) The second technical lemma, stating that the décalage functor has transformed the almost quasi-isomorphisms into actual quasi-isomorphisms, and hence reminiscent of Theorem 2, is:

Lemma 18 *(t4) and (t5) are quasi-isomorphisms.*

We now sketch a proof of the previous two technical lemmas. The arguments are of a similar flavour to what we have already seen in Sects. 2.2 and 6.3, so we will not provide all the details; see [5, Sect. 9] for further details. For the overall logic of the proof, it will be helpful to draw the following commutative diagram of the maps of interest:

$$\begin{array}{ccc}
R\Gamma_{\mathrm{Zar}}(\mathfrak{X}, \widetilde{W_r\Omega}_{\mathfrak{X}/\mathcal{O}}) = R\Gamma_{\mathrm{Zar}}(\mathfrak{X}, \mathbb{A}\Omega_{\mathfrak{X}})/\widetilde{\xi}_r & \xrightarrow{(t7)} & R\Gamma_{\mathrm{Zar}}(\mathfrak{X}, L\eta_{[\zeta_{p^r}]-1} R\nu_* W_r(\widehat{\mathcal{O}}_X^+)) \\
\Big\uparrow {\scriptstyle (t1)\ \mathrm{mod}\ \widetilde{\xi}_r} & & \Big\uparrow {\scriptstyle (t2)} \\
\mathbb{A}\Omega^{\text{proét}}_{R/\mathcal{O}}/\widetilde{\xi}_r = L\eta_\mu\big(R\Gamma_{\text{proét}}(\widehat{X, \mathbb{A}_{\inf,X}})\big)/\widetilde{\xi}_r & \xrightarrow{(t3)} & L\eta_{[\zeta_{p^r}]-1} R\Gamma_{\text{proét}}(X, W_r(\widehat{\mathcal{O}}_X^+)) = \widetilde{W_r\Omega}^{\text{proét}}_{R/\mathcal{O}} \\
\Big\uparrow {\scriptstyle (t4)\ \mathrm{mod}\ \widetilde{\xi}_r} & & \Big\uparrow {\scriptstyle (t5)} \\
\mathbb{A}\Omega^{\square}_{R/\mathcal{O}}/\widetilde{\xi}_r = L\eta_\mu R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p(1)^d, W(R^\flat_\infty))/\widetilde{\xi}_r & \xrightarrow{(t6)} & L\eta_{[\zeta_{p^r}]-1} R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p(1)^d, W_r(R_\infty)) = \widetilde{W_r\Omega}^{\square}_{R/\mathcal{O}}
\end{array}$$

The new maps, namely (t6) and (t7), are simply the base change maps associated to the identifications $\widetilde{\theta}_r : W(R^\flat_\infty)/\widetilde{\xi}_r \xrightarrow{\simeq} W_r(R_\infty)$ and $\widetilde{\theta}_r : \mathbb{A}_{\inf,X}/\widetilde{\xi}_r \xrightarrow{\simeq} W_r(\widehat{\mathcal{O}}_X^+)$. In particular, the diagram commutes by the naturality of global-to-local and base-change maps. We will show that (t1)–(t7) are quasi-isomorphisms. We begin by stating the following abstract description of a certain group cohomology:

Lemma 19 $R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p(1)^d, W_r(R_\infty))$ *is quasi-isomorphic to the derived p-adic completion of a direct sum of Koszul complexes* $K_{W_r(\mathcal{O})}([\zeta^{k_1}]-1, \dots, [\zeta^{k_d}]-1)$, *for varying* $k_i \in \mathbb{Z}[\frac{1}{p}]$.

Proof A self-contained proof of this may be found in [5, Lemma 9.7(i)].

Proof (*Proof that (t5) is a quasi-isom.*) Using Lemma 23 to calculate the cohomology of the Koszul complexes in Lemma 19 (and footnote 9 to exchange cohomology and p-adic completions), it follows that each cohomology group of $R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p(1)^d, W_r(R_\infty))$ is isomorphic to the p-adic completion of a direct sum of copies of

$$W_r(\mathcal{O}), \qquad W_r(\mathcal{O})[[\zeta_{p^j}]-1], \qquad W_r(\mathcal{O})/([\zeta_{p^j}]-1), \qquad j \geq 1,$$

each of which is "good" in the sense of Lemma 2 (wrt. $A = W_r(\mathcal{O})$, $\mathfrak{M} = W_r(\mathfrak{m})$, and $f = [\zeta_{p^r}]-1$)) by [5, Corollary 3.29]. So all cohomology groups $R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p(1)^d, W_r(R_\infty))$ are good, whence Lemma 2 implies that (t5) is a quasi-isomorphism.

Proof (*Proof of Lemma 17(i)*) Since $L\eta_{[\zeta_{p^r}]-1}$ commutes with derived p-adic completion by Remark 3, Lemmas 19 and 24 imply that $\widetilde{W_r\Omega}^{\square}_R = L\eta_{[\zeta_{p^r}]-1}R\Gamma_{\text{cont}}(\mathbb{Z}_p(1)^d, W_r(R_\infty))$ is quasi-isomorphic to the derived p-adic completion of a direct sum of Koszul complexes

$$K_{W_r(\mathcal{O})}\left(\frac{[\zeta_{p^{j_1}}]-1}{[\zeta_{p^r}]-1}, \dots, \frac{[\zeta_{p^{j_d}}]-1}{[\zeta_{p^r}]-1}\right),$$

for varying $j_1, \dots, j_d \le r$. The calculation at the end of the proof of Lemma 15 therefore shows that the cohomology groups of $\widetilde{W_r\Omega}^{\square}_R$ are p-torsion-free. Combining this with quasi-isomorphism (t5) proves Lemma 17(i).

Proof (*Proof of Lemma 17(ii) and that (t2) is a quasi-isom.*) Let R' be a p-adically complete, formally étale R-algebra, and write $R'_\infty := R'\widehat{\otimes}_{\mathcal{O}\langle \underline{T}^{\pm 1}\rangle}\mathcal{O}\langle \underline{T}^{\pm 1/p^\infty}\rangle$. Since Witt vectors preserve étale morphisms [5, Theorem 10.4], the maps $W_r(R_\infty/p^n) \to W_r(R'_\infty/p^n)$ induced by the formally étale map $R_\infty \to R'_\infty$ are étale for all $n \ge 1$, whence the same is true of the maps $W_r(R_\infty)/p^n \to W_r(R'_\infty)/p^n$ (since the systems of ideals $(p^n W_r(B))_{n\ge 1}$ and $(W_r(p^n B))_{n\ge 1}$ are intertwined for any ring B; for a proof see, e.g., [5, Lemma 10.3]). In particular, these latter maps are flat for all $n \ge 1$, whence the canonical map

$$\widetilde{W_r\Omega}^{\square}_{R/\mathcal{O}}\widehat{\otimes}^{\mathbb{L}}_{W_r(R)}W_r(R') \longrightarrow \widetilde{W_r\Omega}^{\square}_{R'/\mathcal{O}}$$

is a quasi-isomorphism. The same is therefore true after replacing $^{\square}$ by $^{\text{proét}}$ (since (t5) is a quasi-isomorphism for both R and R'), and this proves Lemma 17(ii). This is a strong enough coherence result to show that $\widetilde{W_r\Omega}^{\text{proét}}_{R/\mathcal{O}}\widehat{\otimes}^{\mathbb{L}}_{W_r(R)}W_r(\mathcal{O}_{\mathfrak{X}}) \to L\eta_{[\zeta_{p^r}]-1}R\nu_*W_r(\widehat{\mathcal{O}}^+_X)$ is a quasi-isomorphism of complexes of $W_r(\mathcal{O}_{\mathfrak{X}})$-modules, and it follows that (t2) is a quasi-isomorphism (see [5, Corollary 9.11] for further details).

Proof (*Proof that (t6) is a quasi-isom.*) According to footnote 44, it is enough to prove that the cohomology of the complex $R\Gamma_{\text{cont}}(\mathbb{Z}_p(1)^d, W(R^\flat_\infty)) \otimes^{\mathbb{L}}_{\mathbb{A}_{\inf}} \mathbb{A}_{\inf}/\mu\mathbb{A}_{\inf} = R\Gamma_{\text{cont}}(\mathbb{Z}_p(1)^d, W(R^\flat_\infty)/\mu)$ is p-torsion-free.

We first check that this is true after replacing R by $\mathcal{O}\langle\underline{T}^{\pm 1}\rangle$. To show this we first observe that there is an isomorphism of $\mathbb{A}_{\inf}/\mu\mathbb{A}_{\inf}$-algebras

$$\mathbb{A}_{\inf}/\mu\mathbb{A}_{\inf}\langle\underline{U}^{\pm 1/p^\infty}\rangle \xrightarrow{\simeq} W(\mathcal{O}\langle\underline{T}^{\pm 1/p^\infty}\rangle^\flat)/\mu, \quad U_i^k \mapsto [(T_i^k, T_i^{k/p}, T_i^{k/p^2}, \dots)] \quad \left(k \in \mathbb{Z}\left[\tfrac{1}{p}\right]\right),$$

which is proved by quotienting the “standard isomorphism” in the proof of Lemma 16 by $[\zeta_{p^r}]-1$ and then taking $\varprojlim_{r \,\text{wrt}\, F}$. By the same type of Koszul decomposition argument which has been made several times, it now follows that $R\Gamma_{\text{cont}}(\mathbb{Z}_p(1)^d, W(\mathcal{O}\langle\underline{T}^{\pm 1/p^\infty}\rangle^\flat)/\mu)$ is quasi-isomorphic to the derived p-adic completion of

$$\bigoplus_{k_1,\dots,k_d\in\mathbb{Z}\left[\frac{1}{p}\right]} K_{\mathbb{A}_{\mathrm{inf}}/\mu\mathbb{A}_{\mathrm{inf}}}([\varepsilon^{k_1}]-1,\dots,[\varepsilon^{k_2}]-1).$$

The cohomology of each of these Koszul complexes is, by Lemma 23, a finite direct sum of copies of

$$(\mathbb{A}_{\mathrm{inf}}/\mu\mathbb{A}_{\mathrm{inf}})[[\varepsilon^k]-1] \qquad \text{and} \qquad \mathbb{A}_{\mathrm{inf}}/([\varepsilon^k]-1)\mathbb{A}_{\mathrm{inf}}$$

for various $k\in\mathbb{Z}[\frac{1}{p}]$. But these are p-torsion-free since $p, [\varepsilon^k]-1$ is a regular sequence of $\mathbb{A}_{\mathrm{inf}}$ (see footnote 46) for any $k\in\mathbb{Z}[\frac{1}{p}]$ (including $k=1$, to treat the left term).

To treat the case of R itself one uses the framed period ring $\mathbb{A}(R)^\square$ over $\mathbb{A}_{\mathrm{inf}}$ [5, Sect. 9.2]: to summarise its pertinent properties, $\mathbb{A}(R)^\square/\mu$ is a p-adically complete, formally étale over $\mathbb{A}_{\mathrm{inf}}/\mu\mathbb{A}_{\mathrm{inf}}\langle\underline{U}^{\pm 1}\rangle$, and equipped with a Γ-equivariant homomorphism

$$\mathbb{A}_{\mathrm{inf}}/\mu\mathbb{A}_{\mathrm{inf}}\langle\underline{U}^{\pm 1/p^\infty}\rangle \otimes_{\mathbb{A}_{\mathrm{inf}}/\mu\mathbb{A}_{\mathrm{inf}}\langle\underline{U}^{\pm 1}\rangle} \mathbb{A}(R)^\square \longrightarrow W(R_\infty^\flat)$$

which is an isomorphism modulo any power of p and which is compatible with the above identification $\mathbb{A}_{\mathrm{inf}}/\mu\mathbb{A}_{\mathrm{inf}}\langle\underline{U}^{\pm 1/p^\infty}\rangle \cong W(\mathcal{O}\langle\underline{T}^{\pm 1/p^\infty}\rangle^\flat)/\mu$; note that the Γ-action on $\mathbb{A}_{\mathrm{inf}}/\mu\mathbb{A}_{\mathrm{inf}}\langle\underline{U}^{\pm 1}\rangle$ is trivial. Passing to group cohomology therefore shows that

$$R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p(1)^d, W(\mathcal{O}\langle\underline{T}^{\pm 1/p^\infty}\rangle^\flat)/\mu) \otimes^{\mathbb{L}}_{\mathbb{A}_{\mathrm{inf}}/\mu\mathbb{A}_{\mathrm{inf}}\langle\underline{U}^{\pm 1}\rangle} \mathbb{A}(R)^\square \xrightarrow{\sim} R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p(1)^d, W(R_\infty^\flat)),$$

whose cohomology groups are indeed p-torsion-free since this has already been shown to be true of the group cohomology on the left side and the base change is flat modulo any power of p.

Proof (*Proof that (t4) is a quasi-isom.*) Proving that (t4) is a quasi-isomorphism was done in [5] via a subtle generalisation of the "good" cohomology groups argument of Lemma 2, which required calculating $R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p(1)^d, W(\mathcal{O}\langle\underline{T}^{\pm 1/p^\infty}\rangle^\flat))$ in terms of Koszul complexes[51] (see Lemmas 9.12–9.13 and the first paragraph of Proposition 9.14). Here we will offer a simpler argument which was presented first in [2, Remark 7.11].

We need the following strengthening of Lemma 2: "Let $\mathfrak{M}\subseteq A$ be an ideal of a ring and $f\in\mathfrak{M}$ a non-zero-divisor; if $C\to D$ is a morphism of complexes of A-modules whose cone is killed by $\mathfrak{M}$, and all cohomology groups of $C\otimes^{\mathbb{L}}_A A/fA$ contain no non-zero elements killed by $\mathfrak{M}^2$, then $L\eta_f C\to L\eta_f D$ is a quasi-isomorphism." This follows from the proof of [2, Lemma 6.14] and exploits the relation between $L\eta$ and the Bockstein construction.

[51]Here we explain why the analogous calculations we have already seen do not generalise to this case. Although there is an identification $\mathbb{A}_{\mathrm{inf}}\langle\underline{U}^{\pm 1/p^\infty}\rangle \xrightarrow{\sim} W(\mathcal{O}\langle\underline{T}^{\pm 1/p^\infty}\rangle^\flat)$, the convergence of the power series on the left is with respect to the $\langle p,\xi\rangle$-adic topology. But neither $R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p(1)^d,\cdot)$ nor $L\eta_\mu$ commute with derived $\langle p,\xi\rangle$-adic completion!

Applying this in the case $A = \mathbb{A}_{\inf}$, $f = \mu$, and $\mathfrak{M} = W(\mathfrak{m}^\flat)$, the proof immediately reduces to showing that the cohomology of $R\Gamma_{\text{cont}}(\mathbb{Z}_p(1)^d, W(R_\infty^\flat)/\mu)$ contains no non-zero elements killed by $W(\mathfrak{m}^\flat)^2$. But the decomposition from the previous proof showed that each of these cohomology groups was the p-adic completion of a direct sum of copies of the p-torsion-free modules

$$(\mathbb{A}_{\inf}/\mu\mathbb{A}_{\inf})[[\varepsilon^k]-1] \qquad \text{and} \qquad \mathbb{A}_{\inf}/([\varepsilon^k]-1)\mathbb{A}_{\inf}$$

for various $k \in \mathbb{Z}[\frac{1}{p}]$; so it is enough to show for any $k \in \mathbb{Z}[\frac{1}{p}]$ (including $k = 1$, to treat the left term) that $\mathbb{A}_{\inf}/([\varepsilon^k]-1)\mathbb{A}_{\inf}$ contains no non-zero elements killed by $W(\mathfrak{m}^\flat)^2$. But the maps $\widetilde{\theta}_r$ induce an isomorphism $\mathbb{A}_{\inf}/([\varepsilon^k]-1)\mathbb{A}_{\inf} \xrightarrow{\simeq} \varprojlim_{r \text{ wrt } F} W_r(\mathcal{O})/([\zeta^{k/p^r}]-1)$, and each $W_r(\mathcal{O})/([\zeta^{k/p^r}]-1)$ contains no non-zero elements killed by $W_r(\mathfrak{m})^2 = W_r(\mathfrak{m})$ (recall that $W_r(\mathfrak{m})$ is an ideal for almost mathematics, c.f., footnote 20), as we already saw above in the proof that (t5) is a quasi-isomorphism.

Proof (*Proof that (t1), (t3), and (t7) are quasi-isoms.*) Since we now know that (t4) is a quasi-isomorphism, the commutativity of the diagram implies that (t3) is a quasi-isomorphism. Using this quasi-isomorphism, and by taking $\mathrm{Rlim}_{r \text{ wrt } F}$ of the quasi-isomorphisms (t2), it can be shown that (t1) is a quasi-isomorphism [5, Proposition 9.14]. Finally, the commutativity of the diagram implies that (t7) is also a quasi-isomorphism.

This finishes the proofs of the technical lemmas, but we note in addition the following consequence which was needed at the start of the proof of Theorem 6:

Corollary 4 *If $\mathfrak{X}$ is a smooth p-adic formal scheme over $\mathcal{O}$, then the complex of $\mathbb{A}_{\inf}$-modules $R\Gamma_{\text{Zar}}(\mathfrak{X}, \mathbb{A}\Omega_{\mathfrak{X}})$ is derived ξ-adically complete.*

Proof By picking a cover of $\mathfrak{X}$ by small opens, we may suppose that $\mathfrak{X} = \operatorname{Spf} R$ is a small affine as above. Then the complex $R\Gamma_{\text{cont}}(\mathbb{Z}_p(1)^d, W(R_\infty^\flat))$ is derived ξ-adically complete since $W(R_\infty^\flat)$ is ξ-adically complete,[52] whence $\mathbb{A}\Omega_{R/\mathcal{O}}^\square$ is derived ξ-adically complete since $L\eta_\mu$ preserves the completeness by footnote 32. Now quasi-isomorphisms (t1) and (t4) complete the proof.

7.2 Reduction to a Torus and to Theorem 10

We continue to suppose that R is a p-adically complete, formally smooth $\mathcal{O}$-algebra, with notation $\mathfrak{X} = \operatorname{Spf} R$ and $X = \operatorname{Sp} R[\frac{1}{p}]$ as in Sect. 7.1. We wish to apply the construction of Sect. 6.2 (with base perfectoid ring $A = \mathcal{O}$) to the derived p-adic completion

$$D_{R/\mathcal{O}}^{\text{proét}} := R\Gamma_{\text{proét}}\widehat{(X, \mathbb{A}_{\inf,X})},$$

[52] If A is any perfectoid ring then $W(A^\flat)$ is Ker θ-adically complete.

and must therefore check that the necessary hypotheses are fulfilled:

Lemma 20 *$D^{\text{proét}}_{R/\mathcal{O}}$ is a coconnective algebra object in $D(\mathbb{A}_{\text{inf}})$ which is equipped with a φ-semi-linear quasi-isomorphism $\varphi_{\text{proét}} : D^{\text{proét}}_{R/\mathcal{O}} \xrightarrow{\sim} D^{\text{proét}}_{R/\mathcal{O}}$. If R is small, then it moreover satisfies assumptions ($\mathcal{W}1$)–($\mathcal{W}3$) from Sect. 6.2 and there exist $W_r(\mathcal{O})$-algebra homomorphisms $\lambda_{r,\text{proét}} : W_r(R) \to H^0(D^{\text{proét}}_{R/\mathcal{O}}/\widetilde{\xi}_r)$ (natural in R) making the diagrams of Theorem 9 commute.*

Proof $D^{\text{proét}}_{R/\mathcal{O}}$ is clearly a coconnective algebra object in $D(\mathbb{A}_{\text{inf}})$, and it is equipped with a φ-semi-linear quasi-isomorphism $\varphi_{\text{proét}}$ induced by the Frobenius automorphism of $\mathbb{A}_{\text{inf},X}$, similarly to Lemma 12.

Moreover, $H^0(D^{\text{proét}}_{R/\mathcal{O}}) = H^0_{\text{proét}}(X, \mathbb{A}_{\text{inf},X})$ is μ-torsion-free, since $\mathbb{A}_{\text{inf},X}$ is a μ-torsion-free sheaf on $X_{\text{proét}}$ by Corollary 3; this proves that assumption ($\mathcal{W}1$) holds. It remains to check ($\mathcal{W}2$) and ($\mathcal{W}3$), as well as prove the existence of the maps λ_r; for this we must now assume that R is small (but we do not fix any framing). Hypotheses ($\mathcal{W}2$) and ($\mathcal{W}3$) are then exactly the p-torsion-freeness and quasi-isomorphism (t3) of Lemma 17.

Finally, the canonical maps of Zariski sheaves of rings $\mathcal{O}_{\mathfrak{X}} \to \nu_*\widehat{\mathcal{O}}^+_X \to R\nu_*\widehat{\mathcal{O}}^+_X$ on $\mathfrak{X}$ induce analogous maps on Witt vectors (see footnote 19), namely $W_r(\mathcal{O}_{\mathfrak{X}}) \to \nu_* W_r(\widehat{\mathcal{O}}^+_X) \to R\nu_* W_r(\widehat{\mathcal{O}}^+_X)$, which are compatible with R, F, V on each term. Applying $H^0(\mathfrak{X}, -)$ to the composition then yields the following arrow which is also compatible with R, F, V:

$$\lambda_{r,\text{proét}} : W_r(R) = H^0_{\text{Zar}}(\mathfrak{X}, W_r(\mathcal{O}_{\mathfrak{X}})) \longrightarrow H^0_{\text{proét}}(X, W_r(\widehat{\mathcal{O}}^+_X)) \overset{\widetilde{\theta}_r}{\cong} H^0(D^{\text{proét}}_{R/\mathcal{O}}/\widetilde{\xi}_r).$$

The isomorphism $\widetilde{\theta}_r$ is compatible with R, F, V on the left according to a sheaf version of the second set of diagrams in Corollary 6; therefore, overall, these maps $\lambda_{r,\text{proét}}$ make the diagrams of Theorem 9 commute, and they are clearly natural in R, as desired.

Continuing to assume that R is small, the previous lemma states that all hypotheses of Theorem 9 are satisfied for $D^{\text{proét}}_{R/\mathcal{O}}$, and so there are associated universal maps of Witt complexes, natural in R,

$$\lambda^\bullet_{r,\text{proét}} : W_r\Omega^\bullet_{R/\mathcal{O}} \longrightarrow \mathcal{W}^\bullet_r(D^{\text{proét}}_{R/\mathcal{O}}) = H^\bullet(\mathbb{A}\Omega^{\text{proét}}_{R/\mathcal{O}}/\widetilde{\xi}_r).$$

By Remark 10, these factor through the p-adic completion of the left side, i.e.,

$$\widehat{\lambda}^n_{r,\text{proét}} : (W_r\Omega^n_{R/\mathcal{O}})^\wedge_p \longrightarrow H^n(\mathbb{A}\Omega^{\text{proét}}_{R/\mathcal{O}}/\widetilde{\xi}_r).$$

The p-adic Cartier isomorphism will follow from showing that these maps are isomorphisms:

Lemma 21 *The following implications hold:*

$$\widehat{\lambda}^n_{r,\text{proét}} \textit{ is an isomorphism when } R = \mathcal{O}\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle.$$
$$\Downarrow$$
$$\widehat{\lambda}^n_{r,\text{proét}} \textit{ is an isomorphism for every small, formally smooth } \mathcal{O}\textit{-algebra } R.$$
$$\Downarrow$$
The p-adic Cartier isomorphism (Theorem 4) is true.

Proof The first implication is a consequence of the domain and codomain of $\widehat{\lambda}^n_{r,\text{proét}}$ behaving well under formally étale base change, according to Remark 8(v) and (vi) and Lemma 17(ii).

For the second implication it is convenient to briefly change the point of view and notation, by fixing a smooth p-adic formal scheme $\mathfrak{X}$ over $\mathcal{O}$ and letting Spf $R \subseteq \mathfrak{X}$ denote any small affine open. We then consider the composition

$$H^n(\mathbb{A}\Omega^{\text{proét}}_{R/\mathcal{O}}/\widetilde{\xi}_r) \xrightarrow[(t1) \bmod \widetilde{\xi}_r]{\cong} H^n_{\text{Zar}}(\text{Spf } R, \widetilde{W_r\Omega}_{\mathfrak{X}/\mathcal{O}}) \xrightarrow[\text{edge map}]{} \mathcal{H}^n(\widetilde{W_r\Omega}_{\mathfrak{X}/\mathcal{O}})(\text{Spf } R)$$

and note that the edge map is an isomorphism by the coherence result of Lemma 17(ii).[53] Since $(W_r\Omega^n_{R/\mathcal{O}})\widehat{\ }_p = W_r\Omega^n_{\mathfrak{X}/\mathcal{O}}(\text{Spf } R)$ (Remark 8(vi)), the middle assumption therefore leads to isomorphisms

$$W_r\Omega^n_{\mathfrak{X}/\mathcal{O}}(\text{Spf } R) \xrightarrow{\simeq} \mathcal{H}^n(\widetilde{W_r\Omega}_{\mathfrak{X}/\mathcal{O}})(\text{Spf } R)$$

naturally as Spf $R \subseteq \mathfrak{X}$ varies over all small affine opens; that proves the p-adic Cartier isomorphism.

To complete the proof of the p-adic Cartier isomorphism we must prove the top statement in Lemma 21, namely the following:

Proposition 5 *The universal maps*

$$\widehat{\lambda}^n_{r,\text{proét}} : (W_r\Omega^n_{R/\mathcal{O}})\widehat{\ }_p \longrightarrow H^n(\mathbb{A}\Omega^{\text{proét}}_{R/\mathcal{O}}/\widetilde{\xi}_r)$$

are isomorphisms in the special case that $R := \mathcal{O}\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle$.

Proof The proof will consist merely of assembling results we have already established: indeed, the technical lemmas of Sect. 7.1 imply that $H^n(\mathbb{A}\Omega^{\text{proét}}_{R/\mathcal{O}}/\widetilde{\xi}_r)$ can be calculated in terms of group cohomology, which we identified with the de Rham–Witt complex in Theorem 10.

Note first that the map[54] $\mathbb{A}_{\inf}[\underline{U}^{\pm 1/p^\infty}] \to W(\mathcal{O}\langle \underline{T}^{\pm 1/p^\infty}\rangle^\flat)$, $U_i^k \mapsto [(T_i^k, T_i^{k/p}, T_i^{k/p^2}, \dots)]$, when base changed along $\widetilde{\theta}_r$, yields an inclusion $W_r(\mathcal{O})[\underline{U}^{\pm 1/p^\infty}] \hookrightarrow W_r(\mathcal{O}\langle \underline{T}^{\pm 1/p^\infty}\rangle)$, $U_i^k \mapsto T_i^{k/p^r}$ which identifies the right with the p-adic completion

[53]Here we are of course using the trivial identification $\widetilde{W_r\Omega}_{\mathfrak{X}/\mathcal{O}}|_{\text{Spf } R} = \widetilde{W_r\Omega}_{\text{Spf } R/\mathcal{O}}$ in order to appeal to the affine results in Sect. 7.1.

[54]This map is injective and identifies the right with the $\langle p, \xi\rangle$-adic completion of the left, i.e., with $\mathbb{A}_{\inf}\langle \underline{U}^{\pm 1/p^\infty}\rangle$, but we do not need this.

of the left, i.e., with $W(A^\flat)\langle\underline{U}^{\pm 1/p^\infty}\rangle$; indeed, this follows easily from the "standard/modified isomorphisms" which appeared in the proof of Lemma 16. The map $\mathbb{A}_{\inf}[\underline{U}^{\pm 1/p^\infty}] \to W(\mathcal{O}\langle\underline{T}^{\pm 1/p^\infty}\rangle^\flat)$ is obviously also compatible with the actions of the groups $\mathbb{Z}^d \subseteq \mathbb{Z}_p(1)^d$ (induced by our fixed choice of p-power roots of unity) on the left (from Sect. 6.3) and right, thereby inducing the first of the following maps:

$$\begin{array}{ccccc} R\Gamma(\mathbb{Z}^d, \mathbb{A}_{\inf}[\underline{U}^{\pm 1/p^\infty}]) & \longrightarrow & R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p(1)^d, W(\mathcal{O}\langle\underline{T}^{\pm 1/p^\infty}\rangle^\flat)) & \longrightarrow & R\Gamma_{\mathrm{pro\acute{e}t}}(\widehat{\mathrm{Sp}\, R\left[\tfrac{1}{p}\right]}, \mathbb{A}_{\inf,X}) \\ \| & & \text{def.}\,\| & & \| \\ D^{\mathrm{grp}}_{\mathcal{O},d} & & D^{\mathrm{cont}} & & D^{\mathrm{pro\acute{e}t}}_{R/\mathcal{O}} \end{array}$$

Here $D^{\mathrm{grp}} := D^{\mathrm{grp}}_{\mathcal{O},d}$ was the object of study of Sect. 6.3, and the second map is the Cartan–Leray almost quasi-isomorphism which has already appeared, for example just after the statement of Lemma 17. Both maps in the previous line are morphisms of commutative algebra objects in $D(\mathbb{A}_{\inf})$, compatible with the Frobenius on each object (in particular, with φ_{grp} on the left and $\varphi_{\mathrm{pro\acute{e}t}}$ on the right).

Moreover, we claim that the composition makes the following diagram of structure maps commute for each $r \geq 0$:

$$\begin{array}{ccc} H^0(D^{\mathrm{grp}}/\widetilde{\xi}_r) & \longrightarrow & H^0(D_{\mathrm{pro\acute{e}t}}/\widetilde{\xi}_r) \\ \uparrow{\scriptstyle \lambda_{r,\mathrm{grp}}} & & \uparrow{\scriptstyle \lambda_{r,\mathrm{pro\acute{e}t}}} \\ W_r(\mathcal{O}[\underline{T}^{\pm 1}]) & \hookrightarrow & W_r(\mathcal{O}\langle\underline{T}^{\pm 1}\rangle) \end{array}$$

The proof of this compatibility is a straightforward chase through the definitions of the structure maps $\lambda_{r,\mathrm{grp}}$ and $\lambda_{r,\mathrm{pro\acute{e}t}}$. We first identify the top row via $\widetilde{\theta}_r$ with the composition of the top row of the following diagram:

$$\begin{array}{ccccc} W_r(\mathcal{O})[\underline{U}^{\pm 1/p^\infty}]^{\mathbb{Z}^d} & \xrightarrow{U_i^k \mapsto T_i^{k/p^r}} & W_r(\mathcal{O}\langle\underline{T}^{\pm 1/p^\infty}\rangle)^{\mathbb{Z}_p(1)^d} & \longrightarrow & H^0_{\mathrm{pro\acute{e}t}}(\mathrm{Spf}\, R, W_r(\widehat{\mathcal{O}}^+_X)) \\ \uparrow{\scriptstyle \lambda_{r,\mathrm{grp}}} & & & \nwarrow & \uparrow{\scriptstyle \lambda_{r,\mathrm{pro\acute{e}t}}} \\ W_r(\mathcal{O}[\underline{T}^{\pm 1}]) & & \hookrightarrow & & W_r(\mathcal{O}\langle\underline{T}^{\pm 1}\rangle) \end{array}$$

The diagonal arrow here is the obvious inclusion (it is actually an isomorphism); since the Cartan–Leray map (i.e., top right horizontal arrow) is one of $W_r(\mathcal{O}\langle T^{\pm 1}\rangle)$-algebras and $\lambda_{r,\mathrm{pro\acute{e}t}}$ was defined to be precisely the algebra structure map, the resulting triangle commutes. Commutativity of the remaining trapezium is tautological: the definition of $\lambda_{r,\mathrm{grp}}$ in the proof of Lemma 16 was exactly to make this diagram (or, more precisely, the analogous diagram with $W_r(\mathcal{O}[\underline{T}^{\pm 1/p^\infty}])$ instead of $W_r(\mathcal{O}\langle\underline{T}^{\pm 1/p^\infty}\rangle)$) commute.

By the naturality of Theorem 9, the following diagram therefore commutes:

$$\begin{array}{ccc} \mathcal{W}_r^n(D^{\mathrm{grp}}) & \longrightarrow & \mathcal{W}_r^n(D^{\mathrm{pro\acute{e}t}}_{R/\mathcal{O}}) \\ \uparrow{\scriptstyle \lambda^n_{r,\mathrm{grp}}} & & \uparrow{\scriptstyle \lambda^n_{r,\mathrm{pro\acute{e}t}}} \\ W_r\Omega^n_{\mathcal{O}[\underline{T}^{\pm1}]/\mathcal{O}} & \longrightarrow & W_r\Omega^n_{\mathcal{O}\langle\underline{T}^{\pm1}\rangle/\mathcal{O}} \end{array}$$

The bottom horizontal arrow here becomes an isomorphism after p-adic completion,[55] and $\lambda^n_{r,\mathrm{grp}}$ was proved to be an isomorphism in Theorem 10; so to complete the proof it remains to show that the top horizontal arrow identifies $\mathcal{W}_r^n(D^{\mathrm{pro\acute{e}t}}_{R/\mathcal{O}})$ with the p-adic completion of $\mathcal{W}_r^n(D^{\mathrm{grp}})$. But the top horizontal arrow is precisely H^n of the composition

$$L\eta_{[\zeta_{p^r}]-1}(D^{\mathrm{grp}}/\widetilde{\xi}_r) \longrightarrow L\eta_{[\zeta_{p^r}]-1}(D^{\mathrm{cont}}/\widetilde{\xi}_r) \longrightarrow L\eta_{[\zeta_{p^r}]-1}(D^{\mathrm{pro\acute{e}t}}_{R/\mathcal{O}}/\widetilde{\xi}_r),$$

where the second arrow is the quasi-isomorphism (t5) of Lemma 18. Meanwhile, the first arrow identifies the middle term with the derived p-adic completion of the left: indeed, $L\eta_{[\zeta_{p^r}]-1}$ commutes with p-adic completion by Remark 3, so it is enough to check that $D^{\mathrm{cont}}/\widetilde{\xi}_r = R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p(1)^d, W_r(\mathcal{O}\langle\underline{T}^{\pm1/p^\infty}\rangle))$ is the derived p-adic completion of $D^{\mathrm{grp}}/\widetilde{\xi}_r = R\Gamma(\mathbb{Z}^d, W_r(\mathcal{O})[\underline{U}^{\pm1/p^\infty}])$; but this follows from $W_r(\mathcal{O}\langle\underline{T}^{\pm1/p^\infty}\rangle)$ being the p-adic completion of $W_r(\mathcal{O}[\underline{T}^{\pm1/p^\infty}])$. So, finally, recall that the cohomology groups of $L\eta_{[\zeta_{p^r}]-1}(D^{\mathrm{grp}}/\widetilde{\xi}_r)$ are p-torsion-free (since D^{grp} satisfies ($\mathcal{W}$2) and ($\mathcal{W}$3)), whence H^n of its derived p-adic completion is the same as the naive p-adic completion of its H^n.

This completes the proof of the p-adic Cartier isomorphism and these notes.

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Appendix 1: $\mathbb{A}_{\mathrm{inf}}$ and Its Modules

The base ring for the cohomology theory constructed in [5] is Fontaine's infinitesimal period ring $\mathbb{A}_{\mathrm{inf}} := W(\mathcal{O}^\flat)$, where $\mathcal{O}$ is the ring of integers of a complete, non-archimedean, algebraically closed field $\mathbb{C}$ of mixed characteristic. Since $\mathcal{O}$ is a perfectoid ring (Example 2), the general theory developed in Sect. 3 (including Sect. 3.3) applies in particular to $\mathcal{O}$. Our goal here is firstly to present a few results which are particular to $\mathcal{O}$ in order to familiarise the reader, who may be encountering

[55]By Remark 8(vi), the p-adic completions may be identified respectively with $\varprojlim_s W_r\Omega^n_{(\mathcal{O}[\underline{T}^{\pm1}]/p^s)/(\mathcal{O}/p^s\mathcal{O})}$ and $\varprojlim_s W_r\Omega^n_{(\mathcal{O}\langle\underline{T}^{\pm1}\rangle/p^s)/(\mathcal{O}/p^s\mathcal{O})}$, which are clearly the same.

these objects for the first time, with $\mathcal{O}$ and $\mathbb{A}_{\text{inf}}$; then we will explain some of the finer theory of modules over $\mathbb{A}_{\text{inf}}$.

We begin by recalling from [24, Sect. 3] that $\mathcal{O}^\flat$ is the ring of integers of $\mathbb{C}^\flat := \operatorname{Frac} \mathcal{O}^\flat$ (footnote 10 shows that $\mathcal{O}^\flat$ is an integral domain), which is a non-archimedean, algebraically closed field of characteristic $p > 0$, with the same residue field k as $\mathcal{O}$. The absolute value on $\mathbb{C}^\flat$ is given by multiplicatively extending the absolute value on $\mathcal{O}^\flat$ given by

$$\mathcal{O}^\flat = \varprojlim_{x \mapsto x^p} \mathcal{O} \xrightarrow{x \mapsto x^{(0)}} \mathcal{O} \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0},$$

where the first arrow uses the convention introduced just before Lemma 5, and the second arrow is the absolute value on $\mathcal{O}$. The reader may wish to check that this is indeed an absolute value, i.e., satisfies the ultrametric inequality, that $\mathbb{C}^\flat$ is complete under it, and that the ring of integers is exactly $\mathcal{O}^\flat$. The existence of the canonical projection $\mathcal{O}^\flat \to \mathcal{O}/p\mathcal{O}$ implies that $\mathcal{O}^\flat$ and $\mathcal{O}$ have the same residue field. Hensel's lemma shows that $\mathbb{C}^\flat$ is algebraically closed.[56]

Now we turn to $\mathbb{A}_{\text{inf}}$. Let $t \in \mathbb{A}_{\text{inf}}$ be any element whose image in $\mathbb{A}_{\text{inf}}/p\mathbb{A}_{\text{inf}} = \mathcal{O}^\flat$ belongs to $\mathfrak{m}^\flat \setminus \{0\}$; examples include $t = [\pi]$, where $\pi \in \mathfrak{m}^\flat \setminus \{0\}$, and $t = \xi$, where ξ is any generator of $\operatorname{Ker} \theta$. Then p, t is a regular sequence, and $\mathbb{A}_{\text{inf}}$ is a $\langle p, t \rangle$-adically complete local ring whose maximal ideal equals the radical of $\langle p, t \rangle$; in short, $\mathbb{A}_{\text{inf}}$ "appears two-dimensional and Cohen–Macaulay".

In fact, as we will explain the result of this appendix, modules (more precisely, finitely presented modules which become free after inverting p) over $\mathbb{A}_{\text{inf}}$ even behave as though $\mathbb{A}_{\text{inf}}$ were a two-dimensional, regular local ring.[57] Further details may be found in [5, Sect. 4.2].

Remark 11 In light of the goal, it is sensible to recall the structure of modules over any two-dimensional regular local ring Λ, such as $\Lambda = \mathcal{O}_K[[T]]$ where $\mathcal{O}_K$ is a

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[56] We sketch the proof here, which is obtained by reversing the roles of $\mathcal{O}$ and $\mathcal{O}^\flat$ in [24, Proposition 3.8]. Let $p^\flat := (p, p^{1/p}, p^{1/p^2}, \dots) \in A^\flat$, whose absolute value $|p^\flat| = |p|$ we may normalise to p^{-1} for simplicity of notation. It is sufficient to prove the following, which allows a root to any given polynomial to be built by successive approximation: If $f(X) \in \mathcal{O}^\flat[X]$ is a monic irreducible polynomial of degree $d \geq 1$, and $\alpha \in \mathcal{O}^\flat$ satisfies $|f(\alpha)| \leq p^{-n}$ for some $n \geq 0$, then there exists $\varepsilon \in \mathcal{O}$ satisfying $|\varepsilon| \leq p^{-n/d}$ and $|f(\alpha + \varepsilon)| \leq p^{-(n+1)}$. Well, given such $f(X)$ and α, use the fact that $\mathbb{C}^\flat$ and $\mathbb{C}$ have the same value group (this is easy to prove), which is divisible since $\mathbb{C}$ is algebraically closed, to find $c \in \mathcal{O}^\flat$ such that $c^{-d} f(\alpha)$ is a unit of $\mathcal{O}^\flat$. Then $g(X) := c^{-d} f(\alpha + cX)$ is a monic irreducible polynomial in $\mathbb{C}^\flat[X]$ whose constant coefficient lies in $\mathcal{O}^\flat$ (even $\mathcal{O}^{\flat\times}$); a standard consequence of Hensel's lemma is then that $g(X) \in \mathcal{O}^\flat[X]$. Next observe that the canonical projection $\mathcal{O}^\flat \to \mathcal{O}/p\mathcal{O}$ has kernel $p^\flat \mathcal{O}^\flat$ (*Proof.* Either argue using valuations, or extract a more general result from the proof of Lemma 8.), whence every monic polynomial in $\mathcal{O}^\flat/p^\flat\mathcal{O}^\flat$ has a root. So, by lifting a root we find $\beta \in \mathcal{O}^\flat$ satisfying $g(\beta) \in p^\flat\mathcal{O}^\flat$; this implies that $f(\alpha + c\beta) \in f(\beta)p^\flat\mathcal{O}^\flat$, and so $\varepsilon := c\beta$ has the desired property.

[57] However, $\mathbb{A}_{\text{inf}}$ is not Noetherian, it is usually not coherent [20], and the presence of certain infinitely generated, non-topologically-closed ideals implies that it has infinite Krull dimension [21].

discrete valuation ring. Let $\pi, t \in \Lambda$ be a system of local parameters and $\mathfrak{m} = \langle \pi, t \rangle$ its maximal ideal.

(i) Most importantly, any vector bundle on the punctured spectrum $\operatorname{Spec} \Lambda \setminus \{\mathfrak{m}\}$ extends uniquely to a vector bundle on $\operatorname{Spec} \Lambda$.

(ii) Finitely generated modules over Λ are perfect, i.e., admit finite length resolutions by finite free Λ-modules. (*Proof.* Immediate from the regularity of Λ.)

(iii) If M is any finitely generated Λ-module, then there is a functorial short exact sequence

$$0 \longrightarrow M_{\mathrm{tor}} \longrightarrow M \longrightarrow M_{\mathrm{free}} \longrightarrow \overline{M} \longrightarrow 0$$

of Λ-modules, where M_{tor} is torsion, M_{free} is finite free, and $\overline{M}$ is killed by a power of $\mathfrak{m}$.

Proof. M_{tor} is by definition the torsion submodule of M, whence M/M_{tor} restricts to a torsion-free coherent sheaf on the punctured spectrum $\operatorname{Spec} \Lambda \setminus \{\mathfrak{m}\}$; but the punctured spectrum is a regular one-dimensional scheme, so this torsion-free coherent sheaf is necessary a vector bundle, and so extends to a vector bundle on $\operatorname{Spec} \Lambda$ by (i); this vector bundle corresponds to a finite free Λ-module M_{free} which contains M/M_{tor}, with the ensuing quotient $\overline{M}$ being supported at the closed point of $\operatorname{Spec} \Lambda$.

(iv) Finite projective modules over $\Lambda[\frac{1}{\pi}]$ are finite free.

Proof. Let N be a finite projective $\Lambda[\frac{1}{\pi}]$-module, and pick a finitely generated Λ-module $N' \subseteq N$ satisfying $N'[\frac{1}{\pi}] = N$. Then $N'_{\mathfrak{p}}$ is a projective module over $\Lambda_{\mathfrak{p}}$ for every non-maximal prime ideal $\mathfrak{p} \subseteq \Lambda$: indeed either $\pi \notin \mathfrak{p}$, in which case $N'_{\mathfrak{p}}$ is a localisation of the projective module N, or $\mathfrak{p} = \langle \pi \rangle$, in which case $\Lambda_{\mathfrak{p}}$ is a discrete valuation ring and it is sufficient to note that $N'_{\mathfrak{p}}$ obviously has no π-torsion. This means that N' restricts to a vector bundle on the punctured spectrum, whose unique extension to a finite free Λ-module N'' satisfies $N''[\frac{1}{\pi}] = N$.

Theorem 11 *(i) (Kedlaya) Any vector bundle on the punctured spectrum*

$$\operatorname{Spec} \mathbb{A}_{\mathrm{inf}} \setminus \{\textit{the max. ideal of } \mathbb{A}_{\mathrm{inf}}\}$$

extends uniquely to a vector bundle on $\operatorname{Spec} \mathbb{A}_{\mathrm{inf}}$.

(ii) If M is a finitely presented $\mathbb{A}_{\mathrm{inf}}$-module such that $M[\frac{1}{p}]$ is finite free over $\mathbb{A}_{\mathrm{inf}}[\frac{1}{p}]$, then M is perfect (again, this means that M admits a finite length resolution by finite free $\mathbb{A}_{\mathrm{inf}}$-modules).

(iii) If M is a finitely presented $\mathbb{A}_{\mathrm{inf}}$-module such that $M[\frac{1}{p}]$ is finite free over $\mathbb{A}_{\mathrm{inf}}[\frac{1}{p}]$, then there is functorial short exact sequence of $\mathbb{A}_{\mathrm{inf}}$-modules

$$0 \longrightarrow M_{\mathrm{tor}} \longrightarrow M \longrightarrow M_{\mathrm{free}} \longrightarrow \overline{M} \longrightarrow 0$$

such that: M_{tor} is a perfect $\mathbb{A}_{\mathrm{inf}}$-module killed by a power of p; M_{free} is a finite free $\mathbb{A}_{\mathrm{inf}}$-module; and $\overline{M}$ is a perfect $\mathbb{A}_{\mathrm{inf}}$-module killed by a power of the ideal $\langle p, t\rangle$.

(iv) Finite projective modules over $\mathbb{A}_{\mathrm{inf}}[\frac{1}{p}]$ are finite free.

Proof We have nothing to say about (i) here, and refer instead to [5, Lemma 4.6]. We will also only briefly comment on the remaining parts of the theorem, since these self-contained results may be easily read in [5, Sect. 4.2].

(ii) By clearing denominators in a basis for $M[\frac{1}{p}]$ to construct a finite free $\mathbb{A}_{\mathrm{inf}}$-module $M' \subseteq M$ satisfying $M'[\frac{1}{p}] = M[\frac{1}{p}]$, we may reduce to the case that M is killed by a power of p, i.e., M is a $\mathbb{A}_{\mathrm{inf}}/p^r\mathbb{A}_{\mathrm{inf}} = W_r(\mathcal{O}^\flat)$-module for some $r \gg 0$. By an induction on r, using that $W_r(\mathcal{O}^\flat)$ can be shown to be coherent [5, Proposition 3.24] (this is not a trivial result), one can reduce to the case $r = 1$, in which case it easily follows from the classification of finitely presented modules over the valuation ring $\mathcal{O}^\flat$: they have the shape $(\mathcal{O}^\flat)^n \oplus \mathcal{O}^\flat/a_1\mathcal{O}^\flat \oplus \cdots \oplus \mathcal{O}^\flat/a_m\mathcal{O}^\flat$, for some $n \geq 1$ and $a_i \in \mathcal{O}^\flat$, and so in particular are perfect.

(iii) This is proved similarly to the analogous assertion in the previous remark.

(iv) This is proved exactly as the analogous assertion in the previous remark, once it is checked that the localisation $\mathbb{A}_{\mathrm{inf},\langle p\rangle}$ is a discrete valuation ring.

Corollary 5 *Let M be a finitely presented $\mathbb{A}_{\mathrm{inf}}$-module such that $M[\frac{1}{p}]$ is finite free over $\mathbb{A}_{\mathrm{inf}}[\frac{1}{p}]$. If either $M \otimes_{\mathbb{A}_{\mathrm{inf}}} W(k)$ or $M \otimes_{\mathbb{A}_{\mathrm{inf}}} \mathcal{O}$ is p-torsion-free (equivalently, finite free over $W(k)$ or $\mathcal{O}$ respectively), then M is a finite free $\mathbb{A}_{\mathrm{inf}}$-module.*

Proof It follows easily from the hypothesis that the map $M \to M_{\mathrm{free}}$ in Theorem 11(iii) becomes an isomorphism after tensoring by $W(k)$ or $\mathcal{O}$; hence $M[\frac{1}{p}]$ and $M \otimes_{\mathbb{A}_{\mathrm{inf}}} k$ have the same rank over $\mathbb{A}_{\mathrm{inf}}[\frac{1}{p}]$ and k respectively. But an easy Fitting ideal argument shows that if N is a finitely presented module over a local integral domain R satisfying $\dim_{\mathrm{Frac}\,R}(N \otimes_R \mathrm{Frac}\,R) = \dim_{k(R)}(N \otimes_R k(R))$, then N is finite free over R.

To state and prove the next corollary we use the elements $\xi, \xi_r, \mu \in \mathbb{A}_{\mathrm{inf}}$ constructed in Sect. 3.3:

Corollary 6 *Let M be a finitely presented $\mathbb{A}_{\mathrm{inf}}$-module, and assume:*

- *$M[\frac{1}{p\mu}]$ is a finite free $\mathbb{A}_{\mathrm{inf}}[\frac{1}{p\mu}]$-module of the same rank as the $W(k)$-module $M \otimes_{\mathbb{A}_{\mathrm{inf}}} W(k)$.*
- *There exists a Frobenius-semi-linear endomorphism of M which becomes an isomorphism after inverting ξ.*

Then $M[\frac{1}{p}]$ is finite free over $\mathbb{A}_{\mathrm{inf}}[\frac{1}{p}]$.

Proof We must show that each Fitting ideal of the $\mathbb{A}_{\mathrm{inf}}[\frac{1}{p}]$-module $M[\frac{1}{p}]$ is either 0 or $\mathbb{A}_{\mathrm{inf}}[\frac{1}{p}]$; indeed, this means exactly that $M[\frac{1}{p}]$ is finite projective over $\mathbb{A}_{\mathrm{inf}}[\frac{1}{p}]$, which is sufficient by Theorem 11(iv). Since Fitting ideals behave well under base change, it is equivalent to prove that the first non-zero Fitting ideal $J \subseteq \mathbb{A}_{\text{inf}}$ of M contains a power of p. Again using that Fitting ideals base change well, our hypotheses imply that $J\mathbb{A}_{\text{inf}}[\frac{1}{p\mu}] = \mathbb{A}_{\text{inf}}[\frac{1}{p\mu}]$ and $JW(k) \neq 0$; that is, J contains a power of $p\mu$ and $J + W(\mathfrak{m}^\flat)$ contains a power of p, where $W(\mathfrak{m}^\flat) := \text{Ker}(\mathbb{A}_{\text{inf}} \to W(k))$. Because of the existence of the Frobenius on M, we also know that J and $\varphi(J)$ are equal up to a power of $\varphi(\xi)$. In conclusion, we may pick $N \gg$ such that

(i) $(p\mu)^N \in J$;
(ii) $p^N \in J + W(\mathfrak{m})$;
(iii) $\varphi(\xi)^N \varphi(J) \subseteq J$ and $\varphi(\xi)^N J \subseteq \varphi(J)$.

Since $W(\mathfrak{m})$ is the p-adic completion of the ideal generated by $\varphi^{-r}(\mu^N)$, for all $r \geq 0$,[58] observation (ii) lets us write $p^N = \alpha + \beta\varphi^{-r}(\mu^N) + \beta' p^{N+1}$ for some $\alpha \in J$ and $\beta, \beta' \in \mathbb{A}_{\text{inf}}$, and $r \geq 0$. Since $1 - \beta' p$ is invertible, we may easily suppose that $\beta' = 0$, i.e., $p^N = \alpha + \beta\varphi^{-r}(\mu^N)$. Multiplying through by $p^N \xi_r^N$ gives $\xi_r^N p^{2N} = p^N \xi_r^N \alpha + \beta p^N \mu^N$, which belongs to J by (i) and (ii).

We claim, for any $a, i \geq 1$, that

$$\xi_r^a p^{\text{some power}} \in J \implies \xi_r^{a-1} p^{\text{some other power}} \in J.$$

A trivial induction then shows that J contains a power of p, thereby completing the proof, and so it remains only to prove this claim. Suppose $\xi_r^a p^b \in J$ for some $a, b, i \geq 1$. Then $\varphi^r(\xi_r)^a p^b \in \varphi^r(J)$, and so $\varphi^r(\xi_r)^{a+N} p^b \in J$ (since an easy generalisation of (iii) implies that $\varphi^r(\xi_r)^N \varphi^r(J) \subseteq J$). But $\varphi^r(\xi_r) \equiv p^r \mod \xi_r$, so we may write $\varphi^r(\xi_r)^{a+N} = p^{r(a+N)} + \alpha\xi_r$ for some $\alpha \in \mathbb{A}_{\text{inf}}$ and thus deduce that $J \ni (p^{r(a+N)} + \alpha\xi_r)p^b = p^{r(a+N)+b} + \alpha\xi_r p^b$. Now multiply through by ξ_r^{a-1} and use the supposition to obtain $\xi_r^{a-1} p^{r(a+N)+b} \in J$, as required.

We will also need the following to eliminate the appearance of higher Tors in the crystalline specialisation of the $\mathbb{A}_{\text{inf}}$-cohomology theory:

Lemma 22 *Let M be an $\mathbb{A}_{\text{inf}}$-module such that $M[\frac{1}{p}]$ is flat over $\mathbb{A}_{\text{inf}}[\frac{1}{p}]$. Then* $\text{Tor}_*^{\mathbb{A}_{\text{inf}}}(M, W(k)) = 0$ *for* $* > 1$.

Proof Let $[\mathfrak{m}^\flat] \subseteq W(\mathfrak{m}^\flat)$ be the ideal of $\mathbb{A}_{\text{inf}}$ which is generated by Teichmüller lifts of elements of $\mathfrak{m}^\flat$. We first observe that $\mathbb{A}_{\text{inf}}/[\mathfrak{m}^\flat]$ is p-torsion-free and has Tor-dimension $= 1$ over $\mathbb{A}_{\text{inf}}$: indeed, $[\mathfrak{m}^\flat]$ is the increasing union of the ideals $[\pi]\mathbb{A}_{\text{inf}}$, for $\pi \in \mathfrak{m}^\flat \setminus \{0\}$, and the claims are true for $\mathbb{A}_{\text{inf}}/[\pi]\mathbb{A}_{\text{inf}}$ since $p, [\pi]$ is a regular sequence of $\mathbb{A}_{\text{inf}}$.

Next, since $W_r(\mathfrak{m}^\flat)$ is generated by the analogous Teichmüller lifts in $W_r(\mathcal{O}^\flat) = \mathbb{A}_{\text{inf}}/p^r\mathbb{A}_{\text{inf}}$, for any $r \geq 1$ (c.f., footnote 20), the quotient $W(\mathfrak{m}^\flat)/[\mathfrak{m}^\flat]$ is p-divisible. Combined with the previous observation, it follows that $W(\mathfrak{m}^\flat)/[\mathfrak{m}^\flat]$ is uniquely p-divisible, i.e., an $\mathbb{A}_{\text{inf}}[\frac{1}{p}]$-module, whence

58By an easy induction using that p is a non-zero-divisor in $\mathbb{A}_{\text{inf}}/W(\mathfrak{m})$, this follows from the fact that the maximal ideal of $\mathbb{A}_{\text{inf}}/p\mathbb{A}_{\text{inf}} = \mathcal{O}^\flat$ is generated by the elements $\varphi^{-r}(\varepsilon) - 1$, for all $r \geq 0$.

$$\mathrm{Tor}_*^{\mathbb{A}_{\inf}}(W(\mathfrak{m}^\flat)/[\mathfrak{m}^\flat], M) = \mathrm{Tor}_*^{\mathbb{A}_{\inf}\left[\frac{1}{p}\right]}\left(W(\mathfrak{m}^\flat)/[\mathfrak{m}^\flat], M\left[\tfrac{1}{p}\right]\right),$$

which vanishes for $* > 0$ by the hypothesis on M. Combining this with the short exact sequence

$$0 \to W(\mathfrak{m}^\flat)/[\mathfrak{m}^\flat] \to \mathbb{A}_{\inf}/[\mathfrak{m}^\flat] \to \mathbb{A}_{\inf}/W(\mathfrak{m}^\flat) = W(k) \to 0$$

and the initial observation about the Tor-dimension of the middle term completes the proof.

Appendix 2: Two Lemmas on Koszul Complexes

Let R be a ring, and $g_1, \ldots, g_d \in R$. The associated Koszul complex will be denoted by $K_R(g_1, \ldots, g_d) = \bigotimes_{i=1}^d K_R(g_i)$, where $K_R(g_i) := [R \xrightarrow{g_i} R]$. Here we state two useful lemmas concerning such complexes, the second of which describes the behaviour of the décalage functor.

Lemma 23 *Let $g \in R$ be an element which divides $g_1, \ldots, g_d$, and such that g_i divides g for some i. Then there are isomorphisms of R-modules*

$$H^n(K_R(g_1, \ldots, g_d)) \cong R[g]^{\binom{d-1}{n}} \oplus R/gR^{\binom{d-1}{n-1}}$$

for all $n \geq 0$.

Proof [5, Lemma 7.10].

Lemma 24 *Let $f \in R$ be a non-zero-divisor such that, for each i, either f divides g_i or g_i divides f. Then:*

- *If f divides g_i for all i, then $\eta_f K_R(g_1, \ldots, g_d) \cong K_R(g_1/f, \ldots, g_d/f)$.*
- *If g_i divides f for some i, then $\eta_f K_R(g_1, \ldots, g_d)$ is acyclic.*

Proof [5, Lemma 7.9].

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On the Cohomology of the Affine Space

Pierre Colmez and Wiesława Nizioł

Abstract We compute the p-adic geometric pro-étale cohomology of the rigid analytic affine space (in any dimension). This cohomology is non-zero, contrary to the étale cohomology, and can be described by means of differential forms.

1 Introduction

Let K be a complete discrete valuation field of characteristic 0 with perfect residue field of positive characteristic p. Let C be the completion of an algebraic closure $\overline{K}$ of K. We denote by $\mathscr{G}_K$ the absolute Galois group of K (it is also the group of continuous automorphisms of C that fix K).

For $n \geq 1$, let $\mathbf{A}_K^n$ be the rigid analytic affine space over K of dimension n and $\mathbf{A}^n$ be its scalar extension to C. Our main result is the following theorem.

Theorem 1 *For $r \geq 1$, we have isomorphisms of $\mathscr{G}_K$-Fréchet spaces*

$$H^r_{\text{proét}}(\mathbf{A}^n, \mathbf{Q}_p(r)) \simeq \Omega^{r-1}(\mathbf{A}^n)/\operatorname{Ker} d \simeq \Omega^r(\mathbf{A}^n)^{d=0},$$

where Ω denotes the sheaf of differentials.

Remark 2 (i) The p-adic pro-étale cohomology behaves in a remarkably different way from other (more classical) cohomologies. For example, for $i \geq 1$, we have:

- $H^i_{\text{dR}}(\mathbf{A}^n) = H^i_{\text{HK}}(\mathbf{A}^n) = 0$, where $H^\bullet_{\text{HK}}$ is Hyodo-Kato cohomology (see [5] for its definition),
- $H^i_{\text{ét}}(\mathbf{A}^n, \mathbf{Q}_\ell) = H^i_{\text{proét}}(\mathbf{A}^n, \mathbf{Q}_\ell) = 0$, if $\ell \neq p$,
- $H^i_{\text{ét}}(\mathbf{A}^n, \mathbf{Q}_p) = 0$. (Cf. [1] or Remark 11.)

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We listed the $\ell \neq p$ and $\ell = p$ cases of étale cohomology separately because, if $\ell \neq p$, the triviality of the cohomology of $\mathbf{A}^n$ is a consequence of the triviality of the cohomology of the closed ball (which explains why the pro-étale cohomology is also trivial), but the p-adic étale cohomology of the ball is highly nontrivial.

(ii) Using overconvergent syntomic cohomology allows to prove a more general result [2, Theorem 1.8]: if X is a Stein space over K admitting a semistable model over the ring of integers of K, there exists an exact sequence

$$0 \to \Omega^{r-1}(X)/\mathrm{Ker}\, d \to H^r_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) \to (\mathrm{B}^+_{\mathrm{st}} \widehat{\otimes}\, H^r_{\mathrm{HK}}(X))^{N=0,\varphi=p^r} \to 0.$$

However making syntomic cohomology overconvergent is technically demanding and the simple proof below uses special features of the geometry of the affine space.

(iii) Another possible approach (cf. [6]) is to compute the pro-étale cohomology of the relative fundamental exact sequence $0 \to \mathbf{Q}_p(r) \to \mathbb{B}^{\varphi=p^r}_{\mathrm{cris}} \to \mathbb{B}_{\mathrm{dR}}/F^r \to 0$.

Let $\mathring{\mathbf{B}}^n$ be the open unit ball of dimension n. An adaptation of the proof of Theorem 1 shows the following result:

Theorem 3 *For $r \geq 1$, we have isomorphisms of $\mathscr{G}_K$-Fréchet spaces*

$$H^r_{\mathrm{pro\acute{e}t}}(\mathring{\mathbf{B}}^n, \mathbf{Q}_p(r)) \simeq \Omega^{r-1}(\mathring{\mathbf{B}}^n)/\,\mathrm{Ker}\, d \simeq \Omega^r(\mathring{\mathbf{B}}^n)^{d=0}.$$

2 Syntomic Variations

If $r = 1$, one can give an elementary proof of Theorem 1 using Kummer theory, but it does not seem very easy to extend this kind of methods to treat the case $r \geq 2$. Instead we are going to use syntomic methods.

Recall that the étale-syntomic comparison theorem [3, 7] reduces the computation of p-adic étale cohomology to that of syntomic cohomology.[1] The latter is defined as a filtered Frobenius eigenspace of absolute crystalline cohomology (see [4] for a gentle introduction and [7] for a more thorough treatment) and can be thought of as a higher dimensional version of the Fontaine-Lafaille functor. Its computation reduces to a computation of cohomology of complexes built from differential forms, and hence is often doable.

More precisely, if $\mathscr{X}$ is a quasi-compact semistable p-adic formal scheme over $\mathscr{O}_K$, then the Fontaine–Messing period map [4]

$$\alpha^{FM} : \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_C}, \mathbf{Z}_p(r)) \to \tau_{\leq r}\mathrm{R}\Gamma_{\acute{e}t}(\mathscr{X}_C, \mathbf{Z}_p(r)) \quad (1)$$

[1]The computations in [3] are done over K (or over its finite extensions), but working directly over C simplifies a lot the local arguments because there is no need to change the Frobenius and the group Γ of loc. cit. becomes commutative (hence so does its Lie algebra, which makes the arguments using Koszul complexes a lot simpler).

is a p^N-quasi-isomorphism[2] for a constant $N = N(r)$. This generalizes easily to semistable p-adic formal schemes over $\mathcal{O}_C$: the rational étale and pro-étale cohomology of such schemes are computed by the syntomic complexes $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\mathcal{O}_C}, \mathbf{Z}_p(r))_{\mathbf{Q}}$ and $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\mathcal{O}_C}, \mathbf{Q}_p(r))$, respectively, where the latter complex is defined by taking $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\mathcal{O}_C}, \mathbf{Z}_p(r))_{\mathbf{Q}}$ locally and then glueing.

The purpose of this section is to construct a particularly simple complex that, morally, computes the syntomic, and hence (pro-)étale as well, cohomology of the (canonical formal model of the) affine space and the open ball, but does not use a model of the whole space, only of closed balls of increasing radii.

Period rings. —Let $C^\flat$ be the tilt of C and let $\mathrm{A}_{\mathrm{cris}} \subset \mathrm{B}^+_{\mathrm{cris}} = \mathrm{A}_{\mathrm{cris}}[\frac{1}{p}] \subset \mathrm{B}^+_{\mathrm{dR}}$ be the usual Fontaine rings.

Let $\theta : \mathrm{B}^+_{\mathrm{dR}} \to C$ be the canonical projection (its restriction to $\mathrm{A}_{\mathrm{cris}}$ induces a projection $\mathrm{A}_{\mathrm{cris}} \to \mathcal{O}_C$), and let $\mathrm{F}^\bullet_\theta \mathrm{B}^+_{\mathrm{dR}}$ be the filtration by the powers of $\mathrm{Ker}\,\theta$ and $\mathrm{F}^\bullet_\theta \mathrm{A}_{\mathrm{cris}}$ be the induced filtration. For $j \in \mathbf{Z}$, let $\mathrm{A}_j = \mathrm{A}_{\mathrm{cris}}/\mathrm{F}^j_\theta$ (hence $\mathrm{A}_j = 0$ for $j \leq 0$ and $\mathrm{A}_1 = \mathcal{O}_C$).

We choose a morphism of groups $\alpha \mapsto p^\alpha$ from $\mathbf{Q}$ to C^* compatible with the analogous morphism on $\mathbf{Z}$. We denote by $\tilde{p}^\alpha$ the element $(p^\alpha, p^{\alpha/p}, p^{\alpha/p^2}, \dots)$ of $C^\flat$ and by $[\tilde{p}^\alpha]$ its Teichmüller lift in $\mathrm{A}_{\mathrm{cris}}$.

Closed balls. —For $\alpha \in \mathbf{Q}_+$, let

$$D_\alpha = \{z = (z_1, \dots, z_n),\ v_p(z_m) \geq -\alpha,\ \text{for } 1 \leq m \leq n\}$$

be the closed ball of valuation $-\alpha$ in $\mathbf{A}^n$, and denote by $\mathcal{O}(D_\alpha)$ (resp. $\mathcal{O}^+(D_\alpha)$) the ring of analytic functions (resp. analytic functions with integral values) on D_α. We have

$$\mathcal{O}(D_\alpha) = C\langle p^\alpha T_1, \dots, p^\alpha T_n\rangle \quad \text{and} \quad \mathcal{O}^+(D_\alpha) = \mathcal{O}_C\langle p^\alpha T_1, \dots, p^\alpha T_n\rangle.$$

Consider the lifts

$$R^+_\alpha = \mathrm{A}_{\mathrm{cris}}\langle[\tilde{p}^\alpha]T_1, \dots, [\tilde{p}^\alpha]T_n\rangle \quad \text{and} \quad R_\alpha = R^+_\alpha[\tfrac{1}{p}]$$

of $\mathcal{O}^+(D_\alpha)$ and $\mathcal{O}(D_\alpha)$, respectively. We extend φ on $\mathrm{A}_{\mathrm{cris}}$ to $\varphi : R_\alpha \to R_\alpha$ by setting $\varphi(T_m) = T_m^p$, for $1 \leq m \leq n$.

Definition 4 Let $r \geq 0$. If $\alpha \in \mathbf{Q}_+$ and $\Lambda = R_\alpha, R^+_\alpha$, we define the complexes

$$\mathrm{Syn}(\Lambda, r) := [\mathrm{HK}_r(\Lambda) \to \mathrm{DR}_r(\Lambda)],$$

where the brackets $[\cdots]$ denote the mapping fiber, and[3]

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[2]It means that the kernel and cokernel of the induced map on cohomology are annihilated by p^N.

[3]The differentials are taken relative to $\mathrm{A}_{\mathrm{cris}}$.

$$\begin{aligned}\mathrm{HK}_r(\Lambda) &:= [\Omega^\bullet_\Lambda \xrightarrow{\varphi - p^r} \Omega^\bullet_\Lambda],\\ F^r\Omega^\bullet_\Lambda &:= (F^r_\theta \Lambda \to F^{r-1}_\theta \Omega^1_\Lambda \to F^{r-2}_\theta \Omega^2_\Lambda \to \cdots),\\ \mathrm{DR}_r(\Lambda) &:= \Omega^\bullet_\Lambda / F^r = (\cdots \to \mathrm{A}_{r-i} \otimes_{\mathrm{A}_{\mathrm{cris}}} \Omega^i_\Lambda \xrightarrow{1\otimes d_i} \mathrm{A}_{r-i-1} \otimes_{\mathrm{A}_{\mathrm{cris}}} \Omega^{i+1}_\Lambda \to \cdots).\end{aligned}$$

The complex $\mathrm{Syn}(\mathbf{A}^n, r)$. —The above complexes for varying α are closely linked:

- The ring morphism $R_0 \to R_\alpha$, $T_m \to [\tilde{p}^\alpha]T_m$, for $1 \le m \le n$, induces an isomorphism of complexes $\mathrm{Syn}(R_0, r) \xrightarrow{\sim} \mathrm{Syn}(R_\alpha, r)$.
- For $\beta \ge \alpha$, the inclusion $\iota_{\beta,\alpha} : R_\beta \hookrightarrow R_\alpha$ induces a morphism of complexes $\mathrm{Syn}(R_\beta, r) \to \mathrm{Syn}(R_\alpha, r)$ thanks to the fact that $\varphi([\tilde{p}^s]) = [\tilde{p}^s]^p$, for all $s \in \mathbf{Q}_+$.

(We have analogous statements replacing R_α by R^+_α.)

The first point comes just from the fact that two closed balls are isomorphic, but the second point, to the effect that we can find liftings of the $\mathscr{O}(D_\alpha)$'s with compatible Frobenius, is a bit of a miracle, and will simplify greatly the computation of the syntomic cohomology of $\mathbf{A}^n$. In particular, it makes it possible to define the complex $\mathrm{Syn}(\mathbf{A}^n, r) := \mathrm{holim}_\alpha \mathrm{Syn}(R_\alpha, r)$ and, similarly, $\mathrm{HK}_r(\mathbf{A}^n)$ and $\mathrm{DR}_r(\mathbf{A}^n)$.

For $i \ge 0$ and $X = \mathbf{A}^n, R_\alpha, R^+_\alpha$, denote by $\mathrm{HK}^i_r(X)$, $\mathrm{DR}^i_r(X)$, and $\mathrm{Syn}^i(X, r)$ the cohomology groups of the corresponding complexes. We have a long exact sequence:

$$\cdots \to \mathrm{DR}^{i-1}_r(X) \to \mathrm{Syn}^i(X, r) \to \mathrm{HK}^i_r(X) \to \mathrm{DR}^i_r(X) \to \cdots$$

Proposition 5 *If $i \le r$, we have natural isomorphisms:*

- $H^i_{\text{ét}}(D_\alpha, \mathbf{Q}_p(r)) \cong \mathrm{Syn}^i(R_\alpha, r)$, *if* $\alpha \in \mathbf{Q}_+$.
- $H^i_{\text{proét}}(\mathbf{A}^n, \mathbf{Q}_p(r)) \cong \mathrm{Syn}^i(\mathbf{A}^n, r)$.

Proof Take $\alpha \in \mathbf{Q}_+$. By the comparison isomorphism (1), to prove the first claim, it suffices to show that the complex $\mathrm{Syn}(R_\alpha, r)$ computes the rational geometric log-syntomic cohomology of $\mathscr{D}_\alpha := \mathrm{Spf}\, \mathscr{O}^+(D_\alpha)$, the formal affine space over $\mathscr{O}_C$, that is a smooth formal model of D_α. To do this, recall that the latter cohomology is computed by the complex

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{D}_\alpha, \mathbf{Z}_p(r))_{\mathbf{Q}} = [\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{D}_\alpha/\mathrm{A}_{\mathrm{cris}})^{\varphi=p^r}_{\mathbf{Q}} \to \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{D}_\alpha/\mathrm{A}_{\mathrm{cris}})_{\mathbf{Q}}/F^r],$$

where $\mathrm{A}_{\mathrm{cris}}$ is equipped with the unique log-structure extending the canonical log-structure on $\mathscr{O}_C/p$. It suffices thus to show that there exists a quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{D}_\alpha/\mathrm{A}_{\mathrm{cris}})_{\mathbf{Q}} \simeq \Omega^\bullet_{R_\alpha}$ that is compatible with the Frobenius[4] and the filtration. But this is clear since $\mathrm{Spf}\, R^+_\alpha$ is a log-smooth lifting of $\mathscr{D}_\alpha$ from $\mathrm{Spf}\, \mathscr{O}_C$ to $\mathrm{Spf}\, \mathrm{A}_{\mathrm{cris}}$ that is compatible with the Frobenius on $\mathrm{A}_{\mathrm{cris}}$ and $\mathscr{O}^+(D_\alpha)/p$.

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[4]Recall that the Frobenius on crystalline cohomology is defined via the isomorphism $\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{D}_\alpha/\mathrm{A}_{\mathrm{cris}})_{\mathbf{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}((\mathscr{D}_{\alpha,}/p)/\mathrm{A}_{\mathrm{cris}})_{\mathbf{Q}}$ from the canonical Frobenius on the second term.

To show the second claim, we note that, for $\beta \geq \alpha$, there is a natural map (an injection) of liftings $(R^+_\beta \to \mathscr{O}^+(D_\beta)) \to (R^+_\alpha \to \mathscr{O}^+(D_\alpha))$. This allows us to use the comparison isomorphism (1) to define the second quasi-isomorphism in the sequence of maps

$$\begin{aligned}\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(\mathbf{A}^n, \mathbf{Q}_p(r)) &\simeq \tau_{\leq r}\,\mathrm{holim}_k\,\mathrm{R}\Gamma_{\acute{e}t}(D_k, \mathbf{Q}_p(r)) \simeq \tau_{\leq r}\,\mathrm{holim}_k\,\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{D}_k, \mathbf{Z}_p(r))_{\mathbf{Q}}\\ &\simeq \tau_{\leq r}\,\mathrm{holim}_k\,\mathrm{Syn}(R_k, r) = \tau_{\leq r}\mathrm{Syn}(\mathbf{A}^n, r).\end{aligned}$$

Here, the first quasi-isomorphism follows from the fact that $\{D_k\}_{k\in\mathbf{N}}$ is an admissible affinoid covering of $\mathbf{A}^n$ and the third one follows from the first claim. This finishes the proof. □

3 Computation of $\mathrm{HK}^i_r(\mathbf{A}^n)$

The group $\mathrm{HK}^i_r(\mathbf{A}^n)$ is, by construction, obtained from the $\mathrm{HK}^i_r(R_\alpha)$'s, but the latter are, individually, hard to compute and quite nasty: for example, $\mathrm{HK}^1_1(R_\alpha)$ is isomorphic to the quotient of $\mathbf{Q}_p\widehat{\otimes}\mathscr{O}(D_\alpha)^*$ by the sub $\mathbf{Q}_p$-vector space generated by $\mathscr{O}(D_\alpha)^*$; hence it is an infinite dimensional topological $\mathbf{Q}_p$-vector space in which 0 is dense. Fortunately Lemma 7 below shows that this is not a problem for the computation of $\mathrm{HK}^i_r(\mathbf{A}^n)$.

For $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbf{N}^n$, we set $|\mathbf{k}| = k_1 + \cdots + k_n$ and $T^{\mathbf{k}} = T_1^{k_1} \cdots T_n^{k_n}$. For $1 \leq j \leq n$, let ω_j be the differential form $\frac{dT_j}{T_j}$, and let ∂_j be differential operator defined by $df = \sum_{j=1}^n \partial_j f\, \omega_j$. For $\mathbf{j} = \{j_1, \ldots, j_i\}$, with $j_1 \leq j_2 \leq \cdots \leq j_i$, we set $\omega_{\mathbf{j}} = \omega_{j_1} \wedge \cdots \wedge \omega_{j_i}$. All elements η of $\Omega^i_{R_\alpha}$ can be written, in a unique way, in the form $\sum_{|\mathbf{j}|=i} a_{\mathbf{j}}\omega_{\mathbf{j}}$, where $a_{\mathbf{j}} \in \left(\prod_{j\in\mathbf{j}} T_j\right) R_\alpha$.

Lemma 6 *Let M be a sub-$\mathbf{Z}_p$-module of $\mathrm{A}_{\mathrm{cris}}$ or $\mathscr{O}_C$. Let $i \geq 1$ and $\mathbf{k} \in \mathbf{N}^n$. For $\omega = T^{\mathbf{k}} \sum_{|\mathbf{j}|=i} a_{\mathbf{j}}\omega_{\mathbf{j}}$, with $a_{\mathbf{j}} \in M$, such that $d\omega = 0$, there exists $\eta = T^{\mathbf{k}} \sum_{|\mathbf{j}|=i-1} b_{\mathbf{j}}\omega_{\mathbf{j}}$, such that $d\eta = \omega$ and $b_{\mathbf{j}} \in p^{-N(\mathbf{k})}M$, with $N(\mathbf{k}) = \inf_{j\in\mathbf{j}} v_p(k_j)$.*

Proof Permuting the T_m's, we can assume that $v_p(k_1) \leq v_p(k_2) \leq \cdots \leq v_p(k_n)$; in particular, $k_1 \neq 0$. Decompose ω as $\left(\omega_1 \wedge T^{\mathbf{k}} \sum_{1\in\mathbf{j}} a_{\mathbf{j}}\omega_{\mathbf{j}\setminus\{1\}}\right) + \omega'$, and set $\eta = \frac{1}{k_1} T^{\mathbf{k}} \sum_{1\in\mathbf{j}} a_{\mathbf{j}}\omega_{\mathbf{j}\setminus\{1\}}$; we have $\omega - d\eta = T^{\mathbf{k}} \sum_{1\notin\mathbf{j}} c_{\mathbf{j}}\omega_{\mathbf{j}}$ and it has a trivial differential. But $d(T^{\mathbf{k}} \sum_{1\notin\mathbf{j}} c_{\mathbf{j}}\omega_{\mathbf{j}}) = k_1 T^{\mathbf{k}} \sum_{1\notin\mathbf{j}} c_{\mathbf{j}}\omega_{\{1\}\cup\mathbf{j}} + \sum_{1\notin I} c'_I\omega_I$, hence $c_{\mathbf{j}} = 0$ for all $\mathbf{j}$, which proves that $d\eta = \omega$ and allows us to conclude. □

Lemma 7 *Let $\alpha \in \mathbf{Q}_+$ and let $\Lambda_\alpha = R^+_\alpha, \mathscr{O}^+(D_\alpha)$. Then $H^0_{\mathrm{dR}}(\Lambda_\alpha) = \mathrm{A}_{\mathrm{cris}}, \mathscr{O}_C$ and $\mathrm{HK}^0_r(R^+_\alpha) = \mathrm{A}_{\mathrm{cris}}^{\varphi=p^r}$, the natural maps*

$$H^i_{\mathrm{dR}}(\Lambda_{\alpha+1}) \to H^i_{\mathrm{dR}}(\Lambda_\alpha), \quad i \geq 1; \quad \mathrm{HK}^i_r(R^+_{\alpha+2}) \to \mathrm{HK}^i_r(R^+_\alpha), \quad i \geq 2.$$

are identically zero, and the image of the map $\mathrm{HK}^1_r(R^+_{\alpha+2}) \to \mathrm{HK}^1_r(R^+_\alpha)$ is annihilated by p^r.

Proof The computation of the H^0's is straightforward. The proof for the first map is similar (but easier) to that of the second one, so we are only going to prove the latter. Take $i \geq 2$. Let (ω^i, ω^{i-1}) be a representative of an element of $\mathrm{HK}^i_r(R^+_{\alpha+2})$. That is to say

$$\omega^i \in \Omega^i_{R^+_{\alpha+2}}, \quad \omega^{i-1} \in \Omega^{i-1}_{R^+_{\alpha+2}}, \quad d\omega^i = 0 \quad \text{and} \quad d\omega^{i-1} + (\varphi - p^r)\omega^i = 0.$$

Since $d\omega^i = 0$, we deduce from Lemma 6 that there exists $\eta^{i-1} \in \Omega^{i-1}_{R^+_{\alpha+1}}$ such that $\iota_{\alpha+2,\alpha+1}\omega^i = d\eta^{i-1}$ (we used here that $\frac{1}{m}[\tilde{p}]^m \in \mathrm{A_{cris}}$). Let $\omega_1^{i-1} = \iota_{\alpha+2,\alpha+1}\omega^{i-1} + (\varphi - p^r)\eta^{i-1}$. Then $d\omega_1^{i-1} = \iota_{\alpha+2,\alpha+1}d\omega^{i-1} + (\varphi - p^r)d\eta^{i-1} = 0$; hence there exists $\eta^{i-2} \in \Omega^{i-2}_{R^+_\alpha}$ such that $\iota_{\alpha+1,\alpha}\omega_1^{i-1} = d\eta^{i-2}$. It follows that $\iota_{\alpha+2,\alpha}(\omega^i, \omega^{i-1}) = d(\iota_{\alpha+1,\alpha}\eta^{i-1}, \eta^{i-2})$, as wanted.

Take now $i = 1$ and use the notation from the above computation. Arguing as above we show that (ω^1, ω^0) is in the same class as $(0, \omega^0)$, with $\omega^0 \in \mathrm{A_{cris}}$. But the map $\varphi - p^r : \mathrm{A_{cris}} \to \mathrm{A_{cris}}$ is p^r-surjective. This proves the last statement of the lemma. □

Remark 8 (i) The same arguments would prove that there exists $N : \mathbf{Q}^*_+ \to \mathbf{N}$ such that, if $\beta > \alpha$ and $i \geq 1$, the images of the natural maps $H^i_{\mathrm{dR}}(R^+_\beta) \to H^i_{\mathrm{dR}}(R^+_\alpha)$, $\mathrm{HK}^i_r(R^+_\beta) \to \mathrm{HK}^i_r(R^+_\alpha)$ are killed by $p^{N(\beta-\alpha)}$. This is sufficient to extend Corollary 9 and Lemma 10 below to the unit ball $\mathring{\mathbf{B}}^n$.

(ii) Note, however, that $N(u) \to +\infty$ when $u \to 0^+$. This prevents the extension of Lemma 10 to the integral de Rham cohomology of $\mathring{\mathbf{B}}^n$ which is good since this integral de Rham cohomology, in degrees $1 \leq i \leq n$, is far from 0 (but its p^∞-torsion is dense).

Corollary 9 *If $i \geq 1$ then $\mathrm{HK}^i_r(\mathbf{A}^n) = 0$.*

Proof Immediate from Lemma 7 and the exact sequence

$$0 \to \mathrm{R}^1 \varprojlim_k \mathrm{HK}^{i-1}_r(R_k) \to \mathrm{HK}^i_r(\mathbf{A}^n) \to \varprojlim_k \mathrm{HK}^i_r(R_k) \to 0 \qquad \square$$

4 Computation of $\mathbf{DR}^i_r(\mathbf{A}^n)$

Lemma 10 *If $1 \leq i \leq r-1$ then $\mathrm{DR}^i_r(\mathbf{A}^n) \simeq \big(\Omega^i(\mathbf{A}^n)/\operatorname{Ker} d\big)(r-i-1)$, if $i \geq r$ then $\mathrm{DR}^i_r(\mathbf{A}^n) = 0$, and, if $r > 0$, we have an exact sequence*

$$0 \to \mathrm{B}^+_{\mathrm{cris}}/F^r_\theta \to \mathrm{DR}^0_r(\mathbf{A}^n) \to \big(\mathscr{O}(\mathbf{A}^n)/C\big)(r-1) \to 0$$

Proof We have an exact sequence

$$0 \to \mathrm{R}^1 \varprojlim_k \mathrm{DR}^{i-1}_r(R_k) \to \mathrm{DR}^i_r(\mathbf{A}^n) \to \varprojlim_k \mathrm{DR}^i_r(R_k) \to 0$$

The $\mathrm{DR}_r^i(R_k)$'s are the cohomology groups of the complex

$$\dots \longrightarrow \mathrm{A}_{r-i} \otimes_{\mathrm{A}_{\mathrm{cris}}} \Omega^i_{R_k} \xrightarrow{1\otimes d_i} \mathrm{A}_{r-i-1} \otimes_{\mathrm{A}_{\mathrm{cris}}} \Omega^{i+1}_{R_k} \longrightarrow \cdots$$

In particular, they are trivially 0 if $i \geq r$, so assume $i \leq r-1$. The kernel of $1 \otimes d_i$ is $F_\theta^{r-i-1}\mathrm{A}_{r-i} \otimes_{\mathrm{A}_{\mathrm{cris}}} \Omega^i_{R_k} + \mathrm{A}_{r-i} \otimes_{\mathrm{A}_{\mathrm{cris}}} (\Omega^i_{R_k})_{d=0}$ while the image of $1 \otimes d_{i-1}$ is $\mathrm{A}_{r-i} \otimes_{\mathrm{A}_{\mathrm{cris}}} d\Omega^{i-1}_{R_k}$. Since $F_\theta^{r-i-1}\mathrm{A}_{r-i}$ is an $\mathscr{O}_C$-module of rank 1 (generated by the image of $\frac{(p-[\tilde{p}])^{r-i-1}}{(r-i-1)!}$), we have $F_\theta^{r-i-1}\mathrm{A}_{r-i} \otimes_{\mathrm{A}_{\mathrm{cris}}} \Omega^i_{R_k} \simeq \Omega^i(D_k)(r-i-1)$, which gives us the exact sequence

$$0 \to \mathrm{A}_{r-i} \otimes_{\mathrm{A}_{\mathrm{cris}}} H^i_{\mathrm{dR}}(R_k) \to \mathrm{DR}_r^i(R_k) \to \big(\Omega^i(D_k)/\operatorname{Ker} d\big)(r-i-1) \to 0.$$

For $i=0$ this gives the sequence in the lemma.

Assume that $i \geq 1$. The natural map $H^i_{\mathrm{dR}}(R_{k+1}) \to H^i_{\mathrm{dR}}(R_k)$ is identically zero by Lemma 7. Hence

$$\mathrm{R}^j \varprojlim_k (\Omega^i(D_k)/\operatorname{Ker} d) \simeq \mathrm{R}^j \varprojlim_k \mathrm{DR}_r^i(R_k), \quad j \geq 0.$$

Now, note that since our systems are indexed by $\mathbf{N}$, $\mathrm{R}^j \varprojlim_k$ is trivial for $j \geq 2$. Since $\mathrm{R}^1 \varprojlim_k \Omega^i(D_k) = 0$, we have $\mathrm{R}^1 \varprojlim_k (\Omega^i(D_k)/\operatorname{Ker} d) = 0$ (and $\mathrm{R}^1 \varprojlim_k d\Omega^i = 0$). It remains to show that $\varprojlim_k (\Omega^i(D_k)/\operatorname{Ker} d) \simeq \Omega^i(\mathbf{A}^n)/\operatorname{Ker} d$. But this amounts to showing that $\mathrm{R}^1 \varprojlim_k \Omega^i(D_k)_{d=0} = 0$. This is clear for $i=0$ and for $i>0$, since the system $\{H^i_{\mathrm{dR}}(R_k)\}_{k\in\mathbf{N}}$ is trivial (by Lemma 7), this follows from the fact that $\mathrm{R}^1 \varprojlim_k d\Omega^{i-1}(D_k) = 0$. □

5 Proof of Theorems 1 and 3

5.1 *Algebraic Isomorphism*

From Proposition 5 we know that $\tau_{\leq r}\mathrm{Syn}(\mathbf{A}^n, r) \simeq \tau_{\leq r}\mathrm{R}\Gamma_{\text{proét}}(\mathbf{A}^n, \mathbf{Q}_p(r))$. From the long exact sequence

$$\cdots \to \mathrm{DR}_r^{i-1}(\mathbf{A}^n) \to \mathrm{Syn}^i(\mathbf{A}^n, r) \to \mathrm{HK}_r^i(\mathbf{A}_n) \to \mathrm{DR}_r^i(\mathbf{A}^n) \to \cdots$$

and Corollary 9 and Lemma 10, we obtain isomorphisms

$$\big(\Omega^{i-1}(\mathbf{A}^n)/\operatorname{Ker} d\big)(r-i) \xrightarrow{\sim} \mathrm{Syn}^i(\mathbf{A}^n, r), \quad r \geq i \geq 2,$$

and the exact sequence

$$0 \to \mathrm{Syn}^0(\mathbf{A}^n, r) \to \mathrm{B}_{\mathrm{cris}}^{+,\varphi=p^r} \to \mathrm{DR}_r^0(\mathbf{A}^n) \to \mathrm{Syn}^1(\mathbf{A}^n, r) \to 0,$$

which, using the fundamental exact sequence

$$0 \to \mathbf{Q}_p(r) \to \mathrm{B}_{\mathrm{cris}}^{+,\varphi=p^r} \to \mathrm{B}_{\mathrm{cris}}^{+}/F_\theta^r \to 0,$$

proves the first isomorphism in Theorem 1 (together with $\mathrm{Syn}^0(\mathbf{A}^n, r) \cong \mathbf{Q}_p(r)$). The second is an immediate consequence of the fact that $H^i_{\mathrm{dR}}(\mathbf{A}^n) = 0$.

Since an open ball is an increasing union of closed balls, Theorem 3 is proved by the same argument (see Remark 8).

Remark 11 (i) Let $j \in \mathbf{N}$. We note that, since $[\tilde{p}]^p \in p\mathrm{A}_{\mathrm{cris}}$, for every $\alpha \in \mathbf{Q}_+$, the maps[5] $\Omega^i(R^+_{\alpha+m})_j \to \Omega^i(R^+_\alpha)_j$, $m \geq pj$, are the zero maps for $i \geq 1$ and the projection on the constant term for $i = 0$. It follows that

$$\mathrm{holim}_k\, \mathrm{HK}_r(R_k^+)_j \simeq (\mathrm{A}_{\mathrm{cris},j} \xrightarrow{\varphi - p^r} \mathrm{A}_{\mathrm{cris},j}), \quad \mathrm{holim}_k\, \mathrm{DR}_r(R_k^+)_j \simeq (\mathrm{A}_{\mathrm{cris}}/F_\theta^r)_j.$$

Computing as above we get $(\mathrm{holim}_{k,\ell}\, \mathrm{Syn}(R_k^+, r)_j) \otimes \mathbf{Q} \simeq \mathbf{Q}_p(r)$. Hence, by the comparison isomorphism (1), $H^i_{\acute{e}t}(\mathbf{A}^n, \mathbf{Q}_p(r)) = 0, i \geq 1$, which allows us to recover the result of Berkovich [1].

(ii) The above argument does not go through for the open unit ball: the integral de Rham complex does not reduce to the constants in that case and $H^i_{\acute{e}t}(\mathring{\mathbf{B}}^n, \mathbf{Q}_p(r))$ is an infinite dimensionnal $\mathbf{Q}_p$-vector space if $1 \leq i \leq n$.

5.2 *Topological Considerations*

It remains to discuss topology. In what follows, we write $\cong$ for an isomorphism of vector spaces and $\equiv$ for an isomorphism of topological vector spaces.

First, note that all the cohomology groups under consideration are cohomology groups of complexes of Fréchet spaces (and even of finite sums of countable products of Banach spaces), since these complexes can be built out of Čech complexes coming from coverings by affinoids, and the corresponding complexes for affinoids involve finitely many Banach spaces. It follows that, a priori, all the groups we are dealing with are cokernels of maps $F_1 \to F_2$ between Fréchet spaces. If such a group injects continuously into a Fréchet space, then it is a Fréchet space (it is separated hence the image of F_1 in F_2 is closed, and our space is a quotient of a Fréchet space by a closed subspace), and if this injection is a bijection then it is an isomorphism of Fréchet spaces by the Open Mapping Theorem.

Now, we have the following commutative diagram:

[5] The subscript j refers to moding out by p^j.

$$\begin{array}{ccc} H^r_{\text{proét}}(\mathbf{A}^n, \mathbf{Q}_p(r)) & \longrightarrow & \varprojlim_k H^r_{\text{ét}}(D_k, \mathbf{Q}_p(r)) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\equiv} \\ \mathrm{Syn}^r(\mathbf{A}^n, r) & \xrightarrow{\ \cong\ } & \varprojlim_k \mathrm{Syn}^r(R_k, r) \end{array}$$

The horizontal maps are the natural maps (and are continuous), the bottom one being an isomorphism by the earlier computations. The left vertical arrow is an isomorphism by Proposition 5 and the right vertical arrow is a topological isomorphism because the period maps (1) are p^N-quasi-isomorphisms, with N depending only on r. Thus proving that $\varprojlim_k \mathrm{Syn}^r(R_k, r)$ is Fréchet would imply that so is $H^r_{\text{proét}}(\mathbf{A}^n, \mathbf{Q}_p(r))$ and that $H^r_{\text{proét}}(\mathbf{A}^n, \mathbf{Q}_p(r)) \equiv \varprojlim_k \mathrm{Syn}^r(R_k, r)$.

For that, consider the map of distinguished triangles

$$\begin{array}{ccccc} \mathrm{Syn}(R_k, r) & \longrightarrow & \mathrm{HK}_r(R_k) & \longrightarrow & \mathrm{DR}_r(R_k) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ \Omega^{\geq r}(D_k)[-r] & \longrightarrow & \Omega^\bullet(D_k) & \longrightarrow & \Omega^{\leq r-1}(D_k) \end{array}$$

in which:

- the top line is the definition of $\mathrm{Syn}(R_k, r)$, the bottom one is the obvious one,
- γ is obtained by applying θ to the terms of the complex $\mathrm{DR}_r(R_k)$,
- β is obtained by composing the natural map $\mathrm{HK}_r(R_k) \to \Omega^\bullet_{R_k}$ with θ,
- α is obtained by composing the natural map $\mathrm{Syn}(R_k, r) \to F^r\Omega^\bullet_{R_k}$ with θ.

All the maps are continuous (including the boundary maps). For $r \geq 2$, taking cohomology and limits we obtain the commutative diagram

$$\begin{array}{ccc} \varprojlim_k \mathrm{DR}^{r-1}_r(R_k) & \xrightarrow[\cong]{\ \partial\ } & \varprojlim_k \mathrm{Syn}^r(R_k, r) \\ \downarrow{\scriptstyle\cong} & & \downarrow \\ \Omega^{r-1}(\mathbf{A}^n)/\operatorname{Ker} d \equiv \varprojlim_k (\Omega^{r-1}(D_k)/\operatorname{Ker} d) & \xrightarrow[\cong]{\ d\ } & \varprojlim_k \Omega^r(D_k)^{d=0} \equiv \Omega^r(\mathbf{A}^n)^{d=0} \end{array}$$

The bottom map is an isomorphism because $\varprojlim_k H^r_{\mathrm{dR}}(D_k) \simeq H^r_{\mathrm{dR}}(\mathbf{A}^n) = 0$. The top map is an isomorphism because, on level k, its kernel and cokernel are controlled by $\mathrm{HK}^{r-1}_r(R_k)$ and $\mathrm{HK}^r_r(R_k)$ respectively, which die in R_{k-2} by Lemma 7, and the left vertical map is an isomorphism by the proof of Lemma 10. The space $\Omega^r(\mathbf{A}^n)$ is Fréchet; it follows that all other spaces are also Fréchet (in particular $\varprojlim_k \mathrm{Syn}^r(R_k, r)$) and that all the maps are topological isomorphisms. This concludes the proof of Theorem 1 if $r \geq 2$.

For $r = 1$, the argument is similar, with $\varprojlim_k \mathrm{DR}^{r-1}_r(R_k)$ in the above diagram replaced by $(\varprojlim_k \mathrm{DR}^{r-1}_r(R_k))/C$.

The proof for the open ball is similar.

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Arithmetic Chern–Simons Theory II

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with "Appendix 2: Conjugation Action on Group Cochains: Categorical Approach" by Behrang Noohi.

Abstract In this paper, we apply ideas of Dijkgraaf and Witten [6, 32] on 3 dimensional topological quantum field theory to arithmetic curves, that is, the spectra of rings of integers in algebraic number fields. In the first three sections, we define

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classical Chern–Simons actions on spaces of Galois representations. In the subsequent sections, we give formulas for computation in a small class of cases and point towards some arithmetic applications.

1 The Arithmetic Chern–Simons Action: Introduction and Definition

The purpose of this paper is to cast in concrete mathematical form the ideas presented in the preprint [17]. The reader is referred to that paper for motivation and speculation. Since there is no plan to submit it for separate publication, we repeat here the basic constructions before going on to a family of examples. This paper adheres, however, to a rather strict mathematical presentation. As we remind the reader below, the analogies in the background have come to be somewhat well-known under the heading of ‘arithmetic topology.’ The emphasis of this paper, however, will be less on analogies, and more on the possibility that specific technical tools of topology and physics can be imported into number theory.

Let $X = \mathrm{Spec}(\mathcal{O}_F)$, the spectrum of the ring of integers in a number field F. We assume that F is totally imaginary. Denote by $\mathbb{G}_{\mathrm{m}}$ the étale sheaf that associates to a scheme the units in the global sections of its coordinate ring. We have the following canonical isomorphism [20, p. 538]:

$$\mathrm{inv} : H^3(X, \mathbb{G}_{\mathrm{m}}) \simeq \mathbb{Q}/\mathbb{Z}. \tag{$*$}$$

This map is deduced from the ‘invariant’ map of local class field theory. We will therefore use the same name for a range of isomorphisms having the same essential nature, for example,

$$\mathrm{inv} : H^3(X, \mathbb{Z}_p(1)) \simeq \mathbb{Z}_p, \tag{$**$}$$

where $\mathbb{Z}_p(1) = \varprojlim_i \mu_{p^i}$, and $\mu_n \subset \mathbb{G}_{\mathrm{m}}$ is the sheaf of nth roots of 1. This follows from the exact sequence

$$0 \to \mu_n \to \mathbb{G}_{\mathrm{m}} \xrightarrow{(\cdot)^n} \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}/(\mathbb{G}_{\mathrm{m}})^n \to 0.$$

That is, according to *loc. cit.*,

$$H^2(X, \mathbb{G}_{\mathrm{m}}) = 0,$$

while by *op. cit.*, p. 551, we have

$$H^i(X, \mathbb{G}_{\mathrm{m}}/(\mathbb{G}_{\mathrm{m}})^n) = 0$$

for $i \geq 1$. If we break up the above into two short exact sequences,

$$0 \to \mu_n \to \mathbb{G}_m \overset{(\cdot)^n}{\to} \mathcal{K}_n \to 0,$$

and

$$0 \to \mathcal{K}_n \to \mathbb{G}_m \to \mathbb{G}_m/(\mathbb{G}_m)^n \to 0,$$

we deduce

$$H^2(X, \mathcal{K}_n) = 0,$$

from which it follows that

$$H^3(X, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

the n-torsion inside $\mathbb{Q}/\mathbb{Z}$. Taking the inverse limit over $n = p^i$ gives the second isomorphism above. The pro-sheaf $\mathbb{Z}_p(1)$ is a very familiar coefficient system for étale cohomology and (**) is reminiscent of the fundamental class of a compact oriented three manifold for singular cohomology. Such an analogy was noted by Mazur around 50 years ago [21] and has been developed rather systematically by a number of mathematicians, notably, Masanori Morishita [23]. Within this circle of ideas is included the analogy between knots and primes, whereby the map

$$\mathrm{Spec}(\mathcal{O}_F/\mathfrak{P}_v) \rightarrowtail X$$

from the residue field of a prime $\mathfrak{P}_v$ should be similar to the inclusion of a knot. Let F_v be the completion of F at the prime v and $\mathcal{O}_{F_v}$ its valuation ring. If one takes this analogy seriously (as did Morishita), the map

$$\mathrm{Spec}(\mathcal{O}_{F_v}) \to X,$$

should be similar to the inclusion of a handle-body around the knot, whereas

$$\mathrm{Spec}(F_v) \to X$$

resembles the inclusion of its boundary torus.[1] Given a finite set S of primes, we consider the scheme

$$X_S := \mathrm{Spec}(\mathcal{O}_F[1/S]) = X \setminus \{\mathfrak{P}_v\}_{v \in S}.$$

Since a link complement is homotopic to the complement of a tubular neighbourhood, the analogy is then forced on us between X_S and a three manifold with boundary given by a union of tori, one for each 'knot' in S. These of course are basic morphisms in 3 dimensional topological quantum field theory [1]. From this perspective, perhaps

[1] It is not clear to us that the topology of the boundary should really be a torus. This is reasonable if one thinks of the ambient space as a three-manifold. On the other hand, perhaps it's possible to have a notion of a knot in a *homology three-manifold* that has an exotic tubular neighbourhood?

the coefficient system $\mathbb{G}_m$ of the first isomorphism should have reminded us of the S^1-coefficient important in Chern–Simons theory [6, 32]. A more direct analogue of $\mathbb{G}_m$ is the sheaf $O_M^\times$ of invertible analytic functions on a complex variety M. However, for compact Kähler manifolds, the comparison isomorphism

$$H^1(M, S^1) \simeq H^1(M, O_M^\times)_0,$$

where the subscript refers to the line bundles with trivial topological Chern class, is a consequence of Hodge theory. This indicates that in the étale setting with no natural constant sheaf of S^1's, the familiar $\mathbb{G}_m$ has a topological nature, and can be regarded as a substitute.[2] One problem, however, is that the $\mathbb{G}_m$-coefficient computed directly gives divisible torsion cohomology, whence the need for considering coefficients like $\mathbb{Z}_p(1)$ in order to get functions of geometric objects having an analytic nature as arise, for example, in the theory of torsors for motivic fundamental groups [4, 13–16].

We now move to the definition of the arithmetic Chern–Simons action. Let

$$\pi := \pi_1(X, \mathfrak{b}),$$

be the profinite étale fundamental group of X, where we take

$$\mathfrak{b} : \mathrm{Spec}(\overline{F}) \to X$$

to be the geometric point coming from an algebraic closure of F. Assume now that the group $\mu_n(\overline{F})$ of nth roots of unity is in F and fix a trivialisation $\zeta_n : \mathbb{Z}/n\mathbb{Z} \simeq \mu_n$. This induces the isomorphism

$$\mathrm{inv} : H^3(X, \mathbb{Z}/n\mathbb{Z}) \simeq H^3(X, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

Now let A be a finite group and fix a class $c \in H^3(A, \mathbb{Z}/n\mathbb{Z})$. Let

$$M(A) := \mathrm{Hom}_{cont}(\pi, A)/A$$

be the set of isomorphism classes of principal A-bundles over X. Here, the subscript refers to continuous homomorphisms, on which A is acting by conjugation. For $[\rho] \in M(A)$, we get a class

$$\rho^*(c) \in H^3(\pi, \mathbb{Z}/n\mathbb{Z})$$

that depends only on the isomorphism class $[\rho]$; if $\rho_2 = \mathrm{Ad}_a \circ \rho_1$ for some $a \in A$, then $\rho_2^*(c) = \rho_1^*(\mathrm{Ad}_a^*(c))$, but c and $\mathrm{Ad}_a^*(c)$ are cohomologous by Lemma 7.2. Denote

[2]Recall, however, that it is of significance in Chern–Simons theory that one side of this isomorphism is purely topological while the other has an analytic structure.

by inv also the composed map

$$H^3(\pi, \mathbb{Z}/n\mathbb{Z}) \longrightarrow H^3(X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow[\simeq]{\text{inv}} \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

We get thereby a function

$$\begin{aligned} CS_c : M(A) &\longrightarrow \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}; \\ [\rho] &\longmapsto \text{inv}(\rho^*(c)). \end{aligned}$$

This is the basic and easy case of the classical Chern–Simons action[3] in the arithmetic setting.

Section 2 sets down some definitions for 'manifolds with boundary,' that is, X_S as above. In fact, it turns out that the Chern–Simons action with boundaries is necessary for the computation of the action even in the 'compact' case, in a manner strongly reminiscent of computations in topology (see [7, Theorem 1.7 (d)], for example). That is, we will compute the Chern–Simons invariant of a representation ρ of π using a suitable decomposition

$$X\text{``} = \text{''} X_S \cup [\cup_v \text{Spec}(\mathcal{O}_{F_v})]$$

and restrictions of π to X_S and the $\text{Spec}(\mathcal{O}_{F_v})$.

To describe the construction, we need more notations. We assume that all primes of F dividing n are in the finite set of primes S. Let

$$\pi_S := \pi_1(X_S, \mathfrak{b})$$

and

$$\pi_v := \text{Gal}(\overline{F}_v / F_v)$$

equipped with maps

$$i_v : \pi_v \to \pi_S$$

given by choices of embeddings $\overline{F} \hookrightarrow \overline{F}_v$. The collection

$$\{i_v\}_{v \in S}$$

will be denoted by i_S. There is a natural quotient map

$$\kappa_S : \pi_S \to \pi.$$

[3]The authors realise that this terminology is likely to be unfamiliar, and maybe even appears pretentious to number-theorists. However, it does seem to encourage the reasonable view that concepts and structures from geometry and physics can be specifically useful in number theory.

Let

$$Y_S(A) := \mathrm{Hom}_{cont}(\pi_S, A)$$

and denote by $\mathcal{M}_S(A)$ the action groupoid whose objects are the elements of $Y_S(A)$ with morphisms given by the conjugation action of A. We also have the local version

$$Y_S^{loc}(A) := \prod_{v \in S} \mathrm{Hom}_{cont}(\pi_v, A)$$

as well as the action groupoid $\mathcal{M}_S^{loc}(A)$ with objects $Y_S^{loc}(A)$ and morphisms given by the action of $A^S := \prod_{v \in S} A$ conjugating the separate components in the obvious sense. Thus, we have the restriction functor

$$r_S : \mathcal{M}_S(A) \to \mathcal{M}_S^{loc}(A),$$

where a homomorphism $\rho : \pi_S \to A$ is restricted to the collection

$$r_S(\rho) = i_S^* \rho := (\rho \circ i_v)_{v \in S}.$$

We will construct, in Sect. 2, a functor L from $\mathcal{M}_S^{loc}(A)$ to the $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsors as a finite arithmetic version of the Chern–Simons line bundle [7] over $\mathcal{M}_S^{loc}(A)$. To a global representation $\rho \in \mathcal{M}_S(A)$, the Chern–Simons action will then associate an element (Eq. (2.3))

$$CS_c([\rho]) \in L(r_S(\rho)).$$

Now, given $[\rho] \in \mathcal{M}(A)$, we pull it back to $[\rho \circ \kappa_S] \in \mathcal{M}_S(A)$ and apply the Chern–Simons action with boundary to get an element

$$CS_c([\rho \circ \kappa_S]) \in L([r_S(\rho \circ \kappa_S)]).$$

On the other hand, for each $v \in S$, we can pull back ρ to a local unramified representation

$$\rho_v^{\mathrm{ur}} : \pi_v^{\mathrm{ur}} \to \pi \to A,$$

where π_v^{ur} is the unramified quotient of π_v. The extra structure of the unramified representation will then allow us to canonically associate an element

$$\sum_{v \in S} (\beta_v) \in L([r_S(\rho \circ \kappa_S)]),$$

which can be interpreted as the Chern–Simons action of $(\rho_v^{\mathrm{ur}})_{v \in S}$ on $\cup_{v \in S} \mathrm{Spec}(\mathcal{O}_{F_v})$.

Theorem 1.1 (The Decomposition Formula) *Let A be a finite group and fix a class $c \in H^3(A, \mathbb{Z}/n\mathbb{Z})$. Then*

$$CS_c([\rho]) = \sum_{v\in S}(\beta_v) - CS_c([\rho \circ \kappa_S])$$

for $[\rho] \in \mathcal{M}(A)$.

Section 4 is devoted to a proof of Theorem 1.1. The key point of this formula is that $CS_c([\rho])$ can be computed as the difference between two trivialisations of the torsor, a ramified global trivialisation and an unramified local trivialisation.

In Sect. 5, we use this theorem to compute the Chern–Simons action for a class of examples. It is amusing to note the form of the action when A is finite cyclic. That is, let $A = \mathbb{Z}/n\mathbb{Z}$, $\alpha \in H^1(A, \mathbb{Z}/n\mathbb{Z})$ the class of the identity, and $\beta \in H^2(A, \mathbb{Z}/n\mathbb{Z})$ the class of the extension

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \xrightarrow{\ n\ } \mathbb{Z}/n^2\mathbb{Z} \longrightarrow A \longrightarrow 0.$$

Then $\beta = \delta\alpha$, where $\delta : H^1(A, \mathbb{Z}/n\mathbb{Z}) = H^1(A, A) \to H^2(A, \mathbb{Z}/n\mathbb{Z})$ is the boundary map arising from the extension. Put

$$c := \alpha \cup \beta = \alpha \cup \delta\alpha \in H^3(A, \mathbb{Z}/n\mathbb{Z}).$$

Then

$$CS_c([\rho]) = \mathrm{inv}[\rho^*(\alpha) \cup \delta\rho^*(\alpha)],$$

in close analogy to the[4] formulas of abelian Chern–Simons theory.

However, our computations are not limited to the case where A is an abelian cyclic group. Along similar lines, we will provide an infinite family of number fields F and representations ρ such that $CS_c([\rho])$ is non-vanishing for $[\rho] \in M(A)$ with a different class $c \in H^3(A, \mathbb{Z}/2\mathbb{Z})$ and both abelian A (see Propositions 5.14, 5.16, and 5.19) and non-abelian A (see Proposition 5.23).

In Sect. 6, we provide arithmetic applications to a class of Galois embedding problems using the fact that the existence of an unramified extension forces a Chern–Simons invariant to be zero.

In this paper, we do not develop a p-adic theory in the case where the boundary is empty. In future papers, we hope to apply local trivialisations using Selmer complexes to remedy this omission and complete the theory begun in Sect. 3. To get actual p-adic functions, one needs of course to come to an understanding of explicit cohomology classes on p-adic Lie groups, possibly by way of the theory of Lazard [18]. Suitable quantisations of the theory of this paper in a manner amenable to arithmetic applications will be explored as well in future work, as in [3], where a precise arithmetic analogue of a 'path-integral formula' for arithmetic linking numbers is proved. In that preprint, a connection is made also to the class invariant homomorphism from additive Galois module structure theory. A pro-p version of this homomorphism is related to p-adic L-functions and heights, providing some evidence for the speculation from [17].

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[4] In fact, every cohomology class in $H^3(A, \mathbb{Z}/n\mathbb{Z})$ can be written as this form (cf. [25, Sect. 1.7]).

2 The Arithmetic Chern–Simons Action: Boundaries

We keep the notations as in the introduction. We will now employ a cocycle $c \in Z^3(A, \mathbb{Z}/n\mathbb{Z})$ to associate a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor to each point of $Y_S^{loc}(A)$ in an A^S-equivariant manner. We use the notation

$$C_S^i := \prod_{v \in S} C^i(\pi_v, \mathbb{Z}/n\mathbb{Z})$$

for the continuous cochains,

$$Z_S^i := \prod_{v \in S} Z^i(\pi_v, \mathbb{Z}/n\mathbb{Z}) \subset C_S^i$$

for the cocycles, and

$$B_S^i := \prod_{v \in S} B^i(\pi_v, \mathbb{Z}/n\mathbb{Z}) \subset Z_S^i \subset C_S^i$$

for the coboundaries. In particular, we have the coboundary map (see Appendix "Appendix 1: Conjugation on Group Cochains" for the sign convention)

$$d : C_S^2 \to Z_S^3.$$

Let $\rho_S := (\rho_v)_{v \in S} \in Y_S^{loc}(A)$ and put

$$c \circ \rho_S := (c \circ \rho_v)_{v \in S},$$

$$c \circ \mathrm{Ad}_a := (c \circ \mathrm{Ad}_{a_v})_{v \in S}$$

for $a = (a_v)_{v \in S} \in A^S$, where Ad_{a_v} refers to the conjugation action. To define the arithmetic Chern–Simons line associated to ρ_S, we need the intermediate object

$$H(\rho_S) := d^{-1}(c \circ \rho_S)/B_S^2 \subset C_S^2/B_S^2.$$

This is a torsor for

$$H_S^2 := \prod_{v \in S} H^2(\pi_v, \mathbb{Z}/n\mathbb{Z}) \simeq \prod_{v \in S} \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

([25, Theorem (7.1.8)]). We then use the sum map

$$\Sigma : \prod_{v \in S} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

to push this out to a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor. That is, define

$$L(\rho_S) := \Sigma_*[H(\rho_S)]. \tag{2.1}$$

The natural map $H(\rho_S) \to L(\rho_S)$ will also be denoted by the sum symbol Σ.

In fact, L extends to a functor from $\mathcal{M}_S^{loc}(A)$ to the category of $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsors. To carry out this extension, we just need to extend H to a functor to H_S^2-torsors. According to Appendices "Appendix 1: Conjugation on Group Cochains" and "Appendix 2: Conjugation Action on Group Cochains: Categorical Approach", for $a = (a_v)_{v\in S} \in A^S$ and each v, there is an element $h_{a_v} \in C^2(A, \mathbb{Z}/n\mathbb{Z})/B^2(A, \mathbb{Z}/n\mathbb{Z})$ such that

$$c \circ \mathrm{Ad}_{a_v} = c + dh_{a_v}.$$

Also,

$$h_{a_v b_v} = h_{a_v} \circ \mathrm{Ad}_{b_v} + h_{b_v}.$$

Hence, given $a : \rho_S \to \rho'_S$, so that $\rho'_S = \mathrm{Ad}_a \circ \rho_S$, we define

$$H(a) : H(\rho_S) \to H(\rho'_S)$$

to be the map induced by

$$x \mapsto x' = x + (h_{a_v} \circ \rho_v)_{v\in S}.$$

Then

$$dx' = dx + (d(h_{a_v} \circ \rho_v))_{v\in S} = (c \circ \rho_v)_{v\in S} + ((dh_{a_v}) \circ \rho_v)_{v\in S} = (c \circ \mathrm{Ad}_{a_v} \circ \rho_v)_{v\in S}.$$

So

$$x' \in d^{-1}(c \circ \rho'_S)/B_S^2,$$

and by the formula above, it is clear that H is a functor.[5] That is, ab will send x to

$$x + h_{ab} \circ \rho_S,$$

while if we apply b first, we get

$$x + h_b \circ \rho_S \in H(\mathrm{Ad}_b \circ \rho_S),$$

which then goes via a to

[5]While the functor H does depend on the choices of h_a, they are intrinsic to A, in that they are cochains on A, not a priori related to the Galois representations. So we may regard them as part of the data defining the field theory, similar to c.

$$x + h_b \circ \rho_S + h_a \circ \mathrm{Ad}_b \circ \rho_S.$$

Thus,

$$H(ab) = H(a)H(b).$$

Defining

$$L(a) = \Sigma_* \circ H(a)$$

turns L into a functor from $\mathcal{M}_S^{loc}$ to $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsors. Even though we are not explicitly laying down geometric foundations, it is clear that L defines thereby an A^S-equivariant $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor on $Y_S^{loc}(A)$, or a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor on the stack $\mathcal{M}_S^{loc}(A)$.

We can compose the functor L with the restriction $r_S : \mathcal{M}_S(A) \to \mathcal{M}_S^{loc}(A)$ to get an A-equivariant functor L^{glob} from $Y_S(A)$ to $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsors.

Lemma 2.1 *Let $\rho \in Y_S(A)$ and $a \in \mathrm{Aut}(\rho)$. Then $L^{glob}(a) = 0$.*

Proof By assumption, $\mathrm{Ad}_a\rho = \rho$, and hence, $dh_a \circ \rho = 0$. That is, $h_a \circ \rho \in H^2(\pi_S, \mathbb{Z}/n\mathbb{Z})$. Hence, by the reciprocity law for $H^2(\pi_S, \mathbb{Z}/n\mathbb{Z})$ ([25, Theorem (8.1.17)]), we get

$$\Sigma_*(h_a \circ \rho) = 0.$$

By the argument of [7, p. 439], we see that there is a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor

$$L^{\mathrm{inv}}([\rho])$$

of invariant sections for the functor L^{glob} depending only on the orbit $[\rho]$. This is the set of families of elements

$$x_{\rho'} \in L^{glob}(\rho')$$

as ρ' runs over $[\rho]$ with the property that every morphism $a : \rho_1 \to \rho_2$ takes x_{ρ_1} to x_{ρ_2}. Alternatively, $L^{\mathrm{inv}}([\rho])$ is the inverse limit of the $L^{glob}(\rho')$ with respect to the indexing category $[\rho]$.

Since

$$H^3(\pi_S, \mathbb{Z}/n\mathbb{Z}) = 0$$

([25, Proposition (8.3.18)]), the cocycle $c \circ \rho$ is a coboundary

$$c \circ \rho = d\beta \tag{2.2}$$

for $\beta \in C^2(\pi_S, \mathbb{Z}/n\mathbb{Z})$. This element defines a class

$$CS_c([\rho]) := \Sigma([i_S^*(\beta)]) \in L^{\mathrm{inv}}([\rho]). \tag{2.3}$$

A different choice β' will be related by

$$\beta' = \beta + z$$

for a 2-cocycle $z \in Z^2(\pi_S, \mathbb{Z}/n\mathbb{Z})$, which vanishes when mapped to $L((\rho \circ i_v)_{v\in S})$ because of the reciprocity sequence

$$0 \longrightarrow H^2(\pi_S, \mathbb{Z}/n\mathbb{Z}) \longrightarrow H^2_S \xrightarrow{\sum_v \mathrm{inv}_v} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \longrightarrow 0.$$

Thus, the class $CS_c([\rho])$ is independent of the choice of β and defines a global section

$$CS_c \in \Gamma(\mathcal{M}_S(A), L^{glob}).$$

Within the context of this paper, a 'global section' should just be interpreted as an assignment of $CS_c([\rho])$ as above for each orbit $[\rho]$.

3 The Arithmetic Chern–Simons Action: The *p*-adic Case

Now fix a prime p and assume all primes of F dividing p are contained in S. Fix a compatible system $(\zeta_{p^n})_n$ of p-power roots of unity, giving us an isomorphism

$$\zeta : \mathbb{Z}_p \simeq \mathbb{Z}_p(1) := \varprojlim_n \mu_{p^n}.$$

In this section, we will be somewhat more careful with this isomorphism. Also, it will be necessary to make some assumptions on the representations that are allowed.

Let A be a p-adic Lie group, e.g., $GL_n(\mathbb{Z}_p)$. Assume A is equipped with an open homomorphism[6] $t : A \to \Gamma := \mathbb{Z}_p^\times$ and define A^n to be the kernel of the composite map

$$A \to \mathbb{Z}_p^\times \to (\mathbb{Z}/p^n\mathbb{Z})^\times =: \Gamma_n.$$

Let

$$A^\infty = \cap_n A^n = \mathrm{Ker}(t).$$

In this section, we denote by $Y_S(A)$ the continuous homomorphisms

$$\rho : \pi_S \to A$$

such that $t \circ \rho$ is a power χ^s of the p-adic cyclotomic character χ of π_S by a p-adic unit s. (We note that s itself is allowed to vary.) Of course this condition will be satisfied by any geometric Galois representations or natural p-adic families containing one.

As before, A acts on $Y_S(A)$ by conjugation. But in this section, we will restrict the action to A^∞ and use the notation $\mathcal{M}_S(A)$ for the corresponding action groupoid.

Similarly, we denote by Y_S^{loc} the collections of continuous homomorphisms

[6] For example, one may choose t to be the determinant when $A = GL_n(\mathbb{Z}_p)$.

$$\rho_S := (\rho_v : \pi_v \to A)_{v \in S}$$

for which there exists a p-adic unit s such that $t \circ \rho_v = (\chi|_{\pi_v})^s$ for all v. $\mathcal{M}_S^{loc}(A)$ then denotes the action groupoid defined by the product $(A^\infty)^S$ of the conjugation action on the ρ_S.

We now fix a continuous cohomology class

$$c \in H^3(A, \mathbb{Z}_p[[\Gamma]]),$$

where

$$\mathbb{Z}_p[[\Gamma]] = \varprojlim_n \mathbb{Z}_p[\Gamma_n].$$

We represent c by a cocycle in $Z^3(A, \mathbb{Z}_p[[\Gamma]])$, which we will also denote by c. Given $\rho \in Y_S(A)$, we can view $\mathbb{Z}_p[[\Gamma]]$ as a continuous representation of π_S, where the action is left multiplication via $t \circ \rho$. We denote this representation by $\mathbb{Z}_p[[\Gamma]]_\rho$. The isomorphism $\zeta : \mathbb{Z}_p \simeq \mathbb{Z}_p(1)$, even though it's not π_S-equivariant, does induce a π_S-equivariant isomorphism

$$\zeta_\rho : \mathbb{Z}_p[[\Gamma]]_\rho \simeq \Lambda := \mathbb{Z}_p[[\Gamma]] \otimes \mathbb{Z}_p(1).$$

Here, $\mathbb{Z}_p[[\Gamma]]$ written without the subscript refers to the action via the cyclotomic character of π_S (with $s = 1$ in the earlier notation). The isomorphism is defined as follows. If $t \circ \rho = \chi^s$, then we have the isomorphism

$$\mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[\Gamma]]_\rho$$

that sends γ to γ^s. On the other hand, we also have

$$\mathbb{Z}_p[[\Gamma]] \simeq \Lambda$$

that sends γ to $\gamma \otimes \gamma\zeta(1)$. Thus, ζ_ρ can be taken as the inverse of the first followed by the second.

Combining these considerations, we get an element

$$\zeta_\rho \circ \rho^* c = \zeta_\rho \circ c \circ \rho \in Z^3(\pi_S, \Lambda).$$

Similarly, if $\rho_S := (\rho_v)_{v \in S} \in Y_S^{loc}$, we can regard $\mathbb{Z}_p[[\Gamma]]_{\rho_v}$ as a representation of π_v for each v, and we get π_v-equivariant isomorphisms

$$\zeta_{\rho_v} : \mathbb{Z}_p[[\Gamma]]_{\rho_v} \simeq \Lambda.$$

We also use the notation

$$\zeta_{\rho_S} : \prod_{v\in S} \mathbb{Z}_p[[\Gamma]]_{\rho_v} \simeq \prod_{v\in S} \Lambda$$

for the isomorphism given by the product of the ζ_{ρ_v}.

It will be convenient to again denote by $C^i_S(\Lambda)$ the product $\prod_{v\in S} C^i(\pi_v, \Lambda)$ and use the similar notations $Z^i_S(\Lambda)$, $B^i_S(\Lambda)$ and $H^i_S(\Lambda)$. The element $\zeta_{\rho_S} \circ \rho_S^* c$ is an element in $Z^3_S(\Lambda)$. We then put

$$H(\rho_S, \Lambda) := d^{-1}(\zeta_{\rho_S} \circ \rho_S^* c)/B^2_S(\Lambda) \subset C^2_S(\Lambda)/B^2_S(\Lambda).$$

This is a torsor for

$$H^2_S(\Lambda) \simeq \prod_{v\in S} H^2(\pi_v, \Lambda).$$

The augmentation map

$$a : \Lambda \to \mathbb{Z}_p(1)$$

for each v can be used to push this out to a torsor

$$a_*(H(\rho_S, \Lambda))$$

for the group

$$\prod_{v\in S} H^2(\pi_v, \mathbb{Z}_p(1)) \simeq \prod_{v\in S} \mathbb{Z}_p,$$

which then can be pushed out with the sum map

$$\Sigma : \prod_{v\in S} \mathbb{Z}_p \to \mathbb{Z}_p$$

to give us a $\mathbb{Z}_p$-torsor

$$L(\rho_S, \mathbb{Z}_p) := \Sigma_*(a_*(H(\rho_S, \Lambda))).$$

As before, we can turn this into a functor $L(\cdot, \mathbb{Z}_p)$ on $\mathcal{M}^{loc}_S(A)$, taking into account the action of $(A^\infty)^S$. By composing with the restriction functor

$$r_S : \mathcal{M}_S(A) \to \mathcal{M}^{loc}_S(A),$$

we also get a $\mathbb{Z}_p$-torsor $L^{glob}(\cdot, \mathbb{Z}_p)$ on $\mathcal{M}_S(A)$.

We now choose an element $\beta \in C^2(\pi_S, \Lambda)$ such that

$$d\beta = \zeta_\rho \circ c \circ \rho \in Z^3(\pi_S, \Lambda) = B^3(\pi_S, \Lambda)$$

to define the p-adic Chern–Simons action

$$CS_c([\rho]) := \Sigma_* a_* i_S^*(\beta) \in L^{glob}([\rho], \mathbb{Z}_p).$$

The argument that this action is independent of β and equivariant is also the same as before, giving us an element

$$CS_c \in \Gamma(\mathcal{M}_S(A), L^{glob}(\cdot, \mathbb{Z}_p)).$$

4 Towards Computation: The Decomposition Formula

In this section, we indicate how one might go about computing the arithmetic Chern–Simons invariant in the unramified case with finite coefficients. That is, we assume that we are in the setting of Sect. 1. We provide a proof of Theorem 1.1 in a slightly generalized setting.

Let $X = \text{Spec}(\mathcal{O}_F)$ and M a continuous representation of $\pi = \pi_1(X, \mathfrak{b})$ regarded as a locally constant sheaf on X. Assume $M = \varprojlim M_i$ with M_i finite representations such that there is a finite set T of primes in $\mathcal{O}_F$ containing all primes dividing the order of any $|M_i|$. Let $U = \text{Spec}(\mathcal{O}_{F,\ T})$, $\pi_T = \pi_1(U, \mathfrak{b})$, and $\pi_v = \text{Gal}(\overline{F}_v/F_v)$ for a prime v of F. Fix natural homomorphisms

$$\kappa_T : \pi_T \to \pi \quad \text{and} \quad \kappa_v : \pi_v \to \pi.$$

We denote by ρ_T (resp. ρ_v) the composition of κ_T (resp. k_v) with

$$\rho \in \text{Hom}_{cont}(\pi, M).$$

Finally, we write $\mathfrak{P}_v$ for the maximal ideal of $\mathcal{O}_F$ corresponding to the prime v and r_v for the restriction map of cochains or cohomology classes from π_T to π_v.

Denote by $C_c^*(\pi_T, M)$ the complex defined as a mapping fiber

$$C_c^*(\pi_T, M) := \text{Fiber}[C^*(\pi_T, M) \to \prod_{v \in T} C^*(\pi_v, M)].$$

So

$$C_c^n(\pi_T, M) = C^n(\pi_T, M) \times \prod_{v \in T} C^{n-1}(\pi_v, M),$$

and

$$d(a, (b_v)_{v \in T}) = (da, (r_v(a) - db_v)_{v \in T})$$

for $(a, (b_v)_{v \in T}) \in C_c^n(\pi_T, M)$. As in [10, p. 18–19], since there are no real places in F, there is a quasi-isomorphism

$$C_c^*(\pi_T, M) \simeq R\Gamma(X, j_! j^*(M)),$$

where $j : U \to X$ is the inclusion. But there is also an exact sequence

$$0 \longrightarrow j_! j^*(M) \longrightarrow M \longrightarrow i_* i^*(M) \longrightarrow 0,$$

where $i : T \to X$ is the closed immersion complementary to j. Thus, we get an exact sequence

$$\prod_{v \in T} H^2(k_v, i^*(M)) \longrightarrow H^3(C_c^*(\pi_T, M)) \longrightarrow H^3(X, M) \longrightarrow \prod_{v \in T} H^3(k_v, i^*(M)),$$

where $k_v := \mathrm{Spec}(O_F/\mathfrak{P}_v)$, from which we get an isomorphism

$$H_c^3(U, M) := H^3(C_c^*(\pi_T, M)) \simeq H^3(X, M),$$

since k_v has cohomological dimension 1.

We interpret this as a statement that the cohomology of X

$$H^3(X, M)$$

can be identified with cohomology of a 'compactification' of U with respect to the 'boundary,' that is, the union of the $\mathrm{Spec}(F_v)$ for $v \in T$. This means that a class $z \in H^3(X, M)$ is represented by $(a, (b_v)_{v \in T})$, where $a \in Z^3(\pi_T, M)$ and $b_v \in C^2(\pi_v, M)$ in such a way that

$$db_v = r_v(a).$$

There is also the exact sequence

$$\longrightarrow H^2(\pi_T, M) \longrightarrow \prod_{v \in T} H^2(\pi_v, M) \longrightarrow H_c^3(U, M) \longrightarrow 0,$$

the last zero being $H^3(U, M) := H^3(\pi_T, M) = 0$. We can use this to compute the invariant of z when $M = \mu_n$. (Note that F contains μ_n and hence it is in fact isomorphic to the constant sheaf $\mathbb{Z}/n\mathbb{Z}$.) We have to lift z to a collection of classes $x_v \in H^2(\pi_v, \mu_n)$ and then take the sum

$$\mathrm{inv}(z) = \sum_v \mathrm{inv}_v(x_v).$$

This is independent of the choice of the x_v by the reciprocity law (cf. [20, p. 541]). The lifting process may be described as follows. The map

$$\prod_{v \in T} H^2(\pi_v, \mu_n) \longrightarrow H_c^3(U, \mu_n)$$

just takes a tuple of 2-cocycles $(x_v)_{v\in T}$ to $(0, (x_v)_{v\in T})$. But by the vanishing of $H^3(U, \mu_n)$, given $z = (a, (b_{-,v})_{v\in T})$, we can find a global cochain $b_+ \in C^2(\pi_T, \mu_n)$ such that $db_+ = a$. We then put

$$x_v := b_{-,v} - r_v(b_+).$$

Note that $(0, (x_v)_{v\in T})$ is cohomologous to $z = (a, (b_{-,v})_{v\in T})$.

As before, we start with a class $c \in H^3(A, \mu_n) \simeq H^3(A, \mathbb{Z}/n\mathbb{Z})$. Then, we get a class

$$z = j^3 \circ \rho^*(c) \in H^3(X, \mu_n),$$

where $j^i : H^i(\pi, \mu_n) \to H^i(X, \mu_n)$ is the natural map from group cohomology to étale cohomology (cf. [22, Theorem 5.3 of Chap. I]). Let w be a cocycle representing $\rho^*(c) \in H^3(\pi, \mu_n)$. Let $I_v \subset \pi_v$ be the inertia subgroup. We now can trivialise $\kappa_v^*(w)$ by first doing it over π_v/I_v to which it factors. That is, the $b_{-,v}$ as above can be chosen as cochains factoring through π_v/I_v. This is possible because $H^3(\pi_v/I_v, \mu_n) = 0$. The class $(\kappa_T^*(w), (b_{-,v})_{v\in T})$ chosen in this way is independent of the choice of the $b_{-,v}$. This is because $H^2(\pi_v/I_v, \mu_n)$ is also zero. The point is that the representation of z as $(\kappa_T^*(w), (b_{-,v})_{v\in T})$ with unramified $b_{-,v}$ is essentially canonical. More precisely, given $\kappa_v^*(w)|_{(\pi_v/I_v)} \in Z^3(\pi_v/I_v, \mu_n)$, there is a canonical

$$b_{-,v} \in C^2(\pi_v/I_v, \mu_n)/B^2(\pi_v/I_v, \mu_n)$$

such that $db_{-,v} = \kappa_v^*(w)|_{(\pi_v/I_v)}$. This can then be lifted to a canonical class in

$$C^2(\pi_v, \mu_n)/B^2(\pi_v, \mu_n).$$

Now we trivialise $\kappa_T^*(w)$ globally as above, that is, by the choice of $b_+ \in C^2(\pi_T, \mu_n)$ such that $db_+ = \kappa_T^*(w)$. Then $(b_{-,v} - b_{+,v})_{v\in T}$ will be cocycles, where $b_{+,v} := r_v(b_+)$, and we compute

$$\mathrm{inv}(z) = \sum_{v\in T} \mathrm{inv}_v(b_{-,v} - b_{+,v}).$$

Thus, for a given homomorphism $\rho : \pi \to A$, it suffices to find various trivialisations of $\rho^*(c)$ after restriction to π_T and to π_v for $v \in T$.

- We are free to choose a finite set T of primes in a convenient way as long as T contains all primes dividing n. And then, for any $v \in T$, solve

$$db_{-,v} = \rho_v^*(c) \in Z^3(\pi_v, \mu_n).$$

  In fact, $b_{-,v}$ comes from an element in $C^2(\pi_v/I_v, \mu_n)$ by inflation, so $b_{-,v}$ is unramified.
- For chosen T, solve

$$db_+ = \rho_T^*(c) \in Z^3(\pi_T, \mu_n),$$

and we set $b_{+,v} = r_v(b_+) \in C^2(\pi_v, \mu_n)$.

Then, we have the decomposition formula

$$CS_c([\rho]) = \sum_{v \in T} \mathrm{inv}_v([b_{-,v} - b_{+,v}]). \tag{†}$$

In the case $M = \mu_n$ and $S = T$, a finite set of primes in O_F containing all primes dividing n, a simple inspection implies that

$$\sum_{v \in T} \mathrm{inv}_v([b_{-,v} - b_{+,v}]) = \sum_{v \in S} (\beta_v) - CS_c([\rho \circ \kappa_S]).$$

Thus, the formula (†) provides a proof of Theorem 1.1. In general, $b_{-,v}$ and $b_{+,v}$ are not cocycles but their difference is. This corresponds to the fact that $\sum_{v \in S} (\beta_v)$ and $CS_c([\rho \circ \kappa_S])$ are not an element of $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ but their difference is.

A few remarks about this method:

1. Underlying this is the fact that the compact support cohomology $H_c^3(U, \mu_n)$ can be computed relative to the somewhat fictitious boundary of U or as relative cohomology $H^3(X, T; \mu_n)$. Choosing the unramified local trivialisations corresponds to this latter representation.

2. To summarise the main idea again, starting from a cocycle $z \in Z^3(\pi, \mu_n)$ we have canonical unramified trivialisations at each v and a non-canonical global ramified trivialisation.

The invariant of z measures the discrepancy between the unramified local trivialisations and a ramified global trivialisation.

The fact that the non-canonicality of the global trivialisation is unimportant follows from the reciprocity law (cf. [20, p. 541]).

3. The description above that computes the invariant by comparing the local unramified trivialisation with the global ramified one is a precise analogue of the so-called 'gluing formula' for Chern–Simons invariants when applied to $\rho^*(c)$ for a representation $\rho : \pi \to \mathbb{Z}/n\mathbb{Z}$ and a 3-cocycle c on $\mathbb{Z}/n\mathbb{Z}$.

5 Examples

In this section, we provide several explicit examples of computation of $CS_c([\rho])$. We still assume that we are in the setting of Sect. 1.

5.1 General Strategy

To compute the arithmetic Chern–Simons invariants, we essentially use the decomposition formula (†) in Sect. 4. The most difficult part in the above method is finding an element $b_+ \in C^2(\pi_T, \mu_n)$ that gives a global trivialisation.

To simplify our problem, we assume that a cocycle $c \in Z^3(A, \mu_n)$ is defined by the cup product:

$$c = \alpha \cup \epsilon,$$

where $\alpha \in Z^1(A, \mu_n) = \mathrm{Hom}(A, \mu_n)$ and $\epsilon \in Z^2(A, \mathbb{Z}/n\mathbb{Z})$ is a cocycle representing an extension

$$E : 0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \Gamma \xrightarrow[\varphi]{} A \longrightarrow 1.$$

We note that if we take a section σ of φ that sends e_A to e_Γ, then

$$\epsilon(x, y) = \sigma(x) \cdot \sigma(y) \cdot \sigma(xy)^{-1} \in \mathrm{Ker}\,\varphi = \mathbb{Z}/n\mathbb{Z}$$

(cf. [30, p. 183]). As discussed in Sect. 1, this assumption is vacuous if $A = \mathbb{Z}/n\mathbb{Z}$.

To find $b_{-,v}$ and $b_{+,v}$ in the decomposition formula (†), we first trivialise ϵ in π_v and π_T, respectively. Namely, let

$$d\gamma_{-,v} = \rho_v^*(\epsilon) \quad \text{and} \quad d\gamma_+ = \rho_T^*(\epsilon).$$

Here, the precise choice of $\gamma_{-,v}$ will be unimportant, except it should be unramified and normalised so that $\gamma_{-,v}(e_A) = 0$. Hence, we will be inexplicit below about this choice. Again, let $\gamma_{+,v} = r_v(\gamma_+)$. Then, we have

$$d(\rho_v^*(\alpha) \cup \gamma_{-,v}) = -\rho_v^*(\alpha) \cup d\gamma_{-,v} = -\rho_v^*(\alpha \cup \epsilon) = -\rho_v^*(c)$$

and

$$d(\rho_T^*(\alpha) \cup \gamma_+) = -\rho_T^*(\alpha) \cup d\gamma_+ = -\rho_T^*(\alpha \cup \epsilon) = -\rho_T^*(c).$$

Therefore, we can find

$$b_{-,v} = -\rho_v^*(\alpha) \cup \gamma_{-,v} \quad \text{and} \quad b_{+,v} = r_v(b_+) = r_v(-\rho_T^*(\alpha) \cup \gamma_+) = -\rho_v^*(\alpha) \cup \gamma_{+,v}.$$

In summary, we get the following formula.

Theorem 5.1 *For ρ and c as above, we have*

$$CS_c([\rho]) := CS_{[c]}([\rho]) = \sum_{v \in T} \mathrm{inv}_v(\rho_v^*(\alpha) \cup \psi_v), \tag{5.1}$$

where $\psi_v = \gamma_{+,v} - \gamma_{-,v} \in Z^1(\pi_v, \mathbb{Z}/n\mathbb{Z}) = H^1(\pi_v, \mathbb{Z}/n\mathbb{Z}) = \mathrm{Hom}(\pi_v, \mathbb{Z}/n\mathbb{Z})$.

So, to evaluate the arithmetic Chern–Simons action, we need to study

- a trivialisation of certain pullback of a 2-cocycle ϵ, and
- the local invariant of a cup product of two characters on π_v.

In the following two subsections, we will see how this idea can be realised.

5.2 Trivialisation of a Pullback of ϵ

As before, let $\epsilon \in Z^2(A, \mathbb{Z}/n\mathbb{Z})$ denote a 2-cocycle representing an extension

$$E:\ 0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \Gamma \underset{\varphi}{\overset{\sigma}{\rightleftarrows}} A \longrightarrow 1$$

with a section σ such that $\sigma(e_A) = e_\Gamma$.

Suppose that we have the following commutative diagram of group homomorphisms:

$$\begin{array}{ccccc} & & \mathrm{Ker}(f) & \hookrightarrow & \widetilde{A} \\ & \swarrow_{\widetilde{f}|_{\mathrm{Ker}(f)}} & & \swarrow_{\widetilde{f}} & \downarrow f \\ \mathbb{Z}/n\mathbb{Z} & \hookrightarrow & \Gamma & \xrightarrow{\varphi} & A. \end{array} \qquad (\star)$$

Then, we can easily trivialise $f^*(\epsilon) \in Z^2(\widetilde{A}, \mathbb{Z}/n\mathbb{Z})$ using the following lemma.

Lemma 5.2 *For any $g \in \widetilde{A}$, let*

$$\gamma(g) := \sigma(f(g)) \cdot \widetilde{f}(g)^{-1}.$$

Then, $\gamma(g) \in \mathrm{Ker}(\varphi) = \mathbb{Z}/n\mathbb{Z}$ and $d\gamma = f^*(\epsilon) \in Z^2(\widetilde{A}, \mathbb{Z}/n\mathbb{Z})$. Furthermore, we have $\gamma(e_{\widetilde{A}}) = 0$ and $\gamma(g \cdot h) = \gamma(g) + \gamma(h)$ for any $g, h \in \mathrm{Ker}(f)$.

Proof First, we note that $\gamma(g) \in \mathrm{Ker}(\varphi)$ because $\varphi \circ \sigma$ is the identity and $\varphi \circ \widetilde{f} = f$. By definition and the fact that $\mathrm{Ker}(\varphi)$ is in the center of Γ,

$$\begin{aligned} d\gamma(x, y) &= \gamma(y) \cdot \gamma(xy)^{-1} \cdot \gamma(x) = \gamma(y) \cdot \gamma(x) \cdot \gamma(xy)^{-1} \\ &= \{\sigma(f(y)) \cdot \widetilde{f}(y)^{-1}\} \cdot \{\sigma(f(x)) \cdot \widetilde{f}(x)^{-1}\} \cdot \{\sigma(f(xy)) \cdot \widetilde{f}(xy)^{-1}\}^{-1} \\ &= \{\sigma(f(y)) \cdot \widetilde{f}(y)^{-1}\} \cdot \sigma(f(x)) \cdot \widetilde{f}(x)^{-1} \cdot \widetilde{f}(x) \cdot \widetilde{f}(y) \cdot \sigma(f(xy))^{-1} \\ &= \sigma(f(x)) \cdot \{\sigma(f(y)) \cdot \widetilde{f}(y)^{-1}\} \cdot \widetilde{f}(y) \cdot \sigma(f(xy))^{-1} \\ &= \sigma(f(x)) \cdot \sigma(f(y)) \cdot \sigma(f(x \cdot y))^{-1} \\ &= f^*(\epsilon)(x, y). \end{aligned}$$

Therefore the first claim follows. Also, $\gamma(e_{\widetilde{A}}) = 0$ because $\sigma(f(e_{\widetilde{A}})) = \sigma(e_A) = e_\Gamma$ and $\widetilde{f}(e_{\widetilde{A}}) = e_\Gamma$. Finally, for any $g \in \mathrm{Ker}(f)$, $\gamma(g) = -\widetilde{f}(g)$, so it is a homomorphism because $\widetilde{f}$ is a homomorphism and the image of $\widetilde{f}|_{\mathrm{Ker}(f)}$, which is contained in $\mathbb{Z}/n\mathbb{Z}$, is abelian.

Remark 5.3 In Diagram ($\star$), we can take $\widetilde{A} = \Gamma$, $f = \varphi$ and $\widetilde{f}$ is the identity. For the rest of this section, we always fix such a choice.

5.3 Local Invariant Computation

In this subsection, we investigate several conditions to ensure

$$\mathrm{inv}_v(\phi \cup \psi) \neq 0 \in \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

where $\phi \in H^1(\pi_v, \mu_n)$=$\mathrm{Hom}(\pi_v, \mu_n)$ and $\psi \in Z^1(\pi_v, \mathbb{Z}/n\mathbb{Z}) = \mathrm{Hom}(\pi_v, \mathbb{Z}/n\mathbb{Z})$.

Lemma 5.4 *Suppose that ϕ is unramified, i.e., ϕ factors through π_v/I_v. Then,*

$$\mathrm{inv}_v(\phi \cup \psi) = 0$$

if one of the following holds.

1. $\phi = 1$*, the trivial character.*
2. ψ *is unramified.*

Proof If $\phi = 1$, then $\phi \cup \psi = 0 \in H^2(\pi_v, \mu_n)$. Thus, $\mathrm{inv}_v(\phi \cup \psi) = 0$. Also, if ψ is unramified, then $\phi \cup \psi$ arises from $H^2(\pi_v/I_v, \mu_n)$ by inflation, which is 0. Therefore, $\phi \cup \psi = 0 \in H^2(\pi_v, \mu_n)$ and the result follows.

If v does not divide n, then we can prove more.

Lemma 5.5 *Assume that v does not divide n. And assume that ϕ is an unramified generator of* $\mathrm{Hom}(\pi_v, \mu_n)$*, i.e., a generator of* $\mathrm{Hom}(\pi_v/I_v, \mu_n)$*. Then,*

$$\mathrm{inv}_v(\phi \cup \psi) \neq 0 \Longleftrightarrow \psi \textit{ is ramified.}$$

Proof Using a fixed primitive nth root ζ of unity, we fix an isomorphism

$$\begin{aligned} \eta : \mathbb{Z}/n\mathbb{Z} &\longrightarrow \mu_n \\ a &\longmapsto \zeta^a \end{aligned}$$

and using η, we get natural isomorphisms

$$\mathrm{Hom}(\pi_v, \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}) \xleftarrow{\frac{1}{n}\cdot(-)} \mathrm{Hom}(\pi_v, \mathbb{Z}/n\mathbb{Z}) \underset{\eta^{-1}\circ(-)}{\overset{\eta\circ(-)}{\rightleftarrows}} \mathrm{Hom}(\pi_v, \mu_n).$$

In this proof, we will regard ϕ as an element of $\mathrm{Hom}(\pi_v, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ and ψ as one of $\mathrm{Hom}(\pi_v, \mu_n)$ using the above isomorphisms.

If ψ is unramified, $\mathrm{inv}_v(\phi \cup \psi) = 0$ by the above lemma. Since $\mu_n \subset F_v$, by the Kummer theory we can find an element $a \in F_v^*$ such that $\delta(a) = \psi$, where $\delta : F_v^*/(F_v^*)^n \simeq H^1(\pi_v,\ \mu_n) = \mathrm{Hom}(\pi_v, \mu_n)$. Let

$$\mathrm{ord}_v : F_v^* \longrightarrow \mathbb{Z}$$

be the normalized valuation on F_v^* that sends a uniformiser ϖ of O_{F_v} to 1. Then,

$$\psi \text{ is ramified} \iff \mathrm{ord}_v(a) \not\equiv 0 \pmod{n}.$$

Since ϕ is an unramified[7] generator, $\phi(\mathrm{Frob}) = \frac{t}{n}$ for some $t \in (\mathbb{Z}/n\mathbb{Z})^\times$, where Frob is a lift of the Frobenius in π_v/I_v to π_v. Then,

$$\mathrm{inv}_v(\phi \cup \psi) = \mathrm{inv}_v(\phi \cup \delta(a)) = \phi(\mathrm{Frob}^{\mathrm{ord}_v(a)}) = \frac{t \cdot \mathrm{ord}_v(a)}{n}.$$

Combining the above two results, we obtain

$$\psi \text{ is ramified} \iff \mathrm{inv}_v(\phi \cup \psi) \neq 0$$

as desired.

Remark 5.6 When $n = 2$, the above lemmas are enough for the computation of local invariants.

5.4 Construction of Examples

From now on, we assume that $n = 2$.

As a corollary of Sect. 5.2, if we have the following commutative diagrams

[7] This is where our assumption that $v \nmid n$ is used.

$$\begin{array}{ccc} \pi_T & \xrightarrow{\widetilde{\rho_+}} & \Gamma \\ {\scriptstyle \kappa_T}\downarrow & \searrow^{\rho_+} & \downarrow{\scriptstyle \varphi} \\ \pi & \xrightarrow[\rho]{} & A \end{array} \quad \text{and} \quad \begin{array}{ccc} \pi_v & \xrightarrow{\widetilde{\rho_v}} & \Gamma \\ {\scriptstyle \kappa_v}\downarrow & \searrow^{\rho_v} & \downarrow{\scriptstyle \varphi} \\ \pi & \xrightarrow[\rho]{} & A, \end{array} \qquad (\star\star)$$

then we get

$$\gamma_+ = (\widetilde{\rho_+})^*(\gamma) \quad \text{and} \quad \gamma_{-,v} = (\widetilde{\rho_v})^*(\gamma).$$

Thus we can explicitly compute $CS_c([\rho])$ using the previous strategy when we are in the following situation:

Assumption 5.7

1. F is a totally imaginary field.
2. $c = \alpha \cup \epsilon$ with $\alpha : A \to \mu_2$ surjective, and ϵ representing an extension

$$E : \; 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \Gamma \longrightarrow A \longrightarrow 1.$$

3. There are Galois extensions of F:

$$F \subset F^{\alpha} \subset F^{\mathrm{ur}} \subset F^{+}$$

   such that

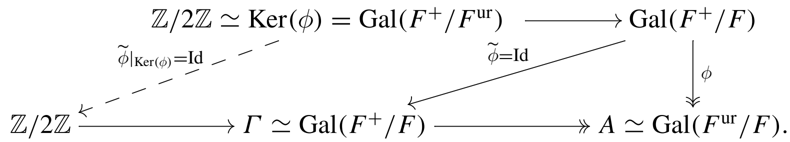
   - $\mathrm{Gal}(F^{\mathrm{ur}}/F)$ is isomorphic to A and F^{ur}/F is unramified everywhere.
   - $\mathrm{Gal}(F^{+}/F)$ is isomorphic to Γ and F^{+}/F is unramified at the primes above 2.
   - F^{α} is the fixed field of the kernel of the composition

$$\mathrm{Gal}(F^{\mathrm{ur}}/F) \xrightarrow{\sim} A \xrightarrow{\alpha} \mu_2$$

     and hence we get a commutative diagram

$$\begin{array}{ccccccc} & & A & & & & \\ & {\scriptstyle \rho}\nearrow & \uparrow{\scriptstyle \simeq} & \searrow & & \overset{\alpha}{\searrow} & \\ \pi & \twoheadrightarrow & \mathrm{Gal}(F^{\mathrm{ur}}/F) & \twoheadrightarrow & \mathrm{Gal}(F^{\alpha}/F) & \xrightarrow{\simeq} & \mu_2. \end{array}$$

Suppose we are in the above assumption. Let S be the set of primes of $\mathcal{O}_F$ ramified in F^{+}, and S_2 the set of primes of $\mathcal{O}_F$ dividing 2. Then by our assumption, $S \cap S_2 = \emptyset$. Let $T = S \cup S_2$. Then, we can find a global trivialisation γ_+ of $\rho_T^*(\epsilon)$ from the following commutative diagram



For each $v \in T$, let $D(v)$ be the decomposition group of $\mathrm{Gal}(F^+/F)$ at v. In other words,

$$D(v) = \{g \in \mathrm{Gal}(F^+/F) : gv = v\} \simeq \mathrm{Gal}(F^+_\nu/F_v),$$

where ν is a prime of F^+ lying above v. And let $I(v)$ be the inertia subgroup of $D(v)$. Then, $I(v) = 0$ if and only if v divides 2. Thus,

$$\gamma_{+,v} \text{ is unramified} \iff v \in S_2.$$

Since $\psi_v := \gamma_{+,v} - \gamma_{-,v}$ and we always take $\gamma_{-,v}$ unramified,

$$\psi_v \text{ is unramified} \iff v \in S_2.$$

Furthermore,

$$\rho_v^*(\alpha) \text{ is trivial} \iff f(D(v)) = 0,$$

where f is the natural projection from $\mathrm{Gal}(F^+/F)$ to $\mathrm{Gal}(F^\alpha/F)$. And $f(D(v)) = 0$ exactly occurs when v splits in F^α. Note that $\rho_v^*(\alpha)$ is always an unramified generator of $\mathrm{Hom}(\pi_v, \mu_2)$ if it is not trivial.

Now we are ready to compute the arithmetic Chern–Simons invariants.

Theorem 5.8 *Suppose we are in Assumption 5.7. Then,*

$$CS_c([\rho]) = \sum_{v \in T} \mathrm{inv}_v(\rho_v^*(\alpha) \cup \psi_v) = \frac{r}{2} \mod \mathbb{Z},$$

where $\psi_v = \gamma_{+,v} - \gamma_{-,v}$ and r is the number of primes in S which are inert in F^α.

Proof The first equality follows from Theorem 5.1. Thus, it suffices to compute $\mathrm{inv}_v(\rho_v^*(\alpha) \cup \psi_v)$ for $v \in T$. By Lemma 5.4, $\mathrm{inv}_v(\rho_v^*(\alpha) \cup \psi_v) = 0$ if either $\rho_v^*(\alpha)$ is trivial or ψ_v is unramified. By the above discussion, $\rho_v^*(\alpha)$ is trivial if and only if $f(D(v)) = 0$, i.e., v splits in F^α; and ψ_v is unramified if and only if $v \in S_2$. Furthermore, if $\rho_v^*(\alpha)$ is not trivial and ψ_v is ramified, then by Lemma 5.5, $\mathrm{inv}_v(\rho_v^*(\alpha) \cup \psi_v) = \frac{1}{2}$. Thus the result follows.

Therefore to provide an example of calculation of the arithmetic Chern–Simons invariants, it suffices to construct a tower of fields satisfying Assumption 5.7, which is essentially the embedding problem in the inverse Galois theory. Instead, we will consider the similar problems over $\mathbb{Q}$, which are much easier to solve (or find from the table). Then, we will construct a tower satisfying Assumption 5.7 from a tower of fields over $\mathbb{Q}$.

Assumption 5.9 Suppose we have a number field L with its subfield K such that

1. $\mathrm{Gal}(L/\mathbb{Q}) \simeq \Gamma$.
2. d_L, the (absolute) discriminant of L, is an odd integer.[8]
3. $\mathrm{Gal}(K/\mathbb{Q}) \simeq A$.
4. $\mathbb{Q}(\sqrt{D})$ is a quadratic subfield of K, where D is a divisor of d_K.[9]
5. $K/\mathbb{Q}(\sqrt{D})$ is unramified at any finite primes.

Then, we have the following.

Proposition 5.10 *Let $F = \mathbb{Q}(\sqrt{-|D| \cdot t})$ be an imaginary quadratic field, where t is a positive squarefree integer prime to D so that $F \cap L = \mathbb{Q}$. Then, there is a tower of fields $F \subset F^{\mathrm{ur}} \subset F^+$ satisfies Assumption 5.7. In fact, we can take*

$$F^{\mathrm{ur}} = KF \quad \text{and} \quad F^+ = LF.$$

Proof First, it is clear that F is totally imaginary. Next, since $F \cap L = \mathbb{Q}$

$$\mathrm{Gal}(LF/F) \simeq \mathrm{Gal}(L/\mathbb{Q}) \simeq \Gamma \quad \text{and} \quad \mathrm{Gal}(KF/F) \simeq \mathrm{Gal}(K/\mathbb{Q}) \simeq A.$$

Since the discriminant of L is odd, L/K is unramified at the primes above 2, and so is LF/KF. Finally, it suffices to show that KF/F is unramified everywhere. Since $K/\mathbb{Q}(\sqrt{D})$ is unramified everywhere, $K/\mathbb{Q}$ is only ramified at the primes dividing D. (Note that the discriminant of K is odd, hence it is unramified at 2.) Moreover, the ramification degree of any prime divisor p of D is 2, and the same is true for $F/\mathbb{Q}$. Since p is odd, KF/F is unramified at the primes above p by Abhyankar's lemma [5, Theorem 1], which implies our claim.

Remark 5.11 Since the ramification indices of any prime divisor p of D are 2 in both $F/\mathbb{Q}$ and $K/\mathbb{Q}$, we can use Abhyankar's lemma in both directions. (Note that our assumption implies that D is odd.) In other words, KF/K is always unramified at the primes dividing D.

The remaining part to check Assumption 5.7 is the choice of F^α. Let

$$B := \{F_1, \ldots, F_m\}$$

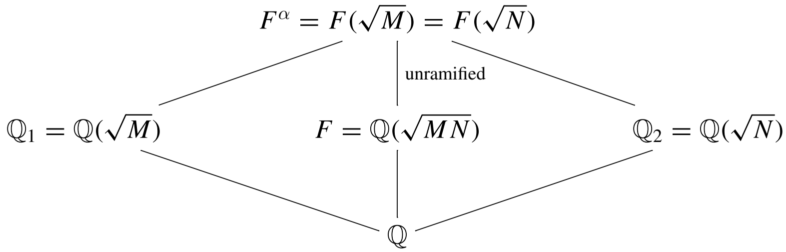
be the set of quadratic subfields of F^{ur}. Then, there is one-to-one correspondence between the set of surjective homomorphisms $\mathrm{Gal}(F^{\mathrm{ur}}/F) \to \mu_2$ and B. Therefore

[8]We may consider when d_L is even. Then later, it is not clear that FL/FK is unramified at the primes above 2. Some choices of t (for F) can make it ramified. Then, it is hard to determine the value of local invariants unless 2 splits in F^α/F.

[9]Here, we always take that d_K is odd because we cannot use Abhyankar's lemma when $p = 2$, and hence we may not remove ramification in the extension FK/F at the primes above 2. In some nice situation, we may directly prove that $F(\sqrt{D})/F$ is unramified at the primes above 2 even though D is even. If so, our assumption on d_K can be removed.

$m = \#\mathrm{Hom}(A, \mu_2) - 1$ and we can define $\alpha_i : A \to \mu_2$ so that $F^{\alpha_i} = F_i$ due to the (chosen) isomorphism $\mathrm{Gal}(F^{\mathrm{ur}}/F) \simeq A$.

Now, suppose $F^\alpha = F(\sqrt{M}) \subset F^{\mathrm{ur}}$ for some divisor M of D. Let $\mathbb{Q}_1 = \mathbb{Q}(\sqrt{M})$ and $\mathbb{Q}_2 = \mathbb{Q}(\sqrt{N})$, where $N = (-|D| \cdot t)/M$. Then, we have the following commutative diagram:



For a prime p, let $\wp$ denote a prime of O_F lying above p. We want to understand the splitting behaviour of $\wp$ in F^α.

Lemma 5.12 *Let p be an odd prime.*

1. *Assume that p divides Dt. Then*

$$\wp \text{ is inert in } F^\alpha \iff p \text{ is inert either in } \mathbb{Q}_1 \text{ or in } \mathbb{Q}_2.$$

2. *If p is inert in F, then $\wp$ always splits in F^α.*
3. *Assume that p splits in F. Then*

$$\wp \text{ splits in } F^\alpha \iff p \text{ splits in } \mathbb{Q}_1.$$

Proof

1. In this case, p is ramified in F, and p is ramified either in $\mathbb{Q}_1$ or in $\mathbb{Q}_2$. Without loss of generality, let p is ramified in $\mathbb{Q}_2$. Then, $\wp$ is inert in F^α if and only if p is inert in $\mathbb{Q}_1$ from the above commutative diagram.
2. Let $\left(\frac{a}{b}\right)$ denote the Legendre symbol. If p is inert in F, then $\left(\frac{MN}{p}\right) = -1$. Therefore either $\left(\frac{M}{p}\right) = 1$ or $\left(\frac{N}{p}\right) = 1$. Without loss of generality, let $\left(\frac{M}{p}\right) = 1$ and $\left(\frac{N}{p}\right) = -1$. Then, p splits in $\mathbb{Q}_1$ and hence there are at least two primes in F^α above p. Since $\wp$ is the unique prime of F above p, $\wp$ splits in F^α.
3. Since $\left(\frac{MN}{p}\right) = 1$, either $\left(\frac{M}{p}\right) = \left(\frac{N}{p}\right) = 1$ or $\left(\frac{M}{p}\right) = \left(\frac{N}{p}\right) = -1$. If $\left(\frac{M}{p}\right) = -1$, then there is only one prime in $\mathbb{Q}_1$ above p. Thus, there are at most two primes in F^α above p. Since p already splits in F, $\wp$ is inert in F^α. On the other hand, if $\left(\frac{M}{p}\right) = 1$, then p splits completely in F^α because p splits completely both in $\mathbb{Q}_1$ and F. Thus, $\wp$ splits in F^α.

Let $D_L = d_L/d_K^2$ be the norm (to $\mathbb{Q}$) of the relative discriminant of L/K. Then, L/K is precisely ramified at the primes dividing D_L, and hence

$$S \subset \{\mathfrak{p} \in \mathrm{Spec}(O_F) : \mathfrak{p} \mid D_L\}.$$

(Note that S is the set of primes in O_F that ramify in F^+.) Let s be the number of prime divisors of (D_L, D), which are inert either in $\mathbb{Q}_1$ or in $\mathbb{Q}_2$. Then, we have the following.

Theorem 5.13 *Assume that we have ρ and c as above. Then,*

$$CS_c([\rho]) \equiv \frac{s}{2} \pmod{\mathbb{Z}}.$$

Proof First, we show that

$$S = \{\mathfrak{p} \in \mathrm{Spec}(O_F) : \mathfrak{p} \mid D_L \text{ but } \mathfrak{p} \nmid t\}.$$

For a prime divisor p of D_L which does not divide t, we show that KF/K is unramified at any primes above p, which implies that LF/KF is ramified at the primes above p. If p does not divide D, then this is done because p is unramified in F. On the other hand, if p divides D, KF/K is unramified at the primes above p by Remark 5.11. Now, assume that p divides (D_L, t), and let $\wp$ be a prime of O_K lying above p. Then, $\wp$ is ramified both in L/K and in KF/K. (Note that since $(t, D) = 1$, $K/\mathbb{Q}$ is unramified at p but $F/\mathbb{Q}$ is ramified at p.) Therefore by the same argument as in Remark 5.11, LF/KF is unramified at the primes above p, which proves the above claim.

Next, by Theorem 5.8 it suffices to compute the number of primes in S which are inert in F^α. Let $\wp \in S$ be a prime above an odd prime p. Assume that p does not divide D. (Then p is unramified in F.) If p is inert in F, then $\wp$ always splits in F^α by Lemma 5.12. If p splits in F and $pO_F = \wp \cdot \wp'$, then $\wp$ is inert (in F^α) if and only if $\wp'$ is inert. Therefore to compute the invariant, the contribution from such split primes can be ignored. So, we may assume that p divides D. Then, there is exactly one (ramified) prime $\wp$ in O_F above p, and our claim follows from Lemma 5.12.

We remark that the computation of s is completely easy because $\mathbb{Q}_1/\mathbb{Q}$ and $\mathbb{Q}_2/\mathbb{Q}$ are just quadratic fields. And this also illustrates that we only need information on the primes dividing (D_L, D) for the computation.

5.5 Case 1: Cyclic Group

Let $A = \mathbb{Z}/2\mathbb{Z}$, and $\Gamma = \mathbb{Z}/4\mathbb{Z}$. Then, we can easily find Galois extensions $L/K/\mathbb{Q}$ in Assumption 5.9 by the theory of cyclotomic fields.

Let p be a prime congruent to 1 modulo 4. Then, we can take L as the degree 4 subfield of $\mathbb{Q}(\mu_p)$, and $K = \mathbb{Q}(\sqrt{p})$. Moreover, $d_L = p^3$ and $d_K = p$.

Let $F = \mathbb{Q}(\sqrt{-p \cdot t})$, where t is a positive squarefree integer prime to p. (Then, $F \cap L = \mathbb{Q}$.)

Proposition 5.14 *Let ρ and c be chosen so that $F^{\alpha} = F^{\mathrm{ur}} = FK$ and $F^{+} = FL$. Then,*

$$CS_c([\rho]) = \frac{1}{2} \Longleftrightarrow \left(\frac{t}{p}\right) = -1.$$

Proof By Theorem 5.13, it suffices to check whether p is inert in $\mathbb{Q}(\sqrt{-t})$. If it is inert, then $CS_c([\rho]) = \frac{1}{2}$, and 0 otherwise. Since $p \equiv 1 \pmod 4$, the result follows.

5.6 Case 2: Non-cyclic Abelian Group

Let $A = V_4 := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the Klein four group, and $\Gamma = \mathcal{Q}_8 = \mathcal{Q}$, the quaternion group. To find Galois extensions $L/K/\mathbb{Q}$ in Assumption 5.9, we first study quaternion extensions of $\mathbb{Q}$ in general.

Proposition 5.15 *Let $L/\mathbb{Q}$ be a Galois extension whose Galois group is isomorphic to $\mathcal{Q}$. Suppose that d_L is odd. Let K be a subfield of L with $\mathrm{Gal}(L/K) \simeq \mathbb{Z}/2\mathbb{Z}$. Then,*

1. *$K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ for some positive squarefree d_1 and d_2.*
2. *$d_1 \equiv d_2 \equiv 1 \pmod 4$.*
3. *Let p be a prime divisor of $d_1 d_2$. Then, p divides $D_L := d_L/d_K^2$.*

Proof Since K is a subfield of L, d_K is also odd. And since $\mathcal{Q}$ has a unique subgroup of order 2, which is normal, $K/\mathbb{Q}$ is Galois and $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Therefore $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, where d_1 and d_2 are products of prime discriminants. If L is totally real, then K must be totally real as well. If L is not totally real, then the complex conjugation generates a subgroup of $\mathrm{Gal}(L/\mathbb{Q})$ of order 2. Since $\mathcal{Q}$ has a unique subgroup of order 2, K must be a fixed field of the complex conjugation, which implies that K is totally real. So, d_1 and d_2 can be taken as positive squarefree integers. Moreover, since they are products of prime discriminants and odd, $d_1 \equiv d_2 \equiv 1 \pmod 4$.

Finally, let p be a prime divisor of d_1, which does not divide d_2. Note that $\mathbb{Q}(\sqrt{d_1}) \subset K \subset L$ and $L/\mathbb{Q}(\sqrt{d_1})$ is a cyclic extension of degree 4. Since p does not divide d_2, $\mathbb{Q}(\sqrt{d_2})/\mathbb{Q}$ is unramified at p and hence $K/\mathbb{Q}(\sqrt{d_2})$ is ramified at the primes dividing p. By [19, Corollary 3], L/K is ramified at the primes above p and hence p divides D_L. By the same argument, the claim follows when p is a divisor of d_2, which does not divide d_1. Let p be a prime divisor of (d_1, d_2). Then, since $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}) = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_1 d_2}) = \mathbb{Q}(\sqrt{d_1}, \sqrt{\frac{d_1 d_2}{p^2}})$ and p does not divide $\frac{d_1 d_2}{p^2}$, the result follows by the same argument as above.

Now, let d_1 and d_2 be two squarefree positive integers such that

- $d_1 \equiv d_2 \equiv 1 \pmod 4$.
- $(d_1, d_2) = 1$.[10]

[10]This is not a vacuous condition. In fact, there is a $\mathcal{Q}$-extension L containing $\mathbb{Q}(\sqrt{21}, \sqrt{33})$ [35].

Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. Suppose that there is a number field L such that

- $L/\mathbb{Q}$ is Galois and $\mathrm{Gal}(L/\mathbb{Q}) \simeq \mathfrak{Q}$.
- L contains K and the discriminant d_L of L is odd.

Let $F = \mathbb{Q}(\sqrt{-d_1 d_2 \cdot t})$, where t is a positive squarefree integer prime to $d_1 d_2$. Then $L \cap F = \mathbb{Q}$ because all quadratic subfields of L are contained in K, which is totally real. Since $\mathrm{Hom}(A, \mu_2)$ is of order 4, there are three quadratic subfield of FK over F:

$$F_1 := F(\sqrt{d_1}),\ \ F_2 := F(\sqrt{d_2}), \text{ and } F_3 := F(\sqrt{d_1 d_2}) = F(\sqrt{-t}).$$

Proposition 5.16 *Let ρ and $c_i = \alpha_i \cup \epsilon$ be chosen so that $F^{\alpha_i} = F_i$, $F^{\mathrm{ur}} = FK$ and $F^+ = FL$. Then,*

$$CS_{c_1}([\rho]) = \frac{1}{2} \iff \prod_{p|d_1} \left(\frac{-d_2 \cdot t}{p}\right) \times \prod_{p|d_2} \left(\frac{d_1}{p}\right) = -1.$$
$$CS_{c_2}([\rho]) = \frac{1}{2} \iff \prod_{p|d_1} \left(\frac{d_2}{p}\right) \times \prod_{p|d_2} \left(\frac{-d_1 \cdot t}{p}\right) = -1.$$
$$CS_{c_3}([\rho]) = \frac{1}{2} \iff \prod_{p|d_1 d_2} \left(\frac{-t}{p}\right) = -1.$$

Proof By the above lemma and Theorem 5.13, it suffices to compute the number of prime divisors of $d_1 d_2$, which are inert in $\mathbb{Q}_1$ or in $\mathbb{Q}_2$.

First, compute $CS_{c_1}([\rho])$. In this case, $\mathbb{Q}_1 = \mathbb{Q}(\sqrt{d_1})$ and $\mathbb{Q}_2 = \mathbb{Q}(\sqrt{-d_2 \cdot t})$. If p is a divisor of d_1, it is inert in $\mathbb{Q}_2$ if and only if

$$\left(\frac{-d_2 \cdot t}{p}\right) = -1.$$

Therefore, the number of such prime divisors of d_1 is odd if and only if

$$\prod_{p|d_1} \left(\frac{-d_2 \cdot t}{p}\right) = -1.$$

Similarly, the number of prime divisors of d_2, which are inert in $\mathbb{Q}_1$, is odd if and only if

$$\prod_{p|d_2} \left(\frac{d_1}{p}\right) = -1.$$

Thus, we have

$$CS_{c_1}([\rho]) = \frac{1}{2} \iff \prod_{p|d_1} \left(\frac{-d_2 \cdot t}{p}\right) \times \prod_{p|d_2} \left(\frac{d_1}{p}\right) = -1.$$

The remaining two cases can easily be done by the same method as above.

We can find Galois extensions $L/K/\mathbb{Q}$ satisfying the above assumptions from the database. Here, we take $L/K/\mathbb{Q}$ from the LMFDB [36] as follows. Let

$$g(x) = x^8 - x^7 + 98x^6 - 105x^5 + 3191x^4 + 1665x^3 + 44072x^2 + 47933x + 328171$$

be an irreducible polynomial over $\mathbb{Q}$, and β be a root of $g(x)$. Let

$$L = \mathbb{Q}(\beta) \quad \text{and} \quad K = \mathbb{Q}(\sqrt{5}, \sqrt{29}).$$

So, $d_1 = 5$ and $d_2 = 29$. Moreover, $D_L = 3^2 \cdot 5^2 \cdot 29^2$.

Let $F = \mathbb{Q}(\sqrt{-5 \cdot 29 \cdot t})$, where t is a positive squarefree integer prime to $5 \cdot 29$.

Corollary 5.17 *Let ρ and $c_i = \alpha_i \cup \epsilon$ be chosen as above. Then,*

$$CS_{c_1}([\rho]) = \frac{1}{2} \iff \left(\frac{t}{5}\right) = -1 \iff t \equiv \pm 2 \pmod 5.$$
$$CS_{c_2}([\rho]) = \frac{1}{2} \iff \left(\frac{t}{29}\right) = -1.$$
$$CS_{c_3}([\rho]) = \frac{1}{2} \iff \left(\frac{t}{5}\right) = -\left(\frac{t}{29}\right).$$

Now, we provide another example. Let $L/K/\mathbb{Q}$ from the the LMFDB [37] as follows. Let

$$g(x) = x^8 - x^7 - 34x^6 + 29x^5 + 361x^4 - 305x^3 - 1090x^2 + 1345x - 395$$

be an irreducible polynomial over $\mathbb{Q}$, and β be a root of $g(x)$. Let

$$L = \mathbb{Q}(\beta) \quad \text{and} \quad K = \mathbb{Q}(\sqrt{5}, \sqrt{21}).$$

So, $d_1 = 5$ and $d_2 = 21$. Moreover, $D_L = 3^2 \cdot 5^2 \cdot 7^2$.

Let $F = \mathbb{Q}(\sqrt{-105 \cdot t})$, where t is a positive squarefree integer prime to 105.

Corollary 5.18 *Let ρ and $c_i = \alpha_i \cup \epsilon$ be chosen as above. Then,*

$$CS_{c_1}([\rho]) = \frac{1}{2} \iff \left(\frac{t}{5}\right) = -1 \iff t \equiv \pm 2 \pmod 5.$$
$$CS_{c_2}([\rho]) = \frac{1}{2} \iff \left(\frac{t}{3}\right) = -\left(\frac{t}{7}\right) \iff 2, 8, 10, 11, 13, 19 \pmod{21}.$$
$$CS_{c_3}([\rho]) = \frac{1}{2} \iff \left(\frac{t}{3}\right) \cdot \left(\frac{t}{5}\right) \cdot \left(\frac{t}{7}\right) = -1.$$

Now, we take $A = V_4$, but $\Gamma = D_4$, the dihedral group of order 8. We found $L/K/\mathbb{Q}$ from the LMFDB [38] as follows. Let

$$g(x) = x^8 - 3x^7 + 4x^6 - 3x^5 + 3x^4 - 3x^3 + 4x^2 - 3x + 1$$

be an irreducible polynomial over $\mathbb{Q}$, and β be a root of $g(x)$. Let

$$L = \mathbb{Q}(\beta) \quad \text{and} \quad K = \mathbb{Q}(\sqrt{-3}, \sqrt{-7}).$$

If we take $D = 21$, then this choice satisfies Assumption 5.9. Moreover, $d_L = 3^6 \cdot 7^4$ and $d_K = 3^2 \cdot 7^2$.

Let $F = \mathbb{Q}(\sqrt{-21 \cdot t})$, where t is a positive squarefree integer prime to 21. (Then, $F \cap L = \mathbb{Q}$ because all imaginary quadratic subfields of L are $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-7})$.) Since $\mathrm{Hom}(A, \mu_2)$ is of order 4, there are three quadratic subfield of FK over F:

$$F_1 := F(\sqrt{-3}),\ \ F_2 := F(\sqrt{-7}), \text{and } F_3 := F(\sqrt{21}).$$

Proposition 5.19 *Let ρ and $c_i = \alpha_i \cup \epsilon$ be chosen so that $F^{\alpha_i} = F_i$, $F^{\mathrm{ur}} = FK$ and $F^+ = FL$. Then,*

$$CS_{c_1}([\rho]) = \frac{1}{2} \Longleftrightarrow \left(\frac{t}{3}\right) = -1 \Longleftrightarrow t \equiv 2 \pmod 3.$$
$$CS_{c_2}([\rho]) = \frac{1}{2} \quad \textit{for all } t.$$
$$CS_{c_3}([\rho]) = \frac{1}{2} \Longleftrightarrow \left(\frac{t}{3}\right) = 1 \Longleftrightarrow t \equiv 1 \pmod 3.$$

Proof Since $D_L = 3^2$, the result follows from Theorem 5.13.

5.7 *Case 3: Non-abelian Group*

Let $A = S_4$, the symmetric group of degree 4. Then, $H^1(A, \mu_2) \simeq \mathbb{Z}/2\mathbb{Z}$ and $H^2(A, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus, there is a unique surjective map $\alpha : A \twoheadrightarrow \mu_2$ and three non-trivial central extensions Γ_i of A by $\mathbb{Z}/2\mathbb{Z}$:

- $\Gamma_1 = 2^+ S_4 \simeq \mathrm{GL}(2, \mathbb{F}_3)$, the general linear group of degree 2 over $\mathbb{F}_3$.
- $\Gamma_2 = 2^- S_4$, the transitive group '16T65' in [33].
- $\Gamma_3 = 2^{\det} S_4$, corresponding to the cup product of the signature with itself.

Let ϵ_i be a cocycle representing the extension

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \Gamma_i \longrightarrow A = S_4 \longrightarrow 0.$$

In this subsection, we will consider the first two cases. There are another descriptions of the groups Γ_1 and Γ_2. Let

$$\mathcal{E} : 1 \longrightarrow \mathrm{SL}(2, \mathbb{F}_3) \longrightarrow \Gamma \longrightarrow \mathbb{F}_3^\times \simeq \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

If $\mathcal{E}$ splits, then $\Gamma \simeq \Gamma_1$, otherwise $\Gamma \simeq \Gamma_2$.

Let $c = \alpha \cup \epsilon_1$. (So, $\Gamma = \Gamma_1$.) Suppose $\mathbb{Q} \subset \mathbb{Q}(\sqrt{D}) \subset K \subset L$ is a tower of fields satisfying Assumption 5.9. Let $F = \mathbb{Q}(\sqrt{-|D| \cdot t})$, where t is a squarefree integer prime to D and greater than 1. Then, $F \cap L = F \cap \mathbb{Q}(\sqrt{D}) = \mathbb{Q}$. (The first equality holds because Γ has a unique subgroup of order 24.)

Proposition 5.20 *Let ρ and c be chosen so that $F^\alpha = F(\sqrt{D})$, $F^{\mathrm{ur}} = FK$ and $F^+ = FL$. Then,*

$$CS_c([\rho]) = 0.$$

Proof Since the extension

$$\mathcal{E} : 1 \longrightarrow \mathrm{SL}(2, \mathbb{F}_3) \longrightarrow \mathrm{GL}(2, \mathbb{F}_3) \longrightarrow \mathbb{F}_3^\times \simeq \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

splits, $\mathrm{Gal}(L/\mathbb{Q}) \simeq \mathrm{Gal}(L/\mathbb{Q}(\sqrt{D})) \rtimes \mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q})$.

Let p be a prime divisor of (D_L, D). By our assumption, p is odd. Let I_p be an inertia subgroup of $\mathrm{Gal}(L/\mathbb{Q}) \simeq \Gamma = \mathrm{GL}(2, \mathbb{F}_3)$. Since L/K and $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$ are ramified at p but $K/\mathbb{Q}(\sqrt{D})$ is not, the ramification index of p in $L/\mathbb{Q}$ is 4, and $I_p \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

On the other hand, since p is odd, $L/\mathbb{Q}$ is tamely ramified at p and hence I_p must be cyclic, which is a contradiction. Therefore $(D_L, D) = 1$ and hence the result follows by Theorem 5.13.

We can find several examples of such towers from the LMFDB. Let

$$g_1(x) = x^8 - 4x^7 + 7x^6 + 7x^5 - 51x^4 + 50x^3 + 61x^2 - 107x - 83$$
$$g_2(x) = x^4 - x - 1$$

be irreducible polynomials over $\mathbb{Q}$ [39, 40], and let L (resp. K) be the the splitting field of $g_1(x)$ (resp. $g_2(x)$). Then, $\mathrm{Gal}(L/\mathbb{Q}) \simeq \mathrm{GL}(2, \mathbb{F}_3)$ and $\mathrm{Gal}(K/\mathbb{Q}) \simeq S_4$. Moreover, $d_L = 3^{24} \cdot 283^{24}$ and $d_K = 283^{12}$. Thus, $D = -283$ satisfies Assumption 5.9. Note that since the discriminant D of $g_2(x)$ is squarefree, $K/\mathbb{Q}(\sqrt{D})$ is unramified everywhere (cf. [12, p. 1]).

Let $F = \mathbb{Q}(\sqrt{-283 \cdot t})$, where t is a squarefree integer prime to 283, and $t > 1$.

Corollary 5.21 *Let ρ and c be chosen so that $F^\alpha = F(\sqrt{-283})$, $F^{\mathrm{ur}} = FK$ and $F^+ = FL$. Then,*

$$CS_c([\rho]) = 0.$$

Now, we consider another case. Let $c = \alpha \cup \epsilon_2$. (So, $\Gamma = \Gamma_2$.) Let L be the splitting field of

$$f(x) = x^{16} + 5x^{15} - 790x^{14} - 4654x^{13} + 234254x^{12} + 1612152x^{11} - 33235504x^{10} - 263221982x^9 + 2331584048x^8 + 21321377994x^7 - 74566280958x^6 - 825209618478x^5 + 922238608476x^4 + 13790070608536x^3 - 6704968288135x^2 - 80794234036917x + 87192014930816.$$

Let K be the splitting field of

$$g(x) = x^4 - x^3 - 4x^2 + x + 2.$$

Then, $\mathrm{Gal}(L/\mathbb{Q}) \simeq \Gamma = \Gamma_2$ and $\mathrm{Gal}(K/\mathbb{Q}) \simeq S_4 = A$.[11] (See [33, 34].)

Lemma 5.22 *We have the following.*

1. $K/\mathbb{Q}(\sqrt{2777})$ *is unramified everywhere.*
2. $\mathbb{Q}(\sqrt{2777})$ *is a unique quadratic subfield of* L.
3. $\mathbb{Q}(\sqrt{2777}) \subset K \subset L$.
4. D_L *is a multiple of* 2777, *i.e.,* L/K *is ramified at the primes above* 2777.

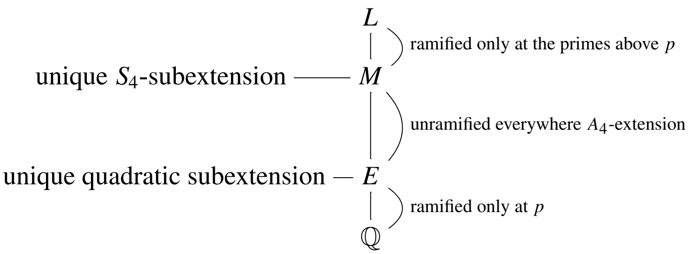
Proof For simplicity, let $E := \mathbb{Q}(\sqrt{2777})$ and $p = 2777$.

1. Since S_4 has a unique subgroup of order 12, K has a unique quadratic subfield K'. Since the discriminant of $g(x)$ is p, a prime, $K' = E$ and K/E is unramified everywhere (cf. [12, p. 1]).
2. Let β_i be the roots of $f(x)$. Then, $L = \cup\, \mathbb{Q}(\beta_i)$. Since the discriminant of the field $\mathbb{Q}[x]/(f(x))$ is p^{12}, $\mathbb{Q}(\beta_i)$ contains E, and so does L. On the other hand, since Γ has also a unique subgroup of order 24, E is a unique quadratic subfield of L.
3. Since

$$f(x) \equiv (x + 1372)^4 \cdot (x + 1791)^4 \cdot (x + 1822)^4 \cdot (x + 2653)^4 \pmod{p},$$

the ramification index of p in $\mathbb{Q}(\beta_i)/\mathbb{Q}$ is 4. Since $L = \cup\, \mathbb{Q}(\beta_i)$ and p is odd, the ramification index of p in $L/\mathbb{Q}$ is 4 by Abhyankar's lemma. Since $L/\mathbb{Q}$ is tamely ramified at p, the inertia subgroup I_p of $\mathrm{Gal}(L/\mathbb{Q}) \simeq \Gamma$ is cyclic of order 4. Since Γ has a unique subgroup C of order 2, I_p contains C. Thus, L/M is ramified at the primes above p, where M is the fixed field of C in L. Since $E/\mathbb{Q}$ is also ramified at p, M/E is unramified at the primes above p, and hence M/E is unramified everywhere.

[11] This example is provided us by Dr. Kwang–Seob Kim.



Now, it suffices to show that $K = M$. Let $N = K \cap M$. Then, since K and M are Galois over E, so is N. Also since the normal subgroups of $\mathrm{Gal}(K/E) \simeq A_4 \simeq \mathrm{Gal}(M/E)$ are either $\{1\}$, V_4 or A_4,

$$\mathrm{Gal}(N/E) \simeq \text{ either } \{1\}, \mathbb{Z}/3\mathbb{Z} \text{ or } A_4.$$

Note that the class group of E is $\mathbb{Z}/3\mathbb{Z}$. Let H be the Hilbert class field of E. Then, the class group of H is V_4. (This can easily be checked because the degree of $H/\mathbb{Q}$ is small.) If $\mathrm{Gal}(N/E) \simeq \{1\}$, then E has two different degree 3 unramified extensions given by K^{V_4} and M^{V_4}, which is a contradiction. If $\mathrm{Gal}(N/E) \simeq \mathbb{Z}/3\mathbb{Z}$, then $N = H$ and N has two different unramified V_4 extensions K and M, which is a contradiction. Thus, $\mathrm{Gal}(N/E) \simeq A_4$ and hence $K = N = M$, as desired.

4. This is proved in (3).

Thus, we can take $D = 2777$. Let $F = \mathbb{Q}(\sqrt{-2777 \cdot t})$ for a positive squarefree integer t prime to 2777. Then, $F \cap L = \mathbb{Q}$ because L has a unique quadratic subfield $\mathbb{Q}(\sqrt{2777})$, which is real.

Proposition 5.23 *Let ρ and c be chosen so that $F^{\alpha} = F(\sqrt{D})$, $F^{\mathrm{ur}} = FK$ and $F^{+} = FL$. Then,*

$$CS_c([\rho]) = \frac{1}{2} \Longleftrightarrow \left(\frac{-t}{2777}\right) = \left(\frac{t}{2777}\right) = -1.$$

Proof Since $(D_L, D) = 2777$ and $F^{\alpha} = F(\sqrt{D}) = F(\sqrt{-t})$, the result follows from Theorem 5.13.

Remark 5.24 Even in the non-abelian case, we have infinite family of non-vanishing arithmetic Chern–Simons invariants!

6 Application

In this section, we give a simple arithmetic application of our computation. Namely, we show non-solvability of a certain case of the embedding problem based on our examples of non-vanishing arithmetic Chern–Simons invariants.

For an odd prime p, let $p^* = (-1)^{\frac{p-1}{2}} p$. Let

$$d_1 = \prod_{i=1}^{s} p_i^* \quad \text{and} \quad d_2 = \prod_{j=1}^{t} q_j^*,$$

where p_i, q_j are distinct odd prime numbers, and $d_1, d_2 > 0$. Let

$$A_i := \left(\frac{d_2}{p_i}\right) = \prod_{1\le j\le t} \left(\frac{q_j}{p_i}\right) \quad \text{and} \quad B_j := \left(\frac{d_1}{q_j}\right) = \prod_{1\le i\le s} \left(\frac{p_i}{q_j}\right).$$

Let

$$\Delta(d_1, d_2) := \prod_{1\le i\le s} A_i \quad \text{and} \quad \Delta(d_2, d_1) := \prod_{1\le j\le t} B_j.$$

Lemma 6.1 $\Delta(d_1, d_2) = \Delta(d_2, d_1)$.

Proof Note that $\Delta(d_1, d_2) = \prod_{\substack{1\le i\le s \\ 1\le j\le t}} \left(\frac{p_i}{q_j}\right)$. Since d_1 is positive, the number of prime divisors of d_1 which are congruent to 3 modulo 4 is even. And the same is true for d_2. Thus by the quadratic reciprocity law,

$$A_i = \prod_{1\le j\le t} \left(\frac{q_j}{p_i}\right) = \prod_{1\le j\le t} \left(\frac{p_i}{q_j}\right).$$

By taking product for all $1 \le i \le s$, we get the result.

Recall that $\mathcal{Q}$ denotes the quaternion group.

Proposition 6.2 *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. If $\Delta(d_1, d_2) = -1$, then there cannot exist a number field L with odd discriminant, such that* $\mathrm{Gal}(L/\mathbb{Q}) \simeq \mathcal{Q}$ *and $K \subset L$.*

A referee of an earlier version of this paper has pointed out that this result can also be obtained using the theorem[12] of Witt in [31, p. 244] (or (7.7) on [8, p. 106]). (In our situation, if such a field L exists, the theorem implies $\Delta(d_1, d_2) = 1$, which gives us a contradiction.) So this proposition should be viewed as a new perspective rather than a new result. In fact, Propositions 6.2 and 6.4 deal with a class of embedding problems wherein the existence of an unramified extension forces a Chern–Simons invariant to be zero. The outline of proof together with the explicit formulas for computing the Chern–Simons invariant should make clear that even the simplest $\mathbb{Z}/2\mathbb{Z}$-valued case is likely to have a non-trivial range of applications. We consider the point of view presented here as a simple and rough analogue of the classical theorem of Herbrand, whereby the existence of certain unramified extensions of cyclotomic fields forces

[12] K extends to a quaternion extension if and only if the Hilbert symbols (d_1, d_2) and $(d_1 d_2, -1)$ agree in the Brauer group.

some L-values to be congruent to zero ([29, Sect. 6.3]). In future papers, we hope to discuss this analogy in greater detail and investigate the possibility of 'converse Herbrand' type results in the spirit of Ribet's theorem [27].

Proof Suppose that there does exist such a field $L/\mathbb{Q}$ satisfying all the given properties above. Choose a prime ℓ such that

- ℓ does not divide $d_1 d_2$.
- $\ell \equiv 3 \pmod 4$.
- $\left(\frac{-\ell}{p_i}\right) = A_i$ and $\left(\frac{-\ell}{q_j}\right) = B_j$ for all i and j.

In fact, $\ell \equiv a \pmod{4d_1d_2}$ for some a with $(a, 4d_1d_2) = 1$, and hence there are infinitely many such primes by Dirichlet's theorem.

Now let $d_3 := \ell^* = -\ell$. And let $F = \mathbb{Q}(\sqrt{d_1d_2d_3})$. Then by direct computation using the quadratic reciprocity law, we get

$$\left(\frac{d_1d_2}{\ell}\right) = \prod_{1\le i\le s}\left(\frac{p_i}{\ell}\right)\prod_{1\le j\le t}\left(\frac{q_j}{\ell}\right) = \prod_{1\le i\le s}\left(\frac{-\ell}{p_i}\right)\prod_{1\le j\le t}\left(\frac{-\ell}{q_j}\right) = \varDelta(d_1,d_2)\cdot\varDelta(d_2,d_1).$$

Thus by the above lemma, we get

$$\left(\frac{d_1d_2}{\ell}\right) = 1.$$

Furthermore, for all i and j

$$\left(\frac{d_2d_3}{p_i}\right) = A_i^2 = 1 \quad \text{and} \quad \left(\frac{d_3d_1}{q_j}\right) = B_j^2 = 1.$$

Therefore by [19, Theorem 1], there is a Galois extension $M/\mathbb{Q}$ such that M/F is unramified everywhere, and $\mathrm{Gal}(M/F) \simeq \mathcal{Q}$. Furthermore $KF = F(\sqrt{d_1}, \sqrt{d_2})$ is the unique subfield of M with $\mathrm{Gal}(M/KF) \simeq \mathbb{Z}/2\mathbb{Z}$.

Let $A = V_4$, and let $c_i = \alpha_i \cup \epsilon$, where $\alpha_i \in H^1(A, \mu_2)$ and $\epsilon \in Z^2(A, \mathbb{Z}/2\mathbb{Z})$ represents the extension $\mathcal{Q}$. Since M/F is an unramified $\mathcal{Q}$-extension, $[\epsilon] = 0 \in H^2(\pi, \mathbb{Z}/2\mathbb{Z})$, where $\pi = \pi_1(\mathrm{Spec}(O_F), \mathfrak{b})$ as before. Thus, $[c_i] = 0 \in H^3(X, \mu_2)$ for all i. This implies that $CS_{c_i}([\rho]) = 0$ for all i, where $\rho \in \mathrm{Hom}(\pi, A)$ factors through

$$\pi \twoheadrightarrow \mathrm{Gal}(KF/F) \simeq A.$$

Take α_1 so that $F^{\alpha_1} = F(\sqrt{d_1})$. Since

$$\prod_{1\le i\le s}\left(\frac{-d_2\cdot\ell}{p_i}\right) \times \prod_{1\le j\le t}\left(\frac{d_1}{q_j}\right) = \prod_{1\le j\le t} B_j = \varDelta(d_2,d_1) = \varDelta(d_1,d_2) = -1$$

by assumption, we get

Table 1 Some biquadratic fields and quaternionic extensions

d_1	d_2	Δ	$\exists L$?	d_1	d_2	Δ	$\exists L$?	d_1	d_2	Δ	$\exists L$?
5	13	-1	No	13	17	1	Yes [44]	17	21	1	Yes [48]
5	17	-1	No	13	21	-1	No	17	29	-1	No
5	21	1	Yes [37]	13	29	1	Yes [45]	17	33	1	Yes [49]
5	29	1	Yes [41]	13	33	-1	No	17	37	-1	No
5	33	-1	No	13	37	-1	No	17	41	-1	No
5	37	-1	No	13	41	-1	No	17	53	1	Yes [50]
5	41	1	Yes [42]	13	53	1	Yes [46]	17	57	-1	No
5	53	-1	No	13	57	-1	No	17	61	-1	No
5	57	-1	No	13	61	1	Yes [47]	17	65	-1	No
5	61	1	Yes [43]	13	69	1	No	17	69	1	Yes [51]

$$CS_{c_1}([\rho]) = \frac{1}{2}$$

by Proposition 5.16, which is a contradiction. Thus, there cannot exist such L.

Remark 6.3 For the explicit construction of quaternion extensions L of $\mathbb{Q}$, see [9] or [28, Theorem 4.5].

In the LMFDB, you can search for $\mathcal{Q}$-extensions L over $\mathbb{Q}$ with odd discriminants. We make a table for readers, which verifies our theorem numerically. Here $\Delta = \Delta(d_1, d_2)$ (Table 1).

When $d_1 = 13$ and $d_2 = 3 \cdot 23 = 69$, there cannot exist such L even though $\Delta(d_1, d_2) = 1$. This follows from the following proposition which is already known to experts (e.g. [28]). For the sake of readers, we provide a complete proof as well.

Proposition 6.4 *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ as above. Let p be a prime divisor of d_i, which is congruent to* 3 *modulo* 4. *If $\left(\frac{d_{3-i}}{p}\right) = 1$, then there cannot exist a number field L such that* $\mathrm{Gal}(L/\mathbb{Q}) \simeq \mathcal{Q}$ *and $K \subset L$.*

Proof Let p be a prime divisor of d_2, which is congruent to 3 modulo 4. Suppose that $\left(\frac{d_1}{p}\right) = 1$ and there exists such a field L. Then by the same argument as in Proposition 5.15, the ramification index of p in $L/\mathbb{Q}$ is 4. Let $\mathcal{O} = \mathbb{Z}[\sqrt{d_1}]$ be the ring of integers of $\mathbb{Q}(\sqrt{d_1})$. Then, since $\left(\frac{d_1}{p}\right) = 1$, $p\mathcal{O} = \wp \cdot \wp'$ for two different maximal ideals $\wp$ and $\wp'$. Thus, $D(\wp) = I(\wp) \simeq \mathbb{Z}/4\mathbb{Z}$, where $D(\wp)$ (resp. $I(\wp)$) is the decomposition group (resp. inertia group) of $\wp$ in $\mathrm{Gal}(L/\mathbb{Q}) \simeq \mathcal{Q}$. Since $\mathcal{O}_\wp \simeq \mathbb{Z}_p$, the $D(\wp) = I(\wp) \simeq \mathbb{Z}/4\mathbb{Z}$ can be regarded as a quotient of $\mathbb{Z}_p^\times \simeq \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$. Because $p - 1 \equiv 2 \pmod 4$, this is a contradiction and hence the result follows.

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7 Appendix 1: Conjugation on Group Cochains

We compute cohomology of a topological group G with coefficients in a topological abelian group M with continuous G-action using the complex whose component of degree i is $C^i(G, M)$, the continuous maps from G^i to M. The differential

$$d : C^i(G, M) \to C^{i+1}(G, M)$$

is given by

$$df(g_1, g_2, \ldots, g_{i+1}) = g_1 f(g_2, \ldots, g_{i+1})$$

$$+ \sum_{k=1}^{i} f(g_1, \ldots, g_{k-1}, g_k g_{k+1}, g_{k+2}, \ldots, g_{i+1}) + (-1)^{i+1} f(g_1, g_2, \ldots, g_i).$$

We denote by

$$B^i(G, M) \subset Z^i(G, M) \subset C^i(G, M)$$

the images and the kernels of the differentials, the coboundaries and the cocycles, respectively. The cohomology is then defined as

$$H^i(G, M) := Z^i(G, M)/B^i(G, M).$$

There is a natural right action of G on the cochains given by

$$a : c \mapsto c^a := a^{-1} c \circ \mathrm{Ad}_a,$$

where Ad_a refers to the conjugation action of a on G^i.

Lemma 7.1 *The G action on cochains commutes with d:*

$$d(c^a) = (dc^a)$$

for all $a \in G$.

Proof If $c \in C^i(G, M)$, then

$$d(c^a)(g_1, g_2, \ldots, g_{i+1}) = g_1 a^{-1} c(\mathrm{Ad}_a(g_2), \ldots, \mathrm{Ad}_a(g_{i+1}))$$

$$+ \sum_{k=1}^{i} a^{-1} c(\mathrm{Ad}_a(g_1), \ldots, \mathrm{Ad}_a(g_{k-1}), \mathrm{Ad}_a(g_k)\mathrm{Ad}_a(g_{k+1}), \mathrm{Ad}_a(g_{k+2}), \ldots, \mathrm{Ad}_a(g_{i+1}))$$

$$+(-1)^{i+1} a^{-1} c(\mathrm{Ad}_a(g_1), \mathrm{Ad}_a(g_2), \ldots, \mathrm{Ad}_a(g_i))$$

$$= a^{-1}\mathrm{Ad}_a(g_1) c(\mathrm{Ad}_a(g_2), \ldots, \mathrm{Ad}_a(g_{i+1}))$$

$$+ \sum_{k=1}^{i} a^{-1} c(\mathrm{Ad}_a(g_1), \ldots, \mathrm{Ad}_a(g_{k-1}), \mathrm{Ad}_a(g_k)\mathrm{Ad}_a(g_{k+1}), \mathrm{Ad}_a(g_{k+2}), \ldots, \mathrm{Ad}_a(g_{i+1}))$$

$$+(-1)^{i+1} a^{-1} c(\mathrm{Ad}_a(g_1), \mathrm{Ad}_a(g_2), \ldots, \mathrm{Ad}_a(g_i))$$

$$= a^{-1}(dc)(\mathrm{Ad}_a(g_1), \mathrm{Ad}_a(g_2), \ldots, \mathrm{Ad}_a(g_{i+1}))$$

$$= (dc)^a(g_1, g_2, \ldots, g_{i+1}).$$

We also use the notation $(g_1, g_2, \ldots, g_i)^a := \mathrm{Ad}_a(g_1, g_2, \ldots, g_i)$. It is well-known that this action is trivial on cohomology. We wish to show the construction of explicit h_a with the property that

$$c^a = c + dh_a$$

for cocycles of degree 1, 2, and 3. The first two are relatively straightforward, but degree 3 is somewhat delicate. In degree 1, first note that $c(e) = c(ee) = c(e) + ec(e) = c(e) + c(e)$, so that $c(e) = 0$. Next, $0 = c(e) = c(gg^{-1}) = c(g) + gc(g^{-1})$, and hence, $c(g^{-1}) = -g^{-1}c(g)$. Therefore,

$$c(aga^{-1}) = c(a) + ac(ga^{-1}) = c(a) + ac(g) + agc(a^{-1}) = c(a) + ac(g) - aga^{-1}c(a).$$

From this, we get

$$c^a(g) = c(g) + a^{-1}c(a) - ga^{-1}c(a).$$

That is,

$$c^a = c + dh_a$$

for the zero cochain $h_a(g) = a^{-1}c(a)$.

Lemma 7.2 *For each $c \in Z^i(G, M)$ and $a \in G$, we can associate an*

$$h_a^{i-1}[c] \in C^{i-1}(G, M)/B^{i-1}(G, M)$$

in such a way that

$$(1)\quad c^a - c = dh_a^{i-1}[c];$$
$$(2)\quad h_{ab}^{i-1}[c] = (h_a^{i-1}[c])^b + h_b^{i-1}[c].$$

Proof This is clear for $i = 0$ and we have shown above the construction of $h_a^0[c]$ for $c \in Z^1(G, M)$ satisfying (1). Let us check the condition (2):

$$h_{ab}^0[c](g) = (ab)^{-1}c(ab)$$

$$= b^{-1}a^{-1}(c(a) + ac(b)) = b^{-1}h_a^0[c](\mathrm{Ad}_b(g)) + h_b^0[c](g) = (h_a^0[c])^b(g) + h_b^0[c](g).$$

We prove the statement using induction on i, which we now assume to be ≥ 2. For a module M, we have the exact sequence

$$0 \to M \to C^1(G, M) \to N \to 0,$$

where $C^1(G, M)$ has the right regular action of G and $N = C^1(G, M)/M$. Here, we give $C^1(G, M)$ the topology of pointwise convergence. There is a canonical linear splitting $s : N \to C^1(G, M)$ with image the group of functions f such that $f(e) = 0$, using which we topologise N. According to [24, Proof of 2.5], the G-module $C^1(G, M)$ is acyclic,[13] that is,

$$H^i(G, C^1(G, M)) = 0$$

for $i > 0$. Therefore, given a cocycle $c \in Z^i(G, M)$, there is an

$$F \in C^{i-1}(G, C^1(G, M))$$

such that its image $f \in C^{i-1}(G, N)$ is a cocycle and $dF = c$. Hence, $d(F^a - F) = c^a - c$. Also, by induction, there is a $k_a \in C^{i-2}(G, N)$ such that $f^a - f = dk_a$ and

[13]The notation there for $C^1(G, M)$ is $F_0^0(G, M)$. One difference is that Mostow uses the complex $E^*(G, M)$ of equivariant homogeneous cochains in the definition of cohomology. However, the isomorphism $E^n \to C^n$ that sends $f(g_0, g_1, \dots, g_n)$ to $f(1, g_1, g_1g_2, \dots, g_1g_2\cdots g_n)$ identifies the two definitions. This is the usual comparison map one uses for discrete groups, which clearly preserves continuity.

$k_{ab} = (k_a)^b + k_b + dl$ for some $l \in C^{i-3}(G, N)$ (zero if $i = 2$). Let $K_a = s \circ k_a$ and put

$$h_a = F^a - F - dK_a.$$

Then the image of h_a in N is zero, so h_a takes values in M, and $dh_a = c^a - c$. Now we check property (2). Note that

$$K_{ab} = s \circ k_{ab} = s \circ (k_a)^b + s \circ k_b + s \circ dl.$$

But $s \circ (k_a)^b - (s \circ k_a)^b$ and $s \circ dl - d(s \circ l)$ both have image in M. Hence, $K_{ab} = K_a^b + K_b + d(s \circ l) + m$ for some cochain $m \in C^{i-2}(G, M)$. From this, we deduce

$$dK_{ab} = (dK_a)^b + dK_b + dm,$$

from which we get

$$h_{ab} = F^{ab} - F - dK_{ab} = (F^a)^b - F^b + F^b - F - (dK_a)^b - dK_b - dm = (h_a)^b + h_b + dm.$$

8 Appendix 2: Conjugation Action on Group Cochains: Categorical Approach

In this section, an alternative and conceptual proof of Lemma 7.2 is outlined. Although not strictly necessary for the purposes of this paper, we believe that a functorial theory of secondary classes in group cohomology will be important in future developments. This point has also been emphasised to M.K. by Lawrence Breen. More details and elaborations will follow in a forthcoming publication by B.N.

8.1 Notation

In what follows G is a group and M is a left G-module. The action is denoted by ${}^a m$. The left conjugation action of $a \in G$ on G is denoted $\mathrm{Ad}_a(x) = axa^{-1}$. We have an induced right action on n-cochains $f\ G^n \to M$ given by

$$f^a(\mathbf{g}) := {}^{a^{-1}}(f(\mathrm{Ad}_a\,\mathbf{g})).$$

Here, $\mathbf{g} \in G^n$ is an n-chain, and $\mathrm{Ad}_a\,\mathbf{g}$ is defined componentwise.

In what follows, $[n]$ stands for the ordered set $\{0, 1, \ldots, n\}$, viewed as a category.

8.2 Idea

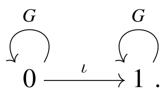
The above action on cochains respects the differential, hence passes to cohomology. It is well known that the induced action on cohomology is trivial. That is, given an n-cocycle f and any element $a \in G$, the difference $f^a - f$ is a coboundary. In this appendix we explain how to construct an $(n-1)$-cochain $h_{a,f}$ such that $d(h_{a,f}) = f^a - f$. The construction, presumably well known, uses standard ideas from simplicial homotopy theory [26, Sect. 1]. The general case of this construction, as well as the missing proofs of some of the statements in this appendix will appear in a separate article.

Let $\mathcal{G}$ denote the one-object category (in fact, groupoid) with morphisms G. For an element $a \in G$, we have an action of a on $\mathcal{G}$ which, by abuse of notation, we will denote again by $\mathrm{Ad}_a : \mathcal{G} \to \mathcal{G}$; it fixes the unique object and acts on morphisms by conjugation by a.

The main point in the construction of the cochain $h_{a,f}$ is that there is a "homotopy" (more precisely, a natural transformation) H_a from the identity functor id: $\mathcal{G} \to \mathcal{G}$ to $\mathrm{Ad}_a : \mathcal{G} \to \mathcal{G}$. The homotopy between id and Ad_a is given by the functor $H_a : \mathcal{G} \times [1] \to \mathcal{G}$ defined by

$$H_a|_0 = \mathrm{id}, \quad H_a|_1 = \mathrm{Ad}_a, \text{ and } H_a(\iota) = a^{-1}.$$

It is useful to visualise the category $\mathcal{G} \times [1]$ as



8.3 Cohomology of Categories

We will use multiplicative notation for morphisms in a category, namely, the composition of $g: x \to y$ with $h: y \to z$ is denoted $gh: x \to z$.

Let $\mathcal{C}$ be a small category and M a left $\mathcal{C}$-module, that is, a functor $M : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$, $x \mapsto M_x$, to the category of abelian groups (or your favorite linear category). Note that when $\mathcal{G}$ is as above, this is nothing but a left G-module in the usual sense. For an arrow $g: x \to y$ in $\mathcal{C}$, we denote the induced map $M_y \to M_x$ by $m \mapsto {}^g m$.

Let $\mathcal{C}^{[n]}$ denote the set of all n-tuples $\mathbf{g}$ of composable arrows in $\mathcal{C}$,

$$\mathbf{g} = \bullet \xrightarrow{g_1} \bullet \xrightarrow{g_2} \cdots \xrightarrow{g_n} \bullet.$$

We refer to such a $\mathbf{g}$ as an n-**cell** in $\mathcal{C}$; this is the same thing as a functor $[n] \to \mathcal{C}$, which we will denote, by abuse of notation, again by $\mathbf{g}$.

An n-**chain** in $\mathcal{C}$ is an element in the free abelian group $\mathrm{C}_n(\mathcal{C}, \mathbb{Z})$ generated by the set $\mathcal{C}^{[n]}$ of n-cells. For an n-cell $\mathbf{g}$ as above, we let $s\mathbf{g} \in \mathrm{Ob}\,\mathcal{C}$ denote the source of g_1.

By an n-**cochain** on $\mathcal{C}$ with values in M we mean a map f that assigns to any n-cell $\mathbf{g} \in \mathcal{C}^{[n]}$ an element in $M_{s\mathbf{g}}$. Note that, by linear extension, we can evaluate f on any n-chain in which all n-cells share a common source point.

The n-cochains form an abelian group $\mathrm{C}^n(\mathcal{C}, M)$. The **cohomology** groups $\mathrm{H}^n(\mathcal{C}, M)$, $n \geq 0$, are defined using the cohomology complex $\mathrm{C}^\bullet(\mathcal{C}, M)$:

$$0 \to \mathrm{C}^0(\mathcal{C}, M) \xrightarrow{d} \mathrm{C}^1(\mathcal{C}, M) \xrightarrow{d} \cdots \xrightarrow{d} \mathrm{C}^n(\mathcal{C}, M) \xrightarrow{d} \mathrm{C}^{n+1}(\mathcal{C}, M) \xrightarrow{d} \cdots$$

where the differential

$$d\colon \mathrm{C}^n(\mathcal{C}, M) \to \mathrm{C}^{n+1}(\mathcal{C}, M)$$

is defined by

$$df(g_1, g_2, \ldots, g_{n+1}) = {}^{g_1}(f(g_2, \ldots, g_{n+1})) + \sum_{1 \leq i \leq n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, g_2, \ldots, g_n).$$

A left G-module M in the usual sense gives rise to a left module on $\mathcal{G}$, which we denote again by M. We sometimes denote $\mathrm{C}^\bullet(\mathcal{G}, M)$ by $\mathrm{C}^\bullet(G, M)$. Note that the corresponding cohomology groups coincide with the group cohomology $\mathrm{H}^n(G, M)$.

The cohomology complex $\mathrm{C}^\bullet(\mathcal{C}, M)$ and the cohomology groups $\mathrm{H}^n(\mathcal{C}, M)$ are functorial in M. They are also functorial in $\mathcal{C}$ in the following sense. A functor $\varphi\colon \mathcal{D} \to \mathcal{C}$ gives rise to a $\mathcal{D}$-module $\varphi^* M := M \circ \varphi\; \mathcal{D}^{op} \to \mathbf{Ab}$. We have a map of complexes

$$\varphi^*\colon \mathrm{C}^\bullet(\mathcal{C}, M) \to \mathrm{C}^\bullet(\mathcal{D}, \varphi^* M), \tag{8.1}$$

which gives rise to the maps

$$\varphi^*\colon \mathrm{H}^n(\mathcal{C}, M) \to \mathrm{H}^n(\mathcal{D}, \varphi^* M)$$

on cohomology, for all $n \geq 0$.

8.4 Definition of the Cochains $h_{a,f}$

The flexibility we gain by working with chains on general categories allows us to import standard ideas from topology to this setting. The following definition of the cochains $h_{a,f}$ is an imitation of a well known construction in topology.

Let $f \in \mathrm{C}^{n+1}(G, M)$ be an $(n+1)$-cochain, and $a \in G$ an element. Let $H_a : \mathcal{G} \times [1] \to \mathcal{G}$ be the corresponding natural transformation. We define $h_{a,f} \in \mathrm{C}^n(G, M)$

by

$$h_{a,f}(\mathbf{g}) = f(H_a(\mathbf{g} \times [1])).$$

Here, $\mathbf{g} \in \mathcal{C}^{[n]}$ is an n-cell in $\mathcal{G}$, so $\mathbf{g} \times [1]$ is an $(n+1)$-chain in $\mathcal{G} \times [1]$, namely, the cylinder over $\mathbf{g}$.

To be more precise, we are using the notation $\mathbf{g} \times [1]$ for the image of the fundamental class of $[n] \times [1]$ in $\mathcal{G} \times [1]$ under the functor $\mathbf{g} \times [1]$ $[n] \times [1] \to \mathcal{G} \times [1]$. We visualize $[n] \times [1]$ as

$$\begin{array}{ccccccc} (0,1) & \longrightarrow & (1,1) & \longrightarrow & \cdots & \longrightarrow & (n,1) \\ \uparrow & & \uparrow & & & & \uparrow \\ (0,0) & \longrightarrow & (1,0) & \longrightarrow & \cdots & \longrightarrow & (n,0) \end{array}$$

Its fundamental class is the alternating sum of the $(n+1)$-cells

$$\begin{array}{ccccccc} & & & & (r,1) & \longrightarrow \cdots \longrightarrow & (n,1) \\ & & & & \uparrow & & \\ (0,0) & \longrightarrow & \cdots & \longrightarrow & (r,0) & & \end{array}$$

in $[n] \times [1]$, for $0 \le r \le n$. Therefore,

$$h_{a,f}(\mathbf{g}) = \sum_{0 \le r \le n} (-1)^r f(g_1, \ldots, g_r, a^{-1}, \mathrm{Ad}_a\, g_{r+1}, \ldots, \mathrm{Ad}_a\, g_n). \tag{8.2}$$

The following proposition can be proved using a variant of Stokes' formula for cochains.

Proposition 8.1 *The graded map* $h_{-,a} \colon \mathrm{C}^{\bullet+1}(G, M) \to \mathrm{C}^{\bullet}(G, M)$ *is a chain homotopy between the chain maps*

$$\mathrm{id}, (-)^a \colon \mathrm{C}^{\bullet}(G, M) \to \mathrm{C}^{\bullet}(G, M).$$

That is,

$$h_{a,df} + d(h_{a,f}) = f^a - f$$

for every $(n+1)$*-cochain* f*. In particular, if* f *is an* $(n+1)$*-cocycle, then* $d(h_{a,f}) = f^a - f$.

8.5 Composing Natural Transformations

Given an $(n+1)$-cochain f, and elements $a, b \in G$, we can construct three n-cochains: $h_{a,f}$, $h_{b,f}$ and $h_{ab,f}$. A natural question to ask is whether these three cochains satisfy a cocycle condition. It turns out that the answer is yes, but only up to a coboundary $dh_{a,b,f}$. Below we explain how $h_{a,b,f}$ is constructed. In fact, we construct cochains $h_{a_1,\ldots,a_k,f}$, for any k elements $a_i \in G$, $1 \le i \le k$, and study their relationship.

Let $f \in \mathrm{C}^{n+k}(G, M)$ be an $(n+k)$-cochain. Let $\mathbf{a} = (a_1, \ldots, a_k) \in G^{\times k}$. Consider the category $\mathcal{G} \times [k]$,

$$\overset{G}{\circlearrowleft} \qquad \overset{G}{\circlearrowleft} \qquad\qquad\qquad\qquad \overset{G}{\circlearrowleft}$$
$$0 \xrightarrow{\iota_0} 1 \xrightarrow{\iota_1} \cdots \xrightarrow{\iota_{k-1}} k.$$

Let $H_{\mathbf{a}} : \mathcal{G} \times [k] \to \mathcal{G}$ be the functor such that $\iota_i \mapsto a_{k-i}^{-1}$ and $H_{\mathbf{a}}|_{\{0\}} = \mathrm{id}_G$. (So, $H_{\mathbf{a}}|_{\{k-i\}} = \mathrm{Ad}_{a_{i+1}\cdots a_k}$.) Define $h_{\mathbf{a},f} \in \mathrm{C}^n(G, M)$ by

$$h_{\mathbf{a},f}(\mathbf{g}) = f(H_{\mathbf{a}}(\mathbf{g} \times [k])). \tag{8.3}$$

Here, $\mathbf{g} \in \mathcal{C}^{[n]}$ is an n-cell in $\mathcal{G}$, so $\mathbf{g} \times [k]$ is an $(n+k)$-chain in $\mathcal{G} \times [k]$.

To be more precise, we are using the notation $\mathbf{g} \times [k]$ for the image of the fundamental class of $[n] \times [k]$ in $\mathcal{G} \times [k]$ under the functor $\mathbf{g} \times [k]\ [n] \times [k] \to \mathcal{G} \times [k]$. We visualize $[n] \times [k]$ as

$$\begin{array}{ccccccc}
(0,k) & \to & (1,k) & \to & \cdots & \to & (n,k) \\
\uparrow & & \uparrow & & & & \uparrow \\
\vdots & & \vdots & & & & \vdots \\
\uparrow & & \uparrow & & & & \uparrow \\
(0,1) & \to & (1,1) & \to & \cdots & \to & (n,1) \\
\uparrow & & \uparrow & & & & \uparrow \\
(0,0) & \to & (1,0) & \to & \cdots & \to & (n,0)
\end{array}$$

Its fundamental class is the $(n+k)$-chain

$$\sum_P (-1)^{|P|} P,$$

where P runs over (length $n+k$) paths starting from $(0,0)$ and ending in (n,k). Note that such paths correspond to (k,n) shuffles; $|P|$ stands for the parity of the shuffle (which is the same as the number of squares above the path in the $n \times k$ grid).

The most economical way to describe the relations between various $h_{\mathbf{a},f}$ is in terms of the cohomology complex of the right module

$$\mathbb{M}^\bullet := \underline{\mathrm{Hom}}\,(\mathrm{C}^\bullet(G, M), \mathrm{C}^\bullet(G, M))\,.$$

Here, $\underline{\mathrm{Hom}}$ stands for the enriched hom in the category of chain complexes, and the right action of G on $\mathbb{M}^\bullet$ is induced from the right action $f \mapsto f^a$ of G on the $\mathrm{C}^\bullet(G, M)$ sitting on the right. The differential on $\mathbb{M}^\bullet$ is defined by

$$d_{\mathbb{M}^\bullet}(u) = (-1)^{|u|} u \circ d_{\mathrm{C}^\bullet(G,M)} - d_{\mathrm{C}^\bullet(G,M)} \circ u,$$

where $|u|$ is the degree of the homogeneous $u \in \mathrm{C}^\bullet(G, M)$.

Note that, for every $\mathbf{a} \in G^{\times k}$, we have $h_{\mathbf{a},f} \in \mathbb{M}^{-k}$. This defines a k-cochain on G of degree $-k$ with values in $\mathbb{M}^\bullet$,

$$h^{(k)} \colon \mathbf{a} \mapsto h_{\mathbf{a},-}, \ \mathbf{a} \in G^{\times k}.$$

We set $h^{(-1)} := 0$. Note that $h^{(0)}$ is the element in $\mathbb{M}^0$ corresponding to the identity map id: $\mathrm{C}^\bullet(G, M) \to \mathrm{C}^\bullet(G, M)$.

The relations between various $h_{\mathbf{a},f}$ can be packaged in a simple differential relation. As in the case $k = 0$ discussed in Proposition 8.1, this proposition can be proved using a variant of Stokes' formula for cochains.

Proposition 8.2 *For every $k \geq -1$, we have $d_{\mathbb{M}^\bullet}(h^{(k+1)}) = d(h^{(k)})$.*

In the above formula, the term $d_{\mathbb{M}^\bullet}(h^{(k+1)})$ means that we apply $d_{\mathbb{M}^\bullet}$ to the values (in $\mathbb{M}^\bullet$) of the cochain $h^{(k+1)}$. The differential on the right hand side of the formula is the differential of the cohomology complex $\mathrm{C}^\bullet(G, \mathbb{M}^\bullet)$ of the (graded) right G-module $\mathbb{M}^\bullet$.

More explicitly, let $f \in \mathrm{C}^{n+k}(G, M)$ be an $(n + k)$-cochain. Then, Proposition 8.2 states that, for every $\mathbf{a} \in G^{\times(k+1)}$, we have the following equality of n-cochains:

$$\begin{aligned}(-1)^{(k+1)} h_{a_1,\ldots,a_{k+1},df} - d h_{a_1,\ldots,a_{k+1},f} = \quad & h_{a_2,\ldots,a_{k+1},f} + \\ & \sum_{1 \leq i \leq k} (-1)^i h_{a_1,\ldots,a_i a_{i+1},\ldots,a_{k+1},f} + \\ & (-1)^{k+1} h^{a_{k+1}}_{a_1,\ldots,a_k,f}.\end{aligned}$$

Corollary 8.3 *Let $f \in \mathrm{C}^{n+k}(G, M)$ be an $(n + k)$-cocycle. Then, for every $\mathbf{a} \in G^{\times(k+1)}$, the n-cochain*

$$h_{a_2,\ldots,a_{k+1},f} + \sum_{1 \leq i \leq k} (-1)^i h_{a_1,\ldots,a_i a_{i+1},\ldots,a_{k+1},f} + (-1)^{k+1} h^{a_{k+1}}_{a_1,\ldots,a_k,f}$$

is a coboundary. In fact, it is the coboundary of $-h_{a_1,\ldots,a_{k+1},f}$.

Example 8.4 Let us examine Corollary 8.3 for small values of k.

(i) For $k = 0$, the statement is that, for every cocycle f, $f - f^a$ is a coboundary. In fact, it is the coboundary of $-h_{f,a}$. We have already seen this in Proposition 8.1.

(ii) For $k = 1$, the statement is that, for every cocycle f, the cochain

$$h_{b,f} - h_{ab,f} + h^b_{a,f}$$

is a coboundary. In fact, it is the coboundary of $-h_{a,b,f}$.

8.6 Explicit Formula for $h_{a_1,\ldots,a_k,f}$

Let $f\colon G^{\times(n+k)} \to M$ be an $(n+k)$-cochain, and $\mathbf{a} := (a_1, a_2, \ldots, a_k) \in G^{\times k}$. Then, by Eq. (8.3), the effect of the n-cochain $h_{a_1,\ldots,a_k,f}$ on an n-tuple $\mathbf{x} := (x_0, x_1, \ldots, x_{n-1}) \in G^{\times n}$ is given by:

$$h_{a_1,\ldots,a_k,f}(x_0, x_1, \ldots, x_{n-1}) = \sum_P (-1)^{|P|} f(\mathbf{x}^P),$$

where $\mathbf{x}^P$ is the $(n+k)$-tuple obtained by the following procedure.

Recall that P is a path from (0, 0) to (n, k) in the n by k grid. The l^{th} component $\mathbf{x}^P_l$ of $\mathbf{x}^P$ is determined by the l^{th} segment on the path P. Namely, suppose that the coordinates of the starting point of this segment are (s, t). Then,

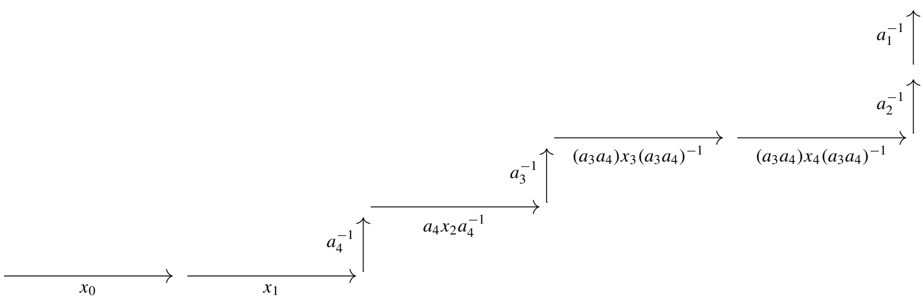
$$\mathbf{x}^P_l = a^{-1}_{k-t}$$

if the segment is vertical, and

$$\mathbf{x}^P_l = (a_{k-t+1} \cdots a_k)x_s(a_{k-t+1} \cdots a_k)^{-1},$$

if the segment is horizontal. Here, we use the convention that $a_0 = 1$.

The following example helps visualize $\mathbf{x}^P$:



The corresponding term is

$$-f(x_0, x_1, a_4^{-1}, a_4x_2a_4^{-1}, a_3^{-1}, (a_3a_4)x_3(a_3a_4)^{-1}, (a_3a_4)x_4(a_3a_4)^{-1}, a_2^{-1}, a_1^{-1}).$$

The sign of the path is determined by the parity of the number of squares in the n by k grid that sit above the path P (in this case 15).

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Some Ring-Theoretic Properties of $\mathbf{A}_{\mathrm{inf}}$

Kiran S. Kedlaya

Abstract The ring of Witt vectors over a perfect valuation ring of characteristic p, often denoted $\mathbf{A}_{\mathrm{inf}}$, plays a pivotal role in p-adic Hodge theory; for instance, Bhatt–Morrow–Scholze have recently reinterpreted and refined the crystalline comparison isomorphism by relating it to a certain $\mathbf{A}_{\mathrm{inf}}$-valued cohomology theory. We address some basic ring-theoretic questions about $\mathbf{A}_{\mathrm{inf}}$, motivated by analogies with two-dimensional regular local rings. For example, we show that in most cases $\mathbf{A}_{\mathrm{inf}}$, which is manifestly not noetherian, is also not coherent. On the other hand, it does have the property that vector bundles over the complement of the closed point in Spec $\mathbf{A}_{\mathrm{inf}}$ do extend uniquely over the puncture; moreover, a similar statement holds in Huber's category of adic spaces.

Keywords Witt vectors · Perfectoid rings

Throughout this paper, let K be a perfect field of characteristic p equipped with a nontrivial valuation v (written additively), e.g., the perfect closure of $\mathbb{F}_p((t))$ with the t-adic valuation. (Note that $K = \mathbb{F}_p$ is excluded by the nontriviality condition.) Unless otherwise specified, we do not assume that K is complete.

A fundamental role is played in p-adic Hodge theory by the ring $\mathbf{A}_{\mathrm{inf}} := W(\mathfrak{o}_K)$, where $\mathfrak{o}_K$ denotes the valuation ring of K and W denotes the functor of p-typical Witt vectors. The ring $\mathbf{A}_{\mathrm{inf}}$ serves as the basis for Fontaine's construction of p-adic period rings and the ensuing analysis of comparison isomorphisms. Recently, Fargues has used $\mathbf{A}_{\mathrm{inf}}$ to give a new description of crystalline representations via a variant of Breuil–Kisin modules [6], while Bhatt–Morrow–Scholze have described the crystalline comparison isomorphism via a direct construction of these modules [3].

We discuss several issues germane to [3] regarding ring-theoretic propositionerties of $\mathbf{A}_{\mathrm{inf}}$, particularly those related to the analogy between $\mathbf{A}_{\mathrm{inf}}$ and two-dimensional regular local rings. In the negative direction, the ring $\mathbf{A}_{\mathrm{inf}}$ is typically not coherent (Theorem 1.2); in the positive direction, vector bundles over the complement of the

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closed point in Spec($\mathbf{A}_{\inf}$) extend over the puncture (Theorem 2.7), and similarly if the Zariski spectrum is replaced by the Huber adic spectrum (Theorem 3.9).

We also discuss briefly some related questions in the case where K is replaced by a more general nonarchimedean Banach ring. These are expected to pertain to a hypothetical relative version of the results of [3].

1 Finite Generation Properties

Definition 1.1 A ring is *coherent* if every finitely generated ideal is finitely presented. Note that an integral domain is coherent if and only if the intersection of any two finitely generated ideals is again finitely generated [5].

A result of Anderson–Watkins [1], building on work of Jøndrup–Small [11] and Vasconcelos [19] (see also [8, Theorem 8.1.9]), asserts that a power series ring over a nondiscrete valuation ring can never be coherent except possibly if the value group is isomorphic to $\mathbb{R}$. Using a similar technique, we have the following.

Theorem 1.2 *Suppose that the value group of K is not isomorphic to $\mathbb{R}$. Then $\mathbf{A}_{\inf}$ is not coherent.*

Proof It suffices to exhibit elements $f, g \in \mathbf{A}_{\inf}$ such that $(f) \cap (g)$ is not finitely generated. Suppose first that the value group of K is archimedean, i.e., the valuation v can be taken to have values in $\mathbb{R}$. Since K is perfect, its value group cannot be discrete, and hence must be dense in $\mathbb{R}$. We can thus choose elements $\overline{x}_0, \overline{x}_1, \ldots \in \mathfrak{o}_K$ such that $v(\overline{x}_0), v(\overline{x}_1), \ldots$ is a decreasing sequence with positive limit $r \notin v(\mathfrak{o}_K)$ and $v(\overline{x}_0/\overline{x}_1) > v(\overline{x}_1/\overline{x}_2) > \cdots$. Put $f := [\overline{x}_0]$ and $g := \sum_{n=0}^{\infty} p^n[\overline{x}_n]$.

Recall that the ring $\mathbf{A}_{\inf}$ admits a theory of *Newton polygons* analogous to the corresponding theory for polynomials or power series over a valuation ring; see [13, Definition 4.2.8] for details. To form the Newton polygon of g, we take the lower convex hull of the set $\{(n, v(\overline{x}_n)) : n = 0, 1, \ldots\}$ in $\mathbb{R}^2$; the slopes of this polygon are equal to $-v(\overline{x}_n/\overline{x}_{n+1})$ for $n = 0, 1, \ldots$. If $h = \sum_{n=0}^{\infty} p^n[\overline{h}_n] \in \mathbf{A}_{\inf}$ is divisible by both f and g, then on one hand, we have $h/f = \sum_{n=0}^{\infty} p^n[\overline{h}_n/\overline{x}_0]$, so $v(\overline{h}_n) \geq v(\overline{x}_0)$ for all n; on the other hand, the Newton polygon of h must include all of the slopes of the Newton polygon of g, so its total width must be at least r. It follows that $v(\overline{h}_0) \geq 2v(\overline{x}_0) - r$.

Conversely, any $\overline{h}_0 \in \mathfrak{o}_K$ with $v(\overline{h}_0) \geq 2v(\overline{x}_0) - r$ extends to some $h \in \mathbf{A}_{\inf}$ divisible by both f and g, e.g., by taking $h = g[\overline{h}_0]/[\overline{x}_0]$. Since $2v(\overline{x}_0) - r \notin v(\mathfrak{o}_K)$, it follows that the image of $(f) \cap (g)$ in $\mathfrak{o}_K$ is an ideal which is not finitely generated; consequently, $(f) \cap (g)$ itself cannot be finitely generated.

Suppose next that the value group of K is not archimedean. We can then choose some nonzero $\overline{x}, \overline{y} \in \mathfrak{o}_K$ such that for every positive integer n, $\overline{x}$ is divisible by $\overline{y}^n$ in $\mathfrak{o}_K$. Let $r_1, r_2, \ldots$ be a decreasing sequence of elements of $\mathbb{Z}[p^{-1}]_{>0}$ whose sum diverges. Put $f := [\overline{x}]$ and $g := \sum_{n=0}^{\infty} p^n[\overline{x}/\overline{y}^{r_1+\cdots+r_n}]$. As above, we see that if $h = \sum_{n=0}^{\infty} p^n[\overline{h}_n] \in \mathbf{A}_{\inf}$ is divisible by both f and g, then on one hand, we have

$v(\overline{h}_n) \geq v(\overline{x})$ for each n; on the other hand, the Newton polygon of h includes all of the slopes of the Newton polygon of g, so its total width must exceed $r_1 + \cdots + r_n$ for each n. It follows that $v(\overline{h}_0) \geq v(\overline{x}) + nv(\overline{y})$ for every positive integer n; conversely, any $\overline{h}_0$ with this property occurs this way for $h = g[\overline{h}_0]/[\overline{x}]$. Again, this means that $(f) \cap (g)$ maps to an ideal of $\mathfrak{o}_K$ which is not finitely generated, so $(f) \cap (g)$ cannot itself be finitely generated. □

Remark 1.3 It is unclear whether the ring $\mathbf{A}_{\inf}$ fails to be coherent even if the value group of K equals $\mathbb{R}$, especially if we also assume that K is spherically complete. It is also unclear whether the ring $\mathbf{A}_{\inf}[p^{-1}]$ is coherent. By contrast, with no restrictions on K, for every positive integer n the quotient $\mathbf{A}_{\inf}/(p^n)$ is coherent [3, proposition 3.24].

Remark 1.4 Let $\mathfrak{m}_K$ be the maximal ideal of K. In order to apply the formalism of almost ring theory (e.g., as developed in [7]) to the ring $\mathbf{A}_{\inf}$, it would be useful to know that the ideal $W(\mathfrak{m}_K)$ of $\mathbf{A}_{\inf}$ has the property that $W(\mathfrak{m}_K) \otimes_{\mathbf{A}_{\inf}} W(\mathfrak{m}_K) \to W(\mathfrak{m}_K)$ is an isomorphism. We do not know whether this holds in general; for example, to prove that this map fails to be surjective, one would have to produce an element of $W(\mathfrak{m}_K)$ which cannot be written as a finite sum of pairwise products, and we do not have a mechanism in mind for precluding the existence of such a presentation. An easier task is to produce elements of $W(\mathfrak{m}_K)$ not lying in the image of the multiplication map $W(\mathfrak{m}_K) \times W(\mathfrak{m}_K) \to W(\mathfrak{m}_K)$, as in the following example communicated to us by Peter Scholze.

Example 1.5 Suppose that $v(K^\times) = \mathbb{Q}$. We first construct a sequence $r_1, r_2, \ldots$ of positive elements of $\mathbb{Q}$ with sum 1 such that every infinite subsequence with infinite complement has irrational sum. To this end, take a sequence $1 = s_0, s_1, s_2, \ldots$ converging to 0 sufficiently rapidly (e.g., doubly exponentially) and put $r_1 = s_0 - s_1, r_2 = s_1 - s_2, \ldots$; any infinite subsequence with infinite complement can be regrouped into sums of consecutive terms, yielding another infinite sequence with rapid decay, and Liouville's criterion implies that the sum of the subsequence is irrational (and even transcendental).

Now choose $x = \sum_{n=0}^{\infty} p^n[\overline{x}_n] \in W(\mathfrak{m}_K)$ with $v(\overline{x}_n) = s_n$; we check that $x \neq yz$ for all $y, z \in W(\mathfrak{m}_K)$. If the equality $x = yz$ were to hold, the Newton polygons of y and z together would comprise the Newton polygon of x; that is, each slope occurs in xy with multiplicity equal to the sum of its multiplicities in the Newton polygons of x and y. Due to the irrationality statement of the previous paragraph, this is impossible if both y and z have infinitely many slopes; consequently, one of the factors, say y, has only finitely many slopes in its Newton polygon. On the other hand, if $y = \sum_{n=0}^{\infty} p^n[\overline{y}_n]$, there cannot exist $c > 0$ such that $v(\overline{y}_n) \geq c$ for all n, as otherwise we would also have $v(\overline{x}_n) \geq c$ for all n. Putting these two facts together, we deduce that $v(\overline{y}_n) = 0$ for some n, a contradiction.

The following related remark was suggested by Bhargav Bhatt.

Remark 1.6 Suppose that the value group of K is archimedean. Consider the following chain of strict inclusions of ideals:

$$0 \subset \bigcup_{\varpi \in \mathfrak{m}_K} [\varpi]\mathbf{A}_{\text{inf}} \subset W(\mathfrak{m}_K) \subset (p) + W(\mathfrak{m}_K)$$

The quotients by the ideals $W(\mathfrak{m}_K)$ and $(p) + W(\mathfrak{m}_K)$ are the integral domains $W(\kappa)$ and κ, where $\kappa := \mathfrak{o}_K/\mathfrak{m}_K$ is the residue field of K; hence these two ideals are prime. The ideal $\bigcup_{\varpi \in \mathfrak{m}_K} [\varpi]\mathbf{A}_{\text{inf}}$ is also prime: it contains $x = \sum_{n=0}^{\infty} p^n[\overline{x}_n]$ if and only if the total multiplicity of all slopes in the Newton polygon of x is strictly less than $v(\overline{x}_0)$.

The previous argument shows that the global (Krull) dimension of $\mathbf{A}_{\text{inf}}$ is at least 3. In fact, one can push this further: by adapting a construction of Arnold [2] that produces arbitrary long chains of prime ideals within the ring of formal power series over a nondiscrete valuation ring, Lang–Ludwig [15] have shown that $\mathbf{A}_{\text{inf}}$ has infinite Krull dimension.[1]

2 Vector Bundles

Recall that for A a two-dimensional regular local ring, the restriction functor from vector bundles on Spec A (i.e., finite free A-modules) to vector bundles on the complement of the closed point is an equivalence of categories. This is usually shown by using the fact that a reflexive module has depth at least 2 [18, Tag 0AVA] in conjunction with the Auslander–Buchsbaum formula [18, Tag 090U] to see that every reflexive A-module is projective.

During the course of Scholze's 2014 Berkeley lectures documented in [17], we explained to him an alternate proof applicable to the case of $\mathbf{A}_{\text{inf}}$; this argument appears as [17, Theorem 14.2.1], and a similar argument is given in [3, Lemma 4.6]. Here, we give a general version of this proof applicable in a variety of cases, which identifies the most essential hypotheses on the ring A.

Hypothesis 2.1 *Throughout Sect. 2, let A be a local ring whose maximal ideal $\mathfrak{p}$ contains a non-zero-divisor π such that $\mathfrak{o} := A/(\pi)$ is (reduced and) a valuation ring with maximal ideal $\mathfrak{m}$. Put $L :=$ Frac $\mathfrak{o}$; in the case $A = \mathbf{A}_{\text{inf}}$ we have $L = K$.*

Definition 2.2 Put $X := \text{Spec}(A)$, $Y := X \setminus \{\mathfrak{p}\}$, and $U := \text{Spec}(A[\pi^{-1}]) \subset X$. Let B be the π-adic completion of $A_{(\pi)}$; note that within $B[\pi^{-1}]$ we have

$$A[\pi^{-1}] \cap B = A. \tag{2.2.1}$$

Let Z be the algebraic stack which is the colimit of the diagram

$$\text{Spec}(A[\pi^{-1}]) \leftarrow \text{Spec}(B[\pi^{-1}]) \rightarrow \text{Spec}(B).$$

[1] Heng Du has subsequently shown that the Krull dimension is at least the cardinality of the continuum. See arXiv:2002.10358.

Lemma 2.3 *For $* \in \{X, Y, Z\}$, let $\mathbf{Vec}_*$ denote the category of vector bundles on $*$.*

(a) The pullback functor $\mathbf{Vec}_X \to \mathbf{Vec}_Y$ is fully faithful.
(b) The pullback functor $\mathbf{Vec}_Y \to \mathbf{Vec}_Z$ is fully faithful.
(c) For $\mathcal{F} \in \mathbf{Vec}_*$ and $M := H^0(*, \mathcal{F})$, the adjunction morphism $\tilde{M}|_* \to \mathcal{F}$ is an isomorphism.

Proof For convenience, we write $\mathcal{O}$ instead of $\mathcal{O}_*$ hereafter. To deduce (a), note that by (2.2.1),

$$H^0(X, \mathcal{O}) = H^0(Y, \mathcal{O}) = H^0(Z, \mathcal{O}) = A.$$

To deduce (b), choose $z \in A$ whose image in $A/(\pi)$ is a nonzero element of $\mathfrak{m}$, so that

$$\operatorname{Spec}(A) = U \cup V, \qquad V := \operatorname{Spec}(A[z^{-1}]);$$

then note that z is invertible in B, and within $B[\pi^{-1}]$ we have

$$A[z^{-1}, \pi^{-1}] \cap B = A[z^{-1}].$$

To deduce (c), note that in case $* = Y$, the injectivity of the maps

$$H^0(U, \mathcal{O}) \to H^0(U \cap V, \mathcal{O}), \qquad H^0(V, \mathcal{O}) \to H^0(U \cap V, \mathcal{O})$$

implies the injectivity of the maps

$$H^0(U, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}), \qquad H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F})$$

and hence the injectivity of the maps

$$M \to H^0(U, \mathcal{F}), \qquad M \to H^0(V, \mathcal{F}).$$

It follows easily that the maps

$$M \otimes_R H^0(U, \mathcal{O}) \to H^0(U, \mathcal{F}), \qquad M \otimes_R H^0(V, \mathcal{O}) \to H^0(V, \mathcal{F})$$

are isomorphisms. The case $* = Z$ is similar. □

The following lemma is taken from [17, Lemma 14.2.3].

Lemma 2.4 *Let κ be the residue field of A, which is also the residue field of $\mathfrak{o}$. Let d be a nonnegative integer. Let N be an $\mathfrak{o}$-submodule of L^d. Then $\dim_\kappa(N \otimes_{\mathfrak{o}} \kappa) \leq d$, with equality if and only if N is a free module of rank d.*

Proof By induction on d, we reduce to the case $d = 1$. We then see that $\dim_\kappa(N \otimes_{\mathfrak{o}} \kappa)$ equals 1 if the set of valuations of elements of N has a least element, in which case N is free of rank 1, and 0 otherwise. □

Lemma 2.5 *For $\mathcal{F} \in \mathbf{Vec}_Z$ of rank d, if the elements $\mathbf{v}_1, \dots, \mathbf{v}_d \in H^0(Z, \mathcal{F})$ generate both $H^0(U, \mathcal{F})$ and $H^0(\mathrm{Spec}(L), \mathcal{F})$, then they also generate $M := H^0(Z, \mathcal{F})$.*

Proof Choose any $\mathbf{v} \in M$. Since $\mathbf{v}_1, \dots, \mathbf{v}_d$ generate $H^0(U, \mathcal{F})$, there exists a unique tuple $(r_1, \dots, r_d)$ over $A[\pi^{-1}]$ such that $\mathbf{v} = \sum_{i=1}^d r_i \mathbf{v}_i$. In particular, there exists a nonnegative integer m such that $\pi^m r_1, \dots, \pi^m r_d \in A$. If $m > 0$, then $\pi^m \mathbf{v}$ is divisible by π in M, so it maps to zero in $H^0(\mathrm{Spec}(L), \mathcal{F})$. Since $\mathbf{v}_1, \dots, \mathbf{v}_d$ form a basis of this module, $\pi^m r_1, \dots, \pi^m r_d$ must be divisible by π in A and so $\pi^{m-1} r_1, \dots, \pi^{m-1} r_d \in A$. By induction, we deduce that $r_1, \dots, r_d \in A$. This proves the claim. □

Lemma 2.6 *For $\mathcal{F} \in \mathbf{Vec}_Z$ of rank d, the module $M := H^0(Z, \mathcal{F})$ is free of rank d over A.*

Proof By Lemma 2.3(c), $M[\pi^{-1}] = H^0(U, \mathcal{F})$ is a projective $A[\pi^{-1}]$-module of rank d, so we can find a finite free $A[\pi^{-1}]$-module F and an isomorphism $F \cong M[\pi^{-1}] \oplus P$ for some finite projective $A[\pi^{-1}]$-module P. By rescaling by a suitably large power of π, we may exhibit a basis of F consisting of elements whose projections to $M[\pi^{-1}]$ all belong to M. This basis then gives rise to an isomorphism $F \cong F_0[\pi^{-1}]$ for F_0 the finite free A-module on the same basis. View

$$\mathrm{Gr}\, M[\pi^{-1}] := \bigoplus_{n \in \mathbb{Z}} (M[\pi^{-1}] \cap \pi^n F_0)/(M[\pi^{-1}] \cap \pi^{n+1} F_0)$$

as a finite projective graded module of rank d over the graded ring

$$\mathrm{Gr}\, A[\pi^{-1}] := \bigoplus_{n \in \mathbb{Z}} \pi^n A/\pi^{n+1} A \cong \mathfrak{o}((\overline{\pi})),$$

then put

$$V := (\mathrm{Gr}\, M[\pi^{-1}]) \otimes_{\mathfrak{o}((\overline{\pi}))} \kappa((\overline{\pi})).$$

Note that for the π-adic topology, the image of M in $\mathrm{Gr}\, M[\pi^{-1}]$ is both open (because M contains a set of module generators of $M[\pi^{-1}]$) and bounded (because the same holds for the dual bundle). Consequently, the image T of M in V is a $\kappa[[\overline{\pi}]]$-sublattice of V. Choose $\mathbf{v}_1, \dots, \mathbf{v}_d \in M$ whose images in V form a basis of T; the images of $\mathbf{v}_1, \dots, \mathbf{v}_d$ in $M \otimes_A \kappa$ are linearly independent, so by Lemma 2.4, $\mathbf{v}_1, \dots, \mathbf{v}_d$ project to a basis of $M \otimes_A \mathfrak{o}$. It follows that $\mathbf{v}_1, \dots, \mathbf{v}_d$ also project to a basis of $M \otimes_A A/(\pi^n)$ for each positive integer n.

Again by considering the dual bundle, we see that the image of F_0 in $M[\pi^{-1}]$ contains $\pi^n M$ for any sufficiently large integer n. Let $\mathbf{e}_1, \dots, \mathbf{e}_m$ be the images in M of the chosen basis of F_0; using the previous paragraph, we can find elements $\mathbf{e}'_1, \dots, \mathbf{e}'_m \in A\mathbf{v}_1 + \cdots + A\mathbf{v}_d$ such that $\mathbf{e}'_j = \sum_i X_{ij} \mathbf{e}_i$ for some matrix X over A with $\det(X) - 1 \in \pi A \subset \mathfrak{p}$. The matrix X is then invertible, whence $\mathbf{v}_1, \dots, \mathbf{v}_d$ generate $M[\pi^{-1}]$. By Lemma 2.5, $\mathbf{v}_1, \dots, \mathbf{v}_d$ generate M, necessarily freely. □

Theorem 2.7 *The pullback functors $\mathbf{Vec}_X \to \mathbf{Vec}_Y \to \mathbf{Vec}_Z$ are equivalences of categories.*

Proof By Lemma 2.3(a), the functors $\mathbf{Vec}_X \to \mathbf{Vec}_Y \to \mathbf{Vec}_Z$ are fully faithful, so it suffices to check that $\mathbf{Vec}_X \to \mathbf{Vec}_Z$ is essentially surjective. For $\mathcal{F} \in \mathbf{Vec}_Z$, by Lemma 2.6, $M = H^0(Z, \mathcal{F})$ is a finite free A-module. By Lemma 2.3(c), we have $\tilde{M}|_Z \cong \mathcal{F}$, proving the claim. □

3 Adic Glueing

We next show that vector bundles on Spec $\mathbf{A}_{\text{inf}}$ can be constructed by glueing not just for a Zariski covering, but for a covering in the setting of adic spaces; this result is used in [17] as part of the construction of mixed-characteristic local shtukas. In the process, we prove a somewhat more general result. Along the way, we will use results of Buzzard–Verberkmoes [4], Mihara [16], and Kedlaya–Liu [13].

We begin by summarizing various definitions from Huber's theory of adic spaces, as described in [10]. See also [12, Lecture 1].

Definition 3.1 We say that a topological ring A is *f-adic* if there exists an open subring A_0 of A (called a *ring of definition*) whose induced topology is the adic topology for some finitely generated ideal of A_0 (called an *ideal of definition*). Such a ring is *Tate* if it contains a topologically nilpotent unit; in certain cases (as in [12, Lecture 1]), one may prefer to instead assume only that the topologically nilpotent elements generate the unit ideal, but we will not do this here.

We will only need to consider f-adic rings which are complete for their topologies, which we refer to as *Huber rings*. Beware that this definition is not entirely standard: some authors use the term *Huber ring* as a synonym for *f-adic ring* without the completeness condition.

For A a Huber ring, let A° denote the subring of power-bounded elements of A; we say that A is *uniform* if A° is bounded in A. (This implies that A is reduced, but not conversely.) A *ring of integral elements* of A is a subring of A° which is open and integrally closed in A.

A *Huber pair* is a pair (A, A^+) in which A is a Huber ring and A^+ is a ring of integral elements of A. To such a pair, we may associate the topological space $\text{Spa}(A, A^+)$ of equivalence classes of continuous valuations on A which are bounded by 1 on A^+. This space may be topologized in such a way that a neighborhood basis is given by subspaces of the form

$$\{v \in \text{Spa}(A, A^+) : v(f_1), \dots, v(f_n) \le v(g) \neq 0\}$$

for some $f_1, \dots, f_n, g \in A$ which generate an open ideal; such spaces are called *rational subspaces* of $\text{Spa}(A, A^+)$. (When A is Tate, every open ideal of A is the unit ideal, and so the condition $v(g) \neq 0$ becomes superfluous.) For this topology, $\text{Spa}(A, A^+)$ is quasicompact and even a *spectral space* in the sense of Hochster [9].

In addition, Huber defines a *structure presheaf* $\mathcal{O}$ on $\text{Spa}(A, A^+)$; in the case where A is Tate and U is the rational subspace defined by some parameters

$f_1, \dots, f_n, g$, the ring $\mathcal{O}(U)$ may be identified with the quotient $A\left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle$ of the Tate algebra $A\langle T_1, \dots, T_n\rangle$ by the closure of the ideal $(gT_1 - f_1, \dots, gT_n - f_n)$.

We say that A is *sheafy* if $\mathcal{O}$ is a sheaf for some choice of A^+; with a bit of work [12, Remark 1.6.9], the same is then true for any A^+. For example, by Proposition 3.3 below, this holds if A is *stably uniform*, meaning that (again for some, and hence any, choice of A^+) for every rational subspace U of $\mathrm{Spa}(A, A^+)$, the ring $\mathcal{O}(U)$ is uniform.

Proposition 3.2 *Let (A, A^+) be a Huber pair with A Tate.*

(a) Choose $f \in A$ and suppose that

$$0 \longrightarrow A \longrightarrow A\langle f\rangle \oplus A\left\langle f^{-1}\right\rangle \stackrel{(x,y)\mapsto x-y}{\longrightarrow} A\left\langle f^{\pm 1}\right\rangle \dashrightarrow 0$$

is exact without the dashed arrow. (It is then also exact with the dashed arrow; e.g., see [12, Lemma 1.8.1].) Then the functor

$$\mathbf{Vec}_{\mathrm{Spec}(A)} \to \mathbf{Vec}_{\mathrm{Spec}(A\langle f\rangle)} \times_{\mathbf{Vec}_{\mathrm{Spec}(A\langle f^{\pm 1}\rangle)}} \mathbf{Vec}_{\mathrm{Spec}(A\langle f^{-1}\rangle)}$$

is an equivalence of categories.

(b) The conclusion of (a) holds whenever A is (Tate and) uniform.

(c) If A is (Tate and) sheafy, then the pullback functor $\mathbf{Vec}_{\mathrm{Spec}(A)} \to \mathbf{Vec}_{\mathrm{Spa}(A,A^+)}$ is an equivalence of categories, with quasi-inverse given by the global sections functor.

Proof For (a), see [12, Lemma 1.9.12]. For (b), see [13, Corollary 2.8.9] or [12, Lemma 1.7.3, Lemma 1.8.1]. For (c), see [13, Theorem 2.7.7] or [12, Theorem 1.4.2]. □

Using Proposition 3.2(a,b), one can deduce the following. However, we give references in lieu of a detailed argument.

Proposition 3.3 *(Buzzard–Verberkmoes, Mihara) Any stably uniform Huber ring is sheafy.*

Proof The original (independent) references are [4, Theorem 7] and [16, Theorem 4.9]. See also [13, Theorem 2.8.10] or [12, Theorem 1.2.13]. □

With these results in mind, we set some more specific notation.

Hypothesis 3.4 *For the remainder of §3, let R be a Huber ring which is perfect of characteristic p and Tate, and let R^+ be a subring of integral elements in R (which is necessarily also perfect). For example, we may take $R = K$, $R^+ = \mathfrak{o}_K$ in case K is complete for a rank 1 valuation. Let $\overline{x} \in R$ be a topologically nilpotent unit; note that necessarily $\overline{x} \in R^+$.*

For the geometric meaning of the following definition, see the proof of Theorem 3.8.

Definition 3.5 Topologize

$$A_1 := W(R^+)[p^{-1}],\ A_2 := W(R^+)[[\overline{x}]^{-1}],\ A_{12} := W(R^+)[(p[\overline{x}])^{-1}]$$

as Huber rings with ring of definition $W(R^+)$ and ideals of definition generated by the respective topologically nilpotent units p, $[\overline{x}]$, $p[\overline{x}]$. Then put

$$B_1 := A_1\left\langle\frac{[\overline{x}]}{p}\right\rangle,\ B_2 := A_2\left\langle\frac{p}{[\overline{x}]}\right\rangle,\ B_{12} := A_{12}\left\langle\frac{[\overline{x}]}{p},\frac{p}{[\overline{x}]}\right\rangle;$$

note that there are canonical isomorphisms of topological rings

$$B_{12} \cong B_1\left\langle\frac{p}{[\overline{x}]}\right\rangle \cong B_2\left\langle\frac{[\overline{x}]}{p}\right\rangle.$$

Also put

$$B_1' := A_2\left\langle\frac{[\overline{x}]}{p}\right\rangle,\ B_2' := A_1\left\langle\frac{p}{[\overline{x}]}\right\rangle;$$

note that there are canonical isomorphisms of underlying rings

$$B_1' \cong B_1[[\overline{x}]^{-1}],\ B_2' \cong B_2[p^{-1}]$$

but these are not homeomorphisms for the implied topologies. For example, in the first isomorphism, the rings of power-bounded elements coincide, but on this common subring the induced topology from B_1' is the $\frac{[\overline{x}]}{p}$-adic topology while the induced topology from $B_1[[\overline{x}]^{-1}]$ is the p-adic topology.

Proposition 3.6 *The following statements hold.*

(a) The Huber rings $C = A_1, A_{12}, B_1, B_2, B_{12}, B_2'$ are stably uniform, and hence sheafy by Proposition 3.3.
(b) The Huber ring $C = A_2$ is uniform. (The same is true for $C = B_1'$, but we will not need this. See also Remark 3.7.)

Proof To prove (a), note that for $C = A_1, A_{12}, B_1, B_{12}, B_2'$, p is a topologically nilpotent unit in C. In these cases, by [13, Theorem 5.3.9], taking the completed tensor product over $\mathbb{Z}_p$ with $\mathbb{Z}_p[p^{p^{-\infty}}]$ yields a perfectoid ring in the sense of [13] (which must be a $\mathbb{Q}_p$-algebra). By splitting from $\mathbb{Z}_p[p^{p^{-\infty}}]$ to $\mathbb{Z}_p$ using the reduced trace, we deduce that C is stably uniform; see [13, Theorem 3.7.4] for further details. For $C = B_2$, p is no longer a unit in C but is still topologically nilpotent, and a similar argument applies using perfectoid rings in the sense of Fontaine; see [14, Corollary 4.1.14] or [12, Lemma 3.1.3].

To prove (b), note that A_2° is p-adically saturated in A_2, $W(R^\circ)$ is contained in A_2°, and the image of $A_2^\circ/(p) \to A_2/(p) \cong R$ is contained in R°. These facts together imply that $A_2^\circ = W(R^\circ)$, which is evidently a bounded subring of A_2. □

Remark 3.7 We believe that A_2 is stably uniform, which would then imply that B'_1 is stably uniform; but we were unable to prove either of these statements. One thing we can observe is that if B'_1 were known to be stably uniform, then combining the preceding results with Proposition 3.2(a) and [12, Theorem 1.2.22] would imply that A_2 is sheafy (and then stably uniform).

We now obtain a comparison between algebraic and adic vector bundles.

Theorem 3.8 *Put $A := W(R^+)$ and let X (resp. Y) be the complement in* $\operatorname{Spec} A$ *(resp.* $\operatorname{Spa}(A, A)$*) of the closed subspace where $p = [\overline{x}] = 0$. Then pullback along the morphism $Y \to X$ of locally ringed spaces defines an equivalence of categories* $\mathbf{Vec}_X \to \mathbf{Vec}_Y$.

Proof For $A_1, A_2, A_{12}, B_1, B_2, B_{12}, B'_1, B'_2$ as in Definition 3.5, we have the following coverings of adic spaces by rational subspaces.

$U \cup V$	U	V	$U \cap V$
Y	$\operatorname{Spa}(B_1, B_1^\circ)$	$\operatorname{Spa}(B_2, B_2^\circ)$	$\operatorname{Spa}(B_{12}, B_{12}^\circ)$
$\operatorname{Spa}(A_1, A_1^\circ)$	$\operatorname{Spa}(B_1, B_1^\circ)$	$\operatorname{Spa}(B'_2, B_2'^\circ)$	$\operatorname{Spa}(B_{12}, B_{12}^\circ)$
$\operatorname{Spa}(A_2, A_2^\circ)$	$\operatorname{Spa}(B'_1, B_1'^\circ)$	$\operatorname{Spa}(B_2, B_2^\circ)$	$\operatorname{Spa}(B_{12}, B_{12}^\circ)$
$\operatorname{Spa}(A_{12}, A_{12}^\circ)$	$\operatorname{Spa}(B'_1, B_1'^\circ)$	$\operatorname{Spa}(B'_2, B_2'^\circ)$	$\operatorname{Spa}(B_{12}, B_{12}^\circ)$

For $i \in \{1, 2, 12\}$, we may apply Propositions 3.2(c) and 3.6(a) to see that the pullback functor $\mathbf{Vec}_{\operatorname{Spec}(B_i)} \to \mathbf{Vec}_{\operatorname{Spa}(B_i, B_i^\circ)}$ is an equivalence. We may also apply Propositions 3.2(a,b) and 3.6(b) to obtain an equivalence

$$\mathbf{Vec}_{\operatorname{Spec}(A_i)} \to \mathbf{Vec}_{\operatorname{Spec}(B_1^?)} \times_{\mathbf{Vec}_{\operatorname{Spec}(B_{12})}} \mathbf{Vec}_{\operatorname{Spec}(B_2^?)}, \qquad B_j^? = \begin{cases} B_j & j \in i \\ B'_j & j \notin i; \end{cases}$$

using the fact that $A_j \to B'_j$ factors through B_j (at the level of rings without topology), it follows that

$$\mathbf{Vec}_{\operatorname{Spec}(A_1)} \times_{\mathbf{Vec}_{\operatorname{Spec}(A_{12})}} \mathbf{Vec}_{\operatorname{Spec}(A_2)} \to \mathbf{Vec}_{\operatorname{Spec}(B_1)} \times_{\mathbf{Vec}_{\operatorname{Spec}(B_{12})}} \mathbf{Vec}_{\operatorname{Spec}(B_2)}$$

is an equivalence. In the 2-commutative diagram

$$\begin{array}{ccc}
\mathbf{Vec}_X & \longrightarrow & \mathbf{Vec}_Y \\
\| & & \| \\
\mathbf{Vec}_{\operatorname{Spec}(A_1)} \times_{\mathbf{Vec}_{\operatorname{Spec}(A_{12})}} \mathbf{Vec}_{\operatorname{Spec}(A_2)} & & \| \\
\downarrow & & \| \\
\mathbf{Vec}_{\operatorname{Spec}(B_1)} \times_{\mathbf{Vec}_{\operatorname{Spec}(B_{12})}} \mathbf{Vec}_{\operatorname{Spec}(B_2)} & \longrightarrow & \mathbf{Vec}_{\operatorname{Spa}(B_1, B_1^\circ)} \times_{\mathbf{Vec}_{\operatorname{Spa}(B_{12}, B_{12}^\circ)}} \mathbf{Vec}_{\operatorname{Spa}(B_2, B_2^\circ)}
\end{array}$$

every arrow except $\mathbf{Vec}_X \to \mathbf{Vec}_Y$ is now known to be an equivalence; we thus obtain the desired result. □

As a corollary, we obtain the following theorem.

Theorem 3.9 *Let v_0 be the valuation on $W(\mathfrak{o}_K)$ induced by the trivial valuation on the residue field of $\mathfrak{o}_K$. Put $A := W(\mathfrak{o}_K)$, $X := \mathrm{Spa}(A, A)$, $Y := X \setminus \{v_0\}$. Let $\mathbf{Mod}_A^{\mathrm{ff}}$ be the category of finite free A-modules. Then the categories $\mathbf{Mod}_A^{\mathrm{ff}}$, $\mathbf{Vec}_X$, $\mathbf{Vec}_Y$ are equivalent via the functor $\mathbf{Mod}_A^{\mathrm{ff}} \to \mathbf{Vec}_X$ taking M to $\tilde{M}$, the pullback functor $\mathbf{Vec}_X \to \mathbf{Vec}_Y$, and the global sections functor $\mathbf{Vec}_Y \to \mathbf{Mod}_A^{\mathrm{ff}}$.*

Proof Combine Theorem 2.7 with Theorem 3.8. □

One might like to parlay Theorem 3.9 into a version with K replaced by R. However, one runs into an obvious difficulty in light of the following standard example in the category of schemes.

Example 3.10 Let k be a field, put $S := k[x, y, z]$, and let M be the S-module

$$\ker(S^3 \to S : (a, b, c) \mapsto ax + by + cz).$$

Put $X := \mathrm{Spec}\, S$, $Y := X \setminus \{(x, y, z)\}$, $Z := X \setminus \overline{\{(x, y)\}}$; then $\tilde{M} \notin \mathbf{Vec}_X$ but $\tilde{M}|_* \in \mathbf{Vec}_*$ for $* \in \{Y, Z\}$. Since $X \setminus Y$ has codimension 3 in X and $Y \setminus Z$ has codimension 2 in Y, $\tilde{M}|_Z$ has a unique extension to an S_2 sheaf (in the sense of Serre) on either X or Y, namely $\tilde{M}$ itself. In particular, $\tilde{M}$ does not lift from $\mathbf{Vec}_Z$ to $\mathbf{Vec}_X$.

With a bit of care, this argument can be translated into an example that shows that Theorem 3.9 indeed fails to generalize to the case where K is replaced by R.

Remark 3.11 For (R, R^+) as in Hypothesis 3.4, let $\mathfrak{p}$ be the radical of the ideal $(p, [\overline{x}])$; it is generated by p and $[\overline{x}]^{p^{-n}}$ for all n. Put

$$X := \mathrm{Spec}(W(R^+)), \qquad Y := X \setminus \{\mathfrak{p}\},$$

and let Z be the algebraic stack which is the colimit of the diagram

$$\mathrm{Spec}(W(R^+)[p^{-1}]) \leftarrow \mathrm{Spec}(W(R)[p^{-1}]) \to \mathrm{Spec}(W(R)).$$

As in Lemma 2.3, we see that the functors $\mathbf{Vec}_X \to \mathbf{Vec}_Y$, $\mathbf{Vec}_Y \to \mathbf{Vec}_Z$ are fully faithful, and that for $* \in \{Y, Z\}$, $\mathcal{F} \in \mathbf{Vec}_*$, $M = H^0(*, \mathcal{F})$, the adjunction morphism $\tilde{M}|_* \to \mathcal{F}$ is an isomorphism. However, one may emulate Example 3.10 so as to produce an object of $\mathbf{Vec}_Y$ and $\mathbf{Vec}_Z$ which does not lift to $\mathbf{Vec}_X$; see Example 3.14 below.

Lemma 3.12 *With notation as in Remark 3.11, for $\mathcal{F} \in \mathbf{Vec}_*$ and $M = H^0(*, \mathcal{F})$, the natural homomorphism $M^\vee \to H^0(*, \mathcal{F}^\vee)$ is an isomorphism. Consequently, the map $M \to M^{\vee\vee}$ is an isomorphism, i.e., M is reflexive.*

Proof From Remark 3.11, we see that the map is injective. To check surjectivity, note that any $f \in H^0(*, \mathcal{F}^\vee)$ restricts to maps $M \to W(R^+)[p^{-1}]$, $M \to W(R)$ which induce the same map $M \to W(R)[p^{-1}]$. We again deduce the claim from the equality $W(R^+)[p^{-1}] \cap W(R) = W(R^+)$. □

Remark 3.13 Recall that for any ring S, a *regular sequence* in S is a finite sequence $s_1, \dots, s_k$ such that for $i = 1, \dots, k$, s_i is not a zero-divisor in $S/(s_1, \dots, s_{i-1})$. If $s_1, \dots, s_k$ is a regular sequence in S, one computes easily that

$$\mathrm{Tor}_k^S(S/(s_1, \dots, s_k), S/(s_1, \dots, s_k)) \cong S/(s_1, \dots, s_k) \neq 0;$$

in particular, $S/(s_1, \dots, s_k)$ has projective dimension at least (and in fact exactly) k as an S-module.

Example 3.14 Let k be a perfect field of characteristic p. Let R^+ be the $(\overline{y}, \overline{z})$-adic completion of the perfect closure of $k[\![\overline{y}, \overline{z}]\!]$. Put $\overline{x} := \overline{yz} \in R^+$. This notation is consistent with Hypothesis 3.4, so we may adopt notation as in Remark 3.11.

Put $I := ([\overline{y}], [\overline{z}], p)W(R^+)$; note that the generators of I form a regular sequence. By Remark 3.13, $W(R^+)/I$ has projective dimension at least 3, I has projective dimension at least 2, and

$$M := \ker(W(R^+)^3 \to I : (a, b, c) \mapsto a[\overline{y}] + b[\overline{z}] + cp)$$

has projective dimension at least 1. In particular, M is not projective.

For $* \in \{Y, Z\}$, the sequence

$$0 \to \tilde{M}|_* \to \mathcal{O}^{\oplus 3} \to \mathcal{O} \to 0 \tag{3.14.1}$$

of sheaves is exact, so $\tilde{M}|_* \in \mathbf{Vec}_*$. Because $H^0(*, \mathcal{O}) = W(R^+)$, applying the functor $H^0(*, \bullet)$ to (3.14.1) yields an isomorphism $H^0(*, \tilde{M}|_*) \cong M$.

However, if $\tilde{M}|_*$ could be extended to an object $\mathcal{F} \in \mathbf{Vec}_X$, we would have $\mathcal{F} \cong \tilde{N}$ for some finite projective $W(R^+)$-module N, and per Remark 3.11 we would have $N \cong H^0(*, \mathcal{F}) = H^0(*, \tilde{M}|_*) \cong M$. This yields a contradiction.

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Sur une q-déformation locale de la théorie de Hodge non-abélienne en caractéristique positive

Michel Gros

Abstract Pour p un nombre premier et q une racine p-ième non triviale de 1, nous présentons les principales étapes de la construction d'une q-déformation locale de la "correspondance de Simpson en caractéristique p" dégagée par Ogus et Vologodsky en 2005. La construction est basée sur l'équivalence de Morita entre un anneau d'opérateurs différentiels q-déformés et son centre. Nous expliquons aussi les liens espérés entre cette construction et celles introduites récemment par Bhatt et Scholze. Pour alléger l'exposition, nous nous limitons au cas de la dimension 1. For p a prime number and q a non trivial pth root of 1, we present the main steps of the construction of a local q-deformation of the "Simpson correspondence in characteristic p" found by Ogus and Vologodsky in 2005. The construction is based on the Morita-equivalence between a ring of q-twisted differential operators and its center. We also explain the expected relations between this construction and those recently done by Bhatt and Scholze. For the sake of readability, we limit ourselves to the case of dimension 1.

Keywords p-adic Hodge theory · q-deformation · Rings of differential operators

1 Introduction

1.1. Ogus et Vologodsky ont dégagé dans [13] un analogue en caractéristique $p > 0$ de la théorie de Hodge non-abélienne, i.e. de la correspondance de Simpson complexe. Soient $\tilde{S}$ un schéma plat sur $\mathbb{Z}/p^2$, $\tilde{X}$, $\tilde{X}'$ deux $\tilde{S}$-schémas lisses de réduction modulo p notées X et X', et $\tilde{\mathrm{F}} : \tilde{X} \to \tilde{X}'$ un $\tilde{S}$-morphisme. Supposons que ces données constituent un relèvement au-dessus de $\tilde{S}$ du morphisme de Frobenius relatif $\mathrm{F}_{X/S} : X \to X'$ associé à X vu comme schéma au-dessus de $S = \tilde{S} \times_{\mathbb{Z}/p^2} \mathbb{Z}/p$. Elles

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permettent alors à Ogus et Vologodsky d'étendre ([13], Theorems 2.8, 2.26) aux $\mathcal{O}_X$-modules munis d'une connexion intégrable dont la p-courbure est supposée seulement *quasi*-nilpotente à la fois le théorème de descente de Cartier ([10], Theorem 5.1) et l'existence d'une décomposition du complexe de De Rham obtenu par Deligne et Illusie ([5], Remark 2.2(ii)) induisant l'opération de Cartier ([10], Theorem 7.2). L'exposé oral d'A. Abbes et le nôtre ont été consacrés aux travaux d'Oyama [14], Shiho [17] et Xu [19] qui ont permis de relever "modulo p^n" cette correspondance d'Ogus et Vologodsky. C'est ici une autre direction qui est explorée.

1.2. Sans rapport avec ce qui précède, Bhatt, Morrow et Scholze ont dégagé ([4], Theorem 1.8) un raffinement entier des théorèmes standards de comparaison entre cohomologies cristalline, de De Rham et étale p-adique pour un schéma formel propre et lisse sur l'anneau des entiers d'une extension non-archimédienne algébriquement close de $\mathbb{C}_p$. Dans l'élaboration de celui-ci apparait un relèvement de l'isomorphisme de Cartier ([4], Theorem 8.3) sur la cohomologie d'un objet ([4], Definition 8.1) d'une certaine catégorie dérivée. Dans des situations géométriques locales bien adaptées ([4], 8.5) auxquelles les auteurs se ramènent pour établir cet isomorphisme, l'existence de ce relèvement découle de l'étude de certains *q-complexes de De Rham* ([4], 7.7) avec q une racine p-ième non triviale de l'unité dans $\mathbb{C}_p$. Ces derniers sont de vrais complexes qui "réalisent" ([4], Sect. 8) les objets des catégories dérivées évoquées ci-dessus. Ils ont eux-aussi une cohomologie se calculant par un relèvement de l'opération de Cartier ([16], Proposition 3.4, (iii); voir aussi [15], Proposition 2.8) qui explique donc localement l'existence de la précédente. Il nous semble plausible que l'extension espérée du théorème de comparaison entier ([4], Theorem 1.8) à des coefficients non constants [18] donne quelque intérêt à essayer d'expliciter une *q-déformation locale de la théorie de Hodge non-abélienne* incluant l'étude de ce type de complexes et les propriétés de leurs cohomologies. Cela devrait peut-être éclaircir un peu d'éventuels liens entre les théories [13] et [4] puis, ultérieurement, ceux avec la correspondance de Simpson p-adique [1].

1.3. Le but de ce rapport est d'esquisser, dans ces situations géométriques locales bien adaptées, une telle variante. Les deux résultats principaux sont, d'une part une q-déformation locale (Théorème 4) de la correspondance développée par Ogus–Vologodsky (Théorème 2) et, d'autre part, sa compatibilité aux cohomologies naturelles du but et de la source (Proposition 8). Un corollaire facile (Corollaire 1) de notre résultat est l'existence de l'opération de Cartier "relevée" ([16], Proposition 3.4 ; voir aussi [15], 2.2). Dans le type de situation géométrique que nous considérons, les résultats principaux d'Ogus–Vologodsky qui nous intéressent ici découlent immédiatement d'une équivalence de Morita, à savoir celle associée à la neutralisation d'une algèbre d'opérateurs différentiels vue comme algèbre d'Azumaya sur son centre ([13], Theorem 2.11). Nous développons simplement un q-analogue de tout le tableau. Plusieurs choix doivent être faits lors des constructions et il est peu probable que ces résultats puissent se globaliser par les techniques standards, purement schématiques, de recollement (voir par exemple [16], Conj. 1.1 et *infra*, [15], 2.2 et

3.4 pour une discussion de problèmes analogues, [4], Rem. 8.4).

1.4. Pour pallier ces difficultés de globalisation des q-complexes de De Rham, Bhatt et Scholze ont introduit très récemment dans [3], à beaucoup d'autres fins aussi (dont celle de réinterpréter les théorèmes de comparaison entiers évoqués plus haut ainsi que les décompositions de Hodge-Tate, ...), de nouvelles techniques, en particulier celles du site prismatique et du site q-cristallin. Ils utilisent pour ce faire la théorie des δ-anneaux et leurs avancées fournissent pour nous un espoir de montrer l'indépendance de tout choix auxiliaire (en particulier d'une coordonnée) dans nos constructions, au moins à isomorphisme près. De toute façon, vu la généralité du cadre dans lequel ils se placent et le potentiel d'applications, il nous a paru indispensable d'en tenir compte et de reconsidérer avec leurs nouveaux outils les questions que nous nous posions au moment de la conférence et de la première version de cet article puis d'indiquer les progrès réalisés depuis lors.

1.5. Ce rapport ne contient pas de démonstrations, pour lesquelles on renvoie à [8] et à [9]. Nous insistons plutôt ici sur la mise en parallèle des théories modulo p ([8]) et q-déformées ([9]) en spécialisant cette dernière au cadre familier (notations, hypothèses, terminologie, ...) de la théorie de Hodge p-adique, ce qui en allège très largement la présentation. Nous résumons très succinctement tout d'abord au §2 les principales étapes suivies dans [8] pour établir la neutralisation (*loc. cit.*, Theorem 4.13) mentionnée ci-dessus. La seule nouveauté par rapport à [8] est le résultat de comparaison cohomologique contenu dans la Proposition 1. Nous passons ensuite au §3, après avoir précisé le cadre géométrique, à la définition des opérateurs différentiels q-déformés. Bien qu'on puisse les définir plus directement (cf. 3.6), c'est par un processus de dualité et donc via la définition de parties principales q-déformées que nous procédons afin de pouvoir raisonner comme dans la théorie modulo p. Dans le §4, nous déterminons le centre de l'algèbre des opérateurs différentiels q-déformés et montrons comment on peut diviser l'action induite par le "Frobenius" (Proposition 5) sur les modules de parties principales q-déformées. Que ceci soit possible est pour l'instant l'aspect le plus miraculeux de toute cette théorie. Nous en déduisons enfin la neutralisation (Théorème 3) d'une complétion centrale de l'algèbre des opérateurs différentiels q-déformés. Le §5 reformule alors l'équivalence de Morita standard qu'on déduit de cette neutralisation en termes de modules munis d'une q-dérivation quasi-nilpotente et de modules de Higgs quasi-nilpotents (5.1) et les conséquences cohomologiques (Proposition 8). Nous terminons enfin au §6 par quelques observations et questions en relation avec [3].

1.6. Ces résultats sont le fruit d'une collaboration avec B. Le Stum et A. Quirós que l'auteur dégage de toute responsabilité pour les erreurs ou imprécisions qui pourraient apparaitre. L'auteur remercie très sincèrement la Fondation Simons et les organisateurs de la session *Simons Symposium on p-adic Hodge Theory* (8-12 Mai 2017), Bhargav Bhatt et Martin Olsson, de lui avoir donné l'opportunité d'avancer sur toutes les questions soulevées par ce projet.

2 Rappels sur la théorie d'Ogus et Vologodsky

2.1. Nous conservons dans cette section les notations et hypothèses de 1.1, résumées par les deux diagrammes suivants :

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\mathrm{F}}} & \tilde{X}' \\ & \searrow & \downarrow \\ & & \tilde{S} \\ & & \downarrow \\ & & \mathbb{Z}/p^2 \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\mathrm{F}_{X/S}} & X' \\ & \searrow & \downarrow \\ & & S \\ & & \downarrow \\ & & \mathbb{Z}/p \end{array} \tag{1}$$

dont celui de droite est donc la réduction modulo p de celui de gauche. Dans ce qui suit, nous allégerons la notation $\mathrm{F}_{X/S}$ en simplement F mais en attirant bien l'attention du lecteur sur le fait que cet allègement n'est pas tout à fait compatible avec les notations adoptées dans [8] (dans *loc. cit.*, F est noté F_X et F y désigne le Frobenius *absolu* de X). Nous supposerons de plus S noethérien pour raccourcir la preuve de la Proposition 1 ci-dessous.

2.2. Nous noterons également simplement $\mathcal{D}_X$ la $\mathcal{O}_X$-algèbre $\mathcal{D}_X^{(0)}$ des opérateurs différentiels de X/S de niveau $m = 0$ introduite par Berthelot ([2], 2.2.1) et utilisée dans ([8], Definition 2.5), parfois dénommée *algèbre des opérateurs PD-différentiels* ou *algèbre des opérateurs différentiels cristallins*. Elle est engendrée par $\mathcal{O}_X$ et par les S-dérivations de $\mathcal{O}_X$ (cf. [2], p. 218, Rem. (i)). Nous noterons $\mathrm{Z}\mathcal{D}_X$ (resp. $\mathrm{Z}\mathcal{O}_X$) le centre de $\mathcal{D}_X$ (resp. le centralisateur dans $\mathcal{D}_X$ de sa sous-algèbre $\mathcal{O}_X$). Nous noterons enfin $\mathrm{S}(\mathcal{T}_{X'})$ la $\mathcal{O}_{X'}$-algèbre (graduée) symétrique du $\mathcal{O}_{X'}$-module $\mathcal{T}_{X'}$ des fonctions sur le fibré cotangent de X'/S. L'application de p-courbure permet (cf. par exemple ([8], Proposition 3.6)) de construire un isomorphisme de $\mathcal{O}_X$-algèbres

$$c : \mathrm{S}(\mathcal{T}_{X'}) \xrightarrow{\sim} \mathrm{F}_*\mathrm{Z}\mathcal{D}_X \ ; \ D \in \mathcal{T}_{X'} \mapsto D^p - D^{[p]}. \tag{2}$$

On peut, de même (cf. *loc. cit.*), identifier $\mathrm{Z}\mathcal{O}_X$ à $\mathrm{F}^*\mathrm{S}(\mathcal{T}_{X'}) = \mathcal{O}_X \otimes_{\mathcal{O}_{X'}} \mathrm{S}(\mathcal{T}_{X'})$.

2.3. L'algèbre $\mathcal{D}_X$ agit de manière naturelle de façon $\mathcal{O}_{X'}$-linéaire sur $\mathcal{O}_X$. Soit $\mathcal{K}_X$ le noyau de la surjection canonique $\mathcal{D}_X \to \mathcal{E}nd_{\mathcal{O}_{X'}}(\mathcal{O}_X)$. C'est un idéal bilatère de $\mathcal{D}_X$. Nous noterons $\widehat{\mathcal{D}_X}$ (resp. $\widehat{\mathrm{Z}\mathcal{D}_X}$, resp. $\widehat{\mathrm{S}(\mathcal{T}_{X'})}$, resp. $\widehat{\mathrm{Z}\mathcal{O}_X}$, resp. $\mathcal{O}_X \otimes_{\mathcal{O}_{X'}} \widehat{\mathrm{S}(\mathcal{T}_{X'})}$) le complété adique de $\mathcal{D}_X$ (resp. $\mathrm{Z}\mathcal{D}_X$, resp. $\mathrm{S}(\mathcal{T}_{X'})$, resp. $\mathrm{Z}\mathcal{O}_X$, resp. $\mathcal{O}_X \otimes_{\mathcal{O}_{X'}} \mathrm{S}(\mathcal{T}_{X'})$) relativement à l'idéal bilatère $\mathcal{K}_X$ (resp. $\mathcal{K}_X \cap \mathrm{S}(\mathcal{T}_{X'})$, resp. $\mathcal{K}_X \cap \mathrm{Z}\mathcal{O}_X$, resp. $\mathcal{K}_X \cap (\mathcal{O}_X \otimes_{\mathcal{O}_{X'}} \mathrm{S}(\mathcal{T}_{X'}))$).

2.4. Dans cette situation restrictive d'existence de $\tilde{\mathrm{F}}$, plusieurs des résultats généraux de ([13], *e.g.* Theorem 2.8) découlent immédiatement du résultat suivant ([8], Theorem 4.13) que nous avons appris de P. Berthelot et dont nous rappelerons brièvement le principe de preuve ci-dessous (2.8, 2.9).

Théorème 1 (*[8],* Theorem 4.13) *Toute donnée de $(\tilde{X}, \tilde{X}', \tilde{\mathrm{F}} : \tilde{X} \to \tilde{X}')$ comme précédemment définit canoniquement un isomorphisme de $\mathcal{O}_X$-algèbres*

$$\widehat{\mathcal{D}_X} \xrightarrow{\sim} \mathcal{E}nd_{\widehat{\mathrm{S}(\mathcal{T}_{X'})}}(\mathcal{O}_X \otimes_{\mathcal{O}_{X'}} \widehat{\mathrm{S}(\mathcal{T}_{X'})}). \tag{3}$$

On remarquera, en prévision de (6), que le but de (3) est simplement $\mathcal{E}nd_{\widehat{Z\mathcal{D}_X}}(\widehat{Z\mathcal{O}_X})$.

2.5. Un lemme classique d'algèbre linéaire ([8], Lem. 5.6) montre alors que les anneaux $\widehat{\mathcal{D}_X}$ et $\widehat{\mathrm{S}(\mathcal{T}_{X'})}$ sont, d'une manière complètement explicite, équivalents au sens de Morita : les deux foncteurs suivants entre les catégories de modules sur ces anneaux, **Mod** $(\widehat{\mathcal{D}_X})$ et **Mod** $(\widehat{\mathrm{S}(\mathcal{T}_{X'})}$, correspondantes sont quasi-inverses l'un de l'autre

$$\mathbb{H} : \mathbf{Mod}\,(\widehat{\mathcal{D}_X}) \to \mathbf{Mod}\,(\widehat{\mathrm{S}(\mathcal{T}_{X'})})\,;\ \mathcal{E} \mapsto \mathcal{H}om_{\widehat{\mathcal{D}_X}}(\mathrm{F}^*\widehat{\mathrm{S}(\mathcal{T}_{X'})}, \mathcal{E}), \tag{4}$$

$$\mathbb{M} : \mathbf{Mod}\,(\widehat{\mathrm{S}(\mathcal{T}_{X'})}) \to \mathbf{Mod}\,(\widehat{\mathcal{D}_X})\,;\ \mathcal{F} \mapsto \mathcal{F} \otimes_{\widehat{\mathrm{S}(\mathcal{T}_{X'})}} \mathrm{F}^*\widehat{\mathrm{S}(\mathcal{T}_{X'})}. \tag{5}$$

Ce résultat fournit, une fois réinterprété (cf. [8], Proposition 5.2) les objets de ces catégories, le résultat suivant :

Théorème 2 (*[8],* Theorem 5.8) *Toute donnée de $(\tilde{X}, \tilde{X}', \tilde{\mathrm{F}} : \tilde{X} \to \tilde{X}')$ comme précédemment définit canoniquement une équivalence entre la catégorie des $\mathcal{O}_X$-modules munis d'une connexion intégrable de p-courbure quasi-nilpotente (cf.* [8], Proposition 5.5*) et la catégorie des $\mathcal{O}_{X'}$-modules munis d'un champ de Higgs quasi-nilpotent (cf.* [8], Proposition 5.4*).*

On vérifie que dans cette équivalence, $\mathcal{O}_X$ muni de sa connexion canonique d, correspond à $\mathcal{O}_{X'}$ muni du champ de Higgs nul.

2.6. Le *complexe de Higgs* d'un $\mathcal{O}_{X'}$-module de Higgs $\mathcal{F} \in$ **Mod** $(\widehat{\mathrm{S}(\mathcal{T}_{X'})})$ est, par définition, le complexe (avec $\mathcal{F}$ placé en degré 0)

$$0 \to \mathcal{F} \xrightarrow{\theta} \mathcal{F} \otimes_{\mathcal{O}_{X'}} \Omega^1_{X'} \xrightarrow{(-)\wedge\theta} \mathcal{F} \otimes_{\mathcal{O}_{X'}} \Omega^2_{X'} \xrightarrow{(-)\wedge\theta} \ldots \tag{6}$$

avec θ l'application $\mathcal{O}_{X'}$-linéaire provenant de la structure naturelle de $\mathrm{S}(\mathcal{T}_{X'})$-module sur $\mathcal{F}$ et, pour alléger, $\Omega^i_{X'}$ le $\mathcal{O}_S$-module des différentielles *relatives* de de degré i de X'/S (noté $\Omega^i_{X'/S}$ lorsque une ambiguïté est possible). On en donnera ci-dessous (10) une autre description. Il résulte facilement de cette équivalence la

Proposition 1 *Si* $\mathcal{E} \in \mathbf{Mod}\,(\widehat{\mathcal{D}_X})$ *et* $\mathcal{F} \in \mathbf{Mod}\,(\widehat{\mathrm{S}(\mathcal{T}_{X'})})$ *se correspondent par l'équivalence ci-dessus, alors l'image directe par* F *du complexe de De Rham de* $\mathcal{E}$ *est quasi-isomorphe au complexe de Higgs de* $\mathcal{F}$.

Le principe de démonstration est le suivant. Pour calculer $\mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}_X}}(\mathcal{O}_X, \mathcal{E})$, on utilise la résolution de Spencer de $\mathcal{O}_X$ par des $\mathcal{O}_X$-modules localement libres sur $\mathcal{D}_X$

$$[\ldots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^2 \mathcal{T}_X \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{T}_X \to \mathcal{D}_X] \to \mathcal{O}_X \to 0. \tag{7}$$

On tensorise alors la partie entre crochets par $\widehat{\mathcal{D}_X}$ en préservant l'exactitude de (7) car $\widehat{\mathcal{D}_X}$ est plat sur $\mathcal{D}_X$ puisque c'est le complété de $\mathcal{D}_X$ relativement à un idéal bilatère engendré par une suite centralisante. On a alors, notant $\Omega^\bullet_X$ pour alléger le complexe $\Omega^\bullet_{X/S}$ des différentielles relatives de X/S, des isomorphismes

$$\mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}_X}}(\mathcal{O}_X, \mathcal{E}) \simeq \mathcal{H}om_{\widehat{\mathcal{D}_X}}(\widehat{\mathcal{D}_X} \otimes_{\mathcal{O}_X} \wedge^\bullet \mathcal{T}_X, \mathcal{E}) \simeq \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_X \tag{8}$$

Pour le complexe de Higgs, on utilise la résolution de Koszul de $\mathcal{O}_{X'}$

$$[\ldots \to \mathrm{S}(\mathcal{T}_{X'}) \otimes_{\mathcal{O}_{X'}} \wedge^2 \mathcal{T}_{X'} \to \mathrm{S}(\mathcal{T}_{X'}) \otimes_{\mathcal{O}_{X'}} \mathcal{T}_{X'} \to \mathrm{S}(\mathcal{T}_{X'})] \to \mathcal{O}_{X'} \to 0. \tag{9}$$

que l'on tensorise par $\widehat{\mathrm{S}(\mathcal{T}_{X'})}$ au-dessus de $\mathrm{S}(\mathcal{T}_{X'})$ en la laissant exacte. On obtient

$$\mathbb{R}\mathcal{H}om_{\widehat{\mathrm{S}(\mathcal{T}_{X'})}}(\mathcal{O}_{X'}, \mathcal{F}) \simeq \mathcal{H}om_{\widehat{\mathrm{S}(\mathcal{T}_{X'})}}(\widehat{\mathrm{S}(\mathcal{T}_{X'})} \otimes_{\mathcal{O}_{X'}} \wedge^\bullet \mathcal{T}_{X'}, \mathcal{F}) \simeq \mathcal{F} \otimes_{\mathcal{O}_{X'}} \Omega^\bullet_{X'}. \tag{10}$$

Les deux foncteurs dérivés (8) et (10) pouvant se calculer à l'aide de résolutions injectives du second argument, la proposition s'ensuit grâce à l'équivalence de catégories donnée par $\mathbb{H}$ et $\mathbb{M}$.

2.7. Notons ici que, par définition de $\mathcal{K}_X$ et grâce au lemme d'algèbre linéaire qu'on vient d'évoquer, les anneaux $\mathcal{D}_X/\mathcal{K}_X$ et $\mathcal{O}_{X'}$ sont équivalents au sens de Morita. C'est, réinterprété dans ce langage, le classique théorème de descente de Cartier ([10], Theorem 7.2). D'autre part, la Proposition 1 fournit exactement, une fois précisé les isomorphismes, la décomposition du complexe de De Rham obtenue par Deligne-Illusie ([5], Rem. 2.2(ii)).

2.8. La démonstration du Théorème 1 procède par dualité. Soient $\mathcal{I} \subset \mathcal{O}_{X \times_S X}$ l'idéal définissant l'immersion diagonale $X \hookrightarrow X \times_S X$, $\mathcal{P}_X$ son *enveloppe à puissances divisées* (notée $\mathcal{P}_{X/S,(0)}$ dans [8], 2.4), $\overline{\mathcal{I}} \subset \mathcal{P}_X$ le *PD-idéal* engendré par $\mathcal{I}$ et, pour n un entier ≥ 0, $\mathcal{P}^n_X = \mathcal{P}_X/\mathcal{I}^{[n+1]}$. On a, par définition,

$$\mathcal{D}_{X,n} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_X, \mathcal{O}_X)\ ;\ \mathcal{D}_X = \cup_{n\geq 0}\mathcal{D}_{X,n}. \tag{11}$$

On remarque alors qu'on a simplement un isomorphisme

$$\widehat{\mathcal{D}_X} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}_X, \mathcal{O}_X). \tag{12}$$

et que $\widehat{\mathcal{D}_X}$ n'est autre que ce qui est classiquement appelé l'*algèbre des opérateurs hyper-PD-différentiels*. D'autre part, notons $\Gamma(\Omega^1_{X'})$ la $\mathcal{O}_{X'}$-algèbre (graduée) à puissances divisées canoniquement associée au $\mathcal{O}_{X'}$-module Ω^1_X ([8], Theorem 1.2). Une vérification d'algèbre linéaire ([8], preuve de Theorem 4.13) fournit un isomorphisme

$$\mathcal{E}nd_{\widehat{\mathrm{S}(\mathcal{T}_{X'})}}(\mathcal{O}_X \otimes_{\mathcal{O}_{X'}} \widehat{\mathrm{S}(\mathcal{T}_{X'})}) \overset{\sim}{\to} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X\times_{X'}X} \otimes_{\mathcal{O}'_X} \Gamma(\Omega^1_{X'}), \mathcal{O}_X) \tag{13}$$

de sorte que le théorème 1 se réduit à la construction d'un isomorphisme d'algèbres de Hopf

$$\mathcal{O}_{X\times_{X'}X} \otimes_{\mathcal{O}_{X'}} \Gamma(\Omega^1_{X'}) \overset{\sim}{\to} \mathcal{P}_X. \tag{14}$$

2.9. Cette construction procède selon les principales étapes suivantes :

- L'application canonique $\mathcal{I} \to \mathcal{P}_X$; $f \to f^{[p]}$ composée avec la projection canonique $\mathcal{P}_X \to \mathcal{I}\mathcal{P}_X$ est une application F^*-linéaire nulle sur $\mathcal{I}^2$ ([8], Lem. 3.1). Elle induit donc ([8], Proposition 3.2) par passage au quotient et linéarisation une application $\mathcal{O}_X$-linéaire
$$\mathrm{F}^*\Omega^1_{X'} \to \mathcal{P}_X/\mathcal{I}\mathcal{P}_X \tag{15}$$
- L'application (15) s'étend en un isomorphisme de $\mathcal{O}_X$-algèbres à puissances divisées ([8], Proposition 3.3)
$$\mathrm{F}^*\Gamma(\Omega^1_{X'}) \overset{\sim}{\to} \mathcal{P}_X/\mathcal{I}\mathcal{P}_X. \tag{16}$$
- La donnée de $(\tilde{X}, \tilde{X}', \tilde{\mathrm{F}} : \tilde{X} \to \tilde{X}')$ permet de factoriser le morphisme (15) en un morphisme de $\mathcal{O}_X$-modules
$$\mathrm{F}^*\Omega^1_{X'} \to \mathcal{P}_X. \tag{17}$$
C'est l'application *Frobenius divisé*, notée $\frac{1}{p!}\tilde{\mathrm{F}}^*$ dans (Proposition 4.8, [8]). On prendra garde ici que la surjection canonique $\mathcal{P}_X \to \mathcal{P}_X/\mathcal{I}\mathcal{P}_X$ n'est pas compatible aux puissances divisées.
- L'application (17) s'étend en un morphisme de $\mathcal{O}_X$-*algèbres à puissances divisées* ([8], Proposition 4.8)
$$\mathrm{F}^*\Gamma(\Omega^1_{X'}) \to \mathcal{P}_X \tag{18}$$
factorisant l'isomorphisme (16).
- L'application (18) s'étend alors canoniquement ([8], Proposition 4.13) en l'isomorphisme de $\mathcal{O}_X$-algèbres de Hopf (14) recherché.

2.10. Il peut être utile au lecteur de savoir que si l'on composait la projection canonique $\mathcal{P}_X \to \mathcal{P}_X/\mathcal{I}\mathcal{P}_X$ avec l'inverse de (16) et qu'on dualisait l'application obtenue, on retrouverait la composée $\mathrm{S}(\mathcal{T}_{X'}) \to \mathrm{F}_*\mathcal{D}_X$ de l'application (2) de p-courbure c et de l'application canonique $\mathrm{Z}\mathcal{D}_X \hookrightarrow \mathcal{D}_X$.

3 Opérateurs différentiels q-déformés

3.1. Soient R un anneau commutatif supposé muni d'un relèvement qu'on notera ici simplement F du Frobenius absolu de R/p et $q \in R$. Soient également A une R-algèbre munie d'un morphisme étale $f : R[t] \to A$ (*i.e.* d'un *framing* au sens de [16], §3 ; [4], §8, ...). On munit $R[t]$ des deux morphismes de R-algèbres σ et F^* induits par $\sigma(t) = qt$ et $\mathrm{F}^*(t) = t^p$. On supposera également, par simplicité, qu'il existe deux morphismes de R-algèbres notés encore $\sigma : A \to A$ et $\mathrm{F}^* : A' := R_{\nwarrow \mathrm{F}} \otimes_R A \to A$, tels que respectivement $\sigma(x) = qx$ et $\mathrm{F}^*(1 \otimes x) = x^p$ avec $x := f(t)$ (élément parfois appelé *coordonnée* sur A). Signalons immédiatement, pour fixer les idées, un exemple particulièrement intéressant pour nous où une telle situation se manifeste. Soient $\overline{\mathbb{Q}}_p$ une clôture algébrique de $\mathbb{Q}_p$, $q \in \overline{\mathbb{Q}}_p$ une racine p-ième de 1 non triviale, K l'extension finie totalement ramifiée de $\mathbb{Q}_p$ engendrée par q et $R := \mathcal{O}_K$ l'anneau des entiers de K muni de $\mathrm{F} = \mathrm{Id_R}$. Alors, la simple donnée de f *étale* comme ci-dessus et des arguments standards suffisent à produire, par passage à la complétion p-adique de A, une situation comme précédemment pour cette dernière.

On s'est limité au cadre de la dimension 1 mais tout ce qui précède et suit vaut en dimension supérieure. On a également fixé une fois pour toutes ce dont on aura besoin mais les données ne seront utilisés qu'au fur et à mesure (la donnée de Frobenius n'est pas requise avant §4).

3.2. Pour u une indéterminée et n un entier ≥ 0, on pose $(n)_u = \frac{u^n-1}{u-1} \in \mathbb{Z}[u]$; $(n)_u! = \prod_{i=1}^n (i)_u \in \mathbb{Z}[u]$; $\binom{n}{k}_u = \frac{(n)_u!}{(k)_u!(n-k)_u!} \in \mathbb{Z}[u]$. Si maintenant q est un élément de R comme dans 3.1, les notations $(n)_q$; $(n)_q!$; $\binom{n}{k}_q$ signifient qu'on a évalué les quantités précédentes en $u = q$ afin d'obtenir des éléments de R. Ayant à éviter plus bas une possible confusion avec la notation standard des puissances divisées par des crochets, nous avons adopté la notation $(n)_q$ avec des parenthèses plutôt que la notation $[n]_q$ de ([6] ou ([16], §1)).

3.3. Soient A comme dans 3.1 et fixons $y \in A$. On va définir tout d'abord l'analogue q-déformé des sections de $\mathcal{P}_X$ (2.8) sur un ouvert affine dans ce cadre. Soit $A\langle\xi\rangle_{q,y}$ le A-module libre de générateurs abstraits notés $\xi^{[n]_{q,y}}$ avec $n \in \mathbb{N}$ (cf. [9], §2). On abrégera, lorsqu'aucune confusion n'en résulte, $\xi^{[0]_{q,y}}$ par 1, $\xi^{[1]_{q,y}}$ par ξ et $\xi^{[n]_{q,y}}$ par $\xi^{[n]}$. On notera $I^{[n+1]_{q,y}}$ le sous-A-module libre de $A\langle\xi\rangle_{q,y}$ engendré par les $\xi^{[k]_{q,y}}$ avec $k > n$.

Proposition 2 *([9],* Proposition 2.2) *Soient $m, n \in \mathbb{N}$. La règle de multiplication*

$$\xi^{[n]}.\xi^{[m]} = \sum_{i=0}^{\min(m,n)} (-1)^i q^{\frac{i(i-1)}{2}} \binom{m+n-i}{m}_q \binom{m}{i}_q y^i \xi^{[m+n-i]} \tag{19}$$

permet de munir $A\langle\xi\rangle_{q,y}$ une structure de A-algèbre commutative et unitaire. L'ensemble $I^{[n+1]_{q,y}}$ est un idéal de $A\langle\xi\rangle_{q,y}$.

On dira alors que $A\langle\xi\rangle_{q,y}$ est l'*anneau des polynômes sur A à puissances divisées q-déformées*. Cette terminologie est justifiée par l'égalité, valide pour tout $n \in \mathbb{N}$,

$$(n)_q!\xi^{[n]} = \prod_{i=0}^{n-1} (\xi + (i)_q y) =: \xi^{(n)} \tag{20}$$

et le fait que les $\xi^{(n)}$ forment, pour $n \in \mathbb{N}$, une base de $A[\xi]$ ([9], Lem. 1.1).

3.4. Soit encore A comme dans 3.1. Supposons désormais que $y = (1-q)x \in A$. Le lecteur remarquera que lorsque $q = 1$, l'algèbre $A\langle\xi\rangle_{q,y}$ n'est autre que la A-algèbre des polynômes à puissances divisées usuelles en ξ.

Définition 1 *(*[9], *Definition 4.2)* Soit $n \in \mathbb{N}$. Le A-module des *parties principales q-déformées* de A d'ordre au plus n (et de niveau 0) et le A-module des parties principales q-déformées de A sont, respectivement,

$$\mathrm{P}^{(0)}_{A/R,\sigma,n} = A\langle\xi\rangle_{q,y}/I^{[n+1]_{q,y}}, \tag{21}$$

$$\mathrm{P}^{(0)}_{A/R,\sigma} = \varprojlim_{n\in\mathbb{N}} A\langle\xi\rangle_{q,y}/I^{[n+1]_{q,y}}. \tag{22}$$

Dans la suite, nous allégerons les notations $\mathrm{P}^{(0)}_{A/R,\sigma,n}$ et $\mathrm{P}^{(0)}_{A/R,\sigma}$ en $\mathrm{P}_{A,\sigma,n}$ et $\mathrm{P}_{A,\sigma}$ respectivement.

3.5. On conserve les hypothèses et notations de 3.4. Rappelons qu'il découle de ([12], Proposition 2.10) qu'il existe un unique endomorphisme R-linéaire ∂_σ de A tel que, pour tous $z_1, z_2 \in A$, on ait

$$\partial_\sigma(z_1 z_2) = z_1\partial_\sigma(z_2) + \sigma(z_1)\partial_\sigma(z_2), \tag{23}$$

i.e. une *σ-dérivation* canonique. Afin de construire les opérateurs différentiels q-déformés par dualité, nous aurons besoin de la définition suivante.

Définition 2 *(*[9], *Definition 4.5)* L'application de Taylor q-déformée (de niveau 0) est l'application

$$\mathcal{T} : A \to \mathrm{P}_{A,\sigma} \tag{24}$$

définie par $\mathcal{T}(z) = \sum_{k=0}^{+\infty} \partial_\sigma^k(z)\xi^{[k]}$ pour tout $z \in A$.

On peut en fait définir $\mathcal{T}$ de manière plus formelle (cf. [9], Definition 4.5) et vérifier que c'est un morphisme d'anneaux, puis décrire cette application grâce à ∂_σ comme on vient de le faire.

Si maintenant M est un A-module à gauche, l'écriture $\mathrm{P}_{A,\sigma,n} \otimes'_R M$ signifie que nous regardons $\mathrm{P}_{A,\sigma,n}$ comme un A-module via l'application $\mathcal{T}$ (24). Autrement dit, pour tous $z \in A$, $s \in M$, $k \in \mathbb{N}$, on a :

$$\xi^{[k]} \otimes' zs = \mathcal{T}(z)\xi^{[k]} \otimes' s \tag{25}$$

Ceci permet de définir, pour chaque $n \in \mathbb{N}$,

$$\mathrm{D}^{(0)}_{A,\sigma,n} = \mathrm{Hom}_A(\mathrm{P}_{A,\sigma,n} \otimes'_A A, A). \tag{26}$$

Pour $n \in \mathbb{N}$, ces A-modules forment un système inductif et permettent donc de considérer

$$\mathrm{D}^{(0)}_{A,\sigma} = \varinjlim_{n\in\mathbb{N}} \mathrm{D}^{(0)}_{A,\sigma,n}. \tag{27}$$

On vérifie alors que la comultiplication

$$\mathrm{P}_{A,\sigma} \to \mathrm{P}_{A,\sigma} \otimes'_A \mathrm{P}_{A,\sigma} \tag{28}$$

définie par $\xi^{[n]} \mapsto \sum_{i=0}^{n} \xi^{[n-i]} \otimes' \xi^{[i]}$ permet de munir $\mathrm{D}^{(0)}_{A,\sigma}$ d'une structure d'anneau (cf. [9], Proposition 5.6).

3.6. L'anneau (27) ainsi construit par dualité n'est autre (cf. [9], Proposition 5.7) que l'extension de Ore $\mathrm{D}_{A/R,\sigma}$ de A par σ et ∂_σ, c'est-à-dire le A-module libre de générateurs abstraits ∂_σ^k ($k \geq 0$) avec la règle de commutation $\partial_\sigma z = \sigma(z)\partial_\sigma + \partial_\sigma(z)$ pour tout $z \in A$. Dans la suite, on utilisera cette notation $\mathrm{D}_{A,\sigma}$ pour l'anneau $\mathrm{D}^{(0)}_{A,\sigma}$ (27) si aucune confusion n'en résulte.

4 *p*-courbure et Frobenius divisé *q*-déformés

4.1. On garde dans tout ce § les notations et hypothèses de 3.1 et l'on suppose de plus que $(p)_q = 0$ dans R et que R est *q-divisible*, c'est-à-dire (cf. [9], 0.3) que pour tout $m \in \mathbb{N}$, $(m)_q$ est inversible dans R s'il est non nul. Ces deux conditions (qui ne sont pas nécessaires simultanément dans tous les énoncés) sont réalisées, par exemple, dans le cas où $R = \mathcal{O}_K$ (3.1) et $q \neq 1$ une racine p-ième de l'unité. On continue de poser $y = (1-q)x$ comme dans 3.4. Soient alors $\mathrm{ZD}_{A,\sigma}$ le centre de l'anneau $\mathrm{D}_{A,\sigma}$ et $\mathrm{Z}A_{A,\sigma}$ le centralisateur dans $\mathrm{D}_{A,\sigma}$ de sa sous-algèbre A.

Proposition 3 (*[9],* Proposition 6.3, Definition 6.5) *Il existe une unique application A-linéaire de A-algèbres*

$$A[\theta] \to \mathrm{D}_{A,\sigma}\ ;\ \theta \mapsto \partial_\sigma^p \tag{29}$$

dite de p-courbure q-déformé (ou simplement p-courbure tordue). Elle induit un isomorphisme de A-algèbres entre $A[\theta]$ et $\mathrm{Z}A_{A,\sigma}$ et de A'-algèbres entre $A'[\theta]$ et $\mathrm{ZD}_{A,\sigma}$.

Cette application est construite par dualité à partir des applications canoniques $\mathrm{P}_{A,\sigma,np} \to \mathrm{P}_{A,\sigma,np}/(\xi)$.

4.2. L'analogue q-déformé du calcul local crucial permettant de prouver l'existence de l'isomorphisme (16) est l'énoncé suivant.

Proposition 4 (*[9],* Definition 2.5, Theorem 2.6) *L'unique application A-linéaire*

$$A\langle\omega\rangle_{1,y^p} \to A\langle\xi\rangle_{q,y}\ ;\ \omega^{[k]} \mapsto \xi^{[pk]} \tag{30}$$

est appelée Frobenius divisé q-déformé (ou simplement Frobenius divisé tordu). C'est un homomorphisme d'anneaux induisant un isomorphisme de A-algèbres

$$A\langle\omega\rangle_{1,y^p} \xrightarrow{\sim} A\langle\xi\rangle_{q,y}/(\xi). \tag{31}$$

4.3. Pour n et i des entiers ≥ 0, on définit (cf. [9], Definition 7.4, Proposition 7.9) des polynômes $A_{n,i}(u)$, $B_{n,i}(u) \in \mathbb{Z}[u]$ par les formules

$$A_{n,i}(u) := \sum_{j=0}^{n} (-1)^{n-j} u^{\frac{p(n-j)(n-j-1)}{2}} \binom{n}{j}_{u^p} \binom{pj}{i}_u \tag{32}$$

et

$$(n)_u! A_{n,i}(u) = (n)_{u^p}!(p)_u^n B_{n,i}(u). \tag{33}$$

Les polynômes $A_{n,i}(u)$ s'introduisent naturellement dans la description de l'action induite par F sur les modules de parties principales q-déformées (cf. [9], Proposition 7.5). La possibilité de définir les polynômes $B_{n,i}(u)$ vient, quant à elle, de l'examen des coefficients des $A_{n,i}(u)$.

Proposition 5 (*[9],* Proposition 7.12) *L'application*

$$[\mathrm{F}^*] : A'\langle\omega\rangle_{1,y} \to A\langle\xi\rangle_{q,y}\ ;\ [\mathrm{F}^*](\omega^{[n]}) = \sum_{i=n}^{pn} B_{n,i}(q) x^{pn-i} \xi^{[i]} \tag{34}$$

est un homomorphisme d'anneaux.

Grâce à cette application, on montre, comme pour l'étape finale de (2.8) la

Proposition 6 (*[9],* Proposition 7.13) *L'application* [F*] *induit un morphisme de A-algèbres*

$$[\mathrm{F}^*] : A[\xi]/(\xi^{(p)_{q,y}}) \ \otimes_{A'} \ A'\langle\omega\rangle_{1,y} \xrightarrow{\sim} A\langle\xi\rangle_{q,y}, \tag{35}$$

qui est un isomorphisme.

C'est l'analogue q-déformé du calcul local crucial permettant de prouver l'existence de l'isomorphisme (14) ([8], Theorem 4.13).

4.4. Par dualité, on déduit de la Proposition 6 la q-déformation suivante de ([8], Proposition 4.8).

Proposition 7 (*[9],* Proposition 8.1) *L'application* [F*] *induit, par dualité, un morphisme de A-modules*

$$\Phi_{A,\sigma} : \mathrm{D}_{A,\sigma} \to \mathrm{Z}A_{A,\sigma} \hookrightarrow \mathrm{D}_{A,\sigma} \ ; \ \partial_\sigma^n \mapsto \sum_{k=0}^{n} B_{k,n}(q) x^{pk-n} \partial_\sigma^{pk}. \tag{36}$$

4.5. Soient, respectivement, $\widehat{\mathrm{D}_{A,\sigma}}$, $\widehat{\mathrm{ZD}_{A,\sigma}}$, $\widehat{\mathrm{Z}A_{A,\sigma}}$ les complétés adiques de $\mathrm{D}_{A,\sigma}$, $\mathrm{ZD}_{A,\sigma}$, $\mathrm{Z}A_{A,\sigma}$ (4.1) relativement à l'élément central $\partial_\sigma^p \in \mathrm{ZD}_{A,\sigma}$.

Théorème 3 (*[9],* Theorem 8.7) *L'application* $\Phi_{A,\sigma}$ *induit un isomorphisme de A-algèbres*

$$\widehat{\mathrm{D}_{A,\sigma}} \xrightarrow{\sim} \mathrm{End}_{\widehat{\mathrm{ZD}_{A,\sigma}}}(\widehat{\mathrm{Z}A_{A,\sigma}}). \tag{37}$$

4.6. Pour $q = 1$, réduisant modulo p, cet isomorphisme redonne l'isomorphisme (3). A un choix de normalisation près (correspondant exactement à la q-déformation de la différence entre diviser par p ou par $p!$ dans la construction du Frobenius divisé en caractéristique p), pour $A = R[t]$, $f = \mathrm{Id}$, l'isomorphisme (37) se décrit explicitement comme dans ([6], §4).

5 Théorie de Hodge non-abélienne q-déformée

On conserve dans ce § les hypothèses et notations générales du §4.

5.1. Le lemme classique d'algèbre linéaire ([8], Lem. 5.6) déjà évoqué en 2.5 montre alors que les anneaux $\widehat{\mathrm{D}_{A,\sigma}}$ et $\widehat{\mathrm{ZD}_{A,\sigma}}$ sont équivalents au sens de Morita. On va traduire cette conséquence en termes plus explicites.

Définition 3 ([9], §8) Soit M un A-module. Une σ-dérivation (de niveau 0) ou simplement *σ-dérivation* de M est une application R-linéaire $\partial_{\sigma,M}$ (=: $\partial_{\sigma,M}^{<1>_0}$) vérifiant, pour tous $r \in A$, $m \in M$, l'égalité (règle de Leibniz q-déformée)

$$\partial_{\sigma,M}(rm) = \partial_\sigma(r)m + \sigma(r)\partial_{\sigma,M}(m). \tag{38}$$

On a une notion évidente de morphismes entre modules munis de σ-dérivations. Rappelons maintenant qu'on dit qu'un endomorphisme u_G d'un groupe abélien G est dit *quasi-nilpotent* si pour tout $g \in G$, il existe $n \in \mathbb{N}$ tel que $u_G^n(g) = 0$.

5.2. Dans le cadre géométrique du §4, l'analogue q-déformé de la correspondance d'Ogus–Vologodsky (4), (5) est l'énoncé suivant.

Théorème 4 (*[9],* Corollary 8.9) *La catégorie des A-modules M munis d'une σ-dérivation quasi-nilpotente σ_M est équivalente à la catégorie des A'-modules H munis d'un endomorphisme A-linéaire quasi-nilpotent u_H.*

L'équivalence est donnée explicitement (comparer avec [8], Proposition 5.7 pour la situation en caractéristique p) par les deux foncteurs suivants quasi-inverses l'un de l'autre

$$\mathbb{H}_q : (M, \sigma_M) \mapsto (H := \{m \in M \mid \Phi_{A,\sigma}(\partial_\sigma^k)(m) = \partial_\sigma^k(m) \text{ pour tout } k \in \mathbb{N}\}, \partial_\sigma^p), \tag{39}$$

$$\mathbb{M}_q : (H, u_H) \mapsto (M := A \otimes_{A'} H, \partial_{\sigma,M}) \tag{40}$$

avec $\partial_{\sigma,M}$ l'unique σ-dérivation de M telle que $\partial_{\sigma,M}(1 \otimes h) = t^{p-1} \otimes u_H(h)$ pour tout $h \in H$. Dans cette équivalence, (A, ∂_σ) (3.5) correspond à $(A', 0)$.

5.3. Formulons maintenant les conséquences cohomologiques de cette équivalence en termes analogues à ceux de la Proposition 1. Si M est un A-module muni d'une σ-dérivation $\partial_{\sigma,M}$, on lui associe son complexe de De Rham q-déformé ou, s'inspirant de la terminologie de [16], *complexe de q-De Rham de M*

$$q\text{-DR}(M/R) : 0 \to M \xrightarrow{\nabla_M} M \otimes_A \Omega^1_{A/R} \to 0 \tag{41}$$

avec M placé en degré 0 et $\nabla_M(m) = \partial_{\sigma,M}(m) \otimes dx$. Bien que cela ne joue pas de rôle à ce niveau, signalons ici qu'il serait beaucoup plus canonique dans cette définition d'utiliser le *R-module des différentielles q-déformées* $\Omega^1_{A/R,\sigma}$ de ([11], Definition 5.3) plutôt que $\Omega^1_{A/R}$ (qui lui est seulement non-canoniquement isomorphe).

D'autre part, pour H un A'-module muni d'un endomorphisme u_H, on peut lui associer son complexe de Higgs

$$\text{Higgs}\,(H/R) : 0 \to H \xrightarrow{\theta_H} H \otimes_{A'} \Omega^1_{A'/R} \to 0 \tag{42}$$

avec H placé en degré 0 et $\theta_H(h) = u_H(h) \otimes dx$.

Proposition 8 (*[9],* Corollary 8.10) *Si $(M, \partial_{\sigma,M})$ est un A-module muni d'une dérivation q-déformée quasi-nilpotente et (H, u_H) un A'-module muni d'un endomorphisme quasi-nilpotent se correspondant suivant les foncteurs $\mathbb{H}_q$ et $\mathbb{M}_q$, alors le complexe* q-DR(M/R) *est quasi-isomorphe au complexe* Higgs (H/R).

Corollaire 1 *Il existe un isomorphisme ("de Cartier q-déformé") de R-modules*

$$C_q : \mathrm{H}^i(q\text{-DR}(A/R)) \xrightarrow{\sim} \Omega^i_{A'/R} \tag{43}$$

pour tout i.

Un suivi des différents morphismes permet de vérifier qu'il s'identifie bien à celui donné dans [16], Proposition 3.4, (iii) pour l'exemple $R = \mathcal{O}_K$ de 3.1.

5.4. La condition de q-divisibilité de 4.1 garde un sens lorsque p est remplacé par une puissance de p et l'hypothèse de q-divisibilité de R correspondante est cruciale pour généraliser à ce cadre les principaux résultats ci-dessus (cf. [9]). On notera ici qu'elle n'est, en général, pas vérifiée pour $R = \mathcal{O}_K$ et $q \neq 1$ une racine p^n-ième ($n > 1$) de l'unité comme dans 3.1.

6 Questions-Travaux en cours

6.1. Pour ce qui est du lien avec [3], les questions que nous nous posons sont toutes celles motivées par l'espoir suivant, dont les termes seront précisés le moment venu :

L'équivalence de catégories du Théorème 4 est un corollaire de l'explicitation locale d'une équivalence canonique, compatible (à torsion près en général) au passage à la cohomologie, entre une catégorie convenable de cristaux sur un site q-cristallin ([3], 16.2) *et une autre de cristaux sur un site prismatique* ([3], 4.1).

Indépendamment de [3], une première étape pourrait consister à reformuler le Théorème 4 comme un cas particulier d'une équivalence entre des catégories de D-modules q-déformés convenables, le modèle "non q-déformé" étant le point de vue proposé par Shiho ([17], Theorem 3.1) consistant à voir la correspondance d'Ogus et Vologodsky comme cas particulier d'un résultat plus général. Pour ce faire, il devrait être utile d'introduire (suivant les mêmes lignes que celles utilisées pour définir (26)) un *anneau d'opérateurs différentiels q-tordus de niveau -1* (avec q "générique") déformant celui introduit par Shiho ([17], §2) et intervenant dans sa généralisation de [13].

Ensuite, dans une seconde étape, pour faire le lien entre [3] et nos constructions, l'idée la plus naturelle est de généraliser la classique équivalence entre catégories de cristaux et catégories de $\mathcal{D}$-modules et sa compatibilité au passage à la cohomologie au cadre des sites évoqués ci-dessus et des anneaux d'opérateurs différentiels q-déformés qui leur correspondent.

Enfin, il restera, dans une dernière étape, à définir dans un cadre géométrique non nécessairement "local", le foncteur canonique entre cristaux qu'on espère pouvoir s'expliciter comme "Frobenius divisé" au niveau des algèbres d'opérateurs différentiels q-déformés de niveau 0 et -1. Ce foncteur devrait simplement être celui induit par le morphisme image inverse déduit du morphisme de sites décrit dans ([3], début de la preuve du Theorem 16.17).

6.2. Donnons, en conservant les notations de 3.1 et en supposant que R modulo $(p)_q$ soit q-divisible, quelques indications sur la première étape et sur la définition de l'anneau $\mathrm{D}^{(-1)}_{A,\sigma}$ d'opérateurs différentiels q-tordus de niveau -1 (d'autres niveaux négatifs, comme dans [17], sont évidemment possibles). La définition de $\mathrm{D}^{(-1)}_{A,\sigma}$ suit celle de $\mathrm{D}^{(0)}_{A,\sigma}$ (27) en remplaçant *formellement* partout $A\langle\xi\rangle_{q,y}$ par $A\langle\frac{\xi}{(p)_q}\rangle_{q^p,y}$ et en modifiant en conséquence (2), etc. On montre alors que la donnée d'une structure de $\mathrm{D}^{(-1)}_{A,\sigma}$-module sur un A-module M est équivalente à la donnée d'une *σ^p-dérivation de niveau -1*, i.e. (comparer avec (3)) d'une application R-linéaire $\partial^{<1>_1}_{\sigma^p,M} : M \to M$ telle que, pour tous $r \in A$, $m \in M$, on ait.

$$\partial^{<1>_1}_{\sigma^p,M}(rm) = (p)_q\partial_{\sigma^p}(r)m + \sigma^p(r)\partial^{<1>_1}_{\sigma^p,M}(m). \tag{44}$$

Il est facile de voir qu'il existe un foncteur "image inverse par Frobenius (relatif)", analogue q-déformé de ([17], Theorem 3.1), de la catégorie des $\mathrm{D}^{(-1)}_{A',\sigma}$-modules dans celle des $\mathrm{D}^{(0)}_{A,\sigma}$-modules dont nous pensons savoir démontrer ([7]) que c'est une équivalence de catégories sur les objets quasi-nilpotents.

En particulier, lorsque $q^p = 1$, un $\mathrm{D}^{(-1)}_{A',\sigma}$-module M n'est pas autre chose qu'un A'-module de Higgs et le théorème 4 serait alors un cas particulier de cette équivalence de catégories plus générale.

6.3. Reprenons les notations de 3.1 et supposons de plus que R soit une algèbre au-dessus de $\mathbb{Z}_p[[q-1]]$ munie d'une structure de δ-anneau ([3], Definition 2.1) telle que $\delta(q) = 0$ (comme pour $\mathbb{Z}_p[[q-1]]$). Supposons également A munie d'une structure de δ-R-algèbre telle que $\delta(x) = 0$, structure qu'on étendra à $A[\xi]$ en posant

$$\delta(\xi) = \sum_{1\le i\le p-1} \frac{1}{p}\binom{p}{i} x^{p-i}\xi^i. \tag{45}$$

Le premier site qui nous intéresse dans 6.1 est le *site q-cristallin*[1]([3], 16.2) de $A/(q-1)$ relativement à $(R,(q-1))$. Le point crucial dans la seconde étape espérée dans 6.1 est une identification de $(A\langle\xi\rangle_{q,y}, I^{[1]_{q,y}})$ (3.3) avec la q-PD-enveloppe ([3], Lemma 16.10) de $(A[\xi],(\xi))$. Précisons le résultat auquel nous parvenons. Pour définition d'une *q-PD-paire* (B, J) (comparer [3], Definition 16.2) ne retenons ici

[1]CAVEAT : Nous empruntons ici et plus bas, abusivement, la terminologie de [3] mais ignorons dans nos rappels certaines des propriétés additionnelles sur les objets requises dans *loc. cit.* si elles ne jouent pas de rôle dans ce que l'on veut expliquer ici (voir d'ailleurs, à ce sujet, les commentaires sur leur éventuel caractère provisoire sous la définition 16.2 de [3]). Les ajustements précis avec les hypothèses de [3], particulièrement ceux nécessitant de prendre en compte complétions et topologies (ne serait-ce que dans la définition des sites) seront donnés dans [7].

(cf. [1]) que la donnée d'une δ-algèbre B au-dessus de (R, δ), sans $(p)_p$-torsion, munie d'un idéal J tel que $\phi(J) \subset (p)_q B$ avec $\phi(b) := b^p + p\delta(b)$ pour tout $b \in B$. Si C est une δ-R-algèbre et I un idéal quelconque de C (auquel cas, on dira que (C, I) est une *δ-paire*, cf. ([3], Definition 3.2)), sa *q-PD-enveloppe*, notée $(C^{[\,]}, I^{[\,]})$, est pour nous ici la q-PD-paire universelle (dont la proposition ci-dessous prouve, pour le cas qui la concerne, l'existence et l'unicité à isomorphisme près) pour le prolongement (unique) à $(C^{[\,]}, I^{[\,]})$ de tout morphisme $(C, I) \to (B, J)$ d'une δ-paire dans une q-PD-paire. On a alors la

Proposition 9 *([7]) Si A est une δ-R-algèbre sans $(p)_q$-torsion, alors la q-PD-enveloppe*[1] *de la δ-paire $(A[\xi], (\xi))$ s'identifie $(A\langle\xi\rangle_{q,y}, I^{[1]_{q,y}})$ (3.3).*

La démonstration consiste à se ramener au cas $R = \mathbb{Z}_p[[q-1]]$ et $A = R[x]$ puis, utilisant l'écriture p-adique de $n = \sum_{r\geq 0} k_r p^r$, de montrer que les $v_n := \xi^{k_0} \prod_{r\geq 0}(\delta^r([\phi](\xi)))^{k_r+1}$ forment, pour $n \in \mathbb{N}$, une base, comme A-module, de $A\langle\xi\rangle_{q,y}$. Ici

$$[\phi] : A\langle\xi\rangle_{q,y} \to A\langle\xi\rangle_{q,y} \ ; \ [\phi](\xi^{[n]_{q,y}}) = \sum_{i=n}^{pn} B_{n,i}(q) x^{pn-i} \xi^{[i]_{q,y}} \tag{46}$$

tient compte, par rapport à [F*] (34), de l'usage du Frobenius absolu dans [3] plutôt que relatif dans [9]. Si maintenant (B, J) est une q-PD-paire, tout morphisme de δ-paires $u : (A[\xi], (\xi)) \to (B, J)$ s'étend alors uniquement à $A\langle\xi\rangle_{q,y}$ par un morphisme d'anneaux envoyant $v_n \in A\langle\xi\rangle_{q,y}$ sur $f^{k_0} \prod_{r\geq 0}(\delta^r(g))^{k_r+1} \in B$ avec $f := u(\xi)$ et $g \in B$ unique tel que $\phi(f) = (p)_q g$.

Signalons que le cas $q = 1$ est celui traité dans ([3], Lem. 2.35) et, pour le lecteur intéressé, l'existence d'un analogue ([15], Lem. 1.3), au moins lorsque $q - 1 \in R^\times$, pour les λ-anneaux.

6.4. Conservons les notations de 6.3 et supposons que $(R, (p)_q)$ soit un *prisme borné* ([3], Definition 3.2) pour pouvoir référer à [3]. Notons $A^{(1)}$ le quotient $A'/(p)_q A'$. Le second site qui nous intéresse dans 6.1 est le *site prismatique* de $A^{(1)}$ relativement à $(R, ((p)_q))$. Comme dans 6.3, le point crucial dans la seconde étape espérée dans 6.1 est de disposer d'une description adéquate (que nous appliquerons *in fine* à A' plutôt qu'à A) de l'*enveloppe prismatique* ([3], Corollary 3.14) de $(A[\xi], (\xi))$ relativement à $(R, ((p)_q))$. Reprenant les termes de la construction donnée dans *loc. cit.*, considérons donc juste ici la question du prolongement universel d'un morphisme de δ-paires (au-dessus de la δ-paire $(R, ((p)_q))$) $u : (A[\xi], (\xi)) \to (B, ((p)_q))$ avec B sans $(p)_q$-torsion à une δ-paire de la forme $(C, ((p)_q))$. Nous montrons qu'un tel objet universel existe et nous l'appellerons (cf. [1]) dans la proposition qui suit *enveloppe prismatique* de $(A[\xi], (\xi))$.

Proposition 10 *([7]) Si A est une δ-R-algèbre sans $(p)_q$-torsion, alors l'enveloppe prismatique*[1] *de la δ-paire $(A[\xi], (\xi))$ s'identifie à $(A\langle\frac{\xi}{(p)_q}\rangle_{q^p,y}, ((p)_q))$ (6.2).*

En effet, la variante de (30) utilisant le Frobenius absolu de A plutôt que relatif comme dans [9] fournit une application (dont (46) est la variante "divisée")

$$\phi : A\langle\xi\rangle_{q,y} \to A\langle\xi\rangle_{q,y} \ ; \ \phi(\xi^{[n]_{q,y}}) = \sum_{i=n}^{pn} A_{n,i}(q)x^{pn-i}\xi^{[i]_{q,y}}. \tag{47}$$

L'équation (33) suffit alors à voir (rappelons au passage que $\phi(q) = q^p$) que $A\langle\frac{\xi}{(p)_q}\rangle_{q^p,y}$ est bien muni d'un relèvement de Frobenius et, par suite, d'une structure de δ-anneau. Enfin, le même argument que pour la Proposition 9 (avec $u(\xi) = (p)_q g$) donne le prolongement cherché de $u : (A[\xi], (\xi)) \to (B, (p)_q B)$ à $(A\langle\frac{\xi}{(p)_q}\rangle_{q^p,y}, ((p)_q))$.

6.5. Pour terminer, remarquons que les arguments de Shiho ([17]) ne nécessitaient pas d'interprétation de ses $\mathrm{D}^{(-1)}$-modules (*loc. cit* §2) quasi-nilpotents en termes de cristaux sur un site mais que, lorsque $q = 1$, le site prismatique ([3], 4.1) en fournit une, qui dans ce cas particulier est juste une variante "avec δ-structures" de celle déjà établie dans ([14], Definition 1.3.1, [19], Definition 7.1) (l'ajout de δ-structures évitant précisément les puissances divisées additionnelles sur les anneaux d'opérateurs différentiels considérés des ces articles). Enfin, compte tenu des considérations topologiques délicates à développer sur les sites considérés dans 6.3-6.4 nous laissons pour ailleurs la discussion d'une possible approche alternative à l'équivalence cherchée dans 6.1 qui serait l'analogue de ([14], Theorem 1.4.3) (équivalence de topos).

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Crystalline $\mathbb{Z}_p$-Representations and A_{inf}-Representations with Frobenius

Takeshi Tsuji

Abstract In the late '80s, Faltings established an integral p-adic Hodge theory with coefficients, in which he generalized Fontaine–Laffaille theory of crystalline $\mathbb{Z}_p$-representations of the absolute Galois group of a p-adic field to the fundamental group of a non-singular algebraic variety over a p-adic field with good reduction. In this paper, we study the theory of coefficients above in the framework of integral p-adic Hodge theory via A_{inf}-cohomology recently introduced by Bhatt, Morrow, and Scholze. We give a local theory (i.e. a theory on an affine open) of A_{inf}-cohomology for a p-torsion free crystalline $\mathbb{Z}_p$-representation of the fundamental group by constructing the associated A_{inf}-representation with Frobenius, which is a variant of the construction by N. Wach of the (φ, Γ)-module associated to a crystalline $\mathbb{Z}_p$-representation of the absolute Galois group.

Keywords Integral p-Adic Hodge Theory · Relative Fontaine–Laffaille Theory · A_{inf}-Cohomology

Mathematics Subject Classification (2010) 14F30 · 14F20 · 14F40

1 Introduction

Let $\mathcal{O}$ be the ring of integers of a complete algebraically closed nonarchimedian extension C of $\mathbb{Q}_p$, let k be the residue field of $\mathcal{O}$, and let A_{inf} be the period ring associated to $\mathcal{O}$ defined by Fontaine (see Sect. 2). Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}$. In [7], Bhatt, Morrow, and Scholze introduced a new cohomology theory $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ lying in the derived category of A_{inf}-modules, and opened a way to compare the integral p-adic étale cohomology $H^i_{\text{ét}}(X_C, \mathbb{Z}_p)$ with the crystalline cohomology $H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$ and the integral de Rham cohomology $H^i_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O})$ for *any* i. In all related preceding work, we can deal with integral p-adic cohomolo-

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gies only when i is smaller compared to p, and we may say that their theory is a breakthrough in the study of integral p-adic cohomologies. However the theory is developed only for the constant coefficients, while an integral p-adic Hodge theory with coefficients (for small i) was established by Faltings [10] in the late '80s. Therefore it is natural to ask whether we have a similar theory for the A_{inf}-cohomology. The purpose of this paper is to give a partial positive answer to this question.

Let K be a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field k, and let O_K be the ring of integers of K. We assume that p is a uniformizer of O_K. Let X be a proper smooth scheme over O_K. In [10], as a theory of coefficients, Faltings generalized the theory of Fontaine–Laffaille on p-torsion crystalline representations of the absolute Galois group of K to locally constant constructible p-torsion sheaves on $X_{K,\text{ét}}$. More precisely, he introduced the category $\mathfrak{MF}^{\nabla}_{[a,b],\mathrm{tor}}(X)$ $(0 \leq b - a \leq p - 2)$ consisting of "p-torsion filtered Frobenius crystals of level with in $[a, b]$", and constructed a fully faithful functor T_{crys} to the category of locally constant constructible p-torsion sheaves on $X_{K,\text{ét}}$. We have an obvious analogue of the theory for smooth $\mathbb{Z}_p$-sheaves on $X_{K,\text{ét}}$. In this paper, we show that there exists a hopeful local theory of A_{inf}-cohomology for a torsion free smooth $\mathbb{Z}_p$-sheaf contained in the essential image of the functor T_{crys} as follows.

We fix some notation used throughout this paper. Let K, k, and O_K be as above. Let $X = \mathrm{Spec}(A)$ be an affine smooth scheme over $\mathrm{Spec}(O_K)$ such that the special fiber $\mathrm{Spec}(A \otimes_{O_K} k)$ is non-empty and geometrically connected, let $\mathcal{A}$ be the p-adic completion of A, which is a noetherian regular domain, and assume that there exist coordinates $t_1, \ldots, t_d \in A^{\times}$ of A over O_K. We choose and fix an algebraic closure $\overline{K}$ of K, $O_{\overline{K}}$ denotes its ring of integers, and G_K denotes the Galois group $\mathrm{Gal}(\overline{K}/K)$. Let $\mathcal{K}$ be the field of fractions of $\mathcal{A}$, let $\overline{\mathcal{K}}$ be an algebraic closure of $\mathcal{K}$ containing $\overline{K}$, and let $\mathcal{K}^{\mathrm{ur}}$ be the union of all finite extensions $\mathcal{L} \subset \overline{\mathcal{K}}$ of $\mathcal{K}$ such that the integral closure of $\mathcal{A}[\frac{1}{p}]$ in $\mathcal{L}$ is étale over $\mathcal{A}[\frac{1}{p}]$. We define $G_{\mathcal{A}}$ (resp. $\Delta_{\mathcal{A}}$) to be the Galois group $\mathrm{Gal}(\mathcal{K}^{\mathrm{ur}}/\mathcal{K})$ (resp. $\mathrm{Gal}(\mathcal{K}^{\mathrm{ur}}/\mathcal{K}\overline{K})$), which is the fundamental group of the generic fiber (resp. the geometric generic fiber) of $\mathrm{Spec}(\mathcal{A})$ with base point $\mathrm{Spec}(\overline{\mathcal{K}})$. Let $\overline{\mathcal{A}}$ be the integral closure of $\mathcal{A}$ in $\mathcal{K}^{\mathrm{ur}}$, and let $A_{\mathrm{inf}}(\overline{\mathcal{A}})$ be the Fontaine's period ring associated to $\overline{\mathcal{A}}$ (see Sect. 2 for details).

For a "free" object M of the category $\mathfrak{MF}^{\nabla}_{[0,p-2]}(\mathcal{A})$, we construct a semilinear $A_{\mathrm{inf}}(\overline{\mathcal{A}})$-representation $TA_{\mathrm{inf}}(M)$ of $G_{\mathcal{A}}$ with Frobenius structure (Sect. 8), generalizing some arguments (Sects. 6 and 7) used in the proof of the theorem of Wach [20] relating (φ, Γ)-theory and Fontaine–Laffaille theory. For the free $\mathbb{Z}_p$-representation $T_{\mathrm{crys}}(M)$ of $G_{\mathcal{A}}$ associated to M, the representation $TA_{\mathrm{inf}}(M)$ is eventually characterized as a unique free $G_{\mathcal{A}}$-stable "lattice" of $T_{\mathrm{crys}}(M) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}(\overline{\mathcal{A}})[\frac{1}{\pi}]$ "trivial" modulo π (see Proposition 76, Theorem 70 and Lemma 64). Here π is the element denoted by μ in [7] and is defined in the paragraph after (1). We prove that the functor TA_{inf} is fully faithful (Theorem 63 (1)), and together with the above characterization, we obtain a new proof of the fully faithfulness of the functor T_{crys} (Theorem 77).

The A_{inf}-cohomology $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ is defined as the cohomology of a complex of $A_{\mathrm{inf}}(\mathcal{O})$-modules on $\mathfrak{X}_{\mathrm{Zar}}$ denoted by $A\Omega_{\mathfrak{X}}$, and its (derived) section on a (small) affine formal scheme $\mathrm{Spf}(R)$ is given by "$A\Omega_R = L\eta_\mu R\Gamma(\Delta, A_{\mathrm{inf}}(\overline{R}))$". A natural candidate of an analogue of $A\Omega_R$ for M is $A\Omega_{\mathcal{A}}(M) := L\eta_\pi R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M))$ (Sect. 15). We see that this has the following relation to the $\mathbb{Z}_p$-representation $T_{\mathrm{crys}}(M)$ of $G_{\mathcal{A}}$ and to the de Rham complex of M similarly to [7, Theorem 14.1]. For the former, the characterization of $TA_{\mathrm{inf}}(M)$ as a lattice of $T_{\mathrm{crys}}(M) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}(\overline{\mathcal{A}})[\frac{1}{\pi}]$ immediately implies that $A\Omega_{\mathcal{A}}(M)$ is isomorphic to $R\Gamma(\Delta_{\mathcal{A}}, T_{\mathrm{crys}}(M) \otimes_{\mathbb{Z}} A_{\mathrm{inf}}(\overline{\mathcal{A}}))$ after inverting π (Theorem 106). As for the latter, there exists a canonical isomorphism independent of the choice of $t_1, \ldots, t_d$ (Theorem 204)

$$A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M \otimes_{\mathcal{A}} \Omega^\bullet_{\mathcal{A}} \xrightarrow{\cong} A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}^L_{A_{\mathrm{inf}}(O_{\overline{K}})} A\Omega_{\mathcal{A}}(M)$$

in the derived category of $A_{\mathrm{crys}}(O_{\overline{K}})$-modules with semilinear action of $\underline{G_K}$(:=G_K with the discrete topology). We can apply the construction of $TA_{\mathrm{inf}}(M)$ also to the period ring $A^{\square}_{\mathrm{inf}}(\mathcal{A})$ associated to a framing (i.e. $t_1, \ldots, t_d$) considered in [7, Sect. 9], and obtain an $A^{\square}_{\mathrm{inf}}(\mathcal{A})$-representation with Frobenius $TA^{\square}_{\mathrm{inf}}(M)$ (Sect. 13). Then we can describe $A\Omega_{\mathcal{A}}(M)$ in terms of the Koszul complex associated to $TA^{\square}_{\mathrm{inf}}(M)$ similarly to [7, Sect. 9], and it allows us to construct the isomorphism above with $\underline{G_K}$-action forgotten (Sect. 15). This construction of the isomorphism heavily relies on the framing, and we prove the independence by giving an alternative construction (different from [7, Sect. 12]), to which the last five sections (Sects. 17–21) are devoted, and which recovers $\underline{G_K}$-action as well.

2 Period Rings

Let σ be the unique lifting of the absolute Frobenius of k to K. As in Sect. 1, let $\overline{K}$ be an algebraic closure of K, and let $O_{\overline{K}}$ be the ring of integers of $\overline{K}$. Let C be the completion of $\overline{K}$, let O_C be the ring of integers of C, and let v_C be the valuation of C normalized by $v_C(p) = 1$.

Let Λ be a normal domain containing $O_{\overline{K}}$. Assume that $\Lambda/p\Lambda \neq 0$ and the absolute Frobenius of $\Lambda/p\Lambda$ is surjective. Let R_Λ be the inverse limit of $\Lambda/p\Lambda \overset{F}{\leftarrow} \Lambda/p\Lambda \overset{F}{\leftarrow} \Lambda/p\Lambda \overset{F}{\leftarrow} \Lambda/p\Lambda \overset{F}{\leftarrow} \cdots$, where F denotes the absolute Frobenius. Then the absolute Frobenius of R_Λ is bijective. We regard R_Λ as a k-algebra by the homomorphism $k \to R_\Lambda; x \mapsto (x^{p^{-n}})_{n\in\mathbb{N}}$. We define the $O_K = W(k)$-algebra $A_{\mathrm{inf}}(\Lambda)$ to be the ring of Witt vectors $W(R_\Lambda)$, which is p-torsion free, and p-adically complete and separated. It has a canonical lifting of the absolute Frobenius of $W(R_\Lambda)/p \cong R_\Lambda$ compatible with σ, which is an automorphism and is denoted by φ in the following. We have a ring homomorphism $\theta\colon A_{\mathrm{inf}}(\Lambda) \to \widehat{\Lambda}$ characterized by $\theta([a]) = \lim_{n\to\infty} \widetilde{a}_n^{p^n}$ for $a = (a_n)_{n\in\mathbb{N}} \in R_\Lambda$, where $\widehat{\Lambda}$ denotes the p-adic completion $\varprojlim_m \Lambda/p^m\Lambda$ of Λ and $\widetilde{a}_n$ denotes a lifting of a_n in Λ for each $n \in \mathbb{N}$ ([12, 1.2.2], [10, II (b)]). By the assumption that the absolute Frobenius of $\Lambda/p\Lambda$ is surjective, we see that θ is surjective ([12, 1.2.2], [10, II (b)], [18, Lemma

A1.1]). The homomorphism θ is compatible with the O_K-algebra structures because $\theta([(x^{p^{-n}})_{n\in\mathbb{N}}]) = \lim_{n\to\infty}[x^{p^{-n}}]^{p^n} = [x]$ for $x \in k$.

Choose a compatible system of p^nth roots β_n ($n \in \mathbb{N}$) of p in $O_{\overline{K}}$: $\beta_{n+1}^p = \beta_n$ ($n \in \mathbb{N}$), $\beta_0 = p$, and define the element $\underline{p}$ of R_Λ to be $(\beta_n \mod p)_{n\in\mathbb{N}}$. The projection to the first component $R_\Lambda \to \Lambda/p\Lambda$; $(a_n)_{n\in\mathbb{N}} \mapsto a_0$ is surjective, its kernel is generated by $\underline{p}$, and $\underline{p}$ is not a zero divisor in R_Λ ([10, II (b)], [18, Lemma A2.1]). This implies that the projection to the $(l+1)$th component $R_\Lambda \to \Lambda/p\Lambda$; $(a_n)_{n\in\mathbb{N}} \mapsto a_l$ is surjective and its kernel is generated by $\underline{p}^{p^l}$ because its composition with the lth power of the absolute Frobenius of R_Λ coincides with the projection to the first component and the absolute Frobenius of R_Λ is bijective. The element $\xi := p - [\underline{p}]$ of $A_{\mathrm{inf}}(\Lambda)$ is a non-zero divisor and generates the ideal $\mathrm{Ker}(\theta)$ ([10, II (b)], [18, Corollary A2.2]). This implies that an element a of $\mathrm{Ker}(\theta)$ generates $\mathrm{Ker}(\theta)$ if and only its image in $A_{\mathrm{inf}}(\Lambda)/p \cong R_\Lambda$ is contained in $\underline{p}R_\Lambda^\times$ ([12, 5.1.2], [18, Corollary A2.4]). We define $\mathrm{Fil}^r A_{\mathrm{inf}}(\Lambda)$ ($r \in \mathbb{N}$) to be $\mathrm{Ker}(\theta)^r$ if $r > 0$ and $A_{\mathrm{inf}}(\Lambda)$ if $r \le 0$.

For $a = (a_n)_{n\in\mathbb{N}} \in R_{O_{\overline{K}}}$, we put $v_R(a) = v_C(\theta([a]))$. Then v_R is a valuation of $R_{O_{\overline{K}}}$, with which $R_{O_{\overline{K}}}$ is a complete valuation ring, and its field of fractions is algebraically closed. We have $v_R(\underline{p}) = p$ because $\theta([\underline{p}]) = p$. This implies that, for any non-zero element a of $R_{O_{\overline{K}}}$, we have $\underline{p}^n \in aR_{O_{\overline{K}}}$ for some $n \in \mathbb{N}$, and therefore the image of a in R_Λ is regular.

Lemma 1 *Let a be an element of $A_{\mathrm{inf}}(O_{\overline{K}})$, and assume that its image $\overline{a}$ in $A_{\mathrm{inf}}(O_{\overline{K}})/pA_{\mathrm{inf}}(O_{\overline{K}}) \cong R_{O_{\overline{K}}}$ is neither zero nor invertible.*

(1) *The (p, a)-adic topology of $A_{\mathrm{inf}}(\Lambda)$ coincides with the $(p, [\underline{p}])$-adic topology.*
(2) *$A_{\mathrm{inf}}(\Lambda)$ is (p, a)-adically complete and separated.*
(3) *$A_{\mathrm{inf}}(\Lambda)/aA_{\mathrm{inf}}(\Lambda)$ is p-torsion free, and p-adically complete and separated.*
(4) *$A_{\mathrm{inf}}(\Lambda)$ and $A_{\mathrm{inf}}(\Lambda)/p^n$ ($n \in \mathbb{N}_{>0}$) are a-torsion free, and a-adically complete and separated.*

Lemma 2 *Let R be a flat $\mathbb{Z}_p$-algebra p-adically complete and separated, and let a be an element of R such that R/pR is a-torsion free, and a-adically complete and separated.*

(1) *R is (a, p)-adically complete and separated.*
(2) *R/a^nR ($n \in \mathbb{N}_{>0}$) are p-torsion free, and p-adically complete and separated.*
(3) *R and R/p^nR ($n \in \mathbb{N}_{>0}$) are a-torsion free, and a-adically complete and separated.*

Lemma 3 *Let R be a commutative ring.*

(1) *Let M be an R-module, and let a and b be two elements of R regular on M. Then we have a canonical isomorphism $(M/aM)[b] \cong (M/bM)[a]$.*
(2) *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules. Let a be an element of R regular on M_3. Then, if two of the three R-modules M_i ($i \in \{1, 2, 3\}$) are a-adically complete and separated, then the remaining one is also a-adically complete and separated.*

Proof (1) We obtain the claim by applying the snake lemma to the multiplication by b on the short exact sequence $0 \to M \xrightarrow{a} M \to M/aM \to 0$.

(2) Since a is regular on M_3, the exact sequence in the claim induces exact sequences $0 \to M_1/a^n \to M_2/a^n \to M_3/a^n \to 0$ ($n \in \mathbb{N}$). By taking the inverse limit over n, we obtain the following homomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varprojlim_n M_1/a^n & \longrightarrow & \varprojlim_n M_2/a^n & \longrightarrow & \varprojlim_n M_3/a^n & \longrightarrow & 0. \end{array}$$

If the two of the three vertical homomorphisms are isomorphisms, then so is the rest. □

Proof of Lemma 2 Since R/p is a-torsion free, and R is p-torsion free and p-adically complete and separated, we see that R/p^n is a-torsion free by induction on n, and that R is a-torsion free by taking the inverse limit. Since a and p are regular in R, we have $R/a^n[p] \cong R/p[a^n] = 0$ by Lemma 3 (1). By applying Lemma 3 (2) to the exact sequence $0 \to R \xrightarrow{a^n} R \to R/a^n \to 0$, we see that R/a^n is p-adically complete and separated.

We have an exact sequence $0 \to R/p^n \xrightarrow{p} R/p^{n+1} \to R/p \to 0$ since R is p-torsion free. Since R/p is a-torsion free and a-adically complete and separated by assumption, we see that R/p^n is a-adically complete and separated by induction on n by applying Lemma 3 (2) to the above exact sequence. Now we have $R \cong \varprojlim_n R/p^n \cong \varprojlim_n (\varprojlim_m R/(p^n, a^m)) \cong \varprojlim_{(n,m)} R/(p^n, a^m)$, i.e., R is (a, p)-adically complete and separated. Finally we have $R \cong \varprojlim_n (\varprojlim_m R/(a^n, p^m)) \cong \varprojlim_n R/a^n$ because R/a^n is p-adically complete and separated. □

Proof of Lemma 1 By the assumption on $\overline{a}$, there exists $m \in \mathbb{N}$ such that $\overline{a}^m \in \underline{p}R_{O_{\overline{K}}}$ and $\underline{p}^m \in \overline{a}R_{O_{\overline{K}}}$. Since $a - [\overline{a}] \in pA_{\mathrm{inf}}(O_{\overline{K}})$, we have the equality $(p, a) = (p, [\overline{a}])$ of ideals of $A_{\mathrm{inf}}(O_{\overline{K}})$, which implies the following inclusions of ideals of $A_{\mathrm{inf}}(\Lambda)$: $(p, a)^m \subset (p, [\underline{p}])$ and $(p, [\underline{p}])^m \subset (p, a)$. Hence the claim (1) holds. The image of $\overline{a}$ in R_Λ is regular as observed before Lemma 1. As the kernel of the projection $R_\Lambda \to \Lambda/p$; $(a_n)_{n\in\mathbb{N}} \mapsto a_l$ is generated by $\underline{p}^{p^l}$ for $l \in \mathbb{N}$, $A_{\mathrm{inf}}(\Lambda)/p = R_\Lambda$ is $\underline{p}$-adically complete and separated. Since $\overline{a}^m \in \underline{p}R_{O_{\overline{K}}}$ and $\underline{p}^m \in \overline{a}R_{O_{\overline{K}}}$, $A_{\mathrm{inf}}(\Lambda)/p$ is $\overline{a}$-adically complete and separated. Thus we can apply Lemma 2 to $A_{\mathrm{inf}}(\Lambda)$ and a, and obtain the claims (2), (3), and (4). □

Lemma 4 *Let Λ' be another normal domain containing Λ such that $\Lambda'/p\Lambda' \neq 0$ and the absolute Frobenius of $\Lambda'/p\Lambda'$ is bijective. Assume that the homomorphism $\Lambda/p\Lambda \to \Lambda'/p\Lambda'$ is injective. Let a and $\overline{a}$ be the same as in Lemma 1. Then the natural homomorphism $A_{\mathrm{inf}}(\Lambda) \to A_{\mathrm{inf}}(\Lambda')$ and its reduction modulo p^m ($m \in \mathbb{N}_{>0}$), modulo a, and modulo (a, p^m) ($m \in \mathbb{N}_{>0}$) are injective. In particular, $A_{\mathrm{inf}}(\Lambda) \to A_{\mathrm{inf}}(\Lambda')$ is strictly compatible with the filtrations* $\mathrm{Fil}^\bullet$. (*Note that $\xi^r \in A_{\mathrm{inf}}(O_{\overline{K}})$ ($r \in \mathbb{N}_{>0}$) satisfy the condition on a in Lemma 1.)*

Proof We assert that the claim for the reduction modulo (a, p) implies those for the other ideals as $(a, p) \Rightarrow (a, p^m) \Rightarrow (p^m), (a) \Rightarrow (0)$. One can prove the first implication by induction on m using the exact sequence $0 \to A_{\mathrm{inf}}(\Lambda^{(\prime)})/(a, p^m) \xrightarrow{p} A_{\mathrm{inf}}(\Lambda^{(\prime)})/(a, p^{m+1}) \to A_{\mathrm{inf}}(\Lambda^{(\prime)})/(a, p) \to 0$ (Lemma 1 (3)). As for the second one, we note that a^n $(n \in \mathbb{N}_{>0})$ also satisfy the condition on a in Lemma 1. Then the claim follows from the fact that $A_{\mathrm{inf}}(\Lambda^{(\prime)})/p^m$ (resp. $A_{\mathrm{inf}}(\Lambda^{(\prime)})/a$) is a (resp. p)-adically complete and separated (Lemma 1 (4) (resp. (3)). Similarly the last implication is a consequence of the fact that $A_{\mathrm{inf}}(\Lambda^{(\prime)})$ is p-adically complete and separated.

Let us prove the claim for the reduction modulo (a, p). Put $\overline{a} = (\overline{a}_n)_{n\in\mathbb{N}}$ $(\overline{a}_n \in O_{\overline{K}}/p)$, and choose an integer m such that $\overline{a}_m \neq 0$, which implies $v_R(\overline{a}) < v_R(\underline{p}^{p^m})$ and $\underline{p}^{p^m} R_{O_{\overline{K}}} \subset \overline{a} R_{O_{\overline{K}}}$ because we have an isomorphism $R_{O_{\overline{K}}}/\underline{p}^{p^m} \xrightarrow{\cong} O_{\overline{K}}/pO_{\overline{K}}$; $(x_n)_{n\in\mathbb{N}} \mapsto x_m$. Let a_m be a lifting of $\overline{a}_m$ in $O_{\overline{K}}$. Then we have isomorphisms $A_{\mathrm{inf}}(\Lambda^{(\prime)})/(p, a) = R_{\Lambda^{(\prime)}}/\overline{a} = R_{\Lambda^{(\prime)}}/(\underline{p}^{p^m}, \overline{a}) \xrightarrow{\cong} \Lambda^{(\prime)}/(p, a_m) = \Lambda^{(\prime)}/a_m$ induced by $R_{\Lambda^{(\prime)}} \to \Lambda^{(\prime)}/p$; $(x_n)_{n\in\mathbb{N}} \mapsto x_m$. The injectivity of $\Lambda/p \to \Lambda'/p$ implies that of $\Lambda/a_m \to \Lambda'/a_m$ because the multiplication by pa_m^{-1} on $\Lambda^{(\prime)}$ induces an injective homomorphism $\Lambda^{(\prime)}/a_m \hookrightarrow \Lambda^{(\prime)}/p$. This completes the proof. □

We endow $A_{\mathrm{inf}}(\Lambda)$ with the $(p, [\underline{p}])$-adic topology. In the following, we assume that we are given a subring Λ_0 of Λ over which Λ is integral and that $\mathrm{Frac}(\Lambda)/\mathrm{Frac}(\Lambda_0)$ is a Galois extension. Let $G(\Lambda/\Lambda_0)$ denote the Galois group $\mathrm{Gal}(\mathrm{Frac}(\Lambda)/\mathrm{Frac}(\Lambda_0))$. Then Λ is a $G(\Lambda/\Lambda_0)$-stable subalgebra of $\mathrm{Frac}(\Lambda)$, and therefore we have a natural action of $G(\Lambda/\Lambda_0)$ on $A_{\mathrm{inf}}(\Lambda)$ with φ and Fil^r. The homomorphism $\theta\colon A_{\mathrm{inf}}(\Lambda) \to \widehat{\Lambda}$ is $G(\Lambda/\Lambda_0)$-equivariant.

Lemma 5 *The action of $G(\Lambda/\Lambda_0)$ on $A_{\mathrm{inf}}(\Lambda)$ is continuous.*

Proof Let n and m be positive integers, and put $l = m + (n-1)$. Then the homomorphism $W_n(R_\Lambda) \to W_n(R_\Lambda)/[\underline{p}^{p^m}]$ factors through $W_n(R_\Lambda/\underline{p}^{p^l})$ because

$$(a_0\underline{p}^{p^l}, a_1\underline{p}^{p^l}, \ldots, a_{n-1}\underline{p}^{p^l}) = \sum_{\nu=0}^{n-1} p^\nu [a_\nu^{p^{-\nu}} \underline{p}^{p^{l-\nu}}] \in [\underline{p}^{p^m}] W_n(R_\Lambda)$$

for $a_\nu \in R_\Lambda$ $(\nu \in \mathbb{N} \cap [0, n-1])$. The action of $G(\Lambda/\Lambda_0)$ on $W_n(R_\Lambda/\underline{p}^{p^l})$ with the discrete topology is continuous because $R_\Lambda/\underline{p}^{p^l} \cong \Lambda/p$; $(a_n)_{n\in\mathbb{N}} \mapsto a_l$. □

Before introducing another period ring $A_{\mathrm{crys}}(\Lambda)$, we give two preliminary lemmas. Let γ be the unique PD-structure on the ideal pO_K of O_K.

Lemma 6 *Let S be an O_K-algebra, and let (I_S, γ_S) be a PD-ideal of S for which $p \in I_S$ and γ_S is compatible with the unique PD-structure γ on pO_K. Let R be an S-algebra, let I be an ideal of R, and let $(\overline{R}, \overline{I}, \delta)$ be the PD-envelope of (R, I) compatible with γ_S ([4, I Définition 2.4.2], [5, 3.19 Theorem]).*

(1) *The O_K-algebra $\overline{R}$ with the ideal $\overline{I}' := \overline{I} + p\overline{R}$ equipped with the PD-structure δ' compatible with δ and γ is the PD-envelope of $(R, I + pR)$ compatible with γ_S.*

Suppose that we are given liftings $\varphi_S \colon S \to S$ and $\varphi_R \colon R \to R$ of the absolute Frobenius of S/pS and R/pR compatible with $\sigma \colon O_K \to O_K$ such that φ_S is a PD-morphism with respect to (I_S, γ_S) and $\varphi_R \circ f = f \circ \varphi_S$ for the structure homomorphism $f \colon S \to R$. By (1)*, φ_R and φ_S induce an endomorphism $\varphi_{\overline{R}}$ of the PD-ring $(\overline{R}, \overline{I}', \delta')$ compatible with φ_S.*

(2) *The reduction mod p of $\varphi_{\overline{R}}$ is the absolute Frobenius of $\overline{R}$.*
(3) *For $r \in \mathbb{N} \cap [0, p-1]$, the rth divided power $\overline{I}'^{[r]}$ of $\overline{I}'$ ([4, I Définition 3.1.1], [5, 3.24 Definition]) satisfies $\varphi_{\overline{R}}(\overline{I}'^{[r]}) \subset p^r\overline{R}$.*

Proof The claim (1) is obvious by the construction of the PD-envelope in the proof of [4, I Théorème 2.4.1] (see [5, 3.20 Remarks (1)]). The R-algebra $\overline{R}$ is generated by $\delta'_n(x)$ $(x \in I + pR, n \in \mathbb{N}_{>0})$ by [4, I Proposition 2.4.3 (ii)] (or [5, 3.20 Remarks (3)]), and the ideal $\overline{I}'^{[r]}$ $(r \in \mathbb{N}_{>0})$ is generated by $\delta'_{n_1}(x_1)\cdots\delta'_{n_s}(x_s)$ $(x_1, \dots, x_s \in I + pR, n_1 + \cdots + n_s \geq r)$ by [4, I Propositions 2.4.3 (ii), 3.1.3]. Let $x \in I + pR$. Then there exists $y \in R$ such that $\varphi_R(x) = x^p + py$. Hence, for $n \in \mathbb{N}_{>0}$, we have $\varphi_{\overline{R}}(\delta'_n(x)) = \delta'_n(x^p + py) = \gamma_n(p)((p-1)!\delta_p(x) + y)^n \in p^{\min\{n,p-1\}}\overline{R}$, and $\delta'_n(x)^p = p!\delta'_p(\delta'_n(x)) \in p\overline{R}$. This implies the claims (2) and (3). □

Lemma 7 *Let $(M_n)_{n\in\mathbb{N}_{>0}}$ be an inverse system consisting of flat $\mathbb{Z}/p^n$-modules such that the transition map induces an isomorphism $M_{n+1} \otimes_{\mathbb{Z}/p^{n+1}} \mathbb{Z}/p^n \xrightarrow{\cong} M_n$ for every $n \in \mathbb{N}_{>0}$. Then the inverse limit $M := \varprojlim_n M_n$ is flat over $\mathbb{Z}_p$, and the natural homomorphism $M/p^n \to M_n$ is an isomorphism for every $n \in \mathbb{N}_{>0}$. In particular, M is p-adically complete and separated.*

Proof Let m be a positive integer. By assumption, the multiplication by p^m on M_{n+m} induces an exact sequence $0 \to M_n \to M_{n+m} \to M_m \to 0$. By taking the inverse limit over n, we obtain an exact sequence $0 \to M \xrightarrow{p^m} M \to M_m \to 0$. □

Let $m \in \mathbb{N}_{>0}$. We define $A_{\mathrm{crys},m}(\Lambda)$ to be the divided power envelope compatible with γ of $A_{\mathrm{inf}}(\Lambda)/p^m$ with respect to the kernel of $(\theta \mod p^m)$. In $A_{\mathrm{crys},m}(\Lambda)$, we have $[\underline{p}]^p = (p-\xi)^p = p!(-\xi)^{[p]} + p\sum_{\nu=1}^{p}\binom{p}{\nu}p^{\nu-1}(-\xi)^{p-\nu} \in pA_{\mathrm{crys},m}(\Lambda)$. The action of $G(\Lambda/\Lambda_0)$ on $A_{\mathrm{inf}}(\Lambda)$ induces its action on the PD-ring $A_{\mathrm{crys},m}(\Lambda)$, which is continuous with respect to the discrete topology of $A_{\mathrm{crys},m}(\Lambda)$ by Lemma 5 because the homomorphism $A_{\mathrm{inf}}(\Lambda)/p^m \to A_{\mathrm{crys},m}(\Lambda)$ factors through $A_{\mathrm{inf}}(\Lambda)/(p^m, [\underline{p}]^{pm})$ and $A_{\mathrm{crys},m}(\Lambda)$ is generated by divided powers of the image of ξ in $A_{\mathrm{crys},m}(\Lambda)$ over $A_{\mathrm{inf}}(\Lambda)/p^m$ ([4, I Proposition 2.4.3 (ii)], [5, 3.20 Remarks (3)]); the image $\overline{\xi} \in A_{\mathrm{crys},m}(\Lambda)$ of ξ is invariant under an open subgroup H of $G(\Lambda/\Lambda_0)$ by Lemma 5, and we have $g(\overline{\xi}^{[n]}) = g(\overline{\xi})^{[n]} = \overline{\xi}^{[n]}$ for $n \in \mathbb{N}_{>0}$ and $g \in H$.

We define $\mathrm{Fil}^r A_{\mathrm{crys},m}(\Lambda)$ $(r \in \mathbb{N}_{>0})$ to be the rth divided power of the divided power ideal of $A_{\mathrm{crys},m}(\Lambda)$ ([4, I Définition 3.1.1], [5, 3.24 Definition]), which is generated by $\xi^{[s]}$ $(s \in \mathbb{N} \cap [r, \infty))$ as an ideal and also as an $A_{\mathrm{inf}}(\Lambda)/p^m$-module ([4, I

Propositions 2.4.3 (ii), 3.1.3 (i)]). It is stable under the action of $G(\Lambda/\Lambda_0)$. We set $\mathrm{Fil}^r A_{\mathrm{crys},m}(\Lambda) = A_{\mathrm{crys},m}(\Lambda)$ for an integer $r \leq 0$. By Lemma 6 (1) and (2), the canonical lifting of the absolute Frobenius on $A_{\mathrm{inf}}(\Lambda)$ induces a lifting φ of the absolute Frobenius of $A_{\mathrm{crys},m}(\Lambda)/p$ to $A_{\mathrm{crys},m}(\Lambda)$ compatible with the $G(\Lambda/\Lambda_0)$-action and the PD-structure on $pA_{\mathrm{crys},m}(\Lambda) + \mathrm{Fil}^1 A_{\mathrm{crys},m}(\Lambda)$ defined by the PD-structure on $\mathrm{Fil}^1 A_{\mathrm{crys},m}(\Lambda)$ and γ. By Lemma 6 (3), we have $\varphi(\mathrm{Fil}^r A_{\mathrm{crys},m}(\Lambda)) \subset p^r A_{\mathrm{crys},m}(\Lambda)$ ($r \in \mathbb{N} \cap [0, p-1]$). We define $A_{\mathrm{crys}}(\Lambda)$ to be the inverse limit of $A_{\mathrm{crys},m}(\Lambda)$ ($m \in \mathbb{N}_{>0}$) endowed with the inverse limit topology of the discrete topology of $A_{\mathrm{crys},m}(\Lambda)$. It is naturally endowed with a continuous action of $G(\Lambda/\Lambda_0)$, a decreasing filtration $\mathrm{Fil}^r A_{\mathrm{crys}}(\Lambda)$ and a σ-semilinear endomorphism φ.

The algebras $A_{\mathrm{crys},m}(\Lambda)$ and $A_{\mathrm{crys}}(\Lambda)$ with Fil^r, φ and $G(\Lambda/\Lambda_0)$-actions can be explicitly constructed as follows. Let $W^{\mathrm{PD}}(R_\Lambda)$ be the divided power envelope of $W(R_\Lambda)$ with respect to $\mathrm{Ker}(\theta)$ compatible with γ. The action of $G(\Lambda/\Lambda_0)$ on $W(R_\Lambda)$ induces its action on $W^{\mathrm{PD}}(R_\Lambda)$. We define $\mathrm{Fil}^r W^{\mathrm{PD}}(R_\Lambda)$ ($r \in \mathbb{N}_{>0}$) to be the rth divided power of the divided power ideal of $W^{\mathrm{PD}}(R_\Lambda)$, which is stable under the action of $G(\Lambda/\Lambda_0)$. We define $\mathrm{Fil}^r W^{\mathrm{PD}}(R_\Lambda)$ to be $W^{\mathrm{PD}}(R_\Lambda)$ for an integer $r \leq 0$. By Lemma 6, φ of $W(R_\Lambda)$ induces a $G(\Lambda/\Lambda_0)$-equivariant endomorphism of $W^{\mathrm{PD}}(R_\Lambda)$ compatible with σ, which is denoted again by φ. We have a unique PD-structure on $\mathrm{Ker}(\theta)[\frac{1}{p}] \subset W(R_\Lambda)[\frac{1}{p}]$ defined by $x \mapsto \frac{x^n}{n!}$ ($n \in \mathbb{N}$) ([4, I 1.2.1], [5, 3.2 Examples 2]), and it is compatible with γ. Hence, by the universal property of divided power envelopes, the homomorphism $W(R_\Lambda) \to W(R_\Lambda)[\frac{1}{p}]$ induces a PD-homomorphism $W^{\mathrm{PD}}(R_\Lambda) \to W(R_\Lambda)[\frac{1}{p}]$ compatible with φ and $G(\Lambda/\Lambda_0)$-actions. This homomorphism is injective, and therefore we may identify $W^{\mathrm{PD}}(R_\Lambda)$ with its image, which is the $W(R_\Lambda)$-subalgebra of $W(R_\Lambda)[\frac{1}{p}]$ generated by $\frac{\xi^n}{n!}$ ($n \in \mathbb{N}$) ([12, 2.3.3], [10, II (b)], [18, Proposition A2.8]). We have $\mathrm{Fil}^r W^{\mathrm{PD}}(R_\Lambda) = W^{\mathrm{PD}}(R_\Lambda) \cap \mathrm{Fil}^r W(R_\Lambda)[\frac{1}{p}]$ for $r \in \mathbb{Z}$ ([18, Lemma A2.9]). In particular, $\mathrm{Fil}^s W^{\mathrm{PD}}(R_\Lambda)/\mathrm{Fil}^r W^{\mathrm{PD}}(R_\Lambda)$ ($r, s \in \mathbb{Z}, s > r$) is p-torsion free. Thus we obtain an explicit construction of $W^{\mathrm{PD}}(R_\Lambda)$. Since $p^m O_K$ is a sub PD-ideal of pO_K, divided power envelopes compatible with γ are compatible with taking the reduction mod p^m ([5, 3.20 Remarks (8)]). Therefore the homomorphism $W(R_\Lambda) \to A_{\mathrm{inf}}(\Lambda)/p^m$ induces an isomorphism of PD-algebras $W^{\mathrm{PD}}(R_\Lambda)/p^m \xrightarrow{\cong} A_{\mathrm{crys},m}(\Lambda)$ compatible with φ and $G(\Lambda/\Lambda_0)$-actions. Since $W^{\mathrm{PD}}(R_\Lambda)/\mathrm{Fil}^r W^{\mathrm{PD}}(R_\Lambda)$ is p-torsion free, we see that the surjective homomorphism $\mathrm{Fil}^r W^{\mathrm{PD}}(R_\Lambda)/p^m \mathrm{Fil}^r W^{\mathrm{PD}}(R_\Lambda) \to \mathrm{Fil}^r A_{\mathrm{crys},m}(\Lambda)$ is an isomorphism for $r \in \mathbb{Z}$. Since $W^{\mathrm{PD}}(R_\Lambda)$ is p-torsion free, the above description of $A_{\mathrm{crys},m}(\Lambda)$ implies that $A_{\mathrm{crys}}(\Lambda)$ and $\mathrm{Fil}^r A_{\mathrm{crys}}(\Lambda)$ ($r \in \mathbb{Z}$) are p-torsion free, and p-adically complete and separated, and we have isomorphisms $A_{\mathrm{crys}}(\Lambda)/p^m \xrightarrow{\cong} A_{\mathrm{crys},m}(\Lambda)$ and $\mathrm{Fil}^r A_{\mathrm{crys}}(\Lambda)/p^m \xrightarrow{\cong} \mathrm{Fil}^r A_{\mathrm{crys},m}(\Lambda)$ ($r \in \mathbb{Z}$) by Lemma 7. We then obtain $\varphi(\mathrm{Fil}^r A_{\mathrm{crys}}(\Lambda)) \subset p^r A_{\mathrm{crys}}(\Lambda)$ for $r \in \mathbb{N} \cap [1, p-1]$ from $\varphi(\mathrm{Fil}^r A_{\mathrm{crys},r}(\Lambda)) \subset p^r A_{\mathrm{crys},r}(\Lambda) = 0$, and see that the topology of $A_{\mathrm{crys}}(\Lambda)$ coincides with the p-adic topology. For $r \in \mathbb{N}$, the multiplication by $\frac{\xi^r}{r!}$ induces an isomorphism $\widehat{\Lambda} \cong W(R_\Lambda)/\mathrm{Fil}^1 W(R_\Lambda) \xrightarrow{\cong} \mathrm{gr}^r_{\mathrm{Fil}} W^{\mathrm{PD}}(R_\Lambda)$ ([18, Proposition A2.9 (2)]). Therefore $\mathrm{gr}^r_{\mathrm{Fil}} W^{\mathrm{PD}}(R_\Lambda)$ ($r \in \mathbb{Z}$) is p-torsion free and p-adically complete and separated, and we obtain an isomorphism $\mathrm{gr}^r_{\mathrm{Fil}}W^{\mathrm{PD}}(R_\Lambda)\xrightarrow{\cong}\mathrm{gr}^r_{\mathrm{Fil}}A_{\mathrm{crys}}(\Lambda)$ ($r\in\mathbb{Z}$) by taking the p-adic completion of the exact sequence $0\to\mathrm{Fil}^{r+1}W^{\mathrm{PD}}(R_\Lambda)\to\mathrm{Fil}^rW^{\mathrm{PD}}(R_\Lambda)\to\mathrm{gr}^r_{\mathrm{Fil}}W^{\mathrm{PD}}(R_\Lambda)\to 0$ ([10, II (b)], [18, Lemma A2.11 (1)]). This implies that $A_{\mathrm{crys}}(\Lambda)/\mathrm{Fil}^rA_{\mathrm{crys}}(\Lambda)$ ($r\in\mathbb{N}_{>0}$) is p-torsion free, and therefore p-adically complete and separated by Lemma 3 (2).

Lemma 8 *Let Λ' be the same as in Lemma 4. Let m and r be positive integers, and let $\mathfrak{a}$ (resp. $\mathfrak{a}'$) be one of the ideals (p^m), Fil^r, (p^m,Fil^r), and (0) of $A_{\mathrm{crys}}(\Lambda)$ (resp. $A_{\mathrm{crys}}(\Lambda')$). Then the natural homomorphism $A_{\mathrm{crys}}(\Lambda)/\mathfrak{a}\to A_{\mathrm{crys}}(\Lambda')/\mathfrak{a}'$ is injective.*

Proof Since $A_{\mathrm{crys}}(\Lambda)$, $A_{\mathrm{crys}}(\Lambda')$, $A_{\mathrm{crys}}(\Lambda)/\mathrm{Fil}^r$, and $A_{\mathrm{crys}}(\Lambda')/\mathrm{Fil}^r$ are p-torsion free, and p-adically complete and separated, it suffices to prove the claim for (p) and (p,Fil^r). The homomorphism $\mathbb{F}_p[T]\to R_\Lambda;\ T\mapsto\underline{p}=(\xi\mod p)$ is flat because it factors through $\mathbb{F}_p[[T]]$ and $\underline{p}$ is regular in R_Λ (Proposition 143 (1)). Therefore, by [5, 3.21 Proposition], we see that $A_{\mathrm{crys}}(\Lambda)/p=A_{\mathrm{crys},1}(\Lambda)$ is a free $R_\Lambda/\underline{p}^p$-module with a basis ($\xi^{[pm]}\mod p$) ($m\in\mathbb{N}$). Note that the PD-polynomial ring $\mathbb{F}_p\langle T\rangle$ is a free $\mathbb{F}_p[T]/(T^p)$-module with a basis $T^{[pm]}$ ($m\in\mathbb{N}$). Combining with the same claim for Λ', we are reduced to showing that $R_\Lambda/\underline{p}^s\to R_{\Lambda'}/\underline{p}^s$ ($s\in\mathbb{N}\cap[1,p]$) is injective, which has been verified in the proof of Lemma 4. □

In the following, we assume, in addition, that

$$\Lambda\text{ is integral over a noetherian normal subring.}\tag{1}$$

For $s\in\mathbb{N}$, let $I^sA_{\mathrm{inf}}(\Lambda)$ (resp. $I^sA_{\mathrm{crys}}(\Lambda)$) be the ideal of $A_{\mathrm{inf}}(\Lambda)$ (resp. $A_{\mathrm{crys}}(\Lambda)$) consisting of x such that $\varphi^\nu(x)\in\mathrm{Fil}^s$ for all $\nu\in\mathbb{N}$. They are stable under the action of $G(\Lambda/\Lambda_0)$ and φ. We have $I^r\cdot I^s\subset I^{r+s}$ ($r,s\in\mathbb{N}$). Let $\varepsilon=(\varepsilon_n)$ be a basis of $\mathbb{Z}_p(1)(O_{\overline{K}})=\varprojlim_{n\in\mathbb{N}}\mu_{p^n}(O_{\overline{K}})$, and let $\underline{\varepsilon}$ denote the element ($\varepsilon_n\mod p$) of $R_{O_{\overline{K}}}$. We have $\varphi_R(\underline{\varepsilon}-1)=\frac{p}{p-1}$ because $v_C((\varepsilon_n-1)^{p^n})=\frac{p}{p-1}$ for every $n\in\mathbb{N}_{>0}$. This implies $(\underline{\varepsilon}-1)^{p-1}\in\underline{p}^p\cdot R^\times_{O_{\overline{K}}}$.

The element $\pi:=[\underline{\varepsilon}]-1\in A_{\mathrm{inf}}(\Lambda)$ is a non-zero divisor and the ideal $I^sA_{\mathrm{inf}}(\Lambda)$ is generated by π^s ([12, 5.1.3 Proposition (i)], [18, Proposition A3.12]). We have $\pi^{p-1}\in pA_{\mathrm{crys}}(\Lambda)$ ([18, Lemma A3.1]). For $s\in\mathbb{N}$, we have the inclusion $\varphi(I^sA_{\mathrm{crys}}(\Lambda))\subset p^sA_{\mathrm{crys}}(\Lambda)$, and if $s\in[0,p-1]$, the natural homomorphism $A_{\mathrm{inf}}(\Lambda)/I^sA_{\mathrm{inf}}(\Lambda)\to A_{\mathrm{crys}}(\Lambda)/I^sA_{\mathrm{crys}}(\Lambda)$ is an isomorphism by [12, 5.3.1 Proposition] and [18, Proposition A3.20]. For the latter, note $\pi=\sum_{n\geq 1}t^{[n]}\equiv t\mod I^1A_{\mathrm{crys}}(O_{\overline{K}})$. This congruence together with [18, Proposition A3.23 and Example A2.7] also implies that, for $r,s\in\mathbb{N}\cap[0,p-1]$ with $s<r$, $(I^sA_{\mathrm{crys}}(\Lambda)\cap\mathrm{Fil}^rA_{\mathrm{crys}}(\Lambda))/(I^{s+1}A_{\mathrm{crys}}(\Lambda)\cap\mathrm{Fil}^rA_{\mathrm{crys}}(\Lambda))$ is generated by $\xi^{r-s}\pi^s$ as an $W(R_\Lambda)$-module. Since $\xi^{r-s}\pi^s\in\mathrm{Fil}^rA_{\mathrm{inf}}(\Lambda)$, we obtain isomorphisms

$$\mathrm{Fil}^rA_{\mathrm{inf}}(\Lambda)/I^sA_{\mathrm{inf}}(\Lambda)\xrightarrow{\cong}\mathrm{Fil}^rA_{\mathrm{crys}}(\Lambda)/I^sA_{\mathrm{crys}}(\Lambda)\tag{2}$$
$$(r,s\in\mathbb{N},0\leq r\leq s\leq p-1).$$

For $s \in \mathbb{N}$, $A_{\mathrm{inf}}(\Lambda)/I^s A_{\mathrm{inf}}(\Lambda)$, $I^s A_{\mathrm{inf}}(\Lambda)$, $A_{\mathrm{crys}}(\Lambda)/I^s A_{\mathrm{crys}}(\Lambda)$ and $I^s A_{\mathrm{crys}}(\Lambda)$ are p-torsion free, and p-adically complete and separated ([18, Lemmas A3.11, A3.19 and A.3.27]).

Put $q := \sum_{a\in\mathbb{F}_p}[\underline{\varepsilon}^{[a]}] \in A_{\mathrm{inf}}(\Lambda)$, $\pi_0 := q - p$ and $q' := \varphi^{-1}(q)$, where $\underline{\varepsilon}^b = (\varepsilon_n^b \mod p)_{n\in\mathbb{N}}$ for $b \in \mathbb{Z}_p$. Then the ideal $\mathrm{Fil}^1 A_{\mathrm{inf}}(\Lambda)$ is generated by q' ([12, 5.2.6 Proposition (ii)], [18, Example A2.7]), which implies that $g(q) \in q \cdot A_{\mathrm{inf}}(\Lambda)^\times$ for $g \in G(\Lambda/\Lambda_0)$.

Lemma 9 (1) *For $a \in \mathbb{Z}_p$, the series $\sum_{n\in\mathbb{N}} \binom{a}{n}\pi^n$ converges to $[\underline{\varepsilon}^a]$ in $A_{\mathrm{inf}}(O_{\overline{K}})$ with respect to the π-adic topology.*

(2) *The ideal $I^{p-1}A_{\mathrm{inf}}(\Lambda)$ is generated by π_0, and we have $\frac{\pi_0}{p} \in I^{p-1}A_{\mathrm{crys}}(\Lambda)$.*

Proof (1) Since $A_{\mathrm{inf}}(O_{\overline{K}})$ is π-adically complete and separated, and $A_{\mathrm{inf}}(O_{\overline{K}})/\pi^l$ is p-adically complete and separated by Lemma 1 (3) and (4), it suffices to prove $[\underline{\varepsilon}^a] \equiv \sum_{n\in\mathbb{N}\cap[0,l-1]} \binom{a}{n}\pi^n \mod \pi^l A_{\mathrm{inf}}(O_{\overline{K}}) + p^m A_{\mathrm{inf}}(O_{\overline{K}})$ for every $l, m \in \mathbb{N}_{>0}$. Put $N := \max\{v_p(n!); n \in \mathbb{N}\cap[0, l-1]\} + m$. Then for any $b \in \mathbb{Z}_p$, we have $[\underline{\varepsilon}^{bp^N}] - 1 = \sum_{n\in\mathbb{N}\cap[1,p^N]} \binom{p^N}{n}([\underline{\varepsilon}^b]-1)^n \in \pi^l A_{\mathrm{inf}}(O_{\overline{K}}) + p^m A_{\mathrm{inf}}(O_{\overline{K}})$ because $[\underline{\varepsilon}^b] - 1 \in I^1 A_{\mathrm{inf}}(O_{\overline{K}}) = \pi A_{\mathrm{inf}}(O_{\overline{K}})$. Hence $[\underline{\varepsilon}^a] \equiv [\underline{\varepsilon}^{a'}] \mod \pi^l A_{\mathrm{inf}}(O_{\overline{K}}) + p^m A_{\mathrm{inf}}(O_{\overline{K}})$ for any $a' \in \mathbb{N}$ such that $a \equiv a' \mod p^N\mathbb{Z}_p$. This completes the proof because $\binom{a'}{n} \in \mathbb{Z}_p$ ($n \in \mathbb{N}\cap[0, l-1]$) converges to $\binom{a}{n} \in \mathbb{Z}_p$ as $a' \in \mathbb{N}$ tends to a.

(2) The second claim follows from the first one and $p^{-1}\pi^{p-1} \in I^{p-1}A_{\mathrm{crys}}(\Lambda)$. For $n \in \mathbb{N}\cap[1, p-1]$, the sum $\sum_{a\in\mathbb{F}_p}[a]^n$ is equal to 0 if $1 \le n \le p-2$, and $p-1$ if $n = p-1$. Therefore $\sum_{a\in\mathbb{F}_p}\binom{[a]}{n}$ vanishes if $1 \le n \le p-2$, and is equal to $\frac{1}{(p-2)!}$ if $n = p-1$. Hence (1) implies that π_0 is of the form $\frac{1}{(p-2)!}\pi^{p-1}(1+\pi c)$, $c \in A_{\mathrm{inf}}(O_{\overline{K}})$. We have $1 + \pi c \in A_{\mathrm{inf}}(O_{\overline{K}})^\times$ because $A_{\mathrm{inf}}(O_{\overline{K}})$ is π-adically complete and separated (Lemma 1 (4)). This completes the proof. □

Lemma 9 (2) implies $g(\pi_0) \in \pi_0 \cdot A_{\mathrm{inf}}(\Lambda)^\times$ for $g \in G(\Lambda/\Lambda_0)$. We have $\varphi(\pi_0) \in \pi_0 q^{p-1} \cdot A_{\mathrm{inf}}(\Lambda)^\times$ because $\pi_0 \in \pi^{p-1}A_{\mathrm{inf}}(\Lambda)^\times$ and $\frac{\pi}{\varphi^{-1}(\pi)}$ generates the ideal $\mathrm{Fil}^1 A_{\mathrm{inf}}(\Lambda)$ ([12, 5.1.2], [18, Example A 2.6]). This implies

$$\varphi(I^{p-1}A_{\mathrm{inf}}(\Lambda)) \subset q^{p-1}A_{\mathrm{inf}}(\Lambda). \tag{3}$$

Let $\mathcal{A}$, $\overline{\mathcal{K}}$, $\mathcal{K}^{\mathrm{ur}}$, $G_{\mathcal{A}}$, and $\overline{\mathcal{A}}$ be as in Sect. 1. Then we see that the absolute Frobenius of $\overline{\mathcal{A}}/p\overline{\mathcal{A}}$ is surjective by showing that the equation $x^{p^2} - px = a$ has a solution in $\overline{\mathcal{A}}$ for every $a \in \overline{\mathcal{A}}$ as follows: Put $\mathcal{L} = \mathcal{K}(a)$, which is a finite extension of $\mathcal{K}$, and let $\mathcal{A}(a)$ be the integral closure of $\mathcal{A}$ in $\mathcal{L}$. Then $\mathcal{A}(a)$ is p-adically complete and separated as it is finite over $\mathcal{A}$, and we have $a \in \mathcal{A}(a)$. The finite free $\mathcal{A}(a)$-algebra $\mathcal{C} := \mathcal{A}(a)[X]/(X^{p^2} - pX - a)$ is étale after inverting p because $(X^{p^2} - pX - a)' = p(-1 + pX^{p^2-1})$, and the image of $-1 + pX^{p^2-1}$ in $\mathcal{C}$ is invertible. Hence, for any solution $x \in \overline{\mathcal{K}}$, the image of the $\mathcal{A}(a)$-homomorphism $\mathcal{C} \to \overline{\mathcal{K}}$ defined by $X \mapsto x$ is contained in $\overline{\mathcal{A}}$. Thus we may apply the above construction of $A_{\mathrm{inf}}(\Lambda)$ and $A_{\mathrm{crys}}(\Lambda)$ to $\Lambda = \overline{\mathcal{A}}$ and $\Lambda_0 = \mathcal{A}$.

Let $\mathcal{B}$ be a flat O_K-algebra p-adically complete and separated such that the homomorphism $O_K/p^m \to \mathcal{B}/p^m$ is smooth for every $m \in \mathbb{N}_{>0}$. Put $\mathcal{B}_m := \mathcal{B}/p^m$,

$O_{K,m} := O_K/p^m$, $\Omega_{\mathcal{B}_m} := \Omega_{\mathcal{B}_m/O_{K,m}}$ for $m \in \mathbb{N}_{>0}$, and $\Omega_{\mathcal{B}} := \varprojlim_m \Omega_{\mathcal{B}_m}$. We assume that there exist $s_1, \ldots, s_e \in \mathcal{B}^\times$ such that $d\log s_i$ $(i \in \mathbb{N} \cap [1, e])$ form a basis of $\Omega_{\mathcal{B}_m}$ for every $m \in \mathbb{N}_{>0}$, and that we are given a surjective O_K-homomorphism $\mathcal{B} \to \mathcal{A}$ and a lifting $\varphi_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{B}$ of the absolute Frobenius of $\mathcal{B}_1$ compatible with σ of O_K. Put $\varphi_{\mathcal{B}_m} := \varphi_{\mathcal{B}} \otimes_{\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z}$ for $m \in \mathbb{N}_{>0}$.

We introduce a period ring $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ associated to $\mathcal{B} \to \mathcal{A}$ and $\overline{\mathcal{A}}/\mathcal{A}$. It is an $A_{\mathrm{crys}}(\overline{\mathcal{A}}) \otimes_{O_K} \mathcal{B}$-algebra equipped with an action of $G_{\mathcal{A}}$, a decreasing filtration, and an integrable connection, and will be used to describe explicitly the $\mathbb{Z}_p$-representation of $G_{\mathcal{A}}$ associated to an object M of $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A}, \Phi)$ (Sect. 4) in terms of the "evaluation" of M/p^m on the PD-envelope of $\mathrm{Spec}(\mathcal{A}/p^m) \hookrightarrow \mathrm{Spec}(\mathcal{B}/p^m)$ (Lemma 37, (38)). See [8, Sect. 6.1] for the case $\mathcal{B} = \mathcal{A}$.

We begin by introducing the PD-envelope mentioned above. Recall that γ denotes the unique PD-structure on the ideal pO_K of O_K. Put $\mathcal{A}_m := \mathcal{A}/p^m$ for $m \in \mathbb{N}_{>0}$. For $m \in \mathbb{N}_{>0}$, we define $\mathcal{P}_m$ to be the divided power envelope compatible with γ of $\mathcal{B}_m$ with respect to the kernel of the homomorphism $\mathcal{B}_m \to \mathcal{A}_m$. We define the decreasing filtration $\mathrm{Fil}^r\mathcal{P}_m$ $(r \in \mathbb{Z})$ of $\mathcal{P}_m$ by ideals to be the rth divided power of the divided power ideal of $\mathcal{P}_m$ if $r > 0$ ([4, I Définition 3.1.1], [5, 3.24 Definition]) and $\mathcal{P}_m$ if $r \leq 0$. We have $\mathrm{Fil}^r\mathcal{P}_m \cdot \mathrm{Fil}^s\mathcal{P}_m \subset \mathrm{Fil}^{r+s}\mathcal{P}_m$ $(r, s \in \mathbb{Z})$. The $O_{K,m}$-algebra $\mathcal{P}_m$ is naturally endowed with an $O_{K,m}$-linear derivation $\nabla_{\mathcal{P}_m}\colon \mathcal{P}_m \to \mathcal{P}_m \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$ compatible with the derivation $d\colon \mathcal{B}_m \to \Omega_{\mathcal{B}_m}$ and integrable as a connection with respect to $\mathcal{B}_m/O_{K,m}$ ([4, IV Sect. 1.3]). We have $\nabla(x^{[n]}) = x^{[n-1]} \otimes dx$ for $x \in \mathrm{Fil}^1\mathcal{P}_m$ and $n \in \mathbb{N}_{>0}$ ([4, IV (1.3.6)]). This implies $\nabla_{\mathcal{P}_m}(\mathrm{Fil}^r\mathcal{P}_m) \subset \mathrm{Fil}^{r-1}\mathcal{P}_m \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$ $(r \in \mathbb{Z})$. By Lemma 6 (1), (2), the lifting of the absolute Frobenius $\varphi_{\mathcal{B}_m}$ induces a lifting of the absolute Frobenius $\varphi_{\mathcal{P}_m}$ on $\mathcal{P}_m$ compatible with $\nabla_{\mathcal{P}_m}$. We have $\varphi_{\mathcal{P}_m}(\mathrm{Fil}^r\mathcal{P}_m) \subset p^r\mathcal{P}_m$ for $r \in \mathbb{N} \cap [0, p-1]$ by Lemma 6 (3). The ring $\mathcal{P}_m$ and their ideals $\mathrm{Fil}^r\mathcal{P}_m$ $(r \in \mathbb{Z})$ are flat over $O_{K,m}$ ([14, I Lemma (1.3) (2)]). We have natural PD-isomorphisms $\mathcal{P}_{m+1} \otimes_{O_{K,m+1}} O_{K,m} \xrightarrow{\cong} \mathcal{P}_m$ compatible with φ and ∇, and isomorphisms $\mathrm{Fil}^r\mathcal{P}_{m+1} \otimes_{O_{K,m+1}} O_{K,m} \xrightarrow{\cong} \mathrm{Fil}^r\mathcal{P}_m$ for $m \in \mathbb{N}_{>0}$ and $r \in \mathbb{Z}$ (loc. cit.). We define the O_K-algebra $\mathcal{P}$ and its filtration by ideals $\mathrm{Fil}^r\mathcal{P}$ $(r \in \mathbb{Z})$ to be $\varprojlim_m \mathcal{P}_m$ and $\varprojlim_m \mathrm{Fil}^r\mathcal{P}_m$. By Lemma 7, the ring $\mathcal{P}$ and its ideals $\mathrm{Fil}^r\mathcal{P}$ $(r \in \mathbb{Z})$ are flat over O_K, and p-adically complete and separated, and we have isomorphisms $\mathcal{P}/p^m \xrightarrow{\cong} \mathcal{P}_m$ and $\mathrm{Fil}^r\mathcal{P}/p^m\mathrm{Fil}^r\mathcal{P} \xrightarrow{\cong} \mathrm{Fil}^r\mathcal{P}_m$ for $m \in \mathbb{N}_{>0}$ and $r \in \mathbb{Z}$. By taking the inverse limit of $\nabla_{\mathcal{P}_m}$ and $\varphi_{\mathcal{P}_m}$, we obtain $\nabla_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{P} \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}$ and $\varphi_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{P}$. We have $\varphi_{\mathcal{P}}(\mathrm{Fil}^r\mathcal{P}) \subset p^r\mathcal{P}$ for $r \in \mathbb{N} \cap [0, p-1]$.

We first introduce a period ring $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ defined over $O_{K,m}$, and then take the inverse limit over m. Put $\overline{\mathcal{A}}_m := \overline{\mathcal{A}}/p^m$ for $m \in \mathbb{N}_{>0}$. For $m \in \mathbb{N}_{>0}$, we define $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ to be the divided power envelope compatible with γ of $(A_{\mathrm{inf}}(\overline{\mathcal{A}}) \otimes_{O_K} \mathcal{B})/p^m$ with respect to the kernel of the surjective homomorphism to $\overline{\mathcal{A}}_m$ induced by $(\theta \mod p^m)\colon A_{\mathrm{inf}}(\overline{\mathcal{A}})/p^m \to \overline{\mathcal{A}}_m$ and $\mathcal{B}_m \to \mathcal{A}_m$. The homomorphism from $\mathcal{B}_m$ (resp. $A_{\mathrm{inf}}(\overline{\mathcal{A}})/p^m$) to $(A_{\mathrm{inf}}(\overline{\mathcal{A}}) \otimes_{O_K} \mathcal{B})/p^m$ induces a homomorphism $\mathcal{P}_m \to \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ (resp. $A_{\mathrm{crys},m}(\overline{\mathcal{A}}) \to \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$) of PD-algebras over $O_{K,m}$. The action of $G_{\mathcal{A}}$ on $A_{\mathrm{inf}}(\overline{\mathcal{A}})$ induces its action on the PD-ring $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$, which is continuous with respect to the discrete topology of $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ by Lemma 5 because the composition $(A_{\inf}(\overline{\mathcal{A}}) \otimes_{O_K} \mathcal{B})/p^m \to A_{\mathrm{crys},m}(\overline{\mathcal{A}}) \otimes_{O_{K,m}} \mathcal{P}_m \to \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ factors through the quotient modulo $[\underline{p}]^{pm}$ (see the paragraph after Lemma 7) and $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ is generated over $(A_{\inf}(\overline{\mathcal{A}}) \otimes_{O_K} \mathcal{B})/p^m$ by divided powers of elements of the kernel of $(A_{\inf}(\overline{\mathcal{A}}) \otimes_{O_K} \mathcal{B})/p^m \to \overline{\mathcal{A}}_m$ ([4, I. Proposition 2.4.3 (ii)], [5, 3.20 Remark (3)]). The homomorphism $\mathcal{P}_m \to \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ (resp. $A_{\mathrm{crys},m}(\overline{\mathcal{A}}) \to \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$) mentioned above is $G_{\mathcal{A}}$-stable (resp. $G_{\mathcal{A}}$-equivariant). We define the decreasing filtration $\mathrm{Fil}^r\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ ($r \in \mathbb{Z}$) of $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ by ideals to be the rth divided power of the divided power ideal if $r > 0$ and $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ if $r \leq 0$. The filtration $\mathrm{Fil}^r\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ ($r \in \mathbb{Z}$) is $G_{\mathcal{A}}$-stable, and we have $\mathrm{Fil}^r\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \cdot \mathrm{Fil}^s\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \subset \mathrm{Fil}^{r+s}\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ for $r, s \in \mathbb{Z}$. The homomorphisms $\mathcal{P}_m \to \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ and $A_{\mathrm{crys},m}(\overline{\mathcal{A}}) \to \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ are compatible with the filtrations because they are PD-homomorphisms. The ring $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ is naturally endowed with a $G_{\mathcal{A}}$-equivariant $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$-linear derivation $\nabla\colon \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \to \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$ compatible with $\nabla\colon \mathcal{P}_m \to \mathcal{P}_m \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$ and integrable as a connection with respect to $\mathcal{B}_m/O_{K,m}$ ([4, IV Sect. 1.3]). We have the inclusion $\nabla(\mathrm{Fil}^r\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})) \subset \mathrm{Fil}^{r-1}\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$ ($r \in \mathbb{Z}$). By Lemma 6 (1), (2), the lifting of the absolute Frobenius $\varphi_{\mathcal{B}}$ and the Frobenius of $A_{\inf}(\overline{\mathcal{A}})$ induce a lifting of the absolute Frobenius φ on $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ compatible with ∇ and the action of $G_{\mathcal{A}}$. The homomorphisms $\mathcal{P}_m \to \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ and $A_{\mathrm{crys},m}(\overline{\mathcal{A}}) \to \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ are compatible with φ's. By Lemma 6 (3), we have the inclusion $\varphi(\mathrm{Fil}^r\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})) \subset p^r\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ for $r \in \mathbb{N} \cap [0, p-1]$. We have a natural PD-homomorphism $\mathscr{A}_{\mathrm{crys},\mathcal{B},m+1}(\overline{\mathcal{A}}) \to \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ compatible with the $G_{\mathcal{A}}$-action, Fil^r, φ, and the homomorphisms from $\mathcal{P}_\bullet$ and $A_{\mathrm{crys},\bullet}(\overline{\mathcal{A}})$.

The ring $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ with the $G_{\mathcal{A}}$-action, Fil^r, ∇ and φ is explicitly described as follows. Let $s_1, \dots, s_e$ be elements of $\mathcal{B}^\times$ such that $d\log s_i$ ($i \in \mathbb{N} \cap [1, e]$) form a basis of $\Omega_{\mathcal{B}_m}$ for every $m \in \mathbb{N}_{>0}$. For each $i \in \mathbb{N} \cap [1, e]$, choose a compatible system of p^nth roots $s_{i,n} \in \overline{\mathcal{A}}^\times$ ($n \in \mathbb{N}$) of the image of s_i in $\mathcal{A}^\times$, let $\underline{s}_i$ be the element $(s_{i,n} \bmod p)_{n\in\mathbb{N}}$ of $R^\times_{\overline{\mathcal{A}}}$, and let $u_{i,m}$ be the image of $[\underline{s}_i] \otimes s_i^{-1} - 1$ in $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$. Then we have $u_{i,m} \in \mathrm{Fil}^1\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ and an isomorphism of PD-algebras over $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$ ([15, Lemma 1.8])

$$A_{\mathrm{crys},m}(\overline{\mathcal{A}})\langle U_1, \dots, U_e\rangle \xrightarrow{\cong} \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}});\ U_i \mapsto u_{i,m}, \tag{4}$$

where the left-hand side is the PD-polynomial ring with variables U_i. This further gives the following explicit description of the filtration on $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$, where $|\underline{n}| = \sum_{1\leq i\leq e} n_i$.

$$\bigoplus_{\underline{n}=(n_i)\in\mathbb{N}^e} \mathrm{Fil}^{r-|\underline{n}|}A_{\mathrm{crys},m}(\overline{\mathcal{A}}) \prod_{1\leq i\leq e} U_i^{[n_i]} \xrightarrow{\cong} \mathrm{Fil}^r\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \qquad (r \in \mathbb{Z}) \tag{5}$$

Let $\varepsilon = (\varepsilon_n) \in \mathbb{Z}_p(1)(O_{\overline{K}})$ and $\underline{\varepsilon} \in R_{O_{\overline{K}}} \subset R_{\overline{\mathcal{A}}}$ be as in the definition of π after (1). For each $i \in \mathbb{N} \cap [1, e]$, we define the continuous map $\eta_i : G_{\mathcal{A}} \to \mathbb{Z}_p$ by $g(s_{i,n}) = s_{i,n}\varepsilon_n^{\eta_i(g)}$ $(n \in \mathbb{N})$. Then the action of $G_{\mathcal{A}}$ on $u_{i,m}$ is given by

$$g(u_{i,m}) = [\underline{\varepsilon}^{\eta_i(g)}]u_{i,m} + ([\underline{\varepsilon}^{\eta_i(g)}] - 1) \quad (g \in G_{\mathcal{A}}). \tag{6}$$

For ∇ and φ, we have

$$\nabla(u_{i,m}^{[n]}) = -u_{i,m}^{[n-1]}(u_{i,m} + 1) \otimes d\log s_i, \tag{7}$$

$$\varphi(u_{i,m}) = (u_{i,m} + 1)^p s_i^p \varphi_{\mathcal{B}}(s_i)^{-1} - 1 \tag{8}$$

for $i \in \mathbb{N} \cap [1, e]$. Since $u_{i,m} + 1 \in \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})^\times$ for every $i \in \mathbb{N} \cap [1, e]$, (5) and (7) imply

$$\mathrm{Fil}^r \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})^{\nabla=0} = \mathrm{Fil}^r A_{\mathrm{crys},m}(\overline{\mathcal{A}}) \quad (r \in \mathbb{Z}). \tag{9}$$

The description (4) implies that $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ is flat over $O_{K,m}$ and the natural homomorphism $\mathscr{A}_{\mathrm{crys},\mathcal{B},m+1}(\overline{\mathcal{A}}) \otimes_{O_{K,m+1}} O_{K,m} \to \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ is an isomorphism.

We define $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ to be the inverse limit of $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ $(m \in \mathbb{N}_{>0})$, which is naturally equipped with a continuous action of $G_{\mathcal{A}}$, a decreasing filtration Fil^r $(r \in \mathbb{Z})$, $\nabla : \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \to \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}$, $\varphi : \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \to \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$, and homomorphisms from $\mathcal{P}$ and $A_{\mathrm{crys}}(\overline{\mathcal{A}})$. We obtain an explicit description of $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ just by taking the inverse limit of the description of $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ given above. We write u_i $(i \in \mathbb{N} \cap [1, e])$ for the element of $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ defined by the compatible system $(u_{i,m})_{m \in \mathbb{N}_{>0}}$. By Lemma 7, the ring $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ is flat over O_K and p-adically complete and separated, and its reduction mod p^m is isomorphic to $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$. The last fact together with $\varphi(\mathrm{Fil}^r \mathscr{A}_{\mathrm{crys},\mathcal{B},r}(\overline{\mathcal{A}})) = 0$ $(r \in \mathbb{N} \cap [1, p-1])$ implies $\varphi(\mathrm{Fil}^r \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})) \subset p^r \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ for $r \in \mathbb{N} \cap [0, p-1]$.

When $\mathcal{B} = \mathcal{A}$ and the surjective homomorphism $\mathcal{B} \to \mathcal{A}$ is the identity map, we write $\mathscr{A}_{\mathrm{crys},m}(\overline{\mathcal{A}})$ and $\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})$ for $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ and $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$, respectively. In this case, we have $\mathcal{P}_m = \mathcal{A}_m$, $\mathrm{Fil}^r \mathcal{P}_m = 0$ $(r \in \mathbb{N}_{>0})$, $\nabla_{\mathcal{P}_m} = d : \mathcal{A}_m \to \Omega_{\mathcal{A}_m}$, and $\varphi_{\mathcal{P}_m} = \varphi_{\mathcal{A}_m}$. By taking the inverse limit over $m \in \mathbb{N}_{>0}$, we obtain $\mathcal{P} = \mathcal{A}$, $\mathrm{Fil}^r \mathcal{P} = 0$ $(r \in \mathbb{N}_{>0})$, $\nabla_{\mathcal{P}} = d : \mathcal{A} \to \Omega_{\mathcal{A}}$, and $\varphi_{\mathcal{P}} = \varphi_{\mathcal{A}}$. When we consider the explicit description (4) of $\mathscr{A}_{\mathrm{crys},m}(\overline{\mathcal{A}})$ and its inverse limit for $\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})$ by using $t_1, \ldots, t_d \in \mathcal{A}^\times$ such that $d\log t_i$ is a basis of $\Omega_{\mathcal{A}_m}$ for every $m \in \mathbb{N}$, we write $t_{i,n}$, $\underline{t}_i$, $v_{i,m}$ and v_i for the elements corresponding to $s_{i,n}$, $\underline{s}_i$, $u_{i,m}$ and u_i.

3 Filtered Crystals

We define filtered crystals on a big crystalline site, and give an interpretation of filtered crystals in terms of modules with integrable connections simply generalizing that for crystals. See Remark 19 for the relation with the work of Ogus in [17].

We first introduce some terminology concerning filtered modules over filtered rings used throughout this paper.

Definition 10 Let E be a topos or site.

(1) *A filtered ring on E* is a pair of a (commutative) ring R on E and a decreasing filtration $\mathrm{Fil}^r R$ ($r \in \mathbb{Z}$) of R by ideals such that $\mathrm{Fil}^0 R = R$ and $\mathrm{Fil}^r R \cdot \mathrm{Fil}^s R \subset \mathrm{Fil}^{r+s} R$ for all $r, s \in \mathbb{Z}$. If E is the topos of sets, it is simply called a filtered ring. *A homomorphism of filtered rings* $(R, \mathrm{Fil}^\bullet R) \to (S, \mathrm{Fil}^\bullet S)$ *on* E is a homomorphism of rings $f: R \to S$ such that $f(\mathrm{Fil}^r R) \subset \mathrm{Fil}^r S$ for every $r \in \mathbb{Z}$.

(2) *A filtered module over a filtered ring* $(R, \mathrm{Fil}^\bullet R)$ *on* E is an R-module M with a decreasing filtration $\mathrm{Fil}^r M$ ($r \in \mathbb{Z}$) by R-submodules such that $\mathrm{Fil}^r R \cdot \mathrm{Fil}^s M \subset \mathrm{Fil}^{r+s} M$ for all $r, s \in \mathbb{Z}$. *A homomorphism of filtered modules over* $(R, \mathrm{Fil}^\bullet R)$ is an R-linear homomorphism sending Fil^r into Fil^r for every $r \in \mathbb{Z}$. For $s \in \mathbb{Z}$, we define the filtered module $R(s)$ over $(R, \mathrm{Fil}^\bullet R)$ by $R(s) = R$ and $\mathrm{Fil}^r R(s) = \mathrm{Fil}^{r-s} R$ ($r \in \mathbb{Z}$).

(3) We say that a filtered module $(M, \mathrm{Fil}^\bullet M)$ over a filtered ring $(R, \mathrm{Fil}^\bullet R)$ on E is *finite filtered free* if there exists an isomorphism $\oplus_{\nu \in \mathbb{N} \cap [1,N]} R(r_\nu) \xrightarrow{\cong} M$ of filtered modules over $(R, \mathrm{Fil}^\bullet R)$ for some $N \in \mathbb{N}$ and $r_\nu \in \mathbb{Z}$ ($\nu \in \mathbb{N} \cap [1, N]$). For integers $a, b \in \mathbb{Z}$ with $a \leq b$, we say that $(M, \mathrm{Fil}^\bullet M)$ is *finite filtered free of level* $[a, b]$ if $a \leq r_\nu \leq b$ for every $\nu \in \mathbb{N} \cap [1, N]$.

(4) Let $f: (R, \mathrm{Fil}^\bullet R) \to (S, \mathrm{Fil}^\bullet S)$ be a homomorphism of filtered rings on E, and let $(M, \mathrm{Fil}^\bullet M)$ be a filtered module over $(R, \mathrm{Fil}^\bullet R)$. We define *the scalar extension of* $(M, \mathrm{Fil}^\bullet M)$ *by* f to be $M \otimes_{R,f} S$ with the decreasing filtration $\mathrm{Fil}^r(M \otimes_{R,f} S)$ defined by the sum of the images of $\mathrm{Fil}^{r-s} M \otimes_{R,f} \mathrm{Fil}^s S$ ($s \in \mathbb{Z}$). This construction is compatible with compositions of homomorphisms of filtered rings.

Definition 11 (1) A *filtered ringed topos* is a pair $(E, (R, \mathrm{Fil}^\bullet R))$ of a topos E and a filtered ring $(R, \mathrm{Fil}^\bullet R)$ on E. Let $\mathrm{MF}(E, (R, \mathrm{Fil}^\bullet R))$ denote the category of filtered modules over $(R, \mathrm{Fil}^\bullet R)$.

(2) A *morphism of filtered ringed topos* $\boldsymbol{f} = (f, \varphi): (E', (R', \mathrm{Fil}^\bullet R')) \to (E, (R, \mathrm{Fil}^\bullet R))$ is a pair of a morphism of topos $f: E' \to E$ and a morphism of filtered rings $\varphi: (f^{-1}(R), f^{-1}(\mathrm{Fil}^\bullet R)) \to (R', \mathrm{Fil}^\bullet R')$.

(3) Let $\boldsymbol{f} = (f, \varphi): (E', (R', \mathrm{Fil}^\bullet R')) \to (E, (R, \mathrm{Fil}^\bullet R))$ be a morphism of filtered ringed topos, and let $(M, \mathrm{Fil}^\bullet M)$ be a filtered module over $(R, \mathrm{Fil}^\bullet R)$. We define the pull-back $\boldsymbol{f}^*(M, \mathrm{Fil}^\bullet M)$ of $(M, \mathrm{Fil}^\bullet M)$ by $\boldsymbol{f}$ to be the scalar extension of the filtered module $(f^{-1}(M), f^{-1}(\mathrm{Fil}^\bullet M))$ over $(f^{-1}(R), f^{-1}(\mathrm{Fil}^\bullet R))$ by φ (Definition 10 (4)). This construction is compatible with compositions of morphisms of filtered ringed topos.

Remark 12 Let $(E, (R, \mathrm{Fil}^\bullet R))$ be a filtered ringed topos, and let $\mathrm{Mod}(E, R)$ be the category of R-modules on E. Then we have a fully faithful functor $\mathrm{Mod}(E, R) \to \mathrm{MF}(E, (R, \mathrm{Fil}^\bullet R))$ defined by $M \mapsto (M, \mathrm{Fil}^\bullet R \cdot M)$. This functor is compatible with the pull-back by a morphism of filtered ringed topos.

Lemma 13 *Let* $\boldsymbol{f} = (f, \varphi): (E', (R', \mathrm{Fil}^\bullet R')) \to (E, (R, \mathrm{Fil}^\bullet R))$ *be a morphism of filtered ringed topos.*

(1) *We have* $\boldsymbol{f}^*(R(s)) \cong R'(s)$ *for every* $s \in \mathbb{Z}$.
(2) *Let* a *and* b *be two integers satisfying* $a \leq b$. *If a filtered module* $(M, \mathrm{Fil}^\bullet M)$ *over* $(R, \mathrm{Fil}^\bullet R)$ *is finite filtered free of level* $[a, b]$, *then so is* $\boldsymbol{f}^*(M, \mathrm{Fil}^\bullet M)$.

Proof The claim (2) follows from (1). The claim (1) is reduced to the case $s = 0$ by shifting the filtration, and then it follows from $\varphi(f^{-1}(\mathrm{Fil}^{r-s} R))\mathrm{Fil}^s R' \subset \mathrm{Fil}^{r-s} R' \cdot \mathrm{Fil}^s R' \subset \mathrm{Fil}^r R'$ and $\varphi(f^{-1}(\mathrm{Fil}^0 R))\mathrm{Fil}^r R' = \mathrm{Fil}^r R'$ for $r, s \in \mathbb{Z}$. □

Lemma 14 *Let* $\boldsymbol{f} = (f, \varphi)\colon (E', (R', \mathrm{Fil}^\bullet R')) \to (E, (R, \mathrm{Fil}^\bullet R))$ *be a morphism of filtered ringed topos, and let* $\boldsymbol{f}^*\colon \mathrm{MF}(E, (R, \mathrm{Fil}^\bullet R)) \to \mathrm{MF}(E', (R', \mathrm{Fil}^\bullet R'))$ *be the pull-back functor* (*Definition 11 (3)*). *Then* $\boldsymbol{f}^*$ *is canonically regarded as a left adjoint of the functor* $\boldsymbol{f}_*\colon \mathrm{MF}(E', (R', \mathrm{Fil}^\bullet R')) \to \mathrm{MF}(E, (R, \mathrm{Fil}^\bullet R))$ *defined by* $\boldsymbol{f}_*(M', \mathrm{Fil}^\bullet M') = (f_* M', f_* \mathrm{Fil}^\bullet M')$.

Proof Let $(M, \mathrm{Fil}^\bullet M)$ (resp. $(M', \mathrm{Fil}^\bullet M')$) be a filtered module over $(R, \mathrm{Fil}^\bullet R)$ (resp. $(R', \mathrm{Fil}^\bullet R')$). Let $\alpha\colon M \to f_* M'$ be an R-linear homomorphism, and let $\beta\colon f^* M = f^{-1}(M) \otimes_{f^{-1}(R)} R' \to M'$ be its left adjoint. Then we have $\alpha(\mathrm{Fil}^r M) \subset f_* \mathrm{Fil}^r M'$ for all $r \in \mathbb{Z}$ if and only if the image of $f^* \mathrm{Fil}^r M \to f^* M \xrightarrow{\beta} M'$ is contained in $\mathrm{Fil}^r M'$ for all $r \in \mathbb{Z}$. The latter condition is equivalent to $\beta(\mathrm{Fil}^r(\boldsymbol{f}^* M)) \subset \mathrm{Fil}^r M'$ for all $r \in \mathbb{Z}$ because $\mathrm{Fil}^{r-s} R' \cdot \mathrm{Fil}^s M' \subset \mathrm{Fil}^r M'$ for every $r, s \in \mathbb{Z}$. □

Definition 15 (1) A *PD-ringed topos* is a pair $(E, (R, J, \gamma))$ of a topos E and a PD-ring (R, J, γ) on E ([4, I Définitions 1.9.1, 1.9.3]).

(2) For a PD-ringed topos $(E, (R, J, \gamma))$, we define the ideal $\mathrm{Fil}^r R$ ($r \in \mathbb{Z}$) of R to be the rth divided power $J^{[r]}$ of J if $r > 0$, and R if $r \leq 0$. Then $(R, \mathrm{Fil}^\bullet R)$ is a filtered ring. By a *filtered module on* $(E, (R, J, \gamma))$, we mean a filtered module over $(R, \mathrm{Fil}^\bullet R)$.

(3) A *morphism of PD-ringed topos* is a pair $\boldsymbol{f} = (f, \varphi)\colon (E', (R', J', \gamma')) \to (E, (R, J, \gamma))$ of a morphism of topos $f\colon E' \to E$ and a morphism of PD-rings $\varphi\colon f^{-1}(R, J, \gamma) \to (R', J', \gamma')$ ([4, I Définition 1.9.3]). It induces a morphism of filtered ringed topos $(E', (R', \mathrm{Fil}^\bullet R')) \to (E, (R, \mathrm{Fil}^\bullet R))$. We define the *pull-back of a filtered module* on $(E, (R, J, \gamma))$ by $\boldsymbol{f}$ to be the pull-back by this morphism of filtered ringed topos (Definition 11 (3)).

Let $(T, \mathcal{J}_T, \gamma_T)$ be a PD-scheme ([4, I Définition 1.9.6]). Then the Zariski topos T_{Zar} with $(\mathcal{O}_T, \mathcal{J}_T, \gamma_T)$ is a PD-ringed topos. By a *filtered* $\mathcal{O}_T$*-module on the PD-scheme* $(T, \mathcal{J}_T, \gamma_T)$, we mean a filtered module on $(T_{\mathrm{Zar}}, (\mathcal{O}_T, \mathcal{J}_T, \gamma_T))$ (Definition 15 (2)). Let $f\colon (T', \mathcal{J}_{T'}, \gamma_{T'}) \to (T, \mathcal{J}_T, \gamma_T)$ be a morphism of PD-schemes. It induces a morphism of PD-ringed topos $f_{\mathrm{Zar}}\colon (T'_{\mathrm{Zar}}, (\mathcal{O}_{T'}, \mathcal{J}_{T'}, \gamma_{T'})) \to (T_{\mathrm{Zar}}, (\mathcal{O}_T, \mathcal{J}_T, \gamma_T))$. For a filtered $\mathcal{O}_T$-module $(\mathcal{M}, \mathrm{Fil}^\bullet \mathcal{M})$ on $(T, \mathcal{J}_T, \gamma_T)$, we define the pull-back $f^*(\mathcal{M}, \mathrm{Fil}^\bullet \mathcal{M})$ by f to be that of $(\mathcal{M}, \mathrm{Fil}^\bullet \mathcal{M})$ by this morphism of PD-ringed topos (Definition 15 (3)).

Let $(S, \mathcal{J}_S, \gamma_S)$ be a PD-scheme on which p is locally nilpotent, and let Z be an S-scheme such that the PD-structure γ_S extends to Z. Let $\mathrm{CRYS}(Z/(S, \mathcal{J}_S, \gamma_S))$ (resp. $(Z/(S, \mathcal{J}_S, \gamma_S))_{\mathrm{CRYS}}$) be the big crystalline site (resp. topos) of Z over

$(S, \mathcal{J}_S, \gamma_S)$, which is equipped with a PD-ring $(\mathcal{O}_{Z/S}, \mathcal{J}_{Z/S})$. We abbreviate $Z/(S, \mathcal{J}_S, \gamma_S)$ to Z/S if there is no risk of confusion.

Similarly to the case of $\mathcal{O}_{Z/S}$-modules on $\mathrm{CRYS}(Z/S)$ ([4, III 4.1.2]), we see, by using Lemma 14, that the category of filtered modules on the PD-ringed topos $(Z/S)_{\mathrm{CRYS}}$ is canonically equivalent to the category of data $(\mathcal{F}_T, \tau_u)$ consisting of a filtered module $\mathcal{F}_T$ on T for each object T of $\mathrm{CRYS}(Z/S)$ and a morphism of filtered modules $\tau_u\colon u^*(\mathcal{F}_T) \to \mathcal{F}_{T'}$ on T' for each morphism $u\colon T' \to T$ of $\mathrm{CRYS}(Z/S)$ satisfying $\tau_{\mathrm{id}} = \mathrm{id}$ and the cocycle condition for compositions of u's, and being an isomorphism when u is an open immersion and the PD-ideal of T' is the pull-back of that of T. We say that a filtered module $\mathcal{F}$ on $\mathrm{CRYS}(Z/S)$ is a *filtered crystal* if, for the corresponding data $(\mathcal{F}_T, \tau_u)$ as above, $\tau_u\colon u^*(\mathcal{F}_T) \to \mathcal{F}_{T'}$ is an isomorphism of filtered modules on T' for every u.

Suppose that we are given a closed immersion ι of Z into a smooth scheme Y over S. Let $Y(r)$ $(r \in \mathbb{N})$ be the fiber product of $r+1$ copies of Y over S, and let $D(r)$ be the PD-envelope compatible with γ_S of the immersion $Z \to Y(r)$ induced by ι. Put $D := D(0)$. Let $p_i\colon D(1) \to D$ $(i \in \{1, 2\})$ (resp. $q_i\colon D(2) \to D$ $(i \in \{1, 2, 3\})$) be the PD-morphism induced by the ith projection $Y(1) \to Y$ (resp. $Y(2) \to Y$). Let $\Delta\colon D \to D(1)$ (resp. $p_{ij}\colon D(2) \to D(1)$ $((i, j) \in \{(1, 2), (2, 3), (1, 3)\})$) be the PD-morphism induced by the diagonal morphism $Y \to Y(1)$ (resp. the morphism $Y(2) \to Y(1)$ defined by the ith and jth projections). The closed immersion $Z \to D(r)$ is a nilimmersion because p is locally nilpotent on S. Hence we may regard a Zariski sheaf on $D(r)$ as a Zariski sheaf on Z, and then also on Y. We have a canonical derivation $\nabla\colon \mathcal{O}_D \to \mathcal{O}_D \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$ which is compatible with the universal derivation $d\colon \mathcal{O}_Y \to \Omega_{Y/S}$, and is an integrable connection on $\mathcal{O}_D$ regarded as an $\mathcal{O}_Y$-module ([4, IV Sect. 1.3]). We have $\nabla(\mathrm{Fil}^r \mathcal{O}_D) \subset \mathrm{Fil}^{r-1}\mathcal{O}_D \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$. The connection ∇ induces morphisms $\nabla^q\colon \mathcal{O}_D \otimes_{\mathcal{O}_Y} \Omega^q_{Y/S} \to \mathcal{O}_D \otimes_{\mathcal{O}_Y} \Omega^{q+1}_{Y/S}$ $(q \in \mathbb{N})$ defined by $a \otimes \omega \mapsto \nabla(a) \wedge \omega + a \otimes d\omega$, and the integrability of ∇ means $\nabla^{q+1} \circ \nabla^q = 0$ $(q \in \mathbb{N})$. We also have $\nabla^{q+q'}(\omega \wedge \eta) = \nabla^q(\omega) \wedge \eta + (-1)^q \omega \wedge \nabla^{q'}(\eta)$ for $\omega \in \mathcal{O}_D \otimes_{\mathcal{O}_Y} \Omega^q_{Y/S}$ and $\eta \in \mathcal{O}_D \otimes_{\mathcal{O}_Y} \Omega^{q'}_{Y/S}$.

We have the following interpretation of a filtered crystal in terms of a filtered module on D with an HPD-stratification, and a filtered module on D with a quasi-nilpotent integrable connection satisfying the Griffiths transversality. See [5, 6.6 Theorem] for the corresponding theorem for crystals on the small crystalline site $\mathrm{Crys}(Z/(S, \mathcal{J}_S, \gamma_S))$. First let us introduce objects appearing in the interpretation.

Definition 16 (i) We define $\mathrm{CF}(Z/(S, \mathcal{J}_S, \gamma_S))$ (or $\mathrm{CF}(Z/S)$ for short) to be the category of filtered crystals on $\mathrm{CRYS}(Z/(S, \mathcal{J}_S, \gamma_S))$.

(ii) We define the category $\mathrm{StratF}(D(\bullet))$ as follows. An object is a filtered module $\mathcal{M}$ on the PD-scheme D equipped with an isomorphism $\varepsilon_{\mathcal{M}}\colon p_2^*(\mathcal{M}) \xrightarrow{\cong} p_1^*(\mathcal{M})$ of filtered modules on $D(1)$ satisfying the following conditions.

(ii-a) The morphism $\mathcal{M} \cong \Delta^* p_2^*(\mathcal{M}) \xrightarrow[\Delta^*(\varepsilon_{\mathcal{M}})]{\cong} \Delta^* p_1^*(\mathcal{M}) \cong \mathcal{M}$ is the identity morphism.

(ii-b) The following diagram is commutative.

$$\begin{array}{ccccc} p_{23}^*p_2^*\mathcal{M} \overset{\sim}{=} q_3^*\mathcal{M} \overset{\sim}{=} p_{13}^*p_2^*\mathcal{M} & & \xrightarrow[p_{13}^*(\varepsilon_{\mathcal{M}})]{\cong} & & p_{13}^*p_1^*\mathcal{M} \overset{\sim}{=} q_1^*\mathcal{M} \overset{\sim}{=} p_{12}^*p_1^*\mathcal{M} \\ & \searrow^{\cong}_{p_{23}^*(\varepsilon_{\mathcal{M}})} & & \nearrow^{\cong}_{p_{12}^*(\varepsilon_{\mathcal{M}})} & \\ & & p_{23}^*p_1^*\mathcal{M} \overset{\sim}{=} q_2^*\mathcal{M} \overset{\sim}{=} p_{12}^*p_2^*\mathcal{M} & & \end{array}$$

A morphism is a morphism of underlying filtered $\mathcal{O}_D$-modules compatible with $\varepsilon_{\mathcal{M}}$'s.

(iii) We define the category $\mathrm{MF}^{\nabla}(Z \hookrightarrow Y/S)$ as follows. An object is a filtered module $\mathcal{M}$ on the PD-scheme D with a quasi-nilpotent integrable connection $\nabla\colon \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$ such that $\nabla(ax) = a\nabla(x) + x \otimes \nabla(a)$ for $a \in \mathcal{O}_D$ and $x \in \mathcal{M}$, and $\nabla(\mathrm{Fil}^r\mathcal{M}) \subset \mathrm{Fil}^{r-1}\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$ for $r \in \mathbb{Z}$ (Griffiths transversality). A morphism is a morphism of underlying filtered $\mathcal{O}_D$-modules compatible with ∇.

Theorem 17 *The three categories* $\mathrm{CF}(Z/(S, \mathcal{J}_S, \gamma_S))$, $\mathrm{StratF}(D(\bullet))$, *and* $\mathrm{MF}^{\nabla}(Z \hookrightarrow Y/S)$ *are naturally equivalent.*

Proof The construction of the equivalence between $\mathrm{CF}(Z/S)$ and $\mathrm{StratF}(D(\bullet))$ is completely parallel to the case without filtrations: For a filtered crystal $\mathcal{F}$ on $\mathrm{CRYS}(Z/S)$, the filtered module $\mathcal{M} := \mathcal{F}_D$ on D with the composition of $p_2^*\mathcal{M} = p_2^*\mathcal{F}_D \underset{\tau_{p_2}}{\xrightarrow{\cong}} \mathcal{F}_{D(1)} \underset{\tau_{p_1}^{-1}}{\xrightarrow{\cong}} p_1^*\mathcal{F}_D = p_1^*\mathcal{M}$ is an object of $\mathrm{StratF}(D(\bullet))$, and this construction is functorial in $\mathcal{F}$. The quasi-inverse of this functor is constructed as follows. Suppose that we are given an object $(\mathcal{M}, \varepsilon_{\mathcal{M}})$ of $\mathrm{StratF}(D(\bullet))$. Let $T = (U \hookrightarrow T, z\colon U \to Z)$ be an object of $\mathrm{CRYS}(Z/S)$ such that T is affine. Since $U \to T$ is a nilimmersion and $Y \to S$ is smooth, there exists a PD-morphism $g\colon T \to D$ over S compatible with the morphism $z\colon U \to Z$. The pull-back of the filtered module $\mathcal{M}$ on D by g is independent of the choice of g up to canonical isomorphism as follows. For two PD-morphisms $g_i : T \to D$ $(i \in \{1, 2\})$ over S compatible with z, their compositions with $p_Y : D \to Y$ induce a morphism $(p_Y \circ g_1, p_Y \circ g_2)\colon T \to Y(1)$ and hence a PD-morphism $g_{12}\colon T \to D(1)$ over S, which satisfies $p_i \circ g_{12} = g_i$. The filtered isomorphism $\varepsilon_{\mathcal{M}}$ induces an isomorphism $g_2^*\mathcal{M} \cong g_{12}^*p_2^*\mathcal{M} \underset{g_{12}^*(\varepsilon_{\mathcal{M}})}{\xrightarrow{\cong}} g_{12}^*p_1^*\mathcal{M} \cong g_1^*\mathcal{M}$ of filtered modules on T. This is the identity morphism if $g_1 = g_2$ by the condition (ii-a) on $\varepsilon_{\mathcal{M}}$ in Definition 16 because g_{12} is the composition of $g_1 = g_2$ and $\Delta\colon D \to D(1)$. For three PD-morphisms $g_i\colon T \to D$ $(i \in \{1, 2, 3\})$, the composition of the isomorphisms $g_3^*\mathcal{M} \overset{\cong}{\to} g_2^*\mathcal{M}$ and $g_2^*\mathcal{M} \overset{\cong}{\to} g_1^*\mathcal{M}$ associated to the pair (g_2, g_3) and (g_1, g_2) coincides with the isomorphism associated to the pair (g_1, g_3) by the condition (ii-b) on $\varepsilon_{\mathcal{M}}$ in Definition 16. Note that the morphism $(p_Y \circ g_1, p_Y \circ g_2, p_Y \circ g_3)\colon T \to Y(2)$ induces a PD-morphism $g_{123}\colon T \to D(2)$, and we have $p_{ij} \circ g_{123} = g_{ij}$ for $(i, j) \in \{(1, 2), (2, 3), (1, 3)\}$. Here $g_{ij}\colon T \to D(1)$ is defined in the same way as g_{12} using g_i and g_j. For each object T of $\mathrm{CRYS}(Z/S)$, we can glue the above pull-backs on all affine open subschemes T and obtain a filtered module $\mathcal{F}_T$ on T. By construction, we have a canonical isomorphism

$\tau_u\colon u^*(\mathcal{F}_T) \xrightarrow{\cong} \mathcal{F}_{T'}$ of filtered modules on T' for each morphism $u\colon T' \to T$ in CRYS(Z/S). It is straightforward to verify that the data $(\mathcal{F}_T, \tau_u)$ is functorial in $\mathcal{M}$ and gives the desired quasi-inverse.

Next let us prove the equivalence between StratF$(D(\bullet))$ and $\mathrm{MF}^{\nabla}(Z \hookrightarrow Y/S)$. Let $(\mathcal{M}, \mathrm{Fil}^{\bullet}\mathcal{M})$ be a filtered module on D. By [5, 6.6 Theorem], we know that there is a canonical bijection between the set of $\mathcal{O}_{D(1)}$-linear isomorphisms $\varepsilon\colon p_2^*\mathcal{M} \xrightarrow{\cong} p_1^*\mathcal{M}$ satisfying the conditions (ii-a) and (ii-b) in Definition 16, and the set of quasi-nilpotent integrable connections $\nabla\colon \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$ on $\mathcal{M}$ satisfying $\nabla(ax) = a\nabla(x) + x \otimes \nabla(a)$ $(a \in \mathcal{O}_D, x \in \mathcal{M})$. Suppose that ε and ∇ correspond to each other. It suffices to verify that ε is a filtered isomorphism if and only if ∇ satisfies the Griffiths transversality. First note that the former is equivalent to $\varepsilon(\mathrm{Fil}^r(p_2^*\mathcal{M})) \subset \mathrm{Fil}^r(p_1^*\mathcal{M})$ for $r \in \mathbb{Z}$ because the inverse of ε is given by $p_1^*\mathcal{M} \cong \iota^* p_2^*\mathcal{M} \xrightarrow[\iota^*(\varepsilon)]{\cong} \iota^* p_1^*\mathcal{M} \cong p_2^*\mathcal{M}$, where ι is the automorphism of $D(1)$ induced by the automorphism of $Y(1) = Y \times_S Y$ exchanging the two components. (See the proof of the independence of $g^*\mathcal{M}$ in the first paragraph.) Since the question is Zariski local on Y, we may assume that there exist $t_1, \dots, t_d \in \Gamma(Y, \mathcal{O}_Y)$ such that dt_ν $(\nu \in \mathbb{N} \cap [1, d])$ form a basis of $\Omega_{Y/S}$. Put $\tau_\nu := p_{2,Y}^*(t_\nu) - p_{1,Y}^*(t_\nu)$ for $\nu \in \mathbb{N} \cap [1, d]$, where $p_{i,Y}$ denotes the composition $D(1) \xrightarrow{p_i} D \to Y$. Then we have an isomorphism of algebras $\mathcal{O}_D\langle T_1, \dots, T_d\rangle \xrightarrow{\cong} \mathcal{O}_{D(1)}$ sending $a \in \mathcal{O}_D$ to $p_1^*(a)$ and $T_\nu^{[n]}$ to $\tau_\nu^{[n]}$ for $n \in \mathbb{N}_{>0}$. For $\underline{n} = (n_\nu) \in \mathbb{N}^d$, put $|\underline{n}| = \sum_{\nu=1}^d n_\nu$, $T^{[\underline{n}]} = \prod_{\nu=1}^d T_\nu^{[n_\nu]}$, and $\tau^{[\underline{n}]} = \prod_{\nu=1}^d \tau_\nu^{[n_\nu]}$. Then the above isomorphism induces an isomorphism $\oplus_{\underline{n} \in \mathbb{N}^d} \mathrm{Fil}^{r-|\underline{n}|}\mathcal{O}_D \cdot T^{[\underline{n}]} \xrightarrow{\cong} \mathrm{Fil}^r\mathcal{O}_{D(1)}$ for $r \in \mathbb{Z}$. Hence we have an isomorphism

$$\bigoplus_{\underline{n} \in \mathbb{N}^d} \mathrm{Fil}^{r-|\underline{n}|}\mathcal{M} \xrightarrow{\cong} \mathrm{Fil}^r(p_1^*\mathcal{M}) = \mathrm{Fil}^r(\mathcal{M} \otimes_{\mathcal{O}_D} \mathcal{O}_{D(1)})$$

sending $(x_{\underline{n}})_{\underline{n} \in \mathbb{N}^d}$ to $\sum_{\underline{n} \in \mathbb{N}^d} x_{\underline{n}} \otimes \tau^{[\underline{n}]}$. (The image is obviously contained in $\mathrm{Fil}^r(p_1^*\mathcal{M})$. The opposite inclusion follows from $\mathrm{Fil}^{r-s-|\underline{n}|}\mathcal{O}_D\tau^{[\underline{n}]} \cdot \mathrm{Fil}^s\mathcal{M} \subset \tau^{[\underline{n}]}$ $\mathrm{Fil}^{r-|\underline{n}|}\mathcal{M}$ in $\mathcal{M} \otimes_{\mathcal{O}_D} \mathcal{O}_{D(1)}$.) On the other hand, the $\mathcal{O}_{D(1)}$-linear isomorphism $\varepsilon\colon p_2^*\mathcal{M} = \mathcal{O}_{D(1)} \otimes_{\mathcal{O}_D} \mathcal{M} \xrightarrow{\cong} p_1^*\mathcal{M} = \mathcal{M} \otimes_{\mathcal{O}_D} \mathcal{O}_{D(1)}$ is described in terms of ∇ as

$$\varepsilon(1 \otimes x) = \sum_{\underline{n}=(n_\nu) \in \mathbb{N}^d} \prod_{\nu=1}^d (\nabla_\nu)^{n_\nu}(x) \otimes \tau^{[\underline{n}]}, \quad x \in \mathcal{M}, \tag{10}$$

where ∇_ν $(\nu \in \mathbb{N} \cap [1, d])$ are the endomorphisms of $\mathcal{M}$ defined by $\nabla(y) = \sum_{\nu=1}^d \nabla_\nu(y) \otimes dt_\nu$ $(y \in \mathcal{M})$. Since the Griffiths transversality is equivalent to $\nabla_\nu(\mathrm{Fil}^r\mathcal{M}) \subset \mathrm{Fil}^{r-1}\mathcal{M}$ $(\nu \in \mathbb{N} \cap [1, d])$, the above observations imply that ∇ satisfies the Griffiths transversality if and only if $\varepsilon(p_2^{-1}(\mathrm{Fil}^r\mathcal{M}))) \subset \mathrm{Fil}^r(p_1^*\mathcal{M})$, where p_2^{-1} denotes the morphism $\mathcal{M} \to p_2^*\mathcal{M} = \mathcal{O}_{D(1)} \otimes_{\mathcal{O}_D} \mathcal{M}; x \mapsto 1 \otimes x$. The latter condition is equivalent to $\varepsilon(\mathrm{Fil}^r(p_2^*\mathcal{M})) \subset \mathrm{Fil}^r(p_1^*\mathcal{M})$ by the definition of the filtration on $p_2^*\mathcal{M}$. (We may also apply Lemma 14 to $(\mathcal{M}, \mathrm{Fil}^\bullet\mathcal{M})$, $p_1^*(\mathcal{M}, \mathrm{Fil}^\bullet\mathcal{M})$, and $p_2\colon D(1) \to D$.) This completes the proof. □

Remark 18 Let $(\mathcal{M}, \nabla)$ be an object of $\mathrm{MF}^\nabla(Z \hookrightarrow Y/S)$, and let $(\mathcal{M}, \varepsilon)$ be the object of $\mathrm{StratF}(D(\bullet))$ associated to $(\mathcal{M}, \nabla)$ by the equivalence of categories in Theorem 17. Suppose that there exist $t_1, \ldots, t_d \in \Gamma(Y, \mathcal{O}_Y)$ such that dt_ν form a basis of $\Omega_{Y/S}$, and define the endomorphisms ∇_ν $(\nu \in \mathbb{N} \cap [1, d])$ of $\mathcal{M}$ by $\nabla(x) = \sum_\nu \nabla_\nu(x) \otimes dt_\nu$. Let $(U \hookrightarrow T, z\colon U \to Z)$ be an object of $\mathrm{CRYS}(Z/S)$, and suppose that we are given two PD-morphisms $g_1, g_2\colon T \to D$ compatible with z. Then, by using (10), we see that the filtered isomorphism

$$g_{12}^*(\varepsilon)\colon g_2^*\mathcal{M} = z^{-1}(\mathcal{M}) \otimes_{z^{-1}(\mathcal{O}_D), g_2^*} \mathcal{O}_T \xrightarrow{\cong} z^{-1}(\mathcal{M}) \otimes_{z^{-1}(\mathcal{O}_D), g_1^*} \mathcal{O}_T = g_1^*\mathcal{M}$$

considered in the first paragraph of the proof of Theorem 17 is given by

$$x \otimes 1 \mapsto \sum_{\underline{n}=(n_\nu)\in\mathbb{N}^d} \prod_{1\le\nu\le d} \nabla_\nu^{n_\nu}(x) \otimes \prod_{1\le\nu\le d} (g_2^*(t_\nu) - g_1^*(t_\nu))^{[n_\nu]}. \tag{11}$$

Note that the differences $g_2^*(t_\nu) - g_1^*(t_\nu)$ are contained in the PD-ideal of $\mathcal{O}_T$. We will also use the following logarithmic variant of the above formula. Suppose that $t_\nu \in \Gamma(Y, \mathcal{O}_Y^\times)$. Then we can define the endomorphisms $\nabla_\nu^{\log}$ $(\nu \in \mathbb{N} \cap [1, d])$ of $\mathcal{M}$ by $\nabla(x) = \sum_\nu \nabla_\nu^{\log}(x) \otimes d\log t_\nu$, where $d\log t_\nu = t_\nu^{-1}dt_\nu$. We have $\nabla_\nu^{\log} = t_\nu\nabla_\nu$, and see $t_\nu^n\nabla_\nu^n = \prod_{j=0}^{n-1}(\nabla_\nu^{\log} - j)$ $(n \in \mathbb{N})$ by induction on n. Hence (11) is rewritten as

$$x \otimes 1 \mapsto \sum_{\underline{n}=(n_\nu)\in\mathbb{N}^d} \prod_{1\le\nu\le d} \prod_{j=0}^{n_\nu-1} (\nabla_\nu^{\log} - j)(x) \otimes \prod_{1\le\nu\le d} (g_2^*(t_\nu)g_1^*(t_\nu)^{-1} - 1)^{[n_\nu]}. \tag{12}$$

Remark 19 Let $(S, \mathcal{J}_S, \gamma_S)$ be a PD-scheme on which p is locally nilpotent, let X be a smooth scheme over S, and let $\mathrm{Crys}(X/S)$ be the small crystalline site of X over $(S, \mathcal{J}_S, \gamma_S)$. We can define filtered crystals on $\mathrm{Crys}(X/S)$ in the same way as those on the big crystalline site, and prove an analogue of Theorem 17. In [17, 3.1.2 Theorem], an interpretation of a filtration Griffiths transversal to an integrable connection in terms of crystals is given. We see that these two claims coincide, i.e. a filtered crystal = a crystal with a filtration G-transversal to $(J_{X/S}, \gamma)$, as follows. Let E be a crystal of $\mathcal{O}_{X/S}$-modules endowed with a decreasing filtration $A^\bullet E$ by $\mathcal{O}_{X/S}$-submodules. If the filtration $A^\bullet E$ satisfy the condition 1 in [17, 3.1.1 Lemma], then $(E, A^\bullet E)$ is a filtered crystal in our sense. (Note that the condition for $f = \mathrm{id}_T$ implies $J_T^{[r]}A^kE_T \subset A^{k+r}E_T$ for $r > 0$.) Therefore, by the last claim in [17, 3.1.1 Lemma], the G-transversality of $A^\bullet$ to $(J_{X/S}, \gamma)$ implies that $(E, A^\bullet E)$ is a filtered crystal in our sense. We can prove that the converse is also true as follows. Suppose that $(E, A^\bullet E)$ is a filtered crystal in our sense. By the definition of a filtered crystal, the filtration $A^\bullet E_T$ on E_T is saturated with respect to (J_T, γ) ([17, 2.1.2 Definition]) for each object T of $\mathrm{Crys}(X/S)$. For any object $U \hookrightarrow T$ in $\mathrm{Crys}(X/S)$, there exists, Zariski locally on T, a morphism $(U \hookrightarrow T) \to (\mathrm{id}\colon U \to U)$ in $\mathrm{Crys}(X/S)$ because $X \to S$ is smooth. This gives a splitting $\mathcal{O}_T = \mathcal{O}_U \oplus J_T$, which induces $E_T \cong E_U \oplus (E_U \otimes_{\mathcal{O}_U} J_T)$. Since $A^k E_T$ is the sum of the images of $A^{k-r} E_U \otimes_{\mathcal{O}_U} J_T^{[r]}$ ($r \in \mathbb{N}$), we see that $A^k E_T$ is the direct sum of $A^k E_U$ and the sum of the images of $A^{k-r} E_U \otimes_{\mathcal{O}_U} J_T^{[r]}$ in $E_U \otimes_{\mathcal{O}_U} J_T$ for $r \in \mathbb{N}_{>0}$. Hence $A^k E_T \cap J_T E_T = A^k E_T \cap (E_U \otimes_{\mathcal{O}_U} J_T)$ is contained in $\sum_{r \in \mathbb{N}_{>0}} J_T^{[r]} A^{k-r} E_T$, i.e., A_T is G'-transversal to (J_T, γ) ([17, 2.1.2 Definition]). Thus we see that $(E, A^\bullet E)$ is G-transversal to $(J_{X/S}, \gamma)$. This argument does not work for the big crystalline site, because the source of an object of the big crystalline site $\mathrm{CRYS}(X/(S, \mathcal{J}_S, \gamma_S))$ is not an open subscheme of X in general.

We obtain the following lemma from Lemma 13 (1) and the Proof of Theorem 17.

Lemma 20 *Let $(\mathcal{M}, \nabla)$ be an object of $\mathrm{MF}^\nabla(Z \hookrightarrow Y/S)$, and let $\mathcal{F}$ be the filtered crystal on $\mathrm{CRYS}(Z/(S, \mathcal{J}_S, \gamma_S))$ associated to $(\mathcal{M}, \nabla)$ by the equivalence of categories in Theorem 17. Suppose that the filtered $\mathcal{O}_D$-module $\mathcal{M}$ is isomorphic to $\oplus_{\nu \in \mathbb{N} \cap [1,N]} \mathcal{O}_D(r_\nu)$ for $N \in \mathbb{N}$ and $r_\nu \in \mathbb{Z}$ ($\nu \in \mathbb{N} \cap [1, N]$) (Definition 10 (2)). Then, for any object $(U \hookrightarrow T, U \to Z)$ of $\mathrm{CRYS}(Z/(S, \mathcal{J}_S, \gamma_S))$ such that T is affine, the filtered $\mathcal{O}_T$-module $\mathcal{F}_T$ is isomorphic to $\oplus_{\nu \in \mathbb{N} \cap [1,N]} \mathcal{O}_T(r_\nu)$.*

Definition 21 We define the categories $\mathrm{C}(Z/(S, \mathcal{J}_S, \gamma_S))$, $\mathrm{Strat}(D(\bullet))$, and $\mathrm{M}^\nabla(Z \hookrightarrow Y/S)$ by replacing filtered modules with modules in Definition 16 (i), (ii), and (iii), respectively.

By simply forgetting filtrations in the proof of Theorem 17, we obtain the following.

Theorem 22 *The three categories $\mathrm{C}(Z/(S, \mathcal{J}_S, \gamma_S))$, $\mathrm{Strat}(D(\bullet))$, and $\mathrm{M}^\nabla(Z \hookrightarrow Y/S)$ are naturally equivalent.*

We discuss the functoriality of the equivalences of categories given in Theorems 17 and 22.

Let $k\colon (S', \mathcal{J}_{S'}, \gamma_{S'}) \to (S, \mathcal{J}_S, \gamma_S)$ be a PD-morphism of PD-schemes on which p is locally nilpotent. Let $f\colon Z \to S$ and $f'\colon Z' \to S'$ be morphisms of schemes such that γ_S and $\gamma_{S'}$ extend to Z and Z', respectively, and let $g\colon Z' \to Z$ be a morphism of schemes such that $f \circ g = k \circ f'$. Then g induces a morphism of PD-ringed topos $g_{\mathrm{CRYS}}\colon ((Z'/S')_{\mathrm{CRYS}}, \mathcal{O}_{Z'/S'}) \to ((Z/S)_{\mathrm{CRYS}}, \mathcal{O}_{Z/S})$ ([4, III (4.2.2)]). The inverse image functor of the underlying morphism of topos is simply given by $(g^*_{\mathrm{CRYS}}(\mathcal{F}))(i_{T'}\colon \overline{T}' \hookrightarrow T', z_{T'}\colon \overline{T}' \to Z') = \mathcal{F}(i_{T'}\colon \overline{T}' \hookrightarrow T', g \circ z_{T'}\colon \overline{T}' \to Z)$. By applying Definition 15 (3) to g_{CRYS}, we obtain a functor

$$g^*_{\mathrm{CRYS}}\colon \mathrm{CF}(Z/(S, \mathcal{J}_S, \gamma_S)) \longrightarrow \mathrm{CF}(Z'/(S', \mathcal{J}_{S'}, \gamma_{S'})). \tag{13}$$

Suppose that we are given a commutative diagram of schemes

$$\begin{array}{ccccc} Z' & \xrightarrow{i'} & Y' & \xrightarrow{h'} & S' \\ \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle k} \\ Z & \xrightarrow{i} & Y & \xrightarrow{h} & S, \end{array}$$

where i and i' are closed immersions, h and h' are smooth, $f = h \circ i$, and $f' = h' \circ i'$. We define $Y(r)$, $D(r)$ ($r \in \mathbb{N}$), and D as before Definition 16 by using $Z \xrightarrow{i} Y \xrightarrow{h} S$ and $(\mathcal{J}_S, \gamma_S)$, and construct $Y'(r)$, $D'(r)$, and D' similarly from $Z' \xrightarrow{i'} Y' \xrightarrow{h'} S'$ and $(\mathcal{J}_{S'}, \gamma_{S'})$. Let $i_D, i_{D(r)}, i_{D'}$ and $i_{D'(r)}$ denote the canonical closed immersions $Z \to D$, $Z \to D(r)$, $Z' \to D'$, and $Z' \to D'(r)$.

We further assume that we are given a morphism

$$\ell\colon D' \longrightarrow Y$$

over the morphism $k\colon S' \to S$ such that $\ell \circ i_{D'} = i \circ g$. (Note that we do not assume that ℓ is induced by a morphism $Y' \to Y$.) If Y' is affine, then D' is affine. Hence, in this case, a morphism ℓ as above always exists because $i_{D'}$ is a nilimmersion and $h\colon Y \to S$ is smooth. For $r \in \mathbb{N}$, let $\ell(r)\colon D'(r) \to Y(r)$ be the unique morphism over k such that the composition with the νth projection $Y(r) \to Y$ coincides with that of the νth projection $D'(r) \to D'$ and ℓ for every $\nu \in \mathbb{N} \cap [1, r+1]$. We have $\ell(r) \circ i_{D'(r)} = i_{Y(r)} \circ g$, where $i_{Y(r)}$ is the immersion $Z \to Y(r)$ induced by i, and the morphisms $\ell(r)$ define a morphism of simplicial schemes $D'(\bullet) \to Y(\bullet)$. Hence $\ell(r)$ ($r \in \mathbb{N}$) induce a morphism of simplicial PD-schemes $\ell_{D(\bullet)}\colon D'(\bullet) \to D(\bullet)$. We write ℓ_D for $\ell_D(0)$.

Let $(\mathcal{M}, \varepsilon_{\mathcal{M}})$ be an object of $\mathrm{StratF}(D(\bullet))$. Let $\mathcal{M}'$ be the filtered $\mathcal{O}_{D'}$-module $\mathcal{M}' := \ell_D^*(\mathcal{M})$. Then, by taking the pull-back of $\varepsilon_{\mathcal{M}}$ by the morphism $\ell_{D(1)}\colon D'(1) \to D(1)$, we obtain an isomorphism of filtered $\mathcal{O}_{D'(1)}$-modules $\varepsilon_{\mathcal{M}'}\colon p_2'^*\mathcal{M}' \xrightarrow{\cong} p_1'^*\mathcal{M}'$, where p_1' and p_2' denote the first and second projections $D'(1) \to D'$. By using the fact that $\ell_{D(\bullet)}$ is a morphism of simplicial PD-schemes, we see that the pair $(\mathcal{M}', \varepsilon_{\mathcal{M}'})$ is an object of $\mathrm{StratF}(D'(\bullet))$. This construction is obviously functorial in $(\mathcal{M}, \varepsilon_{\mathcal{M}})$, and we obtain a functor

$$\ell_{D(\bullet)}^*\colon \mathrm{StratF}(D(\bullet)) \longrightarrow \mathrm{StratF}(D'(\bullet)). \tag{14}$$

Next we will construct a functor

$$\ell^*\colon \mathrm{MF}^{\nabla}(Z \hookrightarrow Y/S) \longrightarrow \mathrm{MF}^{\nabla}(Z' \hookrightarrow Y'/S') \tag{15}$$

Let $\Delta^1\colon D \to D(1)^1$ and $\Delta'^1\colon D' \to D'(1)^1$ be the closed immersions defined by PD-squares of the ideal defining the diagonal maps $D \to D(1)$ and $D' \to D'(1)$. Then the morphism $\ell_{D(1)^1}\colon D'(1)^1 \to D(1)^1$ induced by $\ell_{D(1)}$ gives a morphism

$$\ell_D^*\colon \ell_D^{-1}(\mathcal{O}_D\otimes_{\mathcal{O}_Y}\Omega_{Y/S})\cong \ell_D^{-1}(\mathrm{Ker}(\Delta^{1*}\colon \mathcal{O}_{D(1)^1}\to\mathcal{O}_D))$$
$$\xrightarrow{\ell^*_{D(1)^1}} \mathrm{Ker}(\Delta'^{1*}\colon \mathcal{O}_{D'(1)^1}\to\mathcal{O}_{D'})\cong \mathcal{O}_{D'}\otimes_{\mathcal{O}_{Y'}}\Omega_{Y'/S'}, \qquad (16)$$

and the following diagram is commutative.

$$\begin{array}{ccc} \ell_D^{-1}(\mathcal{O}_D) & \xrightarrow{\ell_D^{-1}(\nabla)} & \ell_D^{-1}(\mathcal{O}_D\otimes_{\mathcal{O}_Y}\Omega_{Y/S}) \\ \downarrow{\scriptstyle \ell_D^*} & & \downarrow{\scriptstyle \ell_D^*} \\ \mathcal{O}_{D'} & \xrightarrow{\nabla} & \mathcal{O}_{D'}\otimes_{\mathcal{O}_{Y'}}\Omega_{Y'/S'} \end{array} \qquad (17)$$

Note that the derivations ∇ on $\mathcal{O}_D$ and $\mathcal{O}_{D'}$ are defined as the differences of the pull-backs by the two projections $D(1)^1 \rightrightarrows D$ and $D'(1)^1 \rightrightarrows D'$, and that the projections are compatible with $\ell_{D(1)^1}$ and ℓ_D.

Let $(\mathcal{M},\nabla)$ be an object of $\mathrm{MF}^\nabla(Z\hookrightarrow Y/S)$. Let $\mathcal{M}'$ be the filtered $\mathcal{O}_{D'}$-module $\ell_D^*(\mathcal{M})$, and let $\ell^{*q}_{D,\mathcal{M}}$ $(q\in\mathbb{N})$ be the morphism $\ell_D^{-1}(\mathcal{M}\otimes_{\mathcal{O}_Y}\Omega^q_{Y/S})\to \mathcal{M}'\otimes_{\mathcal{O}_{Y'}}\Omega^q_{Y'/S'}$ induced by $\ell_D^{-1}(\mathcal{M})\to\mathcal{M}'=\mathcal{O}_{D'}\otimes_{\ell_D^{-1}(\mathcal{O}_D)}\ell_D^{-1}(\mathcal{M})$; $a\mapsto 1\otimes a$ and (16). The morphism $\mathcal{O}_{D'}\times\ell_D^{-1}(\mathcal{M})\to\mathcal{M}'\otimes_{\mathcal{O}_{Y'}}\Omega_{Y'/S'}$ defined by $(a,x)\mapsto a\ell^{*1}_{D,\mathcal{M}}(\ell_D^{-1}(\nabla)(x))+x\otimes\nabla(a)$ is $\ell_D^{-1}(\mathcal{O}_D)$-bilinear and induces a connection $\nabla'\colon\mathcal{M}'\to\mathcal{M}'\otimes_{\mathcal{O}_{D'}}\Omega_{Y'/S'}$ satisfying $\nabla'(ax)=a\nabla'(x)+x\otimes\nabla(a)$ $(a\in\mathcal{O}_{D'}, x\in\mathcal{M}')$. We see that ∇' satisfies the Griffiths transversality by using $\nabla(\mathrm{Fil}^r\mathcal{O}_{D'})\subset \mathrm{Fil}^{r-1}\mathcal{O}_{D'}\otimes_{\mathcal{O}_{Y'}}\Omega_{Y'/S'}$. As for the integrability of ∇', we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{M}' & \xrightarrow{\nabla'} & \mathcal{M}'\otimes_{\mathcal{O}_{Y'}}\Omega^1_{Y'/S'} & \xrightarrow{\nabla'^1} & \mathcal{M}'\otimes_{\mathcal{O}_{Y'}}\Omega^2_{Y'/S'} \\ \uparrow{\scriptstyle \ell^{*0}_{D,\mathcal{M}}} & & \uparrow{\scriptstyle \ell^{*1}_{D,\mathcal{M}}} & & \uparrow{\scriptstyle \ell^{*2}_{D,\mathcal{M}}} \\ \ell_D^{-1}(\mathcal{M}) & \xrightarrow{\ell_D^{-1}(\nabla)} & \ell_D^{-1}(\mathcal{M}\otimes_{\mathcal{O}_Y}\Omega^1_{Y/S}) & \xrightarrow{\ell_D^{-1}(\nabla^1)} & \ell_D^{-1}(\mathcal{M}\otimes_{\mathcal{O}_Y}\Omega^2_{Y/S}), \end{array}$$

where ∇^1 is defined by $\nabla^1(x\otimes\omega)=\nabla(x)\wedge\omega+x\otimes\nabla^1(\omega)$ $(x\in\mathcal{M}, \omega\in\mathcal{O}_D\otimes_{\mathcal{O}_Y}\Omega^1_{Y/S})$, and ∇'^1 is defined similarly. The composition of the lower horizontal morphisms is 0, and that of the upper one is $\mathcal{O}_{D'}$-linear as $\nabla^1\circ\nabla=0$ on $\mathcal{O}_{D'}$. Hence $\nabla'^1\circ\nabla'=0$, i.e., ∇' is integral.

Proposition 23 *Under the notation above, let* $(\mathcal{M},\varepsilon)$ *denote the object of* $\mathrm{StratF}(D(\bullet))$ *corresponding to* $(\mathcal{M},\nabla)$ *by Theorem 17, let* $(\mathcal{M}',\varepsilon')$ *be* $\ell^*_{D(\bullet)}(\mathcal{M},\varepsilon)$ (14)*, and let* $(\mathcal{M}',\nabla'')$ *be the object of* $\mathrm{MF}^\nabla(Z'\hookrightarrow Y'/S')$ *corresponding to* $(\mathcal{M}',\varepsilon')$ *by Theorem 17. Then we have* $\nabla'=\nabla''$*. In particular,* ∇' *is quasi-nilpotent.*

Proof We define $D(1)^1$, $D'(1)^1$, and $\ell_{D(1)^1}$ as before (16). Let ε^1 (resp. ε'^1) be the pull-back of ε (resp. ε') by the morphism $D(1)^1\to D(1)$ (resp. $D'(1)^1\to D'(1)$). Then the homomorphisms $\ell_D^{-1}(\mathcal{O}_{D(1)^1}\otimes_{\mathcal{O}_D}\mathcal{M})\to\mathcal{O}_{D'(1)^1}\otimes_{\mathcal{O}_{D'}}\mathcal{M}'$ and $\ell_D^{-1}(\mathcal{M}\otimes_{\mathcal{O}_D}\mathcal{O}_{D(1)^1})\to\mathcal{M}'\otimes_{\mathcal{O}_{D'}}\mathcal{O}_{D'(1)^1}$ induced by $\ell_{D(1)^1}$ are compatible with $\ell_D^{-1}(\varepsilon^1)$ and ε'^1 by the definition of ε'. Hence the diagram

$$\begin{array}{ccc} \ell_D^{-1}(\mathcal{M}) & \xrightarrow{\ell_D^{-1}(\nabla)} & \ell_D^{-1}(\mathcal{M}\otimes_{\mathcal{O}_Y}\Omega_{Y/S}) \\ \downarrow{\scriptstyle \ell^{*0}_{D,\mathcal{M}}} & & \downarrow{\scriptstyle \ell^{*1}_{D,\mathcal{M}}} \\ \mathcal{M}' & \xrightarrow{\nabla''} & \mathcal{M}'\otimes_{\mathcal{O}_{Y'}}\Omega_{Y'/S'} \end{array}$$

is commutative because ∇ on $\mathcal{M}$ is given by $\varepsilon^1|_{\mathcal{M}}$ minus the inclusion map $\mathcal{M}\hookrightarrow\mathcal{M}\otimes_{\mathcal{O}_D}\mathcal{O}_{D(1)^1}$ and similarly for ∇'' on $\mathcal{M}'$. This implies the claim. □

By Proposition 23, $(\mathcal{M}',\nabla')$ is an object of $\mathrm{MF}^\nabla(Z'\hookrightarrow Y'/S')$ and obtain the desired functor (15).

Proposition 24 *The following diagram is commutative up to canonical isomorphisms, where the horizontal arrows are the equivalences of categories constructed in the proof of Theorem 17.*

$$\begin{array}{ccccc} \mathrm{CF}(Z/(S,\mathcal{J}_S,\gamma_S)) & \rightleftarrows & \mathrm{StratF}(D(\bullet)) & \rightleftarrows & \mathrm{MF}^\nabla(Z\hookrightarrow Y/S) \\ {\scriptstyle (13)}\downarrow{\scriptstyle g^*_{\mathrm{CRYS}}} & & {\scriptstyle (14)}\downarrow{\scriptstyle \ell^*_{D(\bullet)}} & & {\scriptstyle (15)}\downarrow{\scriptstyle \ell^*} \\ \mathrm{CF}(Z'/(S',\mathcal{J}_{S'},\gamma_{S'})) & \rightleftarrows & \mathrm{StratF}(D'(\bullet)) & \rightleftarrows & \mathrm{MF}^\nabla(Z'\hookrightarrow Y'/S') \end{array}$$

Proof Proposition 23 means that the right diagram is commutative. The commutativity of the left diagram follows from the construction of the functor $\ell^*_{D(\bullet)}$ and the explicit description of g^*_{CRYS} recalled above. □

By forgetting filtrations on underlying modules in the proof of Proposition 24, we obtain the following.

Proposition 25 *The following diagram is commutative up to canonical isomorphisms, where the horizontal arrows are the equivalences of categories constructed in the proof of Theorem 22, and the three vertical functors are defined by forgetting filtrations in the construction of (13), (14), and (15).*

$$\begin{array}{ccccc} \mathrm{C}(Z/(S,\mathcal{J}_S,\gamma_S)) & \rightleftarrows & \mathrm{Strat}(D(\bullet)) & \rightleftarrows & \mathrm{M}^\nabla(Z\hookrightarrow Y/S) \\ \downarrow{\scriptstyle g^*_{\mathrm{CRYS}}} & & \downarrow{\scriptstyle \ell^*_{D(\bullet)}} & & \downarrow{\scriptstyle \ell^*} \\ \mathrm{C}(Z'/(S',\mathcal{J}_{S'},\gamma_{S'})) & \rightleftarrows & \mathrm{Strat}(D'(\bullet)) & \rightleftarrows & \mathrm{M}^\nabla(Z'\hookrightarrow Y'/S') \end{array}$$

Finally we discuss the quasi-coherence of filtered crystals and crystals. Note that, for a PD-scheme $(T,\mathcal{J}_T,\gamma_T)$, $\mathrm{Fil}^r\mathcal{O}_T$ $(r\in\mathbb{Z})$ are quasi-coherent ideals of $\mathcal{O}_T$.

Definition 26 (1) We say that a filtered $\mathcal{O}_T$-module $(\mathcal{M},\mathrm{Fil}^\bullet\mathcal{M})$ on a PD-scheme T is *quasi-coherent* if $\mathcal{M}$ and $\mathrm{Fil}^r\mathcal{M}$ $(r\in\mathbb{Z})$ are quasi-coherent $\mathcal{O}_T$-modules.

(2) We say that a crystal $\mathcal{F}$ (resp. a filtered crystal $(\mathcal{F},\mathrm{Fil}^\bullet\mathcal{F})$) on $\mathrm{CRYS}(Z/S)$ is *quasi-coherent* if $\mathcal{F}_T$ (resp. $(\mathcal{F}_T,\mathrm{Fil}^\bullet\mathcal{F}_T)$) is quasi-coherent for every objet T of $\mathrm{CRYS}(Z/S)$. We write $\mathrm{C}_{\mathrm{qc}}(Z/(S,\mathcal{J}_S,\gamma_S))$ (resp. $\mathrm{CF}_{\mathrm{qc}}(Z/(S,\mathcal{J}_S,\gamma_S))$) for

the category of quasi-coherent crystals (resp. quasi-coherent filtered crystals) on $\mathrm{CRYS}(Z/S)$.

(3) We say that an object of $\mathrm{M}^\nabla(Z \hookrightarrow Y/S)$ (resp. $\mathrm{MF}^\nabla(Z \hookrightarrow Y/S)$) is *quasi-coherent* if its underlying $\mathcal{O}_D$-module (resp. filtered $\mathcal{O}_D$-module) is quasi-coherent. We write $\mathrm{M}^\nabla_{\mathrm{qc}}(Z \hookrightarrow Y/S)$ (resp. $\mathrm{MF}^\nabla_{\mathrm{qc}}(Z \hookrightarrow Y/S)$) for the full subcategory consisting of quasi-coherent objects.

Lemma 27 (1) *Let T be a PD-scheme whose underlying scheme is affine. Then the functor $\Gamma(T, -)$ induces an equivalence of categories between the category of quasi-coherent filtered $\mathcal{O}_T$-modules and that of filtered modules over $(\Gamma(T, \mathcal{O}_T), \Gamma(T, \mathrm{Fil}^\bullet\mathcal{O}_T))$.*

(2) *Let $f: T' \to T$ be a morphism of PD-schemes. Then the pull-back of quasi-coherent filtered $\mathcal{O}_T$-modules by f is a quasi-coherent filtered $\mathcal{O}_{T'}$-modules.*

(3) *Let $f: T' \to T$ be a morphism of PD-schemes whose underlying schemes are affine. Then the equivalence of categories in* (1) *for T and T' are compatible with the pull-back by f and the scalar extension by $f^*: \Gamma(T, \mathcal{O}_T) \to \Gamma(T', \mathcal{O}'_T)$.*

Proof (1) Let $(\mathcal{M}, \mathrm{Fil}^\bullet\mathcal{M})$ be a quasi-coherent module with a decreasing filtration by quasi-coherent $\mathcal{O}_T$-submodules. Then, as $\mathrm{Fil}^r\mathcal{O}_T$ is a quasi-coherent ideal of $\mathcal{O}_T$, we see that $\mathrm{Fil}^r\mathcal{O}_T \cdot \mathrm{Fil}^s\mathcal{M} \subset \mathrm{Fil}^{r+s}\mathcal{M}$ if and only if $\Gamma(T, \mathrm{Fil}^r\mathcal{O}_T) \cdot \Gamma(T, \mathrm{Fil}^s\mathcal{M}) \subset \Gamma(T, \mathrm{Fil}^{r+s}\mathcal{M})$.

(2), (3) Let $(\mathcal{M}, \mathrm{Fil}^\bullet\mathcal{M})$ be a quasi-coherent filtered $\mathcal{O}_T$-module, and put $M = \Gamma(T, \mathcal{M})$ and $\mathrm{Fil}^r M = \Gamma(T, \mathrm{Fil}^r\mathcal{M})$ $(r \in \mathbb{Z})$. Then, as $\mathrm{Fil}^s\mathcal{O}_{T'}$ is a quasi-coherent ideal of $\mathcal{O}_{T'}$, the image of $\mathrm{Fil}^s\mathcal{O}_{T'} \otimes_{\mathcal{O}_{T'}} f^*(\mathrm{Fil}^{r-s}\mathcal{M}) \to f^*(\mathcal{M})$ is quasi-coherent. If T and T' are affine, then $\Gamma(T', -)$ of the image coincides with the image of $\mathrm{Fil}^s R_{T'} \otimes_{R_{T'}} (\mathrm{Fil}^{r-s} M \otimes_{R_T} R_{T'}) \to M \otimes_{R_T} R_{T'}$, where $R_T = \mathcal{O}_T(T)$, $R_{T'} = \mathcal{O}_{T'}(T')$, and $\mathrm{Fil}^r R_{T'} = \mathrm{Fil}^r\mathcal{O}_{T'}(T')$. These imply the claims. □

We immediately obtain the following corollary from Lemma 27 (2).

Corollary 28 *The functors g_{CRYS} and ℓ^* appearing in Propositions 24 and 25 preserve quasi-coherent objects.*

By the proof of Theorem 17, we also obtain the following from Lemma 27 (2).

Theorem 29 (1) *The equivalence of categories in Theorem 17 induces that of $\mathrm{CF}_{\mathrm{qc}}(Z/(S, \mathcal{J}_S, \gamma_S))$ and $\mathrm{MF}^\nabla_{\mathrm{qc}}(Z \hookrightarrow Y/S)$.*

(2) *The equivalence of categories in Theorem 22 induces that of $\mathrm{C}_{\mathrm{qc}}(Z/(S, \mathcal{J}_S, \gamma_S))$ and $\mathrm{M}^\nabla_{\mathrm{qc}}(Z \hookrightarrow Y/S)$.*

Assume that Z and Y satisfy the following conditions.

The schemes Z and Y are affine (18)

There exist $t_1, \ldots, t_d \in \Gamma(Y, \mathcal{O}_Y)$ such that $dt_i (1 \le i \le d)$ form a basis of the $\mathcal{O}_Y$ -module $\Omega_{Y/S}$. (19)

The condition (18) implies that D is also affine. Put $R_D := \Gamma(D, \mathcal{O}_D)$, $\mathrm{Fil}^r R_D := \Gamma(D, \mathrm{Fil}^r\mathcal{O}_D)$, $B := \Gamma(Y, \mathcal{O}_Y)$, and $\Omega_B := \Gamma(Y, \Omega_{Y/S})$. Let ∇_{R_D} be the derivation

$R_D \to R_D \otimes_B \Omega_B$ induced by $\nabla\colon \mathcal{O}_D \to \mathcal{O}_D \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$. Choose and fix $t_1, \ldots, t_d$ satisfying (19).

Definition 30 We define the category $\mathrm{MF}^\nabla(R_D, \nabla_{R_D})$ as follows. An object is a filtered module M over $(R_D, \mathrm{Fil}^\bullet R_D)$ with an integrable connection $\nabla\colon M \to M \otimes_B \Omega_B$ satisfying $\nabla(ax) = a\nabla(x) + x \otimes \nabla_{R_D}(a)$ $(a \in R_D, x \in M)$, $\nabla(\mathrm{Fil}^r M) \subset \mathrm{Fil}^{r-1} M \otimes_B \Omega_B$ for $r \in \mathbb{Z}$ (Griffiths transversality), and the following nilpotence: For any $x \in M$, there exists $N \in \mathbb{N}$ such that $\prod_{1\le\nu\le d} \nabla_\nu^{n_\nu}(x) = 0$ for all $(n_\nu) \in \mathbb{N}^d$ with $\sum_\nu n_\nu \ge N$, where the endomorphisms ∇_ν $(\nu \in \mathbb{N} \cap [1, d])$ of M are defined by $\nabla(x) = \sum_\nu \nabla_\nu(x) \otimes dt_\nu$. A morphism is a homomorphism of filtered R_D-modules compatible with ∇.

We define the category $\mathrm{M}^\nabla(R_D, \nabla_{R_D})$ by replacing filtered modules with modules and removing Griffiths transversality.

Proposition 31 *Under the conditions (18) and (19), we have the following equivalences of categories defined by taking the global sections $\Gamma(D, -)$.*

$$\mathrm{MF}^\nabla_{\mathrm{qc}}(Z \hookrightarrow Y/S) \xrightarrow{\cong} \mathrm{MF}^\nabla(R_D, \nabla_{R_D}), \tag{20}$$

$$\mathrm{M}^\nabla_{\mathrm{qc}}(Z \hookrightarrow Y/S) \xrightarrow{\cong} \mathrm{M}^\nabla(R_D, \nabla_{R_D}). \tag{21}$$

Proof Let $(\mathcal{M}, \nabla)$ be an object of $\mathrm{M}^\nabla_{\mathrm{qc}}(Z \hookrightarrow Y/S)$. Then the quasi-nilpotence of ∇ implies the nilpotence of $\Gamma(D, \nabla)$ in Definition 30 because Z is quasi-compact. We also see that ∇ is determined by $\Gamma(D, \nabla)$ because the $\mathcal{O}_D$-module $\mathcal{M}$ is generated by $\Gamma(D, \mathcal{M})$. Therefore we obtain fully faithful functors by taking the global sections on D. Let $(M, \mathrm{Fil}^\bullet M, \nabla)$ be an object of $\mathrm{MF}^\nabla(R_D, \nabla_{R_D})$. By Lemma 27 (1), we have a quasi-coherent filtered module $(\mathcal{M}, \mathrm{Fil}^\bullet\mathcal{M})$ on D whose global sections are $(M, \mathrm{Fil}^\bullet M)$. For any affine open subscheme $\mathrm{Spec}(R')$ of D, the map $M \times R' \to M \otimes_R R' \otimes_B \Omega_B$; $(x, a) \mapsto a\nabla(x) + x \otimes \nabla_{R'}(a)$ is R-bilinear and induces $M \otimes_R R' \to M \otimes_R R' \otimes_B \Omega_B$. Here $\nabla_{R'}\colon R' \to R' \otimes_B \Omega_B$ is the sections of $\nabla\colon \mathcal{O}_D \to \mathcal{O}_D \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$ on $\mathrm{Spec}(R')$. These are compatible with restrictions and define a morphism $\nabla\colon \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$. By using $\nabla(\mathrm{Fil}^r\mathcal{O}_D) \subset \mathrm{Fil}^{r-1}\mathcal{O}_D \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$, and the integrability and quasi-nilpotence of the connection ∇ on $\mathcal{O}_D$, we see that $(\mathcal{M}, \mathrm{Fil}^\bullet\mathcal{M}, \nabla)$ is an object of $\mathrm{MF}^\nabla(Z \hookrightarrow Y/S)$, whose global sections are $(M, \mathrm{Fil}^\bullet M, \nabla)$ by construction. The same argument applies to an object of $\mathrm{M}^\nabla(R_D, \nabla_{R_D})$. Thus we see that the two functors in the proposition are equivalences of categories. □

Let notation and assumption be the same as before Proposition 24, and assume that $Z \to Y \to S$ and $Z' \to Y' \to S'$ satisfy the conditions (18) and (19). We define the categories ${}'\mathrm{MF}^\nabla(R_{D'}, \nabla_{R_{D'}})$ and ${}'\mathrm{M}^\nabla(R_{D'}, \nabla_{R_{D'}})$ by removing the nilpotence condition on ∇ in the definition of $\mathrm{MF}^\nabla(R_{D'}, \nabla_{R_{D'}})$ and $\mathrm{M}^\nabla(R_{D'}, \nabla_{R_{D'}})$ (Definition 30), respectively. Then one can construct functors $\ell^*_{R_D}\colon \mathrm{MF}^\nabla(R_D, \nabla_{R_D}) \to {}'\mathrm{MF}^\nabla(R_{D'}, \nabla_{R_{D'}})$ and $\ell^*_{R_D}\colon \mathrm{M}^\nabla(R_D, \nabla_{R_D}) \to {}'\mathrm{M}^\nabla(R_{D'}, \nabla_{R_{D'}})$ similarly to the construction of (15), and obtain the following.

Proposition 32 *The functor* $\ell_{R_D}^*$ *for* $\mathrm{MF}^{\nabla}(-)$ *is compatible with* ℓ^* *(15) via the equivalence of categories (20) for* $Z \to Y \to S$ *and* $Z' \to Y' \to S'$. *The same claim for* $\mathrm{M}^{\nabla}(-)$ *holds with respect to (21) and* ℓ^* *appearing in Proposition 25. In particular, the functors* $\ell_{R_D}^*$ *factor through* $\mathrm{MF}^{\nabla}(R_{D'}, \nabla_{R_{D'}})$ *and* $\mathrm{M}^{\nabla}(R_{D'}, \nabla_{R_{D'}})$.

4 The Relative Fontaine–Laffaille Theory by Faltings

We define $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A}, \Phi)$ to be the full subcategory of the abelian category $\mathfrak{MF}^{\nabla}_{[0,p-2]}(\mathcal{A})$ introduced by Faltings [10, II (d)] consisting of $(M, \mathrm{Fil}^r M, \nabla, \Phi)$ such that $\mathrm{gr}^r_{\mathrm{Fil}}(M)$ $(r \in \mathbb{Z})$ are free $\mathcal{A}$-modules. Let $\Omega_{\mathcal{A}}$ be the inverse limit of $\Omega_{(A/p^nA)/(O_K/p^nO_K)}$ $(n \in \mathbb{N}_{>0})$, and let $d\colon \mathcal{A} \to \Omega_{\mathcal{A}}$ be the inverse limit of the universal derivations $d\colon A/p^nA \to \Omega_{(A/p^nA)/(O_K/p^nO_K)}$ $(n \in \mathbb{N}_{>0})$. Then an object of $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A}, \Phi)$ is given by the following quadruple $(M, \mathrm{Fil}^r M, \nabla, \Phi)$:

(i) A free $\mathcal{A}$-module of finite type M.

(ii) An integrable connection $\nabla\colon M \to M \otimes_{\mathcal{A}} \Omega_{\mathcal{A}}$ such that $(\nabla \mod p^n)$ is quasi-nilpotent for every $n \in \mathbb{N}_{>0}$, i.e., for any $x \in M$, $\prod_{1\leq i\leq d} \nabla_i^{n_i}(x)$, $(n_i) \in \mathbb{N}^d$ converges to 0 as $\sum_i n_i \to \infty$ with respect to the p-adic topology. Here ∇_i $(i \in \mathbb{N} \cap [1, d])$ denotes the endomorphism of M defined by $\nabla(x) = \sum_{1\leq i\leq d} \nabla_i(x) \otimes dt_i$.

(iii) A decreasing filtration $\mathrm{Fil}^r M$ $(r \in \mathbb{Z})$ of M by $\mathcal{A}$-submodules satisfying the following conditions:

(iii-1) $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{p-1} M = 0$.

(iii-2) $\mathrm{gr}^r_{\mathrm{Fil}} M$ is a free $\mathcal{A}$-module of finite type for every $r \in \mathbb{Z}$.

(iii-3) (Griffiths transversality) $\nabla(\mathrm{Fil}^r M) \subset \mathrm{Fil}^{r-1} M \otimes_{\mathcal{A}} \Omega_{\mathcal{A}}$ $(r \in \mathbb{Z})$.

For $m \in \mathbb{N}_{>0}$, we define $\mathrm{Fil}^r_p(M/p^mM)$ $(r \in \mathbb{Z})$ to be the sum of the images of $p^{[s]}\mathrm{Fil}^{r-s} M$ $(s \in \mathbb{N})$, and ∇_m on M/p^m to be $(\nabla \mod p^m)$. Put $X_m = \mathrm{Spec}(A/p^mA)$ and $\Sigma_m = \mathrm{Spec}(O_K/p^mO_K)$ for $m \in \mathbb{N}_{>0}$. Then, by (ii) and (iii-3), we see that $(M/p^mM, \mathrm{Fil}^{\bullet}_p(M/p^mM), \nabla_m)$ defines an object of $\mathrm{MF}^{\nabla}_{\mathrm{qc}}(X_1 \hookrightarrow X_m/\Sigma_m)$ (Definition 26 (3), (20)). Let $\varphi\colon \mathcal{A} \to \mathcal{A}$ be a lifting of the absolute Frobenius of A/pA compatible with σ of O_K. Then, by applying Propositions 24 and 32 to the absolute Frobenius of X_1 and its lifting to X_m defined by φ for each $m \in \mathbb{N}_{>0}$, we see that the pull-back $(\varphi^*(M), \varphi^*(\nabla))$ of (M, ∇) by φ with the decreasing filtration $\mathrm{Fil}^r_p(\varphi^*(M)) := \sum_{s\in\mathbb{N}} p^{[s]}\varphi^*(\mathrm{Fil}^{r-s} M)$ $(r \in \mathbb{Z})$ is independent of the choice of φ up to canonical isomorphisms (see Remark 33 below). Let $F^*(M)$ denote the filtered $\mathcal{A}$-module with the integrable connection thus obtained.

(iv) An $\mathcal{A}$-linear homomorphism $\Phi\colon F^*(M) \to M$ satisfying the following conditions:

(iv-1) Φ is compatible with the connections.

(iv-2) $\Phi(\mathrm{Fil}^r_p(F^*(M))) \subset p^r F^*(M)$ for $r \in \mathbb{N} \cap [0, p-2]$.

(iv-3) $\sum_{r\in\mathbb{N}\cap[0,p-2]} p^{-r}\Phi(\mathrm{Fil}^r_p(F^*(M))) = M$.

Remark 33 Let φ and φ' be two liftings of the absolute Frobenius of A/pA to $\mathcal{A}$ compatible with σ of O_K. Then, by using (11), we see that the canonical $\mathcal{A}$-linear isomorphism $\varphi^*M = M \otimes_{\mathcal{A},\varphi} \mathcal{A} \xrightarrow{\cong} \varphi'^*M = M \otimes_{\mathcal{A},\varphi'} \mathcal{A}$ is given by $x \otimes 1 \mapsto \sum_{\underline{n}=(n_i)\in\mathbb{N}^d} \prod_{i=1}^d \nabla_i^{n_i}(x) \otimes \prod_{1\leq i\leq d}(\varphi(t_i) - \varphi'(t_i))^{[n_i]}$, where ∇_i $(i \in \mathbb{N} \cap [1, d])$ are defined by $\nabla(x) = \sum_{1\leq i\leq d} \nabla_i(x) \otimes dt_i$, $x \in M$. Note $\varphi(t_i) - \varphi'(t_i) \in p\mathcal{A}$.

Choose and fix a lifting of the absolute Frobenius $\varphi_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}$ compatible with σ of O_K, and define φ of $\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})$ as in Sect. 2. For an object $(M, \mathrm{Fil}^r M, \nabla, \Phi)$ of $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A}, \Phi)$, let $T^*_{\mathrm{crys}}(M)$ be the $\mathbb{Z}_p$-module $\mathrm{Hom}_{\mathcal{A}\text{-lin},\mathrm{Fil},\varphi,\nabla}(M, \mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}}))$ of $\mathcal{A}$-linear maps from M to $\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})$ compatible with the filtrations, φ, and ∇, where φ of M is defined to be the composition of $M \to \varphi^*_{\mathcal{A}}(M) = F^*(M) \xrightarrow{\Phi} M$. Then $T^*_{\mathrm{crys}}(M)$ is a free $\mathbb{Z}_p$-module whose rank is the same as the $\mathcal{A}$-module M (Proposition 66), and the natural action of $G_{\mathcal{A}}$ on $T^*_{\mathrm{crys}}(M)$ is continuous because the action of $G_{\mathcal{A}}$ on $\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})$ is continuous and $\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})/\mathrm{Fil}^r$ $(r \in \mathbb{Z})$ is p-torsion free by (5). Thus we obtain a contravariant $\mathbb{Z}_p$-linear functor

$$T^*_{\mathrm{crys}} \colon \mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A}, \Phi) \longrightarrow \mathrm{Rep}_{\mathrm{free}}(G_{\mathcal{A}}, \mathbb{Z}_p),$$

where $\mathrm{Rep}_{\mathrm{free}}(G_{\mathcal{A}}, \mathbb{Z}_p)$ denotes the category of free $\mathbb{Z}_p$-modules of finite type with continuous action of $G_{\mathcal{A}}$. Furthermore the functor T^*_{crys} is fully faithful (see Theorem 77), and is independent of the choice of $\varphi_{\mathcal{A}}$ up to canonical isomorphisms (see Remark 38).

5 A_{crys}-Representations with φ and Fil

In this section, we introduce a free $A_{\mathrm{crys}}(\overline{\mathcal{A}})$-module $TA_{\mathrm{crys}}(M)$ of finite type with an action of $G_{\mathcal{A}}$, a filtration and a Frobenius endomorphism associated to an object M of $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A}, \Phi)$ ([10, II (e)]).

For $m \in \mathbb{N}_{>0}$, put $\Sigma_m := \mathrm{Spec}(O_K/p^m)$, $\mathcal{A}_m := \mathcal{A}/p^m$, and $X_m := \mathrm{Spec}(\mathcal{A}_m)$, and let γ denote the canonical PD-structure on $p(O_K/p^m)$. To simplify the notation, we write $\mathrm{CRYS}(X_m/\Sigma_m)$ and $(X_m/\Sigma_m)_{\mathrm{CRYS}}$ (resp. $\mathrm{CRYS}(X_1/\Sigma_m)$ and $(X_1/\Sigma_m)_{\mathrm{CRYS}}$) for the big crystalline site and topos of X_m (resp. X_1) over Σ_m with the PD-ideal $(p(O_K/p^m), \gamma)$. Let $F_{\Sigma_m} \colon \Sigma_m \to \Sigma_m$ be the lifting of the absolute Frobenius of Σ_1 defined by $\sigma \colon O_K \to O_K$, It is a PD-morphism with respect to γ. The absolute Frobenius F_{X_1} of X_1 and F_{Σ_m} define a morphism of PD-ringed topos $F_{X_1/\Sigma_m,\mathrm{CRYS}} \colon (X_1/\Sigma_m)_{\mathrm{CRYS}} \to (X_1/\Sigma_m)_{\mathrm{CRYS}}$.

Let $(M, \mathrm{Fil}^\bullet M, \nabla, \Phi)$ be an object of $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A}, \Phi)$. For $m \in \mathbb{N}_{>0}$, let $(M_m, \mathrm{Fil}^\bullet M_m, \nabla, \Phi)$ denote the reduction mod p^m of $(M, \mathrm{Fil}^\bullet M, \nabla, \Phi)$. Then $(M_m, \mathrm{Fil}^\bullet M_m, \nabla)$ defines an object of $\mathrm{MF}^{\nabla}_{\mathrm{qc}}(X_m \overset{\mathrm{id}}{\hookrightarrow} X_m/\Sigma_m)$ (Definition 26 (3)) by (20). By Theorems 17 and 29, this object gives a quasi-coherent filtered crystal $(\mathcal{F}_m, \mathrm{Fil}^\bullet\mathcal{F}_m)$ on $\mathrm{CRYS}(X_m/\Sigma_m)$. Since $(M, \mathrm{Fil}^\bullet M)$ is finite filtered free of level $[0, p-2]$ (Definition 10 (3)), so is $(\Gamma(T, \mathcal{F}_m), \Gamma(T, \mathrm{Fil}^\bullet\mathcal{F}_m))$ for any object $(U \hookrightarrow T, U \to X_m)$ of $\mathrm{CRYS}(X_m/\Sigma_m)$ with T affine, by Lemma 20. The pair (M_m, ∇) defines an object of $\mathrm{M}^{\nabla}_{\mathrm{qc}}(X_1 \hookrightarrow X_m/\Sigma_m)$ (Definition 26 (3), (21)), which gives a quasi-coherent crystal $\mathcal{G}_m$ on $\mathrm{CRYS}(X_1/\Sigma_m)$ by Theorems 22 and 29. The reduction mod p^m of $\Phi\colon F^*M \to M$ equip $\mathcal{G}_m$ with a morphism $\Phi_{\mathcal{G}_m}\colon F^*_{X_1/\Sigma_m,\mathrm{CRYS}}(\mathcal{G}_m) \to \mathcal{G}_m$. By Propositions 25 and 32, we have

$$i^*_{m,\mathrm{CRYS}}(\mathcal{F}_m) = \mathcal{G}_m \tag{22}$$

for the morphism of ringed topos $i_{m,\mathrm{CRYS}}\colon (X_1/\Sigma_m)_{\mathrm{CRYS}} \to (X_m/\Sigma_m)_{\mathrm{CRYS}}$ induced by the closed immersion $i_m\colon X_1 \to X_m$ over id_{Σ_m}. By Propositions 24 (resp. 25) and 32, the pull-back of $(\mathcal{F}_{m+1}, \mathrm{Fil}^\bullet\mathcal{F}_{m+1})$ (resp. $(\mathcal{G}_{m+1}, \Phi_{\mathcal{G}_{m+1}})$) by the morphism of ringed topos induced by $X_m \to X_{m+1}$ (resp. id_{X_1}) and $\Sigma_m \to \Sigma_{m+1}$ is canonically identified with $(\mathcal{F}_m, \mathrm{Fil}^\bullet\mathcal{F}_m)$ (resp. $(\mathcal{G}_m, \Phi_{\mathcal{G}_m})$). This identification is compatible with $i^*_{m,\mathrm{CRYS}}(\mathcal{F}_m) = \mathcal{G}_m$ (22) in the obvious sense.

For $m \in \mathbb{N}_{>0}$, put $\overline{D}_m := \mathrm{Spec}(A_{\mathrm{crys},m}(\overline{\mathcal{A}}))$ and $\overline{X}_m := \mathrm{Spec}(\overline{\mathcal{A}}_m)$, and let $F_{\overline{D}_m}\colon \overline{D}_m \to \overline{D}_m$ be the lifting of the absolute Frobenius of $\overline{D}_1$ defined by φ of $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$. The closed immersion $\overline{X}_m \hookrightarrow \overline{D}_m$ (resp. $\overline{X}_1 \hookrightarrow \overline{D}_m$) is naturally regarded as an object of $\mathrm{CRYS}(X_m/\Sigma_m)$ (resp. $\mathrm{CRYS}(X_1/\Sigma_m)$) endowed with a right action of $G_{\mathcal{A}}$. We define an $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$-module $TA_{\mathrm{crys},m}(M)$ by

$$TA_{\mathrm{crys},m}(M) := \Gamma(\overline{X}_m \hookrightarrow \overline{D}_m, \mathcal{F}_m) \xrightarrow[(22)]{\cong} \Gamma(\overline{X}_1 \hookrightarrow \overline{D}_m, \mathcal{G}_m). \tag{23}$$

For the second isomorphism, note that $\mathcal{F}_m$ is a crystal on $\mathrm{CRYS}(X_m/\Sigma_m)$. The right action of $G_{\mathcal{A}}$ on $\overline{D}_m$ induces its left action on $TA_{\mathrm{crys},m}(M)$. The filtration on $\mathcal{F}_m$ gives a filtration Fil^r $(r \in \mathbb{Z})$ by $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$-submodules on $TA_{\mathrm{crys},m}(M)$, which is stable under the $G_{\mathcal{A}}$-action. The $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$-module $TA_{\mathrm{crys},m}(M)$ with $\mathrm{Fil}^\bullet$ is a filtered module over the filtered ring $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$ which is finite filtered free of level $[0, p-2]$. The Frobenius $\Phi_{\mathcal{G}_m}$ of $\mathcal{G}_m$ and the lifting of Frobenius $F_{\overline{D}_m}$ on $\overline{D}_m$ define a semilinear $G_{\mathcal{A}}$-equivariant endomorphism of $\Gamma(\overline{X}_1 \hookrightarrow \overline{D}_m, \mathcal{G}_m)$ and hence that of $TA_{\mathrm{crys},m}(M)$ as

$$\Gamma(\overline{X}_1 \hookrightarrow \overline{D}_m, \mathcal{G}_m) \longrightarrow \Gamma(\overline{X}_1 \hookrightarrow \overline{D}_m, F^*_{X_1,\mathrm{CRYS}}(\mathcal{G}_m)) \xrightarrow{\Phi_{\mathcal{G}_m}} \Gamma(\overline{X}_1 \hookrightarrow \overline{D}_m, \mathcal{G}_m).$$

Here the first homomorphism is induced by $F_{\overline{X}_1}$ and $F_{\overline{D}_m}$.

Let $\overline{\gamma}$ denote the PD-structure on the ideal $pA_{\mathrm{crys},m}(\overline{\mathcal{A}}) + \mathrm{Fil}^1 A_{\mathrm{crys},m}(\overline{\mathcal{A}})$ of $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$. We write $\mathrm{CRYS}(\overline{X}_m/\overline{D}_m)$ and $(\overline{X}_m/\overline{D}_m)_{\mathrm{CRYS}}$ (resp. $\mathrm{CRYS}(\overline{X}_1/\overline{D}_m)$ and $(\overline{X}_1/\overline{D}_m)_{\mathrm{CRYS}}$) for the big crystalline site and topos of $\overline{X}_m$ (resp. $\overline{X}_1$) over $\overline{D}_m$ with the PD-ideal $(pA_{\mathrm{crys},m}(\overline{\mathcal{A}}) + \mathrm{Fil}^1 A_{\mathrm{crys},m}(\overline{\mathcal{A}}), \overline{\gamma})$. By taking the pull-back of $(\mathcal{F}_m, \mathrm{Fil}^\bullet\mathcal{F}_m)$ (resp. $(\mathcal{G}_m, \Phi_{\mathcal{G}_m})$) under the morphism of ringed topos $(\overline{X}_m/\overline{D}_m)_{\mathrm{CRYS}} \to (X_m/\Sigma_m)_{\mathrm{CRYS}}$ (resp. $(\overline{X}_1/\overline{D}_m)_{\mathrm{CRYS}} \to (X_1/\Sigma_m)_{\mathrm{CRYS}}$), we obtain a quasi-coherent filtered crystal $(\overline{\mathcal{F}}_m, \mathrm{Fil}^\bullet\overline{\mathcal{F}}_m)$ (resp. a quasi-coherent crystal $\overline{\mathcal{G}}_m$ with a morphism $\Phi_{\overline{\mathcal{G}}_m}\colon F^*_{\overline{X}_1/\overline{D}_m,\mathrm{CRYS}}(\overline{\mathcal{G}}_m) \to \overline{\mathcal{G}}_m$) endowed with an action of $G_{\mathcal{A}}$ equivariant with respect to its action on $\overline{X}_m$ (resp. $\overline{X}_1$) and $\overline{D}_m$. Here $F_{\overline{X}_1/\overline{D}_m,\mathrm{CRYS}}$ denotes the morphism of ringed topos $(\overline{X}_1/\overline{D}_m)_{\mathrm{CRYS}} \to (\overline{X}_1/\overline{D}_m)_{\mathrm{CRYS}}$ defined by the absolute Frobenius of $\overline{X}_1$ and $F_{\overline{D}_m}$. Note that $F_{\overline{D}_m}$ is a PD-morphism with respect to $\overline{\gamma}$.

Since $\overline{X}_m \hookrightarrow \overline{D}_m$ (resp. $\overline{X}_1 \hookrightarrow \overline{D}_m$) is a final object of $\mathrm{CRYS}(\overline{X}_m/\overline{D}_m)$ (resp. $\mathrm{CRYS}(\overline{X}_1/\overline{D}_m)$), we have canonical $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$-linear isomorphisms

$$TA_{\mathrm{crys},m}(M) \cong \Gamma((\overline{X}_m/\overline{D}_m)_{\mathrm{CRYS}}, \overline{\mathcal{F}}_m) \cong \Gamma((\overline{X}_1/\overline{D}_m)_{\mathrm{CRYS}}, \overline{\mathcal{G}}_m). \quad (24)$$

The filtration (resp. the Frobenius endomorphism) and the action of $G_{\mathcal{A}}$ on the middle (resp. the right) term induced by the corresponding structures on $\overline{\mathcal{F}}_m$ (resp. $\overline{\mathcal{G}}_m$) are compatible with those structures on $TA_{\mathrm{crys},m}(M)$ defined after (23).

Let $\mathcal{B} \to \mathcal{A}$, $\varphi_{\mathcal{B}}$, $O_{K,m}$ $\mathcal{A}_m$, $\mathcal{B}_m$, $\Omega_{\mathcal{B}_m}$, $\varphi_{\mathcal{B}_m}$, and $(\mathcal{P}_m, \mathrm{Fil}^\bullet\mathcal{P}_m, \nabla_{\mathcal{P}_m}, \varphi_{\mathcal{P}_m})$ be as in Sect. 2. For $m \in \mathbb{N}_{>0}$, let Y_m be $\mathrm{Spec}(\mathcal{B}_m)$, and let F_{Y_m} be $\mathrm{Spec}(\varphi_{\mathcal{B}_m})\colon Y_m \to Y_m$. We give a description of $TA_{\mathrm{crys},m}(M)$ in terms of $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ and the sections of $\mathcal{F}_m$ and $\mathcal{G}_m$ on $X_1, X_m \hookrightarrow \mathrm{Spec}(\mathcal{P}_m)$ (29). The latter sections coincide with M_m when $\mathcal{A} = \mathcal{B}$. We use the description for a general $\mathcal{B}$ in Sects. 16 and 20. We first introduce some notation concerning the sections on $\mathrm{Spec}(\mathcal{P}_m)$. Let $(M_{\mathcal{P}_m}, \mathrm{Fil}^\bullet M_{\mathcal{P}_m})$ be the sections of $(\mathcal{F}_m, \mathrm{Fil}^\bullet\mathcal{F}_m)$ on the object $(X_m \hookrightarrow \mathrm{Spec}(\mathcal{P}_m))$ of $\mathrm{CRYS}(X_m/\Sigma_m)$, which is a filtered module over $(\mathcal{P}_m, \mathrm{Fil}^\bullet\mathcal{P}_m)$ finite filtered free of level $[0, p-2]$. Since $Y_m \to \Sigma_m$ is smooth, the proof of Theorem 17 shows that $(M_m, \mathrm{Fil}^\bullet M_{\mathcal{P}_m})$ is equipped with an integrable connection $\nabla\colon M_{\mathcal{P}_m} \to M_{\mathcal{P}_m} \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$ compatible with $\nabla_{\mathcal{P}_m}$ and satisfying $\nabla(\mathrm{Fil}^r M_{\mathcal{P}_m}) \subset \mathrm{Fil}^{r-1}M_{\mathcal{P}_m} \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$. By (22) and the fact that $\mathcal{F}_m$ is a crystal, the $\mathcal{P}_m$-module $M_{\mathcal{P}_m}$ is canonically isomorphic to the sections of $\mathcal{G}_m$ on the object $(X_1 \hookrightarrow \mathrm{Spec}(\mathcal{P}_m))$ of $\mathrm{CRYS}(X_1/\Sigma_m)$. By applying Propositions 25 and 32 to $X_1 \hookrightarrow X_m$ and id_{Y_m} over id_{Σ_m}, we see that the connection on $M_{\mathcal{P}_m}$ associated to $\mathcal{G}_m$ by Theorem 17 coincides with ∇ above associated to $\mathcal{F}_m$. Therefore Propositions 25 and 32 applied to F_{X_1} and F_{Y_m} over F_{Σ_m} imply that $\Phi_{\mathcal{G}_m}$ induces a $\varphi_{\mathcal{P}_m}$-semilinear endomorphism $\varphi\colon M_{\mathcal{P}_m} \to M_{\mathcal{P}_m}$ compatible with ∇, i.e. $(\varphi \otimes \varphi_{\mathcal{B}}) \circ \nabla = \nabla \circ \varphi$ on $M_{\mathcal{P}_m}$.

We give a slightly different construction of $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$. Let $Y_{\overline{D}_m}$ be $Y_m \times_{\Sigma_m} \overline{D}_m$, and let $\overline{E}_m$ be the PD-envelope compatible with $\overline{\gamma}$ of the closed immersion $\overline{X}_m \to Y_{\overline{D}_m}$ defined by the morphisms $\overline{X}_m \hookrightarrow \overline{D}_m$ and $\overline{X}_m \to X_m \hookrightarrow Y_m$. The right action of $G_{\mathcal{A}}$ on $\overline{D}_m$ induces its right action on $\overline{E}_m$. By Lemma 6 (1), (2), the lifting $F_{Y_m} \times_{F_{\Sigma_m}} F_{\overline{D}_m}\colon Y_{\overline{D}_m} \to Y_{\overline{D}_m}$ of the absolute Frobenius of $Y_{\overline{D}_1}$ induces a lifting $F_{\overline{E}_m}\colon \overline{E}_m \to \overline{E}_m$ of the absolute Frobenius of $\overline{E}_m \times_{\Sigma_m} \Sigma_1$. We also have an $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$-linear derivation $\nabla\colon \Gamma(\overline{E}_m, \mathcal{O}_{\overline{E}_m}) \to \Gamma(\overline{E}_m, \mathcal{O}_{\overline{E}_m}) \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$ ([4, IV Sect. 1.3]). The morphism $Y_{\overline{D}_m} \to Y_m$ induces a homomorphism of PD-algebras $\mathcal{P}_m \to \Gamma(\overline{E}_m, \mathcal{O}_{\overline{E}_m})$ stable under the $G_{\mathcal{A}}$-action and compatible with $\mathrm{Fil}^\bullet$, ∇ and φ. Here the filtration (resp. φ) of the right-hand side is defined by $\Gamma(\overline{E}_m, \mathrm{Fil}^r\mathcal{O}_{\overline{E}_m})$ (resp. $\Gamma(\overline{E}_m, F_{\overline{E}_m})$). By using the fact that $\overline{D}_m$ is the PD-envelope of $\mathrm{Spec}(O_{\overline{K}}/p^m) \hookrightarrow \mathrm{Spec}(A_{\inf}(O_{\overline{K}})/p^m)$ compatible with the PD-structure γ on

pO_K, we can verify that the PD-thickening $\overline{X}_m = \mathrm{Spec}(\overline{\mathcal{A}}_m) \hookrightarrow \mathrm{Spec}(\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}))$ over $\overline{D}_m$ with $\overline{\gamma}$ satisfies the universal property of the PD-envelope $\overline{X}_m \to \overline{E}_m$. This implies that $\mathrm{id}_{\overline{X}_m}$, the morphism $Y_{\overline{D}_m} \to \mathrm{Spec}(\mathcal{B} \otimes_{O_K} A_{\mathrm{inf}}(O_{\overline{K}})/p^m)$, and the PD-morphism $(\overline{D}_m, \overline{\gamma}) \to (\Sigma_m, \gamma)$ induce a PD-isomorphism

$$\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \cong \Gamma(\overline{E}_m, \mathcal{O}_{\overline{E}_m}) \tag{25}$$

compatible with the $G_{\mathcal{A}}$-actions, $\mathrm{Fil}^\bullet$, ∇, φ, and the homomorphisms from $\mathcal{P}_m$ and $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$.

By (25), the sections of $(\overline{\mathcal{F}}_m, \mathrm{Fil}^\bullet\overline{\mathcal{F}}_m)$ on the object $\overline{X}_m \hookrightarrow \overline{E}_m$ of the crystalline site $\mathrm{CRYS}(\overline{X}_m/\overline{D}_m)$ give a filtered module $(\Gamma(\overline{E}_m, \overline{\mathcal{F}}_m), \Gamma(\overline{E}_m, \mathrm{Fil}^\bullet\overline{\mathcal{F}}_m))$ over the filtered ring $(\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}), \mathrm{Fil}^\bullet\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}))$, which is finite filtered free of level $[0, p-2]$ and is naturally endowed with an action of $G_{\mathcal{A}}$. By Theorem 29, (20), and Proposition 32, it is equipped with a $G_{\mathcal{A}}$-equivariant integrable connection $\nabla\colon \Gamma(\overline{E}_m, \overline{\mathcal{F}}_m) \to \Gamma(\overline{E}_m, \overline{\mathcal{F}}_m) \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$ compatible with ∇ on $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ and satisfying $\nabla(\Gamma(\overline{E}_m, \mathrm{Fil}^r\overline{\mathcal{F}}_m)) \subset \Gamma(\overline{E}_m, \mathrm{Fil}^{r-1}\overline{\mathcal{F}}_m) \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$ $(r \in \mathbb{Z})$. By Propositions 25 and 32 applied to $\overline{X}_1 \hookrightarrow \overline{X}_m$ and $\mathrm{id}_{Y_{\overline{D}_m}}$ over $\mathrm{id}_{\overline{D}_m}$, the $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$-module $\Gamma(\overline{E}_m, \overline{\mathcal{F}}_m)$ with the $G_{\mathcal{A}}$-action and ∇ is canonically isomorphic to the sections $\Gamma(\overline{E}_m, \overline{\mathcal{G}}_m)$ of $\overline{\mathcal{G}}_m$ on the object $\overline{X}_1 \hookrightarrow \overline{E}_m$ of $\mathrm{CRYS}(\overline{X}_1/\overline{D}_m)$ equipped with the action of $G_{\mathcal{A}}$ and ∇ defined similarly to $\Gamma(\overline{E}_m, \overline{\mathcal{F}}_m)$. The Frobenius $\Phi_{\overline{\mathcal{G}}_m}$ of $\overline{\mathcal{G}}_m$ and $F_{\overline{E}_m}$ induce a $G_{\mathcal{A}}$-equivariant endomorphism φ of $\Gamma(\overline{E}_m, \overline{\mathcal{G}}_m)$ semilinear with respect to φ of $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$. By applying Propositions 25 and 32 to $F_{\overline{X}_1}$ and $F_{Y_{\overline{D}_m}}$ over $F_{\overline{D}_m}$, we see that it is compatible with ∇.

By (24) and the description of global sections of a crystal in terms of horizontal sections of the corresponding module with an integrable connection on the PD-envelope of an embedding into a smooth scheme ([4, Proposition 4.1.4], [5, 7.1 Theorem]), we obtain a canonical $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$-linear $G_{\mathcal{A}}$-equivariant isomorphisms

$$TA_{\mathrm{crys},m}(M) \cong \Gamma(\overline{E}_m, \overline{\mathcal{F}}_m)^{\nabla=0} \cong \Gamma(\overline{E}_m, \overline{\mathcal{G}}_m)^{\nabla=0}. \tag{26}$$

The filtration (resp. the Frobenius endomorphism) on the middle (resp. right) term is compatible with that on $TA_{\mathrm{crys},m}(M)$. Since $\overline{X}_m \hookrightarrow \overline{D}_m$ and $\overline{X}_1 \hookrightarrow \overline{D}_m$ are final objects of $\mathrm{CRYS}(\overline{X}_m/\overline{D}_m)$ and $\mathrm{CRYS}(\overline{X}_1/\overline{D}_m)$, respectively, we see, by using Lemma 27 (3), that the isomorphisms (26) induce $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$-linear $G_{\mathcal{A}}$-equivariant isomorphisms

$$TA_{\mathrm{crys},m}(M) \otimes_{A_{\mathrm{crys},m}(\overline{\mathcal{A}})} \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \xrightarrow{\cong} \Gamma(\overline{E}_m, \overline{\mathcal{F}}_m) \cong \Gamma(\overline{E}_m, \overline{\mathcal{G}}_m) \tag{27}$$

compatible with ∇, $\mathrm{Fil}^\bullet$ and φ. Here, on the left-hand side, the filtration and φ are defined by the tensor products of those on $TA_{\mathrm{crys},m}(M)$ and $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$, and the connection is defined by $\mathrm{id} \otimes \nabla$. For the compatibility with ∇, we use Propositions 24, 25, and 32 applied to $\mathrm{id}_{\overline{X}_n}$ $(n = 1, m)$, $\mathrm{id}_{\overline{D}_m}$, and $Y_{\overline{D}_m} \to \overline{D}_m$.

Let ι_m be the canonical PD-homomorphism $\mathcal{P}_m \to \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$. We endow $M_{\mathcal{P}_m} \otimes_{\mathcal{P}_m,\iota_m} \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ with the tensor products of the filtrations, integrable connections and φ's on $M_{\mathcal{P}_m}$ and $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$, and also with the $G_{\mathcal{A}}$-action via $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$. By Propositions 24 (resp. Proposition 25) and 32 applied to $\overline{X}_m \to X_m$ (resp. $\overline{X}_1 \to X_1$), $\overline{D}_m \to \Sigma_m$, and $Y_{\overline{D}_m} \to Y_m$, we can compute the sections of $\overline{\mathcal{F}}_m$ (resp. $\overline{\mathcal{G}}_m$) on $\overline{X}_m \hookrightarrow \overline{E}_m$ (resp. $\overline{X}_1 \hookrightarrow \overline{E}_m$) by pulling back $(M_{\mathcal{P}_m}, \mathrm{Fil}^\bullet M_{\mathcal{P}_m}, \nabla)$ (resp. $(M_{\mathcal{P}_m}, \varphi_{\mathcal{P}_m})$) under the PD-morphism $\overline{E}_m \to \mathrm{Spec}(\mathcal{P}_m)$ induced by $Y_{\overline{D}_m} \to Y_m$, and obtain $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$-linear isomorphisms

$$M_{\mathcal{P}_m} \otimes_{\mathcal{P}_m,\iota_m} \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \cong \Gamma(\overline{E}_m, \overline{\mathcal{F}}_m) \cong \Gamma(\overline{E}_m, \overline{\mathcal{G}}_m) \tag{28}$$

such that the filtration (resp. the Frobenius endomorphism), the integrable connection, and the action of $G_{\mathcal{A}}$ on the middle (resp. the right) term are compatible with those on the left term. Combining with (26), we obtain the following $G_{\mathcal{A}}$-equivariant $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$-linear filtered isomorphisms compatible with φ.

$$T A_{\mathrm{crys},m}(M) \cong (M_{\mathcal{P}_m} \otimes_{\mathcal{P}_m,\iota_m} \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}))^{\nabla=0} \tag{29}$$

By (27), the isomorphism (29) induces an $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$-linear $G_{\mathcal{A}}$-equivariant filtered isomorphism

$$T A_{\mathrm{crys},m}(M) \otimes_{A_{\mathrm{crys},m}(\overline{\mathcal{A}})} \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \xrightarrow{\cong} M_{\mathcal{P}_m} \otimes_{\mathcal{P}_m,\iota_m} \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \tag{30}$$

compatible with ∇ and φ.

We give another description of $T A_{\mathrm{crys},m}(M)$ as a filtered module over $A_{\mathrm{crys}}(\overline{\mathcal{A}})$ with φ (see (32)), which depends on the choice of the coordinates $s_1, \ldots, s_e$ of $\mathcal{B}^\times$ over O_K and a compatible system of p^nth roots $s_{i,n} \in \overline{\mathcal{A}}^\times$ $(n \in \mathbb{N})$ of the image of s_i in $\mathcal{A}^\times$ (used in the definition of $u_i \in \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ in Sect. 2).

For any ideal $\mathfrak{a}$ of $A_{\mathrm{inf}}(O_{\overline{K}})$ satisfying $(p, [\underline{p}])^n \subset \mathfrak{a} \subset (p, [\underline{p}])$ for some $n \in \mathbb{N}_{>0}$, we have a commutative square of O_K-algebras

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \overline{\mathcal{A}}/p\overline{\mathcal{A}} \\ {\scriptstyle S_i \mapsto s_i} \uparrow & \overset{\beta^{(0)}_{\mathfrak{a}}}{\searrow} & \uparrow \\ O_K[S_1, \ldots, S_e] & \xrightarrow[S_i \mapsto [\underline{s}_i]]{} & A_{\mathrm{inf}}(\overline{\mathcal{A}})/\mathfrak{a}, \end{array} \tag{31}$$

where the right vertical homomorphism is induced by θ and the upper horizontal one is the composition $\mathcal{B} \to \mathcal{A} \to \overline{\mathcal{A}}/p$. Since the homomorphism of O_K-algebras $O_{K,m}[S_1, \ldots, S_e] \to \mathcal{B}_m$; $S_i \mapsto s_i$ is étale for $m \in \mathbb{N}_{>0}$, we see that there exists a unique homomorphism of O_K-algebras $\beta^{(0)}_{\mathfrak{a}} \colon \mathcal{B} \to A_{\mathrm{inf}}(\overline{\mathcal{A}})/\mathfrak{a}$ such that the two triangles in (31) are commutative. Since $\theta([\underline{s}_i]) \in \widehat{\overline{\mathcal{A}}}^\times$ is the image of s_i under $\mathcal{B} \to \mathcal{A} \to \widehat{\overline{\mathcal{A}}}$, the homomorphism $\beta^{(0)}_{(p^m,\xi)}$ is the composition of $\mathcal{B} \to \mathcal{A} \to \overline{\mathcal{A}}/p^m$.

We define $\beta^{(0)}\colon \mathcal{B} \to A_{\inf}(\overline{\mathcal{A}})$ to be the inverse limit of $\beta^{(0)}_{\mathfrak{a}}$ over $\mathfrak{a}$. By the above description of $\beta^{(0)}_{(p^m,\xi)}$, the reduction mod p^m of $\beta^{(0)}$ induces a PD-homomorphism $\beta_m\colon \mathcal{P}_m \to A_{\mathrm{crys},m}(\overline{\mathcal{A}})$. Let $\beta\colon \mathcal{P} \to A_{\mathrm{crys}}(\overline{\mathcal{A}})$ be the inverse limit of β_m $(m \in \mathbb{N}_{>0})$. By definition, the composition of β with $A_{\mathrm{crys}}(\overline{\mathcal{A}}) \to A_{\mathrm{crys}}(\overline{\mathcal{A}})/\mathrm{Fil}^1 \cong \widehat{\overline{\mathcal{A}}}$ is the composition of $\mathcal{P} \to \mathcal{A} \hookrightarrow \widehat{\overline{\mathcal{A}}}$.

Lemma 34 (1) *The homomorphism β coincides with the composition of the PD-homomorphism $\mathcal{P} \to \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ and the PD-homomorphism $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \to A_{\mathrm{crys}}(\overline{\mathcal{A}})$ over $A_{\mathrm{crys}}(\overline{\mathcal{A}})$ defined by $u_i^{[n]} \mapsto 0$ $(i \in \mathbb{N}\cap[1,e], n \in \mathbb{N}_{>0})$.*
(2) *If $\varphi_{\mathcal{B}}(s_i) = s_i^p$ for all $i \in \mathbb{N}\cap[1,e]$, then $\beta^{(0)}$ and β are compatible with φ, i.e. $\varphi\circ\beta^{(0)} = \beta^{(0)}\circ\varphi_{\mathcal{B}}$ and $\varphi\circ\beta = \beta\circ\varphi_{\mathcal{P}}$.*

Proof (1) Let $m \in \mathbb{N}_{>0}$. By the universal property of PD-envelopes, it suffices to prove that the reduction mod p^m of $\mathcal{B} \to \mathcal{P} \to \mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}}) \to A_{\mathrm{crys}}(\overline{\mathcal{A}})$ coincides with that of $\mathcal{B} \xrightarrow{\beta^{(0)}} A_{\inf}(\overline{\mathcal{A}}) \to A_{\mathrm{crys}}(\overline{\mathcal{A}})$. Since $O_{K,m}[S_1,\ldots,S_e] \to \mathcal{B}_m;\ S_i \mapsto s_i$ is étale, we see that both of the above homomorphisms are the unique homomorphism of O_K-algebras $\mathcal{B}_m \to A_{\mathrm{crys},m}(\overline{\mathcal{A}})$ sending s_i to $[\underline{s}_i]$ such that the composition with $A_{\mathrm{crys},m}(\overline{\mathcal{A}}) \to A_{\mathrm{crys},1}(\overline{\mathcal{A}})/\mathrm{Fil}^1 \cong \overline{\mathcal{A}}/p$ coincides with $\mathcal{B}_m \to \mathcal{A}_m \to \overline{\mathcal{A}}/p$.

(2) By the definition of $\varphi_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{P}$ and φ of $A_{\mathrm{crys}}(\overline{\mathcal{A}})$, the claim for β is reduced to that for $\beta^{(0)}$. By $\varphi([\underline{s}_i]) = [\underline{s}_i]^p$ and the compatibility of φ of $A_{\inf}(\overline{\mathcal{A}})$ with the absolute Frobenius of $\overline{\mathcal{A}}/p$ and $\sigma\colon O_K \to O_K$, the commutative square (31) for $\mathfrak{a} = (p,[\underline{p}])^n$ is compatible with $\varphi_{\mathcal{B}}$, φ of $A_{\inf}(\overline{\mathcal{A}})/\mathfrak{a}$, the absolute Frobenius of $\overline{\mathcal{A}}/p$, and the endomorphism of $O_K[S_1,\ldots,S_e]$ defined by σ and $S_i \mapsto S_i^p$. This implies $\varphi\circ\beta^{(0)} = \beta^{(0)}\circ\varphi_{\mathcal{B}}$. □

Let $m \in \mathbb{N}_{>0}$. The homomorphism β_m induces morphisms $(\overline{X}_m \hookrightarrow \overline{D}_m) \to (X_m \hookrightarrow \mathrm{Spec}(\mathcal{P}_m))$ and $(\overline{X}_1 \hookrightarrow \overline{D}_m) \to (X_1 \hookrightarrow \mathrm{Spec}(\mathcal{P}_m))$ in $\mathrm{CRYS}(X_m/\Sigma_m)$ and $\mathrm{CRYS}(X_1/\Sigma_m)$, respectively. Hence, we may compute $TA_{\mathrm{crys},m}(M)$ by taking the pull-back of $M_{\mathcal{P}_m}$ via the above morphisms (Lemma 27 (3)) and obtain the following $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$-linear filtered isomorphism

$$TA_{\mathrm{crys},m}(M) \cong M_{\mathcal{P}_m} \otimes_{\mathcal{P}_m,\beta_m} A_{\mathrm{crys},m}(\overline{\mathcal{A}}). \tag{32}$$

This is compatible with φ if $\varphi_{\mathcal{B}}(s_i) = s_i^p$ for all $i \in \mathbb{N}\cap[1,e]$ by Lemma 34 (2). Note that this isomorphism is not $G_{\mathcal{A}}$-equivariant because β is not compatible with the action of $G_{\mathcal{A}}$.

The two descriptions (29) and (32) of $TA_{\mathrm{crys},m}(M)$ are related with each other as follows. By combining (30) and (32), we obtain a filtered $\mathscr{A}_{\mathrm{crys},m}(\overline{\mathcal{A}})$-linear isomorphism

$$M_{\mathcal{P}_m} \otimes_{\mathcal{P}_m,\beta_m} \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \xrightarrow{\cong} M_{\mathcal{P}_m} \otimes_{\mathcal{P}_m,\iota_m} \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \tag{33}$$

compatible with the integrable connections, and also with φ if $\varphi_{\mathcal{B}}(s_i) = s_i^p$ for all $i \in \mathbb{N}\cap[1,e]$. On the left-hand side, the filtration and φ are defined by the tensor products of those on $M_{\mathcal{P}_m}$ and $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$, and the connection is defined by $\mathrm{id}\otimes\nabla$. By tracing the construction of (30) and (32), we see that the isomorphism (33) is obtained by taking $\Gamma(\overline{E}_n, -)$ of the canonical isomorphism between the two descriptions of $(\overline{\mathcal{F}}_m)_{\overline{X}_m\hookrightarrow\overline{E}_m}$ (or $(\overline{\mathcal{G}}_m)_{\overline{X}_1\hookrightarrow\overline{E}_m}$) as the pull-backs of $M_{\mathcal{P}_m}$ by the PD-morphisms $\overline{E}_m\to\overline{D}_m\xrightarrow{\mathrm{Spec}(\beta_m)}\mathrm{Spec}(\mathcal{P}_m)$ and $\overline{E}_m\to\mathrm{Spec}(\mathcal{P}_m)$ (induced by $Y_{\overline{D}_m}\to Y_m$). By (12) and $\beta(s_i)=[\underline{s}_i]$, the isomorphism (33) and its inverse are explicitly given by

$$x\otimes 1\mapsto\sum_{\underline{n}=(n_i)\in\mathbb{N}^e}\nabla^{\log}_{\underline{n}}(x)\otimes\prod_i u_{i,m}^{[n_i]},\tag{34}$$

$$\sum_{\underline{n}=(n_i)\in\mathbb{N}^e}\nabla^{\log}_{\underline{n}}(x)\otimes\prod_i(u'_{i,m})^{[n_i]}\leftarrow x\otimes 1\tag{35}$$

for $x\in M_{\mathcal{P}_m}$, where $u'_{i,m}$ =(the image of $[\underline{s}_i]^{-1}\otimes s_i-1$ in $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$)=$(1+u_{i,m})^{-1}-1$, $\nabla^{\log}_i$ ($i\in\mathbb{N}\cap[1,e]$) denotes the endomorphism of $M_{\mathcal{P}_m}$ defined by $\nabla(x)=\sum_{1\le i\le e}\nabla^{\log}_i(x)\otimes d\log s_i$, and $\nabla^{\log}_{\underline{n}}=\prod_{1\le i\le e}\prod_{0\le j\le n_i-1}(\nabla^{\log}_i-j)$.

We take the inverse limit $\varprojlim_m$ of what we have constructed. First we define the $A_{\mathrm{crys}}(\overline{\mathcal{A}})$-module $TA_{\mathrm{crys}}(M)$ by

$$TA_{\mathrm{crys}}(M):=\varprojlim_m TA_{\mathrm{crys},m}(M).\tag{36}$$

The semilinear action of $G_{\mathcal{A}}$ on $TA_{\mathrm{crys},m}(M)$ defines its action on $TA_{\mathrm{crys}}(M)$. We define the decreasing filtration $\mathrm{Fil}^r TA_{\mathrm{crys}}(M)$ ($r\in\mathbb{Z}$) to be the inverse limit $\varprojlim_m\mathrm{Fil}^r TA_{\mathrm{crys},m}(M)$, which is stable under the action of $G_{\mathcal{A}}$. The pair $(TA_{\mathrm{crys}}(M),\mathrm{Fil}^\bullet TA_{\mathrm{crys}}(M))$ is obviously a filtered module over the filtered ring $(A_{\mathrm{crys}}(\overline{\mathcal{A}}),\mathrm{Fil}^\bullet A_{\mathrm{crys}}(\overline{\mathcal{A}}))$. The Frobenius endomorphism φ on each $TA_{\mathrm{crys},m}(M)$ defines a semilinear $G_{\mathcal{A}}$-equivariant endomorphism φ of $TA_{\mathrm{crys}}(M)$.

We define the $\mathcal{P}$-module $M_{\mathcal{P}}$ and its decreasing filtration $\mathrm{Fil}^r M_{\mathcal{P}}$ ($r\in\mathbb{Z}$) to be the inverse limits of $M_{\mathcal{P}_m}$ and $\mathrm{Fil}^r M_{\mathcal{P}_m}$ ($m\in\mathbb{N}_{>0}$). The pair $(M_{\mathcal{P}},\mathrm{Fil}^\bullet M_{\mathcal{P}})$ is a filtered module over the filtered ring $(\mathcal{P},\mathrm{Fil}^\bullet\mathcal{P})$. We define $\nabla\colon M_{\mathcal{P}}\to M_{\mathcal{P}}\otimes_{\mathcal{B}}\Omega_{\mathcal{B}}$ to be the inverse limit of ∇ on $M_{\mathcal{P}_m}$ ($m\in\mathbb{N}_{>0}$), which is an integrable connection on $M_{\mathcal{P}}$ compatible with $\nabla_{\mathcal{P}}$ on $\mathcal{P}$ and satisfies $\nabla(\mathrm{Fil}^r M_{\mathcal{P}})\subset\mathrm{Fil}^{r-1}M_{\mathcal{P}}\otimes_{\mathcal{B}}\Omega_{\mathcal{B}}$ ($r\in\mathbb{Z}$). The Frobenius endomorphism φ of $M_{\mathcal{P}_m}$ for $m\in\mathbb{N}_{>0}$ induces a $\varphi_{\mathcal{P}}$-semilinear endomorphism φ of $M_{\mathcal{P}}$ compatible with ∇, i.e. $(\varphi\otimes\varphi_{\mathcal{B}})\circ\nabla=\nabla\circ\varphi$.

Lemma 35 *The filtered module $M_{\mathcal{P}}$ over $(\mathcal{P},\mathrm{Fil}^\bullet\mathcal{P})$ is finite filtered free of level $[0,p-2]$ (Definition 10 (3)). Moreover the natural homomorphisms $M_{\mathcal{P}}/p^m\to M_{\mathcal{P}_m}$ and $\mathrm{Fil}^r M_{\mathcal{P}}/p^m\to\mathrm{Fil}^r M_{\mathcal{P}_m}$ ($r\in\mathbb{Z}$) are isomorphisms for $m\in\mathbb{N}_{>0}$.*

Proof Let $t_1,\dots,t_d\in A^\times$ be coordinates of A over O_K. Then the homomorphisms $f_m\colon O_{K,m}[T_1,\dots,T_d]\to\mathcal{A}_m;\ T_i\mapsto t_i$ of $O_{K,m}$-algebras have liftings $g_m\colon O_{K,m}[T_1,\dots,T_d]\to\mathcal{P}_m$ such that (g_{m+1} mod p) $=g_m$ ($m\in\mathbb{N}_{>0}$) because $\mathrm{Fil}^1\mathcal{P}_{m+1}=\mathrm{Ker}(\mathcal{P}_{m+1}\to\mathcal{A}_{m+1})\to\mathrm{Fil}^1\mathcal{P}_m=\mathrm{Ker}(\mathcal{P}_m\to\mathcal{A}_m)$ ($m\in\mathbb{N}_{>0}$) are surjective. Since f_m is étale, g_m extends uniquely to a homomorphism of $O_{K,m}$-algebras

$h_m: \mathcal{A}_m \to \mathcal{P}_m$ such that the composition with $\mathcal{P}_m \to \mathcal{A}_m$ is the identity map, and the uniqueness implies that $(h_{m+1} \mod p) = h_m$ $(m \in \mathbb{N}_{>0})$. By Lemma 27 (3), the morphism $(X_m \hookrightarrow \mathrm{Spec}(\mathcal{P}_m)) \to (X_m \xrightarrow{\mathrm{id}_{X_m}} X_m)$ in $\mathrm{CRYS}(X_m/\Sigma_m)$ defined by h_m induces a $\mathcal{P}_m$-linear filtered isomorphisms $M_m \otimes_{\mathcal{A}_m, h_m} \mathcal{P}_m \xrightarrow{\cong} M_{\mathcal{P}_m}$ compatible with m. We obtain the claims by taking the inverse limit over m and using the fact that M is finite filtered free of level $[0, p-2]$. □

By Lemma 35 and Lemma 13 (2), the scalar extension $M_{\mathcal{P}} \otimes_{\mathcal{P},\beta} A_{\mathrm{crys}}(\overline{\mathcal{A}})$ of the filtered module $M_{\mathcal{P}}$ by β is finite filtered free of level $[0, p-2]$, and the natural homomorphisms $(M_{\mathcal{P}} \otimes_{\mathcal{P},\beta} A_{\mathrm{crys}}(\overline{\mathcal{A}}))/p^m \to M_{\mathcal{P}_m} \otimes_{\mathcal{P},\beta_m} A_{\mathrm{crys},m}(\overline{\mathcal{A}})$ are filtered isomorphisms for $m \in \mathbb{N}_{>0}$. By Lemma 34 (2), we can define φ on $M_{\mathcal{P}} \otimes_{\mathcal{P},\beta} A_{\mathrm{crys}}(\overline{\mathcal{A}})$ by $\varphi \otimes \varphi$ if $\varphi_{\mathcal{B}}(s_i) = s_i^p$ for all $i \in \mathbb{N} \cap [1, e]$. By taking the inverse limit of (32) over m, we obtain an $A_{\mathrm{crys}}(\overline{\mathcal{A}})$-linear filtered isomorphism

$$T A_{\mathrm{crys}}(M) \cong M_{\mathcal{P}} \otimes_{\mathcal{P},\beta} A_{\mathrm{crys}}(\overline{\mathcal{A}}), \tag{37}$$

which is also compatible with φ if $\varphi_{\mathcal{B}}(s_i) = s_i^p$ $(i \in \mathbb{N} \cap [1, e])$. This implies the following.

Lemma 36 *The filtered module* $T A_{\mathrm{crys}}(M)$ *over* $(A_{\mathrm{crys}}(\overline{\mathcal{A}}), \mathrm{Fil}^\bullet A_{\mathrm{crys}}(\overline{\mathcal{A}}))$ *is finite filtered free of level* $[0, p-2]$. *Moreover, for every* $m \in \mathbb{N}_{>0}$, *the natural homomorphisms* $T A_{\mathrm{crys}}(M)/p^m \to T A_{\mathrm{crys},m}(M)$ *and* $\mathrm{Fil}^r T A_{\mathrm{crys}}(M)/p^m \to \mathrm{Fil}^r T A_{\mathrm{crys},m}(M)$ $(r \in \mathbb{Z})$ *are isomorphisms.*

Let ι denote the canonical homomorphism $\mathcal{P} \to \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$. We endow $M_{\mathcal{P}} \otimes_{\mathcal{P},\iota} \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ (resp. $T A_{\mathrm{crys}}(M) \otimes_{A_{\mathrm{crys}}(\overline{\mathcal{A}})} \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$) with the tensor products of ∇, $\mathrm{Fil}^\bullet$, and φ (resp. $\mathrm{Fil}^\bullet$, φ, and the $G_{\mathcal{A}}$-action), and with the action of $G_{\mathcal{A}}$ (resp. ∇) via $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$. Then, by Lemmas 35, 36, and 13 (2), $M_{\mathcal{P}} \otimes_{\mathcal{P},\iota} \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ and $T A_{\mathrm{crys}}(M) \otimes_{A_{\mathrm{crys}}(\overline{\mathcal{A}})} \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ are finite filtered free of level $[0, p-2]$ over $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$, and $(M_{\mathcal{P}} \otimes_{\mathcal{P},\iota} \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}))/p^m \to M_{\mathcal{P}_m} \otimes_{\mathcal{P}_m,\iota_m} \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ and $(T A_{\mathrm{crys}}(M) \otimes_{A_{\mathrm{crys}}(\overline{\mathcal{A}})} \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}))/p^m \to T A_{\mathrm{crys},m}(M) \otimes_{A_{\mathrm{crys},m}(\overline{\mathcal{A}})} \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ are filtered isomorphisms. Hence, by taking the inverse limits of (29) and (30), we obtain a $G_{\mathcal{A}}$-equivariant $A_{\mathrm{crys}}(\overline{\mathcal{A}})$-linear filtered isomorphism compatible with φ

$$T A_{\mathrm{crys}}(M) \cong (M_{\mathcal{P}} \otimes_{\mathcal{P},\iota} \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}))^{\nabla=0}, \tag{38}$$

and see that it induces a $G_{\mathcal{A}}$-equivariant $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$-linear filtered isomorphism compatible with ∇ and φ

$$T A_{\mathrm{crys}}(M) \otimes_{A_{\mathrm{crys}}(\overline{\mathcal{A}})} \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \xrightarrow{\cong} M_{\mathcal{P}} \otimes_{\mathcal{P},\iota} \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}). \tag{39}$$

By combining (37) and (39), we obtain an $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$-linear filtered isomorphism

$$M_{\mathcal{P}} \otimes_{\mathcal{P},\beta} \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \xrightarrow{\cong} M_{\mathcal{P}} \otimes_{\mathcal{P},\iota} \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \tag{40}$$

compatible with ∇, and also with φ if $\varphi(s_i)=s_i^p$ $(i\in\mathbb{N}\cap[1,e])$, where $\mathrm{Fil}^\bullet$, φ, and ∇ on the left-hand side are defined to be the pull-back filtration of $\mathrm{Fil}^\bullet M_{\mathcal{P}}$, $\varphi\otimes\varphi$ and $\mathrm{id}\otimes\nabla$. As this is obviously obtained from the isomorphisms (33) by taking the inverse limit over m, (34) and (35) imply that (40) and its inverse are given by

$$x\otimes 1\mapsto \sum_{\underline{n}=(n_i)\in\mathbb{N}^e}\nabla_{\underline{n}}^{\log}(x)\otimes\prod_i u_i^{[n_i]}, \tag{41}$$

$$\sum_{\underline{n}=(n_i)\in\mathbb{N}^e}\nabla_{\underline{n}}^{\log}(x)\otimes\prod_i (u_i')^{[n_i]}\leftarrow x\otimes 1 \tag{42}$$

for $x\in M_{\mathcal{P}}$, where $u_i'=(1+u_i)^{-1}-1$ and the endomorphisms $\nabla_i^{\log}$ $(i\in\mathbb{N}\cap[1,e])$ and $\nabla_{\underline{n}}^{\log}$ $(\underline{n}\in\mathbb{N}^e)$ of $M_{\mathcal{P}}$ are defined in the same way as after (35).

Lemma 37 *We have a canonical $G_{\mathcal{A}}$-equivariant isomorphism*

$$T^*_{\mathrm{crys}}(M)\cong \mathrm{Hom}_{A_{\mathrm{crys}}(\overline{\mathcal{A}})\text{-}lin,\mathrm{Fil}^{\leq p-2},\varphi}(TA_{\mathrm{crys}}(M),A_{\mathrm{crys}}(\overline{\mathcal{A}}))$$

functorial in M. Here $\mathrm{Hom}_{A_{\mathrm{crys}}(\overline{\mathcal{A}})\text{-}lin,\mathrm{Fil}^{\leq p-2},\varphi}$ *denotes the $\mathbb{Z}_p$-module consisting of $A_{\mathrm{crys}}(\overline{\mathcal{A}})$-linear homomorphisms sending* Fil^r *into* Fil^r *for $r\in\mathbb{N}\cap[0,p-2]$ and compatible with φ. (See Sect. 4 for the definition of $T^*_{\mathrm{crys}}(M)$.)*

Proof By the definition of $T^*_{\mathrm{crys}}(M)$, we have a functorial $G_{\mathcal{A}}$-equivariant isomorphism

$$T^*_{\mathrm{crys}}(M)\cong \mathrm{Hom}_{\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})\text{-lin},\mathrm{Fil}^{\leq p-2},\varphi,\nabla}(M\otimes_{\mathcal{A},\iota}\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}}),\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})). \tag{43}$$

By (39) for $\mathcal{B}=\mathcal{A}$ and $\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})^{\nabla=0}=A_{\mathrm{crys}}(\overline{\mathcal{A}})$ (9), we see that the right-hand side of (43) is isomorphic to that of the isomorphism in the claim. $\square$

Remark 38 Since (39) does not depend on the choice of $\varphi_{\mathcal{B}}$, the proof of Lemma 37 shows that $T^*_{\mathrm{crys}}(M)$ regarded as a submodule of $\mathrm{Hom}_{\mathcal{A}\text{-lin},\nabla}(M,\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}}))$ does not dependent on the choice of the lifting of Frobenius $\varphi_{\mathcal{A}}$ of $\mathcal{A}$.

6 Filtered φ-Modules

In this and the next sections, we give a general formulation (Proposition 44) derived from an idea of the theorem of Wach [20, Theorem 3] relating the theory of (φ,Γ)-modules and the Fontaine–Laffaille theory for $\mathbb{Z}_p$-representations of G_K. We treat only p-torsion free modules. We apply the formulation to $A_{\mathrm{inf}}(\Lambda)$, $A_{\mathrm{crys}}(\Lambda)$ and their variants in Sects. 8, 13, 17, and 18. See Proposition 59, 94, 164, and 172 for the precise statements.

Let a be a non-negative integer. Let R be a commutative ring, let q be an element of R, let φ_R be an endomorphism of the ring R, and let $\mathrm{Fil}^r R$ ($r \in \mathbb{N} \cap [0, a]$) be a decreasing filtration of R by ideals such that the following condition is satisfied.

Condition 39 (i) q is not a zero divisor in R.

(ii) $\mathrm{Fil}^0 R = R$, and $\mathrm{Fil}^r R \cdot \mathrm{Fil}^s R \subset \mathrm{Fil}^{r+s} R$ for every $r, s \in \mathbb{N}$ such that $r + s \le a$.

(iii) $\varphi_R(\mathrm{Fil}^r R) \subset q^r R$ for every $r \in \mathbb{N} \cap [0, a]$.

For a negative integer r, we define $\mathrm{Fil}^r R$ to be R. Then we have $\mathrm{Fil}^r R \cdot \mathrm{Fil}^s R \subset \mathrm{Fil}^{r+s} R$ for $r, s \in \mathbb{Z}$ such that $r, s, r + s \le a$.

Definition 40 We define the category $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi)$ as follows. An object is a triplet $(M, \mathrm{Fil}^r M, \varphi_M)$ consisting of the following:

(i) A free R-module M of finite type.

(ii) A decreasing filtration $\mathrm{Fil}^r M$ ($r \in \mathbb{N} \cap [0, a]$) of M satisfying the following condition:

(ii-1) There exist a basis e_ν ($N \in \mathbb{N}, \nu \in \mathbb{N} \cap [1, N]$) of M and $r_\nu \in \mathbb{N} \cap [0, a]$ for each $\nu \in \mathbb{N} \cap [1, N]$ such that $\mathrm{Fil}^r M = \oplus_{\nu \in \mathbb{N} \cap [1,N]} \mathrm{Fil}^{r - r_\nu} R e_\nu$ for $r \in \mathbb{N} \cap [0, a]$.

(iii) A φ_R-semilinear endomorphism $\varphi_M \colon M \to M$ satisfying the following conditions:

(iii-1) $\varphi_M(\mathrm{Fil}^r M) \subset q^r M$ for $r \in \mathbb{N} \cap [0, a]$.

(iii-2) $M = \sum_{r \in \mathbb{N} \cap [0,a]} R \cdot q^{-r} \varphi_M(\mathrm{Fil}^r M)$.

A morphism is an R-linear homomorphism preserving the filtrations and compatible with φ_M's in the obvious sense.

Lemma 41 *Let M, $\mathrm{Fil}^r M$, e_ν and r_ν be as in Definition 40 (i), (ii). Then a φ_R-semilinear endomorphism $\varphi_M \colon M \to M$ satisfies the conditions in Definition 40 (iii) if and only if $\varphi_M(e_\nu) \in q^{r_\nu} M$ for all $\nu \in \mathbb{N} \cap [1, N]$ and $q^{-r_\nu} \varphi_M(e_\nu)$ ($\nu \in \mathbb{N} \cap [1, N]$) form a basis of M.*

Proof Assume that φ_M satisfies the conditions (iii-1) and (iii-2) in Definition 40. Since $e_\nu \in \mathrm{Fil}^{r_\nu} M$, (iii-1) implies $\varphi_M(e_\nu) \in q^{r_\nu} M$. For $r \in \mathbb{N} \cap [0, a]$, we have $\varphi_M(\mathrm{Fil}^{r - r_\nu} R e_\nu) = q^{r_\nu} \varphi_R(\mathrm{Fil}^{r - r_\nu} R) \cdot q^{-r_\nu} \varphi_M(e_\nu)$, and $q^{r_\nu} \varphi_R(\mathrm{Fil}^{r - r_\nu} R) \subset q^r R$ because $\varphi_R(\mathrm{Fil}^s R) \subset q^s R$ ($s \in \mathbb{N} \cap [0, a]$). Therefore the condition (iii-2) implies that $q^{-r_\nu} \varphi_M(e_\nu)$ ($\nu \in \mathbb{N} \cap [1, N]$) generate the R-module M. Since M is a free R-module of rank N, we see that they form a basis of M. As for the sufficiency, (iii-1) follows from the above computation of $\varphi_M(\mathrm{Fil}^{r - r_\nu} R e_\nu)$, and then (iii-2) is obvious because the right-hand side of the equality contains $q^{-r_\nu} \varphi_M(e_\nu)$. □

Lemma 42 *Let $(M, \mathrm{Fil}^r M, \varphi_M)$ and $(M', \mathrm{Fil}^r M', \varphi_{M'})$ be objects of the category $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi)$. Choose a basis e_ν ($N \in \mathbb{N}, \nu \in \mathbb{N} \cap [1, N]$) of M and $r_\nu \in \mathbb{N} \cap [0, a]$ ($\nu \in \mathbb{N} \cap [1, N]$) as in Definition 40 (ii-1). Put $I := \mathbb{N} \cap [1, N]$. By Lemma 41, there exists $A = (a_{\nu\mu}) \in GL_N(R)$ such that $\varphi_M(e_\mu) = q^{r_\mu} \sum_{\nu \in I} a_{\nu\mu} e_\nu$ for all $\mu \in I$. Choose N', e'_ν, r'_ν, and $A' = (a'_{\nu\mu})$ similarly for $(M', \mathrm{Fil}^r M', \varphi_{M'})$, and put $I' := \mathbb{N} \cap [1, N']$. Let $f \colon M \to M'$ be an R-linear homomorphism, and define $B = (b_{\nu\mu}) \in M_{N'N}(R)$ by $f(e_\mu) = \sum_{\nu \in I'} b_{\nu\mu} e'_\nu$ for all $\mu \in I$.*

(1) *We have* $f(\mathrm{Fil}^r M) \subset \mathrm{Fil}^r M'$ *for all* $r \in \mathbb{N} \cap [0, a]$ *if and only if* $b_{\nu\mu} \in \mathrm{Fil}^{r_\mu - r'_\nu} R$ *for all* $(\nu, \mu) \in I' \times I$.

(2) *We have* $f \circ \varphi_M = \varphi_{M'} \circ f$ *if and only if* $BA\mathrm{diag}(q^{r_\nu}; \nu \in I)$ *is equal to* $A'\mathrm{diag}(q^{r'_\nu}; \nu \in I')\varphi_R(B)$.

Proof (1) We can show the necessity by looking at the image of $e_\mu \in \mathrm{Fil}^{r_\mu} M$ under f; for each $\mu \in I$, $f(e_\mu) = \sum_{\nu \in I'} b_{\nu\mu} e'_\nu \in f(\mathrm{Fil}^{r_\mu} M) \subset \mathrm{Fil}^{r_\mu} M'$ implies $b_{\nu\mu} \in \mathrm{Fil}^{r_\mu - r'_\nu} R$ for every $\mu \in I'$. We can verify the sufficiency as $f(\mathrm{Fil}^{r-r_\mu} Re_\mu) = \mathrm{Fil}^{r-r_\mu} R \sum_{\nu \in I'} b_{\nu\mu} e'_\nu \subset \sum_{\nu \in I'} \mathrm{Fil}^{r-r_\mu} R \cdot \mathrm{Fil}^{r_\mu - r'_\nu} Re'_\nu \subset \sum_{\nu \in I'} \mathrm{Fil}^{r-r'_\nu} Re'_\nu = \mathrm{Fil}^r M'$ for $r \in \mathbb{N} \cap [0, a]$ and $\mu \in I$.

(2) The claim follows from the following computation.

$$f \circ \varphi_M(e_1, \ldots, e_N) = f((e_1, \ldots, e_N)A\mathrm{diag}(q^{r_\nu})) = (e'_1, \ldots, e'_{N'})BA\mathrm{diag}(q^{r_\nu}),$$

$$\varphi_M \circ f(e_1, \ldots, e_N) = \varphi_M((e'_1, \ldots, e'_{N'})B) = (e'_1, \ldots, e'_{N'})A'\mathrm{diag}(q^{r'_\nu})\varphi_R(B).$$

□

Let $(R', q', \varphi_{R'}, \mathrm{Fil}^r R')$ be another quadruplet satisfying Condition 39, and let $\kappa\colon R \to R'$ be a ring homomorphism such that $\kappa(q)R' = q'R'$, $\varphi_{R'} \circ \kappa = \kappa \circ \varphi_R$ and $\kappa(\mathrm{Fil}^r R) \subset \mathrm{Fil}^r R'$ $(r \in \mathbb{N} \cap [0, a])$. For $(M, \mathrm{Fil}^r M, \varphi_M) \in \mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi)$, put $M' := R' \otimes_R M$, let $\varphi_{M'}$ be the $\varphi_{R'}$-semilinear endomorphism $\varphi_{R'} \otimes \varphi_M$ of M', and put

$$\mathrm{Fil}^r M' := \sum_{s \in \mathbb{N} \cap [0,r]} (\text{the image of } \mathrm{Fil}^s R' \otimes_R \mathrm{Fil}^{r-s} M)$$

for $r \in \mathbb{N} \cap [0, a]$. Choose a basis e_ν $(N \in \mathbb{N}, \nu \in \mathbb{N} \cap [1, N])$ of M and $r_\nu \in \mathbb{N} \cap [0, a]$ $(\nu \in \mathbb{N} \cap [1, N])$ as in Definition 40 (ii-1). Then M' is a free R'-module with a basis $e'_\nu := 1 \otimes e_\nu$ $(\nu \in \mathbb{N} \cap [1, N])$. For $\nu \in \mathbb{N} \cap [1, N]$ and $r \in \mathbb{N} \cap [0, a]$, we have $\sum_{s \in \mathbb{N} \cap [0,r]} \mathrm{Fil}^s R' \cdot \mathrm{Fil}^{r-s-r_\nu} Re'_\nu = \mathrm{Fil}^{r-r_\nu} R'e'_\nu$; the inclusion $\subset$ is obvious and we obtain the equality by looking at the term for $s = \max\{0, r - r_\nu\}$ in the left-hand side. This implies $\mathrm{Fil}^r M' = \oplus_{\nu \in \mathbb{N} \cap [1,N]} \mathrm{Fil}^{r-r_\nu} R'e'_\nu$ for $r \in \mathbb{N} \cap [0, a]$. Hence, by using Lemma 41 and $\kappa(q)R' = q'R'$, we see that $(M', \mathrm{Fil}^r M', \varphi_{M'})$ is an object of $\mathrm{MF}^{q'}_{[0,a],\mathrm{free}}(R', \varphi')$. This construction is obviously functorial in M, and we obtain a functor

$$\kappa^*\colon \mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi) \longrightarrow \mathrm{MF}^{q'}_{[0,a],\mathrm{free}}(R', \varphi). \tag{44}$$

Let us consider a surjective ring homomorphism $\alpha\colon R \to \overline{R}$ satisfying the following condition. Let J be the kernel of R.

Condition 43 (i) The ideal J is contained in the Jacobson radical of R.

(ii) $\alpha(q)$ is not a zero divisor in $\overline{R}$.

(iii) $\varphi_R(J) \subset J$. (iv) $J \subset \mathrm{Fil}^a R$. (v) $\varphi_R(J) \subset q^{a+1}R$.

(vi) There exists a decreasing sequence of ideals J_n $(n \in \mathbb{N})$ of R contained in J such that $q\varphi_R(q) \cdots \varphi_R^n(q)J \subset J_n$, $q^{-(a+1)}\varphi_R(J_n) \subset J_n$, and $J \to \varprojlim_n J/J_n$ is an isomorphism.

Note that the conditions (ii), (iii) and (v) in Condition 43 imply

$$\varphi_R(J) \subset J \cap q^{a+1}R = q^{a+1}J. \tag{45}$$

Put $\overline{q} = \alpha(q)$ and $\mathrm{Fil}^r\overline{R} := \alpha(\mathrm{Fil}^r R)$ $(r \in \mathbb{N} \cap [0, a])$. By Condition 43 (iii), φ_R induces an endomorphism $\varphi_{\overline{R}}$ of the ring $\overline{R}$. Then the quadruplet $(\overline{R}, \overline{q}, \varphi_{\overline{R}}, \mathrm{Fil}^r\overline{R})$ satisfies Condition 39.

Proposition 44 *The functor* $\alpha^*\colon \mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi) \to \mathrm{MF}^{\overline{q}}_{[0,a],\mathrm{free}}(\overline{R}, \varphi)$ *is an equivalence of categories.*

Proof The essential surjectivity is verified as follows. Let $(\overline{M}, \mathrm{Fil}^r\overline{M}, \varphi_{\overline{M}})$ be an object of $\mathrm{MF}^{\overline{q}}_{[0,a],\mathrm{free}}(\overline{R}, \varphi)$, and choose a basis $\overline{e}_\nu$ $(N \in \mathbb{N}, \nu \in \mathbb{N} \cap [1, N])$ of the $\overline{R}$-module $\overline{M}$ and $r_\nu \in \mathbb{N} \cap [0, a]$ $(\nu \in \mathbb{N} \cap [1, N])$ such that $\mathrm{Fil}^r\overline{M} = \oplus_\nu \mathrm{Fil}^{r-r_\nu}\overline{R}\overline{e}_\nu$ for $r \in \mathbb{N} \cap [0, a]$. Then, by Lemma 41, we have $\varphi_{\overline{M}}(\overline{e}_\mu) = \overline{q}^{r_\mu} \sum_\nu \overline{a}_{\nu\mu}\overline{e}_\nu$ for an invertible matrix $(\overline{a}_{\nu\mu}) \in GL_N(\overline{R})$. Choose a lifting $(a_{\nu\mu}) \in M_N(R)$ of $(\overline{a}_{\nu\mu})$, which is invertible by Condition 43 (i). Then the free module $M := \oplus_\nu Re_\nu$ with φ_M defined by $\varphi_M(e_\mu) = q^{r_\mu} \sum_\nu a_{\nu\mu}e_\nu$ and $\mathrm{Fil}^r M = \oplus_\nu \mathrm{Fil}^{r-r_\nu} Re_\nu$ $(r \in \mathbb{N} \cap [0, a])$ gives the desired lifting of $\overline{M}$ in $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi)$ by Lemma 41.

Let us prove that the functor α^* is fully faithful. Let $(M, \mathrm{Fil}^r M, \varphi_M)$ and $(M', \mathrm{Fil}^r M', \varphi_{M'})$ be objects of $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi)$, and let $(\overline{M}, \mathrm{Fil}^r\overline{M}, \varphi_{\overline{M}})$ and $(\overline{M}', \mathrm{Fil}^r\overline{M}', \varphi_{\overline{M}'})$ be their images in $\mathrm{MF}^{\overline{q}}_{[0,a],\mathrm{free}}(\overline{R}, \varphi)$ under the functor α^* in question. Let $\overline{f}\colon \overline{M} \to \overline{M}'$ be a morphism in $\mathrm{MF}^{\overline{q}}_{[0,a],\mathrm{free}}(\overline{R}, \varphi)$. We show that there exists a unique lifting $f\colon M \to M'$ in $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi)$. Choose a basis e_ν $(N \in \mathbb{N}, \nu \in \mathbb{N} \cap [1, N])$ of M, $r_\nu \in \mathbb{N} \cap [0, a]$ $(\nu \in \mathbb{N} \cap [1, N])$ and $(a_{\nu\mu}) \in GL_N(R)$ such that $\mathrm{Fil}^r M = \oplus_{\nu\in I}\mathrm{Fil}^{r-r_\nu} Re_\nu$ and $\varphi_M(e_\mu) = q^{r_\mu} \sum_{\nu\in I} a_{\nu\mu}e_\nu$, where $I = \mathbb{N} \cap [1, N]$. Choose N', e'_ν, r'_ν and $A' = (a'_{\nu\mu})$ for M' similarly, and put $I' := \mathbb{N} \cap [1, N']$. Let $\overline{e}_\nu$ be the image of e_ν in $\overline{M}$, and let $\overline{e}'_\nu$ be the image of e'_ν in $\overline{M}'$. We define $(\overline{b}_{\nu\mu}) \in M_{N'N}(\overline{R})$ by $\overline{f}(\overline{e}_\mu) = \sum_{\nu\in I'} \overline{b}_{\nu\mu}\overline{e}'_\nu$ $(\mu \in I)$, and choose its lifting $B = (b_{\nu\mu}) \in M_{N'N}(R)$. By Lemma 42 (1) applied to $\overline{f}$ and Condition 43 (iv), we have $b_{\nu\mu} \in \mathrm{Fil}^{r_\mu - r'_\nu} R$. By Lemma 42 (2) applied to $\overline{f}$, we see that

$$C := BA\mathrm{diag}(q^{r_\nu}; \nu \in I) - A'\mathrm{diag}(q^{r'_\nu}; \nu \in I')\varphi_R(B)$$

is contained in $J \cdot M_{N'N}(R)$. For the second term in the right-hand side, we have $q^{r'_\nu}\varphi_R(b_{\nu\mu}) \in q^{r'_\nu}\varphi_R(\mathrm{Fil}^{r_\mu - r'_\nu} R) \subset q^{r_\mu}R$ by Condition 39 (iii). Hence the matrix C is written in the form $C'\mathrm{diag}(q^{r_\nu}; \nu \in I)$, $C' \in M_{N'N}(R)$, and we further see that $C' \in J \cdot M_{N'N}(R)$ because $\overline{R} = R/J$ is $\overline{q}$-torsion free. By applying Lemma 42 to R-linear homomorphisms from M to M' and noting $J \subset \mathrm{Fil}^a R$ (Condition 43 (iv)), we are reduced to showing that there exists a unique $D \in J \cdot M_{N'N}(R)$ such that

$$DA\mathrm{diag}(q^{r_\nu}; \nu \in I) - A'\mathrm{diag}(q^{r'_\nu}; \nu \in I')\varphi_R(D) = C'\mathrm{diag}(q^{r_\nu}; \nu \in I).$$

By (45), this equality is rewritten as $D - qF(D) = C'A^{-1}$, where F denotes the φ_R-semilinear endomorphism of $J \cdot M_{N'N}(R)$ defined by

$$F(X) = A'\mathrm{diag}(q^{r'_\nu}; \nu \in I')q^{-(a+1)}\varphi_R(X)\mathrm{diag}(q^{a-r_\nu}; \nu \in I)A^{-1}.$$

For J_n $(n \in \mathbb{N})$ as in Condition 43 (vi), we have $(qF)^{n+1}(J \cdot M_{N'N}(R)) \subset q\varphi_R(q)\cdots\varphi_R^n(q)J \cdot M_{N'N}(R) \subset J_n \cdot M_{N'N}(R)$, and $J_n \cdot M_{N'N}(R)$ is stable under F. Hence $1 - qF$ induces an automorphism of $J \cdot M_{N'N}(R)/J_n \cdot M_{N'N}(R)$. By taking the inverse limit over $n \in \mathbb{N}$ and using $J \xrightarrow{\cong} \varprojlim_n J/J_n$ (Condition 43 (vi)), we see that $1 - qF$ is an automorphism of $J \cdot M_{N'N}(R)$. This completes the proof. □

Definition 45 We define the category $\mathrm{M}^q_{[0,a],\mathrm{free}}(R,\varphi)$ as follows. An object is a pair (M,φ_M) consisting of the following:

(i) A free R-module M of finite type.

(ii) A φ_R-semilinear endomorphism $\varphi_M\colon M \to M$ satisfying the following condition:

(ii-1) There exist a basis e_ν $(N \in \mathbb{N}, \nu \in \mathbb{N}\cap[1,N])$ of M and $r_\nu \in \mathbb{N}\cap[0,a]$ for each $\nu \in \mathbb{N}\cap[1,N]$ such that $\varphi_M(e_\nu) \in q^{r_\nu}M$ for every $\nu \in \mathbb{N}\cap[1,N]$ and $q^{-r_\nu}\varphi_M(e_\nu)$ $(\nu \in \mathbb{N}\cap[1,N])$ form a basis of M.

By Lemma 41, we have a forgetful functor $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R,\varphi) \to \mathrm{M}^q_{[0,a],\mathrm{free}}(R,\varphi)$.

Lemma 46 *Assume that* $\mathrm{Fil}^r R$ *is the inverse image of* $q^r R$ *under* φ_R *for every* $r \in \mathbb{N}\cap[0,a]$.

(1) *For an object* $(M,\mathrm{Fil}^r M,\varphi_M)$ *of* $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R,\varphi)$*, we have* $\mathrm{Fil}^r M = \varphi_M^{-1}(q^r M)$ *for every* $r \in \mathbb{N}\cap[0,a]$.

(2) *The forgetful functor* $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R,\varphi) \to \mathrm{M}^q_{[0,a],\mathrm{free}}(R,\varphi)$ *is an equivalence of categories.*

Proof (1) Choose e_ν and r_ν $(\nu \in \mathbb{N}\cap[1,N])$ as in Definition 40 (ii-1). For $x = \sum_\nu a_\nu e_\nu \in M$ $(a_\nu \in R)$, we have $\varphi_M(x) = \sum_\nu q^{r_\nu}\varphi_R(a_\nu)q^{-r_\nu}\varphi_M(e_\nu)$. Hence, by Lemma 41, we have $\varphi_M(x) \in q^r M$ if and only if $q^{r_\nu}\varphi_R(a_\nu) \in q^r R$ for every $\nu \in \mathbb{N}\cap[1,N]$. The latter condition is equivalent to $a_\nu \in \mathrm{Fil}^{r-r_\nu}R$ by assumption, i.e., to $x \in \mathrm{Fil}^r M$.

(2) The functor is fully faithful by (1), and it remains to show the essential surjectivity. Let (M,φ_M) be an object of $\mathrm{M}^q_{[0,a],\mathrm{free}}(R,\varphi)$, and choose e_ν and r_ν $(\nu \in \mathbb{N}\cap[1,N])$ as in Definition 45 (ii-1). Then the R-module M with φ_M and $\mathrm{Fil}^r M := \oplus_\nu \mathrm{Fil}^{r-r_\nu}Re_\nu$ $(r \in \mathbb{N}\cap[0,a])$ is an object of $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R,\varphi)$ by Lemma 41. □

Remark 47 Let S be a commutative ring, let t be an element of S which is not a zero divisor, let φ_S be an endomorphism of S, and let $\mathrm{Fil}^r S$ $(r \in \mathbb{N}\cap[0,a])$ be the inverse image of $t^r S$ under φ_S. Then the quadruplet $(S,t,\varphi_S,\mathrm{Fil}^r S)$ satisfies Condition 39, and therefore we can apply Lemma 46 to $(S,t,\varphi_S,\mathrm{Fil}^r S)$.

7 Filtered (φ, G)-Modules

Let $a \in \mathbb{N}$ and $(R, q, \varphi_R, \mathrm{Fil}^r R)$ be the same as in the beginning of Sect. 6. Assume that we are further given an action $\rho_R \colon G \to \mathrm{Aut}(R)$ of a group G on the ring R such that $\rho_R(g)(q)R = qR$, $\rho_R(g) \circ \varphi_R = \varphi_R \circ \rho_R(g)$, and $\rho_R(g)(\mathrm{Fil}^r R) = \mathrm{Fil}^r R$ $(r \in \mathbb{N} \cap [0, a])$ for every $g \in G$. We often abbreviate $\rho_R(g)(\lambda)$ to $g(\lambda)$ for $\lambda \in R$ and $g \in G$ in the following.

Definition 48 We define the category $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi, G)$ as follows. An object is an object $(M, \mathrm{Fil}^r M, \varphi_M)$ of $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi)$ (Definition 40) endowed with a semilinear action $\rho_M \colon G \to \mathrm{Aut}(M)$ of G on the R-module M such that $\rho_M(g)(\mathrm{Fil}^r M) = \mathrm{Fil}^r M$ $(r \in \mathbb{N} \cap [0, a])$ and $\rho_M(g) \circ \varphi_M = \varphi_M \circ \rho_M(g)$ for every $g \in G$. We often abbreviate $\rho_M(g)(x)$ to $g(x)$ for $x \in M$ and $g \in G$ in the following. A morphism is a morphism in $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi)$ whose underlying morphism of R-modules is G-equivariant.

Let $(R', q', \varphi_{R'}, \mathrm{Fil}^r R')$ be another quadruplet endowed with an action $\rho_{R'}$ of a group G', let $\kappa \colon R \to R'$ be a ring homomorphism, and let $\lambda \colon G' \to G$ be a group homomorphism such that $\kappa(q)R' = q'R'$, $\varphi_{R'} \circ \kappa = \kappa \circ \varphi_R$, $\kappa(\mathrm{Fil}^r R) \subset \mathrm{Fil}^r R'$ $(r \in \mathbb{N} \cap [0, a])$ and $\kappa \circ \rho_R(\lambda(g)) = \rho_{R'}(g) \circ \kappa$ for every $g \in G'$. For an object $(M, \mathrm{Fil}^r M, \varphi_M, \rho_M)$ of $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi, G)$, the image $(M', \mathrm{Fil}^r M', \varphi_{M'})$ of $(M, \mathrm{Fil}^r M, \varphi_M)$ under the functor (44) endowed with a semilinear action $\rho_{M'}$ of G' on the R'-module $M' = R' \otimes_R M$ defined by $\rho_{M'}(g) = \rho_{R'}(g) \otimes \rho_M(\lambda(g))$ $(g \in G')$ is an object of $\mathrm{MF}^{q'}_{[0,a],\mathrm{free}}(R', \varphi, G')$. This construction is obviously functorial, and we obtain a functor

$$(\kappa, \lambda)^* \colon \mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi, G) \longrightarrow \mathrm{MF}^{q'}_{[0,a],\mathrm{free}}(R', \varphi, G'). \tag{46}$$

Let $\alpha \colon R \to \overline{R}$ be a surjective ring homomorphism, whose kernel is denoted by J, satisfying Condition 43 and $\rho_R(g)(J) = J$ for every $g \in G$. We define $(\overline{R}, \overline{q}, \mathrm{Fil}^r \overline{R}, \varphi_{\overline{R}})$ as after Condition 43, and let $\rho_{\overline{R}}$ denote the action of G on $\overline{R}$ induced by ρ_R. Then we see that $(\overline{R}, \overline{q}, \varphi_{\overline{R}}, \mathrm{Fil}^r \overline{R}, \rho_{\overline{R}})$ satisfies the same conditions as $(R, q, \varphi_R, \mathrm{Fil}^r R, \rho_R)$.

Proposition 49 *The functor*

$$(\alpha, \mathrm{id}_G)^* \colon \mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi, G) \to \mathrm{MF}^{\overline{q}}_{[0,a],\mathrm{free}}(\overline{R}, \varphi, G)$$

is an equivalence of categories.

Proposition 50 *Let M and M' be objects of $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi)$, and let $\overline{M}$ and $\overline{M}'$ be the images of M and M' under the functor $\alpha^* \colon \mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi) \to \mathrm{MF}^{\overline{q}}_{[0,a],\mathrm{free}}(\overline{R}, \varphi)$. Let $g \in G$, and let $\overline{f}$ be a homomorphism of modules $\overline{M} \to \overline{M}'$ such that $\overline{f}(\lambda x) = g(\lambda)\overline{f}(x)$ $(x \in \overline{M}, \lambda \in \overline{R})$, $\overline{f} \circ \varphi_{\overline{M}} = \varphi_{\overline{M}'} \circ \overline{f}$ and $\overline{f}(\mathrm{Fil}^r \overline{M}) \subset \mathrm{Fil}^r \overline{M}'$ for $r \in \mathbb{N} \cap [0, a]$. Then there exists uniquely a homomorphism of modules $f \colon M \to M'$ such* *that* $f(\lambda x) = g(\lambda) f(x)$ $(x \in M, \lambda \in R)$, $f \circ \varphi_M = \varphi_{M'} \circ f$, $f(\mathrm{Fil}^r M) \subset \mathrm{Fil}^r M'$ $(r \in \mathbb{N} \cap [0, a])$, *and* $(f \bmod J)$ *coincides with* $\overline{f}$.

Proof First note that $\rho_{\overline{R}}(g)$ and $\rho_R(g)$ satisfy the condition on κ given before (44). Giving a morphism $\overline{f}$ as in the proposition is equivalent to giving a morphism $\overline{F}\colon \rho_{\overline{R}}(g)^*(\overline{M}) \to \overline{M}'$ in $\mathrm{MF}^{\overline{q}}_{[0,a],\mathrm{free}}(\overline{R}, \varphi)$. Then, giving a lifting f of $\overline{f}$ is equivalent to giving a morphism $F\colon \rho_R(g)^* M \to M'$ in $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi)$ such that $\alpha^*(F)$ coincides with $\overline{F}$ via the canonical isomorphism $\rho_{\overline{R}}(g)^* \alpha^*(M) \cong \alpha^* \rho_R(g)^*(M)$. Hence the claim follows from Proposition 44. □

Proof of Proposition 49 Let $\overline{\mathcal{C}}_G$, $\mathcal{C}_G$, $\overline{\mathcal{C}}$ and $\mathcal{C}$ denote $\mathrm{MF}^{\overline{q}}_{[0,a],\mathrm{free}}(\overline{R}, \varphi, G)$, $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi, G)$, $\mathrm{MF}^{\overline{q}}_{[0,a],\mathrm{free}}(\overline{R}, \varphi)$, and $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi)$, respectively. Let Ξ_G and Ξ denote the functor $(\alpha, \mathrm{id}_G)^*\colon \mathcal{C}_G \to \overline{\mathcal{C}}_G$ and $\alpha^*\colon \mathcal{C} \to \overline{\mathcal{C}}$, respectively. By Proposition 44, the functor Ξ_G is faithful. Let us prove that it is full. Let M and M' be objects of $\mathcal{C}_G$, and let $\overline{M}$ and $\overline{M}'$ denote their images under Ξ_G. Let $\overline{f}\colon \overline{M} \to \overline{M}'$ be a morphism in $\overline{\mathcal{C}}_G$. By Proposition 44, there exists a unique morphism $f\colon M \to M'$ in $\mathcal{C}$ whose image under Ξ is $\overline{f}$. For any $g \in G$, the composition of $M \xrightarrow{f} M' \xrightarrow{\rho_{M'}(g)} M'$ and that of $M \xrightarrow{\rho_M(g)} M \xrightarrow{f} M'$ are $\rho_R(g)$-semilinear maps compatible with φ and Fil^r. They become equal after the reduction modulo J. Therefore the two compositions are equal by Proposition 50. This means that f is a morphism in $\mathcal{C}_G$. It remains to show that Ξ_G is essentially surjective. Let $\overline{M}$ be an object of $\overline{\mathcal{C}}_G$. By Proposition 44, there exists a lifting M in $\mathcal{C}$ of the object of $\overline{\mathcal{C}}$ underlying $\overline{M}$. By Proposition 50, the action of $g \in G$ on $\overline{M}$ has a unique $\rho_R(g)$-semilinear lifting compatible with φ and Fil^r. The uniqueness implies that these liftings define a semilinear action of G, with which M becomes an object of $\mathcal{C}_G$ whose image under Ξ_G is isomorphic to $\overline{M}$. □

Definition 51 We define the category $\mathrm{M}^q_{[0,a],\mathrm{free}}(R, \varphi, G)$ as follows. An object is an object (M, φ_M) of $\mathrm{M}^q_{[0,a],\mathrm{free}}(R, \varphi)$ (Definition 45) endowed with a semilinear action $\rho_M\colon G \to \mathrm{Aut}(M)$ of G on the R-module M such that $\rho_M(g) \circ \varphi_M = \varphi_M \circ \rho_M(g)$ for every $g \in G$. We often abbreviate $\rho_M(g)(x)$ to $g(x)$ for $x \in M$ and $g \in G$ in the following. A morphism is a morphism in $\mathrm{M}^q_{[0,a],\mathrm{free}}(R, \varphi)$ whose underlying morphism of R-modules is G-equivariant.

Lemma 52 *Assume that* $\mathrm{Fil}^r R$ *is the inverse image of* $q^r R$ *under* φ_R *for every* $r \in \mathbb{N} \cap [0, a]$. *Then the forgetful functor* $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi, G) \to \mathrm{M}^q_{[0,a],\mathrm{free}}(R, \varphi, G)$ *is an equivalence of categories.*

Proof For two objects $(M_i, \mathrm{Fil}^r M_i, \varphi_{M_i}, \rho_{M_i})$ $(i \in \{1, 2\})$ of $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi, G)$, any morphism $(M_1, \varphi_{M_1}, \rho_{M_1}) \to (M_2, \varphi_{M_2}, \rho_{M_2})$ in $\mathrm{M}^q_{[0,a],\mathrm{free}}(R, \varphi, G)$ preserves the filtrations by Lemma 46 (1). Hence the forgetful functor is fully faithful. Let us prove the essential surjectivity. Let (M, φ_M, ρ_M) be an object of $\mathrm{M}^q_{[0,a],\mathrm{free}}(R, \varphi, G)$. Then, by Lemma 46 (2), there exists a decreasing filtration $\mathrm{Fil}^r M$ $(r \in \mathbb{N} \cap [0, a])$ of M with which (M, φ_M) is an object of $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi)$. We have $\rho_M(g)(\mathrm{Fil}^r M) = \mathrm{Fil}^r M$ $(r \in \mathbb{N} \cap [0, a], g \in G)$ by Lemma 46 (1) and $g(q)R = qR$. Therefore

$(M, \mathrm{Fil}^r M, \varphi_M, \rho_M)$ is an object of $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi, G)$. This completes the proof. □

In the rest of this section, we assume that R is a topological ring, G is a topological group, and the action of G on R is continuous. For a free R-module of finite rank M, choosing an isomorphism $R^{\oplus r} \xrightarrow{\cong} M$ of R-modules, we endow M with the topology induced by the product topology of $R^{\oplus r}$ via the isomorphism. Note that this topology is independent of the choice of the isomorphism because an R-linear map between free R-modules of finite rank is continuous with respect to the product topology.

Definition 53 We define $\mathrm{MF}^{q,\mathrm{cont}}_{[0,a],\mathrm{free}}(R, \varphi, G)$ to be the full subcategory of $\mathrm{MF}^q_{[0,a],\mathrm{free}}(R, \varphi, G)$ consisting of M such that the action ρ_M of G on M is continuous, i.e., the map $\mu_M: G \times M \to M; (g, m) \mapsto \rho_M(g)m$ is continuous.

Let M be a free R-module of finite rank endowed with a semilinear action $\rho_M: G \to \mathrm{Aut}(M)$ of G. Choose a basis e_ν ($N \in \mathbb{N}$, $\nu \in \mathbb{N} \cap [1, N]$) of M, and let $c: G \to GL_N(R)$ be the 1-cocycle defined by $\rho_M(g)(e_1, \dots, e_N) = (e_1, \dots, e_N)$ $c(g)$ for $g \in G$. Then the action ρ_M is continuous if and only if $c: G \to M_N(R)$ is continuous; the sufficiency is obvious by the continuity of the map $M_N(R) \times R^N \to R^N; (A, v) \mapsto Av$, and the necessity follows from $c(g)_{\nu\mu} = p_\nu \circ \mu_M \circ (\mathrm{id}_G \times i_\mu)$ $(g, 1)$ ($g \in G, \nu, \mu \in \mathbb{N} \cap [1, N]$), where i_μ (resp. p_ν) denotes the continuous map $R \to M; a \mapsto ae_\mu$ (resp. $M \to R; \sum_\lambda b_\lambda e_\lambda \mapsto b_\nu$). The continuity of c is also equivalent to the following: for any open neighborhood U of 1 in $M_N(R)$, there exists an open neighborhood V of 1 in G such that $c(V) \subset U$. The necessity is clear. As for the sufficiency, suppose that we are given $g \in G$ and an open neighborhood U of $c(g)$ in $M_N(R)$. Then the set $U' := g^{-1}(c(g)^{-1}U)$ is an open neighborhood of 1 in $M_N(R)$, and therefore there exists an open neighborhood V of 1 in G such that $c(V) \subset U'$. The set gV is an open neighborhood of g in G and we have $c(gV) = c(g)g(c(V)) \subset c(g)g(U') = U$.

Let κ and λ be as before (46), and assume that R' is a topological algebra, G' is a topological group, the action of G' on R' is continuous, and the homomorphisms κ and λ are continuous. Then, by using the observation on 1-cocycles in the paragraph above, we see that the functor $(\kappa, \lambda)^*$ (46) induces a functor

$$(\kappa, \lambda)^*: \mathrm{MF}^{q,\mathrm{cont}}_{[0,a],\mathrm{free}}(R, \varphi, G) \longrightarrow \mathrm{MF}^{q',\mathrm{cont}}_{[0,a],\mathrm{free}}(R', \varphi, G'). \tag{47}$$

Let $\alpha: R \to \overline{R}$ be a surjective ring homomorphism, let J be the kernel of α, and assume the following condition.

Condition 54 (a) The conditions (i-v) in Condition 43 are satisfied.
(b) $\rho_R(g)(J) = J$ for every $g \in G$.
(c) J is closed in R.
(d) There exists a decreasing sequence of ideals I_n ($n \in \mathbb{N}$) of R satisfying the following properties.
(d-1) The ideals I_n ($n \in \mathbb{N}$) form a fundamental system of open neighborhoods of 0 in R.

(d-2) The homomorphism $R \to \varprojlim_n R/I_n$ is an isomorphism.

(d-3) $q^{-(a+1)}\varphi_R(I_n \cap J) \subset I_n \cap J$ for every $n \in \mathbb{N}$.

(e) The sequence $\prod_{m=0}^{n} \varphi_R^m(q)$ $(n \in \mathbb{N})$ in R converges to 0.

(f) For every $r \in \mathbb{N} \cap [0, a]$, the homomorphism $q^{-r}\varphi_R \colon \mathrm{Fil}^r R \to R$ is continuous.

(g) The map $G \to R$; $g \mapsto g(q)q^{-1}$ is continuous.

Remark 55 (1) The conditions (f) and (g) in Condition 54 are satisfied if the homomorphism $\varphi_R \colon R \to R$ is continuous and the multiplication by q induces a homeomorphism from R to qR endowed with the topology induced by that of R. Note that the multiplication by q^r induces a homeomorphism $R \to q^r R$ for every $r \in \mathbb{N}$ because the homeomorphism $R \to qR$; $x \mapsto qx$ induces homeomorphisms $q^s R \to q^{s+1} R$ $(s \in \mathbb{N})$.

(2) Condition 54 implies Condition 43 (vi) as follows. By (c), (d-1), and (d-2), we have $J = \varprojlim_n J/(J \cap I_n)$. By (d-1) and (e), there exists a map $\nu \colon \mathbb{N} \to \mathbb{N}$ such that $\prod_{m=0}^{n} \varphi_R^m(q) \in I_{\nu(n)}$ and $\nu(n+1) \geq \nu(n)$ for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} \nu(n) = \infty$. By (d-3), we see that $J_n := J \cap I_{\nu(n)}$ $(n \in \mathbb{N})$ satisfy Condition 43 (vi).

We define $(\overline{R}, \overline{q}, \varphi_{\overline{R}}, \mathrm{Fil}^r \overline{R}, \rho_{\overline{R}})$ as before Proposition 49, and endow $\overline{R}$ with the quotient topology of R. Then the action $\rho_{\overline{R}}$ of G on $\overline{R}$ is continuous.

Proposition 56 *Under the notation and assumption as above, the functor*

$$(\alpha, \mathrm{id}_G)^* \colon \mathrm{MF}^{q,\mathrm{cont}}_{[0,a],\mathrm{free}}(R, \varphi, G) \to \mathrm{MF}^{\overline{q},\mathrm{cont}}_{[0,a],\mathrm{free}}(\overline{R}, \varphi, G)$$

is an equivalence of categories.

Proof Let M be an object of $\mathrm{MF}^{q}_{[0,a],\mathrm{free}}(R, \varphi, G)$, and put $\overline{M} := (\alpha, \mathrm{id}_G)^*(M)$. By Proposition 49, it suffices to prove that the action ρ_M of G on M is continuous if the action $\rho_{\overline{M}}$ of G on $\overline{M}$ is continuous. Assume that $\rho_{\overline{M}}$ is continuous. Choose a basis e_ν $(N \in \mathbb{N}, \nu \in \mathbb{N} \cap [1, N])$ of M, $r_\nu \in \mathbb{N} \cap [0, a]$ $(\nu \in \mathbb{N} \cap [1, N])$ and $A = (a_{\nu\mu}) \in GL_N(R)$ such that $\mathrm{Fil}^r M = \oplus_\nu \mathrm{Fil}^{r-r_\nu} R e_\nu$ $(r \in \mathbb{N} \cap [0, a])$ and $\varphi_M(e_\mu) = q^{r_\mu} \sum_\nu a_{\nu\mu} e_\nu$. Let c be the 1-cocycle $G \to GL_N(R)$ defined by $\rho_M(g)(e_1, \ldots, e_N) = (e_1, \ldots, e_N)c(g)$, and let $\overline{c}$ denote the composition of c with $GL_N(R) \to GL_N(\overline{R})$. It suffices to prove that, for any $n \in \mathbb{N}$, there exists an open neighborhood V of 1 in G such that $c(V) \subset 1 + I_n M_N(R)$. (See the remark after Definition 53.)

By Condition 54 (f), there exists $n' \in \mathbb{N}$ such that $n' \geq n$ and $q^{-r}\varphi_R(\mathrm{Fil}^r R \cap I_{n'}) \subset I_n$ for every $r \in \mathbb{N} \cap [0, a]$. By Condition 54 (g) and the continuity of the actions of G on R and $\overline{M}$, there exists an open neighborhood V of 1 in G such that $\overline{c}(V) \subset 1 + I_{n'} M_N(\overline{R})$, $g(A) \equiv A \bmod I_n M_N(R)$ $(g \in V)$, and $g(q)q^{-1} \equiv 1 \bmod I_n$ $(g \in V)$. We show $c(V) \subset 1 + I_n M_N(R)$. Let $g \in V$. Since $\rho_R(g)(\mathrm{Fil}^r \overline{M}) = \mathrm{Fil}^r \overline{M}$ $(r \in \mathbb{N} \cap [0, a])$, we see that $\overline{c}(g)$ is of the form $1 + \overline{B}$, $\overline{B} = (\overline{b}_{\nu\mu}) \in I_{n'} M_N(\overline{R})$, $\overline{b}_{\nu\mu} \in \mathrm{Fil}^{r_\mu - r_\nu} \overline{R}$ by the same argument as the proof of Lemma 42 (1). Since $J \subset$

$\mathrm{Fil}^a R$, this implies that $c(g)$ is of the form $1 + B + B'$, $B = (b_{\nu\mu}) \in I_{n'} M_N(R)$, $B' = (b'_{\nu\mu}) \in J M_N(R)$, $b_{\nu\mu} \in \mathrm{Fil}^{r_\mu - r_\nu} R$. Put $D := \mathrm{diag}(q^{r_\mu}; \mu \in \mathbb{N} \cap [1, N])$. Then we have

$$\varphi_M \circ \rho_M(g)(e_1, \dots, e_N) = (e_1, \dots, e_N) A D (1 + \varphi_R(B) + \varphi_R(B'))$$
$$\rho_M(g) \circ \varphi_M(e_1, \dots, e_N) = (e_1, \dots, e_N)(1 + B + B') g(AD).$$

Hence the equality $\varphi_M \circ \rho_M(g) = \rho_M(g) \circ \varphi_M$ gives

$$\begin{aligned} & 1 + B + B' = ADg(AD)^{-1} + AD\varphi_R(B)g(AD)^{-1} + AD\varphi_R(B')g(AD)^{-1} \\ \iff\ & B' - AD\varphi_R(B')g(D)^{-1}g(A)^{-1} \\ & \qquad = A\{Dg(D)^{-1} - A^{-1}g(A) + D\varphi_R(B)g(D)^{-1}\}g(A)^{-1} - B. \end{aligned}$$

We derive $B' \in I_n M_N(R)$ from the last equality. We have $Dg(D)^{-1} - 1 = \mathrm{diag}((qg(q)^{-1})^{r_\mu}) - 1 \in I_n M_N(R)$ because $g(q)q^{-1} \in 1 + I_n$. The (ν, μ)-component of $D\varphi_R(B)g(D)^{-1}$ is $q^{r_\nu} g(q)^{-r_\mu} \varphi_R(b_{\nu\mu})$, which is contained in I_n because $b_{\nu\mu} \in \mathrm{Fil}^{r_\mu - r_\nu} R \cap I_{n'}$. Finally we have $B \in I_{n'} M_N(R) \subset I_n M_N(R)$ and $A^{-1}g(A) - 1 \in I_n M_N(R)$. Thus we obtain

$$B' - AD\varphi_R(B')g(D)^{-1}g(A)^{-1} \in I_n M_N(R).$$

By (45) and $q^a g(D)^{-1} \in M_N(R)$, we can define a φ_R-semilinear endomorphism F of $J M_N(R)$ by

$$F(X) = ADq^{-a-1}\varphi_R(X) q^a g(D)^{-1} g(A)^{-1}, \quad X \in J M_N(R),$$

and then we have $(1 - qF)(B') \in (I_n \cap J) M_N(R)$. By Condition 54 (d-3), we see that $(I_n \cap J) M_N(R)$ is stable under F. Hence, by applying $\sum_{l=0}^m (qF)^l$ to $(1 - qF)(B')$, we obtain $(1 - (qF)^{m+1})(B') \in (I_n \cap J) M_N(R)$ for $m \in \mathbb{N}$. We have $(qF)^{m+1}(B') = \prod_{l=0}^m \varphi_R^l(q) F^{m+1}(B')$ and $F^{m+1}(B') \in J M_N(R)$. By Condition 54 (d-1) and (e), there exists $m \in \mathbb{N}$ such that $\prod_{l=0}^m \varphi_R^l(q) \in I_n$, for which we have $(qF)^{m+1}(B') \in I_n J M_N(R) \subset (I_n \cap J) M_N(R)$, and therefore $B' \in (I_n \cap J) M_N(R)$. □

Definition 57 We define $\mathrm{M}^{q,\mathrm{cont}}_{[0,a],\mathrm{free}}(R, \varphi, G)$ to be the full subcategory of $\mathrm{M}^{q}_{[0,a],\mathrm{free}}(R, \varphi, G)$ consisting of M such that the action ρ_M of G on M is continuous.

Lemma 58 *Assume that* $\mathrm{Fil}^r R$ *is the inverse image of* $q^r R$ *under* φ_R *for every* $r \in \mathbb{N} \cap [0, a]$. *Then the forgetful functor* $\mathrm{MF}^{q,\mathrm{cont}}_{[0,a],\mathrm{free}}(R, \varphi, G) \to \mathrm{M}^{q,\mathrm{cont}}_{[0,a],\mathrm{free}}(R, \varphi, G)$ *is an equivalence of categories.*

Proof This immediately follows from Lemma 52. □

8 A_{inf}-Representations with φ

As in Sect. 2, let Λ be a normal domain containing $O_{\overline{K}}$ and integral over a noetherian normal subring (see (1)) such that $\Lambda/p\Lambda \neq 0$ and the absolute Frobenius of $\Lambda/p\Lambda$ is surjective, and let Λ_0 be a subring of Λ such that Λ is integral over Λ_0 and $\mathrm{Frac}(\Lambda)/\mathrm{Frac}(\Lambda_0)$ is a Galois extension. Let $G(\Lambda/\Lambda_0)$ denote its Galois group.

Put $\underline{A}_{\mathrm{inf}}(\Lambda) := A_{\mathrm{inf}}(\Lambda)/I^{p-1}A_{\mathrm{inf}}(\Lambda)$ and $\underline{A}_{\mathrm{crys}}(\Lambda) := A_{\mathrm{crys}}(\Lambda)/I^{p-1}A_{\mathrm{crys}}(\Lambda)$. We define $\mathrm{Fil}^r\underline{A}_{\mathrm{inf}}(\Lambda)$ (resp. $\mathrm{Fil}^r\underline{A}_{\mathrm{crys}}(\Lambda)$) ($r \in \mathbb{Z}$) to be the image of $\mathrm{Fil}^r A_{\mathrm{inf}}(\Lambda)$ (resp. $\mathrm{Fil}^r A_{\mathrm{crys}}(\Lambda)$) in $\underline{A}_{\mathrm{inf}}(\Lambda)$ (resp. $\underline{A}_{\mathrm{crys}}(\Lambda)$). Then, by (2), we have an isomorphism $\underline{A}_{\mathrm{inf}}(\Lambda) \xrightarrow{\cong} \underline{A}_{\mathrm{crys}}(\Lambda)$, and it induces an isomorphism $\mathrm{Fil}^r\underline{A}_{\mathrm{inf}}(\Lambda) \xrightarrow{\cong} \mathrm{Fil}^r\underline{A}_{\mathrm{crys}}(\Lambda)$ for $r \in \mathbb{N}\cap[0, p-1]$. For $r \in \mathbb{N}\cap[0, p-1]$, $\underline{A}_{\mathrm{inf}}(\Lambda)/\mathrm{Fil}^r\underline{A}_{\mathrm{inf}}(\Lambda) \cong \underline{A}_{\mathrm{crys}}(\Lambda)/\mathrm{Fil}^r\underline{A}_{\mathrm{crys}}(\Lambda)$ is isomorphic to $A_{\mathrm{inf}}(\Lambda)/\mathrm{Fil}^r A_{\mathrm{inf}}(\Lambda)$, which is p-torsion free and p-adically complete and separated (Lemma 1 (3)).

We apply the results in Sects. 6 and 7 to $A_{\mathrm{inf}}(\Lambda)$, $A_{\mathrm{crys}}(\Lambda)$ and $\underline{A}_{\mathrm{inf}}(\Lambda)$. The quadruplets

$$\begin{aligned}(A_{\mathrm{crys}}(\Lambda), p, \varphi, (\mathrm{Fil}^r A_{\mathrm{crys}}(\Lambda))_{r\in\mathbb{N}\cap[0,p-2]}),\\ (A_{\mathrm{inf}}(\Lambda), q, \varphi, (\mathrm{Fil}^r A_{\mathrm{inf}}(\Lambda))_{r\in\mathbb{N}\cap[0,p-2]}),\\ (\underline{A}_{\mathrm{inf}}(\Lambda), p, \varphi, (\mathrm{Fil}^r \underline{A}_{\mathrm{inf}}(\Lambda))_{r\in\mathbb{N}\cap[0,p-2]})\end{aligned} \tag{48}$$

satisfy Condition 39 for $a = p-2$. (See before Lemma 9 for the definition of $q \in A_{\mathrm{inf}}(\Lambda)$.) For the second one, we have $\mathrm{Fil}^r A_{\mathrm{inf}}(\Lambda) = (q')^r A_{\mathrm{inf}}(\Lambda) = \varphi^{-1}(q^r A_{\mathrm{inf}}(\Lambda))$ ($r \in \mathbb{N}$). Hence we may apply Lemma 46 and obtain an equivalence of categories

$$\begin{aligned}\mathrm{M}^q_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\Lambda), \varphi) &\xrightarrow{\sim} \mathrm{MF}^q_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\Lambda), \varphi),\\ (M, \varphi_M) &\mapsto (M, \varphi_M, (\varphi_M^{-1}(q^r M))_{r\in\mathbb{N}\cap[0,p-2]}).\end{aligned} \tag{49}$$

For the three quadruplets (48), the homomorphisms $A_{\mathrm{crys}}(\Lambda) \to \underline{A}_{\mathrm{crys}}(\Lambda) \cong \underline{A}_{\mathrm{inf}}(\Lambda)$, $A_{\mathrm{inf}}(\Lambda) \to \underline{A}_{\mathrm{inf}}(\Lambda)$, and $A_{\mathrm{inf}}(\Lambda) \to A_{\mathrm{crys}}(\Lambda)$ satisfy the three conditions on κ assumed before (44). For the second and third homomorphisms, note that we have $q = p(1 + p^{-1}\pi_0)$ and $1 + p^{-1}\pi_0 \in A_{\mathrm{crys}}(O_{\overline{K}})^\times$ because $p^{-1}\pi_0 \in \mathrm{Fil}^1 A_{\mathrm{crys}}(O_{\overline{K}})$ (Lemma 9 (2)) and $x^p = p!x^{[p]} \in pA_{\mathrm{crys}}(O_{\overline{K}})$ for all $x \in \mathrm{Fil}^1 A_{\mathrm{crys}}(O_{\overline{K}})$. Therefore we obtain three functors

$$\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(A_{\mathrm{crys}}(\Lambda), \varphi) \longrightarrow \mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\mathrm{inf}}(\Lambda), \varphi), \tag{50}$$

$$\mathrm{M}^q_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\Lambda), \varphi) \longrightarrow \mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\mathrm{inf}}(\Lambda), \varphi), \tag{51}$$

$$\mathrm{M}^q_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\Lambda), \varphi) \longrightarrow \mathrm{MF}^p_{[0,p-2],\mathrm{free}}(A_{\mathrm{crys}}(\Lambda), \varphi). \tag{52}$$

The composition of (52) and (50) is canonically isomorphic to (51).

The three quadruplets (48) with the actions of $G(\Lambda/\Lambda_0)$ on the underlying algebras satisfy the conditions before Definition 48. See before Lemma 9 for the second quadruplet. We endow $A_{\mathrm{crys}}(\Lambda)$ (resp. $A_{\mathrm{inf}}(\Lambda)$, resp. $\underline{A}_{\mathrm{inf}}(\Lambda)$) with the p

(resp. $(p, [\underline{p}])$, resp. p)-adic topology. Then the actions of $G(\Lambda/\Lambda_0)$ on these rings are continuous (see Lemma 5 and the construction of $A_{\mathrm{crys}}(\Lambda)$ in Sect. 2). Hence we may apply Definition 53 to these three quadruplets with $G(\Lambda/\Lambda_0)$-actions. By applying Lemma 58 to the second quadruplet of (48), we obtain an equivalence of categories

$$\mathrm{M}^{q,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\Lambda), \varphi, G(\Lambda/\Lambda_0)) \xrightarrow{\sim} \mathrm{MF}^{q,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\Lambda), \varphi, G(\Lambda/\Lambda_0)), \tag{53}$$

$$(M, \varphi_M, \rho_M) \mapsto (M, \varphi_M, (\varphi_M^{-1}(q^r M))_{r\in\mathbb{N}\cap[0,p-2]}, \rho_M).$$

The homomorphisms $A_{\mathrm{crys}}(\Lambda) \to \underline{A}_{\mathrm{inf}}(\Lambda)$, $A_{\mathrm{inf}}(\Lambda) \to \underline{A}_{\mathrm{inf}}(\Lambda)$ and $A_{\mathrm{inf}}(\Lambda) \to A_{\mathrm{crys}}(\Lambda)$ are $G(\Lambda/\Lambda_0)$-equivariant, and also continuous because the topology of $A_{\mathrm{inf}}(\Lambda)$ coincides with the (p, π^{p-1})-adic topology (Lemma 1 (1)) and $[\underline{p}]^p = (-\xi + p)^p \in pA_{\mathrm{crys}}(O_{\overline{K}})$. Hence by applying the construction of (47) to these homomorphisms and taking the compositions with (53) for the latter two, we obtain the following three functors, where $G = G(\Lambda/\Lambda_0)$.

$$\mathrm{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A_{\mathrm{crys}}(\Lambda), \varphi, G) \longrightarrow \mathrm{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\underline{A}_{\mathrm{inf}}(\Lambda), \varphi, G) \tag{54}$$

$$\mathrm{M}^{q,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\Lambda), \varphi, G) \longrightarrow \mathrm{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\underline{A}_{\mathrm{inf}}(\Lambda), \varphi, G) \tag{55}$$

$$\mathrm{M}^{q,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\Lambda), \varphi, G) \longrightarrow \mathrm{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A_{\mathrm{crys}}(\Lambda), \varphi, G) \tag{56}$$

The composition of (56) and (54) is canonically isomorphic to (55).

Proposition 59 *The functors (50)–(52) and (54)–(56) are equivalences of categories.*

Proof To simplify the notation, we abbreviate $A_{\mathrm{crys}}(\Lambda)$, $A_{\mathrm{inf}}(\Lambda)$, and $\underline{A}_{\mathrm{inf}}(\Lambda)$ to A_{crys}, A_{inf}, and $\underline{A}_{\mathrm{inf}}$. By Propositions 44, 56, and Remark 55 (2), it suffices to prove that the homomorphisms $A_{\mathrm{crys}} \to \underline{A}_{\mathrm{inf}}$ and $A_{\mathrm{inf}} \to \underline{A}_{\mathrm{inf}}$ satisfy Condition 54 for $a = p-2$. Note that the kernels of these homomorphisms are $I^{p-1}A_{\mathrm{crys}}$ and $I^{p-1}A_{\mathrm{inf}}$, respectively, and that the topology of $\underline{A}_{\mathrm{inf}}$ coincides with the quotient of that of either of A_{inf} and A_{crys}. The condition (a) (ii-v) are verified as follows: $\underline{A}_{\mathrm{inf}}(\Lambda)$ is p-torsion free as mentioned after (2); we see $\varphi(I^{p-1}) \subset I^{p-1}$ and $I^{p-1} \subset \mathrm{Fil}^{p-2}$ by the definition of $I^{p-1}A_{\mathrm{inf}}$ and $I^{p-1}A_{\mathrm{crys}}$, and we have $\varphi(I^{p-1}A_{\mathrm{inf}}) \subset q^{p-1}A_{\mathrm{inf}}$ by (3), and $\varphi(I^{p-1}A_{\mathrm{crys}}) \subset p^{p-1}A_{\mathrm{crys}}$ as recalled before (2). The condition (a) (i) for A_{inf} (resp. A_{crys}) follows from the fact that $I^{p-1}A_{\mathrm{inf}} \subset \pi A_{\mathrm{inf}}$ and A_{inf} is π-adically complete and separated (Lemma 1 (4)) (resp. $I^{p-1}A_{\mathrm{crys}} \subset \mathrm{Fil}^1 A_{\mathrm{crys}} + pA_{\mathrm{crys}}$, A_{crys} is p-adically complete and separated, and $\mathrm{Fil}^1 A_{\mathrm{crys}}/p$ is a nilideal of A_{crys}/p). The condition (b) is obvious.

We first verify the remaining conditions (c-g) for A_{crys}. Since $A_{\mathrm{crys}}/I^{p-1}A_{\mathrm{crys}}$ is p-torsion free, we have $p^n A_{\mathrm{crys}} \cap I^{p-1}A_{\mathrm{crys}} = p^n I^{p-1}A_{\mathrm{crys}}$. This implies (c) and (d) for $I_n = p^n A_{\mathrm{crys}}$ because A_{crys} and $I^{p-1}A_{\mathrm{crys}}$ are p-adically complete and separated as mentioned after (2). The condition (e) and the sufficient condition for (f) and (g) given in Remark 55 (1) are obviously satisfied. Let us prove the conditions (c-g) for A_{inf}. We have $(p^n A_{\mathrm{inf}} + \pi^{n+p-1} A_{\mathrm{inf}}) \cap I^{p-1} A_{\mathrm{inf}} = p^n I^{p-1} A_{\mathrm{inf}} + \pi^n I^{p-1} A_{\mathrm{inf}}$ because $A_{\mathrm{inf}}/I^{p-1}A_{\mathrm{inf}}$ is p-torsion free. Since $I^{p-1}A_{\mathrm{inf}}$ is a free A_{inf}-module of rank 1 and $\varphi(\pi)\pi^{-1} \in A_{\mathrm{inf}}$, the conditions (c) and (d) for $I_n = p^n A_{\mathrm{inf}} + \pi^{n+p-1} A_{\mathrm{inf}}$ follow from Lemma 1 (1) and (2). The condition (e) follows from $\varphi^m(q) = \varphi^{m+1}(q') \in \varphi^{m+1}(\mathrm{Fil}^1 A_{\mathrm{inf}}) \subset \varphi^{m+1}(pA_{\mathrm{inf}} + [\underline{p}]A_{\mathrm{inf}}) \subset pA_{\mathrm{inf}} + [\underline{p}]A_{\mathrm{inf}}$ $(m \in \mathbb{N})$. It remains to verify the sufficient condition for (f) and (g) given in Remark 55 (1). The equality $\varphi([\underline{p}]) = [\underline{p}]^p$ implies that φ of A_{inf} is continuous. The quotient $A_{\mathrm{inf}}/qA_{\mathrm{inf}}$ is p-torsion free by Lemma 1 (3). Hence we have $(p^n A_{\mathrm{inf}} + q^{n+1} A_{\mathrm{inf}}) \cap qA_{\mathrm{inf}} = p^n(qA_{\mathrm{inf}}) + q^n(qA_{\mathrm{inf}})$. By Lemma 1 (1), this shows that $A_{\mathrm{inf}} \to qA_{\mathrm{inf}}; x \mapsto qx$ is a homeomorphism. □

Let M be an object of $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A}, \Phi)$ (Sect. 4). We apply the above results to $TA_{\mathrm{crys}}(M)$ introduced in Sect. 5. Note that $\Lambda = \overline{\mathcal{A}}$ and $\Lambda_0 = \mathcal{A}$ satisfy the conditions in the beginning of this section as observed after (3). Let $t_1, \dots, t_d \in A^{\times}$ be coordinates of A over O_K, i.e. $O_K[T_1, \dots, T_d] \to A; T_i \mapsto t_i$ is étale. Let $\varphi_{\mathcal{A}}$ be the unique lifting $\mathcal{A} \to \mathcal{A}$ of the absolute Frobenius of $\mathcal{A}/p$ compatible with $\sigma\colon O_K \to O_K$ and satisfying $\varphi_{\mathcal{A}}(t_i) = t_i^p$ for all $i \in \mathbb{N} \cap [1, d]$.

Proposition 60 *The free $A_{\mathrm{crys}}(\overline{\mathcal{A}})$-module of finite type $TA_{\mathrm{crys}}(M)$ with* Fil^r $(r \in \mathbb{N} \cap [0, p-2])$, *$\varphi$ and $G_{\mathcal{A}}$-action is an object of* $\mathrm{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A_{\mathrm{crys}}(\overline{\mathcal{A}}), \varphi, G_{\mathcal{A}})$.

Proof By (37) for $(\mathcal{B}, s_1, \dots, s_e) = (\mathcal{A}, t_1, \dots, t_d)$, and $\varphi_{\mathcal{B}} = \varphi_{\mathcal{A}}$, we see that $TA_{\mathrm{crys}}(M)$ with $\mathrm{Fil}^{\bullet}$ and φ is an object of $\mathrm{MF}^{p}_{[0,p-2],\mathrm{free}}(A_{\mathrm{crys}}(\overline{\mathcal{A}}), \varphi)$. By Lemma 36, we have $TA_{\mathrm{crys}}(M)/p^m \xrightarrow{\cong} TA_{\mathrm{crys},m}(M)$ for $m \in \mathbb{N}_{>0}$. Hence the action of $G_{\mathcal{A}}$ on $TA_{\mathrm{crys}}(M)/p^m$ is continuous by (29) because the action of $G_{\mathcal{A}}$ on $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ is continuous (Sect. 2). □

Proposition 61 *The following functor is fully faithful.*

$$TA_{\mathrm{crys}}\colon \mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A}, \Phi) \to \mathrm{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A_{\mathrm{crys}}(\overline{\mathcal{A}}), \varphi, G_{\mathcal{A}})$$

Proof By taking the $G_{\mathcal{A}}$-invariant part of (39) for $\mathcal{B} = \mathcal{A}$, $s_i = t_i$, and $\varphi_{\mathcal{B}} = \varphi_{\mathcal{A}}$, and using Proposition 62 below, we obtain an $\mathcal{A}$-linear filtered isomorphism

$$(TA_{\mathrm{crys}}(M) \otimes_{A_{\mathrm{crys}}(\overline{\mathcal{A}})} \mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}}))^{G_{\mathcal{A}}} \cong M$$

compatible with φ and the integrable connections, and functorial in M. □

Proposition 62 *We have* $\mathcal{A} \xrightarrow{\cong} \mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})^{G_{\mathcal{A}}}$ *and* $\mathrm{Fil}^1 \mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})^{G_{\mathcal{A}}} = 0$.

Proof We can show $\mathrm{gr}^0(\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}}))[\frac{1}{p}]^{G_{\mathcal{A}}} = \mathcal{A}[\frac{1}{p}]$ and $\mathrm{gr}^r(\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}}))[\frac{1}{p}]^{G_{\mathcal{A}}} = 0$ $(r \in \mathbb{N}_{>0})$ in the same way as [19, Proposition 2.12]. Since the filtration Fil^r of $\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})$ is separated by (5) and Lemma 153, we obtain $\mathcal{A}[\frac{1}{p}] \xrightarrow{\cong} \mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})[\frac{1}{p}]^{G_{\mathcal{A}}}$ and $\mathrm{Fil}^1 \mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})^{G_{\mathcal{A}}} = 0$. We can remove $\frac{1}{p}$ because the restriction of the canonical homomorphism $\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}}) \to \widehat{\overline{\mathcal{A}}}$ to $\mathcal{A}$ is the inclusion map and $\widehat{\overline{\mathcal{A}}} \cap \mathcal{A}[\frac{1}{p}] = \mathcal{A}$ in $\widehat{\overline{\mathcal{A}}}[\frac{1}{p}]$.

For the last equality, note that $\overline{\mathcal{A}}/p^n \cong \widehat{\overline{\mathcal{A}}}/p^n$ (Lemma 7) and $\mathcal{A}/p^n \to \overline{\mathcal{A}}/p^n$ is injective since $\mathcal{A}$ is a normal domain and $\overline{\mathcal{A}}$ is integral over $\mathcal{A}$. □

We define $TA_{\mathrm{inf}}(M)$ (resp. $T\underline{A}_{\mathrm{inf}}(M)$) to be the image of $TA_{\mathrm{crys}}(M)$ under a quasi-inverse of the functor (56) (cf. Proposition 59) (resp. under the functor (54)).

Theorem 63 (1) *The following functor is fully faithful.*

$$TA_{\mathrm{inf}}\colon \mathrm{MF}^{\nabla}_{[0,p-2],free}(\mathcal{A},\Phi) \to \mathrm{M}^{q,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\overline{\mathcal{A}}),\varphi,G_{\mathcal{A}})$$

(2) *For an object M of $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A},\Phi)$, we have the following canonical $G_{\mathcal{A}}$-equivariant isomorphisms functorial in M.*

$$\begin{aligned}T^*_{\mathrm{crys}}(M) &\cong \mathrm{Hom}_{\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}),\varphi)}(T\underline{A}_{\mathrm{inf}}(M),\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}))\\ &\cong \mathrm{Hom}_{\mathrm{M}^q_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\overline{\mathcal{A}}),\varphi)}(TA_{\mathrm{inf}}(M),A_{\mathrm{inf}}(\overline{\mathcal{A}}))\end{aligned}$$

Proof The first claim is an immediate consequence of Propositions 61 and 59. The second one follows from Lemma 37 and Proposition 59 for (50) and (52). □

We see that the action of $G_{\mathcal{A}}$ on the underlying $A_{\mathrm{inf}}(\overline{\mathcal{A}})$-module of an object in the essential image of the functor TA_{inf} is "trivial" modulo $I^1A_{\mathrm{inf}}(\overline{\mathcal{A}})$ as follows. Let

$$\overline{\alpha}\colon \mathcal{A} \to A_{\mathrm{crys}}(\overline{\mathcal{A}})/I^1A_{\mathrm{crys}}(\overline{\mathcal{A}}) \underset{(2)}{\cong} A_{\mathrm{inf}}(\overline{\mathcal{A}})/I^1A_{\mathrm{inf}}(\overline{\mathcal{A}}) \tag{57}$$

be the homomorphism induced by $\beta\colon \mathcal{A} \to A_{\mathrm{crys}}(\overline{\mathcal{A}})$ (defined before Lemma 34) for $(\mathcal{B},s_1,\dots,s_e) = (\mathcal{A},t_1,\dots,t_d)$.

Lemma 64 (1) *The homomorphism $\overline{\alpha}$ is $G_{\mathcal{A}}$-equivariant.*
(2) *Let M be an object of $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A},\Phi)$. Then the isomorphism*

$$M \otimes_{\mathcal{A},\overline{\alpha}} A_{\mathrm{inf}}(\overline{\mathcal{A}})/I^1A_{\mathrm{inf}}(\overline{\mathcal{A}}) \cong TA_{\mathrm{inf}}(M) \otimes_{A_{\mathrm{inf}}(\overline{\mathcal{A}})} A_{\mathrm{inf}}(\overline{\mathcal{A}})/I^1A_{\mathrm{inf}}(\overline{\mathcal{A}})$$

induced by (37) for $(\mathcal{B},s_1,\dots,s_e) = (\mathcal{A},t_1,\dots,t_d)$ is $G_{\mathcal{A}}$-equivariant.

Proof For $g \in G_{\mathcal{A}}$, choose $n_i(g) \in \mathbb{Z}_p$ such that $g(\underline{t}_i) = \underline{t}_i\underline{\varepsilon}^{n_i(g)}$ in $R_{\overline{\mathcal{A}}}$, where $\underline{\varepsilon}^b = (\varepsilon_n^b \mod p)_{n\in\mathbb{N}}$ for $b \in \mathbb{Z}_p$. We have $g(v_i) = v_i[\underline{\varepsilon}^{n_i(g)}] + [\underline{\varepsilon}^{n_i(g)}] - 1$ and $[\underline{\varepsilon}^{n_i(g)}] - 1 \in I^1A_{\mathrm{inf}}(\overline{\mathcal{A}})$. Hence the composition δ of the $A_{\mathrm{crys}}(\overline{\mathcal{A}})$-algebra homomorphism $\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}}) \to A_{\mathrm{crys}}(\overline{\mathcal{A}});\ v_i \mapsto 0$ with $A_{\mathrm{crys}}(\overline{\mathcal{A}}) \to A_{\mathrm{crys}}(\overline{\mathcal{A}})/I^1A_{\mathrm{crys}}(\overline{\mathcal{A}}) \cong A_{\mathrm{inf}}(\overline{\mathcal{A}})/I^1A_{\mathrm{inf}}(\overline{\mathcal{A}})$ is $G_{\mathcal{A}}$-equivariant. This implies the claim (1) by Lemma 34 (1). By (41), the scalar extension of (40) for $(\mathcal{B},s_1,\dots,s_e) = (\mathcal{A},t_1,\dots,t_d)$ by δ is the identity map. Hence the claim (2) follows from the $G_{\mathcal{A}}$-equivariance of (39) for $\mathcal{B} = \mathcal{A}$. □

9 Duality for $A_{\mathrm{inf}}/\pi^{p-1}$-Representations with φ

We keep the notation in Sect. 8. In this section, we prove a duality (Proposition 68) for the $\mathbb{Z}_p$-module $\underline{T}^*_{\mathrm{inf}}(\mathcal{M})$ (59) associated to an object $\mathcal{M}$ of the category $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}), \varphi)$, under a certain condition on $\mathcal{M}$, by the same argument as the proof of [10, Theorem 2.6*].

We define the category $\widetilde{\mathrm{MF}}_{[0,p-2]}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}), \varphi)$ as follows. Let $\varphi_{\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})}$ denote the Frobenius φ of $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$. An object is a triplet $(\mathcal{M}, \mathrm{Fil}^r\mathcal{M}, \varphi_r)$ consisting of the following data:

(i) An $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$-module $\mathcal{M}$.

(ii) A decreasing filtration $\mathrm{Fil}^r\mathcal{M}$ ($r \in \mathbb{N} \cap [0, p-2]$) by $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$-submodules such that $\mathrm{Fil}^0\mathcal{M} = \mathcal{M}$ and $\mathrm{Fil}^r\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}) \cdot \mathrm{Fil}^s\mathcal{M} \subset \mathrm{Fil}^{r+s}\mathcal{M}$ for every $r, s \in \mathbb{N} \cap [0, p-2]$ with $r + s \leq p-2$.

(iii) $\varphi_{\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})}$-semilinear endomorphisms $\varphi_r : \mathrm{Fil}^r\mathcal{M} \to \mathcal{M}$ ($r \in \mathbb{N} \cap [0, p-2]$) such that $\varphi_r|_{\mathrm{Fil}^{r+1}\mathcal{M}} = p\varphi_{r+1}$ for every $r \in \mathbb{N} \cap [0, p-3]$.

A morphism is an $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$-linear homomorphism compatible with Fil^r and φ^r ($r \in \mathbb{N} \cap [0, p-2]$) in the obvious sense. We write $\mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathcal{M}, \mathcal{M}')$ for the set of morphisms $\mathcal{M} \to \mathcal{M}'$ in $\widetilde{\mathrm{MF}}_{[0,p-2]}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}), \varphi)$ to simplify the notation.

Let $\mathcal{M} = (\mathcal{M}, \mathrm{Fil}^r\mathcal{M}, \varphi)$ be an object of $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}), \varphi)$. For $m \in \mathbb{N}_{>0}$, the $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$-module $\mathcal{M}/p^m$ endowed with the filtration defined by the images of the injective homomorphisms $\mathrm{Fil}^r\mathcal{M}/p^m\mathrm{Fil}^r\mathcal{M} \hookrightarrow \mathcal{M}/p^m\mathcal{M}$ ($r \in \mathbb{N} \cap [0, p-2]$) and the reduction mod p^m of $p^{-r}\varphi_{\mathrm{Fil}^r\mathcal{M}} : \mathrm{Fil}^r\mathcal{M} \to \mathcal{M}$ for $r \in \mathbb{N} \cap [0, p-2]$, is an object of $\widetilde{\mathrm{MF}}_{[0,p-2]}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}), \varphi)$. In particular, we may regard $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})/p^m$ as an object of $\widetilde{\mathrm{MF}}_{[0,p-2]}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}), \varphi)$.

Proposition 65 *Let $\mathcal{M}$ be an object of $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}), \varphi)$. Then the $\mathbb{Z}_p$-module $\mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathcal{M}, \underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}))$ is free with the same rank as the $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$-module $\mathcal{M}$, and the following natural homomorphism is an isomorphism.*

$$\mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathcal{M}, \underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}))/p \to \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathcal{M}/p, \underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})/p)$$

By combining Propositions 63 (2) and 65, we obtain the following.

Proposition 66 *For an object M of $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A}, \Phi)$, $T^*_{\mathrm{crys}}(M)$ is a free $\mathbb{Z}_p$-module with the same rank as the $\mathcal{A}$-module M.*

Lemma 67 *Let α be an element of $O_{\overline{K}}$ such that $\alpha^p O_{\overline{K}} = pO_{\overline{K}}$, and let $n \in \mathbb{N}_{>0}$. Let $(\beta_{\mu\nu}) \in GL_n(\overline{\mathcal{A}})$, $\gamma_\mu \in \overline{\mathcal{A}}$ ($\mu \in \mathbb{N} \cap [1, n]$), and $r_\mu \in \mathbb{N} \cap [0, p-2]$ ($\mu \in \mathbb{N} \cap [1, n]$). Put $I := \mathbb{N} \cap [1, n]$. We consider the following equations*

$$x_\nu^p = \alpha^{pr_\nu}\left(\gamma_\nu + \sum_{\mu \in I}\beta_{\mu\nu}x_\mu\right) \quad (\nu \in I), \qquad x_\nu \in \alpha^{r_\nu}\overline{\mathcal{A}} \quad (\nu \in I). \tag{58}$$

For $m \in \mathbb{N}$, the solutions of the equations (58) modulo $\alpha^{p(r_\nu+1)+m}$ are determined modulo α^{p+m}, and each such solution considered modulo α^{p+m} has a unique lifting to the solution of (58). Furthermore the equations (58) have p^n solutions.

Proof (cf. [10, Proof of Theorem 2.4]) For $r \in \mathbb{N} \cap [0, p-2]$, $m \in \mathbb{N}$, $a \in \overline{\mathcal{A}}$, and $b \in \overline{\mathcal{A}}$, we have

$$(\alpha^r a + \alpha^{p+m} b)^p \equiv \alpha^{pr} a^p \mod \alpha^{p(r+1)+m+1}\overline{\mathcal{A}}.$$

This implies the first claim. Let us prove the second claim. Let $\mathcal{A}'$ be a normalization of $\mathcal{A}$ in a finite normal $\mathcal{A}_K$-subalgebra of $\overline{\mathcal{A}}_K$ such that $\alpha, \gamma_\nu, \beta_{\mu\nu} \in \mathcal{A}'$. Since $\overline{\mathcal{A}}$ is a union of such $\mathcal{A}'$, $\mathcal{A}'$ is p-adically complete and separated, and $\mathcal{A}'/\alpha^r \to \overline{\mathcal{A}}/\alpha^r$ ($r \in \mathbb{N}_{>0}$) is injective, it suffices to prove that a solution $x_\nu = \overline{a}_\nu \in \mathcal{A}'/\alpha^{p+m}$ ($\nu \in I$) of (58) modulo $\alpha^{p(r_\nu+1)+m}$ has a unique lifting to a solution in $\mathcal{A}'/\alpha^{p+m+1}$ of (58) modulo $\alpha^{p(r_\nu+1)+m+1}$. Choose a lifting $a_\nu \in \mathcal{A}'$ of $\overline{a}_\nu$. Let $c_\nu \in \mathcal{A}'$ ($\nu \in I$). By using the above congruence, we see that $x_\nu = a_\nu + \alpha^{p+m} c_\nu$ ($\nu \in I$) is a solution of (58) modulo $\alpha^{p(r_\nu+1)+m+1}$ if and only if

$$\alpha^{pr_\nu+p+m} \sum_{\mu \in I} \beta_{\mu\nu} c_\mu \equiv a_\nu^p - \alpha^{pr_\nu}(\gamma_\nu + \sum_{\mu \in I} \beta_{\mu\nu} a_\mu) \qquad \mod \alpha^{p(r_\nu+1)+m+1}\mathcal{A}'.$$

Since the right-hand side is contained in $\alpha^{p(r_\nu+1)+m}\mathcal{A}'$ and $(\beta_{\mu\nu})$ is invertible, these equations for c_ν have a unique solution modulo α.

Let us prove the last claim. Put $f_\nu = X_\nu^p - \alpha^{pr_\nu}(\gamma_\nu + \sum_{\mu \in I} \beta_{\mu\nu} X_\mu)$ for $\nu \in I$. Choose an $\mathcal{A}$-subalgebra $\mathcal{A}'$ of $\overline{\mathcal{A}}$ as above, and let B be the finite algebra $\mathcal{A}'[X_1, \ldots, X_n]/(f_\nu, \nu \in I)$, which is free of rank p^n as an $\mathcal{A}'$-module, and let $\overline{X}_\nu$ be the image of X_ν in B. Since $\overline{X}_\nu^p \in \alpha^{pr_\nu} B$ and $\mathcal{A}'$ is p-adically complete and separated, we see that the matrix

$$\left(\frac{\partial f_\nu}{\partial X_\mu}(\overline{X}_1, \ldots, \overline{X}_n)\right)_{\mu\nu} = (p\alpha^{-pr_\nu}\overline{X}_\nu^{p-1}\delta_{\mu\nu} - \beta_{\mu\nu})_{\mu\nu} \cdot (\alpha^{pr_\nu}\delta_{\mu\nu})_{\mu\nu}$$

is invertible in $B_K = B \otimes_{O_K} K$. Hence B_K is a finite étale $\mathcal{A}'_K$-algebra, and $B_K \otimes_{\mathcal{A}'_K} \overline{\mathcal{A}}_K$ is isomorphic to the product of p^n-copies of $\overline{\mathcal{A}}_K$ by the definition of $\overline{\mathcal{A}}$. Hence the equations (58) have p^n solutions in $\overline{\mathcal{A}}_K$. Since B is finite over $\mathcal{A}'$, each solution $a_\nu \in \overline{\mathcal{A}}_K$ is contained in $\overline{\mathcal{A}}$ and therefore $a_\nu \in \alpha^{r_\nu}\overline{\mathcal{A}}$ by the equations (58). □

Proof of Proposition 65 (cf. [10, Proof of Theorem 2.4]). For $m \in \mathbb{N}_{>0}$, let T_m be $\mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathcal{M}/p^m, \underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})/p^m)$. Then we have an exact sequence $0 \to T_1 \to T_{m+1} \xrightarrow{\pi_m} T_m$ because $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$ and $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})/\mathrm{Fil}^r \cong A_{\mathrm{inf}}(\overline{\mathcal{A}})/\mathrm{Fil}^r$ ($r \in \mathbb{N} \cap [0, p-2]$) are p-torsion free. Hence it suffices to prove (i) $\dim_{\mathbb{F}_p} T_1 = \mathrm{rank}_{\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})} \mathcal{M}$, and (ii) the homomorphism π_m above is surjective for every m.

Put $n := \mathrm{rank}_{\underline{A}_{\inf}(\overline{\mathcal{A}})}\mathcal{M}$ and $I := \mathbb{N}\cap[1,n]$. Choose $e_\nu \in \mathcal{M}$, $r_\nu \in \mathbb{N}\cap[0,p-2]$ ($\nu\in I$) and $(a_{\mu\nu})\in GL_n(\underline{A}_{\inf}(\overline{\mathcal{A}}))$ such that $\mathrm{Fil}^r\mathcal{M} = \oplus_{\nu\in I}\mathrm{Fil}^{r-r_\nu}\underline{A}_{\inf}(\overline{\mathcal{A}})e_\nu$ and $\varphi(e_\nu) = p^{r_\nu}\sum_{\mu\in I}a_{\mu\nu}e_\mu$. Since $(\underline{\varepsilon}-1)^{p-1}\in \underline{p}^pR^\times_{O_{\overline{K}}}$, we have isomorphisms

$$\underline{A}_{\inf}(\overline{\mathcal{A}})/p\underline{A}_{\inf}(\overline{\mathcal{A}}) \cong R_{\overline{\mathcal{A}}}/(\underline{\varepsilon}-1)^{p-1}R_{\overline{\mathcal{A}}} \xrightarrow{\cong} \overline{\mathcal{A}}/p\overline{\mathcal{A}},$$

where the second isomorphism is given by the projection to the second component $R_{\overline{\mathcal{A}}}\to\overline{\mathcal{A}}/p\overline{\mathcal{A}}$. Let ι denote the composition of the isomorphisms above, and let $\alpha\in O_{\overline{K}}$ be a lifting of $\iota(q') = \sum_{a\in\mathbb{F}_p}\varepsilon_2^{[a]}\in O_{\overline{K}}/pO_{\overline{K}}$. Then we have $\alpha^pO_{\overline{K}} = pO_{\overline{K}}$ as $v_p(\sum_{a\in\mathbb{F}_p}\varepsilon_2^{[a]}) = p^{-1}$ ([18, Example A2.7]), and ι induces an isomorphism $\mathrm{Fil}^r\underline{A}_{\inf}(\overline{\mathcal{A}})/p \xrightarrow{\cong} \alpha^r\overline{\mathcal{A}}/p\overline{\mathcal{A}}$ for $r\in\mathbb{N}\cap[p-2]$. Recall that we have $\varphi(q') = q = p$ in $\underline{A}_{\inf}(\overline{\mathcal{A}})$ (Lemma 9 (2)) and $\mathrm{Fil}^r\underline{A}_{\inf}(\overline{\mathcal{A}}) = q'^r\underline{A}_{\inf}(\overline{\mathcal{A}})$ ($r\in\mathbb{N}\cap[p-2]$). This implies

$$\varphi_r(a(q')^r) = \varphi(a) \quad (a\in\underline{A}_{\inf}(\overline{\mathcal{A}}),\ r\in\mathbb{N}\cap[0,p-1]). \tag{$*$}$$

Proof of (i): Let f be an $\underline{A}_{\inf}(\overline{\mathcal{A}})/p$-linear map $\mathcal{M}/p\to\underline{A}_{\inf}(\overline{\mathcal{A}})/p$. Then f is contained in T_1 if and only if $f(e_\nu)\in(q')^{r_\nu}\underline{A}_{\inf}(\overline{\mathcal{A}})/p$ ($\nu\in I$) and $\varphi_{r_\nu}(f(e_\nu)) = \sum_{\mu\in I}a_{\mu\nu}f(e_\mu)$. If we put $x_\nu = \iota\circ f(e_\nu)$ ($\nu\in I$), then by ($*$), this is equivalent to $x_\nu\in\alpha^{r_\nu}\overline{\mathcal{A}}/p\overline{\mathcal{A}}$ and $\widetilde{x}_\nu^p = \alpha^{pr_\nu}\sum_{\mu\in I}\iota(a_{\mu\nu})x_\mu$ in $\overline{\mathcal{A}}/\alpha^{p(r_\nu+1)}\overline{\mathcal{A}}$ for every $\nu\in I$, where $\widetilde{x}_\nu$ is a lifting of x_ν in $\overline{\mathcal{A}}$. By applying Lemma 67 to $\gamma_\mu = 0$ and a lifting $(\beta_{\mu\nu})\in GL_n(\overline{\mathcal{A}})$ of $(\iota(a_{\mu\nu}))$, we see that the above equations have p^n solutions.

Proof of (ii): Let f be an element of T_m, and let $\widetilde{f}$ be an $\underline{A}_{\inf}(\overline{\mathcal{A}})$-linear homomorphism $\mathcal{M}/p^{m+1}\to\underline{A}_{\inf}(\overline{\mathcal{A}})/p^{m+1}$ whose reduction mod p^m is f and $\widetilde{f}(e_\nu)\in\mathrm{Fil}^{r_\nu}\underline{A}_{\inf}(\overline{\mathcal{A}})/p^{m+1}$. Let $[p^m]$ denote the injective homomorphism $\underline{A}_{\inf}(\overline{\mathcal{A}})/p\to\underline{A}_{\inf}(\overline{\mathcal{A}})/p^{m+1}$ induced by the multiplication by p^m on $\underline{A}_{\inf}(\overline{\mathcal{A}})$. Then we have $\varphi_{r_\nu}(\widetilde{f}(e_\nu)) - \sum_{\mu\in I}a_{\mu\nu}\widetilde{f}(e_\mu)\in[p^m](\underline{A}_{\inf}(\overline{\mathcal{A}})/p)$. Let $\overline{\gamma}_\nu\in\overline{\mathcal{A}}/p\overline{\mathcal{A}}$ be its image under $\iota\circ[p^m]^{-1}$. Let $x_\nu\in\overline{\mathcal{A}}/p\overline{\mathcal{A}}$ and define the $\underline{A}_{\inf}(\overline{\mathcal{A}})$-linear homomorphism $\widetilde{f}'\colon \mathcal{M}/p^{m+1}\to\underline{A}_{\inf}(\overline{\mathcal{A}})/p^{m+1}$ by $\widetilde{f}'(e_\nu) = \widetilde{f}(e_\nu) - [p^m]\circ\iota^{-1}(x_\nu)$. Since $\underline{A}_{\inf}(\overline{\mathcal{A}})/\mathrm{Fil}^r\underline{A}_{\inf}(\overline{\mathcal{A}})$ is p-torsion free, $\widetilde{f}'(e_\nu)\in\mathrm{Fil}^{r_\nu}\underline{A}_{\inf}(\overline{\mathcal{A}})/p^{m+1}$ if and only if $\iota^{-1}(x_\mu)\in\mathrm{Fil}^{r_\nu}A_{\inf}(\overline{\mathcal{A}})/p$, which is equivalent to $x_\nu\in\alpha^{r_\nu}\overline{\mathcal{A}}/p\overline{\mathcal{A}}$. If $x_\nu = \alpha^{r_\nu}\cdot(y_\nu \bmod p)$ ($y_\nu\in\overline{\mathcal{A}}$), then we have $\varphi_{r_\nu}(\widetilde{f}'(e_\nu)) = \varphi_{r_\nu}(\widetilde{f}(e_\nu)) - [p^m]\circ\iota^{-1}(y_\nu^p \bmod p)$ by ($*$). Since $\sum_{\mu\in I}a_{\mu\nu}\widetilde{f}'(e_\mu) = \sum_{\mu\in I}a_{\mu\nu}\widetilde{f}(e_\mu) - [p^m]\circ\iota^{-1}(\sum_{\mu\in I}\iota(a_{\mu\nu})x_\mu)$, we see that $\widetilde{f}'$ belongs to T_{m+1} if and only if $x_\nu\in\alpha^{r_\nu}\overline{\mathcal{A}}/p\overline{\mathcal{A}}$ and $\widetilde{x}_\nu^p = \alpha^{pr_\nu}(\overline{\gamma}_\nu + \sum_{\mu\in I}\iota(a_{\mu\nu})x_\mu)$ in $\overline{\mathcal{A}}/\alpha^{p(r_\nu+1)}\overline{\mathcal{A}}$ for every $\nu\in I$, where $\widetilde{x}_\nu$ is a lifting of x_ν in $\overline{\mathcal{A}}$. By applying Lemma 67 to a lifting $\gamma_\nu\in\overline{\mathcal{A}}$ of $\overline{\gamma}_\nu$ and a lifting $(\beta_{\mu\nu})\in GL_n(\overline{\mathcal{A}})$ of $(\iota(a_{\mu\nu}))$, we see that the above equations have p^n solutions. □

For an object $\mathcal{M}$ of $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\inf}(\overline{\mathcal{A}}),\varphi)$, we define $\underline{T}^*_{\inf}(\mathcal{M})$ by

$$\underline{T}^*_{\inf}(\mathcal{M}) := \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathcal{M},\underline{A}_{\inf}(\overline{\mathcal{A}})). \tag{59}$$

For an object $\mathcal{M} = (\mathcal{M}, \mathrm{Fil}^r\mathcal{M}, \varphi_{\mathcal{M}})$ of $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}), \varphi)$, we define an object $\mathcal{M}^* = (\mathcal{M}^*, \mathrm{Fil}^r\mathcal{M}^*, \varphi_{\mathcal{M}^*})$ of $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}), \varphi)$ as follows. The underlying $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$-module is $\mathrm{Hom}_{\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})}(\mathcal{M}, \underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}))$. The decreasing filtration is defined by

$$\mathrm{Fil}^r\mathcal{M}^* = \{f \in \mathcal{M}^* \mid f(\mathrm{Fil}^{(p-2)-r+s}\mathcal{M}) \subset \mathrm{Fil}^s\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}) \text{ for all } s \in \mathbb{N} \cap [0, p-2]\}$$

for $r \in \mathbb{N} \cap [0, p-2]$, where $\mathrm{Fil}^r\mathcal{M} = \mathrm{Fil}^{r-(p-2)}\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}) \cdot \mathrm{Fil}^{p-2}\mathcal{M}$ for $r \in \mathbb{N}_{\geq p-2}$. By taking the dual of the $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$-linearization $\Phi_{\mathcal{M}}\colon \varphi^*(\mathcal{M}) \to \mathcal{M}$ of $\varphi_{\mathcal{M}}$, we obtain an $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$-linear homomorphism $\Psi_{\mathcal{M}^*}\colon \mathcal{M}^* \to \varphi^*(\mathcal{M}^*)$. Since $\Phi_{\mathcal{M}}$ is injective and its cokernel is annihilated by p^{p-2}, the same holds for $\Psi_{\mathcal{M}^*}$. Hence there exists a unique $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$-linear homomorphism $\Phi_{\mathcal{M}^*}\colon \varphi^*(\mathcal{M}^*) \to \mathcal{M}$ such that $\Phi_{\mathcal{M}^*} \circ \Psi_{\mathcal{M}^*} = p^{p-2} \cdot \mathrm{id}$ and $\Psi_{\mathcal{M}^*} \circ \Phi_{\mathcal{M}^*} = p^{p-2} \cdot \mathrm{id}$. We can verify that $\mathcal{M}^*$ with $\mathrm{Fil}^r\mathcal{M}^*$ and the $\varphi_{\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})}$-semilinear endomorphism $\varphi_{\mathcal{M}^*}$ induced by $\Phi_{\mathcal{M}^*}$ is an object of $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}), \varphi)$ as follows: Choose $e_\nu \in \mathcal{M}$ ($\nu \in \mathbb{N} \cap [1, N]$), $r_\nu \in \mathbb{N} \cap [0, p-2]$ ($\nu \in \mathbb{N} \cap [1, N]$) and $P = (p_{\nu\mu}) \in GL_N(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}))$ such that $\mathrm{Fil}^r\mathcal{M} = \oplus_\nu \mathrm{Fil}^{r-r_\nu}\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})e_\nu$ ($r \in \mathbb{N} \cap [0, p-2]$) and $\varphi(e_\mu) = p^{r_\mu}\sum_\nu p_{\nu\mu}e_\nu$. Let $e^*_\nu \in \mathcal{M}^*$ be the dual basis of e_ν. Put $P^* = (p^*_{\nu\mu}) = ({}^tP)^{-1}$ and $r^*_\nu = p-2-r_\nu$.

Let us determine $\mathrm{Fil}^r\mathcal{M}^*$. We have $\mathrm{Fil}^r\mathcal{M} = \oplus_{\nu\in\mathbb{N}\cap[1,N]}\mathrm{Fil}^{r-r_\nu}\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})e_\nu$ for all $r \in \mathbb{N}$. For $f \in \mathcal{M}^*$ and $r \in \mathbb{N} \cap [0, p-2]$, we assert the following: The image of $\mathrm{Fil}^{(p-2)-r+s-r_\nu}\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})e_\nu$ under f is contained in $\mathrm{Fil}^s\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$ for every $s \in \mathbb{N} \cap [0, p-2]$ if and only if $f(e_\nu) \in \mathrm{Fil}^{r-r^*_\nu}\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$. The sufficiency follows from $\{(p-2)-r+s-r_\nu\} + (r-r^*_\nu) = s$. The necessity is trivial if $r - r^*_\nu \leq 0$. If $r - r^*_\nu > 0$, it follows from the condition for $s = r - r^*_\nu \in \mathbb{N} \cap [0, p-2]$, for which $(p-2) - r + s - r_\nu = 0$. Thus we obtain

$$\mathrm{Fil}^r\mathcal{M}^* = \oplus_{\nu\in\mathbb{N}\cap[1,N]}\mathrm{Fil}^{r-r^*_\nu}\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})e^*_\nu. \tag{60}$$

Next let us give an explicit description of φ of $\mathcal{M}^*$. The homomorphism $\psi_{\mathcal{M}^*}$ is represented by the matrix ${}^t(p^{r_\mu}p_{\nu\mu})_{\nu\mu} = (p^{r_\nu}p_{\mu\nu})_{\nu\mu}$ with respect to the bases $(\varphi^*(e^*_\nu))_\nu$ and $(e^*_\nu)_\nu$ of $\varphi^*(\mathcal{M}^*)$ and $\mathcal{M}^*$, and we have $(p^{r_\nu}p_{\mu\nu})_{\nu\mu} \cdot (p^{r^*_\mu}p^*_{\nu\mu})_{\nu\mu} = \mathrm{diag}(p^{r_\nu+r^*_\nu}) = p^{p-2} \cdot 1_N$. Hence we have

$$\varphi_{\mathcal{M}^*}(e^*_\mu) = p^{r^*_\mu}\sum_{\nu\in\mathbb{N}\cap[1,N]}p^*_{\nu\mu}e^*_\nu. \tag{61}$$

We define the object $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})(p-2)$ of $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}), \varphi)$ to be $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$ with $\mathrm{Fil}^r\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}) = \underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$ ($r \in \mathbb{N} \cap [0, p-2]$) and $\varphi_{\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})(p-2)} = p^{p-2}\varphi_{\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})}$. We see that the image of $e = \sum_\nu e_\nu \otimes e^*_\nu \in \mathcal{M} \otimes_{\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})} \mathcal{M}^*$ under $\varphi_{\mathcal{M}} \otimes \varphi_{\mathcal{M}^*}$ is $p^{p-2}e$ by using the above description of $\varphi_{\mathcal{M}}$ and $\varphi_{\mathcal{M}^*}$. Hence, for $f \in \underline{T}^*_{\mathrm{inf}}(\mathcal{M})$ and $g \in \underline{T}^*_{\mathrm{inf}}(\mathcal{M}^*)$, the composition of $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}) \xrightarrow{-\cdot\mathrm{id}_{\mathcal{M}}} \mathrm{Hom}_{\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})}(\mathcal{M}, \mathcal{M}) \cong \mathcal{M} \otimes_{\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})}$

$\mathcal{M}^* \xrightarrow{f\otimes g} \underline{A}_{\inf}(\overline{\mathcal{A}})$ belongs to $\underline{T}^*_{\inf}(\underline{A}_{\inf}(\overline{\mathcal{A}})(p-2))$. Therefore this construction defines a $\mathbb{Z}_p$-bilinear map

$$\underline{T}^*_{\inf}(\mathcal{M}) \times \underline{T}^*_{\inf}(\mathcal{M}^*) \longrightarrow \underline{T}^*_{\inf}(A_{\inf}(\overline{\mathcal{A}})(p-2)). \tag{62}$$

We define $t_1, \dots, t_d \in A^\times$ and $\varphi_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}$ as before Proposition 60. Let $\alpha\colon \mathcal{A} \to A_{\inf}(\overline{\mathcal{A}})$ be the homomorphism $\beta^{(0)}$ (defined before Lemma 34) for $(\mathcal{B}, s_1, \dots, s_e) = (\mathcal{A}, t_1, \dots, t_d)$. Then the composition of α with the homomorphism $A_{\inf}(\overline{\mathcal{A}}) \to A_{\inf}(\overline{\mathcal{A}})/I^1A_{\inf}(\overline{\mathcal{A}})$ coincides with $\overline{\alpha}$ (57). We have $\varphi\circ\alpha = \alpha\circ\varphi_{\mathcal{A}}$ by Lemma 34 (2).

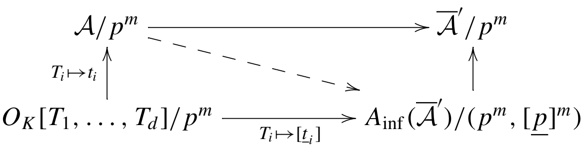
Proposition 68 (cf. [10, Proof of Theorem 2.6*]) *Let $\mathcal{M}$ be an object of $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\inf}(\overline{\mathcal{A}}), \varphi)$ isomorphic to $M \otimes_{\mathcal{A},\overline{\alpha}} \underline{A}_{\inf}(\overline{\mathcal{A}})$ for some object M of $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\mathcal{A}, \varphi)$. Then the paring (62) is perfect.*

As mentioned in the proof of [10, Theorem 2.6*], we can prove Proposition 68 by reducing it to the case of complete discrete valuation ring with algebraically closed residue field. Let $\mathfrak{p}$ be the prime ideal $p\mathcal{A}$ of $\mathcal{A}$. For each $i \in \mathbb{N}\cap[1,d]$, we choose a compatible system of p^nth roots $t_{i,n} \in \overline{\mathcal{A}}^\times$ of t_i: $t_{i,n+1}^p = t_{i,n}$ $(n \in \mathbb{N})$, $t_{i,0} = t_i$. Then $\mathcal{A}_{\mathfrak{p},\infty} := \mathcal{A}_{\mathfrak{p}}[t_{1,n}, \dots, t_{d,n}; n \in \mathbb{N}] (\subset \mathcal{K}^{\mathrm{ur}})$ is a discrete valuation ring with perfect residue field. Let $\mathcal{A}'$ be the p-adic completion of the maximal unramified extension of the p-adic completion of $\mathcal{A}_{\mathfrak{p},\infty}$. Put $\mathcal{K}' := \mathcal{A}'[\frac{1}{p}]$, let $\overline{\mathcal{K}}'$ be an algebraic closure of $\mathcal{K}'$, and let $\overline{\mathcal{A}}'$ be the integral closure of $\mathcal{A}'$ in $\overline{\mathcal{K}}'$. Choose an extension $\mathcal{K}^{\mathrm{ur}} \to \overline{\mathcal{K}}'$ of $\mathcal{A}_\infty := \mathcal{A}[t_{1,n}, \dots, t_{d,n}; n \in \mathbb{N}] \subset \mathcal{A}_{\mathfrak{p},\infty}$. This induces homomorphisms $\overline{\mathcal{A}} \to \overline{\mathcal{A}}'$ and $A_{\inf}(\overline{\mathcal{A}}) \to A_{\inf}(\overline{\mathcal{A}}')$. Since $\mathcal{A}'$ is canonically isomorphic to the Witt rings of its residue field, there is a canonical homomorphism $\alpha'\colon \mathcal{A}' \cong W(\mathcal{A}'/p) \to A_{\inf}(\overline{\mathcal{A}}') = W(R_{\overline{\mathcal{A}}'})$ induced by $\mathcal{A}'/p \to R_{\overline{\mathcal{A}}'}; x \mapsto (x^{p^{-n}})_{n\in\mathbb{N}}$, and $\mathcal{A}'$ has a canonical lifting of the absolute Frobenius $\varphi_{\mathcal{A}'}$. The homomorphism α' is compatible with φ's.

Lemma 69 (1) *The homomorphisms α' and $\alpha\colon \mathcal{A} \to A_{\inf}(\overline{\mathcal{A}})$ is compatible with $\mathcal{A} \to \mathcal{A}'$ and $A_{\inf}(\overline{\mathcal{A}}) \to A_{\inf}(\overline{\mathcal{A}}')$.*
(2) *The homomorphism $\mathcal{A} \to \mathcal{A}'$ is compatible with $\varphi_{\mathcal{A}}$ and $\varphi_{\mathcal{A}'}$.*

Proof We define $\underline{t}_i \in R_{\overline{\mathcal{A}}}$ to be $(t_{i,n} \mod p)_{n\in\mathbb{N}}$ for $i \in \mathbb{N}\cap[1,d]$. Let k' be the residue field $\mathcal{A}'/p\mathcal{A}'$ of $\mathcal{A}'$. Then the isomorphism $W(k') \cong \mathcal{A}'$ sends $[t_i]$ to $\lim_{n\to\infty}(t_{i,n})^{p^n} = t_i$. Hence we have $\alpha'(t_i) = [\underline{t}_i]$ and $\varphi_{\mathcal{A}'}(t_i) = t_i^p$.

(1) The above observation implies that the following diagram of O_K-algebras is commutative for both $\mathcal{A}/p^m \to \mathcal{A}'/p^m \xrightarrow{\alpha'} A_{\inf}(\overline{\mathcal{A}}')/(p^m, [\underline{p}]^m)$ and $\mathcal{A}/p^m \xrightarrow{\alpha} A_{\inf}(\overline{\mathcal{A}})/p^m \to A_{\inf}(\overline{\mathcal{A}}')/(p^m, [\underline{p}]^m)$.



This implies the claim because the left vertical homomorphism is étale and $A_{\mathrm{inf}}(\overline{\mathcal{A}}')$ is $(p, [\underline{p}])$-adically complete and separated (Lemma 1 (2)).

(2) The following diagram is commutative for both $\mathcal{A}/p^m \xrightarrow{\varphi_{\mathcal{A}}} \mathcal{A}/p^m \to \mathcal{A}'/p^m$ and $\mathcal{A}/p^m \to \mathcal{A}'/p^m \xrightarrow{\varphi_{\mathcal{A}'}} \mathcal{A}'/p^m$, where the top (resp. bottom) horizontal map is defined by the composition of $\mathcal{A}/p^m \xrightarrow{\mathrm{pr}} \mathcal{A}/p \to \mathcal{A}'/p$ with the absolute Frobenius of $\mathcal{A}'/p$ (resp. $T_i \mapsto t_i^p$ and $\sigma\colon O_K \to O_K$).

$$\begin{array}{ccc} \mathcal{A}/p^m & \longrightarrow & \mathcal{A}'/p \\ {\scriptstyle T_i\mapsto t_i}\uparrow & \searrow & \uparrow{\scriptstyle \mathrm{pr}} \\ O_K[T_1,\ldots,T_d]/p^m & \longrightarrow & \mathcal{A}'/p^m \end{array}$$

Since the left vertical homomorphism is étale, the two maps are the same. □

Proof of Proposition 68 One can construct a pairing

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathcal{M}/p, \underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})/p) \times \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathcal{M}^*/p, \underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})/p) \\ &\qquad \longrightarrow \mathrm{Hom}_{\mathrm{Fil},\varphi}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})(p-2)/p, \underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})/p) \end{aligned} \tag{63}$$

in the same way as (62), i.e., by sending (f, g) to the composition of

$$\begin{aligned} \underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})(p-2)/p \xrightarrow{-\cdot \mathrm{id}_{\mathcal{M}/p}} \mathrm{End}_{\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})/p}(\mathcal{M}/p) \cong \mathcal{M}/p \otimes_{\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})/p} \mathcal{M}^*/p \\ \xrightarrow{f\otimes g} \underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})/p. \end{aligned}$$

We can verify that the composition belongs to $\mathrm{Hom}_{\mathrm{Fil},\varphi}$ by using $f(\overline{e}_\nu) \in \mathrm{Fil}^{r_\nu} \underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})/p$, $g(\overline{e}_\nu^*) \in \mathrm{Fil}^{r_\nu^*} \underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})/p$ and $\varphi_{p-2}((f \otimes g)(\overline{e}_\nu \otimes \overline{e}_\nu^*)) = (f \otimes g)(\varphi_{r_\nu}(\overline{e}_\nu) \otimes \varphi_{r_\nu^*}(\overline{e}_\nu^*))$, where we choose e_ν and e_ν^* as before (60) and set $\overline{e}_\nu = (e_\nu \mod p)$ and $\overline{e}_\nu^* = (e_\nu^* \mod p)$. The pairing (63) is obviously compatible with (62). By Proposition 65, it suffices to prove that the pairing (63) is perfect.

For an object $\mathcal{N}$ of $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}), \varphi)$ and the object $\mathcal{N}' = \mathcal{N} \otimes_{\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})} \underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}')$ of $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}'), \varphi)$, the homomorphism

$$\mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathcal{N}/p, \underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})/p) \longrightarrow \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathcal{N}'/p, \underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}')/p) \tag{64}$$

is an isomorphism by the description of elements of the source (resp. target) in terms of sections of a finite étale $\overline{\mathcal{A}}_K$ (resp. $\overline{\mathcal{A}}'_K$)-algebra given by the proofs of Proposition 65 and Lemma 67.

Let $\mathcal{M}'$ be the object $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}') \otimes_{\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})} \mathcal{M}$ of $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}'), \varphi)$. Then, by using (60) and (61), we see that the canonical $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}')$-linear isomorphism $(\mathcal{M}')^* \cong \underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}') \otimes_{\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})} \mathcal{M}^*$ gives an isomorphism in $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}}'), \varphi)$, and the paring (63) for $\mathcal{M}$ is compatible with that for $\mathcal{M}'$ via the isomorphisms (64) for $\mathcal{M}$

and $\mathcal{M}^*$. By Lemma 69 (2), $M' := M \otimes_{\mathcal{A}} \mathcal{A}'$ has a natural structure as an object of $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\mathcal{A}', \varphi)$, and by Lemma 69 (1), we have an isomorphism $M' \otimes_{\mathcal{A}',\alpha'} \underline{A}_{\inf}(\overline{\mathcal{A}}') \cong \mathcal{M}'$ in $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\inf}(\overline{\mathcal{A}}'), \varphi)$. Hence, by replacing $O_K \to A$, $\mathcal{M}$, and M with $\mathcal{A}' \xrightarrow{\mathrm{id}} \mathcal{A}'$, $\mathcal{M}'$ and M', we may assume that $O_K = A = \mathcal{A}$ and the residue field k of O_K is algebraically closed.

Let $\widetilde{\mathrm{MF}}_{[0,p-2]}(k, \varphi)$ be the category defined by replacing $(\underline{A}_{\inf}(\overline{\mathcal{A}}), \mathrm{Fil}^\bullet \underline{A}_{\inf}(\overline{\mathcal{A}}))$ and $\varphi_{\underline{A}_{\inf}(\overline{\mathcal{A}})}$ with $(k, 0)$ and the absolute Frobenius of k in the definition of $\widetilde{\mathrm{MF}}_{[0,p-2]}(\underline{A}_{\inf}(\overline{\mathcal{A}}), \varphi)$ given in the beginning of Sect. 9, and let $\mathrm{MF}_{[0,p-2]}(k, \varphi)$ be its full subcategory consisting of objects N satisfying $\dim_k N < \infty$ and $N = \sum_{r\in\mathbb{N}\cap[0,p-2]} \varphi_r(\mathrm{Fil}^r N)$. It is known that the category $\mathrm{MF}_{[0,p-2]}(k, \varphi)$ is an artinian abelian category ([13, 1.8 Proposition]). Let N be an object of the category $\mathrm{MF}_{[0,p-2]}(k, \varphi)$. Then one can prove that $\mathrm{Hom}_{\mathrm{Fil},\varphi}(N, \underline{A}_{\inf}(O_{\overline{K}})/p)$ is an $\mathbb{F}_p$-vector space of dimension equal to the dimension of N over k in the same way as the proof of Proposition 65 (see [20, 2.2.3.1 Examples (b), 2.3.1.2.3 (b) Proposition 1]). This together with the left exactness of $\mathrm{Hom}_k(-, \underline{A}_{\inf}(O_{\overline{K}})/p)$ implies that the functor $\mathrm{Hom}_{\mathrm{Fil},\varphi}(-, \underline{A}_{\inf}(O_{\overline{K}})/p)$ defined on $\mathrm{MF}_{[0,p-2]}(k, \varphi)$ is exact.

For an object N of $\mathrm{MF}_{[0,p-2]}(k, \varphi)$, we define an object N^* of $\mathrm{MF}_{[0,p-2]}(k, \varphi)$ as follows. The underlying k-vector space is $N^* = \mathrm{Hom}_k(N, k)$ and the filtration is defined by $\mathrm{Fil}^r N^* = \{f \in N^* | f(\mathrm{Fil}^{p-1-r} N) = 0\}$ $(r \in \mathbb{Z})$. Then we have a canonical isomorphism $\mathrm{gr}^r_{\mathrm{Fil}} N^* \cong \mathrm{Hom}_k(\mathrm{gr}^{p-2-r}_{\mathrm{Fil}} N, k)$ $(r \in \mathbb{N} \cap [0, p-2])$, and we define the frobenius $\varphi_r\colon \mathrm{Fil}^r N^* \to N^*$ $(r \in \mathbb{N} \cap [p-2])$ by the inverse of the dual of the isomorphism $\Phi\colon \oplus_{s\in\mathbb{N}\cap[0,p-2]} \varphi^*(\mathrm{gr}^s_{\mathrm{Fil}} N) \xrightarrow{\cong} N$ induced by φ_r of N, i.e., the composition of

$$\mathrm{Fil}^r N^* \to \varphi^*(\mathrm{gr}^r_{\mathrm{Fil}} N^*) \hookrightarrow \bigoplus_{s\in\mathbb{N}\cap[0,p-2]} \varphi^*(\mathrm{gr}^s_{\mathrm{Fil}} N^*) \cong \bigoplus_{s\in\mathbb{N}\cap[0,p-2]} (\varphi^*(\mathrm{gr}^{p-2-s}_{\mathrm{Fil}} N))^* \xrightarrow[(\Phi^*)^{-1}]{\cong} N^*.$$

Let $k(p-2)$ be the object of $\mathrm{MF}_{[0,p-2]}(k, \varphi)$ defined by k with $\mathrm{Fil}^{p-2} k = k$ and $\varphi_{p-2}(1) = 1$. Then one can define the following pairing similarly to (63).

$$\mathrm{Hom}_{\mathrm{Fil},\varphi}(N, \underline{A}_{\inf}(O_{\overline{K}})/p) \times \mathrm{Hom}_{\mathrm{Fil},\varphi}(N^*, \underline{A}_{\inf}(O_{\overline{K}})/p) \longrightarrow \mathrm{Hom}_{\mathrm{Fil},\varphi}(k(p-2), \underline{A}_{\inf}(O_{\overline{K}})/p) \quad (65)$$

We define an object M/p of $\mathrm{MF}_{[0,p-2]}(k, \varphi)$ to be the k-vector space M/p equipped with the filtration $\mathrm{Fil}^r M/p$ $(r \in \mathbb{N} \cap [0, p-2])$ and the reduction mod p of $p^{-r}\varphi\colon \mathrm{Fil}^r M \to M$ for $r \in \mathbb{N} \cap [0, p-2]$. We have a natural isomorphism $\mathrm{Hom}_{\mathrm{Fil},\varphi}(L, \underline{A}_{\inf}(O_{\overline{K}})/p) \cong \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathcal{L}, \underline{A}_{\inf}(O_{\overline{K}})/p)$ for $(L, \mathcal{L}) = (M/p, \mathcal{M}/p)$, $((M/p)^*, \mathcal{M}^*/p)$, $(k(p-2), \underline{A}_{\inf}(\overline{\mathcal{A}})(p-2)/p)$. They are compatible with the paring (63) and the pairing (65) for $N = M/p$. Hence it suffices to prove that the pairing (65) for an object N of $\mathrm{MF}_{[0,p-2]}(k, \varphi)$ is perfect. Since $N \mapsto$

$\mathrm{Hom}_{\mathrm{Fil},\varphi}(N, \underline{A}_{\mathrm{inf}}(O_{\overline{K}})/p)$ is exact as observed above and the paring (65) is functorial in N, it suffices to prove the claim when N is a simple object.

By [13, 4.4 Proposition (ii)], there exists $n \in \mathbb{N}_{>0}$, $e_\nu \in N$ and $r_\nu \in \mathbb{N} \cap [0, p-2]$ ($\nu \in \mathbb{Z}/n\mathbb{Z}$) such that $N = \oplus_{\nu \in \mathbb{Z}/n\mathbb{Z}} k e_\nu$, $\mathrm{Fil}^r N = \oplus_{\nu \in \mathbb{Z}/n\mathbb{Z}, r_\nu \geq r} k e_\nu$ and $\varphi_{r_\nu}(e_\nu) = e_{\nu+1}$. Let e_ν^* ($\nu \in \mathbb{Z}/n\mathbb{Z}$) be the dual basis of N^*, and put $r_\nu^* := p - 2 - r_\nu$. Then we have $\mathrm{Fil}^r N^* = \oplus_{\nu \in \mathbb{Z}/n\mathbb{Z}, r_\nu^* \geq r} k e_\nu^*$ and $\varphi_{r_\nu^*}(e_\nu^*) = e_{\nu+1}^*$. As in the proof of Proposition 65, let ι be the composition of $A_{\mathrm{inf}}(O_{\overline{K}})/p \cong R_{O_{\overline{K}}}/(\underline{\varepsilon} - 1)^{p-1} \xrightarrow{\cong} O_{\overline{K}}/p$, and let $\alpha \in O_{\overline{K}}$ be a lifting of $\iota(q')$. Then by the same argument as in the proof of Proposition 65 and Lemma 67, an element f (resp. g) of $\mathrm{Hom}_{\mathrm{Fil},\varphi}(N, \underline{A}_{\mathrm{inf}}(O_{\overline{K}})/p)$ (resp. $\mathrm{Hom}_{\mathrm{Fil},\varphi}(N^*, \underline{A}_{\mathrm{inf}}(O_{\overline{K}})/p)$) is given by $f(e_\nu) = \iota^{-1}(x_\nu \mod p)$ (resp. $g(e_\nu^*) = \iota^{-1}(y_\nu \mod p)$), where (x_ν) (resp. (y_ν)) is a solution of the equations $x_\nu^p = \alpha^{pr_\nu} x_{\nu+1}$, $x_\nu \in \alpha^{r_\nu} O_{\overline{K}}$ (resp. $y_\nu^p = \alpha^{pr_\nu^*} y_{\nu+1}$, $y_\nu \in \alpha^{r_\nu^*} O_{\overline{K}}$). The pairing $h \in \mathrm{Hom}_{\mathrm{Fil},\varphi}(k(p-2), \underline{A}_{\mathrm{inf}}(O_{\overline{K}})/p)$ of f and g is given by $h(1) = \iota^{-1}(\sum_{\nu \in \mathbb{Z}/n\mathbb{Z}} x_\nu y_\nu) \in \underline{A}_{\mathrm{inf}}(O_{\overline{K}})/p$. Let $\alpha_n \in O_{\overline{K}}$ be a $(p^n - 1)$th root of α. Then the solutions of the above equations are given by $x_\nu = \alpha_n^{\sum_{s=0}^{n-1} p^{n-s} r_{\nu+s}} \zeta^{p^\nu}$, $\zeta \in \mu_{p^n-1}(O_{\overline{K}}) \cup \{0\}$ (resp. $y_\nu = \alpha_n^{\sum_{s=0}^{n-1} p^{n-s} r_{\nu+s}^*} \eta^{p^\nu}$, $\eta \in \mu_{p^n-1}(O_{\overline{K}}) \cup \{0\}$). The pairing h above is given by $\iota(h(1)) = \alpha_1^{p(p-2)} \sum_{\nu=0}^{n-1} (\zeta\eta)^{p^\nu}$, where α_1 is the $(p-1)$th root $(\alpha_n)^{(p^n-1)(p-1)^{-1}}$ of α. Since $\mathbb{F}_{p^n}/\mathbb{F}_p$ is a finite separable extension, the trace map $\mathrm{Tr}_{\mathbb{F}_{p^n}/\mathbb{F}_p} : \mathbb{F}_{p^n} \to \mathbb{F}_p$ does not vanish. Hence for each $\zeta \neq 0$, there exists $\eta \neq 0$ such that $\sum_{\nu=0}^{n-1} (\zeta\eta)^{p^\nu} \in O_{\overline{K}}^\times$. This completes the proof because $\alpha_1^{p(p-2)} \notin pO_{\overline{K}}$. □

10 Period Map

Let M be an object of $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A}, \Phi)$. We define $T_{\mathrm{crys}}(M)$ to be the dual $\mathrm{Hom}_{\mathbb{Z}_p}(T^*_{\mathrm{crys}}(M), \mathbb{Z}_p)$ of $T^*_{\mathrm{crys}}(M)$, which is a free $\mathbb{Z}_p$-module whose rank is equal to $\mathrm{rk}_{\mathcal{A}} M$ (Proposition 66). By Theorem 63 (2), we have a $G_{\mathcal{A}}$-equivariant $\mathbb{Z}_p$-linear injective homomorphism $T^*_{\mathrm{crys}}(M) \to \mathrm{Hom}_{A_{\mathrm{inf}}(\overline{\mathcal{A}})}(TA_{\mathrm{inf}}(M), A_{\mathrm{inf}}(\overline{\mathcal{A}}))$. By taking the dual of its $A_{\mathrm{inf}}(\overline{\mathcal{A}})$-linearization, we obtain an $A_{\mathrm{inf}}(\overline{\mathcal{A}})$-linear $G_{\mathcal{A}}$-equivariant homomorphism compatible with φ, where φ of the target is defined by $\mathrm{id} \otimes \varphi_{A_{\mathrm{inf}}(\overline{\mathcal{A}})}$.

$$TA_{\mathrm{inf}}(M) \longrightarrow T_{\mathrm{crys}}(M) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}(\overline{\mathcal{A}}) \tag{66}$$

In this section, we prove the following theorem.

Theorem 70 *The homomorphism (66) is injective and its cokernel is annihilated by* π^{p-2}.

For an object $\mathcal{M}$ of $\mathrm{M}^q_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\overline{\mathcal{A}}), \varphi)$, we define $T^*_{\mathrm{inf}}(\mathcal{M})$ by

$$T^*_{\mathrm{inf}}(\mathcal{M}) := \mathrm{Hom}_{\mathrm{M}^q_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\overline{\mathcal{A}}),\varphi)}(\mathcal{M}, A_{\mathrm{inf}}(\overline{\mathcal{A}})). \tag{67}$$

Proposition 71 *For an object $\mathcal{M}$ of $\mathrm{M}^q_{[0,p-2],\mathrm{free}}(A_{\inf}(\overline{\mathcal{A}}),\varphi)$, $T^*_{\inf}(\mathcal{M})$ is a free $\mathbb{Z}_p$-module whose rank is the same as the free $A_{\inf}(\overline{\mathcal{A}})$-module underlying $\mathcal{M}$.*

Proof The claim immediately follows from Propositions 59 and 65. □

For an object $\mathcal{M}=(\mathcal{M},\varphi_{\mathcal{M}})$ of $\mathrm{M}^q_{[0,p-2],\mathrm{free}}(A_{\inf}(\overline{\mathcal{A}}),\varphi)$, we define an object $\mathcal{M}^*=(\mathcal{M}^*,\varphi_{\mathcal{M}^*})$ of $\mathrm{M}^q_{[0,p-2]}(A_{\inf}(\overline{\mathcal{A}}),\varphi)$ as follows. The underlying $A_{\inf}(\overline{\mathcal{A}})$-module is $\mathrm{Hom}_{A_{\inf}(\overline{\mathcal{A}})}(\mathcal{M},A_{\inf}(\overline{\mathcal{A}}))$. Let $\Psi_{\mathcal{M}^*}\colon \mathcal{M}^*\to\varphi^*(\mathcal{M}^*)$ be the dual of the $A_{\inf}(\overline{\mathcal{A}})$-linearization $\Phi_{\mathcal{M}}\colon\varphi^*(\mathcal{M})\to\mathcal{M}$ of $\varphi_{\mathcal{M}}$. Since $\Phi_{\mathcal{M}}$ is injective and its cokernel is annihilated by q^{p-2}, the same holds for $\Psi_{\mathcal{M}^*}$. Hence there exists a unique $A_{\inf}(\overline{\mathcal{A}})$-linear homomorphism $\Phi_{\mathcal{M}^*}\colon\varphi^*(\mathcal{M}^*)\to\mathcal{M}^*$ such that $\Phi_{\mathcal{M}^*}\circ\Psi_{\mathcal{M}^*}=q^{p-2}\cdot\mathrm{id}$ and $\Psi_{\mathcal{M}^*}\circ\Phi_{\mathcal{M}^*}=q^{p-2}\cdot\mathrm{id}$. We can verify that $\mathcal{M}^*$ with the $\varphi_{A_{\inf}(\overline{\mathcal{A}})}$-semilinear endomorphism $\varphi_{\mathcal{M}^*}$ induced by $\Phi_{\mathcal{M}^*}$ is an object of $\mathrm{M}^q_{[0,p-2]}(A_{\inf}(\overline{\mathcal{A}}),\varphi)$ as follows: Choose $e_\nu\in\mathcal{M}$ ($N\in\mathbb{N}$, $\nu\in\mathbb{N}\cap[1,N]$), $r_\nu\in\mathbb{N}\cap[0,p-2]$ ($\nu\in\mathbb{N}\cap[1,N]$) and $P=(p_{\nu\mu})\in GL_N(A_{\inf}(\overline{\mathcal{A}}))$ such that $\mathcal{M}=\oplus_\nu A_{\inf}(\overline{\mathcal{A}})e_\nu$ and $\varphi_{\mathcal{M}}(e_\mu)=q^{r_\mu}\sum_\nu p_{\nu\mu}e_\nu$. Let $e^*_\nu\in\mathcal{M}^*$ be the dual basis of e_ν. Put $P^*=(p^*_{\nu\mu}):=({}^tP)^{-1}$ and $r^*_\nu=p-2-r_\nu$. Then, by the same argument as before (61), we obtain

$$\varphi_{\mathcal{M}^*}(e^*_\mu)=q^{r^*_\mu}\sum_{\nu\in\mathbb{N}\cap[1,N]}p^*_{\nu\mu}e^*_\nu. \tag{68}$$

We define the object $A_{\inf}(\overline{\mathcal{A}})(p-2)$ of $\mathrm{M}^q_{[0,p-2],\mathrm{free}}(A_{\inf}(\overline{\mathcal{A}}),\varphi)$ to be $A_{\inf}(\overline{\mathcal{A}})$ endowed with the Frobenius defined by $\varphi(1)=q^{p-2}$. We see that the image of $e=\sum_\nu e_\nu\otimes e^*_\nu\in\mathcal{M}\otimes_{A_{\inf}(\overline{\mathcal{A}})}\mathcal{M}^*$ under $\varphi_{\mathcal{M}}\otimes\varphi_{\mathcal{M}^*}$ is $q^{p-2}e$ by using the above description of $\varphi_{\mathcal{M}}$ and $\varphi_{\mathcal{M}^*}$. Hence, for $f\in T^*_{\inf}(\mathcal{M})$ and $g\in T^*_{\inf}(\mathcal{M}^*)$, the composition of $A_{\inf}(\overline{\mathcal{A}})\xrightarrow{-\cdot\mathrm{id}_{\mathcal{M}}}\mathrm{Hom}_{A_{\inf}(\overline{\mathcal{A}})}(\mathcal{M},\mathcal{M})\cong\mathcal{M}\otimes_{A_{\inf}(\overline{\mathcal{A}})}\mathcal{M}^*\xrightarrow{f\otimes g}A_{\inf}(\overline{\mathcal{A}})$ belongs to $T^*_{\inf}(A_{\inf}(\overline{\mathcal{A}})(p-2))$. This construction defines a $\mathbb{Z}_p$-bilinear map

$$T^*_{\inf}(\mathcal{M})\times T^*_{\inf}(\mathcal{M}^*)\to T^*_{\inf}(A_{\inf}(\overline{\mathcal{A}})(p-2)). \tag{69}$$

We define $\varphi_{\mathcal{A}}\colon\mathcal{A}\to\mathcal{A}$ and $\overline{\alpha}\colon\mathcal{A}\to\underline{A}_{\inf}(\overline{\mathcal{A}})$ as before Proposition 60 and Lemma 64.

Proposition 72 *Let $\mathcal{M}$ be an object of $\mathrm{M}^q_{[0,p-2],\mathrm{free}}(A_{\inf}(\overline{\mathcal{A}}),\varphi)$. Suppose that there exist $M\in\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\mathcal{A},\varphi)$ and an isomorphism $M\otimes_{\mathcal{A},\overline{\alpha}}\underline{A}_{\inf}(\overline{\mathcal{A}})\cong\mathcal{M}\otimes_{A_{\inf}(\overline{\mathcal{A}})}\underline{A}_{\inf}(\overline{\mathcal{A}})$ in $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}_{\inf}(\overline{\mathcal{A}}),\varphi)$. Then the paring (69) is perfect.*

Proof Let $\underline{\mathcal{M}}$ be the image of $\mathcal{M}$ under (51). Then the image of $A_{\inf}(\overline{\mathcal{A}})(p-2)$ (resp. $\mathcal{M}^*$) under (51) is canonically isomorphic to $\underline{A}_{\inf}(\overline{\mathcal{A}})(p-2)$ (resp. $\underline{\mathcal{M}}^*$) defined before Proposition 68, the homomorphisms $T^*_{\inf}(\mathcal{N})\to\underline{T}^*_{\inf}(\underline{\mathcal{N}})$ for $(\mathcal{N},\underline{\mathcal{N}})=(\mathcal{M},\underline{\mathcal{M}}),(\mathcal{M}^*,\underline{\mathcal{M}}^*),(A_{\inf}(\overline{\mathcal{A}})(p-2),\underline{A}_{\inf}(\overline{\mathcal{A}})(p-2))$ are isomorphisms by Proposition 59, and these isomorphisms are compatible with the parings (62) and (69). Hence the claim follows from Proposition 68. □

For an object $\mathcal{M}$ of $\mathrm{M}^q_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\overline{\mathcal{A}}),\varphi)$, the inclusion map $T^*_{\mathrm{inf}}(\mathcal{M})\hookrightarrow\mathcal{M}^*$ induces an $A_{\mathrm{inf}}(\overline{\mathcal{A}})$-linear map

$$A_{\mathrm{inf}}(\overline{\mathcal{A}})\otimes_{\mathbb{Z}_p}T^*_{\mathrm{inf}}(\mathcal{M})\longrightarrow\mathcal{M}^*. \tag{70}$$

Lemma 73 *The image of the homomorphism (70)*

$$A_{\mathrm{inf}}(\overline{\mathcal{A}})\otimes_{\mathbb{Z}_p}T^*_{\mathrm{inf}}(A_{\mathrm{inf}}(\overline{\mathcal{A}})(p-2))\longrightarrow A_{\mathrm{inf}}(\overline{\mathcal{A}})(p-2)^*=A_{\mathrm{inf}}(\overline{\mathcal{A}})$$

for $\mathcal{M}=A_{\mathrm{inf}}(\overline{\mathcal{A}})(p-2)$ *is* $I^{p-2}A_{\mathrm{inf}}(\overline{\mathcal{A}})=\pi^{p-2}A_{\mathrm{inf}}(\overline{\mathcal{A}})$.

Proof Let J be an $A_{\mathrm{inf}}(\overline{\mathcal{A}})$-submodule of $A_{\mathrm{inf}}(\overline{\mathcal{A}})$, and let $\underline{J}$ be the image of J in $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$. Then we have $J\subset I^{p-2}A_{\mathrm{inf}}(\overline{\mathcal{A}})$ if and only if $\underline{J}\subset I^{p-2}\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$ because the inverse image of $I^{p-2}\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$ by $A_{\mathrm{inf}}(\overline{\mathcal{A}})\to\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$ is $I^{p-2}A_{\mathrm{inf}}(\overline{\mathcal{A}})$. Suppose that this holds. Then, since $I^{p-2}A_{\mathrm{inf}}(\overline{\mathcal{A}})=A_{\mathrm{inf}}(\overline{\mathcal{A}})\pi^{p-2}$, the homomorphism $I^{p-2}A_{\mathrm{inf}}(\overline{\mathcal{A}})\otimes_{A_{\mathrm{inf}}(\overline{\mathcal{A}})}A_{\mathrm{inf}}(\overline{\mathcal{A}})/I^1A_{\mathrm{inf}}(\overline{\mathcal{A}})\to I^{p-2}\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$ is an isomorphism, and $I^1A_{\mathrm{inf}}(\overline{\mathcal{A}})$ is contained in the Jacobson radical of $A_{\mathrm{inf}}(\overline{\mathcal{A}})$, we see that $J=I^{p-2}A_{\mathrm{inf}}(\overline{\mathcal{A}})$ if and only if $\underline{J}=I^{p-2}\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$ by Nakayama's lemma. As the homomorphism $T^*_{\mathrm{inf}}(A_{\mathrm{inf}}(\overline{\mathcal{A}})(p-2))\to\underline{T}^*_{\mathrm{inf}}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})(p-2))$ is an isomorphism by Proposition 59, it suffice to prove the corresponding claim for $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})(p-2)$. We have $\varphi(\pi)\pi^{-1}=1+[\underline{\varepsilon}]+\cdots+[\underline{\varepsilon}]^{p-1}\equiv p\mod\pi A_{\mathrm{inf}}(\overline{\mathcal{A}})$, which implies $\varphi(\pi^{p-2})\equiv p^{p-2}\pi^{p-2}\mod I^{p-1}A_{\mathrm{inf}}(\overline{\mathcal{A}})$. Since $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})/\pi^{p-2}=A_{\mathrm{inf}}(\overline{\mathcal{A}})/I^{p-2}A_{\mathrm{inf}}(\overline{\mathcal{A}})$ is p-torsion free, we see that the free $\mathbb{Z}_p$-module of rank 1 $\underline{T}^*_{\mathrm{inf}}(\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})(p-2))$ is generated by the $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$-linear map $\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})\to\underline{A}_{\mathrm{inf}}(\overline{\mathcal{A}})$; $1\mapsto\pi^{p-2}$. This completes the proof. □

Proof of Theorem 70 Put $\mathcal{M}:=TA_{\mathrm{inf}}(M)$. We assert that the following diagram is commutative, where we abbreviate $A_{\mathrm{inf}}(\overline{\mathcal{A}})$ to A_{inf}.

$$\begin{array}{ccc}
(A_{\mathrm{inf}}\otimes_{\mathbb{Z}_p}T^*_{\mathrm{inf}}(\mathcal{M}))\times(A_{\mathrm{inf}}\otimes_{\mathbb{Z}_p}T^*_{\mathrm{inf}}(\mathcal{M}^*)) & \xrightarrow{(69)} & A_{\mathrm{inf}}\otimes_{\mathbb{Z}_p}T^*_{\mathrm{inf}}(A_{\mathrm{inf}}(p-2)) \\
\downarrow{\scriptstyle(70)} & & \downarrow{\scriptstyle(70)} \\
\mathcal{M}^*\times\mathcal{M}^{**} & \longrightarrow & (A_{\mathrm{inf}}(p-2))^*=A_{\mathrm{inf}}.
\end{array}$$

Since the two pairings are A_{inf}-bilinear and the vertical maps are A_{inf}-linear, it suffices to prove that the images of $(f,g)\in T^*_{\mathrm{inf}}(\mathcal{M})\times T^*_{\mathrm{inf}}(\mathcal{M}^*)$ under the two compositions coincide. The image of (f,g) under (69) sends $1\in A_{\mathrm{inf}}(p-2)$ to the image of $\mathrm{id}_{\mathcal{M}}$ under $\mathrm{Hom}_{A_{\mathrm{inf}}}(\mathcal{M},\mathcal{M})\xrightarrow{f\circ-}\mathcal{M}^*\xrightarrow{g}A_{\mathrm{inf}}$, which coincides with the image $g(f)$ of $(f,g)\in\mathcal{M}^*\times\mathcal{M}^{**}$ under the lower paring. By Proposition 72 and (37) for $(\mathcal{B},s_1,\ldots,s_e)=(\mathcal{A},t_1,\ldots,t_d)$, the upper pairing is perfect. Hence Lemma 73 and Proposition 71 imply that (70) for $\mathcal{M}$ is injective and its cokernel is annihilated by π^{p-2}. We obtain the claim by taking the dual and using $T^*_{\mathrm{crys}}(M)\cong T^*_{\mathrm{inf}}(\mathcal{M})$ (Theorem 63 (2)). □

11 Fully Faithfulness of T_{crys} and $A_{\inf}$-Representations with φ

We derive the fully faithfulness of the functor T_{crys} from Theorem 70, Lemma 64 (2) and Theorem 63.

Lemma 74 *We have* $(A_{\inf}(\overline{\mathcal{A}})/I^1A_{\inf}(\overline{\mathcal{A}}))(r)^{G_{\mathcal{A}}} = 0$ *for every non-zero integer* r.

Proof Let χ_{cyc} be the cyclotomic character $G_{\mathcal{A}} \to \mathbb{Z}_p^\times$. Let $a \in A_{\inf}(\overline{\mathcal{A}})$ such that $\chi_{\mathrm{cyc}}^r(g)g(a) - a \in I^1A_{\inf}(\overline{\mathcal{A}})$ for every $g \in G_{\mathcal{A}}$. By taking φ^m ($m \in \mathbb{N}$), we obtain $\chi_{\mathrm{cyc}}^r(g)g(\varphi^m(a)) - \varphi^m(a) \in \varphi^m(I^1A_{\inf}(\overline{\mathcal{A}})) \subset \mathrm{Fil}^1A_{\inf}(\overline{\mathcal{A}})$. Since $A_{\inf}(\overline{\mathcal{A}})/\mathrm{Fil}^1A_{\inf}(\overline{\mathcal{A}}) \cong \widehat{\overline{\mathcal{A}}}$ and $\widehat{\overline{\mathcal{A}}}(r)^{G_{\mathcal{A}}} = 0$ (see [19, Proposition 2.12] for example), we have $\varphi^m(a) \in \mathrm{Fil}^1A_{\inf}(\overline{\mathcal{A}})$. Hence $a \in I^1A_{\inf}(\overline{\mathcal{A}})$. □

Lemma 75 *For* $r \in \mathbb{Z}$, *the multiplication by* π^{-r} *induces a* $G_{\mathcal{A}}$*-equivariant* $A_{\inf}(\overline{\mathcal{A}})$*-linear isomorphism* $A_{\inf}(\overline{\mathcal{A}})(-r) \xrightarrow{\cong} \pi^{-r}A_{\inf}(\overline{\mathcal{A}})/\pi^{-r+1}A_{\inf}(\overline{\mathcal{A}})$.

Proof The claim follows from the following equalities for $g \in G_{\mathcal{A}}$ (Lemma 9 (1)).

$$g(\pi) = \sum_{n=1}^{\infty} \binom{\chi_{\mathrm{cyc}}(g)}{n} \pi^n = \chi_{\mathrm{cyc}}(g)\pi(1 + \pi a_g), \quad a_g \in A_{\inf}(O_{\overline{K}}).$$

Note that $1 + \pi A_{\inf}(O_{\overline{K}}) \subset A_{\inf}(O_{\overline{K}})^\times$ since $A_{\inf}(O_{\overline{K}})$ is π-adically complete and separated (Lemma 1 (4)). □

Proposition 76 (1) *Let* $\mathcal{M}$ *be a free* $A_{\inf}(\overline{\mathcal{A}})$*-module of finite rank endowed with a semilinear action of* $G_{\mathcal{A}}$. (*We do not assume the continuity of the action of* $G_{\mathcal{A}}$). *Then there exists at most one* $G_{\mathcal{A}}$*-stable free* $A_{\inf}(\overline{\mathcal{A}})$*-submodule* $\mathcal{M}'$ *of* $\mathcal{M}[\frac{1}{\pi}]$ *satisfying the following properties.*

(a) *The homomorphism* $\mathcal{M}'[\frac{1}{\pi}] \to \mathcal{M}[\frac{1}{\pi}]$ *is an isomorphism.*

(b) *There exists an* $A_{\inf}(\overline{\mathcal{A}})$*-linear* $G_{\mathcal{A}}$*-equivariant isomorphism* $\mathcal{M}' \otimes_{A_{\inf}(\overline{\mathcal{A}})} A_{\inf}(\overline{\mathcal{A}})/I^1A_{\inf}(\overline{\mathcal{A}}) \cong (A_{\inf}(\overline{\mathcal{A}})/I^1A_{\inf}(\overline{\mathcal{A}}))^{\oplus n}$ *for some* $n \in \mathbb{N}$.

(2) *Let* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be free* $A_{\inf}(\overline{\mathcal{A}})$*-modules of finite rank endowed with semilinear action of* $G_{\mathcal{A}}$ *satisfying the condition* (b) *in* (1). *Then any* $G_{\mathcal{A}}$*-equivariant* $A_{\inf}(\overline{\mathcal{A}})[\frac{1}{\pi}]$*-linear homomorphism* $f\colon \mathcal{M}_1[\frac{1}{\pi}] \to \mathcal{M}_2[\frac{1}{\pi}]$ *satisfies* $f(\mathcal{M}_1) \subset \mathcal{M}_2$.

Proof We obtain (1) from (2) by applying (2) to the identity map of $\mathcal{M}[\frac{1}{\pi}]$. Let us prove (2). Suppose that $f(\mathcal{M}_1) \not\subset \mathcal{M}_2$. Then there exists $r \in \mathbb{N}_{>0}$ such that $f(\mathcal{M}_1) \subset \pi^{-r}\mathcal{M}_2$ and $f(\mathcal{M}_1) \not\subset \pi^{-r+1}\mathcal{M}_2$. Then, by Lemma 75, the homomorphism f induces a non-zero $G_{\mathcal{A}}$-equivariant $A_{\inf}(\overline{\mathcal{A}})$-linear homomorphism

$$\overline{f}\colon \mathcal{M}_1 \otimes_{A_{\inf}(\overline{\mathcal{A}})} A_{\inf}(\overline{\mathcal{A}})/I^1A_{\inf}(\overline{\mathcal{A}}) \longrightarrow \mathcal{M}_2 \otimes_{A_{\inf}(\overline{\mathcal{A}})} A_{\inf}(\overline{\mathcal{A}})/I^1A_{\inf}(\overline{\mathcal{A}})(-r).$$

By the assumption on $\mathcal{M}_1$, the source is generated by $G_{\mathcal{A}}$-invariant elements as an $A_{\inf}(\overline{\mathcal{A}})$-module. However the $G_{\mathcal{A}}$-invariant part of the target is 0 by the assumption on $\mathcal{M}_2$ and Lemma 74. This contradicts $\overline{f} \neq 0$. □

Now one can derive the following theorem of Faltings from the fully faithfulness of the functor TA_{inf} (Theorem 63 (1)).

Theorem 77 *The functor T_{crys} is fully faithful.*

Proof For an object M of $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A},\Phi)$, we have an $A_{\mathrm{inf}}(\overline{\mathcal{A}})[\frac{1}{\pi}]$-linear $G_{\mathcal{A}}$-equivariant isomorphism

$$c_M\colon TA_{\mathrm{inf}}(M)[\tfrac{1}{\pi}] \xrightarrow{\cong} T_{\mathrm{crys}}(M)\otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}(\overline{\mathcal{A}})[\tfrac{1}{\pi}]$$

functorial in M by Theorem 70. By Theorem 63 (1), this implies that the functor T_{crys} is faithful. Let M_1 and M_2 be objects of $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A},\Phi)$, and let $f\colon T_{\mathrm{crys}}(M_1)\to T_{\mathrm{crys}}(M_2)$ be a $G_{\mathcal{A}}$-equivariant $\mathbb{Z}_p$-linear homomorphism. Then, by Proposition 76 and Lemma 64 (2), the $G_{\mathcal{A}}$-equivariant $A_{\mathrm{inf}}(\overline{\mathcal{A}})$-linear homomorphism $f\otimes\mathrm{id}\colon T_{\mathrm{crys}}(M_1)\otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}(\overline{\mathcal{A}})\to T_{\mathrm{crys}}(M_2)\otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}(\overline{\mathcal{A}})$ induces a $G_{\mathcal{A}}$-equivariant $A_{\mathrm{inf}}(\overline{\mathcal{A}})$-linear homomorphism $f'\colon TA_{\mathrm{inf}}(M_1)\to TA_{\mathrm{inf}}(M_2)$ via c_{M_1} and c_{M_2}. Since the homomorphism (66) is compatible with φ, we see that f' is compatible with φ. By Theorem 63 (1), there exists a morphism $g\colon M_1\to M_2$ in $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A},\Phi)$ such that $TA_{\mathrm{inf}}(g)=f'$. Now the functoriality of c_M implies $f=T_{\mathrm{crys}}(g)$. □

12 Period Rings Associated to a Framing

We recall the period rings associated to the framing $\square\colon O_K[T_1^{\pm1},\dots,T_d^{\pm1}]\to A;\ T_i\mapsto t_i$ introduced in [7, Sect. 9], and then summarize their basic properties.

Let G_K denote the Galois group $\mathrm{Gal}(\overline{K}/K)$. Since $\mathrm{Spec}(A/pA)\to\mathrm{Spec}(k)$ is geometrically connected (Sect. 1), $\mathcal{A}\otimes_{O_K}O_{\overline{K}}$ is a normal domain, the homomorphism $\mathcal{A}\otimes_{O_K}O_{\overline{K}}\to\overline{\mathcal{A}}$ is injective and the homomorphism $G_{\mathcal{A}}\to G_K$ is surjective. Let ${}_1\mathcal{A}$ denote $\mathcal{A}\otimes_{O_K}O_{\overline{K}}$ in the following. Let $\mathcal{K}_\infty$ be the extension of $\mathcal{K}\overline{K}$ obtained by adjoining all p^nth roots of t_i in $\overline{\mathcal{K}}$ for all $i\in\mathbb{N}\cap[1,d]$, and let $\widetilde{\Gamma}_{\mathcal{A}}$ denote the Galois group $\mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$. We have $\mathcal{K}_\infty\subset\mathcal{K}^{\mathrm{ur}}$ because $t_i\in\mathcal{A}^\times$ for $i\in\mathbb{N}\cap[1,d]$. We define $\mathcal{A}_\infty$ to be the integral closure $\overline{\mathcal{A}}\cap\mathcal{K}_\infty$ of $\mathcal{A}$ in $\mathcal{K}_\infty$. Choose a compatible system of p^nth roots of t_i: $t_{i,n}\in\mathcal{A}_\infty$ $(n\in\mathbb{N})$, $t_{i,n+1}^p=t_{i,n}$, $t_{i,0}=t_i$, and define the 1-cocycle $\chi_i\colon\widetilde{\Gamma}_{\mathcal{A}}\to\mathbb{Z}_p(1):=\varprojlim_n\mu_{p^n}(O_{\overline{K}})$ by $\chi_i(g)=(\zeta_n)_n$, $g(t_{i,n})=t_{i,n}\zeta_n$, and let $\underline{\chi}_i$ denote the composition of χ_i with $\mathbb{Z}_p(1)\hookrightarrow R^\times_{O_{\overline{K}}};\ (\zeta_n)\mapsto(\zeta_n \mod p)$.

Lemma 78 *The following ${}_1\mathcal{A}$-homomorphism defined by $1\otimes T_i^{p^{-n}}\mapsto t_{i,n}$ is an isomorphism*

$$\mathcal{A}\otimes_{O_K[T_1,\dots,T_d]}O_{\overline{K}}[T_1^{p^{-\infty}},\dots,T_d^{p^{-\infty}}]\longrightarrow\mathcal{A}_\infty.$$

Proof Let L be a finite extension of K contained in $\overline{K}$, let O_L be the integral closure of O_K in L, let $\mathfrak{m}_L$ be the maximal ideal of O_L, and let k_L be the residue field $O_L/\mathfrak{m}_L$ of O_L. Let n be a positive integer. Then the

$O_L[T_1^{p^{-n}},\dots,T_d^{p^{-n}}]$-algebra $B:=A\otimes_{O_K[T_1,\dots,T_d]}O_L[T_1^{p^{-n}},\dots,T_d^{p^{-n}}]$ is étale and its reduction mod $\mathfrak{m}_L$ is a regular domain because $\mathrm{Spec}(A/pA\otimes_k k_L)$ is a connected scheme étale over $\mathrm{Spec}(k_L[T_1,\dots,T_d])$ and hence $\mathrm{Frac}(A/p\otimes_k k_L)\otimes_{k_L(T_1,\dots,T_d)} k_L(T_1^{p^{-n}},\dots,T_d^{p^{-n}})$ is a purely inseparable extension of $\mathrm{Frac}(A/p\otimes_k k_L)$. This implies that the p-adic completion $\widehat{B}\cong\mathcal{A}\otimes_{O_K[T_1,\dots,T_d]}O_L[T_1^{p^{-n}},\dots,T_d^{p^{-n}}]$ of B is a regular domain finite over $\mathcal{A}$, and therefore the $\mathcal{A}\otimes_{O_K}O_L$-homomorphism $\mathcal{A}\otimes_{O_K[T_1,\dots,T_d]}O_L[T_1^{p^{-n}},\dots,T_d^{p^{-n}}]\to\mathcal{K}_\infty$ defined by $1\otimes T_i^{p^{-n}}\mapsto t_{i,n}$ is injective, and its image is the integral closure of $\mathcal{A}$ in $\mathcal{K}L(t_{1,n},\dots,t_{d,n})$. Varying L and n, we obtain the claim. $\square$

By Lemma 78, the absolute Frobenius of $\mathcal{A}_\infty/p$ is surjective. Hence the $O_{\overline{K}}$-algebra $\mathcal{A}_\infty$ and its subalgebra $\mathcal{A}$ satisfy the conditions on Λ and Λ_0 in Sect. 2, which are summarized in the beginning of Sect. 8. By applying §2 to $(\Lambda,\Lambda_0)=(\mathcal{A}_\infty,\mathcal{A})$, we obtain $A_{\mathrm{crys}}(\mathcal{A}_\infty)$ and $A_{\inf}(\mathcal{A}_\infty)$ with Fil^r, φ and $\widetilde{\Gamma}_{\mathcal{A}}$-action. We can apply the results on $A_{\mathrm{crys}}(\Lambda)$ and $A_{\inf}(\Lambda)$ in Sect. 8 to $A_{\mathrm{crys}}(\mathcal{A}_\infty)$ and $A_{\inf}(\mathcal{A}_\infty)$.

Let $\mathscr{S}_{\inf}$ (resp. $\mathscr{S}_{\mathrm{crys}}$) denote the set of ideals $\mathfrak{a}$ of $A_{\inf}(O_{\overline{K}})$ (resp. $A_{\mathrm{crys}}(O_{\overline{K}})$) with $(p,[\underline{p}])^n\subset\mathfrak{a}\subset pA_{\inf}(O_{\overline{K}})+\mathrm{Fil}^1A_{\inf}(O_{\overline{K}})=(p,\xi)=(p,[\underline{p}])$ (resp. $(p^n)\subset\mathfrak{a}\subset pA_{\mathrm{crys}}(O_{\overline{K}})+\mathrm{Fil}^1A_{\mathrm{crys}}(O_{\overline{K}})$) for some $n\in\mathbb{N}_{>0}$. Let $a\in A_{\inf}(O_{\overline{K}})\backslash pA_{\inf}(O_{\overline{K}})$. If the image of a under $A_{\inf}(O_{\overline{K}})\to A_{\inf}(O_{\overline{K}})/p\cong R_{O_{\overline{K}}}$ is contained in $\underline{p}R_{O_{\overline{K}}}$, then $\{(p,a)^n\,|\,n\in\mathbb{N}_{>0}\}$ is a cofinal subset of $\mathscr{S}_{\inf}$ by Lemma 1 (1). It is trivial that $\{p^nA_{\mathrm{crys}}(O_{\overline{K}});\,n\in\mathbb{N}_{>0}\}$ is a cofinal subset of $\mathscr{S}_{\mathrm{crys}}$. For $\bullet\in\{\inf,\mathrm{crys}\}$, $\mathfrak{a}\in\mathscr{S}_\bullet$ and Λ as in the beginning of Sect. 2, we define $A_{\bullet,\mathfrak{a}}(\Lambda)$ to be $A_\bullet(\Lambda)/\mathfrak{a}A_\bullet(\Lambda)$. We identify $A_{\bullet,(p^m,\mathrm{Fil}^1)}(\Lambda)$ with Λ/p^m via the isomorphism induced by $\theta\colon A_\bullet(\Lambda)\to\widehat{\Lambda}$.

Let $\bullet\in\{\inf,\mathrm{crys}\}$ and let $\mathfrak{a}\in\mathscr{S}_\bullet$. Let $A_{\bullet,\mathfrak{a}}(O_{\overline{K}})[\underline{U}^{\pm1}]$ denote the $A_{\bullet,\mathfrak{a}}(O_{\overline{K}})$-algebra $A_{\bullet,\mathfrak{a}}(O_{\overline{K}})[U_1^{\pm1},U_2^{\pm1},\dots,U_d^{\pm1}]$. For $g\in\widetilde{\Gamma}_{\mathcal{A}}$, we define the isomorphism $\rho_{\mathfrak{a}}(g)\colon A_{\bullet,\mathfrak{a}}(O_{\overline{K}})[\underline{U}^{\pm1}]\xrightarrow{\cong}A_{\bullet,g(\mathfrak{a})}(O_{\overline{K}})[\underline{U}^{\pm1}]$ compatible with the action of g on $A_\bullet(O_{\overline{K}})$ by $\rho_{\mathfrak{a}}(g)(U_i)=U_i[\underline{\chi}_i(g)]$. Then we have $\rho_{\mathfrak{a}}(1)=\mathrm{id}$ and $\rho_{h(\mathfrak{a})}(g)\rho_{\mathfrak{a}}(h)=\rho_{\mathfrak{a}}(gh)$. We define the homomorphism $\varphi_{\mathfrak{a}}\colon A_{\bullet,\mathfrak{a}}(O_{\overline{K}})[\underline{U}^{\pm1}]\to A_{\bullet,\varphi(\mathfrak{a})}(O_{\overline{K}})[\underline{U}^{\pm1}]$ compatible with φ of $A_\bullet(O_{\overline{K}})$ by $\varphi(U_i)=U_i^p$. We have $\varphi_{g(\mathfrak{a})}\circ\rho_{\mathfrak{a}}(g)=\rho_{\varphi(\mathfrak{a})}(g)\circ\varphi_{\mathfrak{a}}$ for $g\in\widetilde{\Gamma}_{\mathcal{A}}$. For $\mathfrak{a},\mathfrak{b}\in\mathscr{S}_\bullet$ with $\mathfrak{b}\subset\mathfrak{a}$, the projection $A_{\bullet,\mathfrak{b}}(O_{\overline{K}})[\underline{U}^{\pm1}]\to A_{\bullet,\mathfrak{a}}(O_{\overline{K}})[\underline{U}^{\pm1}]$ is compatible with these structures. For a positive integer m, let $O_{\overline{K}}/p^m[\underline{T}^{\pm1}]$ denote the $O_{\overline{K}}$-algebra $O_{\overline{K}}/p^m[T_1^{\pm1},\dots,T_d^{\pm1}]$ endowed with the action of $\widetilde{\Gamma}_{\mathcal{A}}$ defined by the action on $O_{\overline{K}}/p^m$ via G_K and the trivial action on $O_K/p^m[T_1^{\pm1},\dots,T_d^{\pm1}]$. Then we identify $A_{\bullet,(p^m,\mathrm{Fil}^1)}(O_{\overline{K}})[\underline{U}^{\pm1}]$ with $O_{\overline{K}}/p^m[\underline{T}^{\pm1}]$ via the isomorphism defined by $U_i\mapsto T_i$, which is compatible with the action of $\widetilde{\Gamma}_{\mathcal{A}}$ because $[\underline{\varepsilon}^a]-1\in\mathrm{Fil}^1A_{\mathrm{crys}}(O_{\overline{K}})$ for $a\in\mathbb{Z}_p$.

For $\mathfrak{a}\in\mathscr{S}_\bullet$, we define $A_{\bullet,\mathfrak{a}}(O_{\overline{K}})[\underline{U}^{\pm1}]\to A^{\square}_{\bullet,\mathfrak{a}}(\mathcal{A})$ to be the unique étale lifting of the étale homomorphism of $O_{\overline{K}}$-algebras $A_{\bullet,(p,\mathrm{Fil}^1)}(O_{\overline{K}})[\underline{U}^{\pm1}]=O_{\overline{K}}/p[\underline{T}^{\pm1}]\to{}_1\mathcal{A}/p$; $T_i\mapsto t_i$. Note that the kernel of $A_{\bullet,\mathfrak{a}}(O_{\overline{K}})\to A_{\bullet,(p,\mathrm{Fil}^1)}(O_{\overline{K}})$ is nilpotent. For $\mathfrak{a},\mathfrak{b}\in\mathscr{S}_\bullet$ with $\mathfrak{b}\subset\mathfrak{a}$, we have an isomorphism

$$A^{\square}_{\bullet,\mathfrak{b}}(\mathcal{A})\otimes_{A_{\bullet,\mathfrak{b}}(O_{\overline{K}})}A_{\bullet,\mathfrak{a}}(O_{\overline{K}})\xrightarrow{\cong}A^{\square}_{\bullet,\mathfrak{a}}(\mathcal{A})\tag{71}$$

compatible with the composition for $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathscr{S}_\bullet$ with $\mathfrak{c} \subset \mathfrak{b} \subset \mathfrak{a}$. For $\mathfrak{a} \in \mathscr{S}_\bullet$ and $g \in \widetilde{\Gamma}_{\mathcal{A}}$, $\rho_{\mathfrak{a}}(g)$ and the action of g on ${}_1\mathcal{A}/p$ and $O_{\overline{K}}/p[\underline{T}^{\pm 1}]$ induce an automorphism $\rho^{\square}_{\mathfrak{a}}(g)\colon A^{\square}_{\bullet,\mathfrak{a}}(\mathcal{A}) \xrightarrow{\cong} A^{\square}_{\bullet,g(\mathfrak{a})}(\mathcal{A})$. We have $\rho^{\square}_{\mathfrak{a}}(1) = \mathrm{id}$ and $\rho^{\square}_{h(\mathfrak{a})}(g) \circ \rho^{\square}_{\mathfrak{a}}(h) = \rho^{\square}_{\mathfrak{a}}(gh)$. Similarly $\varphi_{\mathfrak{a}}$ and the absolute Frobenius of ${}_1\mathcal{A}/p$ and $O_{\overline{K}}/p[\underline{U}^{\pm 1}]$ induce an endomorphism $\varphi^{\square}_{\mathfrak{a}}\colon A^{\square}_{\bullet,\mathfrak{a}}(\mathcal{A}) \to A^{\square}_{\bullet,\varphi(\mathfrak{a})}(\mathcal{A})$. We have $\varphi^{\square}_{g(\mathfrak{a})} \circ \rho^{\square}_{\mathfrak{a}}(g) = \rho^{\square}_{\varphi(\mathfrak{a})}(g) \circ \varphi^{\square}_{\mathfrak{a}}$ for $g \in \widetilde{\Gamma}_{\mathcal{A}}$. We identify the étale lifting $A_{\bullet,(p^m,\mathrm{Fil}^1)}(O_{\overline{K}})[\underline{U}^{\pm 1}] \to A^{\square}_{\bullet,(p^m,\mathrm{Fil}^1)}(\mathcal{A})$ with the étale lifting $O_{\overline{K}}/p^m[\underline{T}^{\pm 1}] \to {}_1\mathcal{A}/p^m$ via the unique isomorphism, which is compatible with the action of $\widetilde{\Gamma}_{\mathcal{A}}$.

We define $A^{\square}_{\bullet}(\mathcal{A})$ to be the inverse limit of $A^{\square}_{\bullet,\mathfrak{a}}(\mathcal{A})$ ($\mathfrak{a} \in \mathscr{S}_\bullet$) endowed with the inverse limit topology of the discrete topology of $A^{\square}_{\bullet,\mathfrak{a}}(\mathcal{A})$. The homomorphisms $\rho^{\square}_{\mathfrak{a}}(g)$ and $\varphi^{\square}_{\mathfrak{a}}$ are obviously compatible with the isomorphisms (71), and $\{\varphi(\mathfrak{a}); \mathfrak{a} \in \mathscr{S}_\bullet\}$ is cofinal in $\mathscr{S}_\bullet$ because $\varphi([\underline{p}]) = [\underline{p}]^p$ for $\bullet = \inf$. Hence $\rho^{\square}_{\mathfrak{a}}(g)$ and $\varphi^{\square}_{\mathfrak{a}}$ induce the action $\rho^{\square}$ of $\widetilde{\Gamma}_{\mathcal{A}}$ on $A^{\square}_{\bullet}(\mathcal{A})$ and the endomorphism φ of $A^{\square}_{\bullet}(\mathcal{A})$. The endomorphism φ of $A^{\square}_{\bullet}(\mathcal{A})$ is $\widetilde{\Gamma}_{\mathcal{A}}$-equivariant.

Lemma 79 (1) *$A^{\square}_{\inf}(\mathcal{A})$ and $A^{\square}_{\mathrm{crys}}(\mathcal{A})$ are p-torsion free and we have the following isomorphisms.*

$$A^{\square}_{\inf}(\mathcal{A})/p^m \xrightarrow{\cong} \varprojlim_{\mathfrak{a} \in \mathscr{S}_{\inf},\ p^m \in \mathfrak{a}} A^{\square}_{\inf,\mathfrak{a}}(\mathcal{A}), \quad A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m \xrightarrow{\cong} A^{\square}_{\mathrm{crys},(p^m)}(\mathcal{A}).$$

In particular, $A^{\square}_{\inf}(\mathcal{A})$ and $A^{\square}_{\mathrm{crys}}(\mathcal{A})$ are p-adically complete and separated.

(2) *For $\bullet \in \{\inf, \mathrm{crys}\}$ and $\mathfrak{a} \in \mathscr{S}_\bullet$, the homomorphism $A^{\square}_{\bullet}(\mathcal{A})/\mathfrak{a}A^{\square}_{\bullet}(\mathcal{A}) \to A^{\square}_{\bullet,\mathfrak{a}}(\mathcal{A})$ is an isomorphism.*

Proof Put $a := [\underline{p}]$. We abbreviate $A^{\square}_{\inf}(\mathcal{A})$, $A^{\square}_{\inf,\mathfrak{a}}(\mathcal{A})$, etc. to $A^{\square}_{\inf}$, $A^{\square}_{\inf,\mathfrak{a}}$, etc.

(1) Since $A_{\mathrm{crys}}(O_{\overline{K}})$ and $A_{\inf}(O_{\overline{K}})/a^m$ ($m \in \mathbb{N}$) are p-torsion free (see Lemma 1 (3) for the latter) and $A^{\square}_{\bullet,\mathfrak{a}}$ is flat over $A_{\bullet,\mathfrak{a}}(O_{\overline{K}})$ for $\bullet \in \{\inf, \mathrm{crys}\}$ and $\mathfrak{a} \in \mathscr{S}_\bullet$, we obtain the following exact sequences by using (71).

$$0 \longrightarrow A^{\square}_{\inf,(p^l,a^l)} \xrightarrow{p^m} A^{\square}_{\inf,(p^{l+m},a^l)} \longrightarrow A^{\square}_{\inf,(p^m,a^l)} \longrightarrow 0,$$
$$0 \longrightarrow A^{\square}_{\mathrm{crys},(p^l)} \xrightarrow{p^m} A^{\square}_{\mathrm{crys},(p^{l+m})} \longrightarrow A^{\square}_{\mathrm{crys},(p^m)} \longrightarrow 0.$$

We obtain the claims by taking the inverse limits over l.

(2) The claim for $\bullet = \mathrm{crys}$ follows from the claim (1) and (71) because $\{(p^m) \mid m \in \mathbb{N}_{>0}\} \subset \mathscr{S}_{\mathrm{crys}}$ is cofinal. In the case $\bullet = \inf$, the claim is reduced to the case $\mathfrak{a} = (p^m, a^n)$ ($m, n \in \mathbb{N}_{>0}$) by the same argument. Since $A_{\inf}(O_{\overline{K}})/p^m$ is a-torsion free by Lemma 1 (4), we have exact sequences

$$0 \longrightarrow A^{\square}_{\inf,(p^m,a^l)} \xrightarrow{a^n} A^{\square}_{\inf,(p^m,a^{l+n})} \longrightarrow A^{\square}_{\inf,(p^m,a^n)} \longrightarrow 0.$$

We obtain the desired claim by taking the inverse limit over l and using (1). □

For $\mathfrak{a} \in \mathscr{S}_{\mathrm{inf}}$ such that $p \in \mathfrak{a}$, we have $\varphi(\mathfrak{a}) \subset \mathfrak{a}$ and the composition of $\varphi^{\square}_{\mathfrak{a}}$ with the projection map $A^{\square}_{\bullet,\varphi(\mathfrak{a})}(\mathcal{A}) \to A^{\square}_{\bullet,\mathfrak{a}}(\mathcal{A})$ is the absolute Frobenius of $A^{\square}_{\mathrm{inf},\mathfrak{a}}(\mathcal{A})$ by its construction. Therefore the first isomorphism in Lemma 79 (1) for $m = 1$ implies that φ of $A^{\square}_{\mathrm{inf}}(\mathcal{A})$ is a lifting of the absolute Frobenius of $A^{\square}_{\mathrm{inf}}(\mathcal{A})/pA^{\square}_{\mathrm{inf}}(\mathcal{A})$. Similarly, since $\varphi^{\square}_{(p)}$ of $A^{\square}_{\mathrm{crys},(p)}(\mathcal{A})$ is the absolute Frobenius by its construction, the second isomorphism in Lemma 79 (1) shows that φ of $A^{\square}_{\mathrm{crys}}(\mathcal{A})$ is a lifting of the absolute Frobenius of $A^{\square}_{\mathrm{crys}}(\mathcal{A})/pA^{\square}_{\mathrm{crys}}(\mathcal{A})$.

Lemma 80 *Let a be an element of $A_{\mathrm{inf}}(O_{\overline{K}})$, and assume that its image $\overline{a}$ in $A_{\mathrm{inf}}(O_{\overline{K}})/pA_{\mathrm{inf}}(O_{\overline{K}}) \cong R_{O_{\overline{K}}}$ is neither zero nor invertible.*

(1) *The topology of $A^{\square}_{\mathrm{inf}}(\mathcal{A})$ coincides with the (p, a)-adic topology.*
(2) *$A^{\square}_{\mathrm{inf}}(\mathcal{A})$ is (p, a)-adically complete and separated.*
(3) *$A^{\square}_{\mathrm{inf}}(\mathcal{A})/aA^{\square}_{\mathrm{inf}}(\mathcal{A})$ is p-torsion free, and p-adically complete and separated.*
(4) *$A^{\square}_{\mathrm{inf}}(\mathcal{A})$ and $A^{\square}_{\mathrm{inf}}(\mathcal{A})/p^n$ ($n \in \mathbb{N}_{>0}$) are a-torsion free, and a-adically complete and separated.*
(5) *We have an isomorphism $A^{\square}_{\mathrm{inf}}(\mathcal{A})/aA^{\square}_{\mathrm{inf}}(\mathcal{A}) \cong \varprojlim_{m\in\mathbb{N}_{>0}} A^{\square}_{\mathrm{inf},(p^m,a)}(\mathcal{A})$.*

Proof By Lemma 79 (2), the topology of $A^{\square}_{\mathrm{inf}}(\mathcal{A})$ coincides with the $(p, \underline{[p]})$-adic topology. Hence the claim (1) follows from Lemma 1 (1). The claim (5) follows from (3) and Lemma 79 (2). It remains to prove the claims (2), (3), and (4). By Lemma 2 and Lemma 79 (1), it suffices to prove that $A^{\square}_{\mathrm{inf}}(\mathcal{A})/p$ is a-torsion free, and a-adically complete and separated. By replacing a with a suitable power of a if necessary, we may assume $\overline{a} \in \underline{p}R_{O_{\overline{K}}}$. Then $\{(p^m, a^n) \mid m, n \in \mathbb{N}_{>0}\}$ is a cofinal subset of $\mathscr{S}_{\mathrm{inf}}$. Hence, from Lemma 79, we obtain $A^{\square}_{\mathrm{inf}}(\mathcal{A})/p \cong \varprojlim_n A^{\square}_{\mathrm{inf},(p,a^n)}(\mathcal{A}) \cong \varprojlim_n (A^{\square}_{\mathrm{inf}}(\mathcal{A})/p)/a^n$. Since $A_{\mathrm{inf}}(O_{\overline{K}})/p$ is a-torsion free by Lemma 1 (4), and $A^{\square}_{\mathrm{inf},\mathfrak{a}}(\mathcal{A})$ is flat over $A_{\mathrm{inf},\mathfrak{a}}(O_{\overline{K}})$ for $\mathfrak{a} \in \mathscr{S}_{\mathrm{inf}}$, the multiplication by a on $A^{\square}_{\mathrm{inf},(p,a^{l+1})}(\mathcal{A})$ ($l \in \mathbb{N}_{>0}$) together with (71) induces an injective homomorphism $A^{\square}_{\mathrm{inf},(p,a^l)}(\mathcal{A}) \to A^{\square}_{\mathrm{inf},(p,a^{l+1})}(\mathcal{A})$. By taking the inverse limit over l and using Lemma 79 (1), we see that a is regular on $A^{\square}_{\mathrm{inf}}(\mathcal{A})/p$. □

Powers of ξ and π satisfy the assumption on a in Lemma 80. We define the filtrations $\mathrm{Fil}^r A^{\square}_{\mathrm{inf}}(\mathcal{A})$ and $I^r A^{\square}_{\mathrm{inf}}(\mathcal{A})$ ($r \in \mathbb{Z}$) by $\xi^r A^{\square}_{\mathrm{inf}}(\mathcal{A})$ and $\pi^r A^{\square}_{\mathrm{inf}}(\mathcal{A})$, respectively, if $r \geq 0$, and $A^{\square}_{\mathrm{inf}}(\mathcal{A})$ if $r < 0$. By Lemma 80 (5), we have

$$A^{\square}_{\mathrm{inf}}(\mathcal{A})/\mathrm{Fil}^1 A^{\square}_{\mathrm{inf}}(\mathcal{A}) \cong \varprojlim_{m\in\mathbb{N}_{>0}} A^{\square}_{\mathrm{inf},(p^m,\xi)}(\mathcal{A}) = \varprojlim_{m\in\mathbb{N}_{>0}} {}_1\mathcal{A}/p^m = {}_1\widehat{\mathcal{A}}. \tag{72}$$

For an ideal J of $A_{\mathrm{crys}}(O_{\overline{K}})$, we define the ideal $\overline{J}A^{\square}_{\mathrm{crys}}(\mathcal{A})$ of $A^{\square}_{\mathrm{crys}}(\mathcal{A})$ to be the topological closure of $JA^{\square}_{\mathrm{crys}}(\mathcal{A})$ in $A^{\square}_{\mathrm{crys}}(\mathcal{A})$, which is the inverse limit of $J(A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m A^{\square}_{\mathrm{crys}}(\mathcal{A}))$ ($m \in \mathbb{N}_{>0}$) by Lemma 79 (1).

Lemma 81 *Let J be an ideal of $A_{\mathrm{crys}}(O_{\overline{K}})$ contained in $\mathrm{Fil}^1 A_{\mathrm{crys}}(O_{\overline{K}})$ such that $A_{\mathrm{crys}}(O_{\overline{K}})/J$ is p-torsion free. Then $A^{\square}_{\mathrm{crys}}(\mathcal{A})/\overline{J}A^{\square}_{\mathrm{crys}}(\mathcal{A})$ is p-torsion free and, p-adically complete and separated. Moreover the natural homomorphism $(A^{\square}_{\mathrm{crys}}(\mathcal{A})/\overline{J}A^{\square}_{\mathrm{crys}}(\mathcal{A}))/p^m \to A^{\square}_{\mathrm{crys},(p^m,J)}(\mathcal{A})$ is an isomorphism for $m \in \mathbb{N}_{>0}$.*

Proof By Lemma 79 (1) and (71), we have exact sequences

$$0 \longrightarrow J(A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m) \longrightarrow A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m \longrightarrow A^{\square}_{\mathrm{crys},(p^m,J)}(\mathcal{A}) \longrightarrow 0.$$

By taking $\varprojlim_m$, we obtain $A^{\square}_{\mathrm{crys}}(\mathcal{A})/\overline{J}A^{\square}_{\mathrm{crys}}(\mathcal{A}) \xrightarrow{\cong} \varprojlim_m A^{\square}_{\mathrm{crys},(p^m,J)}(\mathcal{A})$. Let $A^{\square}_{\mathrm{crys},J}(\mathcal{A})$ denote the target algebra. Since $A_{\mathrm{crys}}(O_{\overline{K}})/J$ is p-torsion free and $A_{\mathrm{crys},\mathfrak{a}}(O_{\overline{K}}) \to A^{\square}_{\mathrm{crys},\mathfrak{a}}(\mathcal{A})$ is flat for $\mathfrak{a} \in \mathscr{S}_{\mathrm{crys}}$, we have exact sequences

$$0 \longrightarrow A^{\square}_{\mathrm{crys},(p^l,J)}(\mathcal{A}) \xrightarrow{p^m} A^{\square}_{\mathrm{crys},(p^{l+m},J)}(\mathcal{A}) \longrightarrow A^{\square}_{\mathrm{crys},(p^m,J)}(\mathcal{A}) \longrightarrow 0.$$

By taking $\varprojlim_l$, we see that $A^{\square}_{\mathrm{crys},J}(\mathcal{A})$ is p-torsion free, and $A^{\square}_{\mathrm{crys},J}(\mathcal{A})/p^m \to A^{\square}_{\mathrm{crys},(p^m,J)}(\mathcal{A})$ is an isomorphism. This completes the proof. □

For $r \in \mathbb{N}_{>0}$, $\mathrm{Fil}^r A_{\mathrm{crys}}(O_{\overline{K}})$ and $I^r A_{\mathrm{crys}}(O_{\overline{K}})$ satisfy the assumption on J in Lemma 81. We define $\mathrm{Fil}^r A^{\square}_{\mathrm{crys}}(\mathcal{A})$ and $I^r A^{\square}_{\mathrm{crys}}(\mathcal{A})$ to be the topological closure of $\mathrm{Fil}^r A_{\mathrm{crys}}(O_{\overline{K}})A^{\square}_{\mathrm{crys}}(\mathcal{A})$ and $I^r A_{\mathrm{crys}}(O_{\overline{K}})A^{\square}_{\mathrm{crys}}(\mathcal{A})$ in $A^{\square}_{\mathrm{crys}}(\mathcal{A})$, respectively. For $r \in \mathbb{Z}$, $r \leq 0$, we define them to be $A^{\square}_{\mathrm{crys}}(\mathcal{A})$. By Lemma 81, we have isomorphisms

$$\begin{aligned} A^{\square}_{\mathrm{crys}}(\mathcal{A})/\mathrm{Fil}^1 A^{\square}_{\mathrm{crys}}(\mathcal{A}) &\cong \varprojlim_m (A^{\square}_{\mathrm{crys}}(\mathcal{A})/\mathrm{Fil}^1 A^{\square}_{\mathrm{crys}}(\mathcal{A}))/p^m \\ &\cong \varprojlim_m A^{\square}_{\mathrm{crys},(p^m,\mathrm{Fil}^1)}(\mathcal{A}) = \varprojlim_m {}_1\mathcal{A}/p^m{}_1\mathcal{A} \cong {}_1\widehat{\mathcal{A}}. \end{aligned} \quad (73)$$

Let us compare $A^{\square}_{\mathrm{inf}}(\mathcal{A})$ and $A^{\square}_{\mathrm{crys}}(\mathcal{A})$. Let $\mathfrak{a} \in \mathscr{S}_{\mathrm{inf}}$ and $\mathfrak{a}' := \mathfrak{a}A_{\mathrm{crys}}(O_{\overline{K}}) \in \mathscr{S}_{\mathrm{crys}}$. Since the isomorphisms $A_{\bullet,(p,\mathrm{Fil}^1)}(O_{\overline{K}}) \cong O_{\overline{K}}/p$ ($\bullet \in \{\mathrm{inf}, \mathrm{crys}\}$) are compatible with the canonical homomorphism $A_{\mathrm{inf}}(O_{\overline{K}}) \to A_{\mathrm{crys}}(O_{\overline{K}})$, the homomorphism $\kappa_{\mathfrak{a}}\colon A_{\mathrm{inf},\mathfrak{a}}(O_{\overline{K}})[\underline{U}^{\pm 1}] \to A_{\mathrm{crys},\mathfrak{a}'}(O_{\overline{K}})[\underline{U}^{\pm 1}]$ defined by $A_{\mathrm{inf}}(O_{\overline{K}}) \to A_{\mathrm{crys}}(O_{\overline{K}})$ and $U_i \mapsto U_i$ induces a homomorphism $\kappa^{\square}_{\mathfrak{a}}\colon A^{\square}_{\mathrm{inf},\mathfrak{a}}(\mathcal{A}) \to A^{\square}_{\mathrm{crys},\mathfrak{a}'}(\mathcal{A})$, which gives an isomorphism

$$A^{\square}_{\mathrm{inf},\mathfrak{a}}(\mathcal{A}) \otimes_{A_{\mathrm{inf},\mathfrak{a}}(O_{\overline{K}})} A_{\mathrm{crys},\mathfrak{a}'}(O_{\overline{K}}) \xrightarrow{\cong} A^{\square}_{\mathrm{crys},\mathfrak{a}'}(\mathcal{A}). \quad (74)$$

The homomorphisms $\kappa^{\square}_{\mathfrak{a}}$ ($\mathfrak{a} \in \mathscr{S}_{\mathrm{inf}}$) are compatible with the action of $\widetilde{\Gamma}_{\mathcal{A}}$, the homomorphisms φ, and (71). The set $\{\mathfrak{a}' \mid \mathfrak{a} \in \mathscr{S}_{\mathrm{inf}}\}$ is cofinal in $\mathscr{S}_{\mathrm{crys}}$ because $\xi^p = p!\xi^{[p]} \in pA_{\mathrm{crys}}(O_{\overline{K}})$. Hence they induce a homomorphism

$$\kappa\colon A^{\square}_{\mathrm{inf}}(\mathcal{A}) \longrightarrow A^{\square}_{\mathrm{crys}}(\mathcal{A}) \quad (75)$$

compatible with the action of $\widetilde{\Gamma}_{\mathcal{A}}$ and φ.

Lemma 82 *The homomorphism κ induces isomorphisms*

$$\mathrm{Fil}^r A^{\square}_{\inf}(\mathcal{A})/I^s A^{\square}_{\inf}(\mathcal{A}) \xrightarrow{\cong} \mathrm{Fil}^r A^{\square}_{\mathrm{crys}}(\mathcal{A})/I^s A^{\square}_{\mathrm{crys}}(\mathcal{A}) \quad (r, s \in \mathbb{N}, r \leq s \leq p-1).$$

Proof It suffices to prove that $A^{\square}_{\inf}(\mathcal{A})/I^r \to A^{\square}_{\mathrm{crys}}(\mathcal{A})/I^r$ and $A^{\square}_{\inf}(\mathcal{A})/\mathrm{Fil}^r \to A^{\square}_{\mathrm{crys}}(\mathcal{A})/\mathrm{Fil}^r$ ($r \in \mathbb{N} \cap [0, p-1]$) are isomorphisms. These algebras are p-adically complete and separated by Lemmas 80 (3) and 81. Hence, by using Lemmas 79 (2) and 81, we are reduced to showing $A^{\square}_{\inf,\mathfrak{a}_{\inf}}(\mathcal{A}) \xrightarrow{\cong} A^{\square}_{\mathrm{crys},\mathfrak{a}_{\mathrm{crys}}}(\mathcal{A})$ for $\mathfrak{a}_\bullet = p^m A_\bullet(O_{\overline{K}}) + I^r A_\bullet(O_{\overline{K}})$ and $\mathfrak{a}_\bullet = p^m A_\bullet(O_{\overline{K}}) + \mathrm{Fil}^r A_\bullet(O_{\overline{K}})$ ($m \in \mathbb{N}_{>0}$). By (74), this follows from $A_{\inf,\mathfrak{a}_{\inf}}(O_{\overline{K}}) \xrightarrow{\cong} A_{\mathrm{crys},\mathfrak{a}_{\mathrm{crys}}}(O_{\overline{K}})$, which is an immediate consequence of (2). □

Lemma 83 *For $m \in \mathbb{N}_{>0}$, $(\kappa \mod p^m)\colon A^{\square}_{\inf}(\mathcal{A})/p^m \to A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m$ is canonically isomorphic to the PD-envelope of $A^{\square}_{\inf}(\mathcal{A})/p^m$ with respect to the kernel of $A^{\square}_{\inf}(\mathcal{A})/p^m \to {}_1\mathcal{A}/p^m$ compatible with the PD-structure on pO_K. Furthermore* $\mathrm{Fil}^r A_{\mathrm{crys}}(O_{\overline{K}})(A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m)$ *(used in the definition of* $\mathrm{Fil}^r A^{\square}_{\mathrm{crys}}(\mathcal{A})$*) for $r \in \mathbb{N}_{>0}$ corresponds to the rth divided power of the divided power ideal of the PD-envelope.*

Proof Let D_m be the PD-envelope considered in the lemma. By Lemma 79 (2), we have $A^{\square}_{\inf}(\mathcal{A})/(p^m, \xi) \cong A^{\square}_{\inf,(p^m,\xi)}(\mathcal{A}) \cong {}_1\mathcal{A}/p^m$. Put $\mathfrak{a}_m := p^m A_{\inf}(O_{\overline{K}}) + \xi^{pm} A_{\inf}(O_{\overline{K}})$. The image of ξ^{pm} in D_m is zero because the image of ξ in D_m is contained in the PD-ideal of D_m and $p^m D_m = 0$. Hence D_m is isomorphic to the PD-envelope of $A^{\square}_{\inf}(\mathcal{A})/\mathfrak{a}_m A^{\square}_{\inf}(\mathcal{A}) \cong A^{\square}_{\inf,\mathfrak{a}_m}(\mathcal{A})$ with respect to $\xi A^{\square}_{\inf,\mathfrak{a}_m}(\mathcal{A})$ compatible with the PD-structure γ on pO_K ([5, 3.20 Remarks (7)]). Similarly $A_{\mathrm{crys}}(O_{\overline{K}})/p^m \cong A_{\mathrm{crys},m}(O_{\overline{K}})$ is isomorphic to the PD-envelope of $A_{\inf,\mathfrak{a}_m}(O_{\overline{K}})$ with respect to $\xi A_{\inf,\mathfrak{a}_m}(O_{\overline{K}})$ compatible with γ. Since $A_{\inf,\mathfrak{a}_m}(O_{\overline{K}}) \to A^{\square}_{\inf,\mathfrak{a}_m}(\mathcal{A})$ is flat, we obtain a PD-isomorphism $D_m \cong A^{\square}_{\inf,\mathfrak{a}_m}(\mathcal{A}) \otimes_{A_{\inf,\mathfrak{a}_m}(O_{\overline{K}})} A_{\mathrm{crys}}(O_{\overline{K}})/p^m$ ([5, 3.21 Proposition]), whose right-hand side is isomorphic to $A^{\square}_{\mathrm{crys},(p^m)}(\mathcal{A}) \cong A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m$ by (74) and Lemma 79 (1). □

Next we compare $A^{\square}_\bullet(\mathcal{A})$ with $A_\bullet(\mathcal{A}_\infty)$. Let $\underline{t}_i$ be the element $(t_{i,n} \mod p)_{n\in\mathbb{N}}$ of $R_{\mathcal{A}_\infty}$. For $\mathfrak{a} \in \mathscr{S}_\bullet$, let $\iota_{\mathfrak{a},\infty}\colon A_{\bullet,\mathfrak{a}}(O_{\overline{K}})[\underline{U}^{\pm 1}] \to A_{\bullet,\mathfrak{a}}(\mathcal{A}_\infty)$ be the homomorphism induced by $A_\bullet(O_{\overline{K}}) \to A_\bullet(\mathcal{A}_\infty)$ and $U_i \mapsto [\underline{t}_i]$. Then the square of the diagram below is commutative because $\theta([\underline{t}_i]) = \lim_{n\to\infty} t_{i,n}^{p^n} = t_i$ in $\widehat{\mathcal{A}}_\infty$, and there exists a unique homomorphism $\iota^{\square}_{\mathfrak{a},\infty}\colon A^{\square}_{\bullet,\mathfrak{a}}(\mathcal{A}) \to A_{\bullet,\mathfrak{a}}(\mathcal{A}_\infty)$ which makes the two triangles commutative.

$$\begin{array}{ccc} A_{\bullet,\mathfrak{a}}(\mathcal{A}_\infty) & \longrightarrow & \mathcal{A}_\infty/p \\ \uparrow{\scriptstyle \iota_{\mathfrak{a},\infty}} & \nwarrow{\scriptstyle \iota^{\square}_{\mathfrak{a},\infty}} & \uparrow \\ A_{\bullet,\mathfrak{a}}(O_{\overline{K}})[\underline{U}^{\pm 1}] & \longrightarrow & A^{\square}_{\bullet,\mathfrak{a}}(\mathcal{A}) \end{array} \qquad (76)$$

Here the right vertical homomorphism is the composition of $A^{\square}_{\bullet,\mathfrak{a}}(\mathcal{A}) \to {}_1\mathcal{A}/p \to \mathcal{A}_\infty/p$. For $g \in \widetilde{\Gamma}_{\mathcal{A}}$, the squares of the diagrams (76) for $\mathfrak{a}$ and $g(\mathfrak{a})$ are compatible with the action of g because $g([\underline{t}_i]) = [\underline{\chi}_i(g)][\underline{t}_i]$ by the definition of $\underline{\chi}_i(g)$. Hence $\iota^{\square}_{\mathfrak{a},\infty}$ and $\iota^{\square}_{g(\mathfrak{a}),\infty}$ are also compatible with the action of g. Similarly the diagrams (76) for $\mathfrak{a}$ and $\varphi(\mathfrak{a})$ are compatible with φ's because $\varphi([\underline{t}_i]) = [\underline{t}_i]^p$. Therefore $\iota^{\square}_{\mathfrak{a},\infty}$ and $\iota^{\square}_{\varphi(\mathfrak{a}),\infty}$ are compatible with φ's. For $m \in \mathbb{N}_{>0}$, $\iota^{\square}_{(p^m,\mathrm{Fil}^1),\infty}$ coincides with the homomorphism ${}_1\mathcal{A}/p^m \to \mathcal{A}_\infty/p^m$ induced by the inclusion ${}_1\mathcal{A} \subset \mathcal{A}_\infty$. Since $\iota_{\mathfrak{a},\infty}$ ($\mathfrak{a} \in \mathscr{S}_\bullet$) define a morphism of inverse systems of algebras indexed by $\mathscr{S}_\bullet$, we see that $\iota^{\square}_{\mathfrak{a},\infty}$ ($\mathfrak{a} \in \mathscr{S}_\bullet$) define a morphism of inverse systems, whose inverse limit gives an $A_\bullet(O_{\overline{K}})$-algebra homomorphism

$$\iota^{\square}_\infty \colon A^{\square}_\bullet(\mathcal{A}) \longrightarrow A_\bullet(\mathcal{A}_\infty) \tag{77}$$

compatible with the action of $\widetilde{\Gamma}_{\mathcal{A}}$, Fil^r and φ; the compatibility with Fil^r follows from the fact that Fil^r is generated by ξ^r (resp. topologically generated by $\mathrm{Fil}^r A_{\mathrm{crys}}(O_{\overline{K}})$) if $\bullet = \inf$ (resp. crys). Composing with the homomorphism $A_\bullet(\mathcal{A}_\infty) \to A_\bullet(\overline{\mathcal{A}})$ induced by the inclusion map $\mathcal{A}_\infty \to \overline{\mathcal{A}}$, we obtain a homomorphism

$$\iota^{\square} \colon A^{\square}_\bullet(\mathcal{A}) \longrightarrow A_\bullet(\overline{\mathcal{A}}) \tag{78}$$

compatible with the action of $G_{\mathcal{A}}$, Fil^r, and φ.

For $r = ap^{-n} \in \mathbb{Z}[\frac{1}{p}]$ ($a \in \mathbb{Z}, n \in \mathbb{N}$), we define t_i^r to be $t_{i,n}^a$, which depends only on r. For $\underline{r} = (r_1, \ldots, r_d) \in \mathbb{Z}[\frac{1}{p}]^d$, let $\underline{t}^{\underline{r}}$ denote the element $(\prod_{1\le i\le d} t_i^{p^{-n}r_i} \mod p)_{n\in\mathbb{N}}$ of $R_{\mathcal{A}_\infty}$.

Lemma 84 *For* $\mathfrak{a} \in \mathscr{S}_{\mathrm{inf}}$*,* $A_{\mathrm{inf},\mathfrak{a}}(\mathcal{A}_\infty)$ *is a free* $A^{\square}_{\mathrm{inf},\mathfrak{a}}(\mathcal{A})$*-module with a basis* $[\underline{t}^{\underline{r}}]$ ($\underline{r} \in (\mathbb{Z}[\frac{1}{p}] \cap [0, 1[)^d$).

Proof Recall that we have $A^{\square}_{\mathrm{inf},\mathfrak{a}}(\mathcal{A}) = A^{\square}_{\mathrm{inf}}(\mathcal{A})/\mathfrak{a}A^{\square}_{\mathrm{inf}}(\mathcal{A})$ by Lemma 79 (2). It suffices to prove the claim for $\mathfrak{a} = (p^m, \xi^n)$ ($m, n \in \mathbb{N}_{>0}$). If $n = 1$, we have $A_{\mathrm{inf},\mathfrak{a}}(\mathcal{A}_\infty) \cong \mathcal{A}_\infty/p^m$ and $A^{\square}_{\mathrm{inf},\mathfrak{a}}(\mathcal{A}) \cong {}_1\mathcal{A}/p^m$. Hence the claim follows from Lemma 78. By Lemmas 80 (4) and 1 (4), ξ is regular on $A_{\mathrm{inf}}(\mathcal{A}_\infty)/p^m$ and $A^{\square}_{\mathrm{inf}}(\mathcal{A})/p^m$. Hence we have exact sequences $0 \to A_{\mathrm{inf}}(\mathcal{A}_\infty)/(p^m, \xi^n) \xrightarrow{\xi} A_{\mathrm{inf}}(\mathcal{A}_\infty)/(p^m, \xi^{n+1}) \to A_{\mathrm{inf}}(\mathcal{A}_\infty)/(p^m, \xi) \to 0$ and $0 \to A^{\square}_{\mathrm{inf}}(\mathcal{A})/(p^m, \xi^n) \xrightarrow{\xi} A^{\square}_{\mathrm{inf}}(\mathcal{A})/(p^m, \xi^{n+1}) \to A^{\square}_{\mathrm{inf}}(\mathcal{A})/(p^m, \xi) \to 0$. Therefore the claim for a general n follows from that in the case $n = 1$ by induction on n. □

Corollary 85 (1) *Let* $a \in A_{\mathrm{inf}}(O_{\overline{K}})$ *be the same as in Lemma 80, and let* $m \in \mathbb{N}_{>0}$*. Let* $\mathfrak{a}$ *be one of the ideals* (p^m, a)*,* (a)*,* (p^m)*, and* (0) *of* $A_{\mathrm{inf}}(O_{\overline{K}})$*. Then the homomorphism* $A^{\square}_{\mathrm{inf}}(\mathcal{A})/\mathfrak{a} \to A_{\mathrm{inf}}(\mathcal{A}_\infty)/\mathfrak{a}$ *induced by* $\iota^{\square}_\infty$ *is injective. In particular,* $\iota^{\square}_\infty \colon A^{\square}_{\mathrm{inf}}(\mathcal{A}) \to A_{\mathrm{inf}}(\mathcal{A}_\infty)$ *is strictly compatible with the filtrations* Fil^r *and* I^r*.*

(2) *Let $m, r \in \mathbb{N}_{>0}$, and let $\mathfrak{a}$ (resp. $\mathfrak{a}'$) be one of the ideals (p^m), Fil^r, (p^m, Fil^r), and (0) of $A^{\square}_{\mathrm{crys}}(\mathcal{A})$ (resp. $A_{\mathrm{crys}}(\mathcal{A}_\infty)$). Then the homomorphism $A^{\square}_{\mathrm{crys}}(\mathcal{A})/\mathfrak{a} \to A_{\mathrm{crys}}(\mathcal{A}_\infty)/\mathfrak{a}'$ induced by $\iota^{\square}_\infty$ is injective.*

Proof (1) The claim for $\mathfrak{a} = (p^m, a)$ follows from Lemmas 84 and 79 (2) because $(p^m, a^n) \in \mathscr{S}_{\inf}$ for $n \in \mathbb{N}_{>0}$ such that $(a \mod p)^n \in \underline{p} R_{O_{\overline{K}}}$. Then we obtain the injectivity for $\mathfrak{a} = (a)$ (resp. (p^m), resp. (0)) from the fact that $A^{\square}_{\inf}(\mathcal{A})/\mathfrak{a}$ and $A_{\inf}(\mathcal{A}_\infty)/\mathfrak{a}$ are p (resp. a, resp. (p, a))-adically complete and separated by Lemmas 80 and 1.

(2) Since $A_{\mathrm{crys}}(\mathcal{A}_\infty)$, $A_{\mathrm{crys}}(\mathcal{A}_\infty)/\mathrm{Fil}^r$, $A^{\square}_{\mathrm{crys}}(\mathcal{A})$, and $A^{\square}_{\mathrm{crys}}(\mathcal{A})/\mathrm{Fil}^r$ are p-torsion free, and p-adically complete and separated (see Lemmas 79 (1) and 81 for the latter two), it suffices to prove the claim for (p) and (p, Fil^r). By the proof of Lemma 8, $A_{\mathrm{crys}}(\mathcal{A}_\infty)/p$ (resp. $(A_{\mathrm{crys}}(\mathcal{A}_\infty)/\mathrm{Fil}^r)/p$) is isomorphic to the scalar extension of $A_{\inf}(\mathcal{A}_\infty)/(p, \xi^p)$ by $A_{\inf}(O_{\overline{K}})/(p, \xi^p) \to A_{\mathrm{crys}}(O_{\overline{K}})/p$ (resp. $(A_{\mathrm{crys}}(O_{\overline{K}})/\mathrm{Fil}^r)/p$). An obvious analogue for $A^{\square}_{\mathrm{crys}}(\mathcal{A})$ and $A^{\square}_{\inf}(\mathcal{A})$ holds by (74), Lemmas 79, and 81. Hence the claim follows from Lemma 84 for $\mathfrak{a} = pA_{\inf}(O_{\overline{K}}) + \xi^p A_{\inf}(O_{\overline{K}})$. □

Corollary 86 *The actions of $\widetilde{\Gamma}_{\mathcal{A}}$ on $A^{\square}_{\inf}(\mathcal{A})$ and $A^{\square}_{\mathrm{crys}}(\mathcal{A})$ are continuous.*

Proof The claim immediately follows from Corollary 85 because the actions of $\widetilde{\Gamma}_{\mathcal{A}}$ on $A_{\inf}(\mathcal{A}_\infty)$ and $A_{\mathrm{crys}}(\mathcal{A}_\infty)$ are continuous (see Lemma 5 and the construction of $A_{\mathrm{crys}}(\Lambda)$ in Sect. 2). □

In the following, we write $[\underline{t}_i]$ also for the image of U_i in $A^{\square}_{\inf}(\mathcal{A})$ and $A^{\square}_{\mathrm{crys}}(\mathcal{A})$.

If we forget the action of $\widetilde{\Gamma}_{\mathcal{A}}$, we have the following description of $A^{\square}_{\inf,\mathfrak{a}}(\mathcal{A})$ ($\mathfrak{a} \in \mathscr{S}_{\inf}$). We have a commutative square of O_K-algebras

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & {}_1\mathcal{A}/p \\ {\scriptstyle T_i \mapsto t_i}\uparrow & \overset{\alpha_{\mathfrak{a}}}{\searrow} & \uparrow \\ O_K[\underline{T}^{\pm 1}] & \xrightarrow[T_i \mapsto [\underline{t}_i]]{} & A^{\square}_{\inf,\mathfrak{a}}(\mathcal{A}). \end{array} \tag{79}$$

Since the homomorphisms $O_K/p^m[\underline{T}^{\pm 1}] \to \mathcal{A}/p^m$ ($m \in \mathbb{N}_{>0}$) are étale and the kernel of the right vertical surjective homomorphism is nilpotent, there exists a unique homomorphism of O_K-algebras $\alpha_{\mathfrak{a}} \colon \mathcal{A} \to A^{\square}_{\inf,\mathfrak{a}}(\mathcal{A})$ such that the two triangles in (79) are commutative. Since the image of $[\underline{t}_i] \in A^{\square}_{\inf}(\mathcal{A})$ in $A^{\square}_{\inf,(p^m,\xi)}(\mathcal{A}) = {}_1\mathcal{A}/p^m$ is t_i, the homomorphism $\alpha_{(p^m,\xi)}$ is the canonical homomorphism $\mathcal{A} \to {}_1\mathcal{A}/p^m$. Let

$$\alpha \colon \mathcal{A} \longrightarrow A^{\square}_{\inf}(\mathcal{A}) \tag{80}$$

be the inverse limit of $\alpha_{\mathfrak{a}}$ ($\mathfrak{a} \in \mathscr{S}_{\inf}$).

Lemma 87 (1) *Let $\varphi_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}$ be the unique lifting of the absolute Frobenius of $\mathcal{A}/p$ compatible with $\sigma \colon O_K \to O_K$ such that $\varphi_{\mathcal{A}}(t_i) = t_i^p$ for all $i \in \mathbb{N} \cap [1, d]$. Then we have $\varphi \circ \alpha = \alpha \circ \varphi_{\mathcal{A}}$.*

(2) *For* $\mathfrak{a} \in \mathscr{S}_{\mathrm{inf}}$*, the homomorphism* $\alpha_{\mathfrak{a}}$ *induces the following isomorphism of* $A_{\mathrm{inf},\mathfrak{a}}(O_{\overline{K}})$*-algebras.*

$$\mathcal{A} \otimes_{O_K} A_{\mathrm{inf},\mathfrak{a}}(O_{\overline{K}}) \xrightarrow{\cong} A^{\square}_{\mathrm{inf},\mathfrak{a}}(\mathcal{A})$$

Proof (1) For $\mathfrak{a} \in \mathscr{S}_{\mathrm{inf}}$, the squares (79) for $\mathfrak{a}$ and $\varphi(\mathfrak{a})$ are compatible with $\varphi_{\mathcal{A}}$, the absolute Frobenius of ${}_1\mathcal{A}/p$, $\varphi^{\square}_{\mathfrak{a}}\colon A^{\square}_{\mathrm{inf},\mathfrak{a}}(\mathcal{A}) \to A^{\square}_{\mathrm{inf},\varphi(\mathfrak{a})}(\mathcal{A})$, and the endomorphism of $O_K[\underline{T}^{\pm 1}]$ defined by σ and $T_i \mapsto T_i^p$. This implies $\varphi^{\square}_{\mathfrak{a}} \circ \alpha_{\mathfrak{a}} = \alpha_{\varphi(\mathfrak{a})} \circ \varphi_{\mathcal{A}}$. We obtain $\varphi \circ \alpha = \alpha \circ \varphi_{\mathcal{A}}$ by taking the inverse limit over $\mathfrak{a} \in \mathscr{S}_{\mathrm{inf}}$.

(2) The homomorphism in the claim is the unique homomorphism between two étale liftings of $O_{\overline{K}}/p[\underline{T}^{\pm 1}] \to {}_1\mathcal{A}/p$ over $A_{\mathrm{inf},\mathfrak{a}}(O_{\overline{K}})[\underline{U}^{\pm 1}]$. Hence it is an isomorphism. $\square$

By definition, $A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m \cong A^{\square}_{\mathrm{crys},(p^m)}(\mathcal{A})$ (Lemma 79 (1)) is a smooth ring over $A_{\mathrm{crys}}(O_{\overline{K}})/p^m$ with coordinates $[\underline{t}_i]$ ($i \in \mathbb{N} \cap [1, d]$). We define $\Omega_{A^{\square}_{\mathrm{crys}}(\mathcal{A})}$ to be the inverse limit of $\Omega_{(A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m)/(A_{\mathrm{crys}}(O_{\overline{K}})/p^m)}$ ($m \in \mathbb{N}$), which is a free $A^{\square}_{\mathrm{crys}}(\mathcal{A})$-module with a basis $d\log[\underline{t}_i]$ ($i \in \mathbb{N} \cap [1, d]$). By taking the inverse limit of the canonical derivation $d\colon A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m \to \Omega_{(A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m)/(A_{\mathrm{crys}}(O_{\overline{K}})/p^m)}$, we obtain a derivation $d\colon A^{\square}_{\mathrm{crys}}(\mathcal{A}) \to \Omega_{A^{\square}_{\mathrm{crys}}(\mathcal{A})}$. Let

$$\alpha\colon \mathcal{A} \longrightarrow A^{\square}_{\mathrm{crys}}(\mathcal{A}) \tag{81}$$

denote the composition of $\alpha\colon \mathcal{A} \to A^{\square}_{\mathrm{inf}}(\mathcal{A})$ (80) with $\kappa\colon A^{\square}_{\mathrm{inf}}(\mathcal{A}) \to A^{\square}_{\mathrm{crys}}(\mathcal{A})$ (75). By Lemma 87 and (74), we have isomorphisms of $A_{\mathrm{crys}}(O_{\overline{K}})$-algebras compatible with φ

$$\begin{aligned} A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K}\mathcal{A} &:= \varprojlim_m (A_{\mathrm{crys}}(O_{\overline{K}}) \otimes_{O_K} \mathcal{A})/p^m \\ &\cong \varprojlim_m A_{\mathrm{crys},(p^m)}(O_{\overline{K}}) \otimes_{A_{\mathrm{inf},\mathfrak{a}_m}(O_{\overline{K}})} (A_{\mathrm{inf},\mathfrak{a}_m}(O_{\overline{K}}) \otimes_{O_K} \mathcal{A}) \\ &\cong \varprojlim_m A_{\mathrm{crys},(p^m)}(O_{\overline{K}}) \otimes_{A_{\mathrm{inf},\mathfrak{a}_m}(O_{\overline{K}})} A^{\square}_{\mathrm{inf},\mathfrak{a}_m}(\mathcal{A}) \xrightarrow{\cong} A^{\square}_{\mathrm{crys}}(\mathcal{A}), \end{aligned} \tag{82}$$

where $\mathfrak{a}_m$ denotes the ideal of $A_{\mathrm{inf}}(O_{\overline{K}})$ generated by p^m and $[\underline{p}]^{pm}$. The homomorphism $\alpha^*\colon \Omega_{\mathcal{A}} \to \Omega_{A^{\square}_{\mathrm{crys}}(\mathcal{A})}$ induced by ($\alpha \mod p^m$) ($m \in \mathbb{N}$) sends $d\log t_i$ to $d\log[\underline{t}_i]$, and the following diagram is commutative.

$$\begin{array}{ccc} A^{\square}_{\mathrm{crys}}(\mathcal{A}) & \xrightarrow{\quad d \quad} & \Omega_{A^{\square}_{\mathrm{crys}}(\mathcal{A})} \\ {\scriptstyle (82)}\uparrow\cong & & \cong\uparrow{\scriptstyle (\mathrm{id}\otimes\alpha^*)\circ((82)\otimes\mathrm{id})} \\ A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K}\mathcal{A} & \xrightarrow{\mathrm{id}\widehat{\otimes}d} & A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K}\mathcal{A} \otimes_{\mathcal{A}} \Omega_{\mathcal{A}} \end{array} \tag{83}$$

We define derivations $d_i^{\log}\colon A_{\text{crys}}^{\square}(\mathcal{A}) \to A_{\text{crys}}^{\square}(\mathcal{A})$ ($i \in \mathbb{N} \cap [1, d]$) by $d(x) = \sum_{1 \le i \le d} d_i^{\log}(x) \otimes d\log[\underline{t}_i]$ ($x \in A_{\text{crys}}^{\square}(\mathcal{A})$). We give a certain relation between the derivation $d_i^{\log}$ and the action of $\Gamma_{\mathcal{A}}$ on $A_{\text{crys}}^{\square}(\mathcal{A})$ (Proposition 91).

Lemma 88 *The actions of $\Gamma_{\mathcal{A}}$ on $A_{\inf}^{\square}(\mathcal{A})/\pi$ and on $A_{\text{crys}}^{\square}(\mathcal{A})/\pi$ are trivial.*

Proof Let $g \in \Gamma_{\mathcal{A}}$. We have $[\underline{\chi}_i(g)] \equiv 1 \mod \pi A_{\inf}(O_{\overline{K}})$ for $i \in \mathbb{N} \cap [1, d]$, and the action of g on $O_{\overline{K}}$ is trivial. Hence the actions of g on ${}_1\mathcal{A}/p$, $O_{\overline{K}}/p[\underline{T}^{\pm 1}]$ and $A_{\inf,(p^m,\pi)}(O_{\overline{K}})[\underline{U}^{\pm 1}]$ ($m \in \mathbb{N}$) are trivial. This implies that the action of g on $A_{\inf,(p^m,\pi)}^{\square}(\mathcal{A})$ is also trivial. By taking the inverse limit over $m \in \mathbb{N}$ and using Lemma 80 (5), we see that the action of g on $A_{\inf}^{\square}(\mathcal{A})/\pi$ is trivial. Let $F_{g,\inf}$ be the $A_{\inf}(O_{\overline{K}})$-linear endomorphism $\pi^{-1}(g-1)$ of $A_{\inf}^{\square}(\mathcal{A})$. Since $A_{\text{crys}}^{\square}(\mathcal{A}) \cong \varprojlim_m A_{\text{crys},(p^m)}(O_{\overline{K}}) \otimes_{A_{\inf,(p^m,[\underline{p}]^{pm})}(O_{\overline{K}})} A_{\inf,(p^m,[\underline{p}]^{pm})}^{\square}(\mathcal{A}) \cong \varprojlim_m A_{\text{crys}}(O_{\overline{K}})/p^m \otimes_{A_{\inf}(O_{\overline{K}})} A_{\inf}^{\square}(\mathcal{A})$ by (74) and Lemma 79 (2), $F_{g,\inf}$ induces an $A_{\text{crys}}(O_{\overline{K}})$-linear endomorphism $F_{g,\text{crys}}$ of $A_{\text{crys}}^{\square}(\mathcal{A})$ satisfying $\pi F_{g,\text{crys}} = g - 1$. This completes the proof. □

Lemma 89 *Let R be the subring $\mathbb{Z}_p[T, \frac{T^{p-1}}{p}, \frac{1}{p}(\frac{T^{p-1}}{p})^p]$ of the polynomial ring $\mathbb{Q}_p[T]$, and let $\widehat{R}$ be the p-adic completion $\varprojlim_n R/p^nR$ of R. We equip R and $\widehat{R}$ with the p-adic topology.*

(1) *We have $(n!)^{-1}T^{n-1} \in R$ for $n \in \mathbb{N}_{>0}$, and it converges to 0 as $n \to \infty$.*
(2) $\log(1+T) = \sum_{n \in \mathbb{N}_{>0}} (-1)^{n-1} n^{-1} T^n$ *converges in $\widehat{R}$ and is contained in $T\widehat{R}$.*
(3) *We have* $(1+T)^l = \exp(l \log(1+T)) := \sum_{n \in \mathbb{N}} (n!)^{-1} l^n (\log(1+T))^n$ *in $\widehat{R}$ for $l \in \mathbb{N}$.*
(4) *We have* $\log(1+T) \in T \cdot \widehat{R}^{\times}$.
(5) *We have* $p^{-n}\{(1+T)^{p^n} - 1\} \in R$ $(n \in \mathbb{N})$ *and* $\lim_{n \to \infty} p^{-n}\{(1+T)^{p^n} - 1\} = \log(1+T)$ *in $\widehat{R}$.*

Proof (1) Put $n - 1 = (p-1)a + b$ ($a, b \in \mathbb{N}$, $b \in \mathbb{N} \cap [0, p-2]$). Then the claim follows from $(p^a)^{-1}T^{n-1} = (p^{-1}T^{p-1})^a T^b \in \mathbb{Z}_p[T, \frac{T^{p-1}}{p}]$, $v_p(n!) \le \frac{n-1}{p-1}$, and $(p^{-1}T^{p-1})^n \to 0$ in $\widehat{R}$ as $n \to \infty$.

(2) This immediately follows from (1).

(3) By (1) and (2), each term of the power series is contained in $\widehat{R}$ and the sum converges in $\widehat{R}$. For $m \in \mathbb{N}_{>0}$, let f_m be the composition of the inclusion map $R \hookrightarrow \mathbb{Q}_p[T]$ with the projection map $\mathbb{Q}_p[T] \to \mathbb{Q}_p[T]/(T^m)$. Then the image of f_m is a finitely generated $\mathbb{Z}_p$-module, which is p-adically complete and separated. Hence it induces a homomorphism $\widehat{f}_m\colon \widehat{R} \to \mathbb{Q}_p[T]/(T^m)$. By taking the inverse limit over m, we obtain an injective homomorphism $\widehat{f}\colon \widehat{R} \to \mathbb{Q}_p[[T]]$. Since the image of each term of the power series $\sum_{n>0} (-1)^{n-1} n^{-1} T^n$ and $\sum_{n>0} (n!)^{-1} l^n (\log(1+T))^n$ under $\widehat{f}_m$ is 0 for $n \ge m$, it suffices to prove the claim in $\mathbb{Q}_p[[T]]$ with respect to the T-adic topology, which is well-known.

(4) We have $\log(1+T) \in T\widehat{R}$ by (2), and then the claim (1) implies that $\sum_{n>0} (n!)^{-1} (\log(1+T))^{n-1}$ converges to an element of $\widehat{R}$. By (3), we obtain $T =$

$\exp(\log(1+T))-1 \in \log(1+T)\cdot\widehat{R}$. Hence $\log(1+T)\in T\widehat{R}^{\times}$ because $\widehat{R}$ is an integral domain.

(5) By (3), we have

$$p^{-m}\{(1+T)^{p^m}-1\}=\log(1+T)+p^m\sum_{n\geq 2}p^{m(n-2)}(n!)^{-1}(\log(1+T))^n,$$

and, by (1) and (4), the infinite sum $\sum_{n\geq 2}$ in the right-hand side converges to an element of $\widehat{R}$. This implies the claim. □

Let $\varepsilon=(\varepsilon_n)_{n\in\mathbb{N}}\in\mathbb{Z}_p(1)(O_{\overline{K}})$ and $\underline{\varepsilon}\in R_{O_{\overline{K}}}$ be as in the definition of π after (1), and let t be the element $\log[\underline{\varepsilon}]$ of $A_{\mathrm{crys}}(O_{\overline{K}})$.

Lemma 90 (1) *For $l\in\mathbb{Z}$, we have $p^{-n}([\underline{\varepsilon}^{p^n l}]-1)\in A_{\mathrm{crys}}(O_{\overline{K}})$ $(n\in\mathbb{N})$, and it converges to $l\cdot t$ in $A_{\mathrm{crys}}(O_{\overline{K}})$ as $n\to\infty$.*
(2) *We have $\pi\in t\cdot A_{\mathrm{crys}}(O_{\overline{K}})^{\times}$.*

Proof Let $\widehat{R}$ be as in Lemma 89. We have $p^{-1}([\underline{\varepsilon}^l]-1)^{p-1}$, $(p!)^{-1}(p^{-1}([\underline{\varepsilon}^l]-1)^{p-1})^p\in A_{\mathrm{crys}}(O_{\overline{K}})$ because $[\underline{\varepsilon}^l]-1\in\pi A_{\mathrm{inf}}(O_{\overline{K}})$ and $p^{-1}\pi^{p-1}\in \mathrm{Fil}^1A_{\mathrm{crys}}(O_{\overline{K}})$ ([18, Lemma A3.1]). Hence we can define a continuous homomorphism $\kappa_l\colon\widehat{R}\to A_{\mathrm{crys}}(O_{\overline{K}})$ by $T\mapsto[\underline{\varepsilon}^l]-1$, and Lemma 89 (5) implies the claim (1). We obtain the claim (2) from Lemma 89 (4) by using κ_1. □

For $i\in\mathbb{N}\cap[1,d]$, let γ_i be the element of $\Gamma_{\mathcal{A}}$ characterized by $\gamma_i(t_{j,n})=t_{j,n}$ (if $j\neq i$), $\varepsilon_n t_{i,n}$ (if $j=i$) for all $n\in\mathbb{N}$ and $j\in\mathbb{N}\cap[1,d]$; the existence of γ_i follows from Lemma 78.

Proposition 91 (1) *For $\gamma\in\Gamma_{\mathcal{A}}$ and $x\in A^{\square}_{\mathrm{crys}}(\mathcal{A})$, we have $p^{-n}(\gamma^{p^n}-1)(x)\in A^{\square}_{\mathrm{crys}}(\mathcal{A})$ and $\nabla_\gamma(x):=\lim_{n\to\infty}p^{-n}(\gamma^{p^n}-1)(x)$ converges.*
(2) *We have $\nabla_{\gamma_i}=td_i^{\log}$ for $i\in\mathbb{N}\cap[1,d]$.*
(3) *For $x\in A^{\square}_{\mathrm{crys}}(\mathcal{A})$ and $i\in\mathbb{N}\cap[1,d]$, we have $(n!)^{-1}(td_i^{\log})^n(x)\in A^{\square}_{\mathrm{crys}}(\mathcal{A})$, and $\exp(td_i^{\log})(x)=\sum_{n\in\mathbb{N}}(n!)^{-1}(td_i^{\log})^n(x)$ converges to $\gamma_i(x)$.*

Proof Let R and $\widehat{R}$ be as in Lemma 89. By $p^{-1}\pi^{p-1}$, $p^{-1}(p^{-1}\pi^{p-1})^p\in A_{\mathrm{crys}}(O_{\overline{K}})$ (see the proof of Lemma 90) and Lemma 88, we can define an action of R on $A^{\square}_{\mathrm{crys}}(\mathcal{A})$ by $Tx=(\gamma-1)(x)$. Since $A^{\square}_{\mathrm{crys}}(\mathcal{A})$ is p-adically complete and separated (Lemma 79 (1)), this action extends to an action of $\widehat{R}$ on $A^{\square}_{\mathrm{crys}}(\mathcal{A})$. Hence, by Lemma 89 (5), the claim (1) holds and ∇_γ coincides with the action of $\log(1+T)$. Then the first (resp. second) claim in (3) follows from the claim (2) and Lemma 89 (1) and (4) (resp. (3)).

It remains to prove (2). We see that ∇_γ is an $A_{\mathrm{crys}}(O_{\overline{K}})$-linear derivation by taking the limit $m\to\infty$ of $p^{-m}(\gamma^{p^m}-1)(x\cdot y)=p^{-m}(\gamma^{p^m}-1)(x)\cdot\gamma^{p^m}(y)+x\cdot p^{-m}(\gamma^{p^m}-1)(y)$ for $x,y\in A^{\square}_{\mathrm{crys}}(\mathcal{A})$ and $m\in\mathbb{N}$. Hence, by the universal property of the canonical derivation of $A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m$ over $A_{\mathrm{crys}}(O_{\overline{K}})/p^m$ for $m\in\mathbb{N}_{>0}$, there exists a unique $A^{\square}_{\mathrm{crys}}(\mathcal{A})$-linear homomorphism $f_\gamma\colon\Omega_{A^{\square}_{\mathrm{crys}}(\mathcal{A})}\to A^{\square}_{\mathrm{crys}}(\mathcal{A})$ such that

$f_\gamma \circ d = \nabla_\gamma$. When $\gamma = \gamma_i$, we have $f_{\gamma_i}(d[\underline{t}_j]) = \nabla_{\gamma_i}([\underline{t}_j]) = 0$ if $j \neq i$ because $\gamma_i([\underline{t}_j]) = [\underline{t}_j]$, and $f_{\gamma_i}(d[\underline{t}_i]) = \lim_{n\to\infty} p^{-n}([\underline{\varepsilon}^{p^n}] - 1)[\underline{t}_i] = t[\underline{t}_i]$ by Lemma 90 (1). Hence $\nabla_{\gamma_i} = t d_i^{\log}$. □

We can construct a period ring $\mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})$ in the same way as $\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})$ defined in Sect. 2 using $A^{\square}_{\mathrm{crys}}(\mathcal{A})$ instead of $A_{\mathrm{crys}}(\overline{\mathcal{A}})$ as follows. Let γ be the divided power structure of the ideal pO_K of O_K. For $m \in \mathbb{N}_{>0}$, we define $\mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A})$ to be the divided power envelope compatible with γ of $(A^{\square}_{\mathrm{inf}}(\mathcal{A}) \otimes_{O_K} A)/p^m$ with respect to the kernel of the homomorphism to ${}_1\mathcal{A}/p^m$. Then one can define the action of $\widetilde{\Gamma}_{\mathcal{A}}$ on $\mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A})$ using its action on $A^{\square}_{\mathrm{crys}}(\mathcal{A})$, and the $\widetilde{\Gamma}_{\mathcal{A}}$-stable filtration $\mathrm{Fil}^r\mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A})$ ($m \in \mathbb{N}$) using the divided power ideal of $\mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A})$ in the same way as those for $\mathscr{A}_{\mathrm{crys},m}(\overline{\mathcal{A}})$. We also have a natural $\widetilde{\Gamma}_{\mathcal{A}}$-equivariant derivation $\nabla\colon \mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A}) \to \mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A}) \otimes_{\mathcal{A}} \Omega_{\mathcal{A}}$ compatible with the derivation $d\colon \mathcal{A} \to \Omega_{\mathcal{A}}$. It is integrable as a connection with respect to $(A/p^m)/(O_K/p^m)$ and satisfies $\nabla(\mathrm{Fil}^r\mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A})) \subset \mathrm{Fil}^{r-1}\mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A}) \otimes_{\mathcal{A}} \Omega_{\mathcal{A}}$. The lifting of the absolute Frobenius $\varphi_{\mathcal{A}}$ of $\mathcal{A}$ characterized by $\varphi_{\mathcal{A}}(t_i) = t_i^p$ and φ of $A^{\square}_{\mathrm{inf}}(\mathcal{A})$ induce a lifting of the absolute Frobenius φ of $\mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A})$, which is compatible with ∇ and the $\widetilde{\Gamma}_{\mathcal{A}}$-action and satisfies $\varphi(\mathrm{Fil}^r\mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A})) \subset p^r\mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A})$ for $r \in \mathbb{N} \cap [0, p-1]$. Let $v_{i,m}$ denote the image of $[\underline{t}_i] \otimes t_i^{-1} - 1$ in $\mathrm{Fil}^1\mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A})$. Then, by Lemma 83, we have an isomorphism of PD-algebras over $A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m$ (cf. (4), (5))

$$A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m\langle V_1, V_2, \ldots V_d\rangle \xrightarrow{\cong} \mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A}) \tag{84}$$

sending V_i to $v_{i,m}$, via which $\mathrm{Fil}^r\mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A})$ is isomorphic to the direct sum of $\mathrm{Fil}^{r-|\underline{n}|}A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m \prod_{1\le i\le d} V_i^{[n_i]}$ ($\underline{n} = (n_i) \in \mathbb{N}^d$). Combining with Corollary 86, we see that the action of $\widetilde{\Gamma}_{\mathcal{A}}$ on $\mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A})$ is continuous. We define $\mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})$ to be the inverse limit of $\mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A})$ ($m \in \mathbb{N}_{>0}$), which is naturally endowed with a continuous action of $\widetilde{\Gamma}_{\mathcal{A}}$, a decreasing filtration $\mathrm{Fil}^r\mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})$ ($r \in \mathbb{Z}$), and a lifting of the absolute Frobenius φ. Let v_i denote the image of $[\underline{t}_i] \otimes t_i^{-1} - 1$ in $\mathrm{Fil}^1\mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})$.

We have the following analogue of Lemma 34 (1).

Lemma 92 *The homomorphism $\alpha\colon \mathcal{A} \to A^{\square}_{\mathrm{crys}}(\mathcal{A})$ (81) coincides with the composition α' of the canonical homomorphism $\mathcal{A} \to \mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})$ with the homomorphism $\mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A}) \to A^{\square}_{\mathrm{crys}}(\mathcal{A})$ over $A^{\square}_{\mathrm{crys}}(\mathcal{A})$ defined by $v_i^{[n]} \mapsto 0$ ($i \in \mathbb{N} \cap [1, d]$, $n \in \mathbb{N}_{>0}$).*

Proof For $m \in \mathbb{N}_{>0}$, we have a commutative square of O_K/p^m-algebras

$$\begin{array}{ccc}
\mathcal{A}/p^m & \longrightarrow & {}_1\mathcal{A}/p \\
{\scriptstyle T_i\mapsto t_i}\uparrow & \searrow & \uparrow \\
O_K/p^m[\underline{T}^{\pm 1}] & \xrightarrow[T_i\mapsto[\underline{t}_i]]{} & A^{\square}_{\mathrm{crys},(p^m)}(\mathcal{A}).
\end{array}$$

Since $O_K/p^m[\underline{T}^{\pm 1}] \to \mathcal{A}/p^m$ is étale and the kernel of the right vertical surjective homomorphism is a nilideal, there exists a unique dotted homomorphism making the two triangles commutative. The reductions mod p^m of α and α' both satisfy the condition and hence coincide. By taking $\varprojlim_m$, we obtain $\alpha = \alpha'$. $\square$

The homomorphism $\iota^{\square}$ (78) for $\bullet = \inf$ induces an $\mathcal{A}$-PD-homomorphism $\mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A}) \to \mathscr{A}_{\mathrm{crys},m}(\overline{\mathcal{A}})$ which is compatible with $\iota^{\square}$ for $\bullet = \mathrm{crys}$, Fil^r, φ, ∇, and the actions of the Galois groups via $G_{\mathcal{A}} \to \widetilde{\Gamma}_{\mathcal{A}}$. It is injective by (4), (84), Corollary 85 (2) and Lemma 8. By taking the inverse limit over $m \in \mathbb{N}_{>0}$, we obtain an injective homomorphism

$$\mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A}) \hookrightarrow \mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}}) \tag{85}$$

compatible with all structures. We obtain the following proposition from Proposition 62.

Proposition 93 *We have* $\mathcal{A} \xrightarrow{\cong} \mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})^{\widetilde{\Gamma}_{\mathcal{A}}}$ *and* $\mathrm{Fil}^1\mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})^{\widetilde{\Gamma}_{\mathcal{A}}} = 0.$

13 $A^{\square}_{\mathrm{crys}}$-Representations with φ and Fil

We keep the notation and the assumption in Sect. 12. As one can easily guess, one can apply all the arguments in Sect. 5 with $(\mathcal{B}, s_1, \dots, s_e) = (\mathcal{A}, t_1, \dots, t_d)$ and in Sect. 8 to $\mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})$, $A^{\square}_{\mathrm{crys}}(\mathcal{A})$ and $A^{\square}_{\inf}(\mathcal{A})$ except for those related to T^*_{crys} as follows.

Let $(M, \mathrm{Fil}^{\bullet}M, \nabla, \Phi)$ be an object of $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A}, \Phi)$ (Sect. 4). We follow the notation introduced in the second and third paragraphs in Sect. 5. For $m \in \mathbb{N}_{>0}$, put $D^{\square}_m := \mathrm{Spec}(A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m)$ and ${}_1X_m := \mathrm{Spec}({}_1\mathcal{A}/p^m)$. By Lemma 83, the closed immersions ${}_1X_m \hookrightarrow D^{\square}_m$ and ${}_1X_1 \hookrightarrow D^{\square}_m$ are naturally regarded as objects of $\mathrm{CRYS}(X_m/\Sigma_m)$ and $\mathrm{CRYS}(X_1/\Sigma_m)$, respectively. By the proof of Lemma 83, the PD-structure on $\mathrm{Ker}(A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m \to {}_1\mathcal{A}/p)$ is induced by that of $\mathrm{Ker}(A_{\mathrm{crys}}(O_{\overline{K}})/p^m \to O_{\overline{K}}/p)$. Hence the endomorphism φ of $A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m$ is a PD-homomorphism with respect to the PD-structure. Similarly to $TA_{\mathrm{crys},m}(M)$ (23), we can define an $A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m$-module $TA^{\square}_{\mathrm{crys},m}(M)$ with a semilinear $\widetilde{\Gamma}_{\mathcal{A}}$-action, a $\widetilde{\Gamma}_{\mathcal{A}}$-equivariant filtration $\mathrm{Fil}^{\bullet}$, and a $\widetilde{\Gamma}_{\mathcal{A}}$-equivariant semilinear endomorphism φ by evaluating $\mathcal{F}_m$ and $\mathcal{G}_m$ on the objects ${}_1X_m \hookrightarrow D^{\square}_m$ and ${}_1X_1 \hookrightarrow D^{\square}_m$ of $\mathrm{CRYS}(X_m/\Sigma_m)$ and $\mathrm{CRYS}(X_1/\Sigma_m)$ as follows.

$$TA^{\square}_{\mathrm{crys},m}(M) := \Gamma({}_1X_m \hookrightarrow D^{\square}_m, \mathcal{F}_m) \cong \Gamma({}_1X_1 \hookrightarrow D^{\square}_m, \mathcal{G}_m) \tag{86}$$

Let $\gamma^{\square}$ be the PD-structure on the ideal $p(A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m) + \mathrm{Fil}^1A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m$ of $A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m$ compatible with the PD-structures on pO_K and $\mathrm{Fil}^1A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m$. Let ι_m denote the canonical homomorphism $\mathcal{A}_m \to \mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A})$. Then, by using ${}_1X_1 \hookrightarrow {}_1X_m \hookrightarrow D^{\square}_m$, $\gamma^{\square}$, $X_m \times_{\Sigma_m} D^{\square}_m$, and $\mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A})$ instead of $\overline{X}_1 \hookrightarrow \overline{X}_m \hookrightarrow \overline{D}_m$, $\overline{\gamma}$,

$Y_m \times_{\Sigma_m} \overline{D}_m$, and $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$, we obtain the following analogues of (29) and (30): a $\widetilde{\Gamma}_{\mathcal{A}}$-equivariant $A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m$-linear filtered isomorphism compatible with φ

$$TA^{\square}_{\mathrm{crys},m}(M) \cong (M_m \otimes_{\mathcal{A}_m,\iota_m} \mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A}))^{\nabla=0}, \tag{87}$$

and a $\widetilde{\Gamma}_{\mathcal{A}}$-equivariant $\mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A})$-linear filtered isomorphism compatible with ∇ and φ

$$TA^{\square}_{\mathrm{crys},m}(M) \otimes_{A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m} \mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A}) \xrightarrow{\cong} M_m \otimes_{\mathcal{A}_m,\iota_m} \mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A}). \tag{88}$$

Let α_m be the reduction mod p^m of the homomorphism $\alpha\colon \mathcal{A} \to A^{\square}_{\mathrm{crys}}(\mathcal{A})$ (81), which is compatible with φ by Lemma 87 (1). Then, by using α_m instead of β_m, we obtain the following analogues of (32) and (33): an $A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m$-linear filtered isomorphism compatible with φ

$$TA^{\square}_{\mathrm{crys},m}(M) \cong M_m \otimes_{\mathcal{A}_m,\alpha_m} A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m \tag{89}$$

and an $\mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A})$-linear filtered isomorphism compatible with ∇ and φ

$$c^{\square}_{M_m} : M_m \otimes_{\mathcal{A}_m,\alpha_m} \mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A}) \xrightarrow{\cong} M_m \otimes_{\mathcal{A}_m,\iota_m} \mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A}), \tag{90}$$

which is obtained by combining (89) with (88), and is explicitly given by

$$c^{\square}_{M_m}(x \otimes 1) = \sum_{\underline{n}\in\mathbb{N}^d} \nabla^{\log}_{\underline{n}}(x) \otimes \underline{v}^{[\underline{n}]}_m, \quad (c^{\square}_{M_m})^{-1}(x \otimes 1) = \sum_{\underline{n}\in\mathbb{N}^d} \nabla^{\log}_{\underline{n}}(x) \otimes \underline{v}'^{[\underline{n}]}_m \tag{91}$$

for $x \in M_m$, where $v'_{i,m} = [\underline{t}_i]^{-1} \otimes t_i - 1 = (1 + v_{i,m})^{-1} - 1 \in \mathscr{A}^{\square}_{\mathrm{crys},m}(\mathcal{A})$, $\nabla(x) = \sum_{1\le i\le d} \nabla^{\log}_i(x) \otimes d\log t_i$, $\nabla^{\log}_{\underline{n}}(x) = \prod_{1\le i\le d} \prod_{0\le j\le n_i-1} (\nabla^{\log}_i - j)$, $\underline{v}^{[\underline{n}]}_m = \prod_{1\le i\le d} v^{[n_i]}_{i,m}$, and $\underline{v}'^{[\underline{n}]}_m = \prod_{1\le i\le d} v'^{[n_i]}_{i,m}$ for $\underline{n} = (n_i)_{1\le i\le d} \in \mathbb{N}^d$.

We define the $A^{\square}_{\mathrm{crys}}(\mathcal{A})$-module $TA^{\square}_{\mathrm{crys}}(M)$ by

$$TA^{\square}_{\mathrm{crys}}(M) := \varprojlim_m TA^{\square}_{\mathrm{crys},m}(M),$$

which is naturally endowed with a semilinear action of $\widetilde{\Gamma}_{\mathcal{A}}$, a $\widetilde{\Gamma}_{\mathcal{A}}$-stable decreasing filtration Fil^r ($r \in \mathbb{Z}$) compatible with that of $A^{\square}_{\mathrm{crys}}(\mathcal{A})$, and a $\widetilde{\Gamma}_{\mathcal{A}}$-equivariant endomorphism φ compatible with φ of $A^{\square}_{\mathrm{crys}}(\mathcal{A})$. By taking the inverse limit of (89), we obtain an $A^{\square}_{\mathrm{crys}}(\mathcal{A})$-linear isomorphism compatible with φ

$$TA^{\square}_{\mathrm{crys}}(M) \cong M \otimes_{\mathcal{A},\alpha} A^{\square}_{\mathrm{crys}}(\mathcal{A}). \tag{92}$$

Since M is finite filtered free of level $[0, p-2]$ (Definition 10 (3)), we see that the product filtration on the right-hand side of (92) is the inverse limit of the product filtration on $M_m \otimes_{\mathcal{A}_m,\alpha_m} A^{\square}_{\mathrm{crys}}(\mathcal{A})/p^m$ ($m \in \mathbb{N}_{>0}$), and therefore gives the filtration on $TA^{\square}_{\mathrm{crys}}(M)$ via the isomorphism (92). This shows that $TA^{\square}_{\mathrm{crys}}(M)$ is a finite filtered free module of level $[0, p-2]$ over the filtered ring $A^{\square}_{\mathrm{crys}}(\mathcal{A})$. Let ι denote the canonical homomorphism $\mathcal{A} \to \mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})$. By taking the inverse limit of (87) and (88), we obtain a $\widetilde{\Gamma}_{\mathcal{A}}$-equivariant $A^{\square}_{\mathrm{crys}}(\mathcal{A})$-linear filtered isomorphism compatible with φ

$$TA^{\square}_{\mathrm{crys}}(M) \cong (M \otimes_{\mathcal{A},\iota} \mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A}))^{\nabla=0}, \tag{93}$$

and see that it induces a $\widetilde{\Gamma}_{\mathcal{A}}$-equivariant $\mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})$-linear filtered isomorphism compatible with ∇ and φ

$$TA^{\square}_{\mathrm{crys}}(M) \otimes_{A^{\square}_{\mathrm{crys}}(\mathcal{A})} \mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A}) \xrightarrow{\cong} M \otimes_{\mathcal{A},\iota} \mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A}). \tag{94}$$

By taking the inverse limit of (90), we see that (92) and (94) induce an $\mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})$-linear filtered isomorphism compatible with ∇ and φ

$$c^{\square}_M \colon M \otimes_{\mathcal{A},\alpha} \mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A}) \xrightarrow{\cong} M \otimes_{\mathcal{A},\iota} \mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A}), \tag{95}$$

which is explicitly given by

$$c^{\square}_M(x \otimes 1) = \sum_{\underline{n} \in \mathbb{N}^d} \nabla^{\log}_{\underline{n}}(x) \otimes \underline{v}^{[\underline{n}]}, \quad (c^{\square}_M)^{-1}(x \otimes 1) = \sum_{\underline{n} \in \mathbb{N}^d} \nabla^{\log}_{\underline{n}}(x) \otimes \underline{v}'^{[\underline{n}]} \tag{96}$$

for $x \in M$, where $v'_i = (1 + v_i)^{-1} - 1 \in \mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})$, $\underline{v}^{[\underline{n}]} = \prod_{1 \le i \le d} v_i^{[n_i]}$, $\underline{v}'^{[\underline{n}]} = \prod_{1 \le i \le d} v_i'^{[n_i]}$, and the endomorphisms $\nabla^{\log}_i$ and $\nabla^{\log}_{\underline{n}}$ of M are defined in the same way as after (91).

Put $\underline{A}^{\square}_{\inf}(\mathcal{A}) := A^{\square}_{\inf}(\mathcal{A})/I^{p-1}A^{\square}_{\inf}(\mathcal{A}) \cong A^{\square}_{\mathrm{crys}}(\mathcal{A})/I^{p-1}A^{\square}_{\mathrm{crys}}(\mathcal{A})$ (Lemma 82). The Frobenius endomorphism φ of $A^{\square}_{\inf}(\mathcal{A})$ induces an endomorphism of $\underline{A}^{\square}_{\inf}(\mathcal{A})$, which is also denoted by φ. It is compatible with φ of $A^{\square}_{\mathrm{crys}}(\mathcal{A})$ because so is the homomorphism κ (75). For $r \in \mathbb{Z}$ with $r \le p-1$, we define $\mathrm{Fil}^r \underline{A}^{\square}_{\inf}(\mathcal{A})$ to be the image of $\mathrm{Fil}^r A^{\square}_{\inf}(\mathcal{A})$ in $\underline{A}^{\square}_{\inf}(\mathcal{A})$, which coincides with that of $\mathrm{Fil}^r A^{\square}_{\mathrm{crys}}(\mathcal{A})$ (Lemma 82). The three quadruplets

$$\begin{gathered}(A^{\square}_{\mathrm{crys}}(\mathcal{A}), p, \varphi, (\mathrm{Fil}^r A^{\square}_{\mathrm{crys}}(\mathcal{A}))_{r \in \mathbb{N} \cap [0,p-2]}),\\ (A^{\square}_{\inf}(\mathcal{A}), q, \varphi, (\mathrm{Fil}^r A^{\square}_{\inf}(\mathcal{A}))_{r \in \mathbb{N} \cap [0,p-2]}),\\ (\underline{A}^{\square}_{\inf}(\mathcal{A}), p, \varphi, (\mathrm{Fil}^r \underline{A}^{\square}_{\inf}(\mathcal{A}))_{r \in \mathbb{N} \cap [0,p-2]})\end{gathered} \tag{97}$$

satisfy Condition 39 for $a = p-2$ by Lemmas 79 (1), 80, the definition of Fil^r on $A^{\square}_{\inf}(\mathcal{A})$ and $A^{\square}_{\mathrm{crys}}(\mathcal{A})$, $\varphi(\mathrm{Fil}^r A_{\mathrm{crys}}(O_{\overline{K}})) \subset p^r A_{\mathrm{crys}}(O_{\overline{K}})$ ($r \in \mathbb{N} \cap [0, p-2]$) and $\mathrm{Fil}^1 A_{\inf}(O_{\overline{K}}) = q' A_{\inf}(O_{\overline{K}})$. We have $\varphi^{-1}(q^r A^{\square}_{\inf}(\mathcal{A})) = \mathrm{Fil}^r A^{\square}_{\inf}(\mathcal{A})$ ($r \in \mathbb{N}$) because the homomorphisms $A^{\square}_{\inf}(\mathcal{A})/q^r \to A_{\inf}(\mathcal{A}_\infty)/q^r$ and $A^{\square}_{\inf}(\mathcal{A})/(q')^r \to$

$A_{\mathrm{inf}}(\mathcal{A}_\infty)/(q')^r$ are injective for $r \in \mathbb{N}$ by Corollary 85 (1). Hence we may apply Lemma 46 to the second quadruplet and obtain an equivalence of categories

$$\mathrm{M}^q_{[0,p-2],\mathrm{free}}(A^\square_{\mathrm{inf}}(\mathcal{A}),\varphi) \xrightarrow{\sim} \mathrm{MF}^q_{[0,p-2],\mathrm{free}}(A^\square_{\mathrm{inf}}(\mathcal{A}),\varphi), \qquad (98)$$
$$(M,\varphi_M) \mapsto (M,\varphi_M,(\varphi_M^{-1}(q^rM))_{r\in\mathbb{N}\cap[0,p-2]}).$$

For the quadruplets (97), the homomorphisms $A^\square_{\mathrm{crys}}(\mathcal{A}) \to \underline{A}^\square_{\mathrm{inf}}(\mathcal{A})$, $A^\square_{\mathrm{inf}}(\mathcal{A}) \to \underline{A}^\square_{\mathrm{inf}}(\mathcal{A})$ and $A^\square_{\mathrm{inf}}(\mathcal{A}) \to A^\square_{\mathrm{crys}}(\mathcal{A})$ satisfy the conditions on κ assumed before (44) by the definition of I^{p-1} for $A^\square_{\mathrm{crys}}(\mathcal{A})$ and $A^\square_{\mathrm{inf}}(\mathcal{A})$ and $q = p(1+\frac{\pi_0}{p}) \in p\cdot A_{\mathrm{crys}}(O_{\overline{K}})^\times$ as mentioned before (50)–(52). By applying the construction of (44) to these homomorphisms and taking the composition with (98), we obtain three functors

$$\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(A^\square_{\mathrm{crys}}(\mathcal{A}),\varphi) \longrightarrow \mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}^\square_{\mathrm{inf}}(\mathcal{A}),\varphi), \qquad (99)$$
$$\mathrm{M}^q_{[0,p-2],\mathrm{free}}(A^\square_{\mathrm{inf}}(\mathcal{A}),\varphi) \longrightarrow \mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\underline{A}^\square_{\mathrm{inf}}(\mathcal{A}),\varphi), \qquad (100)$$
$$\mathrm{M}^q_{[0,p-2],\mathrm{free}}(A^\square_{\mathrm{inf}}(\mathcal{A}),\varphi) \longrightarrow \mathrm{MF}^p_{[0,p-2],\mathrm{free}}(A^\square_{\mathrm{crys}}(\mathcal{A}),\varphi). \qquad (101)$$

The quadruplets (97) with the action of $\widetilde{\Gamma}_{\mathcal{A}}$ on the underlying algebras satisfy the conditions before Definition 48. We endow $A^\square_{\mathrm{crys}}(\mathcal{A})$, $A^\square_{\mathrm{inf}}(\mathcal{A})$ and $\underline{A}^\square_{\mathrm{inf}}(\mathcal{A})$ with the p (resp. $(p,[\underline{p}])$, resp. p)-adic topology. Then the action of $\widetilde{\Gamma}_{\mathcal{A}}$ on these rings are continuous (Corollary 86), and we can apply Definition 53 to the quadruplets (97). By applying Lemma 58 to the second one, we obtain an equivalence of categories

$$\mathrm{M}^{q,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A^\square_{\mathrm{inf}}(\mathcal{A}),\varphi,\widetilde{\Gamma}_{\mathcal{A}}) \xrightarrow{\sim} \mathrm{MF}^{q,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A^\square_{\mathrm{inf}}(\mathcal{A}),\varphi,\widetilde{\Gamma}_{\mathcal{A}}), \qquad (102)$$
$$(M,\varphi_M,\rho_M) \mapsto (M,\varphi_M,(\varphi_M^{-1}(q^rM))_{r\in\mathbb{N}\cap[0,p-2]},\rho_M).$$

The homomorphisms $A^\square_{\mathrm{crys}}(\mathcal{A}) \to \underline{A}^\square_{\mathrm{inf}}(\mathcal{A})$, $A^\square_{\mathrm{inf}}(\mathcal{A}) \to \underline{A}^\square_{\mathrm{inf}}(\mathcal{A})$ and $A^\square_{\mathrm{inf}}(\mathcal{A}) \to A^\square_{\mathrm{crys}}(\mathcal{A})$ are $\widetilde{\Gamma}_{\mathcal{A}}$-equivariant, and also continuous because the topology of $A^\square_{\mathrm{inf}}(\mathcal{A})$ coincides with the (p,π^{p-1})-adic topology (Lemma 1 (1)) and $[\underline{p}]^p = (-\xi+p)^p \in pA^\square_{\mathrm{crys}}(\mathcal{A})$. Hence by applying the construction of (47) to these homomorphisms and taking the compositions with (102), we obtain the following three functors.

$$\mathrm{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A^\square_{\mathrm{crys}}(\mathcal{A}),\varphi,\widetilde{\Gamma}_{\mathcal{A}}) \longrightarrow \mathrm{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\underline{A}^\square_{\mathrm{inf}}(\mathcal{A}),\varphi,\widetilde{\Gamma}_{\mathcal{A}}) \qquad (103)$$
$$\mathrm{M}^{q,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A^\square_{\mathrm{inf}}(\mathcal{A}),\varphi,\widetilde{\Gamma}_{\mathcal{A}}) \longrightarrow \mathrm{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\underline{A}^\square_{\mathrm{inf}}(\mathcal{A}),\varphi,\widetilde{\Gamma}_{\mathcal{A}}) \qquad (104)$$
$$\mathrm{M}^{q,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A^\square_{\mathrm{inf}}(\mathcal{A}),\varphi,\widetilde{\Gamma}_{\mathcal{A}}) \longrightarrow \mathrm{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A^\square_{\mathrm{crys}}(\mathcal{A}),\varphi,\widetilde{\Gamma}_{\mathcal{A}}) \qquad (105)$$

Proposition 94 *The functors (99)–(101) and (103)–(105) are equivalences of categories.*

Proof By Propositions 44, 56, and Remark 55 (2), it suffices to prove that the homomorphisms $A^{\square}_{\mathrm{crys}}(\mathcal{A}) \to \underline{A}^{\square}_{\mathrm{inf}}(\mathcal{A})$ and $A^{\square}_{\mathrm{inf}}(\mathcal{A}) \to \underline{A}^{\square}_{\mathrm{inf}}(\mathcal{A})$ satisfy Condition 54 for $a = p-2$. Note that the topology of $\underline{A}^{\square}_{\mathrm{inf}}(\mathcal{A})$ coincides with the quotient of that of either of $A^{\square}_{\mathrm{inf}}(\mathcal{A})$ and $A^{\square}_{\mathrm{crys}}(\mathcal{A})$. The conditions (ii-v) in (a) and (b) follow from the definition of the filtrations I^r and Fil^r and some fundamental properties already mentioned; for the condition (v), we use $\varphi(I^{p-1}A_{\mathrm{crys}}(O_{\overline{K}})) \subset p^{p-1}A_{\mathrm{crys}}(O_{\overline{K}})$ and $\varphi(I^{p-1}A_{\mathrm{inf}}(O_{\overline{K}})) \subset q^{p-1}A_{\mathrm{inf}}(O_{\overline{K}})$. We first verify the remaining conditions for $A^{\square}_{\mathrm{crys}}(\mathcal{A})$. The condition (a)(i) follows from the following observation: $A^{\square}_{\mathrm{crys}}(\mathcal{A})$ is p-adically complete and separated by Lemma 79 (1), and $\mathrm{Fil}^1A_{\mathrm{crys}}(O_{\overline{K}}) \cdot A^{\square}_{\mathrm{crys}}(\mathcal{A})/p$ is a nilideal of $A^{\square}_{\mathrm{crys}}(\mathcal{A})/p$ because the PD-ideal $\mathrm{Fil}^1A_{\mathrm{crys}}(O_{\overline{K}})/p \cong \mathrm{Fil}^1A_{\mathrm{crys},1}(O_{\overline{K}})$ of $A_{\mathrm{crys}}(O_{\overline{K}})/p \cong A_{\mathrm{crys},1}(O_{\overline{K}})$ is a nilideal. For the conditions (c-g), we are reduced to showing that $I^{p-1}A^{\square}_{\mathrm{crys}}(\mathcal{A})$ is p-adically complete and separated by the same argument as the proof of Proposition 59. By Lemma 3 (2), this follows from the fact that $A^{\square}_{\mathrm{crys}}(\mathcal{A})$ and $A^{\square}_{\mathrm{crys}}(\mathcal{A})/I^{p-1}A^{\square}_{\mathrm{crys}}(\mathcal{A})$ are p-torsion free, and p-adically complete and separated (Lemmas 79 (1) and 81). Let us verify the conditions (a)(i) and (c-g) for $A^{\square}_{\mathrm{inf}}(\mathcal{A})$. By Lemma 80 (4), $A^{\square}_{\mathrm{inf}}(\mathcal{A})$ is π-adically complete and separated. This implies the condition (a)(i). We can verify the conditions (c-g) in the same way as the proof of Proposition 59 by using Lemma 80 instead of Lemma 1. □

We have $TA^{\square}_{\mathrm{crys}}(M)/p^m \xrightarrow{\cong} TA^{\square}_{\mathrm{crys},m}(M)$ by (89) and (92). By (87) and the same argument as the proof of Proposition 60, we see that $TA^{\square}_{\mathrm{crys}}(M)$ is an object of $\mathrm{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A^{\square}_{\mathrm{crys}}(\mathcal{A}), \varphi, \widetilde{\Gamma}_{\mathcal{A}})$. Furthermore, combining (94) with Proposition 93, we obtain the following proposition in the same way as the proof of Proposition 61.

Proposition 95 *The following functor is fully faithful.*

$$TA^{\square}_{\mathrm{crys}}\colon \mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A}, \Phi) \longrightarrow \mathrm{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A^{\square}_{\mathrm{crys}}(\mathcal{A}), \varphi, \widetilde{\Gamma}_{\mathcal{A}})$$

By comparing (94) with (39) for $\mathcal{B} = \mathcal{A}$ via the homomorphism (85) and using $\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})^{\nabla=0} = A_{\mathrm{crys}}(\overline{\mathcal{A}})$ (9), we obtain the following canonical isomorphism in $\mathrm{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A_{\mathrm{crys}}(\overline{\mathcal{A}}), \varphi, G_{\mathcal{A}})$ functorial in M.

$$TA^{\square}_{\mathrm{crys}}(M) \otimes_{A^{\square}_{\mathrm{crys}}(\mathcal{A}),\iota^{\square}} A_{\mathrm{crys}}(\overline{\mathcal{A}}) \xrightarrow{\cong} TA_{\mathrm{crys}}(M) \tag{106}$$

We define the functor

$$TA^{\square}_{\mathrm{inf}}\colon \mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A}, \Phi) \longrightarrow \mathrm{M}^{q,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A^{\square}_{\mathrm{inf}}(\mathcal{A}), \varphi, \widetilde{\Gamma}_{\mathcal{A}})$$

to be the composition of $TA^{\square}_{\mathrm{crys}}$, and a quasi-inverse of (105) (see Proposition 94). By Proposition 95, we obtain the following.

Theorem 96 *The functor $TA^{\square}_{\mathrm{inf}}$ is fully faithful.*

We also obtain the canonical isomorphism

$$TA^{\square}_{\mathrm{inf}}(M) \otimes_{A^{\square}_{\mathrm{inf}}(\mathcal{A})} A_{\mathrm{inf}}(\overline{\mathcal{A}}) \xrightarrow{\cong} TA_{\mathrm{inf}}(M) \tag{107}$$

in the category $\mathrm{M}^{q,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\overline{\mathcal{A}}), \varphi, G_{\mathcal{A}})$ functorial in M.

One can show the following analogue of Lemma 64 by the same argument using Lemma 92.

Lemma 97 (1) *The homomorphism $\overline{\alpha}\colon \mathcal{A} \to A^{\square}_{\mathrm{inf}}(\mathcal{A})/I^1A^{\square}_{\mathrm{inf}}(\mathcal{A})$ induced by α (80) is $\widetilde{\Gamma}_{\mathcal{A}}$-equivariant.*

(2) *The following isomorphism induced by (92) is $\widetilde{\Gamma}_{\mathcal{A}}$-equivariant.*

$$M \otimes_{\mathcal{A},\overline{\alpha}} A^{\square}_{\mathrm{inf}}(\mathcal{A})/I^1A^{\square}_{\mathrm{inf}}(\mathcal{A}) \cong TA^{\square}_{\mathrm{inf}}(M) \otimes_{A^{\square}_{\mathrm{inf}}(\mathcal{A})} A^{\square}_{\mathrm{inf}}(\mathcal{A})/I^1A^{\square}_{\mathrm{inf}}(\mathcal{A})$$

Lemma 98 *The actions of $\Gamma_{\mathcal{A}}$ on $TA^{\square}_{\mathrm{inf}}(M)/\pi$ and $TA^{\square}_{\mathrm{crys}}(M)/\pi$ are trivial.*

Proof This follows from Lemmas 88 and 97 (2). □

14 Preliminaries on Décalage Functor and Continuous Group Cohomology

In this section, we summarize basic facts on a slight modification $L\eta^{+}_{\mathcal{I}}$ of the décalage functor $L\eta_{\mathcal{I}}$ introduced in [7, Sect. 6] and an interpretation of continuous group cohomology of semilinear representations into the language of topos. Every complex to which we apply the functor $L\eta^{+}_{\mathcal{I}}$ in later sections has the same image under $L\eta_{\mathcal{I}}$ and $L\eta^{+}_{\mathcal{I}}$ by the $\mathcal{I}$-torsion freeness of H^0 and the vanishing of H^q ($q < 0$). Therefore this modification is not crucial for our discussions. We introduce $L\eta^{+}_{\mathcal{I}}$ simply because it admits a natural functor $L\eta^{+}_{\mathcal{I}} \to L\eta^{+}_{\mathcal{I}'}$ when $\mathcal{I} \subset \mathcal{I}'$ (cf. Lemma 102) and it is convenient for dealing with Frobenius structures. In the rest of this paper, we choose and fix a universe $\mathcal{U}$ such that the underlying set of the field K is a $\mathcal{U}$-set, and a topos means a $\mathcal{U}$-topos. Note that the underlying sets of the groups G_K and $G_{\mathcal{A}}$ are $\mathcal{U}$-sets.

Let $f\colon (E', \mathcal{O}') \to (E, \mathcal{O})$ be a morphism of ringed topos. Then the direct image functor $f_*\colon \mathrm{Mod}(E', \mathcal{O}') \to \mathrm{Mod}(E, \mathcal{O})$ and the inverse image functor $f^*\colon \mathrm{Mod}(E, \mathcal{O}) \to \mathrm{Mod}(E', \mathcal{O}')$ have a right derived functor $Rf_*\colon D(E', \mathcal{O}') \to D(E, \mathcal{O})$ and a left derived functor $Lf^*\colon D(E, \mathcal{O}) \to D(E', \mathcal{O}')$, respectively, and Lf^* is canonically regarded as a left adjoint of Rf_* (cf. [1, Tag07A5]). Following [7, Sect. 6.1], we say that a complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules on a ringed topos $(E, \mathcal{O})$ is *strongly K-flat* if $\mathcal{F}^q$ is a flat $\mathcal{O}$-module for every $q \in \mathbb{Z}$ and for every acyclic complex of $\mathcal{O}$-modules $\mathcal{G}^\bullet$, the total complex $\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{G}^\bullet)$ is acyclic. For a strongly K-flat complex of $\mathcal{O}$-modules $\mathcal{F}^\bullet$, we have an isomorphism $Lf^*(\mathcal{F}^\bullet) \xrightarrow{\cong} f^*(\mathcal{F}^\bullet)$.

For a complex of $\mathcal{O}$-modules $\mathcal{F}^\bullet$, there exists a quasi-isomorphism $\mathcal{G}^\bullet \to \mathcal{F}^\bullet$ such that $f^*(\mathcal{G}^\bullet)$ is strongly K-flat for every f as above.

Let $(E, \mathcal{O})$ be a ringed topos, let $\mathcal{I}$ be an ideal of $\mathcal{O}$, and suppose that there exists a conservative family $\mathcal{P}$ of points of E such that for every $p \in \mathcal{P}$, $p^{-1}(\mathcal{I})$ is generated by a regular element of $p^{-1}(\mathcal{O})$. For a complex of $\mathcal{O}$-modules $\mathcal{F}^\bullet$, we define a subcomplex $\eta_{\mathcal{I}}^+\mathcal{F}^\bullet$ of $\mathcal{F}^\bullet$ by $\eta_{\mathcal{I}}^+\mathcal{F}^q = \mathcal{F}^q$ $(q < 0)$ and $\eta_{\mathcal{I}}^+\mathcal{F}^q = \mathrm{Ker}(\mathcal{I}^q\mathcal{F}^q \xrightarrow{d^q_{\mathcal{F}^\bullet}} \mathcal{F}^{q+1}/\mathcal{I}^{q+1}\mathcal{F}^q)$ $(q \geq 0)$. This construction defines a functor $\eta_{\mathcal{I}}^+ \colon K(E, \mathcal{O}) \to K(E, \mathcal{O})$; a homotopy between two morphisms of complexes of $\mathcal{O}$-modules gives a homotopy after taking $\eta_{\mathcal{I}}^+$. The inclusion morphism $\eta_{\mathcal{I}}^+\mathcal{F}^\bullet \hookrightarrow \mathcal{F}^\bullet$ induces $H^q(\eta_{\mathcal{I}}^+\mathcal{F}^\bullet) = H^q(\mathcal{F}^\bullet)$ for $q \leq 0$. For an integer $q \geq 1$, the equality $Z^q(\mathcal{I}^q\mathcal{F}^\bullet) = Z^q(\eta_{\mathcal{I}}^+\mathcal{F}^\bullet)$ induces a surjective homomorphism

$$H^q(\mathcal{I}^q\mathcal{F}^\bullet) \longrightarrow H^q(\eta_{\mathcal{I}}^+\mathcal{F}^\bullet) \tag{108}$$

because $\mathcal{I}^q\mathcal{F}^{q-1} \subset (\eta_{\mathcal{I}}^+\mathcal{F})^{q-1}$.

Lemma 99 ([7, Lemma 6.4])

(1) *Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules, let q be a positive integer and assume that the morphism $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}^q \to \mathcal{F}^q$ is a monomorphism. Then the homomorphism (108) induces an isomorphism*

$$H^q(\mathcal{I}^q\mathcal{F}^\bullet)/(H^q(\mathcal{I}^q\mathcal{F}^\bullet)[\mathcal{I}]) \xrightarrow{\cong} H^q(\eta_{\mathcal{I}}^+\mathcal{F}^\bullet).$$

(2) *Let $f \colon \mathcal{F}_1^\bullet \to \mathcal{F}_2^\bullet$ be a quasi-isomorphism of complexes of $\mathcal{O}$-modules. If the morphisms $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}_\nu^q \to \mathcal{F}_\nu^q$ $(\nu \in \{1, 2\}, q \in \mathbb{N} \cap [1, \infty[)$ are injective, then the morphism $\eta_{\mathcal{I}}^+ f$ is also a quasi-isomorphism.*

Proof By taking the inverse image by each point $p \in \mathcal{P}$, we are reduced to the case E is the topos of $\mathcal{U}$-sets and $\mathcal{I}$ is an ideal of a ring $\mathcal{O}$ generated by a regular element. Then it is straightforward to verify the claims. □

Let L be a full subcategory of $K(E, \mathcal{O})$ consisting of strongly K-flat complexes. Then L forms a triangulated subcategory of $K(E, \mathcal{O})$ (cf. [1, Tag06YL]). Since every complex of $\mathcal{O}$-modules $\mathcal{F}^\bullet$ has a quasi-isomorphism $\mathcal{G}^\bullet \to \mathcal{F}^\bullet$ from a strongly K-flat complex, the functor $L_{\mathrm{Qis}} \to D(E, \mathcal{O})$ is an equivalence between triangulated categories, where Qis denotes the set of quasi-isomorphisms in L. Let U be a quasi-inverse. By Lemma 99 (2), the functor $\eta_{\mathcal{I}}^+$ induces a functor $\overline{\eta_{\mathcal{I}}^+} \colon L_{\mathrm{Qis}} \to D(E, \mathcal{O})$. We define the functor

$$L\eta_{\mathcal{I}}^+ \colon D(E, \mathcal{O}) \to D(E, \mathcal{O}) \tag{109}$$

to be the composition $\overline{\eta_{\mathcal{I}}^+} \circ U$.

Lemma 100 (1) *There exists a canonical morphism of functors*

$$\xi \colon L\eta_{\mathcal{I}}^+ \circ Q \to Q \circ \eta_{\mathcal{I}}^+$$

such that $\xi(\mathcal{F}^\bullet)$ is an isomorphism for every object $\mathcal{F}^\bullet$ of L. Here Q denotes the canonical functor $K(E,\mathcal{O}) \to D(E,\mathcal{O})$. Furthermore the pair $(L\eta^+_{\mathcal{I}}, \xi)$ with the above property is unique up to unique isomorphisms.

(2) *For a complex of $\mathcal{O}$-modules $\mathcal{F}^\bullet$ such that $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}^q \to \mathcal{F}^q$ is a monomorphism for every positive integer q, the morphism $\xi(\mathcal{F}^\bullet)$ is an isomorphism.*

Proof (1) Let $\mathcal{F}^\bullet \in K(E,\mathcal{O})$, and define $\mathcal{G}^\bullet \in L$ by $UQ(\mathcal{F}^\bullet) = \mathcal{G}^\bullet$. Then we have a canonical isomorphism $Q(\mathcal{F}^\bullet) \xrightarrow{\cong} Q(\mathcal{G}^\bullet)$ in $D(E,\mathcal{O})$. Choose its presentation $\mathcal{F}^\bullet \xleftarrow{\sim} \mathcal{H}^\bullet \xrightarrow{\sim} \mathcal{G}^\bullet$, $\mathcal{H}^\bullet \in L$ in $K(E,\mathcal{O})$. Then by Lemma 99 (2), we obtain morphisms $L\eta^+_{\mathcal{I}} Q(\mathcal{F}^\bullet) = \overline{\eta}^+_{\mathcal{I}}\mathcal{G}^\bullet = Q\eta^+_{\mathcal{I}}(\mathcal{G}^\bullet) \xleftarrow{\cong} Q\eta^+_{\mathcal{I}}(\mathcal{H}^\bullet) \to Q\eta^+_{\mathcal{I}}(\mathcal{F}^\bullet)$. We also see that the last morphism is an isomorphism if $\mathcal{F}^\bullet$ belongs to L. One can verify that the composition of the above morphisms is independent of the choice of the presentation, and functorial in $\mathcal{F}^\bullet$. The last claim on the uniqueness of $(L\eta^+_{\mathcal{I}}, \xi)$ immediately follows from the equivalence $L_{\mathrm{Qis}} \xrightarrow{\sim} D(E,\mathcal{O})$.

(2) This is an immediate consequence of Lemma 99 (2). □

Let $\mathcal{F}^\bullet \in D(E,\mathcal{O})$. For $q \in \mathbb{N}$, the exact sequence $0 \to \mathcal{I}^{q+1}/\mathcal{I}^{q+2} \to \mathcal{I}^q/\mathcal{I}^{q+2} \to \mathcal{I}^q/\mathcal{I}^{q+1} \to 0$ induces a distinguished triangle $\mathcal{I}^{q+1}/\mathcal{I}^{q+2} \otimes^L_{\mathcal{O}} \mathcal{F}^\bullet \to \mathcal{I}^q/\mathcal{I}^{q+2} \otimes^L_{\mathcal{O}} \mathcal{F}^\bullet \to \mathcal{I}^q/\mathcal{I}^{q+1} \otimes^L_{\mathcal{O}} \mathcal{F}^\bullet \xrightarrow{+1}$ in $D(E,\mathcal{O})$, and then a morphism $\mathrm{Bock}^q\colon H^q(\mathcal{I}^q/\mathcal{I}^{q+1} \otimes^L_{\mathcal{O}} \mathcal{F}^\bullet) \to H^{q+1}(\mathcal{I}^{q+1}/\mathcal{I}^{q+2} \otimes^L_{\mathcal{O}} \mathcal{F}^\bullet)$. We have $\mathrm{Bock}^{q+1} \circ \mathrm{Bock}^q = 0$ for $q \in \mathbb{N}$. We define the complex of $\mathcal{O}$-modules $\mathrm{Bock}^+(\mathcal{F}^\bullet)$ concentrated in degree ≥ 0 by $\mathrm{Bock}^+(\mathcal{F})^q = H^q(\mathcal{I}^q/\mathcal{I}^{q+1} \otimes^L_{\mathcal{O}} \mathcal{F}^\bullet)$ and $d^q_{\mathrm{Bock}^+(\mathcal{F})} = \mathrm{Bock}^q$ for $q \in \mathbb{N}$. This construction is functorial in $\mathcal{F}^\bullet$ and gives a functor $\mathrm{Bock}^+\colon D(E,\mathcal{O}) \to C(E,\mathcal{O}/\mathcal{I})$.

Proposition 101 (1) *Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules. If $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}^q \to \mathcal{F}^q$ is injective for every integer $q \geq -1$, the morphisms $\eta^+_{\mathcal{I}}\mathcal{F}^q \to Z^q(\mathcal{I}^q\mathcal{F}^\bullet/\mathcal{I}^{q+1}\mathcal{F}^\bullet)$ $(q \in \mathbb{N})$ defined by $\eta^+_{\mathcal{I}}\mathcal{F}^q \subset \mathcal{I}^q\mathcal{F}^q$ induce a quasi-isomorphism of complexes of $\mathcal{O}$-modules*

$$\tau_{\geq 0}(\mathcal{O}/\mathcal{I} \otimes_{\mathcal{O}} \eta^+_{\mathcal{I}} K^\bullet) \xrightarrow{\sim} \mathrm{Bock}^+(\mathcal{F}^\bullet).$$

(2) *We have the following canonical isomorphism of functors from $D(E,\mathcal{O})$ to $D(E,\mathcal{O}/\mathcal{I})$.*

$$\tau_{\geq 0}(\mathcal{O}/\mathcal{I} \otimes^L_{\mathcal{O}} L\eta^+_{\mathcal{I}}(-)) \xrightarrow{\cong} \mathrm{Bock}^+(-)$$

(3) *For $\mathcal{F}^\bullet \in D(E,\mathcal{O})$ such that $H^0(\mathcal{F}^\bullet)$ is $\mathcal{I}$-torsion free and $H^q(\mathcal{F}^\bullet) = 0$ for every integer $q < 0$, the canonical morphism $\mathcal{O}/\mathcal{I} \otimes^L_{\mathcal{O}} L\eta^+_{\mathcal{I}}(\mathcal{F}) \to \tau_{\geq 0}(\mathcal{O}/\mathcal{I} \otimes^L_{\mathcal{O}} L\eta^+_{\mathcal{I}}(\mathcal{F}))$ is a quasi-isomorphism.*

Proof (1) It is obvious that we obtain a morphism of sheaves $\mathcal{O}/\mathcal{I} \otimes_{\mathcal{O}} (\eta^+_{\mathcal{I}}\mathcal{F})^q \to H^q(\mathcal{I}^q/\mathcal{I}^{q+1} \otimes_{\mathcal{O}} \mathcal{F}^\bullet)$ for each $q \in \mathbb{N}$. To show that it induces the desired quasi-isomorphism, we may take the stalk at each $p \in \mathcal{P}$, and assume that E is the topos of $\mathcal{U}$-sets and $\mathcal{I}$ is generated by a regular element a. By [7, Proposition 6.12], we obtain a morphism of complexes which is a quasi-isomorphism in degree ≥ 1. For $q \in \mathbb{N}$, we have

$$\mathcal{O}/\mathcal{I} \otimes_{\mathcal{O}} (\eta^+_{\mathcal{I}}\mathcal{F})^q \cong \{a^q\mathcal{F}^q \cap d^{-1}(a^{q+1}\mathcal{F}^{q+1})\}/\{a^{q+1}\mathcal{F}^q \cap d^{-1}(a^{q+2}\mathcal{F}^{q+1})\},$$
$$H^q(\mathcal{I}^q/\mathcal{I}^{q+1} \otimes_{\mathcal{O}} \mathcal{F}^\bullet) \cong \{a^q\mathcal{F}^q \cap d^{-1}(a^{q+1}\mathcal{F}^{q+1})\}/\{a^{q+1}\mathcal{F}^q + d(a^q\mathcal{F}^{q-1})\}.$$

Using this description, we obtain

$$H^0(\mathcal{O}/\mathcal{I} \otimes_{\mathcal{O}} \eta^+_{\mathcal{I}}\mathcal{F}^\bullet) \cong \{d^{-1}(a^2\mathcal{F}^1)\}/\{(a\mathcal{F}^0 \cap d^{-1}(a^2\mathcal{F}^1)) + d(\mathcal{F}^{-1})\},$$
$$\mathrm{Ker}(\mathrm{Bock}^0) \cong \{a\mathcal{F}^0 + d^{-1}(a^2\mathcal{F}^1)\}/\{a\mathcal{F}^0 + d(\mathcal{F}^{-1})\}.$$

This shows the quasi-isomorphism in degree 0 because $d(\mathcal{F}^{-1}) \subset d^{-1}(a^2\mathcal{F}^1)$.

(2) Let $\mathcal{F}^\bullet \in D(E, \mathcal{O})$. By choosing a quasi-isomorphism $\mathcal{G}^\bullet \to \mathcal{F}^\bullet$ from a strongly K-flat complex, we obtain quasi-isomorphisms $\tau_{\geq 0}(\mathcal{O}/\mathcal{I} \otimes^L L\eta^+_{\mathcal{I}}\mathcal{F}^\bullet) \xleftarrow{\sim} \tau_{\geq 0}(\mathcal{O}/\mathcal{I} \otimes^L L\eta^+_{\mathcal{I}}\mathcal{G}^\bullet) \xrightarrow{\sim} \tau_{\geq 0}(\mathcal{O}/\mathcal{I} \otimes_{\mathcal{O}} \eta^+_{\mathcal{I}}\mathcal{G}^\bullet) \xrightarrow{\sim} \mathrm{Bock}^+(\mathcal{G}^\bullet) \xrightarrow{\sim} \mathrm{Bock}^+(\mathcal{F}^\bullet)$. It is straightforward to verify that the composition of them is independent of the choice of $\mathcal{G}^\bullet \to \mathcal{F}^\bullet$, and functorial in $\mathcal{F}^\bullet$.

(3) Put $\mathcal{G}^\bullet := L\eta^+_{\mathcal{I}}\mathcal{F}^\bullet$. Then we have $H^q(\mathcal{G}^\bullet) = H^q(\mathcal{F}^\bullet)$ and $H^q(\mathcal{I} \otimes^L_{\mathcal{O}} \mathcal{G}^\bullet) \cong \mathcal{I} \otimes_{\mathcal{O}} H^q(\mathcal{G}^\bullet)$ for $q \leq 0$. Hence we see that $H^q(\mathcal{O}/\mathcal{I} \otimes^L_{\mathcal{O}} \mathcal{G}^\bullet) = 0$ for $q < 0$ by using the distinguished triangle $\mathcal{I} \otimes^L_{\mathcal{O}} \mathcal{G}^\bullet \to \mathcal{G}^\bullet \to \mathcal{O}/\mathcal{I} \otimes^L_{\mathcal{O}} \mathcal{G}^\bullet \xrightarrow{+}$. □

Lemma 102 *Let $f\colon (E', \mathcal{O}') \to (E, \mathcal{O})$ be a morphism of ringed topos, and let $\mathcal{I}'$ be an ideal of $\mathcal{O}'$ such that the image of $f^{-1}(\mathcal{I}) \to \mathcal{O}'$ is contained in $\mathcal{I}'$ and that the pair $(\mathcal{O}', \mathcal{I}')$ satisfies the same condition as $(\mathcal{O}, \mathcal{I})$.*

(1) *There exists a morphism of functors*

$$\alpha\colon Lf^* \circ L\eta^+_{\mathcal{I}} \to L\eta^+_{\mathcal{I}'} \circ Lf^*$$

determined by the following property: For every strongly K-flat complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules such that $f^*(\mathcal{F}^\bullet)$ is also a strongly K-flat complex, $\alpha(\mathcal{F}^\bullet)$ coincides with the composition of $Lf^*L\eta^+_{\mathcal{I}}\mathcal{F}^\bullet \to f^*\eta^+_{\mathcal{I}}\mathcal{F}^\bullet \to \eta^+_{\mathcal{I}'}f^*(\mathcal{F}^\bullet) \xleftarrow{\cong} L\eta^+_{\mathcal{I}'}Lf^*(\mathcal{F}^\bullet)$, where the middle morphism is induced by the morphism $f^*\eta^+_{\mathcal{I}}\mathcal{F}^\bullet \to f^*\mathcal{F}^\bullet$. If f is flat and $\mathcal{I}'$ is generated by the image of $f^{-1}(\mathcal{I}) \to \mathcal{O}'$, then the morphism α is an isomorphism.

(2) *Assume that the functor $f_*\colon \mathrm{Mod}(E', \mathcal{O}') \to \mathrm{Mod}(E, \mathcal{O})$ is exact. Then there exists a morphism of functors*

$$\beta\colon L\eta^+_{\mathcal{I}} \circ f_* \to f_* \circ L\eta^+_{\mathcal{I}'}$$

determined by the following property: For every strongly K-flat complex of $\mathcal{O}'$-modules $\mathcal{F}^\bullet$, $\beta(\mathcal{F}^\bullet)$ coincides with the composition of $L\eta^+_{\mathcal{I}}f_*\mathcal{F}^\bullet \to \eta^+_{\mathcal{I}}f_*\mathcal{F}^\bullet \to f_*\eta^+_{\mathcal{I}'}\mathcal{F}^\bullet \xleftarrow{\cong} f_*L\eta^+_{\mathcal{I}'}\mathcal{F}^\bullet$. If $E = E'$, the morphism of topos underlying f is the identity functor, and $\mathcal{I}' = \mathcal{I}\mathcal{O}'$, then the morphism β is an isomorphism.

Proof (1) Let L (resp. L') be the full subcategory of $K(E, \mathcal{O})$ (resp. $K(E', \mathcal{O}')$) consisting of strongly K-flat complexes $\mathcal{F}^\bullet$ with $f^*(\mathcal{F}^\bullet)$ also strongly K-flat (resp. strongly K-flat complexes). Let $f^*_{L,L'}$, $\eta^+_{\mathcal{I},L}$, and $\eta^+_{\mathcal{I}',L'}$ denote the functors $L \to L'$, $L \to K(E, \mathcal{O})$, and $L' \to K(E', \mathcal{O}')$ induced by the functors f^*, $\eta^+_{\mathcal{I}}$ and $\eta^+_{\mathcal{I}'}$ for complexes. Then we are reduced to constructing a morphism $f^*\eta^+_{\mathcal{I},L} \to \eta^+_{\mathcal{I}',L'} f^*_{L,L'}$. For $\mathcal{F}^\bullet \in L$, we see that the morphism $f^{-1}(\mathcal{F}^\bullet) \to f^*\mathcal{F}^\bullet$ induces a morphism between subcomplexes $f^{-1}(\eta^+_{\mathcal{I}}\mathcal{F}^\bullet) \to \eta^+_{\mathcal{I}'} f^*\mathcal{F}^\bullet$, and then $f^*\eta^+_{\mathcal{I}}\mathcal{F}^\bullet \to \eta^+_{\mathcal{I}'} f^*\mathcal{F}^\bullet$ in $C^+(E', \mathcal{O}')$. We further see that this is an isomorphism under the assumption in the last claim.

(2) Let L' be the full subcategory of $K(E', \mathcal{O}')$ consisting of K-flat complexes. By Lemma 100, it suffices to construct a morphism γ from the composition of $L' \xrightarrow{f_*} K(E, \mathcal{O}) \xrightarrow{\eta^+_{\mathcal{I}}} K(E, \mathcal{O})$ to the composition of $L' \xrightarrow{\eta^+_{\mathcal{I}'}} K(E', \mathcal{O}') \xrightarrow{f_*} K(E, \mathcal{O})$ and show that γ is an isomorphism under the assumption in the last claim. Note that $\mathcal{I} \otimes_{\mathcal{O}} f_*\mathcal{F}^\bullet \to f_*\mathcal{F}^\bullet$ ($\mathcal{F}^\bullet \in L'$) is injective under the assumption in the last claim. For $\mathcal{F}^\bullet \in L'$, we have $\mathcal{I}^q f_*\mathcal{F}^q \subset f_*(\mathcal{I}'^q\mathcal{F}^q)$ in $f_*\mathcal{F}^q$ for each q, and it implies $\eta^+_{\mathcal{I}} f_*\mathcal{F}^\bullet \subset f_*\eta^+_{\mathcal{I}'}\mathcal{F}^\bullet$ in $f_*\mathcal{F}^\bullet$. The two subcomplexes coincide under the assumption in the last claim. □

Next we discuss continuous group cohomology of semilinear representations. For an ordered set Λ and a category C, let C^Λ denote the category $\mathrm{Func}(\Lambda, C)$ of functors from Λ to C, where we regard Λ as a category whose object is an element of Λ and $\sharp\mathrm{Hom}_\Lambda(a, b) = 1$ if $b \geq a$ and 0 otherwise. For example, $C^{\mathbb{N}^\circ}$ is the category of inverse systems of objects of C indexed by $\mathbb{N}$, and $C^{\mathbb{N}}$ is the category of inductive systems of objects of C indexed by $\mathbb{N}$. For a topos E, we have a morphism of topos

$$\underleftarrow{l} : E^{\mathbb{N}^\circ} \to E$$

defined by $\underleftarrow{l}_*((\mathcal{F}_n)_{n\in\mathbb{N}}) = \varprojlim_{n\in\mathbb{N}} \mathcal{F}_n$ and $\underleftarrow{l}^*(\mathcal{G}) = (\mathcal{G} \xleftarrow{\mathrm{id}} \mathcal{G} \xleftarrow{\mathrm{id}} \mathcal{G} \xleftarrow{\mathrm{id}} \cdots)$, and a morphism of topos

$$\underrightarrow{l} : E \to E^{\mathbb{N}}$$

defined by $\underrightarrow{l}_*(\mathcal{F}) = (\mathcal{F} \xrightarrow{\mathrm{id}} \mathcal{F} \xrightarrow{\mathrm{id}} \mathcal{F} \xrightarrow{\mathrm{id}} \cdots)$ and $\underrightarrow{l}^*((\mathcal{G}_n)_{n\in\mathbb{N}}) = \varinjlim_n \mathcal{G}_n$.

For a profinite group (resp. group) G whose underlying set is a $\mathcal{U}$-set, let G-**Sets** be the category of $\mathcal{U}$-sets with discrete topology endowed with a continuous action of G (resp. $\mathcal{U}$-sets with an action of G). Then the category G-**Sets** is a $\mathcal{U}$-topos. For a closed normal subgroup (resp. a normal subgroup) N of G and the quotient $H = G/N$, we have a morphism of topos

$$\mathrm{inv}_N : G\text{-}\mathbf{Sets} \to H\text{-}\mathbf{Sets}$$

defined by $\mathrm{inv}_{N*}(\mathcal{F}) = \mathcal{F}^N := \{x \in \mathcal{F} \mid g(x) = x \text{ for all } g \in N\}$ and $\mathrm{inv}_N^*(\mathcal{G}) = (\mathcal{G}$ with the action of G via $G \to H)$.

For a profinite group G, let $\underline{G}$ denote the underlying abstract group. Then we have a morphism of topos

$$\iota\colon \underline{G}\text{-}\mathbf{Sets} \to G\text{-}\mathbf{Sets}$$

defined by $\iota_*\mathcal{F} = \mathcal{F}^{\mathrm{cont}}$ and $\iota^*\mathcal{G} = \mathcal{G}$, where $\mathcal{F}^{\mathrm{cont}}$ denotes the subset of $\mathcal{F}$ consisting of elements invariant by an open subgroup of G.

Let G be a profinite group, let N be a closed normal subgroup of G, and let H be the quotient G/N. Let $R_\star$, $S_\star$, and S be commutative ring objects of $G\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}$, $H\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}$ and $\underline{H}\text{-}\mathbf{Sets}$. For example, $R_\star = (R_n)_{n\in\mathbb{N}}$ is an inverse system of commutative rings R_n with continuous action of G indexed by $n \in \mathbb{N}$. Assume that we are given the following extensions of the morphisms of topos $\mathrm{inv}_N^{\mathbb{N}^\circ}$ and $\underleftarrow{l}$ to those of ringed topos.

$$\mathrm{inv}_N^{\mathbb{N}^\circ}\colon (G\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, R_\star) \to (H\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, S_\star), \quad \underleftarrow{l}\colon (\underline{H}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, S_\star) \to (\underline{H}\text{-}\mathbf{Sets}, S),$$

i.e. H-equivariant ring homomorphisms $(S_n)_{n\in\mathbb{N}} \to (R_n^N)_{n\in\mathbb{N}}$ and $S \to \varprojlim_n S_n$. Under this setting, we define the functor

$$R\Gamma(N, -)\colon D(G\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, R_\star) \longrightarrow D(\underline{H}\text{-}\mathbf{Sets}, S) \tag{110}$$

to be the composition of

$$\begin{aligned} D(G\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, R_\star) &\xrightarrow{R\,\mathrm{inv}_{N*}^{\mathbb{N}^\circ}} D(H\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, S_\star) \\ &\xrightarrow{\iota^{\mathbb{N}^\circ *}} D(\underline{H}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, S_\star) \xrightarrow{R\,\underleftarrow{l}_*} D(\underline{H}\text{-}\mathbf{Sets}, S). \end{aligned} \tag{111}$$

One can compute the underlying complex of the image of a complex bounded below under $R\Gamma(N, -)$ by first restricting the action of G to N and then taking $R\Gamma(N, -)$, as follows, and similarly for the composition of $R\Gamma(N, -)$ with a décalage functor.

Lemma 103 (1) *The following diagram is commutative up to canonical isomorphisms, where the left vertical functor is induced by the functor restricting the action of G to N, the remaining three vertical functors are induced by the functors forgetting the action of $\underline{H}$, and we abbreviate* **Sets** *to* **S**.

$$\begin{array}{ccccccc} D^+(G\text{-}\mathbf{S}^{\mathbb{N}^\circ}, R_\star) & \xrightarrow{R\,\mathrm{inv}_{N*}^{\mathbb{N}^\circ}} & D^+(H\text{-}\mathbf{S}^{\mathbb{N}^\circ}, S_\star) & \xrightarrow{\iota^{\mathbb{N}^\circ *}} & D^+(\underline{H}\text{-}\mathbf{S}^{\mathbb{N}^\circ}, S_\star) & \xrightarrow{R\,\underleftarrow{l}\,*} & D^+(\underline{H}\text{-}\mathbf{S}, S) \\ \downarrow F & & \downarrow F & & \downarrow F & & \downarrow F \\ D^+(N\text{-}\mathbf{S}^{\mathbb{N}^\circ}, R_\star) & \xrightarrow{R\,\mathrm{inv}_{N*}^{\mathbb{N}^\circ}} & D^+(\mathbf{S}^{\mathbb{N}^\circ}, S_\star) & = & D^+(\mathbf{S}^{\mathbb{N}^\circ}, S_\star) & \xrightarrow{R\,\underleftarrow{l}\,*} & D^+(\mathbf{S}, S) \end{array}$$

(2) *Let I be an ideal of S generated by a regular element after forgetting the action of $\underline{H}$. Then the following diagram is commutative up to a canonical isomorphism.*

$$\begin{array}{ccc} D(\underline{H}\text{-}\mathbf{Sets}, S) & \xrightarrow{L\eta_I^+} & D(\underline{H}\text{-}\mathbf{Sets}, S) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ D(\mathbf{Sets}, S) & \xrightarrow{L\eta_I^+} & D(\mathbf{Sets}, S) \end{array}$$

Proof (1) By the universal property of derived functors, we obtain a morphism $F \circ R\mathrm{inv}_{N*}^{\mathbb{N}^\circ} \to R\mathrm{inv}_{N*}^{\mathbb{N}^\circ} \circ F$. By [3, V$^{\mathrm{bis}}$ Corollaire (1.13.12)], the nth component of $R\mathrm{inv}_{N*}^{\mathbb{N}^\circ}$ is given by $R\mathrm{inv}_{N*}$ for S_n and R_n. Hence the proof for the left-square is reduced to the case without $\mathbb{N}^\circ$, where the claim is well-known (e.g. [2, Proposition V.11.5]). The claim for the middle square is obvious. For the right square, it suffices to prove that, for every injective object $\mathcal{F}_\star$ of $\mathrm{Mod}(\underline{H}\text{-}\mathbf{Sets}, S_\star)$, the transition morphisms $\mathcal{F}_{n+1} \to \mathcal{F}_n$ $(n \in \mathbb{N})$ are split surjective. Let $n \in \mathbb{N}$. Let $\mathcal{G}_\star$ (resp. $\mathcal{H}_\star$) be the object of $\mathrm{Mod}(\underline{H}\text{-}\mathbf{Sets}, S_\star)$ obtained from $\mathcal{F}_\star$ by replacing $\mathcal{F}_m$ $(m > n)$ with 0 (resp. $\mathcal{F}_m$ $(m > n+1)$ and $\mathcal{F}_{n+1}$ with 0 and $\mathcal{F}_n$). We define the transition morphism $\mathcal{H}_{n+1} \to \mathcal{H}_n$ to be the identity. Then we have monomorphisms $i\colon \mathcal{G}_\star \to \mathcal{H}_\star$ and $j\colon \mathcal{G}_\star \to \mathcal{F}_\star$ defined by the identity morphisms and 0 maps. Since $\mathcal{F}_\star$ is injective, there exists a morphism $k\colon \mathcal{H}_\star \to \mathcal{F}_\star$ such that $k \circ i = j$, and then we see that the composition of $k_{n+1}\colon \mathcal{H}_{n+1} = \mathcal{F}_n \to \mathcal{F}_{n+1}$ and the transition morphism $\mathcal{F}_{n+1} \to \mathcal{F}_n$ is the identity.

(2) The forgetful functor $\underline{H}$-**Sets** $\to$ **Sets** is exact and has the right adjoint sending X to $\mathrm{Map}(\underline{H}, X)$ with the left $\underline{H}$-action defined by $(hf)(-) = f(-h)$ $(f \in \mathrm{Map}(\underline{H}, X), h \in H)$. Hence the claim follows from Lemma 102 (1). □

Corollary 104 *Let $\mathcal{F}_\star^\bullet$ be a complex of $R_\star$-modules bounded below on G-$\mathbf{Sets}^{\mathbb{N}^\circ}$ such that the transition map $\mathcal{F}_{n+1}^q \to \mathcal{F}_n^q$ is surjective for every $n \in \mathbb{N}$ and $q \in \mathbb{Z}$. Then the image of $R\Gamma(N, \mathcal{F}_\star^\bullet)$ under the forgetful functor $F\colon D^+(\underline{H}\text{-}\mathbf{Sets}) \to D^+(\mathbf{Sets})$ is canonically isomorphic to $\mathrm{Tot}(C_{\mathrm{cont}}^\bullet(N, \mathcal{F}^\bullet))$, where $C_{\mathrm{cont}}^\bullet(N, -)$ denotes the continuous inhomogeneous cochain complex and $\mathcal{F}^\bullet$ denotes the inverse limit of $\mathcal{F}_n^\bullet$ $(n \in \mathbb{N})$ equipped with the inverse limit of the discrete topologies.*

Assume that we have the following commutative diagrams of ringed topos such that the underlying morphisms of topos of the vertical arrows are the identity functors and the morphism $\underline{g}$ is induced by g.

$$\begin{array}{ccccccc} (G\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, R'_\star) & \xrightarrow{\mathrm{inv}_N^{\mathbb{N}^\circ}} & (H\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, S'_\star) & \quad & (\underline{H}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, S'_\star) & \xrightarrow{\underleftarrow{l}} & (\underline{H}\text{-}\mathbf{Sets}, S') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & {\scriptstyle \underline{g}}\downarrow & & \downarrow{\scriptstyle h} \\ (G\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, R_\star) & \xrightarrow{\mathrm{inv}_N^{\mathbb{N}^\circ}} & (H\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, S_\star), & & (\underline{H}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, S_\star) & \xrightarrow{\underleftarrow{l}} & (\underline{H}\text{-}\mathbf{Sets}, S) \end{array} \tag{112}$$

Then we have the following three morphisms of functors denoted by $\Rightarrow$ in the diagram; the middle one is the base change morphism; we abbreviate **Sets** to **S**.

$$
\begin{array}{ccccccc}
D(G\text{-}\mathbf{S}^{\mathbb{N}^\circ}, R'_\star) & \xrightarrow{R\,\mathrm{inv}^{\mathbb{N}^\circ}_{N*}} & D(H\text{-}\mathbf{S}^{\mathbb{N}^\circ}, S'_\star) & \xrightarrow{\iota^{\mathbb{N}^\circ *}} & D(\underline{H}\text{-}\mathbf{S}^{\mathbb{N}^\circ}, S'_\star) & \xrightarrow{R\,\underleftarrow{l}_*} & D(\underline{H}\text{-}\mathbf{S}, S') \\
{\scriptstyle f_*}\downarrow & \overset{\sim}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle g_*} & \nearrow\!\!\!\nearrow & \downarrow{\scriptstyle \underline{g}_*} & \overset{\sim}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle h_*} \\
D(G\text{-}\mathbf{S}^{\mathbb{N}^\circ}, R_\star) & \xrightarrow{R\,\mathrm{inv}^{\mathbb{N}^\circ}_{N*}} & D(H\text{-}\mathbf{S}^{\mathbb{N}^\circ}, S_\star) & \xrightarrow{\iota^{\mathbb{N}^\circ *}} & D(\underline{H}\text{-}\mathbf{S}^{\mathbb{N}^\circ}, S_\star) & \xrightarrow{R\,\underleftarrow{l}_*} & D(\underline{H}\text{-}\mathbf{S}, S)
\end{array}
$$

By composing the three morphisms, we obtain the following morphism of functors from $D(G\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, R'_\star)$ to $D(\underline{H}\text{-}\mathbf{Sets}, S)$.

$$R\Gamma(N, -) \circ f_* \longrightarrow h_* \circ R\Gamma(N, -) \tag{113}$$

Assume further that we are given ideals $I \subset S$ and $I' \subset S'$ each of which is generated by a regular element if we forget the action of $\underline{H}$, such that $IS' \subset I'$. Then by combining (113) with Lemma 102 (2), we obtain the following morphism of functors from $D(G\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, R'_\star)$ to $D(\underline{H}\text{-}\mathbf{Sets}, S)$.

$$L\eta_I^+ \circ R\Gamma(N, -) \circ f_* \longrightarrow h_* \circ L\eta_{I'}^+ \circ R\Gamma(N, -) \tag{114}$$

Similarly we have the following three morphisms of functors denoted by $\Rightarrow$ in the diagram; the left and right ones are the base change morphisms.

$$
\begin{array}{ccccccc}
D(G\text{-}\mathbf{S}^{\mathbb{N}^\circ}, R_\star) & \xrightarrow{R\,\mathrm{inv}^{\mathbb{N}^\circ}_{N*}} & D(H\text{-}\mathbf{S}^{\mathbb{N}^\circ}, S_\star) & \xrightarrow{\iota^{\mathbb{N}^\circ *}} & D(\underline{H}\text{-}\mathbf{S}^{\mathbb{N}^\circ}, S_\star) & \xrightarrow{R\,\underleftarrow{l}_*} & D(\underline{H}\text{-}\mathbf{S}, S) \\
{\scriptstyle Lf^*}\downarrow & \swarrow\!\!\!\swarrow & \downarrow{\scriptstyle Lg^*} & \overset{\sim}{\swarrow\!\!\!\swarrow} & \downarrow{\scriptstyle L\underline{g}^*} & \swarrow\!\!\!\swarrow & \downarrow{\scriptstyle Lh^*} \\
D(G\text{-}\mathbf{S}^{\mathbb{N}^\circ}, R'_\star) & \xrightarrow{R\,\mathrm{inv}^{\mathbb{N}^\circ}_{N*}} & D(H\text{-}\mathbf{S}^{\mathbb{N}^\circ}, S'_\star) & \xrightarrow{\iota^{\mathbb{N}^\circ *}} & D(\underline{H}\text{-}\mathbf{S}^{\mathbb{N}^\circ}, S'_\star) & \xrightarrow{R\,\underleftarrow{l}_*} & D(\underline{H}\text{-}\mathbf{S}, S')
\end{array}
$$

Composing these three morphisms and then using Lemma 102 (1), we obtain the following morphism of functors from $D(G\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, R_\star)$ to $D(\underline{H}\text{-}\mathbf{Sets}, S')$.

$$Lh^* \circ L\eta_I^+ \circ R\Gamma(N, -) \longrightarrow L\eta_{I'}^+ \circ R\Gamma(N, -) \circ Lf^* \tag{115}$$

Lemma 105 *Suppose that we are given a compatible system of morphisms of ringed topos φ from the diagram (112) to itself whose underlying morphisms of topos are the identity functors, and ideals $\widetilde{I} \subset S$ and $\widetilde{I}' \subset S'$ satisfying the same conditions as I and I' such that the images of I and I' under $S \to \varphi_* S$ and $S' \to \varphi_* S'$ are contained in $\varphi_*\widetilde{I}$ and $\varphi_*\widetilde{I}'$, respectively. Then the base change morphisms $Lf^* \circ \varphi_* \to \varphi_* \circ Lf^*$ and $Lh^* \circ \varphi_* \to \varphi_* \circ Lh^*$ are compatible with (115) for f and h, and (114) for φ's, i.e., the following diagram is commutative up to a canonical isomorphism.*

$$\begin{array}{ccc} Lh^* \circ L\eta_I^+ \circ R\Gamma(N,-) \circ \varphi_* & \longrightarrow & L\eta_{I'}^+ \circ R\Gamma(N,-) \circ Lf^* \circ \varphi_* \\ \downarrow & & \downarrow \\ Lh^* \circ \varphi_* \circ L\eta_{\widetilde{I}}^+ \circ R\Gamma(N,-) & & L\eta_{I'}^+ \circ R\Gamma(N,-) \circ \varphi_* \circ Lf^* \\ \downarrow & & \downarrow \\ \varphi_* \circ Lh^* \circ L\eta_{\widetilde{I}}^+ \circ R\Gamma(N,-) & \longrightarrow & \varphi_* \circ L\eta_{\widetilde{I'}}^+ \circ R\Gamma(N,-) \circ Lf^* \end{array}$$

Proof The compatibility with $R\Gamma(N,-)$ follows from the compatibility of base change morphisms for commutative squares of ringed topos with compositions of direct image functors and with those of inverse image functors. The compatibility with $L\eta_J^+$ ($J = I, I', \widetilde{I}, \widetilde{I'}$) is reduced to the same claim concerning the corresponding functors for sheaves because the morphism $\varphi_* \circ L\eta_{\widetilde{I'}}^+ \circ Lh^*K^\bullet \to \varphi_* \circ \eta_{\widetilde{I'}}^+ \circ h^*K^\bullet$ is a quasi-isomorphism for a strongly K-flat complex on $D(\underline{H}\text{-}\mathbf{Sets}, S)$. □

15 Galois Cohomology of A_{inf}-Representations and de Rham Complexes

Let $\Delta_{\mathcal{A}}$ denote $\mathrm{Gal}(\mathcal{K}^{\mathrm{ur}}/\overline{K}\mathcal{K})$, and let M be an object of $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A}, \Phi)$. In this section, we study the relation between the twisted Galois cohomology $L\eta_\pi^+ R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M))$ and the de Rham complex of M by using the almost purity theorem by Faltings. We choose and fix a framing $\square\colon O_K[T_1^{\pm 1}, \dots, T_d^{\pm 1}] \to A$. The computation of the twisted Galois cohomology in this section is done via the twisted Galois cohomology of $TA_{\mathrm{inf}}^{\square}(M)$ (Sect. 13) (following the idea in [7, Sects. 9, 12.1]), and therefore, the construction of the comparison isomorphisms (125) and (126) with de Rham complexes given in this section heavily depends on the choice of the framing $\square$. In later sections, we will give alternative ways to construct the comparison maps by using the period rings $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$; the construction is different from [7, Sect. 12.2].

For an object $\mathcal{M}^\bullet$ of $K(\underline{G_K}\text{-}\mathbf{Sets}, A_{\mathrm{inf}}(O_{\overline{K}}))$ or $K(A_{\mathrm{inf}}(O_{\overline{K}})\text{-Mod})$ (resp. $D(\underline{G_K}\text{-}\mathbf{Sets}, A_{\mathrm{inf}}(O_{\overline{K}}))$ or $D(A_{\mathrm{inf}}(O_{\overline{K}})\text{-Mod})$), we write $\eta_\pi^+\mathcal{M}^\bullet$ (resp. $L\eta_\pi^+\mathcal{M}^\bullet$) for $\eta^+_{\pi A_{\mathrm{inf}}(O_{\overline{K}})}\mathcal{M}^\bullet$ (resp. $L\eta^+_{\pi A_{\mathrm{inf}}(O_{\overline{K}})}\mathcal{M}^\bullet$) (Sect. 14) in the following.

Let $\mathscr{I}_{\mathrm{inf,G}}$ be the subset of $\mathscr{I}_{\mathrm{inf}}$ (Sect. 12) consisting of G_K-invariant ideals, which is cofinal in $\mathscr{I}_{\mathrm{inf}}$. For $\mathfrak{a} \in \mathscr{I}_{\mathrm{inf,G}}$, put $A_{\mathrm{inf},\mathfrak{a}}(\overline{\mathcal{A}}) := A_{\mathrm{inf}}(\overline{\mathcal{A}})/\mathfrak{a}$ (Sect. 12) and $TA_{\mathrm{inf},\mathfrak{a}}(M) := TA_{\mathrm{inf}}(M)/\mathfrak{a}$, each of which has a natural action of $G_{\mathcal{A}}$, and also a Frobenius endomorphism φ if $\varphi(\mathfrak{a}) \subset \mathfrak{a}$. Let $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathscr{I}_{\mathrm{inf,G}}$ which forms a fundamental system of open neighborhoods of 0 in $A_{\mathrm{inf}}(O_{\overline{K}})$. Then we obtain a ring object $A_{\mathrm{inf},\mathfrak{a}_\star}(\overline{\mathcal{A}})$ on $G_{\mathcal{A}}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}$ and an $A_{\mathrm{inf},\mathfrak{a}_\star}(\overline{\mathcal{A}})$-module $TA_{\mathrm{inf},\mathfrak{a}_\star}(M)$ on $G_{\mathcal{A}}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}$. We define the object $R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M))$ of $D(\underline{G_K}\text{-}\mathbf{Sets}, A_{\mathrm{inf}}(O_{\overline{K}}))$ to be the image of $TA_{\mathrm{inf},\mathfrak{a}_\star}(M)$ under the functor (Sect. 14)

$$R\Gamma(\Delta_{\mathcal{A}}, -)\colon D(G_{\mathcal{A}}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, A_{\mathrm{inf},\mathfrak{a}_\star}(\overline{\mathcal{A}})) \longrightarrow D(\underline{G_K}\text{-}\mathbf{Sets}, A_{\mathrm{inf}}(O_{\overline{K}})).$$

By Corollary 104, we have a canonical isomorphism

$$R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M)) \cong C^{\bullet}_{\mathrm{cont}}(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M))$$

in $D(A_{\mathrm{inf}}(O_{\overline{K}})\text{-Mod})$. This implies that the definition of $R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M))$ above is independent of the choice of $(\mathfrak{a}_n)_{n\in\mathbb{N}}$. Choose $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ such that $\varphi(\mathfrak{a}_n) \subset \mathfrak{a}_n$ for every $n \in \mathbb{N}$ (e.g. $\mathfrak{a}_n = (p, [\underline{p}])^{n+1}$). Then φ on $A_{\mathrm{inf},\mathfrak{a}_n}(\overline{\mathcal{A}})$ ($n \in \mathbb{N}$) defines a morphism of ringed topos $\varphi\colon (G_{\mathcal{A}}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, A_{\mathrm{inf},\mathfrak{a}_\star}(\overline{\mathcal{A}})) \to (G_{\mathcal{A}}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, A_{\mathrm{inf},\mathfrak{a}_\star}(\overline{\mathcal{A}}))$, and φ of $TA_{\mathrm{inf},\mathfrak{a}_n}(M)$ for each $n \in \mathbb{N}$ gives a morphism of $A_{\mathrm{inf},\mathfrak{a}_\star}(\overline{\mathcal{A}})$-modules $TA_{\mathrm{inf},\mathfrak{a}_\star}(M) \to \varphi_*(TA_{\mathrm{inf},\mathfrak{a}_\star}(M))$ on $\underline{G_{\mathcal{A}}}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}$. Using (113), we obtain a morphism in $D(\underline{G_K}\text{-}\mathbf{Sets}, A_{\mathrm{inf}}(O_{\overline{K}}))$

$$\varphi\colon R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M)) \longrightarrow \varphi_* R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M)),$$

where the morphism of ringed topos φ from $(\underline{G_K}\text{-}\mathbf{Sets}, A_{\mathrm{inf}}(O_{\overline{K}}))$ to itself is defined by φ of $A_{\mathrm{inf}}(O_{\overline{K}})$. We define $R\Gamma(\Delta_{\mathcal{A}}, T_{\mathrm{crys}}(M) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}(\overline{\mathcal{A}}))$ with φ in the same way. Then we immediately obtain the following claim from Theorem 70.

Theorem 106 *We have a canonical isomorphism in* $D(\underline{G_K}\text{-}\mathbf{Sets}, A_{\mathrm{inf}}(O_{\overline{K}})[\frac{1}{\pi}])$

$$(L_\pi\eta^+ R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M)))[\tfrac{1}{\pi}] \xrightarrow{\cong} R\Gamma(\Delta_{\mathcal{A}}, T_{\mathrm{crys}}(M) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}(\overline{\mathcal{A}}))[\tfrac{1}{\pi}]$$

which is functorial in M*, where* $[\frac{1}{\pi}]$ *means* $\otimes^L_{A_{\mathrm{inf}}(O_{\overline{K}})} A_{\mathrm{inf}}(O_{\overline{K}})[\frac{1}{\pi}]$.

Let ${}_1\mathcal{A}$, $\mathcal{K}_\infty$, $t_{i,n}$, $\widetilde{\Gamma}_{\mathcal{A}}$, χ_i, $\underline{\chi}_i$ and $\mathcal{A}_\infty$ be as in the beginning of Sect. 12, and let $\Gamma_{\mathcal{A}}$ be $\mathrm{Gal}(\mathcal{K}_\infty/\overline{K}\mathcal{K})$. By Lemma 78, we have an isomorphism $\Gamma_{\mathcal{A}} \xrightarrow{\cong} \prod_{1\leq i\leq d}\mathbb{Z}_p(1)$ defined by $\gamma \mapsto (\chi_i(\gamma))_i$. Recall that we have a $\widetilde{\Gamma}_{\mathcal{A}}$-equivariant homomorphism (77) $\iota^{\square}_\infty\colon A^{\square}_{\mathrm{inf}}(\mathcal{A}) \to A_{\mathrm{inf}}(\mathcal{A}_\infty)$ compatible with φ and Fil^r.

We define $T\widetilde{A}^{\square}_{\mathrm{inf}}(M)$ to be $TA^{\square}_{\mathrm{inf}}(M) \otimes_{A^{\square}_{\mathrm{inf}}(\mathcal{A})} A_{\mathrm{inf}}(\mathcal{A}_\infty)$, which is a free $A_{\mathrm{inf}}(\mathcal{A}_\infty)$-module of finite type naturally endowed with a semilinear action of $\widetilde{\Gamma}_{\mathcal{A}}$ and a semilinear $\widetilde{\Gamma}_{\mathcal{A}}$-equivariant endomorphism φ. The action of $\widetilde{\Gamma}_{\mathcal{A}}$ is continuous with respect to the $(p, [\underline{p}])$-adic topology of $T\widetilde{A}^{\square}_{\mathrm{inf}}(M)$ by Lemma 5 for $(\Lambda, \Lambda_0) = (\mathcal{A}_\infty, \mathcal{A})$. By (107), we have a canonical $A_{\mathrm{inf}}(\overline{\mathcal{A}})$-linear $G_{\mathcal{A}}$-equivariant isomorphism compatible with φ

$$T\widetilde{A}^{\square}_{\mathrm{inf}}(M) \otimes_{A_{\mathrm{inf}}(\mathcal{A}_\infty)} A_{\mathrm{inf}}(\overline{\mathcal{A}}) \xrightarrow{\cong} TA_{\mathrm{inf}}(M). \tag{116}$$

We define $R\Gamma(\Gamma_{\mathcal{A}}, T\widetilde{A}^{\square}_{\mathrm{inf}}(M))$ and $R\Gamma(\Gamma_{\mathcal{A}}, TA^{\square}_{\mathrm{inf}}(M))$ with φ in the same way as $R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M))$. In the rest of this section, we forget the action of $\underline{G_K}$ on these cohomology groups and study them as an object of $D(A_{\mathrm{inf}}(O_{\overline{K}})\text{-Mod})$.

We obtain the following proposition from Faltings' almost purity theorem.

Proposition 107 *Let $\mathcal{I}$ be the ideal of $A_{\mathrm{inf}}(O_{\overline{K}})$ generated by $[\underline{p}^{p^{-l}}]$ ($l \in \mathbb{N}$). Then, for $m, n \in \mathbb{N}_{>0}$, the cohomology of the cone of the following natural morphism is annihilated by $\mathcal{I}$.*

$$R\Gamma(\Gamma_{\mathcal{A}}, T\widetilde{A}_{\mathrm{inf}}^{\square}(M)/(p^m, [\underline{p}]^n)) \longrightarrow R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M)/(p^m, [\underline{p}]^n))$$

Proof Note $\mathcal{I}^2 = \mathcal{I}$. Let Λ be one of $\mathcal{A}_\infty$ and $\overline{\mathcal{A}}$. Since $A_{\mathrm{inf}}(\Lambda)/[\underline{p}]^n$ is p-torsion free and $A_{\mathrm{inf}}(\Lambda)/p$ is $[\underline{p}]$-torsion free by Lemma 1 (3) and (4), the claim is reduced to the case $m = n = 1$. Put $H_{\mathcal{A}} = \mathrm{Gal}(\mathcal{K}^{\mathrm{ur}}/\mathcal{K}_\infty)$, $\underline{T} = TA_{\mathrm{inf}}(M)/(p, [\underline{p}])$, and $\underline{\widetilde{T}}^{\square} = T\widetilde{A}_{\mathrm{inf}}^{\square}(M)/(p, [\underline{p}])$. Since $A_{\mathrm{inf}}(\Lambda)/(p, [\underline{p}]) \cong R_\Lambda/\underline{p} \xrightarrow{\cong} \Lambda/p$; $(a_n)_{n\in\mathbb{N}} \mapsto a_0$ and $\underline{\widetilde{T}}^{\square} \otimes_{\mathcal{A}_\infty/p} \overline{\mathcal{A}}/p \xrightarrow{\cong} \underline{T}$ by (116), the almost purity theorem by Faltings ([11, 2b, 2c], [9, 2.4. Theorem (ii)], [2, Proposition V.12.8]) implies that $\mathcal{I} \cdot H^q(H_{\mathcal{A}}, \underline{T}) = 0$ ($q \in \mathbb{N}_{>0}$) and the kernel and the cokernel of $\underline{\widetilde{T}}^{\square} \to H^0(H_{\mathcal{A}}, \underline{T})$ are annihilated by $\mathcal{I}$. Hence the Hochschild-Serre spectral sequence for $H_{\mathcal{A}} \subset G_{\mathcal{A}}$ and $\underline{T}$ gives the desired claim. □

Let $\mathfrak{m}_{\overline{K}}$ be the maximal ideal of $O_{\overline{K}}$, and let $\overline{k}$ be the residue filed $O_{\overline{K}}/\mathfrak{m}_{\overline{K}}$ of $O_{\overline{K}}$. Then the homomorphism $R_{O_{\overline{K}}} \to \overline{k}$; $(a_n)_{n\in\mathbb{N}} \mapsto (a_0 \mod \mathfrak{m}_{\overline{K}})$ induces a homomorphism $A_{\mathrm{inf}}(O_{\overline{K}}) = W(R_{O_{\overline{K}}}) \to W(\overline{k})$.

Lemma 108 *For any $x \in \mathrm{Ker}(A_{\mathrm{inf}}(O_{\overline{K}}) \to W(\overline{k}))$ and $n \in \mathbb{N}_{>0}$, there exists $m \in \mathbb{N}$ such that $x \in [\underline{p}^{p^{-m}}]A_{\mathrm{inf}}(O_{\overline{K}}) + p^n A_{\mathrm{inf}}(O_{\overline{K}})$.*

Proof Since the kernel of $R_{O_{\overline{K}}} \to \overline{k}$ is generated by $\underline{p}^{p^{-r}}$ ($r \in \mathbb{N}$), there exist $l \in \mathbb{N}$ and $x_1, \dots, x_{n-1} \in R_{O_{\overline{K}}}$ such that $x \equiv (\underline{p}^{p^{-l}}x_1, \underline{p}^{p^{-l}}x_2, \dots, \underline{p}^{p^{-l}}x_{n-1}) = \sum_{\nu=0}^{n-1} p^\nu [\underline{p}^{p^{-l-\nu}}][x_\nu^{p^{-\nu}}]$ modulo $p^n A_{\mathrm{inf}}(O_{\overline{K}})$. Hence the claim holds for $m = l + n - 1$. □

Lemma 109 *Let $(K_m)_{m\in\mathbb{N}}$ be a complex bounded below of inverse systems of $A_{\mathrm{inf}}(O_{\overline{K}})/p^m$-modules. Assume $[\underline{p}^{p^{-l}}] \cdot H^q(K_m) = 0$ for any $m \in \mathbb{N}$, $q \in \mathbb{Z}$, and $l \in \mathbb{N}$. Then, for any $y \in \mathrm{Ker}(A_{\mathrm{inf}}(O_{\overline{K}}) \to W(\overline{k}))$, the multiplication by y on $R\varprojlim_m K_m$ is zero in the derived category of $A_{\mathrm{inf}}(O_{\overline{K}})$-modules.*

Proof Put $J := \mathrm{Ker}(A_{\mathrm{inf}}(O_{\overline{K}}) \to W(\overline{k}))$. Then $J/p^m \to A_{\mathrm{inf}}(O_{\overline{K}})/p^m$ is injective and its image is generated by $[\underline{p}^{p^{-n}}]$ ($n \in \mathbb{N}$) by Lemma 108. Since $[\underline{p}]$ is regular in $A_{\mathrm{inf}}(O_{\overline{K}})/p^m = W_m(R_{O_{\overline{K}}})$, $J/p^m J = \varinjlim_n [\underline{p}^{p^{-n}}](A_{\mathrm{inf}}(O_{\overline{K}})/p^m)$ is flat over $A_{\mathrm{inf}}(O_{\overline{K}})/p^m$, and $H^q(J/p^m \otimes_{A_{\mathrm{inf}}(O_{\overline{K}})/p^m} K_m) \cong J/p^m \otimes_{A_{\mathrm{inf}}(O_{\overline{K}})/p^m} H^q(K_m)$ ($q \in \mathbb{Z}$). By using $(J/p^m) \cdot (J/p^m) = J/p^m$ and $J/p^m \cdot H^q(K_m) = 0$ ($q \in \mathbb{Z}$), we see that the right-hand sides of the above isomorphisms vanish, and therefore $R\varprojlim_m (J/p^m \otimes_{A_{\mathrm{inf}}(O_{\overline{K}})/p^m} K_m) = 0$. This implies the claim because, for any $y \in J$, the multiplication by y on $(K_m)_{m\in\mathbb{N}}$ factors through $(J/p^m \otimes_{A_{\mathrm{inf}}(O_{\overline{K}})/p^m} K_m)_{m\in\mathbb{N}}$. □

Corollary 110 *The cone of the natural morphism*

$$R\Gamma(\Gamma_{\mathcal{A}}, T\widetilde{A}_{\mathrm{inf}}^{\square}(M)) \longrightarrow R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M))$$

is annihilated by any element of $\mathrm{Ker}(A_{\mathrm{inf}}(O_{\overline{K}}) \to W(\overline{k}))$ *in the derived category of* $A_{\mathrm{inf}}(O_{\overline{K}})$*-modules.*

Proof By Proposition 107, we can apply Lemma 109 to the cone of

$$(C_{\mathrm{cont}}(\varGamma_{\mathcal{A}}, T\widetilde{A}^{\square}_{\mathrm{inf}}(M)/\mathfrak{a}_m))_{m\in\mathbb{N}} \longrightarrow (C^{\bullet}_{\mathrm{cont}}(\varDelta_{\mathcal{A}}, TA_{\mathrm{inf}}(M)/\mathfrak{a}_m))_{m\in\mathbb{N}},$$

where $\mathfrak{a}_m = p^m A_{\mathrm{inf}}(O_{\overline{K}}) + [\underline{p}]^m A_{\mathrm{inf}}(O_{\overline{K}})$. □

Lemma 111 (cf. [16, the proof of Lemma 18], [6, Lemma 5.14]) *Let R be a commutative ring, and let I be an ideal of R generated by a regular element.*

(1) *Let C be a complex of I-torsion free R-modules. If $H^0(C)$ is I-torsion free and $H^q(C) = 0$ for every integer $q < 0$, then the inclusion map $\eta_I^+ C \to \eta_I C$ is a quasi-isomorphism, where η_I is defined as in [7, Sect. 6].*
(2) *Let J be an ideal of R containing I. Let $f: C_1 \to C_2$ be a morphism of complexes of I-torsion free R-modules, and let C_3 be the mapping cone of f. Suppose that* (i) *$J \cdot H^q(C_3) = 0$ and* (ii) *$H^q(C_1/aC_1)[J^2] = 0$ for all $q \in \mathbb{Z}$. Then the morphism $\eta_I C_1 \to \eta_I C_2$ is a quasi-isomorphism.*

Proof (1) This immediately follows from $H^q(\eta_I C) \cong I^q \otimes_R (H^q(C)/H^q(C)[I])$ and $H^q(\eta_I^+ C) \cong H^q(C)$ (if $q \leq 0$), $H^q(\eta_I C)$ (if $q > 0$).

(2) Let a be a generator of I. The homomorphism $f[\frac{1}{a}]\colon C_1[\frac{1}{a}] \to C_2[\frac{1}{a}]$ is a quasi-isomorphism by the assumption (i). Since $C_i[\frac{1}{a}]/\eta_I C_i = \varinjlim_n (a^{-n}\eta_I C_i)/\eta_I C_i$ and the multiplication by a^{-n} on $\eta_I C_i$ induces an isomorphism $\eta_I C_i/a\eta_I C_i \xrightarrow{\cong} a^{-n}\eta_I C_i/a^{-n+1}\eta_I C_i$ for every $n \in \mathbb{N}_{>0}$, it suffices to prove that the morphism $\eta_I C_1/a\eta_I C_1 \to \eta_I C_2/a\eta_I C_2$ induced by f is a quasi-isomorphism.

Set $\overline{C}_i := C_i/aC_i$ ($i \in \{1,2,3\}$), let g be the morphism $C_2 \to C_3$, and let $\overline{f}$ be the morphism $\overline{C}_1 \to \overline{C}_2$ induced by f. We may identify $\overline{C}_3$ with the mapping cone of $\overline{f}$. Let Bock_i^q denote the boundary map $H^q(\overline{C}_i) \to H^{q+1}(\overline{C}_i)$ associated to the short exact sequence $0 \to \overline{C}_i \xrightarrow{a} C_i/a^2C_i \to \overline{C}_i \to 0$, which is compatible with the morphisms induced by f and g. Recall that we have quasi-isomorphisms $(\eta_I C_i)/a \to (H^{\bullet}(\overline{C}_i), \mathrm{Bock}_i^{\bullet})$ ([7, Proposition 6.12]), which are compatible with the morphisms induced by f and g. We have $H^q(\eta_I C_3) = H^q(C_3)/(H^q(C_3)[a]) = 0$ for all $q \in \mathbb{Z}$ by the assumption (i). Therefore the complex $(H^{\bullet}(\overline{C}_3), \mathrm{Bock}_3^{\bullet})$ is acyclic. We have a long exact sequence

$$\cdots \to H^{q-1}(\overline{C}_3) \to H^q(\overline{C}_1) \to H^q(\overline{C}_2) \to H^q(\overline{C}_3) \to \cdots.$$

The exact sequence $0 \to H^r(C_3)/a \to H^r(\overline{C}_3) \to H^{r+1}(C_3)[a] \to 0$ and the assumption (i) imply $J^2 \cdot H^r(\overline{C}_3) = 0$. Hence, by the assumption (ii), the above exact sequence splits into short exact sequences

$$0 \to H^q(\overline{C}_1) \to H^q(\overline{C}_2) \to H^q(\overline{C}_3) \to 0.$$

Hence the morphism $(H^\bullet(\overline{C}_1), \mathrm{Bock}_1^\bullet) \to (H^\bullet(\overline{C}_2), \mathrm{Bock}_2^\bullet)$ induced by f is a quasi-isomorphism. This completes the proof. □

Proposition 112 *The following morphism is an isomorphism*

$$L\eta_\pi^+ R\Gamma(\Gamma_{\mathcal{A}}, T\widetilde{A}_{\inf}^{\square}(M)) \to L\eta_\pi^+ R\Gamma(\Delta_{\mathcal{A}}, TA_{\inf}(M)).$$

Proof Put $T := T\widetilde{A}_{\inf}^{\square}(M)$. By Lemma 1 (4), T/p^mT $(m \in \mathbb{N}_{>0})$ are π-torsion free. Hence we have an exact sequence

$$0 \longrightarrow C^\bullet_{\mathrm{cont}}(\Gamma_{\mathcal{A}}, (T/p^m)/\pi^m) \xrightarrow{\pi} C^\bullet_{\mathrm{cont}}(\Gamma_{\mathcal{A}}, (T/p^m)/\pi^{m+1}) \\ \longrightarrow C^\bullet_{\mathrm{cont}}(\Gamma_{\mathcal{A}}, (T/p^m)/\pi) \longrightarrow 0.$$

Since the inverse system $\{C^\bullet_{\mathrm{cont}}(\Gamma_{\mathcal{A}}, (T/\pi^m)/p^m)\}_{m\in\mathbb{N}}$ satisfies the Mittag-Leffler condition and $T/\pi T$ is p-adically complete and separated by Lemma 1 (3), we obtain an isomorphism $C^\bullet_{\mathrm{cont}}(\Gamma_{\mathcal{A}}, T)/\pi \xrightarrow{\cong} C^\bullet_{\mathrm{cont}}(\Gamma_{\mathcal{A}}, T/\pi)$ by taking $\varprojlim_m$. Let $\mathcal{I}$ be as in Proposition 107. By Lemma 111 for $R = A_{\inf}(O_{\overline{K}})$, $I = \pi R$ and $J = \mathcal{I} + I$, Corollary 110, and the π-torsion freeness of $T\widetilde{A}_{\inf}^{\square}(M)$ and $TA_{\inf}(M)$ (Lemma 1 (4)), it suffices to show $H^q(\Gamma_{\mathcal{A}}, T\widetilde{A}_{\inf}^{\square}(M)/\pi))[\mathcal{I}] = 0$. Note that $\mathcal{I}^2 = \mathcal{I}$. We prove this in Lemma 115 below. □

Let us compute $R\Gamma(\Gamma_{\mathcal{A}}, T\widetilde{A}_{\inf}^{\square}(M)/\pi)$ and $R\Gamma(\Gamma_{\mathcal{A}}, T\widetilde{A}_{\inf}^{\square}(M))$.

Lemma 113 (1) *Let r and s be integers prime to p, and let μ and ν be integers such that $\mu \geq \nu$. Then we have $[\underline{\varepsilon}^{rp^{-\nu}}] - 1 \in ([\underline{\varepsilon}^{sp^{-\mu}}] - 1)A_{\inf}(O_{\overline{K}})$.*

(2) *Let $\mathcal{I}$ be the ideal of $A_{\inf}(O_{\overline{K}})$ defined in Proposition 107. For $\nu \in \mathbb{Z}$ and $m \in \mathbb{N}_{>0}$, $(A_{\inf}^{\square}(\mathcal{A})/([\underline{\varepsilon}^{p^{-\nu}}] - 1)A_{\inf}^{\square}(\mathcal{A}))/p^m$ has no non-trivial $\mathcal{I}$-torsion element.*

Proof (1) By applying φ^μ, we are reduced to the case $\mu = 0$ and $\nu \leq 0$. Since ε_n^s is a primitive p^nth root of 1, $[\underline{\varepsilon}^s] - 1$ generates $I^1A_{\inf}(O_{\overline{K}})$. Therefore we may assume $s = 1$ and then the claim is obvious.

(2) Since the homomorphism $A_{\inf}^{\square}(\mathcal{A})/([\underline{\varepsilon}^{p^{-\nu}}] - 1, p^m) \to A_{\inf}(\mathcal{A}_\infty)/([\underline{\varepsilon}^{p^{-\nu}}] - 1, p^m)$ induced by $\iota_\infty^{\square}$ is injective by Corollary 85 (1), it suffices to prove that its target is $\mathcal{I}$-torsion free. By applying φ^ν, we are reduced to the case $\nu = 0$. Since $A_{\inf}(\mathcal{A}_\infty)/([\underline{\varepsilon}] - 1)$ is p-torsion free by Lemma 1 (3), it is enough to show that $A_{\inf}(\mathcal{A}_\infty)/([\underline{\varepsilon}] - 1, p) = R_{\mathcal{A}_\infty}/(\underline{\varepsilon} - 1)$ is $\mathcal{I}$-torsion free. This follows from the isomorphism $R_{\mathcal{A}_\infty}/(\underline{\varepsilon} - 1) \cong \mathcal{A}_\infty/(\varepsilon_1 - 1)$ induced by the projection to the second component, and Lemma 114 below. □

Lemma 114 (cf. [18, Lemma A3.14]) *Let Λ be a normal domain containing $O_{\overline{K}}$, and assume that $\Lambda/p\Lambda \neq 0$ and Λ is integral over a noetherian normal subalgebra Λ_0. Then $\Lambda/a\Lambda$ has no non-trivial $\mathfrak{m}_{\overline{K}}$-torsion for any $a \in \mathfrak{m}_{\overline{K}}$.*

Proof We may assume that a is an Nth root of p for some $N \in \mathbb{N}_{>0}$. For an extension $\mathcal{E}$ of $\mathrm{Frac}\,\Lambda_0$ contained in $\mathrm{Frac}\,\Lambda$, let $\Lambda_{\mathcal{E}}$ be the integral closure of Λ_0 in $\mathcal{E}$. If $a \in \Lambda_{\mathcal{E}}$, the homomorphism $\Lambda_{\mathcal{E}}/a \to \Lambda/a$ is injective because Λ is integral over $\Lambda_{\mathcal{E}}$ and

$\Lambda_{\mathcal{E}}$ is integrally closed in $\Lambda_{\mathcal{E}}[\frac{1}{a}]$. If $\mathcal{E}$ is a finite extension of $\mathrm{Frac}\Lambda_0$, $\Lambda_{\mathcal{E}}$ is a noetherian normal domain finite over Λ_0 because $\mathcal{E}$ is a separable extension of $\mathrm{Frac}\Lambda_0$ and Λ_0 is a noetherian normal domain. Let x be an element of Λ such that $m_{\overline{K}}x \subset a\Lambda$. Put $\mathcal{L} = \mathrm{Frac}(\Lambda_0[a, x])$ and $\mathcal{L}_n = \mathcal{L}(\varepsilon_n)$ $(n \in \mathbb{N}_{\geq 0})$, which are finite extensions of $\mathrm{Frac}\Lambda_0$. It suffices to prove $v_{\mathfrak{p}}(x) \geq v_{\mathfrak{p}}(a)$ for every prime ideal $\mathfrak{p}$ of $\Lambda_{\mathcal{L}}$ of height 1. Choose such a $\mathfrak{p}$. For each $n \in \mathbb{N}_{>0}$, there exists a prime ideal $\mathfrak{p}_n$ of $\Lambda_{\mathcal{L}_n}$ of height 1 lying above $\mathfrak{p}$. As $(\varepsilon_n - 1)x \in a\Lambda_{\mathcal{L}_n}$ by assumption, we have $v_{\mathfrak{p}_n}(x) + \frac{1}{p^{n-1}(p-1)} v_{\mathfrak{p}_n}(p) \geq v_{\mathfrak{p}_n}(a)$. This implies $v_{\mathfrak{p}}(x) + \frac{1}{p^{n-1}(p-1)} v_{\mathfrak{p}}(p) \geq v_{\mathfrak{p}}(a)$. By taking $\lim_{n\to\infty}$, we obtain $v_{\mathfrak{p}}(x) \geq v_{\mathfrak{p}}(a)$. □

For $c \in \mathbb{N}_{>0}$ and a module T with endomorphisms δ_i $(i \in \mathbb{N} \cap [1, c])$ commuting with each other, we define the complex $K(T; \delta_1, \ldots, \delta_c)$ as follows: Let E be $\mathbb{Z}^c$ and let $e_1, \ldots, e_c$ denote the standard basis of E. We define the degree q-part of the complex to be $T \otimes_{\mathbb{Z}} \wedge^q_{\mathbb{Z}} E$ and define the differential $d^q : T \otimes_{\mathbb{Z}} \wedge^q_{\mathbb{Z}} E \to T \otimes_{\mathbb{Z}} \wedge^{q+1}_{\mathbb{Z}} E$ by $x \otimes y \mapsto \sum_{1\leq\nu\leq c} \delta_\nu(x) \otimes (e_\nu \wedge y)$.

Let $\varepsilon = (\varepsilon_n)$ be as in the definition of π given after (1). For $i \in \mathbb{N} \cap [1, d]$, let γ_i be the unique element of $\Gamma_{\mathcal{A}}$ satisfying $\chi_j(\gamma_i) = 0$ (if $j \neq i$), ε (if $j = i$). For a $\mathbb{Z}/p^m$-module T with an action of $\Gamma_{\mathcal{A}}$ continuous with respect to the discrete topology of T, we have the following canonical isomorphism functorial in T.

$$R\Gamma(\Gamma_{\mathcal{A}}, T) \cong K(T; \gamma_1 - 1, \ldots, \gamma_d - 1)$$

Lemma 115 *Let $\mathcal{I}$ be the ideal of $A_{\mathrm{inf}}(O_{\overline{K}})$ defined as in Proposition 107. Then we have $H^q(\Gamma_{\mathcal{A}}, T\widetilde{A}^{\square}_{\mathrm{inf}}(M)/\pi)[\mathcal{I}] = 0$ for every $q \in \mathbb{N}$.*

Proof Set $T := T A^{\square}_{\mathrm{inf}}(M)/\pi$ to simplify the notation. Note that the action of $\Gamma_{\mathcal{A}}$ on T is trivial by Lemma 98. By Lemmas 84 and 79 (2), we have

$$(T\widetilde{A}^{\square}_{\mathrm{inf}}(M)/\pi)/p^m = \bigoplus_{\underline{r}\in(\mathbb{Z}[\frac{1}{p}]\cap[0,1[)^d} T/p^m[\underline{t}^{\underline{r}}].$$

For $\underline{r} = \underline{0}$, we have $H^q(\Gamma_{\mathcal{A}}, T/p^m) \cong T/p^m \otimes_{\mathbb{Z}} \wedge^q E$ and the homomorphism $H^q(\Gamma_{\mathcal{A}}, T/p^{m+1}) \to H^q(\Gamma_{\mathcal{A}}, T/p^m)$ is surjective. For $\underline{r} \neq \underline{0}$, choose $\nu \in \mathbb{N}_{>0}$ such that $\underline{r} \in (\mathbb{Z}p^{-\nu})^d \backslash (\mathbb{Z}p^{-\nu+1})^d$. Then, by [7, Lemma 7.10], Lemmas 113 (1), and 80 (4), $H^q(\Gamma_{\mathcal{A}}, T/p^m[\underline{t}^{\underline{r}}]) \cong H^q(K(T/p^m[\underline{t}^{\underline{r}}]; \gamma_1 - 1, \ldots, \gamma_d - 1))$ is isomorphic to the direct sum of $\binom{d-1}{q-1}$ copies of $(T/p^m)/\varphi^{-\nu}(\pi)$ and $\binom{d-1}{q}$ copies of $(T/p^m)[\varphi^{-\nu}(\pi)] = (\pi\varphi^{-\nu}(\pi)^{-1}) \cdot (T/p^m)$, and the map $H^q(\Gamma_{\mathcal{A}}, T/p^{m+1}[\underline{t}^{\underline{r}}]) \to H^q(\Gamma_{\mathcal{A}}, T/p^m[\underline{t}^{\underline{r}}])$ is surjective. By taking the sum over $\underline{r}$, we see that the homomorphism $H^q(\Gamma_{\mathcal{A}}, (T\widetilde{A}^{\square}_{\mathrm{inf}}(M)/\pi)/p^{m+1}) \to H^q(\Gamma_{\mathcal{A}}, (T\widetilde{A}^{\square}_{\mathrm{inf}}(M)/\pi)/p^m)$ is surjective, and by Lemma 113 (2), $H^q(\Gamma_{\mathcal{A}}, (T\widetilde{A}^{\square}_{\mathrm{inf}}(M)/\pi)/p^m)[\mathcal{I}] = 0$. This completes the proof. □

Proposition 116 *The morphism* $R\Gamma(\Gamma_{\mathcal{A}}, TA^{\square}_{\inf}(M)) \to R\Gamma(\Gamma_{\mathcal{A}}, T\widetilde{A}^{\square}_{\inf}(M))$ *induces an isomorphism*

$$L\eta^{+}_{\pi} R\Gamma(\Gamma_{\mathcal{A}}, TA^{\square}_{\inf}(M)) \xrightarrow{\cong} L\eta^{+}_{\pi} R\Gamma(\Gamma_{\mathcal{A}}, T\widetilde{A}^{\square}_{\inf}(M)).$$

Proof For $\underline{r} \in (\mathbb{Z}[\frac{1}{p}] \cap [0,1[)^d$, let $T\widetilde{A}^{\square}_{\inf}(M)_{\underline{r}}$ be the $\widetilde{\Gamma}_{\mathcal{A}}$-stable $A^{\square}_{\inf}(\mathcal{A})$-submodule $[\underline{t}^{\underline{r}}] \cdot TA^{\square}_{\inf}(M)$ of $T\widetilde{A}^{\square}_{\inf}(M)$, which is a free $A^{\square}_{\inf}(\mathcal{A})$-module of rank 1 by Lemma 84. For $x \in TA^{\square}_{\inf}(M)$ and $i \in \mathbb{N} \cap [1,d]$ such that $r_i \neq 0$, we have

$$\begin{aligned}&(\gamma_i - 1)([\underline{t}^{\underline{r}}]x) = [\underline{t}^{\underline{r}}]([\underline{\varepsilon}^{r_i}]\gamma_i(x) - x)\\ =&[\underline{t}^{\underline{r}}](([\underline{\varepsilon}^{r_i}] - 1)x + [\underline{\varepsilon}^{r_i}](\gamma_i(x) - x)) = [\underline{t}^{\underline{r}}]([\underline{\varepsilon}^{r_i}] - 1)(x + [\underline{\varepsilon}^{r_i}]\eta_{r_i} \cdot \pi^{-1}(\gamma_i - 1)(x)),\end{aligned}$$

where $\eta_{r_i} = \pi([\underline{\varepsilon}^{r_i}] - 1)^{-1} \in A_{\inf}(O_{\overline{K}})$ and $\pi^{-1}(\gamma_i - 1)(x) \in TA^{\square}_{\inf}(M)$ by Lemma 98. Since $A^{\square}_{\inf}(\mathcal{A})$ is (p, η_{r_i})-adically complete (Lemma 80 (2)), the $A_{\inf}(O_{\overline{K}})$-linear $\Gamma_{\mathcal{A}}$-equivariant endomorphism $g_{\underline{r},i}$ of $TA^{\square}_{\inf}(M)$ defined by $g_{\underline{r},i}(x) = x + [\underline{\varepsilon}^{r_i}]\eta_{r_i}\pi^{-1}(\gamma_i - 1)(x)$ is an isomorphism. The endomorphism $h_{\underline{r},i}$ of $T\widetilde{A}^{\square}_{\inf}(M)_{\underline{r}}$ defined by $h_{\underline{r},i}([\underline{t}^{\underline{r}}]x) = [\underline{t}^{\underline{r}}]\eta_{r_i} g^{-1}_{\underline{r},i}(x)$ is $A_{\inf}(O_{\overline{K}})$-linear, commutes with the action of γ_j $(j \neq i)$, and satisfies $(\gamma_i - 1) \circ h_{\underline{r},i} = h_{\underline{r},i} \circ (\gamma_i - 1) = \pi \cdot \mathrm{id}$.

For $\underline{r} = (r_i)_{1 \leq i \leq d} \in (\mathbb{Z}[\frac{1}{p}] \cap [0,1[)^d$ with $\underline{r} \neq \underline{0}$, let $i(\underline{r})$ be the smallest $i \in \mathbb{N} \cap [1,d]$ such that $r_i \neq 0$. For $i \in \mathbb{N} \cap [1,d]$ and $\mathfrak{a} \in \mathscr{S}_{\inf,\mathrm{G}}$, we define $T\widetilde{A}^{\square}_{\inf,\mathfrak{a}}(M)_i$ to be the direct sum of $T\widetilde{A}^{\square}_{\inf}(M)_{\underline{r}}/\mathfrak{a}$ over $\underline{r} \neq \underline{0}$ with $i(\underline{r}) = i$, and $T\widetilde{A}^{\square}_{\inf}(M)_i$ to be its inverse limit over $\mathfrak{a} \in \mathscr{S}_{\inf,\mathrm{G}}$. By Lemma 84, we have an $A^{\square}_{\inf}(\mathcal{A})$-linear $\widetilde{\Gamma}_{\mathcal{A}}$-equivariant isomorphism $T\widetilde{A}^{\square}_{\inf}(M) \cong TA^{\square}_{\inf}(M) \oplus (\oplus_{1 \leq i \leq d} T\widetilde{A}^{\square}_{\inf}(M)_i)$. Therefore it suffices to prove $\pi \cdot H^q(\Gamma_{\mathcal{A}}, T\widetilde{A}^{\square}_{\inf}(M)_i) = 0$ for $i \in \mathbb{N} \cap [1,d]$. Note that this implies $H^0(\Gamma_{\mathcal{A}}, T\widetilde{A}^{\square}_{\inf}(M)_i) = 0$ as $A_{\inf}(\mathcal{A}_\infty)$ is π-torsion free (Lemma 1 (4)).

Let $i \in \mathbb{N} \cap [1,d]$ and choose a permutation $(j_1, j_2, \ldots, j_d)$ of $(1, 2, \ldots, d)$ such that $j_1 = i$. Put $\mathfrak{a}_n := (p^n, [\underline{p}]^n) \in \mathscr{S}_{\inf,\mathrm{G}}$ for $n \in \mathbb{N}_{>0}$. Then we have an isomorphism

$$\begin{aligned}R\Gamma(\Gamma_{\mathcal{A}}, T\widetilde{A}^{\square}_{\inf}(M)_i) &\cong R\varprojlim_n \big(K(T\widetilde{A}^{\square}_{\inf,\mathfrak{a}_n}(M)_i; \gamma_{j_1} - 1, \ldots, \gamma_{j_d} - 1)\big)\\ &\cong K(T\widetilde{A}^{\square}_{\inf}(M)_i; \gamma_{j_1} - 1, \ldots, \gamma_{j_d} - 1)\\ &\cong \mathrm{Cone}(-(\gamma_i - 1)\colon K_i \to K_i)[-1],\end{aligned}$$

where $K_i = K(T\widetilde{A}^{\square}_{\inf}(M)_i; \gamma_{j_2} - 1, \ldots, \gamma_{j_d} - 1)$. By taking the inverse limit over $\mathfrak{a} \in \mathscr{S}_{\inf,\mathrm{G}}$ of the direct sum of $(h_{\underline{r},i} \bmod \mathfrak{a})$ for $\underline{r} \neq \underline{0}$ with $i(\underline{r}) = i$, we obtain an endomorphism h_i of $T\widetilde{A}^{\square}_{\inf}(M)_i$ such that $(\gamma_i - 1) \circ h_i = h_i \circ (\gamma_i - 1) = \pi \cdot \mathrm{id}$ and $h_i \circ \gamma_j = \gamma_j \circ h_i$ $(j \neq i)$. Therefore $\gamma_i - 1$ on K_i is injective and the cone of $-(\gamma_i - 1)\colon K_i \to K_i$ is quasi-isomorphic to its cokernel, which is annihilated by π. □

Lemma 117 *There exists a canonical isomorphism*

$$R\Gamma(\Gamma_{\mathcal{A}}, TA^{\square}_{\mathrm{inf}}(M))) \cong K(TA^{\square}_{\mathrm{inf}}(M); \gamma_1 - 1, \dots, \gamma_d - 1)$$

compatible with φ *and functorial in* M.

Proof Let $\mathfrak{a}_n = (p^n, [\underline{p}]^n) \in \mathscr{S}_{\mathrm{inf,G}}$ for $n \in \mathbb{N}_{>0}$. Then we have the following isomorphism compatible with φ. Note $\varphi(\mathfrak{a}_n) \subset \mathfrak{a}_n$ for $n \in \mathbb{N}_{>0}$.

$$\begin{aligned} R\Gamma(\Gamma_{\mathcal{A}}, TA^{\square}_{\mathrm{inf}}(M)) &\cong R\varprojlim_n \bigl(K(TA^{\square}_{\mathrm{inf}}(M)/\mathfrak{a}_n; \gamma_1 - 1, \dots, \gamma_d - 1)\bigr) \\ &\cong K(TA^{\square}_{\mathrm{inf}}(M); \gamma_1 - 1, \dots, \gamma_d - 1). \end{aligned}$$

□

By combining Propositions 112, 116, and Lemma 117, we obtain the following.

Proposition 118 *There exists a canonical isomorphism*

$$\eta^{+}_{\pi} K(TA^{\square}_{\mathrm{inf}}(M); \gamma_1 - 1, \dots, \gamma_d - 1) \xrightarrow{\cong} L\eta^{+}_{\pi} R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M)) \qquad (117)$$

in $D^{+}(A_{\mathrm{inf}}(O_{\overline{K}})\text{-Mod})$ *compatible* φ *and functorial in* M.

We will show that the source of (117) becomes isomorphic to the de Rham complex of $A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M := \varprojlim_m (A_{\mathrm{crys}}(O_{\overline{K}}) \otimes_{O_K} M)/p^m$ after taking $R\varprojlim_m ((A_{\mathrm{crys}}(O_{\overline{K}})/p^m) \otimes^{L}_{A_{\mathrm{inf}}(O_{\overline{K}})} -)$.

Lemma 119 *We have* $\pi^{[n]}\pi^{-1} \in \mathbb{Z}_p[\pi, \frac{\pi^{p-1}}{p}] \subset A_{\mathrm{crys}}(O_{\overline{K}})$ *for* $n \in \mathbb{N}_{>0}$, *and it converges to* 0 *as* $n \to \infty$ *with respect to the* p*-adic topology of* $A_{\mathrm{crys}}(O_{\overline{K}})$.

Proof This follows from $p^{-1}\pi^{p-1}, p^{-1}(p^{-1}\pi^{p-1}) \in A_{\mathrm{crys}}(O_{\overline{K}})$ (see the proof of Lemma 90) and Lemma 89 (1). □

Put $t := \log[\underline{\varepsilon}] \in A_{\mathrm{crys}}(O_{\overline{K}})$ as before Lemma 90.

Proposition 120 (1) *For* $\gamma \in \Gamma_{\mathcal{A}}$ *and* $x \in TA^{\square}_{\mathrm{crys}}(M)$, *we have* $p^{-n}(\gamma^{p^n} - 1)(x) \in TA^{\square}_{\mathrm{crys}}(M)$, *and* $\nabla_{\gamma}(x) := \lim_{n\to\infty} p^{-n}(\gamma^{p^n} - 1)(x)$ *converges to an element of* $t \cdot TA^{\square}_{\mathrm{crys}}(M)$ *with respect to the* p*-adic topology of* $TA^{\square}_{\mathrm{crys}}(M)$. *The endomorphism* ∇_{γ} $(\gamma \in \Gamma_{\mathcal{A}})$ *of* $TA^{\square}_{\mathrm{crys}}(M)$ *is* $A_{\mathrm{crys}}(O_{\overline{K}})$*-linear, is* $\Gamma_{\mathcal{A}}$*-equivariant, and commutes with* φ. *Moreover we have* $\nabla_{\gamma} \circ \nabla_{\gamma'} = \nabla_{\gamma'} \circ \nabla_{\gamma}$ *for* $\gamma, \gamma' \in \Gamma_{\mathcal{A}}$.

(2) *The homomorphism* $\nabla\colon TA^{\square}_{\mathrm{crys}}(M) \to TA^{\square}_{\mathrm{crys}}(M) \otimes_{A^{\square}_{\mathrm{crys}}(\mathcal{A})} \Omega_{A^{\square}_{\mathrm{crys}}(\mathcal{A})}$ *defined by* $\nabla(x) := \sum_{1\le i\le d} t^{-1}\nabla_{\gamma_i}(x) \otimes d\log[\underline{t}_i]$ *is an integrable connection with respect to* $d\colon A^{\square}_{\mathrm{crys}}(\mathcal{A}) \to \Omega_{A^{\square}_{\mathrm{crys}}(\mathcal{A})}$, *is* $\Gamma_{\mathcal{A}}$*-equivariant, and commutes with* φ: $(\varphi \otimes \varphi) \circ \nabla = \nabla \circ \varphi$.

(3) *For* $\gamma \in \Gamma_{\mathcal{A}}$ *and* $x \in TA^{\square}_{\mathrm{crys}}(M)$, *we have* $(n!)^{-1}(\nabla_{\gamma})^n(x) \in TA^{\square}_{\mathrm{crys}}(M)$ *and* $\exp(\nabla_{\gamma})(x) := \sum_{n\in\mathbb{N}}(n!)^{-1}(\nabla_{\gamma})^n(x)$ *converges to* $\gamma(x)$.

(4) *For* $\gamma \in \Gamma_{\mathcal{A}}$ *and* $x \in TA^{\square}_{\mathrm{inf}}(M)$, *we have* $t^{-1}\nabla_{\gamma}(x) \equiv \pi^{-1}(\gamma-1)(x)$ *mod* $I^1A^{\square}_{\mathrm{crys}}(\mathcal{A}) \cdot TA^{\square}_{\mathrm{crys}}(M)$.

Proof (1), (3) Let $\widehat{R}$ be as in Lemma 89. By Lemma 98, $\gamma(\pi) = \pi$, and $p^{-1}\pi^{p-1}$, $p^{-1}(p^{-1}\pi^{p-1})^p \in A_{\mathrm{crys}}(O_{\overline{K}})$ (proof of Lemma 90), we can define a continuous action of $\widehat{R}$ on $TA^{\square}_{\mathrm{crys}}(M)$ by $Tx = (\gamma-1)(x)$. By Lemma 89 (5), we see that $p^{-n}(\gamma^{p^n}-1)(x) \in TA^{\square}_{\mathrm{crys}}(M)$ and it converges to $\log(1+T)\cdot x$ as $n \to \infty$. By Lemmas 89 (4), 98, and 90 (2), we obtain $\nabla_{\gamma}(x) \in (\gamma-1)(TA^{\square}_{\mathrm{crys}}(M)) \subset \pi \cdot TA^{\square}_{\mathrm{crys}}(M) = t \cdot TA^{\square}_{\mathrm{crys}}(M)$. Since the action of γ on $TA^{\square}_{\mathrm{crys}}(M)$ is $A_{\mathrm{crys}}(O_{\overline{K}})$-linear, is $\Gamma_{\mathcal{A}}$-equivariant, and commutes with φ, so does the endomorphism ∇_{γ} and we have $\nabla_{\gamma} \circ \nabla_{\gamma'} = \nabla_{\gamma'} \circ \nabla_{\gamma}$ for any $\gamma' \in \Gamma_{\mathcal{A}}$. The first (resp. second) claim in (3) follows from Lemma 89 (1) and (4) (resp. (3)).

(2) For $a \in A^{\square}_{\mathrm{crys}}(\mathcal{A})$ and $x \in TA^{\square}_{\mathrm{crys}}(M)$, $\nabla_{\gamma_i}(ax)$ is equal to

$$\lim_{n\to\infty}\{p^{-n}(\gamma_i^{p^n}-1)(a)\cdot\gamma_i^{p^n}(x) + a\cdot p^{-n}(\gamma_i^{p^n}-1)(x)\} = td_i^{\log}(a)x + a\nabla_{\gamma_i}(x)$$

by Proposition 91 (2). Hence the claim follows from $\nabla_{\gamma_i}\circ\nabla_{\gamma_j} = \nabla_{\gamma_j}\circ\nabla_{\gamma_i}$, $\gamma\circ\nabla_{\gamma_i} = \nabla_{\gamma_i}\circ\gamma$ ($\gamma \in \Gamma_{\mathcal{A}}$), and $\varphi\circ\nabla_{\gamma_i} = \nabla_{\gamma_i}\circ\varphi$ proven in (1). Note that the action of $\Gamma_{\mathcal{A}}$ on t and $d\log[\underline{t}_i]$ is trivial, and that we have $\varphi(t) = pt$ and $\varphi(d\log[\underline{t}_i]) = pd\log[\underline{t}_i]$.

(4) By Lemmas 90 (2) and 119, we have $(n!)^{-1}\pi^{n-1}, (n!)^{-1}t^{n-1} \in A_{\mathrm{crys}}(O_{\overline{K}})$ for $n \in \mathbb{N}_{>0}$, and they converge to 0 as $n \to \infty$. By the claim (3), we have $(\gamma-1)(x) = t\{(t^{-1}\nabla_{\gamma}(x) + \sum_{n=2}^{\infty}\frac{t^{n-1}}{n!}(t^{-1}\nabla_{\gamma})^n(x)\}$ and $t = \pi(1+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1}{n}\pi^{n-1})$, and the two series $\sum_{n=2}^{\infty}$ converge to elements of $I^1A^{\square}_{\mathrm{crys}}(\mathcal{A})\cdot TA^{\square}_{\mathrm{crys}}(M)$ and $I^1A^{\square}_{\mathrm{crys}}(\mathcal{A})$ as $I^1A^{\square}_{\mathrm{crys}}(\mathcal{A})$ is closed in $A^{\square}_{\mathrm{crys}}(\mathcal{A})$ by definition. □

By (82), the isomorphism (92) induces an isomorphism compatible with φ

$$A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K}M := \varprojlim_m (A_{\mathrm{crys}}(O_{\overline{K}})\otimes_{O_K}M)/p^m \xrightarrow{\cong} TA^{\square}_{\mathrm{crys}}(M). \tag{118}$$

Proposition 121 *The following diagram is commutative.*

$$\begin{array}{ccc} A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K}M & \xrightarrow{\mathrm{id}\widehat{\otimes}\nabla} & (A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K}M)\otimes_{\mathcal{A}}\Omega_{\mathcal{A}} \\ {\scriptstyle (118)}\downarrow\cong & & \cong\downarrow{\scriptstyle (\mathrm{id}\otimes\alpha^*)\circ((118)\otimes\mathrm{id})} \\ TA^{\square}_{\mathrm{crys}}(M) & \xrightarrow{\nabla} & TA^{\square}_{\mathrm{crys}}(M)\otimes_{A^{\square}_{\mathrm{crys}}(\mathcal{A})}\Omega_{A^{\square}_{\mathrm{crys}}(\mathcal{A})}. \end{array}$$

Proof It suffices to prove the claim for the restriction on M. Let $x \in M$, let y be its image in $TA^{\square}_{\mathrm{crys}}(M)$ under (118), and let z_i ($i \in \mathbb{N}\cap[1,d]$) (resp. z) $\in M\otimes_{\mathcal{A},\iota}\mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})$ be the image of $\nabla_{\gamma_i}(y)$ (resp. y) under the isomorphism (94). We have $z = c^{\square}_M(x\otimes 1)$ (see (95)). Hence, by (96), we have

$$z_i = \lim_{m\to\infty} p^{-m}(1\otimes\gamma_i^{p^m}-1)(z) = \lim_{m\to\infty} p^{-m}(1\otimes\gamma_i^{p^m}-1)(\sum_{\underline{n}\in\mathbb{N}^d}\nabla_{\underline{n}}^{\log}(x)\otimes\underline{v}^{[\underline{n}]}).$$

For $m, n \in \mathbb{N}$, we have

$$\begin{aligned}(\gamma_i^{p^m}-1)(v_i^{[n]}) &= ([\underline{\varepsilon}^{p^m}]v_i + ([\underline{\varepsilon}^{p^m}]-1))^{[n]} - v_i^{[n]}\\ &= ([\underline{\varepsilon}^{p^m n}]-1)v_i^{[n]} + \sum_{l=1}^{n}([\underline{\varepsilon}^{p^m}]v_i)^{[n-l]}([\underline{\varepsilon}^{p^m}]-1)^{[l]},\end{aligned}$$

and $\gamma_i(v_j) = v_j$ $(j\neq i)$. By Lemma 90 (1), we obtain

$$\lim_{m\to\infty} p^{-m}(\gamma_i^{p^m}-1)(v_i^{[n]}) = t(nv_i^{[n]} + v_i^{[n-1]})\ \ (\text{if } n>0),\quad 0\ \ (\text{if } n=0).$$

Hence we have

$$\begin{aligned}z_i &= t\sum_{\underline{n}\in\mathbb{N}^d,\underline{n}\geq\underline{1}_i}\nabla_{\underline{n}}^{\log}(x)\otimes\underline{v}^{[\underline{n}-n_i\underline{1}_i]}(n_i v_i^{[n_i]} + v_i^{[n_i-1]})\\ &= t\sum_{\underline{n}\in\mathbb{N}^d}(n_i\nabla_{\underline{n}}^{\log} + \nabla_{\underline{n}+\underline{1}_i}^{\log})(x)\otimes\underline{v}^{[\underline{n}]}\\ &= t\sum_{\underline{n}\in\mathbb{N}^d}\nabla_{\underline{n}}^{\log}(\nabla_i^{\log}(x))\otimes\underline{v}^{[\underline{n}]} = c_M^{\square}(\nabla_i^{\log}(x)\otimes t).\end{aligned}$$

This implies that $\nabla_{\gamma_i}(y)$ is the image of $t\otimes\nabla_i^{\log}(x)$ under (118). □

Put $\Omega_{\mathcal{A}}^q := \wedge_{\mathcal{A}}^q\Omega_{\mathcal{A}}$ and $\Omega_{A_{\mathrm{crys}}^{\square}(\mathcal{A})}^q := \wedge_{A_{\mathrm{crys}}^{\square}(\mathcal{A})}^q\Omega_{A_{\mathrm{crys}}^{\square}(\mathcal{A})}$ $(q\in\mathbb{N})$. By Proposition 121, the isomorphism (118) induces an isomorphism of de Rham complexes compatible with φ

$$(A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K}M)\otimes_{\mathcal{A}}\Omega_{\mathcal{A}}^{\bullet} \cong TA_{\mathrm{crys}}^{\square}(M)\otimes_{A_{\mathrm{crys}}^{\square}(\mathcal{A})}\Omega_{A_{\mathrm{crys}}^{\square}(\mathcal{A})}^{\bullet}. \qquad (119)$$

Put $\nabla_i^{\log} := t^{-1}\nabla_{\gamma_i}$ $(i\in\mathbb{N}\cap[1,d])$ on $TA_{\mathrm{crys}}^{\square}(M)$. By Lemmas 119 and 90 (2), we can define an $A_{\mathrm{crys}}(O_{\overline{K}})$-linear endomorphism F_i of $TA_{\mathrm{crys}}^{\square}(M)$ by

$$F_i(x) = \sum_{n\in\mathbb{N}_{>0}} t^{-1}t^{[n]}(\nabla_i^{\log})^{n-1}(x). \qquad (120)$$

Lemma 122 (1) *F_i is an automorphism congruent to* id *modulo $I^1A_{\mathrm{crys}}^{\square}(\mathcal{A})$.*
(2) *We have $t\cdot F_i\circ\nabla_i^{\log} = \gamma_i - 1$ on $TA_{\mathrm{crys}}^{\square}(M)$ for $i\in\mathbb{N}\cap[1,d]$.*
(3) *For $i, j\in\mathbb{N}\cap[1,d]$, we have $F_iF_j = F_jF_i$, $F_i\nabla_j^{\log} = \nabla_j^{\log}F_i$ and $F_i\gamma_j = \gamma_jF_i$.*
(4) *We have $\varphi\circ F_i = F_i\circ\varphi$ for $i\in\mathbb{N}\cap[1,d]$.*

Proof (1) Since $A_{\mathrm{crys}}(O_{\overline{K}})$ is p-adically complete and separated, and the elements π and $p^{-1}\pi^{p-1}$ of $\mathrm{Fil}^1 A_{\mathrm{crys}}(O_{\overline{K}})$ are nilpotent in $A_{\mathrm{crys}}(O_{\overline{K}})/p$, it suffices to prove that F_i becomes the identity map modulo $(p, \pi, p^{-1}\pi^{p-1})$ and also modulo $I^1 A^{\square}_{\mathrm{crys}}(\mathcal{A})$. This follows from Lemmas 119 and 90 (2). For the second case, note that $I^1 A^{\square}_{\mathrm{crys}}(\mathcal{A})$ is closed in $A^{\square}_{\mathrm{crys}}(\mathcal{A})$ by definition.

(2) This follows from Proposition 120 (3) and $\nabla_{\gamma_i} = t\nabla_i^{\log}$.

(3) One can verify the first two equalities by explicit computation using $\nabla_i^{\log}\nabla_j^{\log} = \nabla_j^{\log}\nabla_i^{\log}$. The last one follows from $\nabla_{\gamma_i} \circ \gamma_j = \gamma_j \circ \nabla_{\gamma_i}$ (Proposition 120 (1)).

(4) The claim follows from $\varphi \circ \nabla_{\gamma_i} = \nabla_{\gamma_i} \circ \varphi$ (Proposition 120 (1)). □

For $I = \{i_1 < \cdots < i_q\} \subset \mathbb{N} \cap [1, d]$, we define $d\log[\underline{t}_I] \in \Omega^q_{A^{\square}_{\mathrm{crys}}(\mathcal{A})}$ and $e_I \in \wedge^q E$, where $E = \mathbb{Z}^d$, to be $d\log[\underline{t}_{i_1}] \wedge \ldots \wedge d\log[\underline{t}_{i_q}]$ and $e_{i_1} \wedge \ldots \wedge e_{i_q}$. Let F_I denote the composition $F_{i_q} \circ F_{i_{q-1}} \circ \cdots \circ F_{i_1}$, which is an $A_{\mathrm{crys}}(O_{\overline{K}})$-linear automorphism of $TA^{\square}_{\mathrm{crys}}(M)$. We define the isomorphism $F^q \colon TA^{\square}_{\mathrm{crys}}(M) \otimes_{A^{\square}_{\mathrm{crys}}(\mathcal{A})} \Omega^q_{A^{\square}_{\mathrm{crys}}(\mathcal{A})} \xrightarrow{\cong} TA^{\square}_{\mathrm{crys}}(M) \otimes_{\mathbb{Z}} \wedge^q E$ by $F^q(x \otimes d\log[\underline{t}_I]) = F_I(x) \otimes e_I$ for $x \in TA^{\square}_{\mathrm{crys}}(M)$ and $I \subset \mathbb{N} \cap [1, d]$ with $\sharp I = q$.

Proposition 123 *The isomorphisms*

$$G^q := t^q F^q \colon TA^{\square}_{\mathrm{crys}}(M) \otimes_{A^{\square}_{\mathrm{crys}}(\mathcal{A})} \Omega^q_{A^{\square}_{\mathrm{crys}}(\mathcal{A})} \xrightarrow{\cong} t^q K^q(TA^{\square}_{\mathrm{crys}}(M); \gamma_1 - 1, \ldots, \gamma_d - 1)$$

for $q \in \mathbb{N}$ induce an isomorphism of complexes compatible with φ

$$G \colon TA^{\square}_{\mathrm{crys}}(M) \otimes_{A^{\square}_{\mathrm{crys}}(\mathcal{A})} \Omega^{\bullet}_{A^{\square}_{\mathrm{crys}}(\mathcal{A})} \xrightarrow{\cong} \eta^+_{\pi} K(TA^{\square}_{\mathrm{crys}}(M); \gamma_1 - 1, \ldots, \gamma_d - 1).$$

Proof By Lemmas 98 and 90 (2), the degree q-part of the complex $\eta^+_{\pi} K(TA^{\square}_{\mathrm{crys}}(M); \gamma_1 - 1, \ldots, \gamma_d - 1)$ is given by $t^q TA^{\square}_{\mathrm{crys}}(M) \otimes_{\mathbb{Z}} \wedge^q E$. For $x \in TA^{\square}_{\mathrm{crys}}(M)$ and $I \subset \mathbb{N} \cap [1, d]$, we have

$$\begin{aligned} G^{q+1} \circ \nabla^q(x d\log[\underline{t}_I]) &= G^{q+1}\left(\textstyle\sum_{i \in I^c} \nabla_i^{\log}(x) d\log[\underline{t}_i] \wedge d\log[\underline{t}_I]\right) \\ &= t^{q+1} \textstyle\sum_{i \in I^c} F_{I \cup \{i\}} \circ \nabla_i^{\log}(x) \otimes e_i \wedge e_I \\ d^q \circ G^q(x d\log[\underline{t}_I]) &= d^q(t^q F_I(x) \otimes e_I) = t^q \textstyle\sum_{i \in I^c} (\gamma_i - 1) \circ F_I(x) \otimes e_i \wedge e_I, \end{aligned}$$

where $I^c = (\mathbb{N} \cap [1, d]) \backslash I$ and $q = \sharp I$. Lemma 122 (2) and (3) imply that these two elements coincide. The compatibility with φ follows from Lemma 122 (4), $\varphi(d\log[\underline{t}_i]) = p d\log[\underline{t}_i]$, and $\varphi(t) = pt$. □

Proposition 124 *We have the following canonical isomorphisms, which are compatible with φ except the third one. We abbreviate $A_{\mathrm{crys}}(O_{\overline{K}})$, $A_{\mathrm{inf}}(O_{\overline{K}})$, and the Koszul complex $K(TA^{\square}_{\mathrm{inf}}(M); \gamma_1 - 1, \ldots, \gamma_d - 1)$ to A_{crys}, A_{inf}, and $K_{\gamma}(TA^{\square}_{\mathrm{inf}}(M))$, respectively, and we regard O_C as an A_{inf}-algebra via θ.*

$$R\varprojlim_m (A_{\mathrm{crys}}/p^m \otimes^L_{A_{\mathrm{inf}}} \eta^+_\pi K_\gamma(TA^{\square}_{\mathrm{inf}}(M))) \xrightarrow{\cong} A_{\mathrm{crys}}\widehat{\otimes}_{O_K} M \otimes_{\mathcal{A}} \Omega^\bullet_{\mathcal{A}} \tag{121}$$

$$A_{\mathrm{inf}}/\pi \otimes^L_{A_{\mathrm{inf}}} \eta^+_\pi K_\gamma(TA^{\square}_{\mathrm{inf}}(M)) \xrightarrow{\cong} \varprojlim_m ((A_{\mathrm{inf}}/\pi)/p^m \otimes_{O_K} M \otimes_{\mathcal{A}} \Omega^\bullet_{\mathcal{A}}) \tag{122}$$

$$O_C \otimes^L_{A_{\mathrm{inf}}} \eta^+_\pi K_\gamma(TA^{\square}_{\mathrm{inf}}(M)) \longrightarrow \varprojlim_m (O_C/p^m \otimes_{O_K} M \otimes_{\mathcal{A}} \Omega^\bullet_{\mathcal{A}}) \tag{123}$$

$$R\varprojlim_m (W_m(\overline{k}) \otimes^L_{A_{\mathrm{inf}}} \eta^+_\pi K_\gamma(TA^{\square}_{\mathrm{inf}}(M))) \xrightarrow{\cong} \varprojlim_m (W_m(\overline{k}) \otimes_{O_K} M \otimes_{\mathcal{A}} \Omega^\bullet_{\mathcal{A}}) \tag{124}$$

Proof We have $\eta^+_\pi K_\gamma(TA^{\square}_{\mathrm{crys}}(M)) \cong A_{\mathrm{crys}}\widehat{\otimes}_{O_K} M \otimes_{\mathcal{A}} \Omega^\bullet_{\mathcal{A}}$ compatible with φ by (119) and Proposition 123. By Lemma 98, we have $(\eta^+_\pi K_\gamma(TA^{\square}_{\mathrm{inf}}(M)))^q = \pi^q K_\gamma(TA^{\square}_{\mathrm{inf}}(M))^q$ and $(\eta^+_\pi K_\gamma(TA^{\square}_{\mathrm{crys}}(M)))^q = \pi^q K_\gamma(TA^{\square}_{\mathrm{crys}}(M))^q$ for $q \in \mathbb{N}$. In particular, they are free of finite type over $A^{\square}_{\mathrm{inf}}(\mathcal{A})$ and $A^{\square}_{\mathrm{crys}}(\mathcal{A})$, respectively, and the latter is the scalar extension of the former.

Let S be one of A_{crys}/p^m, A_{inf}/π, O_C, and $W_m(\overline{k})$. Then we have an isomorphism

$$S \otimes^L_{A_{\mathrm{inf}}} \eta^+_\pi K_\gamma(TA^{\square}_{\mathrm{inf}}(M)) \xrightarrow{\cong} S \otimes_{A_{\mathrm{inf}}} \eta^+_\pi K_\gamma(TA^{\square}_{\mathrm{inf}}(M))$$

by Lemma 125 below and the fact that π and ξ are regular on A_{inf} and $A^{\square}_{\mathrm{inf}}(\mathcal{A})$ (Lemmas 1 (4), 80 (4)). Since $A^{\square}_{\mathrm{inf}}(\mathcal{A})/\pi$ and $A^{\square}_{\mathrm{inf}}(\mathcal{A})/\xi$ are p-adically complete and separated by Lemma 80 (3), the target of the above isomorphism is p-adically complete and separated if $S = A_{\mathrm{inf}}/\pi$ or O_C.

Let $\mathcal{S}$ be one of A_{crys}, A_{inf}/π, O_C, and $W(\overline{k})$, and put $\mathcal{S}_m := \mathcal{S}/p^m$ $(m \in \mathbb{N})$. Then the homomorphism $A_{\mathrm{inf}} \to \mathcal{S}$ naturally factors through A_{crys}; in the case $\mathcal{S} = W(\overline{k})$, note that the image of $\mathrm{Ker}(\theta) = \xi A_{\mathrm{inf}}$ in $W(\overline{k})$ is $pW(\overline{k})$. Hence we have $\mathcal{S}_m \otimes_{A_{\mathrm{inf}}} A^{\square}_{\mathrm{inf}}(\mathcal{A}) \xrightarrow{\cong} \mathcal{S}_m \otimes_{A_{\mathrm{crys}}} A^{\square}_{\mathrm{crys}}(\mathcal{A})$ by Lemma 79 (2) and (74). Thus we see that the source of the morphism in question relevant to $\mathcal{S}$ is isomorphic to

$$\begin{aligned}\varprojlim_m (\mathcal{S}_m \otimes_{A_{\mathrm{crys}}} \eta^+_\pi K_\gamma(TA^{\square}_{\mathrm{crys}}(M))) &\cong \varprojlim_m (\mathcal{S}_m \otimes_{A_{\mathrm{crys}}} (A_{\mathrm{crys}}\widehat{\otimes}_{\mathcal{A}} M) \otimes_{\mathcal{A}} \Omega^\bullet_{\mathcal{A}}) \\ &\cong \varprojlim_m (\mathcal{S}_m \otimes_{O_K} M \otimes_{\mathcal{A}} \Omega^\bullet_{\mathcal{A}}).\end{aligned}$$

□

Lemma 125 (1) *For $m \in \mathbb{N}_{>0}$, we have an isomorphism*

$$A_{\mathrm{crys}}(O_{\overline{K}})/p^m \otimes^L_{A_{\mathrm{inf}}(O_{\overline{K}})} A^{\square}_{\mathrm{inf}}(\mathcal{A}) \cong A_{\mathrm{crys}}(O_{\overline{K}})/p^m \otimes_{A_{\mathrm{inf}}(O_{\overline{K}})} A^{\square}_{\mathrm{inf}}(\mathcal{A}).$$

(2) *For $m \in \mathbb{N}_{>0}$, we have an isomorphism*

$$W_m(\overline{k}) \otimes^L_{A_{\mathrm{inf}}(O_{\overline{K}})} A^{\square}_{\mathrm{inf}}(\mathcal{A}) \cong W_m(\overline{k}) \otimes_{A_{\mathrm{inf}}(O_{\overline{K}})} A^{\square}_{\mathrm{inf}}(\mathcal{A}).$$

Proof Since the images of p^m and π^{pm} in $A_{\mathrm{crys}}(O_{\overline{K}})/p^m$ and $W_m(\overline{k})$ vanish, and p^m, π^{pm} form a regular sequence on $A_{\mathrm{inf}}(O_{\overline{K}})$ and $A_{\mathrm{inf}}^{\square}(\mathcal{A})$ (Lemmas 1 (4), 79 (1), 80 (4)), we have $A_{\mathrm{inf}}(O_{\overline{K}})/(p^m,\pi^{pm})\otimes^L_{A_{\mathrm{inf}}(O_{\overline{K}})}A_{\mathrm{inf}}^{\square}(\mathcal{A})\xrightarrow{\cong}A_{\mathrm{inf}}^{\square}(\mathcal{A})/(p^m,\pi^{pm})$. Hence we may replace $A_{\mathrm{inf}}(O_{\overline{K}})$ and $A_{\mathrm{inf}}^{\square}(\mathcal{A})$ with $A_{\mathrm{inf}}(O_{\overline{K}})/(p^m,\pi^{pm})=A_{\mathrm{inf},\mathfrak{a}_m}(O_{\overline{K}})$ and $A_{\mathrm{inf}}^{\square}(\mathcal{A})/(p^m,\pi^{pm})\cong A_{\mathrm{inf},\mathfrak{a}_m}^{\square}(\mathcal{A})$ (Lemma 79 (2)), where $\mathfrak{a}_m=p^mA_{\mathrm{inf}}(O_{\overline{K}})+\pi^{pm}A_{\mathrm{inf}}(O_{\overline{K}})\in\mathscr{S}_{\mathrm{inf}}$. Then the claim follows from the smoothness of $A_{\mathrm{inf},\mathfrak{a}}(O_{\overline{K}})\to A_{\mathrm{inf},\mathfrak{a}}^{\square}(\mathcal{A})$ ($\mathfrak{a}\in\mathscr{S}_{\mathrm{inf}}$). □

Put $A_{\mathrm{inf}}(O_{\overline{K}})/\pi\widehat{\otimes}_{O_K}M:=\varprojlim_m(A_{\mathrm{inf}}(O_{\overline{K}})/\pi)/p^m\otimes_{O_K}M$. By combining Propositions 118 and 124, we obtain the following isomorphisms in $D^+(A_{\mathrm{inf}}(O_{\overline{K}})/\pi\text{-Mod})$ and $D^+(A_{\mathrm{crys}}(O_{\overline{K}})\text{-Mod})$ compatible with φ, where we abbreviate $A_{\mathrm{inf}}(O_{\overline{K}})$ and $A_{\mathrm{crys}}(O_{\overline{K}})$ to A_{inf} and A_{crys}, respectively.

$$(A_{\mathrm{inf}}/\pi\widehat{\otimes}_{O_K}M)\otimes_{\mathcal{A}}\Omega^{\bullet}_{\mathcal{A}}\xrightarrow{\cong}A_{\mathrm{inf}}/\pi\otimes^L_{A_{\mathrm{inf}}}L\eta^{+}_{\pi}R\Gamma(\Delta_{\mathcal{A}},TA_{\mathrm{inf}}(M))\tag{125}$$

$$(A_{\mathrm{crys}}\widehat{\otimes}_{O_K}M)\otimes_{\mathcal{A}}\Omega^{\bullet}_{\mathcal{A}}\xrightarrow{\cong}R\varprojlim_m(A_{\mathrm{crys}}/p^m\otimes^L_{A_{\mathrm{inf}}}L\eta^{+}_{\pi}R\Gamma(\Delta_{\mathcal{A}},TA_{\mathrm{inf}}(M)))\tag{126}$$

In the following sections, we give an alternative construction of these morphisms without forgetting $\underline{G_K}$-action, i.e., in $D^+(\underline{G_K}\text{-}\mathbf{Sets},A_{\mathrm{inf}}(O_{\overline{K}})/\pi)$ and $D^+(\underline{G_K}\text{-}\mathbf{Sets},A_{\mathrm{crys}}(O_{\overline{K}}))$.

16 Comparison Theorem with de Rham Complex over A_{inf}/π

In this section, we show that the morphism (125) is independent of the choice of the framing □ and is defined in the derived category $D^+(\underline{G_K}\text{-}\mathbf{Sets},A_{\mathrm{inf}}(O_{\overline{K}})/\pi)$ by giving another construction of the morphism when $p\geq 5$ (Theorems 136 and 139). This independence can also be derived from the corresponding claim for (126), which is proved in later sections for any p but in a much more complicated way.

We define the filtration I^r ($r\in\mathbb{Z}$) on $TA_{\mathrm{inf}}(M)$ by $I^rA_{\mathrm{inf}}(\overline{\mathcal{A}})\cdot TA_{\mathrm{inf}}(M)$. For $q\in\mathbb{N}$, let $\mathrm{Bock}^q_I\colon H^q(\Delta_{\mathcal{A}},\mathrm{gr}^q_ITA_{\mathrm{inf}}(M))\to H^{q+1}(\Delta_{\mathcal{A}},\mathrm{gr}^{q+1}_ITA_{\mathrm{inf}}(M))$ be the boundary map associated to the exact sequence $0\to\mathrm{gr}^{q+1}_ITA_{\mathrm{inf}}(M)\to I^qTA_{\mathrm{inf}}(M)/I^{q+2}TA_{\mathrm{inf}}(M)\to\mathrm{gr}^q_ITA_{\mathrm{inf}}(M)\to 0$. By the same argument as the proof of Proposition 112, we obtain an isomorphism $C^{\bullet}_{\mathrm{cont}}(\Delta_{\mathcal{A}},TA_{\mathrm{inf}}(M))/\pi^n\xrightarrow{\cong}C^{\bullet}_{\mathrm{cont}}(\Delta_{\mathcal{A}},TA_{\mathrm{inf}}(M)/\pi^n)$. By Corollary 104, we obtain an isomorphism

$$R\Gamma(\Delta_{\mathcal{A}},TA_{\mathrm{inf}}(M))\otimes^L_{A_{\mathrm{inf}}(O_{\overline{K}})}A_{\mathrm{inf}}(O_{\overline{K}})/\pi^n\xrightarrow{\cong}R\Gamma(\Delta_{\mathcal{A}},TA_{\mathrm{inf}}(M)/\pi^n)$$

in $D^+(\underline{G_K}\text{-}\mathbf{Sets},A_{\mathrm{inf}}(O_{\overline{K}})/\pi^n)$. Therefore we have a canonical isomorphism (Proposition 101)

$$L\eta_\pi^+ R\Gamma(\Delta_{\mathcal{A}}, TA_{\rm inf}(M))\otimes^L_{A_{\rm inf}(O_{\overline{K}})} A_{\rm inf}(O_{\overline{K}})/\pi \xrightarrow{\cong} (H^\bullet(\Delta_{\mathcal{A}}, {\rm gr}^\bullet_I TA_{\rm inf}(M)), {\rm Bock}^\bullet_I) \quad (127)$$

in the derived category $D^+(\underline{G_K}\text{-}\mathbf{Sets}, A_{\rm inf}(O_{\overline{K}})/\pi)$ of $A_{\rm inf}(O_{\overline{K}})/\pi$-modules with $\underline{G_K}$-action. We give an alternative construction of (125) via the target of (127).

Let $\mathcal{B}$ be a flat O_K-algebra p-adically complete and separated such that the homomorphisms $O_K/p^m \to \mathcal{B}/p^m$ ($m \in \mathbb{N}_{>0}$) are smooth, and suppose that we are given a surjective O_K-homomorphism $\mathcal{B} \to \mathcal{A}$ and $s_1, \ldots, s_e \in \mathcal{B}^\times$ such that $d\log s_i$ ($i \in \mathbb{N} \cap [1, e]$) form a basis of $\Omega_{(\mathcal{B}/p^m)/(O_K/p^m)}$ for every $m \in \mathbb{N}_{>0}$. Let $\varphi_{\mathcal{B}} \colon \mathcal{B} \to \mathcal{B}$ be the unique lifting of the absolute Frobenius of $\mathcal{B}/p$ compatible with σ of O_K such that $\varphi_{\mathcal{B}}(s_i) = s_i^p$. We will compare the two framings defined by $t_1, \ldots, t_d \in A^\times$ and by $t'_1, \ldots, t'_d \in A^\times$ via $\mathcal{B} := \varprojlim_n (A \otimes_{O_K} A)/p^n$ with the product homomorphism $\mathcal{B} \to \mathcal{A}$ and $t_i \otimes 1, 1 \otimes t'_i \in \mathcal{B}^\times$ ($i \in \mathbb{N} \cap [1, d]$).

Put $O_{K,m} := O_K/p^m$, $\mathcal{A}_m := \mathcal{A}/p^m$, $\mathcal{B}_m := \mathcal{B}/p^m$, $\Omega_{\mathcal{B}_m} := \Omega_{\mathcal{B}_m/O_{K,m}}$, $\Omega_{\mathcal{B}} := \varprojlim_m \Omega_{\mathcal{B}_m}$, and $\varphi_{\mathcal{B}_m} := \varphi_{\mathcal{B}} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m$ as in Sect. 2. We define $\mathcal{P}_m$ with ${\rm Fil}^r$, $\nabla_{\mathcal{P}_m}$ and $\varphi_{\mathcal{P}_m}$, and $\mathscr{A}_{{\rm crys},\mathcal{B},m}(\overline{\mathcal{A}})$ with $G_{\mathcal{A}}$-action, ${\rm Fil}^r$, ∇ and φ as in Sect. 2 by using $\mathcal{B} \to \mathcal{A}$ and $\varphi_{\mathcal{B}}$. We define $u_{i,m} \in \mathscr{A}_{{\rm crys},\mathcal{B},m}(\overline{\mathcal{A}})$ as before the explicit description (4) of $\mathscr{A}_{{\rm crys},\mathcal{B},m}(\overline{\mathcal{A}})$ by using s_i and a compatible system of p^nth roots $s_{i,n} \in \overline{\mathcal{A}}^\times$ ($n \in \mathbb{N}$) of the image of s_i in $\mathcal{A}^\times$ for $i \in \mathbb{N} \cap [1, e]$. By the choice of $\varphi_{\mathcal{B}}$, we have $\varphi(u_i) = (u_i + 1)^p - 1$ for $i \in \mathbb{N} \cap [1, e]$.

We define the decreasing filtration $I^r\mathscr{A}_{{\rm crys},\mathcal{B},m}(\overline{\mathcal{A}})$ ($r \in \mathbb{Z}$) of $\mathscr{A}_{{\rm crys},\mathcal{B},m}(\overline{\mathcal{A}})$ by

$$I^r\mathscr{A}_{{\rm crys},\mathcal{B},m}(\overline{\mathcal{A}}) := \bigoplus_{\underline{n}=(n_i)\in\mathbb{N}^e} I^{r-|\underline{n}|}A_{{\rm crys},m}(\overline{\mathcal{A}}) \prod_i u_{i,m}^{[n_i]}, \quad (128)$$

where $|\underline{n}| = \sum_i n_i$, and $I^sA_{{\rm crys},m}(\overline{\mathcal{A}})$ ($s \in \mathbb{Z}$) denotes the image of $I^sA_{\rm crys}(\overline{\mathcal{A}})$ in $A_{{\rm crys},m}(\overline{\mathcal{A}})$. Note that this definition depends on the choice of s_i and $s_{i,n}$.

Lemma 126 (1) *The filtration $I^r\mathscr{A}_{{\rm crys},\mathcal{B},m}(\overline{\mathcal{A}})$ ($r \in \mathbb{Z}$) depends only on s_i.*

(2) *$I^r\mathscr{A}_{{\rm crys},\mathcal{B},m}(\overline{\mathcal{A}})$ ($r \in \mathbb{Z}$) are ideals of $\mathscr{A}_{{\rm crys},\mathcal{B},m}(\overline{\mathcal{A}})$. For $r, r' \in \mathbb{Z}$, we have $I^r\mathscr{A}_{{\rm crys},\mathcal{B},m}(\overline{\mathcal{A}}) \cdot I^{r'}\mathscr{A}_{{\rm crys},\mathcal{B},m}(\overline{\mathcal{A}}) \subset I^{r+r'}\mathscr{A}_{{\rm crys},\mathcal{B},m}(\overline{\mathcal{A}})$.*

(3) *The filtration $I^r\mathscr{A}_{{\rm crys},\mathcal{B},m}(\overline{\mathcal{A}})$ ($r \in \mathbb{Z}$) is stable under the action of $G_{\mathcal{A}}$ and φ. We have also $\nabla(I^r\mathscr{A}_{{\rm crys},\mathcal{B},m}(\overline{\mathcal{A}})) \subset I^{r-1}\mathscr{A}_{{\rm crys},\mathcal{B},m}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}$ for $r \in \mathbb{Z}$.*

Proof For $i \in \mathbb{N} \cap [1, e]$, if we choose another system $s'_{i,n} \in \overline{\mathcal{A}}^\times$ ($n \in \mathbb{N}$), then the corresponding element $u'_{i,m}$ of $\mathscr{A}_{{\rm crys},\mathcal{B},m}(\overline{\mathcal{A}})$ is $[\underline{\varepsilon}^a](u_{i,m} + 1) - 1$, where $a \in \mathbb{Z}_p$ is defined by $s'_{i,n} = s_{i,n}\varepsilon_n^a$ ($n \in \mathbb{N}$). This implies the claim (1) because $([\underline{\varepsilon}^a] - 1)^{[n]} \in I^nA_{\rm crys}(O_{\overline{K}})$ ($n \in \mathbb{N}$). The claim (2) is obvious, and the claim (3) follows from the formulae (6)–(8). Note that $\varphi_{\mathcal{B}}(s_i) = s_i^p$. □

Set $\Omega_{\mathcal{B}}^q = \wedge_{\mathcal{B}}^q \Omega_{\mathcal{B}}$ ($q \in \mathbb{N}$). We define the decreasing filtration I^r ($r \in \mathbb{Z}$) on $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}^q$ to be $I^{r-q}\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}^q$. Then by Lemma 126 (3), we obtain the following complex of $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$-modules filtered by I^r ($r \in \mathbb{Z}$) and endowed with an action of $G_{\mathcal{A}}$ and an endomorphism φ.

$$A_{\mathrm{crys},m}(\overline{\mathcal{A}}) \longrightarrow \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}^{\bullet} \tag{129}$$

The following lemma is the key to another construction of the comparison map.

Lemma 127 *There is an $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$-linear homotopy compatible with the filtration $I^{\bullet}$ and with m between the identity map and the zero map of the complex (129).*

Proof Set $\omega_i := \nabla(u_{i,m}) = -(1+u_{i,m}) \otimes d\log s_i$. Then one can define the desired $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$-linear filtered homotopy $k^0\colon \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \to A_{\mathrm{crys},m}(\overline{\mathcal{A}})$ and $k^q\colon \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}^q \to \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}^{q-1}$ ($q \in \mathbb{N}_{>0}$) by the following formulae for $\underline{n} = (n_i) \in \mathbb{N}^e$ and $1 \leq i_1 < \cdots < i_q \leq e$.

$$\begin{aligned} &k^0\Big(\prod_i u_i^{[n_i]}\Big) = 1 \ (\text{if } \underline{n} = \underline{0}), \quad 0 \ (\text{otherwise}), \\ &k^q\Big(\prod_i u_i^{[n_i]}\omega_{i_1} \wedge \cdots \wedge \omega_{i_q}\Big) \\ =&\prod_i u_i^{[n_i+\delta_{ii_1}]}\omega_{i_2} \wedge \cdots \wedge \omega_{i_q} \ (\text{if } n_i = 0 \ (1 \leq i < i_1)), \quad 0 \ (\text{otherwise}). \end{aligned}$$

□

Let M be an object of $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A}, \Phi)$ (Sect. 4). We define $M_{\mathcal{P}_m}$ with Fil^r, ∇, and φ as in Sect. 5. We define the filtration I^r ($r \in \mathbb{Z}$) on $TA_{\mathrm{crys},m}(M)$ (23) by $I^r A_{\mathrm{crys}}(\overline{\mathcal{A}}) \cdot TA_{\mathrm{crys},m}(M)$. Similarly we define the filtration I^r ($r \in \mathbb{Z}$) on the de Rham complex $M_{\mathcal{P}_m} \otimes_{\mathcal{P}_m} \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}^{\bullet}$ by $M_{\mathcal{P}_m} \otimes_{\mathcal{P}_m} I^{r-q}\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}^q$. The filtration thus defined on each degree is compatible with the differential maps by Lemma 126 (3). By taking the tensor product of $TA_{\mathrm{crys},m}(M)$ and (129) over $A_{\mathrm{crys},m}(\overline{\mathcal{A}})$ and using Lemma 127 and (30), we obtain a filtered resolution

$$TA_{\mathrm{crys},m}(M) \to M_{\mathcal{P}_m} \otimes_{\mathcal{B}_m} \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}^{\bullet} \tag{130}$$

compatible with the action of $G_{\mathcal{A}}$ and φ. By taking gr_I^q ($q \in \mathbb{N}$) of (130), we obtain a resolution

$$\mathrm{gr}_I^q(TA_{\mathrm{crys},m}(M)) \to \mathrm{gr}_I^q(M_{\mathcal{P}_m} \otimes_{\mathcal{P}_m} \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}^{\bullet}) \tag{131}$$

compatible with the action of $G_{\mathcal{A}}$ and φ.

Set $\mathcal{P}_{A_{\inf},m} := \mathcal{P}_m \otimes_{O_K} A_{\inf}(O_{\overline{K}})/\pi$ and $\mathcal{P}_{A_{\inf}} := \varprojlim \mathcal{P}_{A_{\inf},m}$. Let $\mathcal{P}_{A_{\inf},\star}$-Mod denote the category of inverse systems of $\mathcal{P}_{A_{\inf},m}$-modules for $m \in \mathbb{N}_{>0}$. We regard $(A_{\inf}(\overline{\mathcal{A}})/\pi)/p^m$ as an $\mathcal{P}_{A_{\inf},m}$-algebra by the $G_{\mathcal{A}}$-equivariant homomorphism $\mathcal{P}_{A_{\inf},m} \to \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})/I^1 \cong A_{\inf}(\overline{\mathcal{A}})/(p^m, \pi)$, which depends on the choice of coordinates s_i and coincides with the morphism naturally induced by $\beta_m\colon \mathcal{P}_m \to A_{\mathrm{crys},m}(\overline{\mathcal{A}})$ (Lemma 34 (1)). Then we have a G_K-equivariant morphism

$$\bigoplus_{0\le r\le q}(M_{\mathcal{P}_\star}\otimes_{O_K}\mathrm{gr}_I^{q-r}A_{\mathrm{crys}}(O_{\overline{K}}))\otimes_{\mathcal{B}}\Omega^r_{\mathcal{B}}[-r] \longrightarrow \Gamma(\Delta_{\mathcal{A}},\mathrm{gr}_I^q(M_{\mathcal{P}_\star}\otimes_{\mathcal{P}_\star}\mathscr{A}_{\mathrm{crys},\mathcal{B},\star}(\overline{\mathcal{A}})\otimes_{\mathcal{B}}\Omega^\bullet_{\mathcal{B}})) \quad (132)$$

of complexes of inverse systems of $\mathcal{P}_{A_{\mathrm{inf}},m}$-modules compatible with φ. Using (131) and (132), we obtain a morphism

$$\bigoplus_{0\le r\le q}\varprojlim_m(M_{\mathcal{P}_m}\otimes_{O_K}\mathrm{gr}_I^{q-r}A_{\mathrm{crys}}(O_{\overline{K}}))\otimes_{\mathcal{B}}\Omega^r_{\mathcal{B}}[-r]\to R\Gamma(\Delta_{\mathcal{A}},\mathrm{gr}_I^q TA_{\mathrm{crys}}(M)) \quad (133)$$

in $D^+(\underline{G_K}\text{-}\mathbf{Sets},\mathcal{P}_{A_{\mathrm{inf}}})$ compatible with φ. In particular, we obtain a $\mathcal{P}_{A_{\mathrm{inf}}}$-linear G_K-equivariant homomorphism compatible with φ

$$M_{\mathcal{P},A_{\mathrm{inf}}}\otimes_{\mathcal{B}}\Omega^q_{\mathcal{B}}\to H^q(\Delta_{\mathcal{A}},\mathrm{gr}_I^q TA_{\mathrm{crys}}(M)), \quad (134)$$

where

$$M_{\mathcal{P},A_{\mathrm{inf}}}=\varprojlim_m(M_{\mathcal{P}_m}\otimes_{O_K}A_{\mathrm{inf}}(O_{\overline{K}})/\pi). \quad (135)$$

Lemma 128 (1) *If M is the constant object $\mathcal{A}$, i.e. $\mathcal{A}$ equipped with the filtration* $\mathrm{Fil}^0\mathcal{A}=\mathcal{A}$, $\mathrm{Fil}^1\mathcal{A}=0$ *and the given φ and ∇, then the morphism (134) is compatible with the natural product structures.*

(2) *The following diagram is commutative for $q, q'\in\mathbb{N}$, where T denotes $TA_{\mathrm{crys}}(M)$.*

$$\begin{array}{ccc}
(\mathcal{P}_{A_{\mathrm{inf}}}\otimes_{\mathcal{B}}\Omega^q_{\mathcal{B}})\otimes_{\mathcal{P}_{A_{\mathrm{inf}}}}(M_{\mathcal{P},A_{\mathrm{inf}}}\otimes_{\mathcal{B}}\Omega^{q'}_{\mathcal{B}}) & \xrightarrow{\wedge} & M_{\mathcal{P},A_{\mathrm{inf}}}\otimes_{\mathcal{B}}\Omega^{q+q'}_{\mathcal{B}}\\
\downarrow{\scriptstyle (134)} & & \downarrow{\scriptstyle (134)}\\
H^q(\Delta_{\mathcal{A}},\mathrm{gr}_I^q A_{\mathrm{crys}}(\overline{\mathcal{A}}))\otimes_{\mathcal{P}_{A_{\mathrm{inf}}}}H^{q'}(\Delta_{\mathcal{A}},\mathrm{gr}_I^{q'}T) & \xrightarrow{\cup} & H^{q+q'}(\Delta_{\mathcal{A}},\mathrm{gr}_I^{q+q'}T).
\end{array}$$

Proof The claim (1) is the special case $M=\mathcal{A}$ of the claim (2), which follows from the following commutative diagram. Here we write $\mathrm{DR}_{\mathscr{A}_{\mathrm{crys}},m}(\mathcal{B},M)$ (resp. $\mathrm{DR}_{A_{\mathrm{inf}},m}(\mathcal{B},M)$) for the de Rham complex $M_{\mathcal{P}_m}\otimes_{\mathcal{P}_m}\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})\otimes_{\mathcal{B}}\Omega^\bullet_{\mathcal{B}}$ (resp. $(M_{\mathcal{P}_m}\otimes_{O_K}A_{\mathrm{inf}}(O_{\overline{K}}))\otimes_{\mathcal{B}}\Omega^\bullet_{\mathcal{B}}$). The middle and lower horizontal maps are defined by the wedge products.

$$\begin{array}{ccc}
\mathrm{gr}_I^q A_{\mathrm{crys},m}(\overline{\mathcal{A}}) \otimes_{\mathcal{P}_{A_{\inf},m}} \mathrm{gr}_I^{q'} TA_{\mathrm{crys},m}(M) & \longrightarrow & \mathrm{gr}_I^{q+q'} TA_{\mathrm{crys},m}(M) \\
\downarrow (131) & & \downarrow (131) \\
\mathrm{gr}_I^q(\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}}) \otimes_{\mathcal{P}_{A_{\inf},m}} \mathrm{gr}_I^{q'} \mathrm{DR}_{\mathscr{A}_{\mathrm{crys}},m}(\mathcal{B}, M) & \longrightarrow & \mathrm{gr}_I^{q+q'} \mathrm{DR}_{\mathscr{A}_{\mathrm{crys}},m}(\mathcal{B}, M) \\
\uparrow (132) & & \uparrow (132) \\
\mathrm{gr}_I^q((\mathcal{P}_m \otimes_{O_K} A_{\inf}(O_{\overline{K}})) \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}}) \otimes_{\mathcal{P}_{A_{\inf},m}} \mathrm{gr}_I^{q'} \mathrm{DR}_{A_{\inf},m}(\mathcal{B}, M) & \longrightarrow & \mathrm{gr}_I^{q+q'} \mathrm{DR}_{A_{\inf},m}(\mathcal{B}, M)
\end{array}$$

□

For $q \in \mathbb{N}$, let $\mathrm{Bock}_I^q \colon H^q(\Delta_{\mathcal{A}}, \mathrm{gr}_I^q TA_{\mathrm{crys}}(M)) \to H^{q+1}(\Delta_{\mathcal{A}}, \mathrm{gr}_I^{q+1} TA_{\mathrm{crys}}(M))$ be the boundary map associated to the exact sequence $0 \to \mathrm{gr}_I^{q+1} TA_{\mathrm{crys}}(M) \to I^q TA_{\mathrm{crys}}(M)/I^{q+2} TA_{\mathrm{crys}}(M) \to \mathrm{gr}_I^q TA_{\mathrm{crys}}(M) \to 0$.

Lemma 129 *For $q \in \mathbb{N}$, the following diagram is commutative.*

$$\begin{array}{ccc}
M_{\mathcal{P},A_{\inf}} \otimes_{\mathcal{B}} \Omega^q_{\mathcal{B}} & \xrightarrow{(134)} & H^q(\Delta_{\mathcal{A}}, \mathrm{gr}_I^q(TA_{\mathrm{crys}}(M))) \\
\nabla^q \downarrow & & \downarrow \mathrm{Bock}_I^q \\
M_{\mathcal{P},A_{\inf}} \otimes_{\mathcal{B}} \Omega^{q+1}_{\mathcal{B}} & \xrightarrow{(134)} & H^{q+1}(\Delta_{\mathcal{A}}, \mathrm{gr}_I^{q+1}(TA_{\mathrm{crys}}(M))).
\end{array}$$

Proof To simplify the notation, we write T_m, $\mathrm{DR}_{\mathscr{A}_{\mathrm{crys}},m}$, and $\mathrm{DR}_{A_{\inf},m}$ for $TA_{\mathrm{crys},m}(M)$, $M_{\mathcal{P}_m} \otimes_{\mathcal{P}_m} \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}}$, and $(M_{\mathcal{P}_m} \otimes_{O_K} A_{\inf}(O_{\overline{K}})) \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}}$, respectively. Then we have the following commutative diagram whose three horizontal lines are exact.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{gr}_I^{q+1} T_m & \longrightarrow & \frac{I^q T_m}{I^{q+2} T_m} & \longrightarrow & \mathrm{gr}_I^q T_m & \longrightarrow & 0 \\
 & & \downarrow (131) & & \downarrow (130) & & \downarrow (131) & & \\
0 & \longrightarrow & \mathrm{gr}_I^{q+1}(\mathrm{DR}_{\mathscr{A}_{\mathrm{crys}},m}) & \longrightarrow & \frac{I^q \mathrm{DR}_{\mathscr{A}_{\mathrm{crys}},m}}{I^{q+2} \mathrm{DR}_{\mathscr{A}_{\mathrm{crys}},m}} & \longrightarrow & \mathrm{gr}_I^q(\mathrm{DR}_{\mathscr{A}_{\mathrm{crys}},m}) & \longrightarrow & 0 \\
 & & \uparrow (132) & & \uparrow & & \uparrow (132) & & \\
0 & \longrightarrow & \mathrm{gr}_I^{q+1}(\mathrm{DR}_{A_{\inf},m}) & \longrightarrow & \frac{I^q \mathrm{DR}_{A_{\inf},m}}{I^{q+2} \mathrm{DR}_{A_{\inf},m}} & \longrightarrow & \mathrm{gr}_I^q(\mathrm{DR}_{A_{\inf},m}) & \longrightarrow & 0.
\end{array}$$

Set $T = TA_{\mathrm{crys}}(M)$ and $\mathrm{DR}_{A_{\inf}} := \varprojlim_m \mathrm{DR}_{A_{\inf},m}$. Then the above diagram induces a morphism of distinguished triangles

$$\begin{array}{ccccccc}
R\Gamma(\Delta_{\mathcal{A}}, \mathrm{gr}_I^{q+1} T) & \longrightarrow & R\Gamma(\Delta_{\mathcal{A}}, \frac{I^q T}{I^{q+2} T}) & \longrightarrow & R\Gamma(\Delta_{\mathcal{A}}, \mathrm{gr}_I^q T) & \longrightarrow & \\
\uparrow (133) & & \uparrow & & \uparrow (133) & & \\
\mathrm{gr}_I^{q+1}(\mathrm{DR}_{A_{\inf}}) & \longrightarrow & \frac{I^q(\mathrm{DR}_{A_{\inf}})}{I^{q+2}(\mathrm{DR}_{A_{\inf}})} & \longrightarrow & \mathrm{gr}_I^q(\mathrm{DR}_{A_{\inf}}) & \longrightarrow &
\end{array}$$

in $D^+(G_K\text{-}\mathbf{Sets}, \varprojlim_m \mathcal{P}_m \otimes_{O_K} A_{\rm inf}(O_{\overline{K}})/\pi^2))$. The differential maps of the complex $\mathrm{gr}^r_I(\mathrm{DR}_{A_{\rm inf}})$ are 0, and it is straightforward to verify that the boundary map of the bottom distinguished triangle

$$M_{\mathcal{P},A_{\rm inf}} \otimes_{\mathcal{B}} \Omega^q_{\mathcal{B}} = \mathrm{gr}^q_I(\mathrm{DR}_{A_{\rm inf}})^q \longrightarrow \mathrm{gr}^{q+1}_I(\mathrm{DR}_{A_{\rm inf}})^{q+1} = M_{\mathcal{P},A_{\rm inf}} \otimes_{\mathcal{B}} \Omega^{q+1}_{\mathcal{B}}$$

is given by the differential map ∇ induced by the connection on $M_{\mathcal{P}_m}$. □

By (2) with $(r, s) = (0, p-1)$, and the definition of $TA_{\rm inf}(M)$ before Theorem 63, we have a $G_{\mathcal{A}}$-equivariant isomorphism compatible with the filtrations I^r $(r \in \mathbb{Z})$ and φ

$$TA_{\rm inf}(M)/I^{p-1}TA_{\rm inf}(M) \xrightarrow{\cong} TA_{\rm crys}(M)/I^{p-1}TA_{\rm crys}(M),$$

which induces isomorphisms

$$H^q(\Delta_{\mathcal{A}}, \mathrm{gr}^q_I TA_{\rm inf}(M)) \xrightarrow{\cong} H^q(\Delta_{\mathcal{A}}, \mathrm{gr}^q_I TA_{\rm crys}(M)) \quad (q \in \mathbb{N} \cap [0, p-2]) \tag{136}$$

compatible with Bock^q_I $(q \in \mathbb{N} \cap [0, p-3])$ and φ. Hence (134) induces G_K-equivariant $\mathcal{P}_{A_{\rm inf}}$-linear homomorphisms compatible with φ

$$M_{\mathcal{P},A_{\rm inf}} \otimes_{\mathcal{B}} \Omega^q_{\mathcal{B}} \longrightarrow H^q(\Delta_{\mathcal{A}}, \mathrm{gr}^q_I TA_{\rm inf}(M)) \quad (q \in \mathbb{N} \cap [0, p-2]), \tag{137}$$

which are also compatible with ∇ and Bock^q_I by Lemma 129. We extend this to all degrees when $p \geq 5$.

Remark 130 Let $r \in \mathbb{Z}$, and let $q(r)$ be the largest integer such that $r(p-1)^{-1} \geq q(r)$. Then $\mathrm{gr}^r_I A_{\rm inf}(\overline{\mathcal{A}})$ (resp. $\mathrm{gr}^r_I A_{\rm crys}(\overline{\mathcal{A}})$) is a free $A_{\rm inf}(\overline{\mathcal{A}})/\pi$-module with a basis π^r (resp. $\frac{1}{q(r)!p^{q(r)}}\pi^r$) (see [12, 5.3.1 Proposition], [18, Proposition A3.20], and Lemma 90 (2)). Hence the construction of $TA_{\rm inf}(M)$ implies that we have the following $G_{\mathcal{A}}$-equivariant canonical isomorphism compatible with the product structures and φ.

$$\frac{1}{q(r)!p^{q(r)}}\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathrm{gr}^r_I TA_{\rm inf}(M) \xrightarrow{\cong} \mathrm{gr}^r_I TA_{\rm crys}(M)$$

We assume that $p \geq 5$ in the following. For $q \in \mathbb{N}$, the homomorphism $\mathrm{gr}^1_I A_{\rm inf}(\overline{\mathcal{A}})^{\otimes_{\mathcal{P}_{A_{\rm inf}}} q} \to \mathrm{gr}^q_I A_{\rm inf}(\overline{\mathcal{A}})$ induces a G_K-equivariant $\mathcal{P}_{A_{\rm inf}}$-linear homomorphism compatible with φ

$$\wedge^q_{\mathcal{P}_{A_{\rm inf}}} H^1(\Delta_{\mathcal{A}}, \mathrm{gr}^1_I A_{\rm inf}(\overline{\mathcal{A}})) \longrightarrow H^q(\Delta_{\mathcal{A}}, \mathrm{gr}^q_I A_{\rm inf}(\overline{\mathcal{A}})).$$

Composing this with the qth exterior product of (137) for $M = \mathcal{A}$ and $q = 1 (\leq p-2)$, we obtain a G_K-equivariant $\mathcal{P}_{A_{\rm inf}}$-linear homomorphism compatible with φ

$$\mathcal{P}_{A_{\rm inf}} \otimes_{\mathcal{B}} \Omega^q_{\mathcal{B}} \longrightarrow H^q(\Delta_{\mathcal{A}}, \mathrm{gr}^q_I A_{\rm inf}(\overline{\mathcal{A}})). \tag{138}$$

By taking the cup product of (138) with (137) for $q = 0$, we obtain a G_K-equivariant $\mathcal{P}_{A_{\text{inf}}}$-linear homomorphism compatible with φ

$$M_{\mathcal{P},A_{\text{inf}}} \otimes_{\mathcal{B}} \Omega^q_{\mathcal{B}} \longrightarrow H^q(\Delta_{\mathcal{A}}, \mathrm{gr}^q_I TA_{\text{inf}}(M)), \tag{139}$$

which coincides with (137) if $q \leq p-2$ by Lemma 128.

Proposition 131 *For all $q \in \mathbb{N}$, the homomorphisms (139) are compatible with ∇^q and Bock^q_I.*

Proof Since $p-2 \geq 2$, Lemma 129 implies that the homomorphisms (138) and (139) are compatible with ∇^q and Bock^q_I for $q \in \{0, 1\}$. By the construction of (138) and (139), we see that the following diagram is commutative for $q, q' \in \mathbb{N}$.

$$\begin{array}{ccc} (\mathcal{P}_{A_{\text{inf}}} \otimes_{\mathcal{B}} \Omega^{q'}_{\mathcal{B}}) \otimes_{\mathcal{P}_{A_{\text{inf}}}} (M_{\mathcal{P},A_{\text{inf}}} \otimes_{\mathcal{B}} \Omega^q_{\mathcal{B}}) & \xrightarrow{\wedge} & M_{\mathcal{P},A_{\text{inf}}} \otimes_{\mathcal{B}} \Omega^{q'+q}_{\mathcal{B}} \\ \downarrow{\scriptstyle (138)\otimes(139)} & & \downarrow{\scriptstyle (139)} \\ H^{q'}(\Delta_{\mathcal{A}}, \mathrm{gr}^{q'}_I A_{\text{inf}}(\overline{\mathcal{A}})) \otimes_{\mathcal{P}_{A_{\text{inf}}}} H^q(\Delta_{\mathcal{A}}, \mathrm{gr}^q_I TA_{\text{inf}}(M)) & \xrightarrow{\cup} & H^{q'+q}(\Delta_{\mathcal{A}}, \mathrm{gr}^{q'+q}_I TA_{\text{inf}}(M)) \end{array}$$

One can prove the proposition for a general q by induction on q by using the above commutative diagram for $q' = 1, 2$ and Lemma 132 below applied to $R = A_{\text{inf}}(O_{\overline{K}})$ and

$$C^\bullet_{\text{cont}}(\Delta_{\mathcal{A}}, A_{\text{inf}}(\overline{\mathcal{A}})) \otimes_{A_{\text{inf}}(O_{\overline{K}})} C^\bullet_{\text{cont}}(\Delta_{\mathcal{A}}, TA_{\text{inf}}(M)) \xrightarrow{\cup} C^\bullet_{\text{cont}}(\Delta_{\mathcal{A}}, TA_{\text{inf}}(M)).$$

□

Lemma 132 *Let R be a commutative ring, let a be a regular element of R, and let $\overline{R}$ denote R/aR. Let C_i $(i \in \{1, 2, 3\})$ be complexes of a-torsion free R-modules, and define the decreasing filtration F^q of the complex $C_i[\frac{1}{a}]$ by $F^q C_i = a^q C_i$ for each $i \in \{1, 2, 3\}$. Let Bock^q_i $(q \in \mathbb{Z},\ i \in \{1, 2, 3\})$ be the boundary map $H^q(\mathrm{gr}^q_F C_i) \to H^{q+1}(\mathrm{gr}^{q+1}_F C_i)$ induced by the short exact sequence $0 \to \mathrm{gr}^{q+1}_F C_i \to F^q C_i / F^{q+2} C_i \to \mathrm{gr}^q_F C_i \to 0$. Suppose that we are given a morphism of complexes of R-modules $C_1 \otimes_R C_2 \to C_3$, which induces a morphism of complexes $\mathrm{gr}^q_F C_1 \otimes_{\overline{R}} \mathrm{gr}^{q'}_F C_2 \to \mathrm{gr}^{q+q'}_F C_3$ and an $\overline{R}$-linear map $- \cup -\colon H^q(\mathrm{gr}^q_F C_1) \otimes_{\overline{R}} H^{q'}(\mathrm{gr}^{q'}_F C_2) \to H^{q+q'}(\mathrm{gr}^{q+q'}_F C_3)$ for $q, q' \in \mathbb{Z}$. Then we have*

$$\mathrm{Bock}^{q+q'}_3(x \cup y) = \mathrm{Bock}^q_1(x) \cup y + (-1)^q x \cup \mathrm{Bock}^{q'}_2(y)$$

for $q, q' \in \mathbb{Z}$, $x \in H^q(\mathrm{gr}^q_F C_1)$ and $y \in H^{q'}(\mathrm{gr}^{q'}_F C_2)$.

Proof Straightforward computation. □

By Proposition 131, (139) and (127) induce a morphism

$$M_{\mathcal{P},A_{\mathrm{inf}}} \otimes_{\mathcal{B}} \Omega^{\bullet}_{\mathcal{B}} \longrightarrow L\eta^{+}_{\pi} R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M)) \otimes^{L}_{A_{\mathrm{inf}}(O_{\overline{K}})} A_{\mathrm{inf}}(O_{\overline{K}})/\pi \tag{140}$$

in $D^{+}(\underline{G_K}\text{-}\mathbf{Sets}, A_{\mathrm{inf}}(O_{\overline{K}})/\pi)$ compatible with φ.

Next we show that (140) coincides with (125) in the derived category of $A_{\mathrm{inf}}(O_{\overline{K}})/\pi$-modules when $\mathcal{B} = \mathcal{A}$ and $t_1, \dots, t_d \in \mathcal{A}^{\times}$ are the coordinates defined by a framing $\square$. Since $A^{\square}_{\mathrm{inf}}(\mathcal{A})/\pi$ is p-adically complete and separated and $(A^{\square}_{\mathrm{inf}}(\mathcal{A})/\pi)/p^m \cong A^{\square}_{\mathrm{inf},(p^m,\pi)}(\mathcal{A})$ by Lemmas 80 (3) and 79 (2), we obtain the following isomorphism from the reduction mod $I^1 A^{\square}_{\mathrm{crys}}(\mathcal{A})$ of (92) by using Lemma 82 with $(r, s) = (0, 1)$ and Lemma 87 (2).

$$TA^{\square}_{\mathrm{inf}}(M)/\pi \cong \varprojlim_{m} (M/p^m M \otimes_{O_K} A_{\mathrm{inf}}(O_{\overline{K}})/\pi) = M_{\mathcal{A},A_{\mathrm{inf}}} \tag{141}$$

We write $M_{A_{\mathrm{inf}}}$ for $M_{\mathcal{A},A_{\mathrm{inf}}}$ to simplify the notation. We define $\gamma_i \in \Gamma_{\mathcal{A}}$ as before Lemma 115. Then, by Lemma 98 and (141), we have isomorphisms

$$\begin{aligned} H^q(\Gamma_{\mathcal{A}}, TA^{\square}_{\mathrm{inf}}(M)/\pi) &\cong H^q(K(TA^{\square}_{\mathrm{inf}}(M)/\pi; \gamma_1 - 1 \dots, \gamma_d - 1)) \\ &\cong TA^{\square}_{\mathrm{inf}}(M)/\pi \otimes_{\mathbb{Z}} \wedge^q \mathbb{Z}^d \cong TA^{\square}_{\mathrm{inf}}(M)/\pi \otimes_{\mathcal{A}} \Omega^q_{\mathcal{A}} \cong M_{A_{\mathrm{inf}}} \otimes_{\mathcal{A}} \Omega^q_{\mathcal{A}} \end{aligned} \tag{142}$$

for $q \in \mathbb{N}$; the third isomorphism is defined by $x \otimes e_{i_1} \wedge \cdots \wedge e_{i_q} \mapsto x \otimes d\log t_{i_1} \wedge \cdots \wedge d\log t_{i_q}$ for $x \in TA^{\square}_{\mathrm{inf}}(M)/\pi$ and $1 \le i_1 < \cdots < i_q \le d$, where e_i $(1 \le i \le d)$ denotes the standard basis of $\mathbb{Z}^d$.

Proposition 133 *The following diagram is commutative for $q \in \mathbb{N}$.*

$$\begin{array}{ccc} M_{A_{\mathrm{inf}}} \otimes_{\mathcal{A}} \Omega^q_{\mathcal{A}} & \xrightarrow[\pi^q\cdot(142)]{\cong} & H^q(\Gamma_{\mathcal{A}}, \mathrm{gr}^q_I TA^{\square}_{\mathrm{inf}}(M)) \\ \downarrow{\scriptstyle (139)} & \swarrow & \\ H^q(\Delta_{\mathcal{A}}, \mathrm{gr}^q_I TA_{\mathrm{inf}}(M)) & & \end{array}$$

Proof If M is the constant object $\mathcal{A}$, then the morphisms (142) are compatible with the natural product structures. We also see that the following diagram is commutative.

$$\begin{array}{ccc} (\mathcal{A}_{A_{\mathrm{inf}}} \otimes_{\mathcal{A}} \Omega^q_{\mathcal{A}}) \otimes M_{A_{\mathrm{inf}}} & \xrightarrow{\wedge} & M_{A_{\mathrm{inf}}} \otimes_{\mathcal{A}} \Omega^q_{\mathcal{A}} \\ \downarrow{\scriptstyle (142)} & & \downarrow{\scriptstyle (142)} \\ H^q(\Gamma_{\mathcal{A}}, A^{\square}_{\mathrm{inf}}(\mathcal{A})/\pi) \otimes H^0(\Gamma_{\mathcal{A}}, TA^{\square}_{\mathrm{inf}}(M)/\pi) & \xrightarrow{\cup} & H^q(\Gamma_{\mathcal{A}}, TA^{\square}_{\mathrm{inf}}(M)/\pi) \end{array}$$

By the construction of the morphism (139), it suffices to prove the claim in the case $q = 0$ and in the case $q = 1$ and $M = \mathcal{A}$.

The morphism (139) for $q = 0$ is defined by the composition of

$$M_{A_{\mathrm{inf}}} \longrightarrow M \otimes_{\mathcal{A},\overline{\alpha}} A_{\mathrm{inf}}(\overline{\mathcal{A}})/\pi \cong \mathrm{gr}^0_I(M \otimes_{\mathcal{A}} \mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}}) \otimes_{\mathcal{A}} \Omega^{\bullet}_{\mathcal{A}}) \xrightarrow[(131)^{-1}]{\cong} \mathrm{gr}^0_I T A_{\mathrm{inf}}(M)$$

See (57) and Lemma 34 (1) for $\overline{\alpha}$. The composition of the middle and right isomorphisms coincides with the isomorphism considered in Lemma 64 (2) by its proof. Therefore the composition of the three homomorphisms above is the same as the composition of (141) with $TA^{\square}_{\mathrm{inf}}(M)/\pi = \mathrm{gr}^0_I TA^{\square}_{\mathrm{inf}}(M) \to \mathrm{gr}^0_I TA_{\mathrm{inf}}(M)$. Note that (37) for $(\mathcal{B}, s_i) = (\mathcal{A}, t_i)$ is the scalar extension of (92) by the homomorphism $\iota^{\square}\colon A^{\square}_{\mathrm{crys}}(\mathcal{A}) \to A_{\mathrm{crys}}(\overline{\mathcal{A}})$ (78) because the composition of $\alpha\colon \mathcal{A} \to A^{\square}_{\mathrm{inf}}(\mathcal{A})$ (80) and $\iota^{\square}\colon A^{\square}_{\mathrm{inf}}(\mathcal{A}) \to A_{\mathrm{inf}}(\overline{\mathcal{A}})$ coincides with $\beta^{(0)}$ for $(\mathcal{B}, s_i) = (\mathcal{A}, t_i)$ defined before Lemma 34. (For the last fact, we compare (31) for $(\mathcal{B}, s_i) = (\mathcal{A}, t_i)$ and (79) using the fact that the homomorphism $\iota^{\square}$ sends $[\underline{t}_i]$ to $[\underline{t}_i]$ ($i \in \mathbb{N} \cap [1, d]$) by definition.) Thus we obtain the claim for $q = 0$.

Suppose that $q = 1$ and $M = \mathcal{A}$. Since the homomorphisms in the diagram is $\mathcal{A}_{A_{\mathrm{inf}}}$-linear, it suffices to compute the images of $1 \otimes d\log t_i$. The element $1 \otimes d\log t_i$ of $\mathrm{gr}^1_I(\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}}) \otimes_{\mathcal{A}} \Omega^1_{\mathcal{A}}) \cong (\mathrm{gr}^0_I A_{\mathrm{crys}}(\overline{\mathcal{A}})) \otimes_{\mathcal{A}} \Omega^1_{\mathcal{A}}$ is the image of the element $-1 \otimes v_i$ of $\mathrm{gr}^1_I \mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}}) = \mathrm{gr}^1_I A_{\mathrm{crys}}(\overline{\mathcal{A}}) \oplus (\oplus_{1 \le i \le d} \mathrm{gr}^0_I A_{\mathrm{crys}}(\overline{\mathcal{A}}) v_i)$ under the differential map of the target complex of $\varprojlim_m$ (131). For $\gamma \in \Gamma_{\mathcal{A}}$, we have the following equality in $\mathrm{gr}^1_I \mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})$ by (6): $(\gamma - 1)(v_i) = (1 + v_i)[\underline{\varepsilon}^{\eta_i(\gamma)}] - 1 - v_i = [\underline{\varepsilon}^{\eta_i(\gamma)}] - 1 = \eta_i(\gamma)\pi$. See Lemma 9 (1) for the last equality. Hence the image of $1 \otimes d\log t_i$ under (139) is given by the 1-cocycle $\gamma \mapsto \eta_i(\gamma)\pi$. This coincides with the image of $1 \otimes d\log t_i$ under the map via (142) by Lemma 134 below. □

Lemma 134 *Let M be a p-adically complete and separated module endowed with a trivial action of $\Gamma_{\mathcal{A}}$. Let e_i ($i \in \mathbb{N} \cap [1, d]$) be the standard basis of $\mathbb{Z}^d$. Then, for $x \in M$, the image of $x \otimes e_i$ under the isomorphism below is given by the* 1*-cocycle $\gamma \mapsto \eta_i(\gamma)x$.*

$$M \otimes_{\mathbb{Z}} \mathbb{Z}^d \cong H^1(K(M; \gamma_1 - 1, \ldots, \gamma_d - 1)) \cong H^1(\Gamma_{\mathcal{A}}, M)$$

Proof Let us recall the construction of the isomorphism. Let $\mathcal{M}$ be the module $\mathrm{Map}_{\mathrm{cont}}(\Gamma_{\mathcal{A}}, M)$ consisting of continuous maps $\Gamma_{\mathcal{A}} \to M$ endowed with the continuous action of $\Gamma_{\mathcal{A}}$ defined by $(\gamma f)(\delta) = f(\delta\gamma)$. We have $H^q(\Gamma_{\mathcal{A}}, \mathcal{M}) = 0$ for $q \in \mathbb{N}_{>0}$. We have another action of $\Gamma_{\mathcal{A}}$ on $\mathcal{M}$ defined by $[\gamma]f(\delta) = \gamma f(\gamma^{-1}\delta)$, which commutes with the above action. The Koszul complex with respect to $[\gamma_i] - 1$ gives a $\Gamma_{\mathcal{A}}$-equivariant resolution $M \to K(\mathcal{M}; [\gamma_1] - 1, \ldots, [\gamma_d] - 1)$. We have an obvious isomorphism $M \xrightarrow{\cong} \mathcal{M}^{\Gamma_{\mathcal{A}}}$ sending x to the constant function $c_x\colon \gamma \mapsto x$. For $\gamma \in \Gamma_{\mathcal{A}}$, we have $c_{\gamma x} = [\gamma]c_x$. Hence the $\Gamma_{\mathcal{A}}$-invariant part of the above Koszul complex of $\mathcal{M}$ is isomorphic to the complex $K(M; \gamma_1 - 1, \ldots, \gamma_d - 1)$, and this gives the isomorphism in the claim. The element $c_x \otimes e_i$ of $K^1(\mathcal{M}; [\gamma_1] - 1, \ldots, [\gamma_d] - 1) = \mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Z}^d$ is the image of the element f of $K^0(\mathcal{M}; [\gamma_1] - 1, \ldots, [\gamma_d] - 1) = \mathcal{M}$ defined by $f(\gamma_1^{r_1} \cdots \gamma_d^{r_d}) = -r_i x$ for $(r_1, \ldots, r_d) \in \mathbb{Z}_p^d$. Hence the claim follows from $(-(\gamma_j - 1)f)(\gamma_1^{r_1} \cdots \gamma_d^{r_d}) = x$ (if $j = i$), 0 (otherwise). □

Lemma 135 (1) *For $q \in \mathbb{N}$, the following diagram is commutative.*

$$\begin{array}{ccc}
M_{A_{\mathrm{inf}}} \otimes_{\mathcal{A}} \Omega^q_{\mathcal{A}} & \xrightarrow[\pi^q\cdot(142)]{\cong} & H^q(\Gamma_{\mathcal{A}}, \mathrm{gr}^q_I TA^{\square}_{\mathrm{inf}}(M)) \\
\downarrow{\scriptstyle \nabla^q} & & \downarrow{\scriptstyle \mathrm{Bock}^q_I} \\
M_{A_{\mathrm{inf}}} \otimes_{\mathcal{A}} \Omega^{q+1}_{\mathcal{A}} & \xrightarrow[\pi^{q+1}\cdot(142)]{\cong} & H^{q+1}(\Gamma_{\mathcal{A}}, \mathrm{gr}^{q+1}_I TA^{\square}_{\mathrm{inf}}(M))
\end{array}$$

(2) *The composition of the isomorphisms*

$$M_{A_{\mathrm{inf}}} \otimes_{\mathcal{A}} \Omega^{\bullet}_{\mathcal{A}} \xrightarrow[(1)]{\cong} (H^{\bullet}(\Gamma_{\mathcal{A}}, \mathrm{gr}^{\bullet}_I TA^{\square}_{\mathrm{inf}}(M)), \mathrm{Bock}^{\bullet}_I)$$
$$\xleftarrow[\text{Prop.101}]{\cong} (L\eta^+_{\pi} R\Gamma(\Gamma_{\mathcal{A}}, TA^{\square}_{\mathrm{inf}}(M))) \otimes^L_{A_{\mathrm{inf}}(O_{\overline{K}})} A_{\mathrm{inf}}(O_{\overline{K}})/\pi$$

coincides with the isomorphism obtained from (122) and Lemma 117. Here we obtain the second isomorphism in the same way as (127).

Proof (1) Put $K^{\bullet} := K(TA^{\square}_{\mathrm{inf}}(M); \gamma_1 - 1, \dots, \gamma_d - 1)$ and $I^r K^{\bullet} := \pi^r K^{\bullet}$. Then Bock^q_I is induced by the exact sequence $0 \to \mathrm{gr}^{q+1}_I K^{\bullet} \to I^q K^{\bullet}/I^{q+2}K^{\bullet} \to \mathrm{gr}^q_I K^{\bullet} \to 0$. Therefore, for $x \in TA^{\square}_{\mathrm{inf}}(M)$, the image of $(\pi^q x \mod \pi^{q+1}) \otimes e_{i_1} \wedge \cdots \wedge e_{i_q} \in \mathrm{gr}^q_I K^q = H^q(\mathrm{gr}^q_I K^{\bullet})$ under Bock^q_I is given by

$$\sum_{1 \le i \le d} (\pi^{q+1}(\pi^{-1}(\gamma_i - 1)(x)) \mod \pi^{q+2}) \otimes e_i \wedge e_{i_1} \wedge \cdots \wedge e_{i_q}$$
$$\in \mathrm{gr}^{q+1}_I K^{q+1} = H^{q+1}(\mathrm{gr}^{q+1}_I K^{\bullet}).$$

This implies the claim by Propositions 121 and 120 (4).

(2) The claim follows from the commutative diagram below, where we abbreviate $TA^{\square}_{\mathrm{crys}}(M)$, $TA^{\square}_{\mathrm{inf}}(M)$, $K(-; \gamma_1 - 1, \dots, \gamma_d - 1)$, and $\eta^+_{I^1 A_{\mathrm{inf}}(O_{\overline{K}})}$ to T_{crys}, T_{inf}, $K_{\gamma}(-)$, and η^+_I; the lower left triangle is commutative by Lemma 122 (1) and $t - \pi \in I^2 A_{\mathrm{crys}}(O_{\overline{K}})$.

$$\begin{array}{ccccc}
\dfrac{T_{\mathrm{inf}} \otimes_{\mathbb{Z}} \wedge^q \mathbb{Z}^d}{\pi(T_{\mathrm{inf}} \otimes_{\mathbb{Z}} \wedge^q \mathbb{Z}^d)} & \xrightarrow{\cong} & H^q(K_{\gamma}(T_{\mathrm{inf}}/\pi)) & \xrightarrow[\pi^q\cdot-]{\cong} & H^q(K_{\gamma}(\mathrm{gr}^q_I T_{\mathrm{inf}})) \\
\downarrow{\scriptstyle \cong} & & \searrow{\scriptstyle \pi^q\cdot -} & & \uparrow{\scriptstyle \text{Prop. 101 (1)}} \\
\dfrac{T_{\mathrm{crys}} \otimes_{\mathbb{Z}} \wedge^q \mathbb{Z}^d}{I^1 A^{\square}_{\mathrm{crys}}(\mathcal{A}) \cdot (T_{\mathrm{crys}} \otimes_{\mathbb{Z}} \wedge^q \mathbb{Z}^d)} & \xrightarrow[t^q F^q]{\cong} & \dfrac{(\eta^+_I K_{\gamma}(T_{\mathrm{crys}}))^q}{I^1 A^{\square}_{\mathrm{crys}}(\mathcal{A}) \cdot (\eta^+_I K_{\gamma}(T_{\mathrm{crys}}))^q} & \xleftarrow[\cong]{} & \dfrac{(\eta^+_I K_{\gamma}(T_{\mathrm{inf}}))^q}{\pi(\eta^+_I K_{\gamma}(T_{\mathrm{inf}}))^q}
\end{array}$$

□

Combining Proposition 133 and Lemma 135, we obtain the following theorem.

Theorem 136 *Let* $\square\colon \mathrm{Spec}(A) \to \mathrm{Spec}(O_K[T_1^{\pm 1}, \dots, T_d^{\pm 1}])$ *be a framing, and let t_i denote the image of T_i in $\mathcal{A}$. Then the morphism (140) associated to $\mathcal{B} = \mathcal{A}$ and $t_i \in \mathcal{A}^{\times}$ coincides with the isomorphism (125) defined by using the framing* $\square$.

Proof By Lemma 135 (2), we have the following commutative diagram, where we write B_I^q, $-/\pi$, T, and $T^\square$ for Bock_I^q, $-\otimes^L_{A_{\inf}(O_{\overline{K}})} A_{\inf}(O_{\overline{K}})/\pi$, $TA_{\inf}(M)$, and $TA_{\inf}^\square(M)$, respectively.

$$\begin{array}{ccccc}
\eta_\pi^+ K_\gamma(T^\square)/\pi & \xrightarrow[\text{Lem.117}]{\cong} & L\eta_\pi^+ R\Gamma(\Gamma_{\mathcal{A}}, T^\square)/\pi & \xrightarrow[\text{Props. 116, 112}]{\cong} & L\eta_\pi^+ R\Gamma(\Delta_{\mathcal{A}}, T)/\pi \\
{\scriptstyle (122)}\downarrow\cong & & \downarrow\cong & & {\scriptstyle (127)}\downarrow\cong \\
M_{A_{\inf}} \otimes_{\mathcal{A}} \Omega^\bullet_{\mathcal{A}} & \xrightarrow{\pi^\bullet(142)} & (H^\bullet(\Gamma_{\mathcal{A}}, \mathrm{gr}^\bullet_I T^\square), \mathrm{B}^\bullet_I) & \longrightarrow & (H^\bullet(\Delta_{\mathcal{A}}, \mathrm{gr}^\bullet_I T), \mathrm{B}^\bullet_I)
\end{array}$$

The composition of the bottom horizontal homomorphisms coincides with (139) by Proposition 133. This implies the claim. □

In the rest of this section, we prove a functoriality of the morphism (140) in $(\mathcal{B}, s_1, \dots, s_e)$ and show that the morphism (140) is a quasi-isomorphism and does not depend on the choice of $s_1, \dots, s_e$.

Let $\mathcal{B}'$ be a $\mathcal{B}$-algebra p-adically complete and separated such that the homomorphisms $\mathcal{B}/p^m \to \mathcal{B}'/p^m$ $(m \in \mathbb{N}_{>0})$ are smooth, and suppose that we are given a surjective homomorphism $\mathcal{B}' \to \mathcal{A}$ compatible with the homomorphism $\mathcal{B} \to \mathcal{A}$, and $s'_1, \dots, s'_{e'} \in \mathcal{B}'^\times$ such that $d\log s'_j$ $(j \in \mathbb{N} \cap [1, e'])$ form a basis of $\Omega_{(\mathcal{B}'/p^m)/(\mathcal{B}/p^m)}$ for $m \in \mathbb{N}_{>0}$. Set $\Omega_{\mathcal{B}'} := \varprojlim_m \Omega_{(\mathcal{B}'/p^m)/(O_K/p^m)}$ and $\Omega^q_{\mathcal{B}'} := \wedge^q_{\mathcal{B}'} \Omega_{\mathcal{B}'}$ for $q \in \mathbb{N}$. By applying the construction of $\mathcal{P}_m$, $\mathcal{P}$, $M_{\mathcal{P}_m}$ and $M_{\mathcal{P}}$ for $\mathcal{B} \to \mathcal{A}$ to $\mathcal{B}' \to \mathcal{A}$, we define $\mathcal{P}'_m$, $\mathcal{P}'$, $M_{\mathcal{P}'_m}$, and $M_{\mathcal{P}'}$. We define the filtration I^r $(r \in \mathbb{Z})$ on $\mathscr{A}_{\mathrm{crys},\mathcal{B}',m}(\overline{\mathcal{A}})$ by using s'_j and the image of s_i in $\mathcal{B}'^\times$. The O_K-homomorphism $\mathcal{B} \to \mathcal{B}'$ induces PD-homomorphisms $\mathcal{P}_m \to \mathcal{P}'_m$ and $\mathcal{P} \to \mathcal{P}'$ compatible with φ and ∇. It also induces homomorphisms of $A_{\mathrm{crys}}(\overline{\mathcal{A}})$-algebras $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \to \mathscr{A}_{\mathrm{crys},\mathcal{B}',m}(\overline{\mathcal{A}})$ and $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \to \mathscr{A}_{\mathrm{crys},\mathcal{B}'}(\overline{\mathcal{A}})$ compatible with the homomorphisms $\mathcal{P}_m \to \mathcal{P}'_m$ and $\mathcal{P} \to \mathcal{P}'$ above, the $G_{\mathcal{A}}$-action, φ, ∇, and the filtrations I^r $(r \in \mathbb{Z})$. We have the following canonical $\mathcal{P}'$-linear isomorphism compatible with φ and ∇.

$$M_{\mathcal{P}'_m} \cong M_{\mathcal{P}_m} \otimes_{\mathcal{P}_m} \mathcal{P}'_m \ (m \in \mathbb{N}_{>0}), \quad M_{\mathcal{P}'} \cong M_{\mathcal{P}} \otimes_{\mathcal{P}} \mathcal{P}' \tag{143}$$

Following the notation $M_{\mathcal{P},A_{\inf}}$, we write $M_{\mathcal{P}',A_{\inf}}$ for the inverse limit $\varprojlim_m M_{\mathcal{P}'_m} \otimes_{O_K} A_{\inf}(O_{\overline{K}})/\pi$. By using the morphism $Y'_{\overline{D}_m} := \mathrm{Spec}(\mathcal{B}'/p^m \otimes_{O_K/p^m} A_{\mathrm{crys}}(\overline{\mathcal{A}})/p^m) \to Y_{\overline{D}_m} = \mathrm{Spec}(\mathcal{B}/p^m \otimes_{O_K/p^m} A_{\mathrm{crys}}(\overline{\mathcal{A}})/p^m)$ compatible with the embeddings of $\overline{X}_m = \mathrm{Spec}(\overline{\mathcal{A}}/p^m)$, we see that (30), (130)–(134), (137)–(139) for $\mathcal{B}$ and $\mathcal{B}'$ are all compatible with the natural morphisms from the modules for $\mathcal{B}$ to those for $\mathcal{B}'$. Thus we obtain the following functoriality.

Lemma 137 *Under the notation and the assumption as above, the following diagram is commutative, where the right vertical morphism is induced by (143).*

$$\begin{array}{ccc} M_{\mathcal{P},A_{\mathrm{inf}}} \otimes_{\mathcal{B}} \Omega^{\bullet}_{\mathcal{B}} & \xrightarrow{(140)} & L\eta^{+}_{\pi} R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M)) \otimes^{L}_{A_{\mathrm{inf}}(O_{\overline{K}})} A_{\mathrm{inf}}(O_{\overline{K}})/\pi \\ \downarrow & \nearrow_{(140)} & \\ M_{\mathcal{P}',A_{\mathrm{inf}}} \otimes_{\mathcal{B}'} \Omega^{\bullet}_{\mathcal{B}'} & & \end{array}$$

Lemma 138 *Under the notation and the assumption as above, the morphism* $M_{\mathcal{P},A_{\mathrm{inf}}} \otimes_{\mathcal{B}} \Omega^{\bullet}_{\mathcal{B}} \to M_{\mathcal{P}',A_{\mathrm{inf}}} \otimes_{\mathcal{B}'} \Omega^{\bullet}_{\mathcal{B}'}$ *induced by (143) is a quasi-isomorphism.*

Proof For each $m \in \mathbb{N}_{>0}$, the morphism $M_{\mathcal{P}_m} \otimes_{\mathcal{B}} \Omega^{\bullet}_{\mathcal{B}} \to M_{\mathcal{P}'_m} \otimes_{\mathcal{B}'} \Omega^{\bullet}_{\mathcal{B}'}$ is a quasi-isomorphism because both sides compute $R\Gamma((X_1/\Sigma_m)_{\mathrm{crys}}, \mathcal{F}_m)$ ([5, 7.1 Theorem]). See before (22) for the definition of $\mathcal{F}_m$. We obtain the desired quasi-isomorphism by taking $\otimes_{O_K} A_{\mathrm{inf}}(O_{\overline{K}})/\pi$ and then $R\varprojlim_m$. □

Theorem 139 *The morphism* (140)

$$M_{\mathcal{P},A_{\mathrm{inf}}} \otimes_{\mathcal{B}} \Omega^{\bullet}_{\mathcal{B}} \to L\eta^{+}_{\pi} R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M)) \otimes^{L}_{A_{\mathrm{inf}}(O_{\overline{K}})} A_{\mathrm{inf}}(O_{\overline{K}})/\pi$$

associated to $\mathcal{B}$ *and* $s_i \in \mathcal{B}^{\times}$ *is a quasi-isomorphism and does not depend on the choice of* s_i.

Proof By applying Lemma 137 to $\mathcal{A}, \mathcal{B} \to \varprojlim_n (\mathcal{A} \otimes_{O_K} \mathcal{B})/p^m$ and using Lemma 138, the first claim is reduced to Theorem 136. Let $s'_1, \dots, s'_e$ be another set of coordinates of $\mathcal{B}$ over O_K. Put $\mathcal{B}' := \varprojlim_m (\mathcal{B} \otimes_{O_K} \mathcal{B})/p^m$. Then, by Lemma 137, the morphism (140) associated to $\mathcal{B} \to \mathcal{A}$ and s_i (resp. s'_i) factors through the morphism (140) associated to the product map $\mathcal{B}' \to \mathcal{A}$ and $s_i \otimes 1$, $1 \otimes s'_i$ via the morphism λ (resp. λ') $M_{\mathcal{P},A_{\mathrm{inf}}} \otimes_{\mathcal{B}} \Omega^{\bullet}_{\mathcal{B}} \to M_{\mathcal{P}',A_{\mathrm{inf}}} \otimes_{\mathcal{B}'} \Omega^{\bullet}_{\mathcal{B}'}$ induced by $\mathcal{B} \to \mathcal{B}'; a \mapsto a \otimes 1$ (resp. $1 \otimes a$). The product map $\mathcal{B}' \to \mathcal{B}$ induces a G_K-equivariant $A_{\mathrm{inf}}(O_{\overline{K}})/\pi$-linear morphism $\mu\colon M_{\mathcal{P}',A_{\mathrm{inf}}} \otimes_{\mathcal{B}'} \Omega^{\bullet}_{\mathcal{B}'} \to M_{\mathcal{P},A_{\mathrm{inf}}} \otimes_{\mathcal{B}} \Omega^{\bullet}_{\mathcal{B}}$ such that $\mu \circ \lambda$ and $\mu \circ \lambda'$ are both the identity map. Since λ and λ' are quasi-isomorphisms by Lemma 138, this implies that λ and λ' coincide in the derived category of $A_{\mathrm{inf}}(O_{\overline{K}})/\pi$-modules with semilinear G_K-action. □

17 Period Rings with Truncated Divided Powers

In this section, we introduce and study period rings with truncated divided powers, which are used to give a description of the scalar extension of the modified Galois cohomology $L\eta^{+}_{\pi} R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M))$ by $A_{\mathrm{inf}}(O_{\overline{K}}) \to A_{\mathrm{crys}}(O_{\overline{K}})$ in Sect. 20. See the remark after Proposition 156 and the proof of Proposition 192 for the reason why we need such period rings.

As in the beginning of Sect. 8, let Λ be a normal domain containing $O_{\overline{K}}$, and assume that $\Lambda/p\Lambda \neq 0$, the absolute Frobenius of $\Lambda/p\Lambda$ is surjective, and Λ is integral over a noetherian normal subring. Let R_{Λ}, $A_{\mathrm{inf}}(\Lambda) = W(R_{\Lambda})$, $\theta\colon A_{\mathrm{inf}}(\Lambda) \to \widehat{\Lambda}$, $\underline{p} \in R_{\Lambda}$ and $\xi \in A_{\mathrm{inf}}(\Lambda)$ be as in the second and third paragraphs of Sect. 2. Let

$\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}} \in \mathbb{Z}_p(1)(O_{\overline{K}})$, $\underline{\varepsilon} \in R_\Lambda$, and $\pi \in A_{\inf}(\Lambda)$ be as after (1). The quotient $\eta := \pi(\varphi^{-1}(\pi))^{-1} \in A_{\inf}(\Lambda)$ generates $\mathrm{Ker}(\theta)$, i.e. $\eta \in \xi \cdot A_{\inf}(\Lambda)^\times$ ([12, 5.1.2], [18, Example A 2.6]).

Let N be a positive integer. We define the ring $W^{\mathrm{PD},(N)}(R_\Lambda)$ to be the $W(R_\Lambda)$-subalgebra of $W(R_\Lambda)[p^{-1}]$ generated by $\frac{1}{p^l!}\xi^{p^l}$ ($l \in \mathbb{N} \cap [0, N]$). We have $W^{\mathrm{PD},(N)}(R_\Lambda) \subset W^{\mathrm{PD},(N')}(R_\Lambda)$ for positive integers $N' \geq N$. The ring $W^{\mathrm{PD},(N)}(R_\Lambda)$ is functorial with respect to $\mathbb{Z}_p$-algebra homomorphisms between Λ's.

For $L \in \mathbb{N}$ and $n \in \mathbb{N}$, we define $\{n\}_{(L)} \in \mathbb{N}$ to be $v_p(r!) + qv_p((p^L)!)$, where $n = qp^L + r$ ($q \in \mathbb{N}, r \in \mathbb{N} \cap [0, p^L - 1]$). For $l \in \mathbb{N}$, we have

$$\{p^l\}_{(L)} = \begin{cases} v_p(p^l!) = \frac{p^l-1}{p-1} & \text{if } l \leq L, \\ p^{l-L} v_p(p^L!) = \frac{p^l - p^{l-L}}{p-1} < \frac{p^l-1}{p-1} = v_p(p^l!) & \text{if } l > L. \end{cases} \tag{144}$$

For an element x of a $\mathbb{Q}_p$-algebra and $n \in \mathbb{N}$, we define $x^{[n]_{(L)}}$ to be $p^{-\{n\}_{(L)}} x^n$, and $x^{[n]}$ to be $(n!)^{-1}x^n$. For $n \in \mathbb{N}$, recall that the p-adic valuation of $n!$ is given by

$$v_p(n!) = \sum_{l\geq 0} a_l v_p(p^l!) \quad (n = \sum_{l \geq 0} a_l p^l, a_l \in \mathbb{N} \cap [0, p-1]). \tag{145}$$

Lemma 140 (1) *For $n, n' \in \mathbb{N}$, we have $\{n + n'\}_{(L)} \geq \{n\}_{(L)} + \{n'\}_{(L)}$.*

(2) *For $n \in \mathbb{N}$, define $a \in \mathbb{N}$ and $a_l \in \mathbb{N} \cap [0, p-1]$ for $l \in \mathbb{N}$ by $n = ap^L + \sum_{l\in\mathbb{N}\cap[0,L-1]} a_l p^l = \sum_{l\in\mathbb{N}} a_l p^l$. Then we have*

$$\{n\}_{(L)} = a\{p^L\}_{(L)} + \sum_{l\in\mathbb{N}\cap[0,L-1]} a_l\{p^l\}_{(L)} = \sum_{l\in\mathbb{N}} a_l \{p^l\}_{(L)}. \tag{146}$$

(3) *For $n \in \mathbb{N}$, we have $\{n\}_{(L)} = v_p(n!)$ if $n \leq p^{L+1} - 1$, and $\{n\}_{(L)} < v_p(n!)$ if $n \geq p^{L+1}$.*

(4) *For $n \in \mathbb{N}$, we have $\{n\}_{(L)} = \{n\}_{(L+1)}$ if $n \leq p^{L+1} - 1$, and $\{n\}_{(L)} < \{n\}_{(L+1)}$ if $n \geq p^{L+1}$.*

(5) *For $n \in \mathbb{N}$, we have $0 \leq \{n+1\}_{(L)} - \{n\}_{(L)} \leq L$.*

Proof The claim is trivial when $L = 0$ because $\{n\}_{(0)} = 0$ for every $n \in \mathbb{N}$. We assume $L > 0$. For $n, n' \in \mathbb{N}$, put $n = qp^L + r$ and $n' = q'p^L + r'$ ($q, q' \in \mathbb{N}, r, r' \in \mathbb{N} \cap [0, p^L - 1]$). Then we have $n + n' = (q + q')p^L + (r + r')$, $v_p((r + r')!) \geq v_p(r!) + v_p(r'!)$, and, by (145), $v_p((r + r')!) = v_p(p^L!) + v_p(r''!)$ when $r + r' = p^L + r''$ with $r'' \in \mathbb{N} \cap [0, p^L - 1]$. This implies the claim (1). The claim (2) follows from the definition of $\{n\}_{(L)}$, (145), and (144). We obtain the claim (3) (resp. (4)) by comparing (146) and (145) (resp. (146) for L and $L + 1$) and using (144). For $r = qp^L + r$ ($q \in \mathbb{N}, r \in \mathbb{N} \cap [0, p^L - 1]$), we have $\{n + 1\}_{(L)} - \{n\}_{(L)} = v_p((r + 1)!) - v_p(r!) = v_p(r + 1)$. This implies the claim (4). □

Let $\mathbb{Z}_p\langle T\rangle^{(N)}$ be the $\mathbb{Z}_p[T]$-subalgebra of $\mathbb{Q}_p[T]$ generated by $T^{[p^l]}$ ($l \in \mathbb{N} \cap [0, N]$).

Corollary 141 (1) *The $\mathbb{Z}_p$-algebra $\mathbb{Z}_p\langle T\rangle^{(N)}$ is a free $\mathbb{Z}_p$-module with a basis $T^{[n]_{(N)}}$ ($n \in \mathbb{N}$).*
(2) *We have $T^{[n]} \in \mathbb{Z}_p\langle T\rangle^{(N)}$ for $n \in \mathbb{N} \cap [0, p^{N+1} - 1]$.*
(3) *We have an isomorphism of $\mathbb{F}_p$-algebras*

$$\mathbb{F}_p[T_0, T_1, \dots, T_{N-1}, T_N]/(T_0^p, \dots, T_{N-1}^p) \xrightarrow{\cong} \mathbb{Z}_p\langle T\rangle^{(N)}/p; T_i \mapsto (T^{[p^i]} \mod p).$$

Proof We have $T^{[n]_{(N)}} \in \mathbb{Z}_p\langle T\rangle^{(N)}$ by Lemma 140 (2) and $T^{[n]_{(N)}} T^{[n']_{(N)}} \in \mathbb{Z}_p T^{[n+n']_{(N)}}$ by Lemma 140 (1). This implies (1). We obtain (2) from (1) and Lemma 140 (3). The homomorphism in (3) is well-defined by $(T^{[p^i]})^p \in \mathbb{Z}_p^\times p T^{[p^{i+1}]}$ for $i \in \mathbb{N} \cap [0, N-1]$. By (1) and Lemma 140 (2), $\mathbb{Z}_p\langle T\rangle^{(N)}$ is a free $\mathbb{Z}_p$-module with a basis $(T^{[p^N]})^a \prod_{0 \le i \le N-1} (T^{[p^i]})^{a_i}$ ($a_i \in \mathbb{N} \cap [0, p-1], a \in \mathbb{N}$). This implies the claim (3). □

Corollary 142 *The $W(R_\Lambda)$-algebra $W^{\mathrm{PD},(N)}(R_\Lambda)$ coincides with the $W(R_\Lambda)$-subalgebra of $W(R_\Lambda)[p^{-1}]$ generated by $[\underline{p}]^{[p^l]}$ ($l \in \mathbb{N} \cap [0, N]$).*

Proof By Corollary 141 (2), $W^{\mathrm{PD},(N)}(R_\Lambda)$ (resp. the second $W(R_\Lambda)$-algebra in the claim) contains $\xi^{[n]}$ (resp. $[\underline{p}]^{[n]}$) for $n \in \mathbb{N} \cap [0, p^{N+1} - 1]$. Hence the claim follows from $\xi^{[p^l]} = \sum_{0 \le \nu \le p^l} p^{[\nu]} (-[\underline{p}])^{[p^l - \nu]}$ and $[\underline{p}]^{[p^l]} = \sum_{0 \le \nu \le p^l} p^{[\nu]} (-\xi)^{[p^l - \nu]}$ for $l \in \mathbb{N} \cap [0, N]$. □

Corollary 142 implies that $W^{\mathrm{PD},(N)}(R_\Lambda)$ is stable under the Frobenius automorphism of $W(R_\Lambda)[p^{-1}]$ (induced by that of $W(R_\Lambda)$). Let φ also denote the induced endomorphism of $W^{\mathrm{PD},(N)}(R_\Lambda)$.

Proposition 143 *We regard $W(R_\Lambda)$ as a $\mathbb{Z}_p[T]$-algebra by the homomorphism $\mathbb{Z}_p[T] \to W(R_\Lambda)$; $T \mapsto \xi$ (resp. $T \mapsto [\underline{p}]$).*

(1) *The homomorphism $\mathbb{Z}/p^n\mathbb{Z}[T] \to W_n(R_\Lambda)$ is flat for every $n \in \mathbb{N}_{>0}$.*
(2) *The composition of $W(R_\Lambda) \otimes_{\mathbb{Z}_p[T]} \mathbb{Z}_p\langle T\rangle^{(N)} \to W(R_\Lambda) \otimes_{\mathbb{Z}_p[T]} \mathbb{Q}_p[T] \xrightarrow{\cong} W(R_\Lambda)[p^{-1}]$ induces an isomorphism*

$$W(R_\Lambda) \otimes_{\mathbb{Z}_p[T]} \mathbb{Z}_p\langle T\rangle^{(N)} \xrightarrow{\cong} W^{\mathrm{PD},(N)}(R_\Lambda).$$

Lemma 144 *Let R be a flat $\mathbb{Z}_p$-algebra, and let $a \in R$ such that R/pR is a-torsion free. We regard R as a $\mathbb{Z}_p[T]$-algebra by the homomorphism $\mathbb{Z}_p[T] \to R$; $T \mapsto a$.*

(1) *If R/pR is a-adically complete and separated, then R/p^nR is a flat $\mathbb{Z}/p^n\mathbb{Z}[T]$-algebra for every $n \in \mathbb{N}_{>0}$.*
(2) *Let M be a $\mathbb{Z}_p[T]$-module and suppose that, for every $x \in M$, there exists $m \in \mathbb{N}$ such that $p^m x = 0$ and $T^m x = 0$. Then $\mathrm{Tor}_r^{\mathbb{Z}_p[T]}(M, R) = 0$ for every $r \ge 1$.*
(3) *Let S be a $\mathbb{Z}_p[T]$-subalgebra of $\mathbb{Q}_p[T]$ such that $T^{n_0} \in pS$ for some $n_0 \in \mathbb{N}$. Then the homomorphism $R \otimes_{\mathbb{Z}_p[T]} S \to R \otimes_{\mathbb{Z}_p[T]} \mathbb{Q}_p[T]$ is injective.*

Proof (1) Since p is regular on R, the natural homomorphism $R/p^mR \otimes_{\mathbb{Z}/p^m\mathbb{Z}[T]} (p^m\mathbb{Z}[T]/p^{m+1}\mathbb{Z}[T]) \to p^mR/p^{m+1}R$ is an isomorphism for $m \in \mathbb{N}_{>0}$. Hence, by the local criteria of flatness, it suffices to prove the claim for $n = 1$. By assumption, the homomorphism $\mathbb{F}_p[T] \to R/pR$ extends to a homomorphism $\mathbb{F}_p[[T]] \to R/pR$, which is flat because a is regular on R/pR. Since $\mathbb{F}_p[T] \to \mathbb{F}_p[[T]]$ is flat, $\mathbb{F}_p[T] \to R/pR$ is also flat.

(2) We may assume that M is finitely generated since M is the filtered direct limit of its finitely generated $\mathbb{Z}_p[T]$-submodules and $\mathrm{Tor}_r^{\mathbb{Z}_p[T]}(-, R)$ commutes with filtered direct limits. By considering the graded quotients of the filtration $(T, p)^rM$ $(r \in \mathbb{N})$, which is of finite length, we are reduced to the case $M = \mathbb{Z}_p[T]/(p, T)$. By applying $\mathrm{Tor}_\bullet^{\mathbb{Z}_p[T]}(-, R)$ to the exact sequence $0 \to \mathbb{Z}_p[T] \xrightarrow{p} \mathbb{Z}_p[T] \to \mathbb{F}_p[T] \to 0$, and using the p-torsion freeness of R, we obtain $\mathrm{Tor}_r^{\mathbb{Z}_p[T]}(\mathbb{F}_p[T], R) = 0$ $(r > 0)$. Then, by applying $\mathrm{Tor}_\bullet^{\mathbb{Z}_p[T]}(-, R)$ to the exact sequence $0 \to \mathbb{F}_p[T] \xrightarrow{T} \mathbb{F}_p[T] \to \mathbb{F}_p[T]/(T) \to 0$, and using the a-torsion freeness of R/pR, we obtain $\mathrm{Tor}_r^{\mathbb{Z}_p[T]}(\mathbb{F}_p[T]/(T), R) = 0$ $(r > 0)$.

(3) The $\mathbb{Z}_p[T]$-module $\mathbb{Q}_p[T]/S$ is generated by $(p^{-m} \mod S)$ $(m \in \mathbb{N})$, and we have $p^m \cdot p^{-m} \in S$ and $T^{mn_0} \cdot p^{-m} = (p^{-1}T^{n_0})^m \in S$. Hence we have $\mathrm{Tor}_1^{\mathbb{Z}_p[T]}(\mathbb{Q}_p[T]/S, R) = 0$ by (2). □

Proof of Proposition 143 By Lemma 1 (4), we can apply Lemma 144 (1) and (3) to $R = W(R_\Lambda)$, $a = \xi$ (resp. $[\underline{p}]$), and $S = \mathbb{Z}_p\langle T\rangle^{(N)}$, and obtain the claims, using also Corollary 142 when $a = [\underline{p}]$. □

We define the decreasing filtration $\mathrm{Fil}^r(W(R_\Lambda)[p^{-1}])$ $(r \in \mathbb{Z})$ of $W(R_\Lambda)[p^{-1}]$ by ideals to be $(\mathrm{Fil}^r W(R_\Lambda))[\frac{1}{p}]$. Since $W(R_\Lambda)/\mathrm{Fil}^r = W(R_\Lambda)/\xi^r$ is p-torsion free (Lemma 1 (3)), we have

$$\mathrm{Fil}^r W(R_\Lambda) = W(R_\Lambda) \cap \mathrm{Fil}^r(W(R_\Lambda)[p^{-1}]), \quad r \in \mathbb{Z}. \tag{147}$$

We define $\mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda)$ $(r \in \mathbb{Z})$ to be the filtration of $W^{\mathrm{PD},(N)}(R_\Lambda)$ by ideals induced by that of $W(R_\Lambda)[p^{-1}]$, i.e., $W^{\mathrm{PD},(N)}(R_\Lambda) \cap \mathrm{Fil}^r(W(R_\Lambda)[p^{-1}])$.

Lemma 145 *Let* $r \in \mathbb{N}$.

(1) $\mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda)$ *is generated by* $\xi^{[s]_{(N)}}$ $(s \in \mathbb{N} \cap [r, \infty))$ *as a* $W(R_\Lambda)$*-submodule of* $W^{\mathrm{PD},(N)}(R_\Lambda)$.
(2) *The image of the following injective homomorphism is the* $\widehat{\Lambda}$*-module generated by* $(\xi^{[r]_{(N)}}$ *mod* $\mathrm{Fil}^{r+1})$.

$$\mathrm{gr}^r W^{\mathrm{PD},(N)}(R_\Lambda) \hookrightarrow \mathrm{gr}^r(W(R_\Lambda)[p^{-1}]) \cong \widehat{\Lambda}[p^{-1}] \cdot (\xi^r \textit{ mod } \mathrm{Fil}^{r+1}).$$

(3) *The quotient* $W^{\mathrm{PD},(N)}(R_\Lambda)/\mathrm{Fil}^r$ *is p-torsion free and p-adically complete and separated.*

Proof The claim (2) immediately follows from (1). By using Lemma 3 (2), we obtain (3) from (2) by induction on r. Let us prove (1) by induction on r. The claim for $r = 0$ holds by Corollary 141 (1). Let $r \in \mathbb{N}_{>0}$, and suppose that the claim holds for $r-1$. Then any element of $\mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda)$ is written as $\sum_{n\geq r-1} a_n\xi^{[n]_{(N)}}$ $(a_n \in W(R_\Lambda)$ $(n \geq r-1)$, $a_n = 0$ $(n \gg 0))$. Since $\xi^{[n]_{(N)}} \in \mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda)$ for $n \geq r$, we have $a_{r-1}\xi^{[r-1]_{(N)}} \in \mathrm{Fil}^r W(R_\Lambda)[\frac{1}{p}] = \xi^r W(R_\Lambda)[\frac{1}{p}]$, which implies $a_{r-1} \in \xi W(R_\Lambda)[\frac{1}{p}] \cap W(R_\Lambda) = \xi W(R_\Lambda)$ (147), and therefore $a_{r-1}\xi^{[r-1]_{(N)}} \in W(R_\Lambda)\xi\xi^{[r-1]_{(N)}} \subset W(R_\Lambda)\xi^{[r]_{(N)}}$ by Lemma 140 (5). □

Corollary 146 *We have* $\varphi(\mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda)) \subset p^r W^{\mathrm{PD},(N)}(R_\Lambda)$ *for* $r \in \mathbb{N} \cap [0, p-1]$.

Proof By Lemmas 145 (1) and 140 (2), the ideal $\mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda)$ of $W^{\mathrm{PD},(N)}(R_\Lambda)$ is generated by ξ^r and $\xi^{[p^l]}$ $(l \in \mathbb{N} \cap [1, N])$. By Corollary 142, we have $\varphi(\xi) = p(1 - p^{-1}[\underline{p}]^p) \in pW^{\mathrm{PD},(N)}(R_\Lambda)$. Hence the claim follows from $p^l - v_p(p^l!) = p^l - (p-1)^{-1}(p^l - 1) = (p-1)^{-1}(p^l(p-2)+1) \geq (p-1)^{-1}(p(p-2)+1) = p-1$ for $l \in \mathbb{N}_{>0}$. □

Proposition 147 *The ring* $W^{\mathrm{PD},(N)}(R_\Lambda)$ *coincides with the* $W(R_\Lambda)$*-subalgebra of* $W(R_\Lambda)[p^{-1}]$ *generated by* $(p^{-1}\pi^{p-1})^{[p^l]}$ $(l \in \mathbb{N} \cap [0, N-1])$.

Proof Put $\pi' := \varphi^{-1}(\pi)$ and $\eta := \pi(\pi')^{-1}$. We have $\eta \in \xi W(R_{O_{\overline{K}}})^\times$ ([12, 5.1.2], [18, Example A2.6]). By multiplying $\eta = \{(1+\pi')^p - 1\}(\pi')^{-1} = (\pi')^{p-1} + \sum_{1\leq i\leq p-1}\binom{p}{i}(\pi')^{p-1-i}$ by $\eta^{p-1}p^{-1}$, we obtain $p^{-1}\eta^p = p^{-1}\pi^{p-1} + \eta a$, $a \in W(R_{O_{\overline{K}}})$. This implies the claim for $N = 1$. We have $(p^{-1}\eta^p)^{[n]} \in \mathbb{Z}_p^\times\eta^{[pn]}$ for $n \in \mathbb{N}$, and $\eta^{[n]} \in W^{\mathrm{PD},(N-1)}(R_\Lambda)$ for an integer $N \geq 2$ and $n \in \mathbb{N} \cap [0, p^N - 1]$ by Corollary 141 (2). Hence we have $(p^{-1}\pi^{p-1})^{[p^{N-1}]} = (p^{-1}\eta^p)^{[p^{N-1}]} + b_N = u\cdot\eta^{[p^N]} + b_N$, $b_N \in W^{\mathrm{PD},(N-1)}(R_{O_{\overline{K}}})$, $u \in \mathbb{Z}_p^\times$ for an integer $N \geq 2$, and obtain the claim by induction on N. □

Let $\mathbb{Z}_p[T]\langle\frac{T^{p-1}}{p}\rangle^{(N-1)}$ be the $\mathbb{Z}_p[T]$-subalgebra of $\mathbb{Q}_p[T]$ generated by the elements $(\frac{T^{p-1}}{p})^{[p^l]}$ $(l \in \mathbb{N} \cap [0, N-1])$.

Lemma 148 (1) *The* $\mathbb{Z}_p$*-algebra* $\mathbb{Z}_p[T]\langle\frac{T^{p-1}}{p}\rangle^{(N-1)}$ *is a free* $\mathbb{Z}_p$*-module with a basis* $T^r(\frac{T^{p-1}}{p})^{[n]_{(N-1)}}$ $(n \in \mathbb{N}, r \in \mathbb{N} \cap [0, p-2])$.

(2) *We have* $(\frac{T^{p-1}}{p})^{[n]} \in \mathbb{Z}_p[T]\langle\frac{T^{p-1}}{p}\rangle^{(N-1)}$ *for* $n \in \mathbb{N} \cap [0, p^N - 1]$.

Proof Put $L = N - 1$ to simplify the notation. By Lemma 140 (2), we have $T^r(\frac{T^{p-1}}{p})^{[n]_{(N-1)}} \in \mathbb{Z}_p[T]\langle\frac{T^{p-1}}{p}\rangle^{(N-1)}$ for $n \in \mathbb{N}$ and $r \in \mathbb{N} \cap [0, p-2]$. By using Lemma 140 (1), we see that, for $n, n' \in \mathbb{N}$ and $r, r' \in \mathbb{N} \cap [0, p-2]$, $T^r(\frac{T^{p-1}}{p})^{[n]_{(N-1)}} \cdot T^{r'}(\frac{T^{p-1}}{p})^{[n']_{(N-1)}} \in \mathbb{Z}_pT^{r+r'}(\frac{T^{p-1}}{p})^{[n+n']_{(N-1)}}$, and if $r'' := r + r' \geq p-1$, $T^{r''}(\frac{T^{p-1}}{p})^{[n+n']_{(N-1)}} = pT^{r''-(p-1)}\frac{T^{p-1}}{p}(\frac{T^{p-1}}{p})^{[n+n']_{(N-1)}} \in p\mathbb{Z}_pT^{r''-(p-1)}(\frac{T^{p-1}}{p})^{[n+n'+1]_{(N-1)}}$. This implies (1). The claim (2) follows from (1) and Lemma 140 (3). □

Proposition 149 *We regard $W(R_\Lambda)$ as a $\mathbb{Z}_p[T]$-algebra by the homomorphism $\mathbb{Z}_p[T] \to W(R_\Lambda); T \mapsto \pi$.*

(1) *The homomorphism $\mathbb{Z}/p^n\mathbb{Z}[T] \to W_n(R_\Lambda)$ is flat for every $n \in \mathbb{N}_{>0}$.*
(2) *The composition of $W(R_\Lambda) \otimes_{\mathbb{Z}_p[T]} \mathbb{Z}_p[T]\langle \frac{T^{p-1}}{p} \rangle^{(N-1)} \to W(R_\Lambda) \otimes_{\mathbb{Z}_p[T]} \mathbb{Q}_p[T] \xrightarrow{\cong} W(R_\Lambda)[p^{-1}]$ induces an isomorphism*

$$W(R_\Lambda) \otimes_{\mathbb{Z}_p[T]} \mathbb{Z}_p[T]\langle \frac{T^{p-1}}{p} \rangle^{(N-1)} \xrightarrow{\cong} W^{\mathrm{PD},(N)}(R_\Lambda).$$

Proof By Lemma 1 (4), $W(R_\Lambda)/p$ is π-torsion free, and π-adically complete and separated. Since $W(R_\Lambda)$ is p-torsion free, we obtain the claim by applying Lemma 144 (1) and (3) to $R = W(R_\Lambda), a = \pi$, and $S = \mathbb{Z}_p[T]\langle \frac{T^{p-1}}{p} \rangle^{(N-1)}$, and using Proposition 147. □

We define $I^r(W(R_\Lambda)[p^{-1}])$ $(r \in \mathbb{Z})$ to be the ideal of $W(R_\Lambda)[p^{-1}]$ consisting of elements x such that $\varphi^n(x) \in \mathrm{Fil}^r(W(R_\Lambda)[p^{-1}])$ for all $n \in \mathbb{N}$. By (147), we have $I^r(W(R_\Lambda)[p^{-1}]) = (I^r A_{\text{inf}}(\Lambda))[p^{-1}] = \pi^{\max\{r,0\}} \cdot A_{\text{inf}}(\Lambda)[\frac{1}{p}]$ and

$$I^r A_{\text{inf}}(\Lambda) = A_{\text{inf}}(\Lambda) \cap I^r(W(R_\Lambda)[p^{-1}]). \tag{148}$$

We define the ideal $I^r W^{\mathrm{PD},(N)}(R_\Lambda)$ of $W^{\mathrm{PD},(N)}(R_\Lambda)$ to be $W^{\mathrm{PD},(N)}(R_\Lambda) \cap I^r(W(R_\Lambda)[p^{-1}])$, which coincides with the ideal consisting of elements x such that $\varphi^n(x) \in \mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda)$ for all $n \in \mathbb{N}$.

Lemma 150 *Let $r \in \mathbb{N}$.*

(1) *$I^r W^{\mathrm{PD},(N)}(R_\Lambda)$ is the $W(R_\Lambda)$-submodule of $W^{\mathrm{PD},(N)}(R_\Lambda)$ generated by $\pi^a(\frac{\pi^{p-1}}{p})^{[b]_{(N-1)}}$ $(a \in \mathbb{N} \cap [0, p-2], b \in \mathbb{N}, a + (p-1)b \geq r)$.*
(2) *Let $s \in \mathbb{N} \cap [0, p-2]$ and $q \in \mathbb{N}$, and put $r := s + (p-1)q$. Then the multiplication by $\pi^s(\frac{\pi^{p-1}}{p})^{[q]_{(N-1)}}$ induces an isomorphism*

$$W(R_\Lambda)/I^1 W(R_\Lambda) \xrightarrow{\cong} \mathrm{gr}_I^r W^{\mathrm{PD},(N)}(R_\Lambda).$$

(3) *The quotient $W^{\mathrm{PD},(N)}(R_\Lambda)/I^r W^{\mathrm{PD},(N)}(R_\Lambda)$ is p-torsion free, and p-adically complete and separated.*

Proof The claim (2) follows from the claim (1) and (148) for $r = 1$ because π is not a zero divisor of $W(R_\Lambda)[\frac{1}{p}]$. Since $W(R_\Lambda)/I^1 W(R_\Lambda) = W(R_\Lambda)/\pi$ is p-torsion free, and p-adically complete and separated by Lemma 1 (3), we obtain (3) from (2) by induction on r by using Lemma 3 (2). One can prove the claim (1) in the same way as the proof of Lemma 145 (1) by using Proposition 149 (2), Lemma 148 (1), (148) for $r = 1$, and $\pi \cdot \pi^{p-2}(\frac{\pi^{p-1}}{p})^{[n]_{(N-1)}} \in \mathbb{Z}_p \cdot (\frac{\pi^{p-1}}{p})^{[n+1]_{(N-1)}}$. □

Definition 151 We define the period ring $A_{\text{crys}}^{(N)}(\Lambda)$ to be the inverse limit $\varprojlim_m W^{\mathrm{PD},(N)}(R_\Lambda)/p^m$. We define a decreasing filtration $\mathrm{Fil}^r A_{\text{crys}}^{(N)}(\Lambda)$ $(r \in \mathbb{Z})$ of

$A_{\mathrm{crys}}^{(N)}(\Lambda)$ by ideals to be $\varprojlim_m \mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda)/p^m$. Note that the homomorphism $\mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda)/p^n \to W^{\mathrm{PD},(N)}(R_\Lambda)/p^n$ is injective by Lemma 145 (3). The Frobenius endomorphism φ of $W^{\mathrm{PD},(N)}(R_\Lambda)$ induces that of $A_{\mathrm{crys}}^{(N)}(\Lambda)$, which is again denoted by φ. We endow $A_{\mathrm{crys}}^{(N)}(\Lambda)$ with the p-adic topology, which coincides with the $(p, [\underline{p}])$-adic topology because $[\underline{p}]^p \in pA_{\mathrm{crys}}^{(N)}(\Lambda)$ by Corollary 142.

By Lemma 7, $A_{\mathrm{crys}}^{(N)}(\Lambda)$ and $\mathrm{Fil}^r A_{\mathrm{crys}}^{(N)}(\Lambda)$ are p-torsion free and the natural homomorphisms $A_{\mathrm{crys}}^{(N)}(\Lambda)/p^m \to W^{\mathrm{PD},(N)}(R_\Lambda)/p^m$ and $\mathrm{Fil}^r A_{\mathrm{crys}}^{(N)}(\Lambda)/p^m \to \mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda)/p^m$ are isomorphisms. The latter implies that $A_{\mathrm{crys}}^{(N)}(\Lambda)$ and $\mathrm{Fil}^r A_{\mathrm{crys}}^{(N)}(\Lambda)$ are p-adically complete and separated.

The topological algebra $A_{\mathrm{crys}}^{(N)}(\Lambda)$ with Fil^r and φ is obviously functorial with respect to $\mathbb{Z}_p$-algebra homomorphisms of Λ's. By Corollary 146, we have

$$\varphi(\mathrm{Fil}^r A_{\mathrm{crys}}^{(N)}(\Lambda)) \subset p^r \mathrm{Fil}^r A_{\mathrm{crys}}^{(N)}(\Lambda) \quad (r \in \mathbb{N} \cap [0, p-1]). \tag{149}$$

As it is recalled in Sect. 2, the usual period ring $A_{\mathrm{crys}}(\Lambda)$ is canonically isomorphic to the p-adic completion of the $W(R_\Lambda)$-subalgebra $W^{\mathrm{PD}}(R_\Lambda)$ of $W(R_\Lambda)[p^{-1}]$ generated by $\xi^{[n]}$ ($n \in \mathbb{N}$). Therefore we have natural continuous ring homomorphisms compatible with φ and functorial in Λ

$$\begin{aligned} A_{\mathrm{inf}}(\Lambda) \to A_{\mathrm{crys}}^{(1)}(\Lambda) \to A_{\mathrm{crys}}^{(2)}(\Lambda) \to \\ \cdots \to A_{\mathrm{crys}}^{(N)}(\Lambda) \to A_{\mathrm{crys}}^{(N+1)}(\Lambda) \to \cdots \to A_{\mathrm{crys}}(\Lambda), \end{aligned} \tag{150}$$

which induce isomorphisms

$$\varinjlim_N A_{\mathrm{crys}}^{(N)}(\Lambda)/p^m \xrightarrow{\cong} A_{\mathrm{crys}}(\Lambda)/p^m \qquad (m \in \mathbb{N}_{>0}) \tag{151}$$

because $\varinjlim_N W^{\mathrm{PD},(N)}(R_\Lambda) \xrightarrow{\cong} W^{\mathrm{PD}}(R_\Lambda)$. As it is recalled in Sect. 2, the filtration $\mathrm{Fil}^r A_{\mathrm{crys}}(\Lambda)$ ($r \in \mathbb{Z}$) of $A_{\mathrm{crys}}(\Lambda)$ is given by the p-adic completion of $\mathrm{Fil}^r W^{\mathrm{PD}}(R_\Lambda) = W^{\mathrm{PD}}(R_\Lambda) \cap \mathrm{Fil}^r(W(R_\Lambda)[p^{-1}])$, and we have isomorphisms

$$W^{\mathrm{PD}}(R_\Lambda)/\mathrm{Fil}^r W^{\mathrm{PD}}(R_\Lambda) \xrightarrow{\cong} A_{\mathrm{crys}}(\Lambda)/\mathrm{Fil}^r A_{\mathrm{crys}}(\Lambda) \quad (r \in \mathbb{Z}). \tag{152}$$

Similarly, by taking $\varprojlim_m(- \otimes_{\mathbb{Z}} \mathbb{Z}/p^m)$ of the exact sequence

$$0 \longrightarrow \mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda) \longrightarrow W^{\mathrm{PD},(N)}(R_\Lambda) \longrightarrow \frac{W^{\mathrm{PD},(N)}(R_\Lambda)}{\mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda)} \longrightarrow 0$$

and using Lemma 145 (3), we obtain the following isomorphisms.

$$W^{\mathrm{PD},(N)}(R_\Lambda)/\mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda) \xrightarrow{\cong} A_{\mathrm{crys}}^{(N)}(\Lambda)/\mathrm{Fil}^r A_{\mathrm{crys}}^{(N)}(\Lambda) \quad (r \in \mathbb{Z}). \tag{153}$$

From (152), (153), and (147), we obtain the following.

Lemma 152 *The homomorphisms* $A_{\inf}(\Lambda) \to A_{\mathrm{crys}}^{(N)}(\Lambda) \to A_{\mathrm{crys}}(\Lambda)$ *(150) are strictly compatible with the filtrations* $\mathrm{Fil}^\bullet$.

Lemma 153 *The filtrations* Fil^r $(r \in \mathbb{Z})$ *on* $A_{\inf}(\Lambda)$, $A_{\mathrm{crys}}^{(N)}(\Lambda)$ *and* $A_{\mathrm{crys}}(\Lambda)$ *are separated.*

Proof Since these algebras are p-adically separated and their quotients by Fil^r are p-torsion free, it suffices to prove the claim for the images of Fil^r in the reduction mod p of these algebras. The claim for $A_{\inf}(\Lambda)/p$ follows from Lemma 1 (4) for $a = \xi$. For $A_{\mathrm{crys}}^{(N)}(\Lambda)/p \cong W^{\mathrm{PD},(N)}(R_\Lambda)/p$ and $A_{\mathrm{crys}}(\Lambda)/p \cong A_{\mathrm{crys},1}(\Lambda)$, we obtain the claim from the following isomorphisms sending T_i to $(\xi^{[p^i]} \bmod p)$.

$$R_\Lambda/\underline{p}^p[T_1, \ldots, T_{N-1}, T_N]/(T_1^p, \ldots, T_{N-1}^p) \xrightarrow{\cong} A_{\mathrm{crys}}^{(N)}(\Lambda)/p, \tag{154}$$

$$R_\Lambda/\underline{p}^p[T_i; i \in \mathbb{N}_{>0}]/(T_i^p; i \in \mathbb{N}_{>0}) \xrightarrow{\cong} A_{\mathrm{crys}}(\Lambda)/p = A_{\mathrm{crys},1}(\Lambda). \tag{155}$$

The first one follows from Proposition 143 (2) and Corollary 141 (3). The second one follows from the flatness of $k[T] \to A_{\inf}(\Lambda)/p$; $T \mapsto (\xi \mod p)$ (Proposition 143 (1)), [5, 3.21 Proposition], and $\mathbb{F}_p[T_i; i \in \mathbb{N}]/(T_i^p, i \in \mathbb{N}) \xrightarrow{\cong} \mathbb{F}_p\langle T\rangle/p$; $T_i \mapsto T^{[p^i]}$. □

Corollary 154 *The homomorphisms* $A_{\inf}(\Lambda) \to A_{\mathrm{crys}}^{(N)}(\Lambda) \to A_{\mathrm{crys}}(\Lambda)$ *are injective.*

Proof This is an immediate consequence of Lemmas 152 and 153. □

Remark 155 (1) The ideal $\mathrm{Fil}^1 A_{\mathrm{crys}}^{(N)}(\Lambda)/p^n$ of $A_{\mathrm{crys}}^{(N)}(\Lambda)/p^n$ is not a nilideal. The claim is reduced to the case $n = 1$, and $(\xi^{[p^N]} \mod p)$ is not nilpotent in $A_{\mathrm{crys}}^{(N)}(\Lambda)/p$ by (154).

(2) The reduction mod p of φ of $A_{\mathrm{crys}}^{(N)}(\Lambda)$ does not coincide with the absolute Frobenius. Indeed we have $\varphi(\xi^{[p^N]}) \in pA_{\mathrm{crys}}^{(N)}(\Lambda)$ by (149), but $(\xi^{[p^N]})^p \notin pA_{\mathrm{crys}}^{(N)}(\Lambda)$ as observed in (1).

Proposition 156 (1) *For $s \in \mathbb{N}$, the element $\varphi^{-s}(\pi)$ is regular in $A_{\mathrm{crys}}^{(N)}(\Lambda)$.*

(2) *For $s \in \mathbb{N}$, the quotient $A_{\mathrm{crys}}^{(N)}(\Lambda)/\varphi^{-s}(\pi)$ is p-adically complete and separated, and its p-primary torsion part is annihilated by p^N.*

(3) *For $s \in \mathbb{N}$, the quotients $A_{\mathrm{crys}}^{(N)}(\Lambda)/\varphi^{-s}(\pi)$ and $(A_{\mathrm{crys}}^{(N)}(\Lambda)/\varphi^{-s}(\pi))/p^m$ $(m \in \mathbb{N}_{>0})$ do not have a non-zero element annihilated by the ideal $\sum_{l\in\mathbb{N}}[\underline{p}^{p^{-l}}]A_{\inf}(O_{\overline{K}})$ of $A_{\inf}(O_{\overline{K}})$.*

As it is mentioned before [7, Lemma 12.8], the property (2) in Proposition 156 does not hold for $A_{\mathrm{crys}}(\Lambda)$, and this is the main reason why we introduce $A_{\mathrm{crys}}^{(N)}(\Lambda)$. The same remark applies to $A_{\mathrm{crys}}^{\square}(\mathcal{A})$ (cf. Proposition 168 (2)).

Proof (1), (2) For $m \in \mathbb{N}_{>0}$, let K_m and C_m be the kernel and cokernel of the multiplication by T on $(\mathbb{Z}_p[T]\langle \frac{T^{p-1}}{p}\rangle^{(N-1)})/p^m$. For $n \in \mathbb{N}$, put $\alpha(n) = \{n+1\}_{(N-1)} - \{n\}_{(N-1)} + 1$, which is contained in $\mathbb{N} \cap [1, N]$ by Lemma 140 (5). Then, by Lemma 148 (1), we have

$$K_m = \oplus_{n\in\mathbb{N}}\ p^{m-\alpha(n)}\mathbb{Z}/p^m\mathbb{Z} \cdot T^{p-2}(p^{-1}T^{p-1})^{[n]_{(N-1)}},$$
$$C_m = \mathbb{Z}/p^m \oplus (\oplus_{n\in\mathbb{N}}\ \mathbb{Z}/p^{\alpha(n)}(p^{-1}T^{p-1})^{[n+1]_{(N-1)}})$$

for $m \in \mathbb{N}$ and $m \geq N$. This implies that the homomorphism $K_{m+N} \to K_m$ vanishes for $m \geq N$. By Proposition 149, we have an exact sequence

$$0 \longrightarrow K_m \otimes_{W_m} W_m(R_\Lambda)/\pi \longrightarrow W^{\mathrm{PD},(N)}(R_\Lambda)/p^m \xrightarrow{\pi} W^{\mathrm{PD},(N)}(R_\Lambda)/p^m \longrightarrow C_m \otimes_{W_m} W_m(R_\Lambda)/\pi \longrightarrow 0.$$

By taking $\varprojlim_m$ and using the fact that $W(R_\Lambda)/\pi$ is p-torsion free (Lemma 1 (3)), we obtain the claims (1) and (2) for $s = 0$. We also obtain

$$(A^{(N)}_{\mathrm{crys}}(\Lambda)/\pi)/p^m = (A_{\mathrm{inf}}(\Lambda)/\pi)/p^m \oplus (\oplus_{n\in\mathbb{N}}(A_{\mathrm{inf}}(\Lambda)/\pi)/p^{\alpha(n)}\overline{(\tfrac{\pi^{p-1}}{p})^{[n+1]_{(N-1)}}}).$$

For $s \in \mathbb{N}$, we have $\pi \in \varphi^{-s}(\pi)A_{\mathrm{inf}}(O_{\overline{K}})$. Hence the claim (1) for $s = 0$ implies that $\varphi^{-s}(\pi)$ is regular in $A^{(N)}_{\mathrm{crys}}(\Lambda)$ and the multiplication by $\pi\varphi^{-s}(\pi)^{-1}$ induces an injective homomorphism $A^{(N)}_{\mathrm{crys}}(\Lambda)/\varphi^{-s}(\pi) \hookrightarrow A^{(N)}_{\mathrm{crys}}(\Lambda)/\pi$. The latter shows that the p-primary torsion part of $A^{(N)}_{\mathrm{crys}}(\Lambda)/\varphi^{-s}(\pi)$ is annihilated by p^N. This together with the regularity of $\varphi^{-s}(\pi)$ in $A^{(N)}_{\mathrm{crys}}(\Lambda)$ and $A^{(N)}_{\mathrm{crys}}(\Lambda) = \varprojlim_m A^{(N)}_{\mathrm{crys}}(\Lambda)/p^m$ implies $A^{(N)}_{\mathrm{crys}}(\Lambda)/\varphi^{-s}(\pi) = \varprojlim_m (A^{(N)}_{\mathrm{crys}}(\Lambda)/\varphi^{-s}(\pi))/p^m$ by Lemma 157 below.

(3) Let $\mathcal{I}$ be the ideal $\sum_{l\in\mathbb{N}}[\underline{p}^{p^{-l}}]A_{\mathrm{inf}}(O_{\overline{K}})$. By (2), it suffices to prove the claim for the reduction mod p^m. By the description of $(A^{(N)}_{\mathrm{crys}}(\Lambda)/\pi)/p^m$ above and $\pi \in \varphi^{-s}(\pi)A_{\mathrm{inf}}(O_{\overline{K}})$, we are reduced to proving $(A_{\mathrm{inf}}(\Lambda)/\varphi^{-s}(\pi))/p^m[\mathcal{I}] = 0$. Since $A_{\mathrm{inf}}(\Lambda)/\varphi^{-s}(\pi)$ is p-torsion free (Lemma 1 (3)), it is further reduced to the case $m = 1$. We have isomorphisms $(A_{\mathrm{inf}}(\Lambda)/\varphi^{-s}(\pi))/p = R_\Lambda/(\underline{\varepsilon}^{p^{-s}} - 1) \xrightarrow{\cong} \Lambda/(\varepsilon_{s+1} - 1)$ induced by the projection to the second component, and the claim follows from Lemma 114. □

Lemma 157 *Let R be a commutative ring, and let a be an element of R.*

(1) *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules. Let N be a non-negative integer, and suppose $M_3[a^n] = M_3[a^N]$ for every integer $n \geq N$. Then the following sequence is exact*

$$0 \longrightarrow \varprojlim_n M_1/a^n \longrightarrow \varprojlim_n M_2/a^n \longrightarrow \varprojlim_n M_3/a^n \longrightarrow 0.$$

(2) *Under the same notation and assumption as* (1), *if two of the three R-modules M_i $(i \in \{1, 2, 3\})$ are a-adically complete and separated, so is the rest.*

Proof The claim (2) follows from (1) in the same way as the proof of Lemma 3 (2). By applying the snake lemma to the multiplication by a^n and a^{n+1} on the exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$, we obtain the following commutative diagram whose two lines are exact.

$$\begin{array}{ccccccccc} M_3[a^{n+1}] & \longrightarrow & M_1/a^{n+1} & \longrightarrow & M_2/a^{n+1} & \longrightarrow & M_3/a^{n+1} & \longrightarrow & 0 \\ \downarrow a & & \downarrow \mathrm{pr} & & \downarrow \mathrm{pr} & & \downarrow \mathrm{pr} & & \\ M_3[a^n] & \longrightarrow & M_1/a^n & \longrightarrow & M_2/a^n & \longrightarrow & M_3/a^n & \longrightarrow & 0 \end{array}$$

Let K_n be the kernel of $M_1/a^n \to M_2/a^n$ for $n \in \mathbb{N}_{>0}$. Then, as $M_3[a^{n+N}] \xrightarrow{a^N} M_3[a^n]$ vanishes for every $n \in \mathbb{N}_{>0}$ by assumption, we have $\varprojlim_n N_n = 0$ and $R^1 \varprojlim_n N_n = 0$. This implies the claim (1). □

We also introduce another completion of $W^{\mathrm{PD},(N)}(R_\Lambda)$ to have Proposition 164.

Definition 158 We define the period ring $A_{\mathrm{crys}}^{(N+)}(\Lambda)$ to be the inverse limit $\varprojlim_n W^{\mathrm{PD},(N)}(R_\Lambda)/I^n W^{\mathrm{PD},(N)}(R_\Lambda)$. We define a decreasing filtration by ideals $\mathrm{Fil}^r A_{\mathrm{crys}}^{(N+)}(\Lambda)$ (resp. $I^r A_{\mathrm{crys}}^{(N+)}(\Lambda)$) ($r \in \mathbb{Z}$) of $A_{\mathrm{crys}}^{(N+)}(\Lambda)$ to be the inverse limit of $\mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda)/I^n W^{\mathrm{PD},(N)}(R_\Lambda)$ (resp. $I^r W^{\mathrm{PD},(N)}(R_\Lambda)/I^n W^{\mathrm{PD},(N)}(R_\Lambda)$) ($n \in \mathbb{N} \cap [r, \infty)$). The Frobenius endomorphism φ of $W^{\mathrm{PD},(N)}(R_\Lambda)$ induces that of $A_{\mathrm{crys}}^{(N+)}(\Lambda)$, which is again denoted by φ.

It is obvious that the homomorphism $W^{\mathrm{PD},(N)}(R_\Lambda) \to A_{\mathrm{crys}}^{(N+)}(\Lambda)$ induces the following isomorphisms.

$$W^{\mathrm{PD},(N)}(R_\Lambda)/\mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda) \xrightarrow{\cong} A_{\mathrm{crys}}^{(N+)}(\Lambda)/\mathrm{Fil}^r A_{\mathrm{crys}}^{(N+)}(\Lambda) \quad (r \in \mathbb{Z}) \tag{156}$$

$$W^{\mathrm{PD},(N)}(R_\Lambda)/I^r W^{\mathrm{PD},(N)}(R_\Lambda) \xrightarrow{\cong} A_{\mathrm{crys}}^{(N+)}(\Lambda)/I^r A_{\mathrm{crys}}^{(N+)}(\Lambda) \quad (r \in \mathbb{Z}) \tag{157}$$

Lemma 159 (1) *For $r \in \mathbb{Z}$, we have an isomorphism*

$$\mathrm{Fil}^r A_{\mathrm{crys}}^{(N+)}(\Lambda) \xrightarrow{\cong} \varprojlim_{m,n \in \mathbb{N}, n \geq r} \mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda)/(I^n W^{\mathrm{PD},(N)}(R_\Lambda) + p^m \mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda)).$$

(2) $A_{\mathrm{crys}}^{(N+)}(\Lambda)$ *is p-torsion free, and p-adically complete and separated. For $m \in \mathbb{N}_{>0}$, we have an isomorphism*

$$A_{\mathrm{crys}}^{(N+)}(\Lambda)/p^m \xrightarrow{\cong} \varprojlim_{n \in \mathbb{N}} W^{\mathrm{PD},(N)}(R_\Lambda)/(I^n W^{\mathrm{PD},(N)}(R_\Lambda) + p^m W^{\mathrm{PD},(N)}(R_\Lambda)).$$

(3) *For $r \in \mathbb{Z}$, $I^r A_{\mathrm{crys}}^{(N+)}(\Lambda)$ coincides with the ideal of $A_{\mathrm{crys}}^{(N+)}(\Lambda)$ consisting of elements x such that $\varphi^n(x) \in \mathrm{Fil}^r A_{\mathrm{crys}}^{(N+)}(\Lambda)$ for all $n \in \mathbb{N}$.*

(4) *For $r, s \in \mathbb{Z}$ such that $s > r$, $I^r A_{\rm crys}^{(N+)}(\Lambda)$ and $I^r A_{\rm crys}^{(N+)}(\Lambda)/I^s A_{\rm crys}^{(N+)}(\Lambda)$ are p-adically complete and separated.*

Proof For $r, n \in \mathbb{N}$ with $r \geq n$, the quotient $\mathrm{Fil}^r W^{\mathrm{PD},(N)}(R_\Lambda)/I^n W^{\mathrm{PD},(N)}(R_\Lambda)$ is p-adically complete and separated by Lemmas 145 (3), 150 (3), and 3 (2). This implies the claim (1). We obtain the second claim of (2) and the p-torsion freeness of $A_{\rm crys}^{(N+)}(\Lambda)$ by taking the inverse limit of the exact sequence

$$0 \to \frac{W^{\mathrm{PD},(N)}(R_\Lambda)}{I^n W^{\mathrm{PD},(N)}(R_\Lambda)} \xrightarrow{p^m} \frac{W^{\mathrm{PD},(N)}(R_\Lambda)}{I^n W^{\mathrm{PD},(N)}(R_\Lambda)} \to \frac{W^{\mathrm{PD},(N)}(R_\Lambda)}{I^n W^{\mathrm{PD},(N)}(R_\Lambda)+p^m W^{\mathrm{PD},(N)}(R_\Lambda)} \to 0.$$

Then the claim (1) implies the remaining claim of (2). The claim (3) immediately follows from the definition of the filtrations I^r and Fil^r on $A_{\rm crys}^{(N+)}(\Lambda)$. We can derive the claim (4) from the first claim of (2), Lemma 150 (3), and (157) by using Lemma 3 (2). □

We endow $A_{\rm crys}^{(N+)}(\Lambda)$ with the topology induced by the inverse limit of the discrete topology of $W^{\mathrm{PD},(N)}(R_\Lambda)/(I^n W^{\mathrm{PD},(N)}(R_\Lambda) + p^m W^{\mathrm{PD},(N)}(R_\Lambda))$ via the isomorphism in Lemma 159 (1).

Lemma 160 (1) *The identity map of $W^{\mathrm{PD},(N)}(R_\Lambda)$ induces a continuous injective homomorphism $A_{\rm crys}^{(N)}(\Lambda) \to A_{\rm crys}^{(N+)}(\Lambda)$ compatible with φ and strictly compatible with the filtrations* Fil^r.

(2) *The inclusion map $W^{\mathrm{PD},(N)}(R_\Lambda) \to W^{\mathrm{PD},(N+1)}(R_\Lambda)$ induces a continuous injective homomorphism $A_{\rm crys}^{(N+)}(\Lambda) \to A_{\rm crys}^{(N+1)}(\Lambda)$ compatible with φ and strictly compatible with the filtrations* Fil^r.

Proof (1) By Lemma 159 (1) and the definition of the topology of $A_{\rm crys}^{(N+)}(\Lambda)$ above, we have a continuous homomorphism $A_{\rm crys}^{(N)}(\Lambda) \to A_{\rm crys}^{(N+)}(\Lambda)$ compatible with φ and the filtrations. We see that it is strictly compatible with the filtrations by (156) and (153), and then it is injective by Lemma 153.

(2) For $b = a_N p^N + a_{N-1} p^{N-1} + \cdots + a_1 p + a_0$ ($a_N \in \mathbb{N}, a_1, \ldots, a_{N-1} \in \mathbb{N} \cap [0, p-1]$), we have $\{b\}_{(N)} - \{b\}_{(N-1)} = a_N(\frac{p^N-1}{p-1} - p\frac{p^{N-1}-1}{p-1}) = a_N$ by Lemma 140 (2). Hence, for $m \in \mathbb{N}$, we have $\{b\}_{(N)} - \{b\}_{(N-1)} \geq m$ if $b \geq mp^N$. By Lemma 150 (1), this implies $I^{mp^N(p-1)} W^{\mathrm{PD},(N)}(R_\Lambda) \subset p^m W^{\mathrm{PD},(N+1)}(R_\Lambda)$. Hence we have a continuous map compatible with φ and the filtrations as in the claim. It is strictly compatible with the filtrations Fil^r by (156) and (153). Then, by Lemma 159 (3), the kernel is contained in the intersection of $I^r A_{\rm crys}^{(N+)}(\Lambda)$ ($r \in \mathbb{N}$), which is 0 by the definition of $A_{\rm crys}^{(N+)}(\Lambda)$ and the filtration $I^r A_{\rm crys}^{(N+)}(\Lambda)$ ($r \in \mathbb{N}$). □

Lemma 161 *The topology of $A_{\rm crys}^{(N+)}(\Lambda)$ is induced by the p-adic topology of $A_{\rm crys}(\Lambda)$ via the injective homomorphism $A_{\rm crys}^{(N+)}(\Lambda) \to A_{\rm crys}(\Lambda)$ (Lemma 160 (2) and Corollary 154).*

Proof By the isomorphism $W^{\mathrm{PD}}(R_\Lambda)/p^m \xrightarrow{\cong} A_{\rm crys}(\Lambda)/p^m$, it suffices to prove that the topology of $W^{\mathrm{PD},(N)}(R_\Lambda)$ induced by the p-adic topology of $W^{\mathrm{PD}}(R_\Lambda)$ is the same as that defined by $p^n W^{\mathrm{PD},(N)}(R_\Lambda) + I^l W^{\mathrm{PD},(N)}(R_\Lambda)$ $(n, l \in \mathbb{N}_{>0})$. By the proof of Lemma 160 (2), the kernel of $W^{\mathrm{PD},(N)}(R_\Lambda) \to W^{\mathrm{PD}}(R_\Lambda)/p^m$ contains $p^m W^{\mathrm{PD},(N)}(R_\Lambda) + I^{mp^N(p-1)} W^{\mathrm{PD},(N)}(R_\Lambda)$.

Put $I^r W^{\mathrm{PD}}(R_\Lambda) := I^r(W(R_\Lambda)[p^{-1}]) \cap W^{\mathrm{PD}}(R_\Lambda)$ $(r \in \mathbb{Z})$. Let $m \in \mathbb{N}_{>0}$, and put $l = m(p-1)^2$. Since $W^{\mathrm{PD}}(R_\Lambda)$ is the union of $W^{\mathrm{PD},(N)}(R_\Lambda)$ $(N \in \mathbb{N}_{>0})$, Lemma 150 (1) shows that $W^{\mathrm{PD}}(R_\Lambda)/I^l$ is generated by $\pi^a(\frac{\pi^{p-1}}{p})^{[b]}$ as an $W(R_\Lambda)$-module for $a \in \mathbb{N} \cap [0, p-2]$ and $b \in \mathbb{N}$ such that $a + (p-1)b \le l - 1$. Since $v_p(b!) \le \frac{b}{p-1} \le m$ for b as above, we obtain $p^m(W^{\mathrm{PD}}(R_\Lambda)/I^l) \subset W^{\mathrm{PD},(N)}(R_\Lambda)/I^l$. Hence the multiplication by p^m on $W^{\mathrm{PD}}(R_\Lambda)/I^l$ induces a homomorphism $f_m \colon W^{\mathrm{PD}}(R_\Lambda)/I^l \to W^{\mathrm{PD},(N)}(R_\Lambda)/I^l$, whose composition with the homomorphism $W^{\mathrm{PD},(N)}(R_\Lambda)/I^l \hookrightarrow W^{\mathrm{PD}}(R_\Lambda)/I^l$ is the multiplication by p^m. The kernel of $W^{\mathrm{PD},(N)}(R_\Lambda) \to W^{\mathrm{PD}}(R_\Lambda)/p^{2m}$ is contained in the kernel of the composition of

$$W^{\mathrm{PD},(N)}(R_\Lambda) \to (W^{\mathrm{PD}}(R_\Lambda)/I^l)/p^{2m} \xrightarrow{f_m} (W^{\mathrm{PD},(N)}(R_\Lambda)/I^l)/p^{2m},$$

which is equal to the composition of

$$W^{\mathrm{PD},(N)}(R_\Lambda) \to (W^{\mathrm{PD},(N)}(R_\Lambda)/I^l)/p^{2m} \xrightarrow{p^m} (W^{\mathrm{PD},(N)}(R_\Lambda)/I^l)/p^{2m}.$$

The latter kernel is $I^l W^{\mathrm{PD},(N)}(R_\Lambda) + p^m W^{\mathrm{PD},(N)}(R_\Lambda)$ because $W^{\mathrm{PD},(N)}(R_\Lambda)/I^l$ is p-torsion free. This completes the proof. □

Let Λ_0 be a subring of Λ such that Λ is integral over Λ_0 and $\mathrm{Frac}(\Lambda)/\mathrm{Frac}(\Lambda_0)$ is a Galois extension. Let $G(\Lambda/\Lambda_0)$ denote the Galois group of $\mathrm{Frac}(\Lambda)$ over $\mathrm{Frac}(\Lambda_0)$. Then Λ is a $G(\Lambda/\Lambda_0)$-stable subalgebra of $\mathrm{Frac}(\Lambda)$. Therefore the group $G(\Lambda/\Lambda_0)$ naturally acts on $A_{\mathrm{crys}}^{(N)}(\Lambda)$ and $A_{\mathrm{crys}}^{(N+)}(\Lambda)$ with φ and Fil^r.

Proposition 162 *The actions of $G(\Lambda/\Lambda_0)$ on $A_{\mathrm{crys}}^{(N)}(\Lambda)$ and on $A_{\mathrm{crys}}^{(N+)}(\Lambda)$ are continuous.*

Proof It suffices to prove that the action of $G(\Lambda/\Lambda_0)$ on $W^{\mathrm{PD},(N)}(R_\Lambda)/p^m$ with the discrete topology is continuous. By Corollary 142, $W^{\mathrm{PD},(N)}(R_\Lambda)/p^m$ is a $W(R_\Lambda)/(p^m, [\underline{p}]^{pm})$-algebra generated by the image of $[\underline{p}]^{[p^l]}$ $(l \in \mathbb{N} \cap [1, N])$. Hence the proposition follows from Lemmas 5 and 163 below. □

Lemma 163 *For any $n \in \mathbb{N}$ and $a \in p^n\mathbb{Z}_p$, we have $[\underline{\varepsilon}^a] - 1 \in p^n W^{\mathrm{PD},(1)}(R_\Lambda)$.*

Proof Put $b := p^{-n}a \in \mathbb{Z}_p$ and $x := [\underline{\varepsilon}^b] - 1 \in \mathrm{Fil}^1 W(R_\Lambda)$. Then we have $[\underline{\varepsilon}^{pb}] - 1 = \sum_{\nu=1}^{p-1} \binom{p}{\nu} x^\nu + p! x^{[p]} \in p W^{\mathrm{PD},(1)}(R_\Lambda)$. We obtain $[\underline{\varepsilon}^{p^n b}] - 1 \in p^n W^{\mathrm{PD},(1)}(R_\Lambda)$ by induction on n. □

Since $W^{\mathrm{PD},(N)}(R_\Lambda)/I^m$ $(m \in \mathbb{N})$ are p-torsion free, we have $p^r A_{\mathrm{inf}}^{(N+)}(\Lambda) \cong \varprojlim_m p^r(W^{\mathrm{PD},(N)}(R_\Lambda)/I^m)$ for $r \in \mathbb{N}$. Hence by Corollary 146, we have

$$\varphi(\mathrm{Fil}^r A_{\mathrm{crys}}^{(N+)}(\Lambda)) \subset p^r A_{\mathrm{crys}}^{(N+)}(\Lambda) \quad (r \in \mathbb{N} \cap [0, p-1]). \tag{158}$$

Hence the quadruplet $(A_{\mathrm{crys}}^{(N+)}(\Lambda), p, \varphi, \mathrm{Fil}^r A_{\mathrm{crys}}^{(N+)}(\Lambda))$ satisfies Condition 39 for $a = p-2$, and the action of $G(\Lambda/\Lambda_0)$ on it satisfies the conditions before Definitions 48 and 53 by Proposition 162. Therefore we may apply Definitions 40 and 53 to this quadruplet with $G(\Lambda/\Lambda_0)$-action. We have $q = p(1+\frac{\pi_0}{p})$ and $(\frac{\pi_0}{p})^n \in I^{n(p-1)}W^{\mathrm{PD},(1)}(R_\Lambda)$ $(n \in \mathbb{N})$ by Lemma 9 (2). This implies $1+\frac{\pi_0}{p} \in A_{\mathrm{crys}}^{(N+)}(\Lambda)^\times$. Therefore, by applying the construction of (44) and (47) to the two continuous homomorphisms $A_{\mathrm{inf}}(\Lambda) \to A_{\mathrm{crys}}^{(N+)}(\Lambda)$ and $A_{\mathrm{crys}}^{(N+)}(\Lambda) \to A_{\mathrm{crys}}(\Lambda)$ (Lemma 160, (150)) and composing with the isomorphism (49) (resp. (53)) for the first (resp. third) one, we obtain the following four functors, where $G = G(\Lambda/\Lambda_0)$.

$$\mathrm{M}^q_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\Lambda), \varphi) \longrightarrow \mathrm{MF}^p_{[0,p-2],\mathrm{free}}(A_{\mathrm{crys}}^{(N+)}(\Lambda), \varphi), \tag{159}$$

$$\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(A_{\mathrm{crys}}^{(N+)}(\Lambda), \varphi) \longrightarrow \mathrm{MF}^p_{[0,p-2],\mathrm{free}}(A_{\mathrm{crys}}(\Lambda), \varphi), \tag{160}$$

$$\mathrm{M}^{q,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\Lambda), \varphi, G) \longrightarrow \mathrm{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A_{\mathrm{crys}}^{(N+)}(\Lambda), \varphi, G), \tag{161}$$

$$\mathrm{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A_{\mathrm{crys}}^{(N+)}(\Lambda), \varphi, G) \longrightarrow \mathrm{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A_{\mathrm{crys}}(\Lambda), \varphi, G). \tag{162}$$

We need the following proposition in the proof of Proposition 181.

Proposition 164 *The functors (159)–(162) are equivalences of categories.*

Proof Let $\underline{A}_{\mathrm{crys}}^{(N+)}(\Lambda)$ be $A_{\mathrm{crys}}^{(N+)}(\Lambda)/I^{p-1}A_{\mathrm{crys}}^{(N+)}(\Lambda)$ with φ, Fil^r and $G(\Lambda/\Lambda_0)$-action induced by those of $A_{\mathrm{crys}}^{(N+)}(\Lambda)$. We endow $\underline{A}_{\mathrm{crys}}^{(N+)}(\Lambda)$ with the quotient topology, which coincides with the p-adic topology. Then we have the following commutative diagram of topological rings with φ, Fil^r and continuous $G(\Lambda/\Lambda_0)$-action.

$$\begin{array}{ccccc} A_{\mathrm{inf}}(\Lambda) & \longrightarrow & A_{\mathrm{crys}}^{(N+)}(\Lambda) & \longrightarrow & A_{\mathrm{crys}}(\Lambda) \\ \downarrow & & \downarrow & & \downarrow \\ \underline{A}_{\mathrm{inf}}(\Lambda) & \longrightarrow & \underline{A}_{\mathrm{crys}}^{(N+)}(\Lambda) & \longrightarrow & \underline{A}_{\mathrm{crys}}(\Lambda) \end{array}$$

The bottom left (resp. right) homomorphism is well-defined and injective by (157) and (148) (resp. Lemmas 159 (3), 160 (2), and 152). Since the composition of them is a filtered isomorphism by (2), each of them is also a filtered isomorphism. Hence, by Propositions 59, 44, and 56 together with Remark 55 (2), it suffices to show that the homomorphism $A_{\mathrm{crys}}^{(N+)}(\Lambda) \to \underline{A}_{\mathrm{crys}}^{(N+)}(\Lambda)$ satisfies Condition 54 for $a = p-2$. We abbreviate $A_{\mathrm{crys}}^{(N+)}(\Lambda)$ to $A_{\mathrm{crys}}^{(N+)}$ in the following. The conditions (a) (iii) and (a) (iv) are obviously satisfied, and the conditions (a) (ii) and (a) (v) follow from (157) and (158) with $r = p-1$, respectively. The condition (a) (i) holds because $A_{\mathrm{crys}}^{(N+)} \cong \varprojlim_n A_{\mathrm{crys}}^{(N+)}/I^n A_{\mathrm{crys}}^{(N+)}$ by (157) and $I^n \cdot I^m \subset I^{n+m}$ on $A_{\mathrm{crys}}^{(N+)}$. The conditions (b) and (e) are obvious. The ideals $I_n := p^n A_{\mathrm{crys}}^{(N+)} + I^{n+(p-1)}A_{\mathrm{crys}}^{(N+)}$ $(n \in \mathbb{N})$ satisfy (d-1) and (d-2). We have $I_n \cap I^{p-1}A_{\mathrm{crys}}^{(N+)} = p^n I^{p-1}A_{\mathrm{crys}}^{(N+)} + I^{n+(p-1)}A_{\mathrm{crys}}^{(N+)}$ and $\varphi(I^{n+(p-1)}A_{\mathrm{crys}}^{(N+)}) \subset p^{p-1}(I^{n+(p-1)}A_{\mathrm{crys}}^{(N+)})$ because $A_{\mathrm{crys}}^{(N+)}/I^{p-1}A_{\mathrm{crys}}^{(N+)}$ and $A_{\mathrm{crys}}^{(N+)}/I^{n+(p-1)}A_{\mathrm{crys}}^{(N+)}$ are p-torsion free. Therefore the ideals I_n also satisfy (d-3). By Lemma 159 (4), we also obtain

$$\begin{aligned}
&\varprojlim_{n}(I^{p-1}A_{\mathrm{crys}}^{(N+)}/(I_n\cap I^{p-1}A_{\mathrm{crys}}^{(N+)}))\\
&\cong\varprojlim_{m,n}(I^{p-1}A_{\mathrm{crys}}^{(N+)}/(p^mI^{p-1}A_{\mathrm{crys}}^{(N+)}+I^{n+(p-1)}A_{\mathrm{crys}}^{(N+)}))\\
&\cong\varprojlim_{n} I^{p-1}A_{\mathrm{crys}}^{(N+)}/I^{n+(p-1)}A_{\mathrm{crys}}^{(N+)}\cong I^{p-1}A_{\mathrm{crys}}^{(N+)}.
\end{aligned}$$

Thus the condition (c) holds. The sufficient condition for (f) and (g) given in Remark 55 (1) holds because $pA_{\mathrm{crys}}^{(N+)}\cap(p^{m+1}A_{\mathrm{crys}}^{(N+)}+I^nA_{\mathrm{crys}}^{(N+)})=p(p^mA_{\mathrm{crys}}^{(N+)}+I^nA_{\mathrm{crys}}^{(N+)})$ for $n,m\in\mathbb{N}$. □

18 Period Rings with Truncated Divided Powers Associated to a Framing

We follow the notation in Sects. 12 and 17. We introduce and study period rings with truncated divided powers associated to a framing □. Recall that we have introduced a compatible system of étale homomorphisms $A_{\mathrm{inf},\mathfrak{a}}(O_{\overline{K}})[\underline{U}^{\pm1}]\to A_{\mathrm{inf},\mathfrak{a}}^{\square}(\mathcal{A})$ ($\mathfrak{a}\in\mathscr{S}_{\mathrm{inf}}$) with φ and an action of $\widetilde{\Gamma}_{\mathcal{A}}$. Let $\mathscr{S}_{\mathrm{crys}}^{(N)}$ denote the set of ideals $\mathfrak{a}$ of $A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})$ satisfying $p^n\in\mathfrak{a}\subset pA_{\mathrm{crys}}^{(N)}(O_{\overline{K}})+\mathrm{Fil}^1A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})$ for some $n\in\mathbb{N}_{>0}$. For $\mathfrak{a}\in\mathscr{S}_{\mathrm{crys}}^{(N)}$, we define $A_{\mathrm{crys},\mathfrak{a}}^{(N)}(\Lambda)$ to be the quotient $A_{\mathrm{crys}}^{(N)}(\Lambda)/\mathfrak{a}$, and $A_{\mathrm{crys},\mathfrak{a}}^{\square,(N)}(\mathcal{A})$ to be $A_{\mathrm{inf}}^{\square}(\mathcal{A})\otimes_{A_{\mathrm{inf}}(O_{\overline{K}})}A_{\mathrm{crys},\mathfrak{a}}^{(N)}(O_{\overline{K}})$. For $g\in\widetilde{\Gamma}_{\mathcal{A}}$ and $\mathfrak{a}\in\mathscr{S}_{\mathrm{crys}}^{(N)}$, the isomorphism $\rho^{\square}(g)\colon A_{\mathrm{inf}}^{\square}(\mathcal{A})\xrightarrow{\cong}A_{\mathrm{inf}}^{\square}(\mathcal{A})$ and the action of g on $A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})$ induce an isomorphism $\rho_{\mathfrak{a}}^{\square}(g)\colon A_{\mathrm{crys},\mathfrak{a}}^{\square,(N)}(\mathcal{A})\xrightarrow{\cong}A_{\mathrm{crys},g(\mathfrak{a})}^{\square,(N)}(\mathcal{A})$ satisfying $\rho_{\mathfrak{a}}^{\square}(1)=\mathrm{id}$ and $\rho_{h(\mathfrak{a})}^{\square}(g)\circ\rho_{\mathfrak{a}}^{\square}(h)=\rho_{\mathfrak{a}}^{\square}(gh)$. For $\mathfrak{a}\in\mathscr{S}_{\mathrm{crys}}^{(N)}$, the endomorphisms φ of $A_{\mathrm{inf}}^{\square}(\mathcal{A})$ and $A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})$ induce a homomorphism $\varphi_{\mathfrak{a}}^{\square}\colon A_{\mathrm{crys},\mathfrak{a}}^{\square,(N)}(\mathcal{A})\to A_{\mathrm{crys},\varphi(\mathfrak{a})}^{\square,(N)}(\mathcal{A})$ satisfying $\varphi_{g(\mathfrak{a})}^{\square}\circ\rho_{\mathfrak{a}}^{\square}(g)=\rho_{\varphi(\mathfrak{a})}^{\square}(g)\circ\varphi_{\mathfrak{a}}^{\square}$ for $g\in\widetilde{\Gamma}_{\mathcal{A}}$. For $\mathfrak{a},\mathfrak{b}\in\mathscr{S}_{\mathrm{crys}}^{(N)}$ with $\mathfrak{b}\subset\mathfrak{a}$, we have a canonical isomorphism

$$A_{\mathrm{crys},\mathfrak{b}}^{\square,(N)}(\mathcal{A})\otimes_{A_{\mathrm{crys},\mathfrak{b}}^{(N)}(O_{\overline{K}})}A_{\mathrm{crys},\mathfrak{a}}^{(N)}(O_{\overline{K}})\xrightarrow{\cong}A_{\mathrm{crys},\mathfrak{a}}^{\square,(N)}(\mathcal{A})\tag{163}$$

compatible with $\varphi_{\star}^{\square}$ and the action of $\widetilde{\Gamma}_{\mathcal{A}}$ in the obvious sense. It is also compatible with the composition for $\mathfrak{a},\mathfrak{b},\mathfrak{c}\in\mathscr{S}_{\mathrm{crys}}^{(N)}$ with $\mathfrak{c}\subset\mathfrak{b}\subset\mathfrak{a}$. Let $\mathfrak{a}\in\mathscr{S}_{\mathrm{crys}}^{(N)}$, and let n be a positive integer such that $p^n\in\mathfrak{a}$. Then, since $[\underline{p}]^{pn}\in p^nA_{\mathrm{crys}}^{(N)}(O_{\overline{K}})$ by Corollary 142, we obtain the following canonical isomorphism from Lemma 79 (2).

$$A_{\mathrm{inf},(p^n,[\underline{p}]^{pn})}^{\square}(\mathcal{A})\otimes_{A_{\mathrm{inf},(p^n,[\underline{p}]^{pn})}(O_{\overline{K}})}A_{\mathrm{crys},\mathfrak{a}}^{(N)}(O_{\overline{K}})\xrightarrow{\cong}A_{\mathrm{crys},\mathfrak{a}}^{\square,(N)}(\mathcal{A})\tag{164}$$

In particular, this implies that $A_{\mathrm{crys},\mathfrak{a}}^{\square,(N)}(\mathcal{A})$ is smooth over $A_{\mathrm{crys},\mathfrak{a}}^{(N)}(O_{\overline{K}})$.

We define $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})$ to be the inverse limit of $A^{\square,(N)}_{\mathrm{crys},\mathfrak{a}}(\mathcal{A})$ ($\mathfrak{a} \in \mathscr{S}^{(N)}_{\mathrm{crys}}$) endowed with the inverse limit topology of the discrete topology of $A^{\square,(N)}_{\mathrm{crys},\mathfrak{a}}(\mathcal{A})$. The isomorphisms $\rho^{\square}_{\mathfrak{a}}(g)$ and the homomorphisms $\varphi^{\square}_{\mathfrak{a}}$ induce the action of $\widetilde{\Gamma}_{\mathcal{A}}$ on $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})$ and the endomorphism φ of $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})$.

Lemma 165 *The action of $\widetilde{\Gamma}_{\mathcal{A}}$ on $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})$ is continuous.*

Proof This follows from (164), Lemma 86, and Proposition 162. □

Lemma 166 (1) *$A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})$ is p-torsion free, and p-adically complete and separated.*

(2) *For $\mathfrak{a} \in \mathscr{S}^{(N)}_{\mathrm{crys}}$, the homomorphism $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})/\mathfrak{a} \to A^{\square,(N)}_{\mathrm{crys},\mathfrak{a}}(\mathcal{A})$ is an isomorphism.*

Proof Since $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$ is p-torsion free, and $A^{(N)}_{\mathrm{crys},\mathfrak{a}}(\mathcal{A})$ ($\mathfrak{a} \in \mathscr{S}^{(N)}_{\mathrm{crys}}$) are flat over $A^{(N)}_{\mathrm{crys},\mathfrak{a}}(O_{\overline{K}})$, we obtain exact sequences $0 \to A^{\square,(N)}_{\mathrm{crys},(p^l)}(\mathcal{A}) \xrightarrow{p^m} A^{\square,(N)}_{\mathrm{crys},(p^{m+l})}(\mathcal{A}) \to A^{\square,(N)}_{\mathrm{crys},(p^m)}(\mathcal{A}) \to 0$ for $m, l \in \mathbb{N}_{>0}$ by using (163). By taking the inverse limit over l, we obtain exact sequences $0 \to A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A}) \xrightarrow{p^m} A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A}) \to A^{\square,(N)}_{\mathrm{crys},(p^m)}(\mathcal{A}) \to 0$ for $m \in \mathbb{N}_{>0}$. This completes the proof because $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})$ is the inverse limit of $A^{\square,(N)}_{\mathrm{crys},(p^m)}(\mathcal{A})$ ($m \in \mathbb{N}_{>0}$) by definition and, for any $\mathfrak{a} \in \mathscr{S}^{(N)}_{\mathrm{crys}}$, there exists $m \in \mathbb{N}_{>0}$ such that $p^m \in \mathfrak{a}$. □

Remark 167 Lemma 166 (2) implies that the topology of $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})$ coincides with the p-adic topology.

Proposition 168 (1) *For $s \in \mathbb{N}$, the $A_{\mathrm{inf}}(O_{\overline{K}})$-algebra $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})$ is $\varphi^{-s}(\pi)$-torsion free.*

(2) *For $s \in \mathbb{N}$, the quotient $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})/\varphi^{-s}(\pi)A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})$ is p-adically complete and separated, and its p-primary torsion part is annihilated by p^N.*

(3) *The action of $\Gamma_{\mathcal{A}}$ on $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})/\pi A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})$ is trivial.*

Proof By Lemmas 79 (1) and 80 (4), $A^{\square}_{\mathrm{inf}}(\mathcal{A})$ is p-torsion free, and $A^{\square}_{\mathrm{inf}}(\mathcal{A})/p$ is π-torsion free and π-adically complete and separated. Hence the homomorphism $\mathbb{Z}/p^m[T] \to A^{\square}_{\mathrm{inf}}(\mathcal{A})/p^m; T \mapsto \pi$ is flat by Lemma 144 (1). On the other hand, by Lemma 166 (2) and Proposition 149 (2), we have an isomorphism

$$A^{\square}_{\mathrm{inf}}(\mathcal{A})/p^m \otimes_{\mathbb{Z}_p[T]/p^m} (\mathbb{Z}_p[T]\langle \tfrac{T^{p-1}}{p}\rangle^{(N-1)})/p^m \xrightarrow{\cong} A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})/p^m.$$

Therefore one can prove the claims (1) and (2) by the same argument as the proof of Proposition 156 (1) and (2). Note that $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A}) \cong \varprojlim_m A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})/p^m$ by Lemma 166 (1) and $A^{\square}_{\mathrm{inf}}(\mathcal{A})/\pi$ is p-torsion free by Lemma 80 (3). By Lemma 88 and the above description of $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})/p^m$, we see that the action of $\Gamma_{\mathcal{A}}$ on $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})/\pi = \varprojlim_m (A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})/\pi)/p^m$ is trivial. □

For an ideal J of $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$, we define the ideal $\overline{J}A^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A})$ of $A^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A})$ to be the topological closure of $JA^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A})$ in $A^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A})$, which is the inverse limit of $J(A^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A})/p^m)$ ($m\in\mathbb{N}_{>0}$) by Lemma 166.

Lemma 169 *Let J be an ideal of $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$ such that $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})/J$ is p-torsion free and $J\subset \mathrm{Fil}^1A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$. Then $A^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A})/\overline{J}A^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A})$ is p-torsion free, and p-adically complete and separated. Moreover the natural homomorphism $(A^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A})/\overline{J}A^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A}))/p^m\to A^{\Box,(N)}_{\mathrm{crys},(p^m,J)}(\mathcal{A})$ is an isomorphism for $m\in\mathbb{N}_{>0}$.*

Proof We can prove the lemma in the same way as Lemma 81 by using Lemma 166, (163), and the flatness of $A^{(N)}_{\mathrm{crys},\mathfrak{a}}(O_{\overline{K}})\to A^{\Box,(N)}_{\mathrm{crys},\mathfrak{a}}(\mathcal{A})$ ($\mathfrak{a}\in\mathscr{S}^{(N)}_{\mathrm{crys}}$). □

For $r\in\mathbb{N}_{>0}$, $\mathrm{Fil}^rA^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$ satisfies the assumption on J in Lemma 169 by Lemma 145 (3) and (153). We define $\mathrm{Fil}^rA^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A})$ to be the closure of $\mathrm{Fil}^rA^{(N)}_{\mathrm{crys}}(O_{\overline{K}})A^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A})$ in $A^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A})$. We put $\mathrm{Fil}^rA^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A})=A^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A})$ for $r\in\mathbb{Z}$, $r\leq 0$.

We have natural homomorphisms

$$A^{\Box}_{\inf}(\mathcal{A})\to A^{\Box,(1)}_{\mathrm{crys}}(\mathcal{A})\to A^{\Box,(2)}_{\mathrm{crys}}(\mathcal{A})\to\cdots A^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A})\to A^{\Box,(N+1)}_{\mathrm{crys}}(\mathcal{A})\to\cdots \tag{165}$$

compatible with with the filtrations, φ and the actions of $\widetilde{\Gamma}_{\mathcal{A}}$. The natural homomorphism $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\to A_{\mathrm{crys}}(O_{\overline{K}})$ (150) and the isomorphisms (164) and (74) induce homomorphisms $A^{\Box,(N)}_{\mathrm{crys},\mathfrak{a}}(\mathcal{A})\to A^{\Box}_{\mathrm{crys},\mathfrak{a}'}(\mathcal{A})$ ($\mathfrak{a}\in\mathscr{S}^{(N)}_{\mathrm{crys}}$, $\mathfrak{a}'=\mathfrak{a}A_{\mathrm{crys}}(O_{\overline{K}})\in\mathscr{S}_{\mathrm{crys}}$) compatible with φ and the actions of $\widetilde{\Gamma}_{\mathcal{A}}$. By taking the inverse limit over $\mathfrak{a}\in\mathscr{S}^{(N)}_{\mathrm{crys}}$, we obtain a homomorphism

$$A^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A})\longrightarrow A^{\Box}_{\mathrm{crys}}(\mathcal{A}) \tag{166}$$

compatible with the filtrations $\mathrm{Fil}^\bullet$, φ, the actions of $\widetilde{\Gamma}_{\mathcal{A}}$ and the homomorphisms (165) and (75).

The homomorphism $\iota^{\Box}_{\infty}\colon A^{\Box}_{\inf}(\mathcal{A})\to A_{\inf}(\mathcal{A}_\infty)$ (77) and the homomorphisms $A^{(N)}_{\mathrm{crys},\mathfrak{a}}(O_{\overline{K}})\to A^{(N)}_{\mathrm{crys},\mathfrak{a}}(\mathcal{A}_\infty)$ ($\mathfrak{a}\in\mathscr{S}^{(N)}_{\mathrm{crys}}$) induce an inverse system of homomorphisms $\iota^{\Box,(N)}_{\mathfrak{a},\infty}\colon A^{\Box,(N)}_{\mathrm{crys},\mathfrak{a}}(\mathcal{A})\to A^{(N)}_{\mathrm{crys},\mathfrak{a}}(\mathcal{A}_\infty)$ ($\mathfrak{a}\in\mathscr{S}^{(N)}_{\mathrm{crys}}$) and then a homomorphism $\iota^{\Box,(N)}_{\infty}\colon A^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A})\to A^{(N)}_{\mathrm{crys}}(\mathcal{A}_\infty)$, which is compatible with φ, the filtrations $\mathrm{Fil}^\bullet$, and the actions of $\widetilde{\Gamma}_{\mathcal{A}}$. We also see that the homomorphism $\iota^{\Box,(N)}_{\infty}$ and $\iota^{\Box}_{\infty}\colon A^{\Box}_{\mathrm{crys}}(\mathcal{A})\to A_{\mathrm{crys}}(\mathcal{A}_\infty)$ (77) are compatible with (166) and $A^{(N)}_{\mathrm{crys}}(\mathcal{A}_\infty)\to A_{\mathrm{crys}}(\mathcal{A}_\infty)$ (150).

Proposition 170 (1) *The homomorphism $\iota^{\Box,(N)}_{\mathfrak{a},\infty}\colon A^{\Box,(N)}_{\mathrm{crys},\mathfrak{a}}(\mathcal{A})\to A^{(N)}_{\mathrm{crys},\mathfrak{a}}(\mathcal{A}_\infty)$ is injective for any $\mathfrak{a}\in\mathscr{S}^{(N)}_{\mathrm{crys}}$.*

(2) *The homomorphism $A^{\Box,(N)}_{\mathrm{crys}}(\mathcal{A})/\mathrm{Fil}^r\to A^{(N)}_{\mathrm{crys}}(\mathcal{A}_\infty)/\mathrm{Fil}^r$ and its reduction mod p^m ($m\in\mathbb{N}_{>0}$) are injective.*

(3) *With the notation introduced before Lemma 84, $A^{(N)}_{\mathrm{crys},\mathfrak{a}}(\mathcal{A}_\infty)$ is a free $A^{\Box,(N)}_{\mathrm{crys},\mathfrak{a}}(\mathcal{A})$-module with a basis $[\underline{t}^{\underline{r}}]$ ($\underline{r}\in(\mathbb{Z}[\frac{1}{p}]\cap[0,1[)^d$) for every $\mathfrak{a}\in\mathscr{S}^{(N)}_{\mathrm{crys}}$.*

Proof The claim (1) is an immediate consequence of (3). The claim (3) follows from (164), $A_{\mathrm{inf}}(\mathcal{A}_\infty)\otimes_{A_{\mathrm{inf}}(O_{\overline{K}})} A^{(N)}_{\mathrm{crys},\mathfrak{a}}(O_{\overline{K}})\xrightarrow{\cong} A^{(N)}_{\mathrm{crys},\mathfrak{a}}(\mathcal{A}_\infty)$ (Proposition 143 (2)), and Lemma 84. Let us prove (2). The quotients $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})/\mathrm{Fil}^r$ and $A^{(N)}_{\mathrm{crys}}(\mathcal{A}_\infty)/\mathrm{Fil}^r$ are p-torsion free and p-adically complete and separated by Lemma 169, (153), and Lemma 145 (3). Their reduction mod p^m are isomorphic to $A^{\square,(N)}_{\mathrm{crys},(p^m,\mathrm{Fil}^r)}(\mathcal{A})$ and $A^{(N)}_{\mathrm{crys},(p^m,\mathrm{Fil}^r)}(\mathcal{A}_\infty)$, respectively, by Lemmas 169 and 145 (1). Hence the claim follows from (1). □

Corollary 171 *For $s\in\mathbb{N}$, $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})/\varphi^{-s}(\pi)$ and $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})/(p^m,\varphi^{-s}(\pi))$ ($m\in\mathbb{N}_{>0}$) do not have a non-zero element annihilated by the ideal $\sum_{l\in\mathbb{N}}[\underline{p}^{p^{-l}}]A_{\mathrm{inf}}(O_{\overline{K}})$ of $A_{\mathrm{inf}}(O_{\overline{K}})$.*

Proof By Proposition 168 (2), it suffices to prove the claim for the reduction mod p^m. Since $(\varphi^{-s}(\pi),p^m)\supset(\pi,p^m)\in\mathscr{S}^{(N)}_{\mathrm{crys}}$, Lemma 166 (2) and Proposition 170 (3) imply that the homomorphism $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})/(\varphi^{-s}(\pi),p^m)\to A^{(N)}_{\mathrm{crys}}(\mathcal{A}_\infty)/(\varphi^{-s}(\pi),p^m)$ induced by $\iota^{\square,(N)}_\infty$ is injective. Thus the claim is reduced to Proposition 156 (3). □

We also introduce $A^{\square,(N+)}_{\mathrm{crys}}(\mathcal{A})$ and prove an analogue of Proposition 164 for filtered φ-modules. For $m,n\in\mathbb{N}_{>0}$, let $A^{(N)}_{\mathrm{crys},(I^n,p^m)}(O_{\overline{K}})$ denote the quotient $A^{(N+)}_{\mathrm{crys}}(O_{\overline{K}})/(I^nA^{(N+)}_{\mathrm{crys}}(O_{\overline{K}})+p^mA^{(N+)}_{\mathrm{crys}}(O_{\overline{K}}))$, and let $A^{\square,(N)}_{\mathrm{crys},(I^n,p^m)}(\mathcal{A})$ denote $A^{\square}_{\mathrm{inf}}(\mathcal{A})\otimes_{A_{\mathrm{inf}}(O_{\overline{K}})}A^{(N)}_{\mathrm{crys},(I^n,p^m)}(O_{\overline{K}})$. By Lemma 79 (2), $A^{\square,(N)}_{\mathrm{crys},(I^n,p^m)}(\mathcal{A})$ is isomorphic to $A^{\square}_{\mathrm{inf},(\pi^n,p^m)}(\mathcal{A})\otimes_{A_{\mathrm{inf},(\pi^n,p^m)}(O_{\overline{K}})}A^{(N)}_{\mathrm{crys},(I^n,p^m)}(O_{\overline{K}})$. We define the ring $A^{\square,(N+)}_{\mathrm{crys}}(\mathcal{A})$ to be the inverse limit of $A^{\square,(N)}_{\mathrm{crys},(I^n,p^m)}(\mathcal{A})$ ($m,n\in\mathbb{N}_{>0}$) endowed with the inverse limit of the discrete topologies. The algebra $A^{\square,(N+)}_{\mathrm{crys}}(\mathcal{A})$ is naturally endowed with an action of $\widetilde{\Gamma}_{\mathcal{A}}$, continuous by Corollary 86 and Proposition 162 for $(\Lambda,\Lambda_0)=(O_{\overline{K}},O_K)$, and an endomorphism φ. The homomorphisms $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})/p^m\to A^{(N)}_{\mathrm{crys},(I^n,p^m)}(O_{\overline{K}})$ and $A^{(N)}_{\mathrm{crys},(I^{mp^N(p-1)},p^m)}(O_{\overline{K}})\to A^{(N+1)}_{\mathrm{crys}}(O_{\overline{K}})/p^m$ (see the proof of Lemma 160 (2)) induce continuous homomorphisms

$$A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})\longrightarrow A^{\square,(N+)}_{\mathrm{crys}}(\mathcal{A})\longrightarrow A^{\square,(N+1)}_{\mathrm{crys}}(\mathcal{A}) \tag{167}$$

compatible with the action of $\widetilde{\Gamma}_{\mathcal{A}}$ and φ. We define the filtrations $\mathrm{Fil}^rA^{\square,(N+)}_{\mathrm{crys}}(\mathcal{A})$ and $I^rA^{\square,(N+)}_{\mathrm{crys}}(\mathcal{A})$ to be the inverse limits of $\mathrm{Fil}^rA^{(N+)}_{\mathrm{crys}}(O_{\overline{K}})A^{\square,(N)}_{\mathrm{crys},(I^n,p^m)}(\mathcal{A})$ and $I^rA^{(N+)}_{\mathrm{crys}}(O_{\overline{K}})A^{\square,(N)}_{\mathrm{crys},(I^n,p^m)}(\mathcal{A})$, which are $\widetilde{\Gamma}_{\mathcal{A}}$-stable and satisfy $\mathrm{Fil}^r\cdot\mathrm{Fil}^s\subset\mathrm{Fil}^{r+s}$ and $I^r\cdot I^s\subset I^{r+s}$. The homomorphisms (167) are compatible with the filtrations $\mathrm{Fil}^\bullet$ by definition.

Since $A^{(N+)}_{\mathrm{crys}}(O_{\overline{K}})/I^nA^{(N+)}_{\mathrm{crys}}(O_{\overline{K}})$ is p-torsion free, we have an exact sequence

$$0\longrightarrow A^{\square,(N)}_{\mathrm{crys},(I^n,p^m)}(\mathcal{A})\xrightarrow{p^l}A^{\square,(N)}_{\mathrm{crys},(I^n,p^{m+l})}(\mathcal{A})\longrightarrow A^{\square,(N)}_{\mathrm{crys},(I^n,p^l)}(\mathcal{A})\longrightarrow 0.$$

By taking the inverse limit over m and n, we obtain an exact sequence

$$0 \longrightarrow A_{\rm crys}^{\square,(N+)}(\mathcal{A}) \xrightarrow{p^l} A_{\rm crys}^{\square,(N+)}(\mathcal{A}) \longrightarrow \varprojlim_n A_{{\rm crys},(I^n,p^l)}^{\square,(N)}(\mathcal{A}) \longrightarrow 0. \tag{168}$$

This exact sequence (168) for $l = r$ and Corollary 146 for $\Lambda = O_{\overline{K}}$ imply

$$\varphi({\rm Fil}^r A_{\rm crys}^{\square,(N+)}(\mathcal{A})) \subset p^r A_{\rm crys}^{\square,(N+)}(\mathcal{A}) \quad (r \in \mathbb{N} \cap [0, p-1]). \tag{169}$$

We can apply Definition 40 to $(A_{\rm crys}^{\square,(N+)}(\mathcal{A}), p, \varphi, {\rm Fil}^r A_{\rm crys}^{(N+)}(\mathcal{A}))$ and $a = p-2$, and obtain the category ${\rm MF}^p_{[0,p-2],{\rm free}}(A_{\rm crys}^{\square,(N+)}(\mathcal{A}), \varphi)$. Since $q \in p \cdot A_{\rm crys}^{(N+)}(O_{\overline{K}})^\times$ as observed before Proposition 164, we obtain the following two functors by applying the construction of (44) to $A_{\rm inf}^{\square}(\mathcal{A}) \to A_{\rm crys}^{\square,(N+)}(\mathcal{A}) \to A_{\rm crys}^{\square}(\mathcal{A})$ and composing with (98) for the first one.

$$\mathrm{M}^q_{[0,p-2],{\rm free}}(A_{\rm inf}^{\square}(\mathcal{A}), \varphi) \longrightarrow {\rm MF}^p_{[0,p-2],{\rm free}}(A_{\rm crys}^{\square,(N+)}(\mathcal{A}), \varphi), \tag{170}$$

$${\rm MF}^p_{[0,p-2],{\rm free}}(A_{\rm crys}^{\square,(N+)}(\mathcal{A}), \varphi) \longrightarrow {\rm MF}^p_{[0,p-2],{\rm free}}(A_{\rm crys}^{\square}(\mathcal{A}), \varphi). \tag{171}$$

Proposition 172 *The functors (170) and (171) are equivalences of categories.*

Proof We omit $(\mathcal{A})$ appearing in the notation $A_{\rm crys}^{\square,(N+)}(\mathcal{A})$, $A_{\rm crys}^{\square}(\mathcal{A})$, $A_{\rm inf}^{\square}(\mathcal{A})$ etc. to simplify the notation. We first show that we have a filtered isomorphism $A_{\rm inf}^{\square}/I^{p-1} \xrightarrow{\cong} A_{\rm crys}^{\square,(N+)}/I^{p-1}$. By Lemmas 159 (3), 160 (2), and 152, the homomorphism $A_{\rm crys}^{(N+)}(O_{\overline{K}}) \to A_{\rm crys}(O_{\overline{K}})$ is compatible with the filtrations $I^\bullet$. This implies that the homomorphism $A_{\rm crys}^{\square,(N+)} \to A_{\rm crys}^{\square}$ is also compatible with $I^\bullet$. We have an isomorphism $A^{\square}_{{\rm inf},(\pi^{p-1},p^m)} \xrightarrow{\cong} A^{\square,(N)}_{{\rm crys},(I^{p-1},p^m)}$ because $A_{\rm inf}(O_{\overline{K}})/I^{p-1} \xrightarrow{\cong} A_{\rm crys}^{(N+)}(O_{\overline{K}})/I^{p-1}$ by Lemma 150 (1), (148), and (157). By taking the inverse limit of the exact sequences

$$0 \to I^{p-1}A_{\rm crys}^{(N+)}(O_{\overline{K}})A^{\square,(N)}_{{\rm crys},(I^m,p^m)} \to A^{\square,(N)}_{{\rm crys},(I^m,p^m)} \to A^{\square,(N)}_{{\rm crys},(I^{p-1},p^m)} \to 0$$

for $m \geq p-1$, we obtain an isomorphism $A_{\rm crys}^{\square,(N+)}/I^{p-1} \xrightarrow{\cong} \varprojlim_m A^{\square,(N)}_{{\rm crys},(I^{p-1},p^m)}$. On the other hand, we have an isomorphism $A_{\rm inf}^{\square}/I^{p-1} \xrightarrow{\cong} \varprojlim_m A^{\square}_{{\rm inf},(\pi^{p-1},p^m)}$ by Lemma 80 (5). Hence we have $A_{\rm inf}^{\square}/I^{p-1} \xrightarrow{\cong} A_{\rm crys}^{\square,(N+)}/I^{p-1}$, which is a filtered isomorphism because so is its composition with $A_{\rm crys}^{\square,(N+)}/I^{p-1} \to A_{\rm crys}^{\square}/I^{p-1}$ by Lemma 82.

By Propositions 94 and 44, it remains to prove that the ideal $I^{p-1}A_{\rm crys}^{\square,(N+)}$ of $A_{\rm crys}^{\square,(N+)}$ satisfies Condition 43 with $a = p-2$ and $q = p$. The conditions (iii) and (iv) are obvious by the definition of $I^\bullet$ and ${\rm Fil}^\bullet$. The condition (v) follows from (169) and $I^{p-1} \subset {\rm Fil}^{p-1}$. The condition (ii) holds because $A_{\rm inf}^{\square}/I^{p-1}$ is p-torsion free (Lemma 80 (3)). The kernel of $A_{\rm crys}^{\square,(N+)} \to A^{\square,(N)}_{{\rm crys},(I,p)}$ is contained in the Jacobson radical because the kernel of $A^{\square,(N)}_{{\rm crys},(I^n,p^m)} \to A^{\square,(N)}_{{\rm crys},(I,p)}$ is nilpotent. Hence the condition (i) is satisfied. By (168), we see that $A_{\mathrm{crys}}^{\square,(N+)}$ is p-torsion free and p-adically complete and separated. By Lemma 80 (3), $A_{\mathrm{inf}}^{\square}/I^{p-1}A_{\mathrm{inf}}^{\square}$ is also p-adically complete and separated. This implies that $I^{p-1}A_{\mathrm{crys}}^{\square,(N+)}$ is p-adically complete and separated by Lemma 3 (2). Therefore the condition (vi) holds for $J_n = p^{n+1}I^{p-1}A_{\mathrm{crys}}^{\square,(N+)}$. □

Lemma 173 *The ring $A_{\mathrm{crys}}^{\square,(N+)}(\mathcal{A})$ is isomorphic to the inverse limit of the images of $A_{\mathrm{crys}}^{\square,(N+)}(\mathcal{A})$ in $A_{\mathrm{crys}}^{\square}(\mathcal{A})/p^m$ ($m \in \mathbb{N}$).*

Proof We have $A_{\mathrm{crys}}^{\square}(\mathcal{A})/p^m = A_{\mathrm{crys},(p^m)}^{\square}(\mathcal{A})$ (Lemma 79 (1)). Let $\mathfrak{a}_m$ be the kernel of $A_{\mathrm{crys}}^{(N+)}(O_{\overline{K}}) \to A_{\mathrm{crys}}(O_{\overline{K}})/p^m$. Then, by the proof of Lemma 161, for any $m \in \mathbb{N}_{>0}$, there exists $m' \in \mathbb{N}_{>0}$ such that we have the following maps of quotients of $A_{\mathrm{crys}}^{(N+)}(O_{\overline{K}})$.

$$A_{\mathrm{crys}}^{(N+)}(O_{\overline{K}})/\mathfrak{a}_{m'} \to A_{\mathrm{crys},(I^m,p^m)}^{(N+)}(O_{\overline{K}}), \quad A_{\mathrm{crys},(I^{m'},p^{m'})}^{(N+)}(O_{\overline{K}}) \to A_{\mathrm{crys}}^{(N+)}(O_{\overline{K}})/\mathfrak{a}_m.$$

This implies the claim because $A_{\mathrm{inf},\mathfrak{a}}(O_{\overline{K}}) \to A_{\mathrm{inf},\mathfrak{a}}^{\square}(\mathcal{A})$ ($\mathfrak{a} \in \mathscr{S}_{\mathrm{inf}}$) is flat. □

19 de Rham Complexes with Truncated Divided Powers

Let $\mathcal{B} \to \mathcal{A}$ and $\varphi_{\mathcal{B}}$ be as in the settings for the definition of $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ and $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ in Sect. 2. Let $O_{K,m}$, $\mathcal{A}_m$, $\mathcal{B}_m$, $\Omega_{\mathcal{B}_m}$, $\Omega_{\mathcal{B}}$, $\varphi_{\mathcal{B}_m}$, $(\mathcal{P}_m, \mathrm{Fil}^r\mathcal{P}_m, \nabla_{\mathcal{P}_m}, \varphi_{\mathcal{P}_m})$ ($m \in \mathbb{N}_{>0}$), and $(\mathcal{P}, \mathrm{Fil}^r\mathcal{P}, \nabla_{\mathcal{P}}, \varphi_{\mathcal{P}})$ be the same as in Sect. 2. In this section, we construct "filtered de Rham complexes with truncated divided powers" associated to the "evaluation" of an object M of $\mathrm{MF}_{[0,p-2],\mathrm{free}}^{\nabla}(\mathcal{A}, \Phi)$ on a certain subring of $\mathcal{P}$, and study its relation with $TA_{\mathrm{inf}}(M)$.

Let $\mathcal{J}$ and $\mathcal{J}_m$ ($m \in \mathbb{N}_{>0}$) be the kernel of $\mathcal{B} \to \mathcal{A}$ and $\mathcal{B}_m \to \mathcal{A}_m$, respectively. Let N be a positive integer. We define $\mathcal{P}_m^{(N)}$ to be the $\mathcal{B}_m$-subalgebra of $\mathcal{P}_m$ generated by the elements $x^{[p^l]}$ ($x \in \mathcal{J}_m, l \in \mathbb{N} \cap [1, N]$). Recall that $p\mathcal{P}_m + \mathrm{Fil}^1\mathcal{P}_m$ has a unique PD-structure compatible with the PD-structures on $\mathrm{Fil}^1\mathcal{P}_m$ and on $p\mathbb{Z}_p$. For $x \in p\mathcal{P}_m + \mathrm{Fil}^1\mathcal{P}_m$, we write $x^{[n]}$ ($n \in \mathbb{N}$) for the nth divided power of x with respect to the PD-structure. For $x \in p\mathcal{B}_m + \mathcal{J}_m$ and $n \in \mathbb{N} \cap [0, p^{N+1} - 1]$, we have $x^{[n]} \in \mathcal{P}_m^{(N)}$ by Lemma 140 (2), (3), and $p^{[m]} \in \mathbb{Z}_p$ ($m \in \mathbb{N}$). This implies that $\mathcal{P}_m^{(N)}$ is stable under $\varphi_{\mathcal{P}_m}$ (resp. $\nabla_{\mathcal{P}_m}$) because $\varphi_{\mathcal{B}_m}(p\mathcal{B}_m + \mathcal{J}_m) \subset p\mathcal{B}_m + \mathcal{J}_m$ and $\varphi_{\mathcal{P}_m}$ is a PD-homomorphism with respect to the PD-ideal $p\mathcal{P}_m + \mathrm{Fil}^1\mathcal{P}_m$ (resp. $\nabla_{\mathcal{P}_m}(x^{[n]}) = x^{[n-1]} \otimes d(x)$ for $x \in \mathcal{J}_m$ and $n \in \mathbb{N}$, and $\nabla_{\mathcal{P}_m}$ is a derivation). For $n \in \mathbb{N}$ and $x \in \mathcal{J}_m$, we define $x^{[n]_{(N)}}$ to be $(x^{[p^N]})^{pa}x^{[b]}$, where $n = p^{N+1}a + b$ ($a \in \mathbb{N}, b \in \mathbb{N} \cap [0, p^{N+1} - 1]$). Then we have $x^{[n]_{(N)}} \cdot x^{[n']_{(N)}} \subset \mathbb{Z}_p x^{[n+n']_{(N)}}$ by Lemma 140 (1), and it implies $\nabla_{\mathcal{P}_m}(x^{[n]_{(N)}}) \subset \mathbb{Z}_p x^{[n-1]_{(N)}} \otimes d(x)$ by the definition of $x^{[n]_{(N)}}$ and $x^{[m]} = x^{[m]_{(N)}}$ for $m \in \mathbb{N} \cap [0, p^{N+1} - 1]$. We define $\mathrm{Fil}^r\mathcal{P}_m^{(N)}$ ($r \in \mathbb{N}_{>0}$) to be the ideal of $\mathcal{P}_m^{(N)}$ generated by $\prod_{1 \leq i \leq c} x_i^{[n_i]_{(N)}}$ ($c \in \mathbb{N}_{>0}, x_i \in \mathcal{J}_m, n_i \in \mathbb{N}, \sum_{1 \leq i \leq c} n_i \geq r$).

Put $\mathrm{Fil}^r\mathcal{P}_m^{(N)}=\mathcal{P}_m^{(N)}$ if $r\leq 0$. Then we have $\mathrm{Fil}^r\mathcal{P}_m^{(N)}\mathrm{Fil}^{r'}\mathcal{P}_m^{(N)}\subset\mathrm{Fil}^{r+r'}\mathcal{P}_m^{(N)}$, and $\nabla_{\mathcal{P}_m}(\mathrm{Fil}^r\mathcal{P}_m^{(N)})\subset\mathrm{Fil}^{r-1}\mathcal{P}_m^{(N)}\otimes_{\mathcal{B}_m}\Omega_{\mathcal{B}_m}$ for $r,r'\in\mathbb{Z}$. We also have $\varphi_{\mathcal{P}_m}(\mathrm{Fil}^r\mathcal{P}_m^{(N)})\subset p^r\mathcal{P}_m^{(N)}$ for $r\in\mathbb{N}\cap[0,p-1]$ because $\varphi_{\mathcal{P}_m}(x^{[p^l]})=(x^p+py)^{[p^l]}=p^{[p^l]}((p-1)!x^{[p]}+y)^p\in p^{p-1}\mathcal{P}_m^{(N)}$ and $\varphi_{\mathcal{P}_m}(x^n)=p^n((p-1)!x^{[p]}+y)^n\in p^n\mathcal{P}_m^{(N)}$ for $x\in\mathcal{J}_m$, $l\in\mathbb{N}\cap[1,N]$ and $n\in\mathbb{N}\cap[0,p-1]$. Here $y\in\mathcal{B}_m$ is defined by $\varphi_{\mathcal{B}_m}(x)=x^p+py$. We obviously have $\mathcal{P}_m^{(N)}\subset\mathcal{P}_m^{(N+1)}$, $\mathrm{Fil}^r\mathcal{P}_m^{(N)}\subset\mathrm{Fil}^r\mathcal{P}_m^{(N+1)}$ and $\mathrm{Fil}^r\mathcal{P}_m^{(N)}\subset\mathrm{Fil}^r\mathcal{P}_m$. The surjective homomorphism $\mathcal{P}_{m+1}\to\mathcal{P}_m$ induces surjective homomorphisms $\mathcal{P}_{m+1}^{(N)}\to\mathcal{P}_m^{(N)}$ and $\mathrm{Fil}^r\mathcal{P}_{m+1}^{(N)}\to\mathrm{Fil}^r\mathcal{P}_m^{(N)}$ $(r\in\mathbb{Z})$.

We define $\mathcal{P}^{(N)}$ to be $\varprojlim_m\mathcal{P}_m^{(N)}$ regarded as a $\mathcal{B}$-subalgebra of $\mathcal{P}$, and $\mathrm{Fil}^r\mathcal{P}^{(N)}$ to be $\varprojlim_m\mathrm{Fil}^r\mathcal{P}_m^{(N)}$. Then $(\mathcal{P}^{(N)},\mathrm{Fil}^\bullet\mathcal{P}^{(N)})$ is a filtered ring (Definition 10 (1)), $\mathcal{P}^{(N)}\subset\mathcal{P}$ is stable under $\nabla_{\mathcal{P}}$ and $\varphi_{\mathcal{P}}$, and we have $\nabla(\mathrm{Fil}^r\mathcal{P}^{(N)})\subset\mathrm{Fil}^{r-1}\mathcal{P}^{(N)}\otimes_{\mathcal{B}}\Omega_{\mathcal{B}}$ $(r\in\mathbb{Z})$. The multiplication by p^r on $\mathcal{P}_{m+r}$ uniquely decomposes as $\mathcal{P}_{m+r}\to\mathcal{P}_m\xrightarrow{\cong}p^r\mathcal{P}_{m+r}\hookrightarrow\mathcal{P}_{m+r}$ ([14, I Lemma (1.3) (2)]), and it induces a decomposition $\mathcal{P}_{m+r}^{(N)}\to\mathcal{P}_m^{(N)}\xrightarrow{\cong}p^r\mathcal{P}_{m+r}^{(N)}\hookrightarrow\mathcal{P}_{m+r}^{(N)}$ of the multiplication by p^r on $\mathcal{P}_{m+r}^{(N)}$. By taking the inverse limit with respect to m, we obtain isomorphisms $p^r\mathcal{P}\xrightarrow{\cong}\varprojlim_m p^r\mathcal{P}_m$ and $p^r\mathcal{P}^{(N)}\xrightarrow{\cong}\varprojlim_m p^r\mathcal{P}_m^{(N)}$. In particular, this implies $\varphi_{\mathcal{P}}(\mathrm{Fil}^r\mathcal{P}^{(N)})\subset p^r\mathcal{P}^{(N)}$ for $r\in\mathbb{N}\cap[0,p-1]$.

We use the following proposition in the construction of the comparison morphism (182) in Sect. 20. See (179) and (192).

Proposition 174 *For $N,m\in\mathbb{N}_{>0}$, the homomorphism $\varinjlim_m\mathcal{P}^{(N)}/p^m\mathcal{P}^{(N)}\to\mathcal{P}_m$ is an isomorphism.*

Lemma 175 *For $N,m\in\mathbb{N}_{>0}$, we have $\mathcal{P}_m^{(N)}\cap p\mathcal{P}_m=\mathcal{P}_m^{(N)}\cap p\mathcal{P}_m^{(N+1)}$.*

Proof By definition, the PD-scheme $D_m=\mathrm{Spec}(\mathcal{P}_m)$ is the PD-envelope of the closed immersion $X_m=\mathrm{Spec}(\mathcal{A}_m)\to Y_m=\mathrm{Spec}(\mathcal{B}_m)$ compatible with the PD-structure on pO_K. Let $\mathcal{D}_m$ be the direct image of $\mathcal{O}_{D_m}$ under $D_m\to Y_m$. For $N\in\mathbb{N}_{>0}$, let $\mathcal{D}_m^{(N)}$ be the $\mathcal{O}_{Y_m}$-subalgebra of $\mathcal{D}_m$ generated by local sections $x^{[p^l]}$ $(x\in\mathrm{Ker}(\mathcal{O}_{Y_m}\to\mathcal{O}_{X_m}),l\in\mathbb{N}\cap[0,N])$). By using Lemma 140 (2), (3), we see that, if the ideal $\mathcal{J}_m=\mathrm{Ker}(\mathcal{B}_m\to\mathcal{A}_m)$ is generated by $x_1,\ldots,x_r$, then $\mathcal{D}_m^{(N)}$ is generated by $x_i^{[p^l]}$ $(i\in\mathbb{N}\cap[1,r],l\in\mathbb{N}\cap[0,N])$ over $\mathcal{O}_{Y_m}$. This implies that $\mathcal{D}_m^{(N)}$ is a quasi-coherent $\mathcal{O}_{Y_m}$-subalgebra of $\mathcal{D}_m$, and we have $\mathcal{P}_m^{(N)}=\Gamma(Y_m,\mathcal{D}_m^{(N)})$. Hence it suffices to prove the claim with $\mathcal{P}_m$ and $\mathcal{P}_m^{(\bullet)}$ replaced by $\mathcal{D}_m$ and $\mathcal{D}_m^{(\bullet)}$.

Since this question is Zariski local on X_m, as in the proof of [14, I Lemma (1.3)], we may assume that $\mathcal{J}_m$ is generated by elements $f_1,\ldots,f_s$ such that the $O_{K,m}$-homomorphism $R_m:=O_{K,m}[T_1,\ldots,T_s]\to\mathcal{B}_m;T_i\mapsto f_i$ is flat. Then we have $\mathcal{P}_m=\mathcal{B}_m\otimes_{R_m}R_m^{\mathrm{PD}}$, where R_m^{PD} denotes the PD-polynomial ring $O_{K,m}\langle T_1,\ldots,T_s\rangle$ ([5, 3.21 Proposition]). For $N\in\mathbb{N}_{>0}$, let $R_m^{\mathrm{PD},(N)}$ be the R_m-subalgebra of R_m^{PD} generated by $T_i^{[p^l]}$ $(i\in\mathbb{N}\cap[1,s],l\in\mathbb{N}\cap[0,N])$. Then we have $\mathcal{P}_m^{(N)}=\mathcal{B}_m\otimes_{R_m}R_m^{\mathrm{PD},(N)}$. Thus we are further reduced to proving the claim with $\mathcal{P}_m$ and $\mathcal{P}_m^{(\bullet)}$ replaced by R_m^{PD} and $R_m^{\mathrm{PD},(\bullet)}$.

For $n\in\mathbb{N}$ and $i\in\mathbb{N}\cap[1,s]$, let $T_i^{[n]_{(N)}}\in R_m^{\mathrm{PD}}$ be the image of $p^{-\{n\}_{(N)}}T_i^n\in O_K\langle T_1,\ldots,T_s\rangle$. Then, by the same argument as the proof of Corollary 141 (1), we see that $R_m^{\mathrm{PD},(N)}$ is generated by $\underline{T}^{[\underline{n}]_{(N)}} = \prod_{1\leq i\leq s} T_i^{[n_i]_{(N)}}$ ($\underline{n} = (n_i) \in \mathbb{N}^s$) as an $O_{K,m}$-module. This implies that $R_m^{\mathrm{PD},(N)} \cap pR_m^{\mathrm{PD}}$ (resp. $R_m^{\mathrm{PD},(N)} \cap pR_m^{\mathrm{PD},(N+1)}$) is the $O_{K,m}$-module generated by $p^{\varepsilon_{\underline{n}}}\underline{T}^{[\underline{n}]_{(N)}}$ ($\underline{n} \in \mathbb{N}^s$), where $\varepsilon_{\underline{n}} = 1$ if $\{n_i\}_{(N)} = v_p(n_i!)$ (resp. $\{n_i\}_{(N)} = \{n_i\}_{(N+1)}$) for all $i \in \mathbb{N} \cap [1, s]$, and $\varepsilon_{\underline{n}} = 0$ otherwise. Now the claim follows from $\{n\}_{(N)} < v_p(n!) \Leftrightarrow n \geq p^{N+1} \Leftrightarrow \{n\}_{(N)} < \{n\}_{(N+1)}$ for $n \in \mathbb{N}$ (Lemma 140 (3), (4)). □

Proof of Proposition 174 Since $\mathcal{P}_m = \cup_{N\in\mathbb{N}_{>0}} \mathcal{P}_m^{(N)}$ and $\mathcal{P}^{(N)} \to \mathcal{P}_m^{(N)}$ is surjective, the homomorphism in question is surjective. Let us prove that it is also injective. By taking the inverse limit of the equality in Lemma 175, we obtain $\mathcal{P}^{(N)} \cap p\mathcal{P} = \mathcal{P}^{(N)} \cap p\mathcal{P}^{(N+1)}$. This implies $\mathcal{P}^{(N)} \cap p^m\mathcal{P} \subset p\mathcal{P}^{(N+1)} \cap p^m\mathcal{P} = p(\mathcal{P}^{(N+1)} \cap p^{m-1}\mathcal{P})$ for $m \in \mathbb{N}_{>0}$. Hence we obtain $\mathcal{P}^{(N)} \cap p^m\mathcal{P} \subset p^m\mathcal{P}^{(N+m)}$ ($m \in \mathbb{N}$) by induction on m. Let $\mathcal{K}_m^{(N)}$ be the kernel of $\mathcal{P}^{(N)}/p^m \to \mathcal{P}_m = \mathcal{P}/p^m$. Then the last claim means that the homomorphism $\mathcal{K}_m^{(N)} \to \mathcal{K}_m^{(N+m)}$ vanishes, and therefore we have $\varinjlim_N \mathcal{K}_m^{(N)} = 0$. This completes the proof. □

Let M be an object of $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A}, \Phi)$, and put $M_m := M/p^mM$ and $\mathrm{Fil}^r M_m := \mathrm{Fil}^r M/p^m \subset M_m$. We define a free $\mathcal{P}_m$-module $M_{\mathcal{P}_m}$ of finite type with an integrable connection $\nabla\colon M_{\mathcal{P}_m} \to M_{\mathcal{P}_m} \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$ compatible with $\nabla_{\mathcal{P}_m}$ and a $\varphi_{\mathcal{P}_m}$-semilinear endomorphism $\varphi\colon M_{\mathcal{P}_m} \to M_{\mathcal{P}_m}$ compatible with ∇ as after (24). It is also naturally endowed with a filtration $\mathrm{Fil}^r M_{\mathcal{P}_m}$ ($r \in \mathbb{Z}$) such that $\mathrm{Fil}^r\mathcal{P}_m \cdot \mathrm{Fil}^s M_{\mathcal{P}_m} \subset \mathrm{Fil}^{r+s} M_{\mathcal{P}_m}$ ($r, s \in \mathbb{Z}$), $\nabla(\mathrm{Fil}^r M_{\mathcal{P}_m}) \subset \mathrm{Fil}^{r-1} M_{\mathcal{P}_m} \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$ ($r \in \mathbb{Z}$). Let δ be a homomorphism $\mathcal{A}_m \to \mathcal{P}_m$ over O_K/p^m whose composition with $\mathcal{P}_m \to \mathcal{A}_m$ is the identity map. There exists such δ since $\mathrm{Spec}(\mathcal{A}_m) \to \mathrm{Spec}(O_{K,m})$ is smooth and $\mathrm{Spec}(\mathcal{A}_m) \to \mathrm{Spec}(\mathcal{P}_m)$ is a nilimmersion. By the definition of $M_{\mathcal{P}_m}$, the proof of Theorem 17, Theorem 29, Proposition 31, and Lemma 27 (3), we have a canonical isomorphism

$$M_{\mathcal{P}_m} \cong M_m \otimes_{\mathcal{A}_m,\delta} \mathcal{P}_m \tag{172}$$

of filtered modules over $(\mathcal{P}_m, \mathrm{Fil}^\bullet\mathcal{P}_m)$, where the target is the scalar extension of $(M_m, \mathrm{Fil}^\bullet M_m)$ under the morphism of filtered rings $\delta\colon (\mathcal{A}, 0) \to (\mathcal{P}_m, \mathrm{Fil}^\bullet\mathcal{P}_m)$ (Definition 10 (4)). If we choose another $\delta'\colon \mathcal{A}_m \to \mathcal{P}_m$, then, by Remark 18, the composition of

$$M_m \otimes_{\mathcal{A}_m,\delta'} \mathcal{P}_m \cong M_{\mathcal{P}_m} \cong M_m \otimes_{\mathcal{A}_m,\delta} \mathcal{P}_m \tag{173}$$

is given by

$$x \otimes 1 \mapsto \sum_{\underline{n}\in\mathbb{N}^d} \prod_{1\leq i\leq d} \nabla_i^{n_i}(x) \otimes \prod_{1\leq i\leq d} (\delta'(t_i) - \delta(t_i))^{[n_i]}, \quad x \in M_m, \tag{174}$$

where the endomorphism ∇_i of M_m is defined by $\nabla(y) = \sum_{1\leq i\leq d} \nabla_i(y) \otimes dt_i$.

We will define a canonical free $\mathcal{P}_m^{(1)}$-submodule $M_{\mathcal{P}_m^{(1)}}$ of $M_{\mathcal{P}_m}$ such that $\mathcal{P}_m \otimes_{\mathcal{P}_m^{(1)}} M_{\mathcal{P}_m^{(1)}} \cong M_{\mathcal{P}_m}$.

Proposition 176 *We have* $\nabla_i^{p(p-1)}(M) \subset pM$ *for each* $i \in \mathbb{N} \cap [1, d]$.

Lemma 177 *We have $\nabla_i^p(\mathcal{A}) \subset p\mathcal{A}$ for each $i \in \mathbb{N} \cap [1, d]$.*

Proof This is well-known and proved as follows. Since the relative Frobenius of the étale homomorphism $k[T_1, \dots, T_d] \to \mathcal{A}_1$; $T_i \mapsto t_i$ is an isomorphism, $\mathcal{A}_1$ is a free $(\mathcal{A}_1)^p$-module with a basis $\prod_{1\leq i\leq d} t_i^{n_i}$, $(n_i) \in (\mathbb{N} \cap [0, p-1])^d$. Since ∇_i on $\mathcal{A}_1$ is $(\mathcal{A}_1)^p$-linear, this implies $\nabla_i^p(\mathcal{A}_1) = 0$. □

Proof of Proposition 176 Choose a lifting $\varphi\colon \mathcal{A} \to \mathcal{A}$ of the absolute Frobenius of $\mathcal{A}_1$ compatible with σ of O_K. Noting $\varphi(\Omega_{\mathcal{A}}) \subset p\Omega_{\mathcal{A}}$, we define $\varphi_1\colon \Omega_{\mathcal{A}} \to \Omega_{\mathcal{A}}$ to be $p^{-1}\varphi$. We define a φ-semilinear homomorphism $\varphi_r\colon \mathrm{Fil}^r M \to M$ ($r \in \mathbb{Z} \cap (-\infty, p-2]$) to be $p^{-r}\varphi|_{\mathrm{Fil}^r M}$. Then we have $(\varphi_{r-1} \otimes \varphi_1) \circ \nabla = \nabla \circ \varphi_r$ on $\mathrm{Fil}^r M$. Let $\overline{\nabla}_r\colon \mathrm{gr}^r_{\mathrm{Fil}} M \to \mathrm{gr}^{r-1}_{\mathrm{Fil}} M \otimes_{\mathcal{A}} \Omega_{\mathcal{A}}$ ($r \in \mathbb{N} \cap [0, p-2]$) be the $\mathcal{A}$-linear homomorphism induced by ∇ on M. Put $\mathrm{gr}M := \oplus_{0\leq r\leq p-2}\mathrm{gr}^r_{\mathrm{Fil}} M$, and let $\overline{\nabla}\colon \mathrm{gr}M \to \mathrm{gr}M \otimes_{\mathcal{A}} \Omega_{\mathcal{A}}$ be the $\mathcal{A}$-linear homomorphism defined by the sum of $\overline{\nabla}^r$. Then φ_r induces an $\mathcal{A}_1$-linear isomorphism

$$\overline{\Phi}\colon (\mathrm{gr}M)/p \otimes_{\mathcal{A}_1,\varphi} \mathcal{A}_1 \xrightarrow{\cong} M_1,$$

and the above compatibility between ∇ and φ_r implies that the following diagram is commutative.

$$\begin{array}{ccc} (\mathrm{gr}M)/p & \xrightarrow{\overline{\varphi}} & M_1 \\ \downarrow{\scriptstyle \overline{\nabla} \bmod p} & & \downarrow{\scriptstyle \nabla} \\ (\mathrm{gr}M)/p \otimes_{\mathcal{A}} \Omega_{\mathcal{A}} & \xrightarrow{\overline{\varphi}\otimes\varphi_1} & M_1 \otimes_{\mathcal{A}} \Omega_{\mathcal{A}} \end{array} \tag{175}$$

Here $\overline{\varphi}$ denotes the restriction of $\overline{\Phi}$ on $(\mathrm{gr}M)/p$. For $r \in \mathbb{N} \cap [0, p-2]$, let $G_r(M_1)$ denote the image of $(\oplus_{0\leq s\leq r}\mathrm{gr}^s_{\mathrm{Fil}} M)/p \otimes_{\mathcal{A}_1,\varphi} \mathcal{A}_1$ under $\overline{\Phi}$. Put $G_{-1}(M_1) = 0$. Then, by the above commutative diagram, we see that $G_r(M_1)$ is stable under ∇ and the isomorphism $(\mathrm{gr}^r M)/p \otimes_{\mathcal{A}_1,\varphi} \mathcal{A}_1 \xrightarrow{\cong} \mathrm{gr}^G_r(M_1)$ ($r \in \mathbb{N} \cap [0, p-2]$) induced by $\overline{\Phi}$ is compatible with the connection on $\mathrm{gr}^G_r(M_1)$ and the connection $\mathrm{id} \otimes \nabla$ on the source. By Lemma 177, we have $\nabla_i^p = 0$ on $\mathrm{gr}^G_r(M_1)$ for $r \in \mathbb{N} \cap [0, p-2]$. This implies $\nabla_i^{p(p-1)} = 0$ on M_1. □

Since the ideal $\mathcal{J}_m$ of $\mathcal{B}_m$ is finitely generated, and the PD-ideal of $\mathcal{P}_m$ is a nilideal, the homomorphism $\mathcal{B}_m \to \mathcal{P}_m$ factors through $\mathcal{B}_m/\mathcal{J}_m^f$ for a sufficiently large $f \in \mathbb{N}$. Choose such an f and then a homomorphism $\overline{\delta}\colon \mathcal{A}_m \to \mathcal{B}_m/\mathcal{J}_m^f$ over $O_{K,m}$ whose composition with $\mathcal{B}_m/\mathcal{J}_m^f \to \mathcal{A}_m$ is the identity map. Such a δ exists because $O_{K,m} \to \mathcal{A}_m$ is smooth. We define δ to be the composition of $\overline{\delta}$ with the homomorphism $\mathcal{B}_m/\mathcal{J}_m^f \to \mathcal{P}_m^{(1)}$, and define the $\mathcal{P}_m^{(1)}$-submodule

$$M_{\mathcal{P}_m^{(1)}} \subset M_{\mathcal{P}_m} \tag{176}$$

to be the image of $M_m \otimes_{\mathcal{A}_m,\delta} \mathcal{P}_m^{(1)}$ under the isomorphism (172). We define $\mathrm{Fil}^r M_{\mathcal{P}_m^{(1)}}$ ($r \in \mathbb{Z}$) to be the image of the filtration on the scalar extension of $(M_m, \mathrm{Fil}^\bullet M_m)$ under the morphism of filtered rings $\delta\colon (\mathcal{A}, 0) \to (\mathcal{P}_m^{(1)}, \mathrm{Fil}^\bullet\mathcal{P}_m^{(1)})$ (Definition 10 (4)). We have $\mathrm{Fil}^r\mathcal{P}_m^{(1)} \cdot \mathrm{Fil}^s M_{\mathcal{P}_m^{(1)}} \subset \mathrm{Fil}^{r+s} M_{\mathcal{P}_m^{(1)}}$ $(r, s \in \mathbb{Z})$.

Proposition 178 *The $\mathcal{P}_m^{(1)}$-submodules $M_{\mathcal{P}_m^{(1)}}$ and $\mathrm{Fil}^r M_{\mathcal{P}_m^{(1)}}$ $(r \in \mathbb{Z})$ of $M_{\mathcal{P}_m}$ defined above are independent of the choice of f and $\overline{\delta}$.*

Proof Let f' and $\overline{\delta}'$ be another choice, and let δ' be the composition of $\overline{\delta}'$ and $\mathcal{B}_m/\mathcal{J}_m^{f'} \to \mathcal{P}_m^{(1)}$. By replacing f and f' with $\min\{f, f'\}$, we may assume $f = f'$. Then the image of $\mathrm{Fil}^r M_m$ $(r \in \mathbb{N} \cap [0, p-2])$ under the composition of the isomorphisms (173) is contained in $\mathrm{Fil}^r(M_m \otimes_{\mathcal{A}_m,\delta} \mathcal{P}_m^{(1)})$ because $\delta'(t_i) - \delta(t_i)$ is contained in the image of $\mathcal{J}_m$, $\nabla_i^n(M) \subset p^{\lfloor \frac{n}{p(p-1)} \rfloor} M$ by Proposition 176, and $v_p(n!) = \lfloor \frac{n}{p} \rfloor + v_p(\lfloor \frac{n}{p} \rfloor !) \leq \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p(p-1)} \rfloor$ for $n \in \mathbb{N}_{>0}$. Here $\lfloor x \rfloor$ denotes the largest integer $\leq x$ for $x \in \mathbb{R}$. By exchanging $\overline{\delta}$ and $\overline{\delta}'$ and applying the same argument, we obtain the claim. □

Proposition 179 (1) *We have the inclusions $\nabla(M_{\mathcal{P}_m^{(1)}}) \subset M_{\mathcal{P}_m^{(1)}} \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$ and $\nabla(\mathrm{Fil}^r M_{\mathcal{P}_m^{(1)}}) \subset \mathrm{Fil}^{r-1} M_{\mathcal{P}_m^{(1)}} \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$ for $r \in \mathbb{Z}$.*

(2) *We have $\varphi(M_{\mathcal{P}_m^{(1)}}) \subset M_{\mathcal{P}_m^{(1)}}$.*

(3) *The $\mathcal{P}_m$-linear homomorphism $\mathcal{P}_m \otimes_{\mathcal{P}_m^{(1)}} M_{\mathcal{P}_m^{(1)}} \to M_{\mathcal{P}_m}$ is an isomorphism of filtered modules over $(\mathcal{P}_m, \mathrm{Fil}^\bullet\mathcal{P}_m)$, where the source is the scalar extension of $(M_{\mathcal{P}_m^{(1)}}, \mathrm{Fil}^\bullet M_{\mathcal{P}_m^{(1)}})$ under $(\mathcal{P}_m^{(1)}, \mathrm{Fil}^\bullet\mathcal{P}_m^{(1)}) \to (\mathcal{P}_m, \mathrm{Fil}^\bullet\mathcal{P}_m)$ (Definition 10 (4)).*

(4) *The homomorphism $M_{\mathcal{P}_{m+1}} \to M_{\mathcal{P}_m}$ induces an isomorphism $\mathcal{P}_m^{(1)} \otimes_{\mathcal{P}_{m+1}^{(1)}} M_{\mathcal{P}_{m+1}^{(1)}} \xrightarrow{\cong} M_{\mathcal{P}_m^{(1)}}$ of filtered modules over $(\mathcal{P}_m^{(1)}, \mathrm{Fil}^\bullet\mathcal{P}_m^{(1)})$, where the source is the scalar extension of $(M_{\mathcal{P}_{m+1}^{(1)}}, \mathrm{Fil}^\bullet M_{\mathcal{P}_{m+1}^{(1)}})$ under the homomorphism of filtered rings $(\mathcal{P}_{m+1}^{(1)}, \mathrm{Fil}^\bullet\mathcal{P}_{m+1}^{(1)}) \to (\mathcal{P}_m^{(1)}, \mathrm{Fil}^\bullet\mathcal{P}_m^{(1)})$.*

(5) *There exist $N \in \mathbb{N}$, $r_\nu \in \mathbb{N} \cap [0, p-2]$ $(\nu \in \mathbb{N} \cap [1, N])$, a basis $e_{\nu,m} \in M_{\mathcal{P}_m^{(1)}}$ $(\nu \in \mathbb{N} \cap [1, N])$ of $M_{\mathcal{P}_m^{(1)}}$ and $(a_{\nu\mu,m})_{\nu\mu} \in GL_N(\mathcal{P}_m^{(1)})$ for each $m \in \mathbb{N}_{>0}$ such that the images of $e_{\nu,m+1}$ and $a_{\nu\mu,m+1}$ in $M_{\mathcal{P}_m^{(1)}}$ and $\mathcal{P}_m^{(1)}$ are $e_{\nu,m}$ and $a_{\nu\mu,m}$, $\mathrm{Fil}^r M_{\mathcal{P}_m^{(1)}}$ $(r \in \mathbb{Z})$ is the direct sum of $\mathrm{Fil}^{r-r_\nu}\mathcal{P}_m^{(1)} e_{\nu,m}$ $(\nu \in \mathbb{N} \cap [1, N])$, and $\varphi(e_{\mu,m}) = p^{r_\mu} \sum_{1 \leq \nu \leq N} a_{\nu\mu,m} e_{\nu,m}$.*

Proof The claims (2) and (4) follow from (5) and Lemma 13 (1), and the claim (3) is obvious by the definition of $(M_{\mathcal{P}_m^{(1)}}, \mathrm{Fil}^\bullet M_{\mathcal{P}_m^{(1)}})$. Let us prove (1) and (5). Choose f and $\overline{\delta}$, and define δ as in the definition of $M_{\mathcal{P}_m^{(1)}}$.

(1) By applying the construction of (16) to $\mathrm{Spec}(\delta)\colon \mathrm{Spec}(\mathcal{P}_m) \to \mathrm{Spec}(\mathcal{A}_m)$, we obtain a homomorphism $\delta^1\colon \Omega_{\mathcal{A}_m} \to \mathcal{P}_m \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$ compatible with $d\colon \mathcal{A}_m \to \Omega_{\mathcal{A}_m}$ and $\nabla_{\mathcal{P}_m}\colon \mathcal{P}_m \to \mathcal{P}_m \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$ (17). By Proposition 32, the connection on $M_{\mathcal{P}_m}$ is given by $\nabla(x \otimes y) = \delta^1(\nabla(x)) \otimes y + x \otimes \nabla_{\mathcal{P}_m}(y)$ $(x \in M_m, y \in \mathcal{P}_m)$ via the isomorphism (172). Since $\Omega_{\mathcal{A}_m}$ is generated by $d(\mathcal{A}_m)$ as an $\mathcal{A}_m$-module and $\delta(\mathcal{A}_m) \subset \mathcal{P}_m$ is contained in the image of $\mathcal{B}_m$, the image of the homomorphism $\delta^1\colon \Omega_{\mathcal{A}_m} \to \mathcal{P}_m \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$ is contained in the image of $\Omega_{\mathcal{B}_m}$. Hence the claim follows from $\nabla_{\mathcal{P}_m}(\mathrm{Fil}^r\mathcal{P}_m^{(1)}) \subset \mathrm{Fil}^{r-1}\mathcal{P}_m^{(1)} \otimes_{\mathcal{B}_m} \Omega_{\mathcal{B}_m}$.

(5) Put $\mathcal{J}_m' := p\mathcal{B}_m + \mathcal{J}_m$. Choose an increasing sequence of positive integers f_m $(m \in \mathbb{N}_{>0})$ such that the homomorphism $\mathcal{B}_m \to \mathcal{P}_m$ factors through $\mathcal{B}_m/(\mathcal{J}_m')^{f_m}$.

Choose liftings $\widetilde{t}_1, \dots, \widetilde{t}_d \in \mathcal{B}$ of $t_1, \dots, t_d \in \mathcal{A}$. Then, for each $m \in \mathbb{N}_{>0}$, there exists a unique homomorphism $\overline{\delta}_m \colon \mathcal{A}_m \to \mathcal{B}_m/(\mathcal{J}'_m)^{f_m}$ of $O_{K,m}$-algebras such that the composition with $\mathcal{B}_m/(\mathcal{J}'_m)^{f_m} \to \mathcal{A}_m$ is the identity map and $\overline{\delta}_m(t_i) = \widetilde{t}_i$ ($i \in \mathbb{N} \cap [1, d]$). The uniqueness implies that the composition $\mathcal{A}_{m+1} \xrightarrow{\overline{\delta}_{m+1}} \mathcal{B}_{m+1}/(\mathcal{J}'_{m+1})^{f_{m+1}} \xrightarrow{\mathrm{pr}} \mathcal{B}_m/(\mathcal{J}'_m)^{f_m}$ coincides with $\mathcal{A}_{m+1} \xrightarrow{\mathrm{pr}} \mathcal{A}_m \xrightarrow{\overline{\delta}_m} \mathcal{B}_m/(\mathcal{J}'_m)^{f_m}$. Let δ_m denote the composition of $\overline{\delta}_m$ and $\mathcal{B}_m/(\mathcal{J}'_m)^{f_m} \to \mathcal{P}_m^{(1)}$. Then we can define $M_{\mathcal{P}_m^{(1)}}$ and the filtration on it by using δ_m. Choose a basis e_ν ($N \in \mathbb{N}, \nu \in \mathbb{N} \cap [1, N]$) of M and $r_\nu \in \mathbb{N} \cap [0, p-2]$ ($\nu \in \mathbb{N} \cap [1, N]$) such that $\mathrm{Fil}^r M = \oplus_{r_\nu \geq r} \mathcal{A} e_\nu$ ($r \in \mathbb{Z}$). Let $e_{\nu,m}$ be the image of $(e_\nu \bmod p^m) \otimes 1$ under the isomorphism $M_m \otimes_{\mathcal{A}_m, \delta_m} \mathcal{P}_m \xrightarrow{\cong} M_{\mathcal{P}_m}$ (172) associated to δ_m. Then $e_{\nu,m}$ ($\nu \in \mathbb{N} \cap [1, N]$) is a basis of the $\mathcal{P}_m^{(1)}$-module $M_{\mathcal{P}_m^{(1)}}$, and, by Lemma 13 (1), we have $\mathrm{Fil}^r M_{\mathcal{P}_m^{(1)}} = \oplus_{\nu \in \mathbb{N} \cap [1,N]} \mathrm{Fil}^{r - r_\nu} \mathcal{P}_m^{(1)} e_{\nu,m}$ ($r \in \mathbb{Z}$). For each $\nu \in \mathbb{N} \cap [1, N]$, the elements $e_{\nu,m}$ ($m \in \mathbb{N}_{>0}$) obviously form a compatible system with respect to the homomorphisms $M_{\mathcal{P}_{m+1}^{(1)}} \to M_{\mathcal{P}_m^{(1)}}$ by the choice of δ_m ($m \in \mathbb{N}_{>0}$).

Choose and fix a lifting $\varphi \colon \mathcal{A} \to \mathcal{A}$ of the absolute Frobenius of $\mathcal{A}_1$ compatible with $\sigma \colon O_K \to O_K$. Then Φ on M induces a φ-semilinear endomorphism $\varphi \colon M \to \varphi^*(M) = F^*(M) \xrightarrow{\Phi} M$. By applying Remark 18 to the pull-backs of (M_m, ∇) by $\mathrm{Spec}(\delta_m \circ \varphi)$ and $\mathrm{Spec}(\varphi_{\mathcal{P}_m} \circ \delta_m) \colon \mathrm{Spec}(\mathcal{P}_m) \to \mathrm{Spec}(\mathcal{A}_m)$, we see that the Frobenius endomorphism φ of $M_{\mathcal{P}_m}$ is given by the composition of $\mathcal{P}_m$-linear maps

$$(M_m \otimes_{\mathcal{A}_m, \delta_m} \mathcal{P}_m) \otimes_{\mathcal{P}_m, \varphi_{\mathcal{P}_m}} \mathcal{P}_m \xrightarrow[\kappa_m]{\cong} (M_m \otimes_{\mathcal{A}_m, \varphi} \mathcal{A}_m) \otimes_{\mathcal{A}_m, \delta_m} \mathcal{P}_m \xrightarrow{\Phi \otimes \mathrm{id}_{\mathcal{P}_m}} M_m \otimes_{\mathcal{A}_m, \delta_m} \mathcal{P}_m,$$

where the image of $x \in M_m$ under κ_m (resp. κ_m^{-1}) is $\sum_{\underline{n} \in \mathbb{N}^d} \prod_i \nabla_i^{n_i}(x) \otimes \prod_i (\varphi \delta_m(t_i) - \delta_m \varphi(t_i))^{[n_i]}$ (resp. $\sum_{\underline{n} \in \mathbb{N}^d} \prod_i \nabla_i^{n_i}(x) \otimes \prod_i (\delta_m \varphi(t_i) - \varphi \delta_m(t_i))^{[n_i]}$). The element $\varphi \delta_m(t_i) - \delta_m \varphi(t_i)$ is contained in the image of $p\mathcal{B}_m$ because $\varphi_{\mathcal{B}}$ induces a lifting of Frobenius of $\mathcal{B}_m/(\mathcal{J}'_m)^{f_m}$ compatible with $\varphi_{\mathcal{P}_m}$. Therefore the above morphisms induce $\mathcal{P}_m^{(1)}$-linear maps

$$(M_m \otimes_{\mathcal{A}_m, \delta_m} \mathcal{P}_m^{(1)}) \otimes_{\mathcal{P}_m^{(1)}, \varphi_{\mathcal{P}_m}} \mathcal{P}_m^{(1)} \xrightarrow[\kappa_m^{(1)}]{\cong} (M_m \otimes_{\mathcal{A}_m, \varphi} \mathcal{A}_m) \otimes_{\mathcal{A}_m, \delta_m} \mathcal{P}_m^{(1)} \xrightarrow{\Phi \otimes \mathrm{id}_{\mathcal{P}_m^{(1)}}} M_m \otimes_{\mathcal{A}_m, \delta_m} \mathcal{P}_m^{(1)}.$$

Let $(M, \overline{\mathrm{Fil}}^\bullet M)$ be the scalar extension of $(M, \mathrm{Fil}^\bullet M)$ by the homomorphism of filtered rings $(\mathcal{A}, 0) \to (\mathcal{A}, p^{[\bullet]}\mathcal{A})$ (Definition 10 (4)). We have $\overline{\mathrm{Fil}}^r M = \sum_{s \leq r} p^{[r-s]} \mathrm{Fil}^s M = \oplus_\nu p^{[\max\{r - r_\nu, 0\}]} \mathcal{A} e_\nu$ ($r \in \mathbb{Z}$), $\nabla_i(\overline{\mathrm{Fil}}^r M) \subset \overline{\mathrm{Fil}}^{r-1} M$, and $p^{[r]} \overline{\mathrm{Fil}}^s M \subset \overline{\mathrm{Fil}}^{r+s} M$. Therefore, for each $r \in \mathbb{Z}$, the above description of κ_m and κ_m^{-1} implies that $\kappa_m^{(1)}$ induces an isomorphism between the $\mathcal{P}_m^{(1)}$-submodules generated by the image of $\overline{\mathrm{Fil}}^r M$. By taking the inverse limit of the above morphisms over m, we obtain $\mathcal{P}^{(1)}$-linear maps

$$(M \otimes_{\mathcal{A},\delta} \mathcal{P}^{(1)}) \otimes_{\mathcal{P}^{(1)},\varphi_{\mathcal{P}}} \mathcal{P}^{(1)} \xrightarrow[\kappa^{(1)}]{\cong} (M \otimes_{\mathcal{A},\varphi} \mathcal{A}) \otimes_{\mathcal{A},\delta} \mathcal{P}^{(1)} \xrightarrow{\Phi\otimes \mathrm{id}_{\mathcal{P}^{(1)}}} M \otimes_{\mathcal{A},\delta} \mathcal{P}^{(1)},$$

where $\delta\colon \mathcal{A} \to \mathcal{P}^{(1)}$ denotes the inverse limit of $\delta_m\colon \mathcal{A}_m \to \mathcal{P}^{(1)}_m$. Since $p^r\mathcal{P}^{(1)} = \varprojlim_m p^r\mathcal{P}^{(1)}_m$ ($r \in \mathbb{N}$), we see that $\varprojlim_m$ of the $\mathcal{P}^{(1)}_m$-submodule of $M_m \otimes_{\mathcal{A}_m,\varphi_{\mathcal{P}_m}\circ\delta_m} \mathcal{P}^{(1)}_m$ (resp. $M_m \otimes_{\mathcal{A}_m,\delta_m\circ\varphi} \mathcal{P}^{(1)}_m$) generated by the image of $\overline{\mathrm{Fil}}^r M$ coincides with the $\mathcal{P}^{(1)}$-submodule $\overline{\mathrm{Fil}}^r(M \otimes_{\mathcal{A},\varphi_{\mathcal{P}}\circ\delta} \mathcal{P}^{(1)})$ (resp. $\overline{\mathrm{Fil}}^r(M \otimes_{\mathcal{A},\delta\circ\varphi} \mathcal{P}^{(1)})$) of $M \otimes_{\mathcal{A},\varphi_{\mathcal{P}}\circ\delta} \mathcal{P}^{(1)}$ (resp. $M \otimes_{\mathcal{A},\delta\circ\varphi} \mathcal{P}^{(1)}$) generated by the image of $\overline{\mathrm{Fil}}^r M$. Therefore $\kappa^{(1)}$ induces an isomorphisms between the $\mathcal{P}^{(1)}$-submodules generated by the images of $\overline{\mathrm{Fil}}^r M$. Since $\varphi(\overline{\mathrm{Fil}}^r M) \subset p^r M$ ($r \in \mathbb{N} \cap [0, p-2]$) and $p^{-r}\varphi(\overline{\mathrm{Fil}}^r M)$ ($r \in \mathbb{N} \cap [0, p-2]$) generate M, we see that $M \otimes_{\mathcal{A},\delta} \mathcal{P}^{(1)}$ is generated by $p^{-r}(\Phi \otimes \mathrm{id}_{\mathcal{P}^{(1)}})(\overline{\mathrm{Fil}}^r(M \otimes_{\mathcal{A},\varphi\circ\delta} \mathcal{P}^{(1)})) = p^{-r}(\Phi \otimes \mathrm{id}_{\mathcal{P}^{(1)}}) \circ \kappa^{(1)}(\overline{\mathrm{Fil}}^r(M \otimes_{\mathcal{A},\varphi_{\mathcal{P}}\circ\delta} \mathcal{P}^{(1)}))$ for $r \in \mathbb{N} \cap [0, p-2]$. By the explicit description of $\overline{\mathrm{Fil}}^r M$ in terms of e_ν above, this implies that $e'_\nu := p^{-r_\nu}(\Phi \otimes \mathrm{id}_{\mathcal{P}^{(1)}}) \circ \kappa^{(1)}(e_\nu \otimes 1 \otimes 1)$ ($\nu \in \mathbb{N} \cap [1, N]$) generate $M \otimes_{\mathcal{A},\delta} \mathcal{P}^{(1)}$. Since $M \otimes_{\mathcal{A},\delta} \mathcal{P}^{(1)}$ is free of rank N, e'_ν form its basis and there exists $(a_{\nu\mu}) \in GL_N(\mathcal{P}^{(1)})$ such that $e'_\mu = \sum_\nu a_{\nu\mu}(e_\nu \otimes 1)$. The images $a_{\nu\mu,m}$ of $a_{\nu\mu}$ in $\mathcal{P}^{(1)}_m$ satisfy the desired condition. □

As before (37), we define the $\mathcal{P}$-module $M_{\mathcal{P}}$ and its decreasing filtration $\mathrm{Fil}^r M_{\mathcal{P}}$ ($r \in \mathbb{Z}$) by $\mathcal{P}$-submodules to be the inverse limits of $M_{\mathcal{P}_m}$ and $\mathrm{Fil}^r M_{\mathcal{P}_m}$. We have $\mathrm{Fil}^r\mathcal{P} \cdot \mathrm{Fil}^s M_{\mathcal{P}} \subset \mathrm{Fil}^{r+s} M_{\mathcal{P}}$ for $r, s \in \mathbb{Z}$, i.e., $(M_{\mathcal{P}}, \mathrm{Fil}^\bullet M_{\mathcal{P}})$ is a filtered module over $(\mathcal{P}, \mathrm{Fil}^\bullet\mathcal{P})$ (Definition 10 (2)). We define $\nabla\colon M_{\mathcal{P}} \to M_{\mathcal{P}} \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}$ to be the inverse limit of ∇ on $M_{\mathcal{P}_m}$, which is an integrable connection on $M_{\mathcal{P}}$ compatible with ∇ on $\mathcal{P}$ and satisfies $\nabla(\mathrm{Fil}^r M_{\mathcal{P}}) \subset \mathrm{Fil}^{r-1} M_{\mathcal{P}} \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}$. The Frobenius endomorphism φ of $M_{\mathcal{P}_m}$ for $m \in \mathbb{N}_{>0}$ induces a $\varphi_{\mathcal{P}}$-semilinear endomorphism φ of $M_{\mathcal{P}}$ compatible with ∇.

The inverse limits of $M_{\mathcal{P}^{(1)}_m}$ and $\mathrm{Fil}^r M_{\mathcal{P}^{(1)}_m}$ ($r \in \mathbb{Z}$) give $\mathcal{P}^{(1)}$-submodules $M_{\mathcal{P}^{(1)}}$ and $\mathrm{Fil}^r M_{\mathcal{P}^{(1)}}$ of $M_{\mathcal{P}}$ and $\mathrm{Fil}^r M_{\mathcal{P}}$. The $\mathcal{P}^{(1)}$-module $M_{\mathcal{P}^{(1)}}$ equipped with $\mathrm{Fil}^\bullet M_{\mathcal{P}^{(1)}}$ is a filtered module over $(\mathcal{P}^{(1)}, \mathrm{Fil}^\bullet\mathcal{P}^{(1)})$ (Definition 10 (2)). Let N be a positive integer. We define $M_{\mathcal{P}^{(N)}}$ to be the $\mathcal{P}^{(N)}$-submodule of $M_{\mathcal{P}}$ generated by $M_{\mathcal{P}^{(1)}}$, and its $\mathcal{P}^{(N)}$-submodule $\mathrm{Fil}^r M_{\mathcal{P}^{(N)}}$ ($r \in \mathbb{Z}$) to be the sum of $\mathrm{Fil}^{r-s}\mathcal{P}^{(N)}\mathrm{Fil}^s M_{\mathcal{P}^{(1)}}$ ($s \in \mathbb{N} \cap [0, r]$) if $r \geq 0$ and $M_{\mathcal{P}^{(N)}}$ if $r < 0$. The $\mathcal{P}^{(N)}$-module $M_{\mathcal{P}^{(N)}}$ with $\mathrm{Fil}^\bullet M_{\mathcal{P}^{(N)}}$ is a filtered module over $(\mathcal{P}^{(N)}, \mathrm{Fil}^\bullet\mathcal{P}^{(N)})$ (Definition 10 (2)). By Proposition 179 (1) and (2), we have $\nabla(M_{\mathcal{P}^{(N)}}) \subset M_{\mathcal{P}^{(N)}} \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}$, $\nabla(\mathrm{Fil}^r M_{\mathcal{P}^{(N)}}) \subset \mathrm{Fil}^{r-1} M_{\mathcal{P}^{(N)}} \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}$ for $r \in \mathbb{Z}$, and $\varphi(M_{\mathcal{P}^{(N)}}) \subset M_{\mathcal{P}^{(N)}}$. By Proposition 179 (3) and (5), there exist $e_\nu \in M_{\mathcal{P}^{(1)}}$ ($n \in \mathbb{N}, \nu \in \mathbb{N} \cap [1, n]$), $r_\nu \in \mathbb{N} \cap [0, p-2]$ ($\nu \in \mathbb{N} \cap [1, n]$), and $(a_{\nu\mu}) \in GL_n(\mathcal{P}^{(1)})$ such that $M_{\mathcal{P}^{(N)}}$ (resp. $M_{\mathcal{P}}$) is a free $\mathcal{P}^{(N)}$ (resp. $\mathcal{P}$)-module with a basis e_ν ($\nu \in \mathbb{N} \cap [1, n]$),

$$\mathrm{Fil}^r M_{\mathcal{P}} = \bigoplus_{\nu\in\mathbb{N}\cap[1,n]} \mathrm{Fil}^{r-r_\nu}\mathcal{P}e_\nu, \quad \mathrm{Fil}^r M_{\mathcal{P}^{(N)}} = \bigoplus_{\nu\in\mathbb{N}\cap[1,n]} \mathrm{Fil}^{r-r_\nu}\mathcal{P}^{(N)}e_\nu \qquad (177)$$

for $r \in \mathbb{Z}$, and

$$\varphi(e_\mu) = p^{r_\mu} \sum_{\nu \in \mathbb{N}\cap[1,n]} a_{\nu\mu} e_\nu, \quad \mu \in \mathbb{N}\cap[1,n]. \tag{178}$$

Put $\Omega^q_{\mathcal{B}} := \wedge^q \Omega_{\mathcal{B}}$ ($q \in \mathbb{N}$). The integrable connection ∇ on $M_{\mathcal{P}}$ defines a complex $M_{\mathcal{P}} \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}}$ with a Frobenius endomorphism φ and a decreasing filtration Fil^r ($r \in \mathbb{Z}$) defined by $\mathrm{Fil}^r(M_{\mathcal{P}} \otimes_{\mathcal{B}} \Omega^q_{\mathcal{B}}) = \mathrm{Fil}^{r-q} M_{\mathcal{P}} \otimes_{\mathcal{B}} \Omega^q_{\mathcal{B}}$. The $\mathcal{P}^{(N)}$-submodule $M_{\mathcal{P}^{(N)}}$ of $M_{\mathcal{P}}$ with $\mathrm{Fil}^r M_{\mathcal{P}^{(N)}}$ gives a subcomplex $M_{\mathcal{P}^{(N)}} \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}}$ stable under φ and endowed with a decreasing filtration Fil^r ($r \in \mathbb{Z}$) defined by $\mathrm{Fil}^r(M_{\mathcal{P}^{(N)}} \otimes_{\mathcal{B}} \Omega^q_{\mathcal{B}}) = \mathrm{Fil}^{r-q} M_{\mathcal{P}^{(N)}} \otimes_{\mathcal{B}} \Omega^q_{\mathcal{B}}$. By Proposition 174, we have an isomorphism

$$\varinjlim_N ((M_{\mathcal{P}^{(N)}} \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}}) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m) \xrightarrow{\cong} M_{\mathcal{P}_m} \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}}. \tag{179}$$

For a positive integer N, we define the object $TA^{(N+)}_{\mathrm{crys}}(M)$ of the category $\mathrm{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(A^{(N+)}_{\mathrm{crys}}(\overline{\mathcal{A}}), \varphi, G_{\mathcal{A}})$ to be the image of $TA_{\inf}(M)$ under the functor (161) for $\Lambda = \overline{\mathcal{A}}$ and $\Lambda_0 = \mathcal{A}$. By the definition of $TA_{\inf}(M)$, the image of $TA^{(N+)}_{\mathrm{crys}}(M)$ under the functor (162) for $\Lambda = \overline{\mathcal{A}}$ and $\Lambda_0 = \mathcal{A}$ is canonically isomorphic to $TA_{\mathrm{crys}}(M)$. We study the relation between $M_{\mathcal{P}^{(N)}}$ and $TA^{(N+)}_{\mathrm{crys}}(M)$.

We choose and fix coordinates $s_1, \ldots, s_e \in \mathcal{B}^\times$ of $\mathcal{B}$ over O_K, and let $\varphi_{\mathcal{B}} \colon \mathcal{B} \to \mathcal{B}$ be the unique lifting of the absolute Frobenius of $\mathcal{B}_1$ compatible with $\sigma \colon O_K \to O_K$ such that $\varphi_{\mathcal{B}}(s_i) = s_i^p$ for every $i \in \mathbb{N}\cap[1,e]$. We further choose a compatible system of p^nth roots $s_{i,n} \in \overline{\mathcal{A}}^\times$ of the image of s_i in $\mathcal{A}^\times$ for each $i \in \mathbb{N}\cap[1,e]$, and define the homomorphisms $\beta^{(0)} \colon \mathcal{B} \to A_{\inf}(\overline{\mathcal{A}})$ and $\beta \colon \mathcal{P} \to A_{\mathrm{crys}}(\overline{\mathcal{A}})$ as before Lemma 34.

Lemma 180 *The homomorphism β induces a homomorphism $\beta^{(N)} \colon \mathcal{P}^{(N)} \to A^{(N+)}_{\mathrm{crys}}(\overline{\mathcal{A}})$ for $N \in \mathbb{N}_{>0}$. Moreover $\beta^{(N)}$ is compatible with the filtrations.*

Proof Since the image of $\mathrm{Ker}(\mathcal{B}_m \to \mathcal{A}_m)$ under the reduction mod p^m of $\beta^{(0)}$ is contained in $\mathrm{Ker}(A_{\inf}(\overline{\mathcal{A}})/p^m \to \widehat{\overline{\mathcal{A}}}/p^m) = \xi(A_{\inf}(\overline{\mathcal{A}})/p^m)$, the image of $\mathcal{P}^{(N)}_m$ under $\beta_m \colon \mathcal{P}_m \to A_{\mathrm{crys},m}(\overline{\mathcal{A}})$ is contained in the image of $A^{(N+)}_{\mathrm{crys}}(\overline{\mathcal{A}})$. By Lemma 161, this implies $\beta(\mathcal{P}^{(N)}) \subset A^{(N+)}_{\mathrm{crys}}(\overline{\mathcal{A}})$. The last claim of the lemma follows from $\mathrm{Fil}^r \mathcal{P}^{(N)} \subset \mathrm{Fil}^r \mathcal{P}$ and the fact that $A^{(N+)}_{\mathrm{crys}}(\overline{\mathcal{A}}) \to A_{\mathrm{crys}}(\overline{\mathcal{A}})$ is strictly compatible with the filtrations (Lemmas 160 (2) and 152). □

We can apply Definition 40 with $a = p-2$ to $(\mathcal{P}^{(N)}, p, \varphi_{\mathcal{P}^{(N)}}, \mathrm{Fil}^\bullet \mathcal{P}^{(N)})$ because $\varphi(\mathrm{Fil}^r \mathcal{P}^{(N)}) \subset p^r \mathcal{P}^{(N)}$ ($r \in \mathbb{N}\cap[0,p-2]$), and then (44) to the homomorphism $\beta^{(N)} \colon \mathcal{P}^{(N)} \to A^{(N+)}_{\mathrm{crys}}(\overline{\mathcal{A}})$. By (177) and (178), the $\mathcal{P}^{(N)}$-module $M_{\mathcal{P}^{(N)}}$ with $\mathrm{Fil}^r M_{\mathcal{P}^{(N)}}$ ($r \in \mathbb{N}\cap[0,p-2]$) and φ is an object of $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(\mathcal{P}^{(N)}, \varphi)$.

Proposition 181 *The isomorphism (37) induces an isomorphism*

$$TA^{(N+)}_{\mathrm{crys}}(M) \cong M_{\mathcal{P}^{(N)}} \otimes_{\mathcal{P}^{(N)}, \beta^{(N)}} A^{(N+)}_{\mathrm{crys}}(\overline{\mathcal{A}})$$

in the category $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(A^{(N+)}_{\mathrm{crys}}(\overline{\mathcal{A}}), \varphi)$.

Proof We have isomorphisms

$$TA^{(N+)}_{\mathrm{crys}}(M)\otimes_{A^{(N+)}_{\mathrm{crys}}(\overline{\mathcal{A}})}A_{\mathrm{crys}}(\overline{\mathcal{A}})\xrightarrow{\cong}TA_{\mathrm{crys}}(M),$$

$$(M_{\mathcal{P}^{(N)}}\otimes_{\mathcal{P}^{(N)},\beta^{(N)}}A^{(N+)}_{\mathrm{crys}}(\overline{\mathcal{A}}))\otimes_{A^{(N+)}_{\mathrm{crys}}(\overline{\mathcal{A}})}A_{\mathrm{crys}}(\overline{\mathcal{A}})\xrightarrow{\cong}M_{\mathcal{P}}\otimes_{\mathcal{P},\beta}A_{\mathrm{crys}}(\overline{\mathcal{A}})$$

in the category $\mathrm{MF}^p_{[0,p-2],\mathrm{free}}(A_{\mathrm{crys}}(\overline{\mathcal{A}}),\varphi)$. Hence the claim follows from Proposition 164 for (160). □

We define $u_i\in\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ $(i\in\mathbb{N}\cap[1,e])$ as in Sect. 2 by using s_i and $s_{i,n}$ chosen before Lemma 180. For a positive integer N, we define the subring $\mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ of $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ to be the inverse limit of the image of $\oplus_{\underline{n}\in\mathbb{N}^e}A^{(N+)}_{\mathrm{crys}}(\overline{\mathcal{A}})\prod_{1\leq i\leq e}u_i^{[n_i]}$ in $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ for $m\in\mathbb{N}_{>0}$. By Lemma 161, $\mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ is isomorphic to the inverse limit of $\oplus_{\underline{n}\in\mathbb{N}^e}A^{(N+)}_{\mathrm{crys}}(\overline{\mathcal{A}})/(I^rA^{(N+)}_{\mathrm{crys}}(\overline{\mathcal{A}})+p^mA^{(N+)}_{\mathrm{crys}}(\overline{\mathcal{A}}))\prod_{1\leq i\leq e}u_i^{[n_i]}$ $(r,m\in\mathbb{N}_{>0})$. We define the ideal $\mathrm{Fil}^r\mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ $(r\in\mathbb{Z})$ of $\mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ to be the inverse limit of the image of $\oplus_{\underline{n}\in\mathbb{N}^e}\mathrm{Fil}^{r-|\underline{n}|}A^{(N+)}_{\mathrm{crys}}(\overline{\mathcal{A}})\prod_{1\leq i\leq e}u_i^{[n_i]}$ in $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$. We have $\mathrm{Fil}^r\mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})\cdot\mathrm{Fil}^s\mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})\subset\mathrm{Fil}^{r+s}\mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ for $r,s\in\mathbb{Z}$.

Lemma 182 *We have* $\pi^{-1}([\underline{\varepsilon}^a]-1)^{[n]}\in I^{n-1}W^{\mathrm{PD},(1)}(R_{O_{\overline{K}}})$ *for* $a\in\mathbb{Z}_p$ *and* $n\in\mathbb{N}_{>0}$.

Proof As $[\underline{\varepsilon}^a]-1\in I^1A_{\mathrm{inf}}(O_{\overline{K}})=\pi A_{\mathrm{inf}}(O_{\overline{K}})$, it suffices to prove the claim for $a=1$. We have $(n!)^{-1}T^{n-1}\in\mathbb{Z}_p[T,\frac{T^{p-1}}{p}]$ for $n\in\mathbb{N}_{>0}$ because $v_p(n!)\leq\frac{n-1}{p-1}$. Hence we obtain the claim from Proposition 147 and $I^rW^{\mathrm{PD},(1)}(R_{O_{\overline{K}}})=W^{\mathrm{PD},(1)}(R_{O_{\overline{K}}})\cap\pi^r\cdot W(R_{O_{\overline{K}}})[\frac{1}{p}]$. □

Lemma 183 *Let* N *be a positive integer.*

(1) $\mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ *and* $\mathrm{Fil}^r\mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ $(r\in\mathbb{Z})$ *do not depend on the choice of* $s_{i,n}$ $(i\in\mathbb{N}\cap[1,e],n\in\mathbb{N}_{>0})$.
(2) $\mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ *is stable under the action of* $G_{\mathcal{A}}$, ∇ *and* φ. *The filtration* $\mathrm{Fil}^\bullet\mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ *of* $\mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ *is also* $G_{\mathcal{A}}$*-stable, and we have* $\nabla(\mathrm{Fil}^r\mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}))\subset\mathrm{Fil}^{r-1}\mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})\otimes_{\mathcal{B}}\Omega_{\mathcal{B}}$ $(r\in\mathbb{Z})$.

Proof We have $([\underline{\varepsilon}^a]-1)^{[n]}\in I^nW^{\mathrm{PD},(1)}(R_{O_{\overline{K}}})$ for $a\in\mathbb{Z}_p$ and $n\in\mathbb{N}$ by Lemma 182. Therefore one can verify the claims in the same way as the proof of Lemma 126 (1) and (3). Note $\varphi_{\mathcal{B}}(s_i)=s_i^p$ $(i\in\mathbb{N}\cap[1,e])$. □

Lemma 184 *The canonical homomorphism* $\iota\colon\mathcal{P}\to\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ *of filtered rings induces a homomorphism* $\iota^{(N)}\colon\mathcal{P}^{(N)}\to\mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ *of filtered rings for any positive integer* N.

Proof By (42) applied to $M=\mathcal{A}$, we see that the image of $x\in\mathcal{P}^{(N)}$ under ι is given by $\sum_{\underline{n}\in\mathbb{N}^e}\beta^{(N)}(\nabla^{\log}_{\underline{n}}(x))\prod_{1\leq i\leq e}(u'_i)^{[n_i]}$, which is contained in $\mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ because the image of the infinite sum in $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ is a finite sum for every $m \in \mathbb{N}$, and the image $(\sum_{l\geq 1}(-1)^l l! u_{i,m}^{[l]})^{[n]}$ of $(u_i')^{[n]}$ in $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ is contained in $\oplus_{s\geq n}\mathbb{Z}/p^m u_{i,m}^{[s]}$. The compatibility with the filtrations follows from $\nabla_{\underline{n}}^{\log}(\mathrm{Fil}^r\mathcal{P}^{(N)}) \subset \mathrm{Fil}^{r-|\underline{n}|}\mathcal{P}^{(N)}$ ($\underline{n} \in \mathbb{N}^e$) and $\beta^{(N)}(\mathrm{Fil}^r\mathcal{P}^{(N)}) \subset \mathrm{Fil}^r A_{\mathrm{crys}}^{(N+)}(\overline{\mathcal{A}})$ (Lemma 180). □

Proposition 185 *For any positive integer N, the isomorphism (39) induces an isomorphism*

$$TA_{\mathrm{crys}}^{(N+)}(M) \otimes_{A_{\mathrm{crys}}^{(N+)}(\overline{\mathcal{A}})} \mathscr{A}_{\mathrm{crys},\mathcal{B}}^{(N+)}(\overline{\mathcal{A}}) \xrightarrow{\cong} M_{\mathcal{P}^{(N)}} \otimes_{\mathcal{P}^{(N)},\iota^{(N)}} \mathscr{A}_{\mathrm{crys},\mathcal{B}}^{(N+)}(\overline{\mathcal{A}}). \tag{180}$$

Proof By Proposition 181, it suffices to prove that the isomorphism (40) induces an isomorphism

$$M_{\mathcal{P}^{(N)}} \otimes_{\mathcal{P}^{(N)},\beta^{(N)}} \mathscr{A}_{\mathrm{crys},\mathcal{B}}^{(N+)}(\overline{\mathcal{A}}) \xrightarrow{\cong} M_{\mathcal{P}^{(N)}} \otimes_{\mathcal{P}^{(N)},\iota^{(N)}} \mathscr{A}_{\mathrm{crys},\mathcal{B}}^{(N+)}(\overline{\mathcal{A}}).$$

The images in $(M_{\mathcal{P}} \otimes_{\mathcal{P},\beta} \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}))/p^m$ and $(M_{\mathcal{P}} \otimes_{\mathcal{P},\iota} \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}))/p^m$ of the infinite sums appearing in (41) and (42) become finite sums. Note that the image of $(u_i')^{[n]}$ ($n \in \mathbb{N}$) in $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$ is contained in $\oplus_{s\geq n}\mathbb{Z}/p^m u_{i,m}^{[s]}$ as mentioned in the proof of Lemma 184. This implies that the reduction mod p^m of (40) induces an isomorphism between the images of $M_{\mathcal{P}^{(N)}} \otimes_{\mathcal{P}^{(N)},\beta^{(N)}} \mathscr{A}_{\mathrm{crys},\mathcal{B}}^{(N+)}(\overline{\mathcal{A}})$ and $M_{\mathcal{P}^{(N)}} \otimes_{\mathcal{P}^{(N)},\iota^{(N)}} \mathscr{A}_{\mathrm{crys},\mathcal{B}}^{(N+)}(\overline{\mathcal{A}})$. We obtain the claim by taking the inverse limit. □

Let $t_1, \ldots, t_d \in A^\times$ be coordinates of A over O_K, and let $\varphi_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ be the unique lifting of the absolute Frobenius of $\mathcal{A}/p$ compatible with $\sigma : O_K \to O_K$ such that $\varphi_{\mathcal{A}}(t_i) = t_i^p$ for all $i \in \mathbb{N} \cap [1, d]$. We prove an analogue of Proposition 185 for $TA_{\inf}^{\square}(M)$ (Sect. 13).

We define $TA_{\mathrm{crys}}^{\square,(N+)}(M)$ to be $TA_{\inf}^{\square}(M) \otimes_{A_{\inf}^{\square}(A)} A_{\mathrm{crys}}^{\square,(N+)}(\mathcal{A})$ endowed with semilinear extensions of φ and $\widetilde{\Gamma}_{\mathcal{A}}$-action, and with the product filtrations.

Recall that we have defined a homomorphism $\alpha : \mathcal{A} \to A_{\inf}^{\square}(\mathcal{A})$ (80).

Proposition 186 *The isomorphism (92) induces the following isomorphism in the category* $\mathrm{MF}_{[0,p-2],\mathrm{free}}^{p}(A_{\mathrm{crys}}^{\square,(N+)}(\mathcal{A}), \varphi)$.

$$TA_{\mathrm{crys}}^{\square,(N+)}(M) \cong M \otimes_{\mathcal{A},\alpha} A_{\mathrm{crys}}^{\square,(N+)}(\mathcal{A})$$

Proof The same as the proof of Proposition 181 using Proposition 172. □

We define the subring $\mathscr{A}_{\mathrm{crys}}^{\square,(N+)}(\mathcal{A})$ of $\mathscr{A}_{\mathrm{crys}}^{\square}(\mathcal{A})$ (Sect. 12) to be the inverse limit of the images of $\oplus_{\underline{n}\in\mathbb{N}^d} A_{\mathrm{crys}}^{\square,(N+)}(\mathcal{A}) \prod_{1\leq i\leq d} v_i^{[n_i]}$ in $\mathscr{A}_{\mathrm{crys},m}^{\square}(\mathcal{A})$. It is isomorphic to the inverse limit of $\oplus_{\underline{n}\in\mathbb{N}^d} A_{\mathrm{crys},(I^n,p^m)}^{\square,(N+)}(\mathcal{A}) \prod_{1\leq i\leq d} v_i^{[n_i]}$ over $m, n \in \mathbb{N}$ by Lemma 173.

Lemma 187 (1) $\mathscr{A}_{\mathrm{crys}}^{\square,(N+)}(\mathcal{A})$ *does not depend on the choice of the p^nth roots $t_{i,n}$ of t_i ($i \in \mathbb{N} \cap [1, d]$, $n \in \mathbb{N}_{>0}$) used in the definition of v_i.*

(2) $\mathscr{A}_{\mathrm{crys}}^{\square,(N+)}(\mathcal{A})$ *is stable under the action of $\widetilde{\Gamma}_{\mathcal{A}}$, ∇ and φ.*

Proof The same as Lemma 183. □

Lemma 188 *The canonical homomorphism* $\iota\colon \mathcal{A} \to \mathscr{A}_{\mathrm{crys}}^{\square}(\mathcal{A})$ *factors through* $\mathscr{A}_{\mathrm{crys}}^{\square,(N+)}(\mathcal{A})$.

Proof The same as Lemma 184 by using (96) for $M = \mathcal{A}$. □

Proposition 189 *The isomorphism* (94) *induces an isomorphism*

$$TA_{\mathrm{crys}}^{\square,(N+)}(M) \otimes_{A_{\mathrm{crys}}^{\square,(N+)}(\mathcal{A})} \mathscr{A}_{\mathrm{crys}}^{\square,(N+)}(\mathcal{A}) \xrightarrow{\cong} M \otimes_{\mathcal{A},\iota} \mathscr{A}_{\mathrm{crys}}^{\square,(N+)}(\mathcal{A}). \tag{181}$$

Proof The same as Proposition 185 using Proposition 186 and (96). □

20 Comparison Morphism from de Rham Complex over A_{crys}

We follow the notation in Sects. 14, 15, 17, and 18.

Let $\mathcal{B} \to \mathcal{A}$, $s_1, \ldots, s_e \in \mathcal{B}^\times$, $\Omega_{\mathcal{B}}$, $\varphi_{\mathcal{B}}$, $\mathcal{P}_m$, $\mathcal{P}$, $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$, $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$, $u_{i,m} \in \mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}})$, and $u_i \in \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ be as in Sect. 2. We assume that $\varphi_{\mathcal{B}}$ is the unique lifting of Frobenius satisfying $\varphi_{\mathcal{B}}(s_i) = s_i^p$ for all $i \in \mathbb{N} \cap [1, e]$. Let M be an object of $\mathrm{MF}_{[0,p-2],\mathrm{free}}^{\nabla}(\mathcal{A}, \Phi)$, and define $M_{\mathcal{P}_m}$ and $M_{\mathcal{P}}$ associated to M as in Sect. 5. Let $A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M_{\mathcal{P}}$ be the p-adic completion of $A_{\mathrm{crys}}(O_{\overline{K}}) \otimes_{O_K} M_{\mathcal{P}}$, which is naturally endowed with an action of G_K, φ, and an integrable connection ∇ with respect to $\mathcal{B}/O_K$. Put $\Omega_{\mathcal{B}}^q := \wedge_{\mathcal{B}}^q \Omega_{\mathcal{B}}$ $(q \in \mathbb{N})$.

In this section, we will construct a morphism

$$A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M_{\mathcal{P}} \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}^{\bullet} \longrightarrow A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}^L_{A_{\mathrm{inf}}(O_{\overline{K}})} L\eta_{\pi}^{+} R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M)) \tag{182}$$

compatible with φ in the derived category $D(\underline{G_K}\text{-}\mathbf{Sets}, A_{\mathrm{crys}}(O_{\overline{K}}))$ (see Sect. 14) of $A_{\mathrm{crys}}(O_{\overline{K}})$-modules with semilinear action of G_K. The construction a priori depends on the choice of $s_1, \ldots, s_e \in \mathcal{B}^\times$. We will show certain functoriality of (182) in $(\mathcal{B}, s_1, \ldots, s_e)$ (Proposition 201) and, as its consequence, prove that (182) does not depend on the choice of $s_1, \ldots, s_e$ (Theorem 203). We will prove that (182) is a quasi-isomorphism in Sect. 21 (Theorem 204).

For $N \in \mathbb{N}_{>0}$, we define $TA_{\mathrm{crys}}^{(N)}(M)$ to be $A_{\mathrm{crys}}^{(N)}(\overline{\mathcal{A}}) \otimes_{A_{\mathrm{inf}}(\overline{\mathcal{A}})} TA_{\mathrm{inf}}(M)$, which is a free $A_{\mathrm{crys}}^{(N)}(\overline{\mathcal{A}})$-module of finite type naturally endowed with a semilinear action of $G_{\mathcal{A}}$ and a semilinear $G_{\mathcal{A}}$-equivariant endomorphism φ. The action of $G_{\mathcal{A}}$ is continuous by Proposition 162. We define an object $R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{crys}}^{(N)}(M))$ of $D(\underline{G_K}\text{-}\mathbf{Sets}, A_{\mathrm{crys}}^{(N)}(O_{\overline{K}}))$ in the same way as $R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M))$ defined in Sect. 15 by using the sequence $(p^m A_{\mathrm{crys}}^{(N)}(O_{\overline{K}}))_{m\in\mathbb{N}}$ in $\mathscr{S}_{\mathrm{crys}}^{(N)}$. To construct the morphism (182), we first prove the following description of the target with $A_{\mathrm{crys}}(O_{\overline{K}})$ replaced by $A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})$ in terms of the cohomology of $TA_{\mathrm{crys}}^{(N)}(M)$ (cf. [7, Sect. 12]).

Proposition 190 *For each $N \in \mathbb{N}_{>0}$, the following canonical morphisms in the derived category $D(\underline{G_K}\text{-}\mathbf{Sets}, A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}))$ of $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$-modules with semilinear action of G_k are isomorphisms.*

$$\begin{aligned}&A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}^L_{A_{\mathrm{inf}}(O_{\overline{K}})} L\eta^+_\pi R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M))\\ &\to R\varprojlim_m (L\eta^+_\pi R\Gamma(\Delta_{\mathcal{A}}, TA^{(N)}_{\mathrm{crys}}(M)) \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^m\mathbb{Z}) \leftarrow L\eta^+_\pi R\Gamma(\Delta_{\mathcal{A}}, TA^{(N)}_{\mathrm{crys}}(M)).\end{aligned}$$

Let $\square$ be the framing considered in Sects. 12 and 18. For $N \in \mathbb{N}_{>0}$, we define $TA^{\square,(N)}_{\mathrm{crys}}(M)$ and $T\widetilde{A}^{\square,(N)}_{\mathrm{crys}}(M)$ to be $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A}) \otimes_{A^{\square}_{\mathrm{inf}}(\mathcal{A})} TA^{\square}_{\mathrm{inf}}(M)$ and $A^{(N)}_{\mathrm{crys}}(\mathcal{A}_\infty) \otimes_{A^{\square}_{\mathrm{inf}}(\mathcal{A})} TA^{\square}_{\mathrm{inf}}(M)$, respectively, which are naturally endowed with semilinear actions of $\widetilde{\Gamma}_{\mathcal{A}}$ and $\widetilde{\Gamma}_{\mathcal{A}}$-equivariant semilinear endomorphisms φ. The actions of $\widetilde{\Gamma}_{\mathcal{A}}$ are continuous by Proposition 162 and Lemma 165. We define the cohomology $R\Gamma(\Gamma_{\mathcal{A}}, TA^{\square,(N)}_{\mathrm{crys}}(M))$ and $R\Gamma(\Gamma_{\mathcal{A}}, T\widetilde{A}^{\square,(N)}_{\mathrm{crys}}(M))$ in the same way as $R\Gamma(\Delta_{\mathcal{A}}, TA^{(N)}_{\mathrm{crys}}(M))$ above. We prove Proposition 190 by reducing it to a corresponding claim for $TA^{\square}_{\mathrm{inf}}(M)$ and $TA^{\square,(N)}_{\mathrm{crys}}(M)$.

Proposition 191 (cf. Corollary 110) *For each $N \in \mathbb{N}_{>0}$, the cohomology of the mapping cone of*

$$R\Gamma(\Gamma_{\mathcal{A}}, T\widetilde{A}^{\square,(N)}_{\mathrm{crys}}(M)) \longrightarrow R\Gamma(\Delta_{\mathcal{A}}, TA^{(N)}_{\mathrm{crys}}(M))$$

is annihilated by the kernel of $A_{\mathrm{inf}}(O_{\overline{K}}) \to W(\bar{k})$.

Proof By Lemma 109, it suffices to prove that the cohomology of the cone of

$$R\Gamma(\Gamma_{\mathcal{A}}, T\widetilde{A}^{\square,(N)}_{\mathrm{crys}}(M)/p^m) \to R\Gamma(\Delta_{\mathcal{A}}, TA^{(N)}_{\mathrm{crys}}(M)/p^m)$$

is annihilated by $[\underline{p}^{-l}]$ for all $l \in \mathbb{N}$, and it is reduced to the case $m = 1$. For $\Lambda = \overline{\mathcal{A}}, \mathcal{A}_\infty$, the homomorphism $A_{\mathrm{inf}}(\Lambda) \to A^{(N)}_{\mathrm{crys}}(\Lambda)/p$ factors through the quotient $A_{\mathrm{inf}}(\Lambda)/(p, [\underline{p}]^p) \cong R_\Lambda/\underline{p}^p \xrightarrow{\cong} \Lambda/p$; $(a_n)_{n\in\mathbb{N}} \mapsto a_1$ of $A_{\mathrm{inf}}(\Lambda)$. By using Proposition 143 (2), we obtain a $G_{\mathcal{A}}$-equivariant isomorphism $A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})/p \cong A^{(N)}_{\mathrm{crys}}(\mathcal{A}_\infty)/p \otimes_{\mathcal{A}_\infty/p} \overline{\mathcal{A}}/p$. By (107), we have a $G_{\mathcal{A}}$-equivariant isomorphism $TA^{(N)}_{\mathrm{crys}}(M)/p \cong T\widetilde{A}^{\square,(N)}_{\mathrm{crys}}(M)/p \otimes_{\mathcal{A}_\infty/p} \overline{\mathcal{A}}/p$. Hence the claim follows from the almost purity theorem by Faltings ([11, 2b, 2c], [9, 2.4. Theorem (ii)], [2, Proposition V.12.8]) in the same way as the proof of Proposition 107. $\square$

Proposition 192 (cf. Lemma 115) *For each $N \in \mathbb{N}_{>0}$, the cohomology of*

$$A_{\mathrm{inf}}(O_{\overline{K}})/\pi A_{\mathrm{inf}}(O_{\overline{K}}) \otimes^L_{A_{\mathrm{inf}}(O_{\overline{K}})} R\Gamma(\Gamma_{\mathcal{A}}, T\widetilde{A}^{\square,(N)}_{\mathrm{crys}}(M))$$

has no non-zero element annihilated by the ideal $\sum_{l\in\mathbb{Z}}[\underline{p}^{p^{-l}}]A_{\mathrm{inf}}(O_{\overline{K}})$ of $A_{\mathrm{inf}}(O_{\overline{K}})$.

Proof Put $\mathcal{I} := \sum_{l\in\mathbb{Z}}[\underline{p}^{p^{-l}}]A_{\mathrm{inf}}(O_{\overline{K}})$. We abbreviate $K(-;\gamma_1-1,\dots,\gamma_d-1)$ to $K_\gamma(-)$. We obtain $R\Gamma(\Gamma_{\mathcal{A}}, T\widetilde{A}^{\square,(N)}_{\mathrm{crys}}(M)) \cong K_\gamma(T\widetilde{A}^{\square,(N)}_{\mathrm{crys}}(M))$ by the same argument as the proof of Lemma 117. Since $A^{(N)}_{\mathrm{crys}}(\mathcal{A}_\infty)$ is π-torsion free (Proposition 156 (1)), the complex considered in the proposition is quasi-isomorphic to $K_\gamma(T\widetilde{A}^{\square,(N)}_{\mathrm{crys}}(M)/\pi)$. Put $T := TA^{\square,(N)}_{\mathrm{crys}}(M)/\pi$ and $\widetilde{T} := T\widetilde{A}^{\square,(N)}_{\mathrm{crys}}(M)/\pi$. The action of $\Gamma_{\mathcal{A}}$ on T is trivial by Proposition 168 (3) and Lemma 98. Recall that $A^{(N)}_{\mathrm{crys}}(\mathcal{A}_\infty)/\pi$ and $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})/\pi$ are p-adically complete and separated by Propositions 156 (2) and 168 (2). By Proposition 170 (3) and Lemma 166 (2), $\widetilde{T}/p^m$ is the direct sum of $[\underline{t}^{\underline{r}}]T/p^m$ ($\underline{r}\in(\mathbb{Z}[\frac{1}{p}]\cap[0,1[)^d$), and we have $T/p^m \xrightarrow{\cong} [\underline{t}^{\underline{r}}]T/p^m;\, x\mapsto[\underline{t}^{\underline{r}}]x$. For a non-zero $\underline{r}=(r_i)_{1\le i\le d}\in(\mathbb{Z}[\frac{1}{p}]\cap[0,1[)^d$, let $\nu(\underline{r})$ be the positive integer defined by $\underline{r}\in p^{-\nu(\underline{r})}\mathbb{Z}^d\backslash p^{-\nu(\underline{r})+1}\mathbb{Z}^d$. For $i\in\mathbb{N}\cap[1,d]$, let Λ_i be the subset of $(\mathbb{Z}[\frac{1}{p}]\cap[0,1[)^d$ consisting of $\underline{r}\neq\underline{0}$ such that i is the smallest integer $\in[1,d]$ satisfying $r_i\notin p^{-\nu(\underline{r})+1}\mathbb{Z}$. For $i\in\mathbb{N}\cap[1,d]$, we define $\widetilde{T}_i$ to be the inverse limit of the direct sum of $[\underline{t}^{\underline{r}}]T/p^m$ ($\underline{r}\in\Lambda_i$) with respect to m. Then $\widetilde{T}$ is the direct sum of T and $\widetilde{T}_i$ ($i\in\mathbb{N}\cap[1,d]$), $\widetilde{T}_i$ is $\Gamma_{\mathcal{A}}$-stable, and an element of $\widetilde{T}_i$ is uniquely written as $\sum_{\underline{r}\in\Lambda_i}[\underline{t}^{\underline{r}}]x_{\underline{r}}$ ($x_{\underline{r}}\in T$, $x_{\underline{r}}\to 0$ p-adically as $\nu(\underline{r})\to\infty$).

We have $H^q(K_\gamma(T))\cong T\otimes_{\mathbb{Z}}\wedge^q\mathbb{Z}^d$, and $T[\mathcal{I}]=0$ by Corollary 171. Put $\pi_r=[\underline{\varepsilon}^r]-1$ for $r\in\mathbb{Z}[\frac{1}{p}]$. For $i,j\in\mathbb{N}\cap[1,d]$, $j\neq i$, we define the endomorphism $g_i^{(j)}$ of $\widetilde{T}_i$ by $g_i^{(j)}(\sum_{\underline{r}\in\Lambda_i}[\underline{t}^{\underline{r}}]x_{\underline{r}})=\sum_{\underline{r}\in\Lambda_i}[\underline{t}^{\underline{r}}]\pi_{r_j}\pi_{r_i}^{-1}x_{\underline{r}})$ (Lemma 113 (1)). Then, for each i, the $A_{\mathrm{inf}}(O_{\overline{K}})$-linear endomorphisms $g_i^{(j)}$ ($j\neq i$) and γ_i of $\widetilde{T}_i$ commute with each other, and satisfy $\gamma_j-1=(\gamma_i-1)\circ g_i^{(j)}$ ($j\neq i$). By [7, Lemma 7.10], $H^q(K_\gamma(\widetilde{T}_i))$ is isomorphic to the direct sum of $\binom{d-1}{q}$ copies of $\widetilde{T}_i^{\gamma_i=1}$ and $\binom{d-1}{q-1}$ copies of $\widetilde{T}_i/(\gamma_i-1)\widetilde{T}_i$ as $A_{\mathrm{inf}}(O_{\overline{K}})$-modules. We have $\widetilde{T}_i^{\gamma_i=1}[\mathcal{I}]=0$ by Corollary 171. It remains to prove $\widetilde{T}_i/(\gamma_i-1)\widetilde{T}_i[\mathcal{I}]=0$. Suppose that the image of an element $x=\sum_{\underline{r}\in\Lambda_i}[\underline{t}^{\underline{r}}]x_{\underline{r}}$ ($x_{\underline{r}}\in T$) of $\widetilde{T}_i$ in $\widetilde{T}_i/(\gamma_i-1)\widetilde{T}_i$ is annihilated by $\mathcal{I}$. Then the image of $x_{\underline{r}}$ in $T/\pi_{r_i}T$ is annihilated by $\mathcal{I}$. By Lemma 113 (1), we have $\pi_{r_i}A_{\mathrm{inf}}(O_{\overline{K}})=\varphi^{-\nu(\underline{r})}(\pi)A_{\mathrm{inf}}(O_{\overline{K}})$. Therefore we have $x_{\underline{r}}\in\pi_{r_i}T$ by Corollary 171. Choose $\mu(\underline{r})\in\mathbb{N}$ for each $\underline{r}\in\Lambda_i$ such that $x_{\underline{r}}\in p^{\mu(\underline{r})}T$ and $\mu(\underline{r})\to\infty$ as $\nu(\underline{r})\to\infty$. Put $y_{\underline{r}}:=p^{-\mu(\underline{r})}x_{\underline{r}}\in T$. Then we have $p^Ny_{\underline{r}}\in\pi_{r_i}T$ for $\underline{r}\in\Lambda_i$ with $\mu(\underline{r})>N$ by Proposition 168 (2). For $\underline{r}\in\Lambda_i$ with $\mu(\underline{r})\le N$ (resp. $\mu(\underline{r})>N$), choose $z_{\underline{r}}$ (resp. $w_{\underline{r}}$) $\in T$ such that $x_{\underline{r}}=\pi_{r_i}z_{\underline{r}}$ (resp. $p^Ny_{\underline{r}}=\pi_{r_i}w_{\underline{r}}$). Then $\sum_{\underline{r}\in\Lambda_i,\mu(\underline{r})\le N}[\underline{t}^{\underline{r}}]z_{\underline{r}}+\sum_{\underline{r}\in\Lambda_i\mu(\underline{r})>N}[\underline{t}^{\underline{r}}]p^{\mu(\underline{r})-N}w_{\underline{r}}$ converges to an element z of $\widetilde{T}_i$, and we have $x=(\gamma_i-1)(z)$. □

Corollary 193 *For each $N\in\mathbb{N}_{>0}$, the following morphism in the derived category $D(\underline{G_K}$-**Sets**, $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}))$ (Sect. 14) of $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$-modules with semilinear G_K-action is an isomorphism.*

$$L\eta^+_\pi R\Gamma(\Gamma_{\mathcal{A}}, T\widetilde{A}^{\square,(N)}_{\mathrm{crys}}(M)) \longrightarrow L\eta^+_\pi R\Gamma(\Delta_{\mathcal{A}}, TA^{(N)}_{\mathrm{crys}}(M))$$

Proof We may forget the action of G_K by Lemma 103. By Propositions 191 and 192, we can apply Lemma 111 (2) for $R=A_{\mathrm{inf}}(O_{\overline{K}})$, $I=\pi R$, and $J=\sum_{l\in\mathbb{N}}[\underline{p}^{p^{-l}}]R+\pi R$ to $R\Gamma(\Gamma_{\mathcal{A}}, T\widetilde{A}^{\square,(N)}_{\mathrm{crys}}(M))\to R\Gamma(\Delta_{\mathcal{A}}, TA^{(N)}_{\mathrm{crys}}(M))$. Then we obtain the claim from Lemma 111 (1) because $A_{\mathrm{crys}}^{(N)}(\mathcal{A}_\infty)$ and $A_{\mathrm{crys}}^{(N)}(\overline{\mathcal{A}})$ are π-torsion free by Lemma 156 (1). $\square$

Proposition 194 *For each $N \in \mathbb{N}_{>0}$, the following morphism in the derived category $D(\underline{G_K}$-**Sets**, $A_{\mathrm{crys}}^{(N)}(O_{\overline{K}}))$ (Sect. 14) of $A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})$-modules with semilinear G_k-action is an isomorphism.*

$$L\eta_\pi^+ R\Gamma(\Gamma_{\mathcal{A}}, TA_{\mathrm{crys}}^{\square,(N)}(M)) \longrightarrow L\eta_\pi^+ R\Gamma(\Gamma_{\mathcal{A}}, T\widetilde{A}_{\mathrm{crys}}^{\square,(N)}(M))$$

Proof We can prove the claim in the same way as Proposition 116 by using $p^m A_{\mathrm{crys}}^{(N)}(O_{\overline{K}}) \in \mathscr{S}_{\mathrm{crys}}^{(N)}$ instead of $\mathfrak{a}, \mathfrak{a}_n \in \mathscr{S}_{\mathrm{inf},\mathrm{G}}$, and the following facts. The rings $A_{\mathrm{crys}}^{(N)}(\mathcal{A}_\infty)$ and $A_{\mathrm{crys}}^{\square,(N)}(\mathcal{A})$ are p-torsion free and p-adically complete and separated (Lemma 166 (1)). The action of $\Gamma_{\mathcal{A}}$ on $TA_{\mathrm{crys}}^{\square,(N)}(M)/\pi$ is trivial (Proposition 168 (3), Lemma 98). The element π is regular in $A_{\mathrm{crys}}^{\square,(N)}(\mathcal{A})$ (Proposition 168 (1)). The element $(\pi([\underline{\varepsilon}^r]-1)^{-1})^p$ is contained in $pA_{\mathrm{crys}}^{(N)}(O_{\overline{K}})$ for $r \in \mathbb{Z}[\frac{1}{p}]\backslash\mathbb{Z}$ because $\pi([\underline{\varepsilon}^r]-1)^{-1} \in \xi A_{\mathrm{inf}}(O_{\overline{K}})$. We have $T\widetilde{A}_{\mathrm{crys}}^{\square,(N)}(M)/p^m = \oplus_{\underline{r}\in(\mathbb{Z}[\frac{1}{p}]\cap[0,1[)^d}[\underline{t}^{\underline{r}}]T/p^m$ and $T/p^m \xrightarrow{\cong} [\underline{t}^{\underline{r}}]T/p^m; x \mapsto [\underline{t}^{\underline{r}}]x$, where T denotes $TA_{\mathrm{crys}}^{\square,(N)}(M)$ (Lemma 166 (2), Proposition 170 (3)). $\square$

Proof of Proposition 190 By Propositions 112, 116, Corollary 193, and Proposition 194, it suffices to prove the claim with $\Delta_{\mathcal{A}}$, TA_{inf}, and $TA_{\mathrm{crys}}^{(N)}$ replaced by $\Gamma_{\mathcal{A}}$, $TA_{\mathrm{inf}}^{\square}$, and $TA_{\mathrm{crys}}^{\square,(N)}$. By Lemma 98, Proposition 168 (3), and the same argument as the proof of Lemma 117, we obtain isomorphisms

$$L\eta_\pi^+ R\Gamma(\Gamma_{\mathcal{A}}, T) \cong \eta_\pi^+ K(T; \gamma_1 - 1, \dots, \gamma_d - 1) \cong \pi^\bullet K^\bullet(T; \gamma_1 - 1, \dots, \gamma_d - 1)$$

for $T = TA_{\mathrm{inf}}^{\square}(M)$ (resp. $TA_{\mathrm{crys}}^{\square,(N)}(M)$) in the derived category of $A_{\mathrm{inf}}(O_{\overline{K}})$ (resp. $A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})$)-modules. Therefore it suffices to prove the isomorphism $A_{\mathrm{inf}}^{\square}(\mathcal{A}) \otimes^L_{A_{\mathrm{inf}}(O_{\overline{K}})} A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})/p^m \xrightarrow{\cong} A_{\mathrm{inf}}^{\square}(\mathcal{A}) \otimes_{A_{\mathrm{inf}}(O_{\overline{K}})} A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})/p^m$, which is verified in the same way as the proof of Lemma 125. $\square$

Let N be a positive integer. We define $\mathscr{A}_{\mathrm{crys},\mathcal{B}}^{(N)}(\overline{\mathcal{A}})$ to be the p-adic completion of $\oplus_{\underline{n}\in\mathbb{N}^e} A_{\mathrm{crys}}^{(N)}(\overline{\mathcal{A}}) \prod_i u_i^{[n_i]}$ regarded as a subring of $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$.

Lemma 195 (1) *$\mathscr{A}_{\mathrm{crys},\mathcal{B}}^{(N)}(\overline{\mathcal{A}})$ does not depend on the choice of the compatible system of p^nth roots $s_{i,n}$ of the image of s_i in $\mathcal{A}$ used in the construction of $u_{i,m}$ and u_i.*
(2) *$\mathscr{A}_{\mathrm{crys},\mathcal{B}}^{(N)}(\overline{\mathcal{A}})$ is stable under the action of $G_{\mathcal{A}}$, ∇ and φ.*

Proof We have $([\underline{\varepsilon}^a]-1)^{[n]} \in W^{\mathrm{PD},(1)}(R_{O_{\overline{K}}})$ $(a \in \mathbb{Z}_p, n \in \mathbb{N})$ by Lemma 182, and one can verify the claims in the same way as the proof of Lemma 126 (1) and (3). Note $\varphi_{\mathcal{B}}(s_i) = s_i^p$ $(i \in \mathbb{N} \cap [1, e])$. $\square$

By Corollary 154, the natural homomorphism $\mathscr{A}^{(N)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \to \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ is injective. Let N be an integer ≥ 2. Then the homomorphism $W^{\mathrm{PD},(N-1)}(R_{\overline{\mathcal{A}}}) \to W^{\mathrm{PD},(N)}(R_{\overline{\mathcal{A}}})/p^m$ factors through $W^{\mathrm{PD},(N-1)}(R_{\overline{\mathcal{A}}})/(I^{mp^{N-1}(p-1)}, p^m)$ (see the proof of Lemma 160 (2)), and therefore induces a homomorphism

$$\mathscr{A}^{((N-1)+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \longrightarrow \mathscr{A}^{(N)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \tag{183}$$

compatible with the action of $G_{\mathcal{A}}$, ∇, φ, and $A^{((N-1)+)}_{\mathrm{crys}}(\overline{\mathcal{A}}) \to A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})$ (Lemma 160 (2)). The homomorphism (183) is injective by Lemma 160 (2). By taking the scalar extension of (180) (with N replaced by $N-1$) under (183), we obtain the following.

Proposition 196 *For an integer $N \geq 2$, the isomorphism (39) induces an isomorphism*

$$TA^{(N)}_{\mathrm{crys}}(M) \otimes_{A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})} \mathscr{A}^{(N)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \xrightarrow{\cong} M_{\mathcal{P}^{(N-1)}} \otimes_{\mathcal{P}^{(N-1)}} \mathscr{A}^{(N)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}). \tag{184}$$

Let λ be one of π and $\varphi^{-1}(\pi)$. In order to construct the morphism (182) and show its compatibility with φ, we introduce a subcomplex $\mathscr{A}^{(N)}_{\mathrm{crys},\lambda}\Omega^{\bullet}_{\mathcal{B}}(\overline{\mathcal{A}})$ of $\lambda^{-e}\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega^{\bullet}_{\mathcal{B}}$ giving a resolution of $A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})$ such that the complex $\mathscr{A}^{(N)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega^{\bullet}_{\mathcal{B}}$ is contained in $\eta^{+}_{\lambda}(\mathscr{A}^{(N)}_{\mathrm{crys},\lambda}\Omega^{\bullet}_{\mathcal{B}}(\overline{\mathcal{A}}))$.

For $i \in \mathbb{N} \cap [1, e]$, we define $\mathscr{A}^{i,1,(N)}_{\mathrm{crys},\mathcal{B},\lambda}(\overline{\mathcal{A}})$ (resp. $\mathscr{A}^{i,0,(N)}_{\mathrm{crys},\mathcal{B},\lambda}(\overline{\mathcal{A}})$) to be the p-adic completion of $\oplus_{n\in\mathbb{N}}\lambda^{-1}A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})u_i^{[n]}$ (resp. $A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}}) \oplus (\oplus_{n\in\mathbb{N}_{>0}}\lambda^{-1}A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})u_i^{[n]})$) regarded as an $A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})$-submodule of $\lambda^{-1}\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$. Put $\omega_i := \nabla(u_i) = -(u_i + 1)d\log s_i \in \mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}$. The subcomplex $\mathscr{A}^{i,0,(N)}_{\mathrm{crys},\mathcal{B},\lambda}(\overline{\mathcal{A}}) \xrightarrow{\nabla} \mathscr{A}^{i,1,(N)}_{\mathrm{crys},\mathcal{B},\lambda}(\overline{\mathcal{A}})\omega_i$ of $\lambda^{-1}\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega^{\bullet}_{\mathcal{B}}$ gives a resolution of $A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})$. We define $\mathscr{A}^{(N)}_{\mathrm{crys},\lambda}\Omega^{\bullet}_{\mathcal{B}}(\overline{\mathcal{A}})$ to be the p-adic completion of the tensor product of the above complexes for $i \in \mathbb{N} \cap [1, e]$ over $A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})$, which may be regarded as a subcomplex of $\lambda^{-e}\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega^{\bullet}_{\mathcal{B}}$. We define $\mathscr{A}^{J,(N)}_{\mathrm{crys},\mathcal{B},\lambda}(\overline{\mathcal{A}}) \subset \lambda^{-e}\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$ for $J \subset \mathbb{N} \cap [1, e]$ to be the p-adic completion of

$$\bigoplus_{\underline{n}\in\mathbb{N}^e} \lambda^{-\sharp((\mathrm{Supp}\,\underline{n})\cup J)} A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}}) \prod_i u_i^{[n_i]},$$

where Supp $\underline{n}$ denotes the subset of $\mathbb{N} \cap [1, e]$ consisting of i with $n_i > 0$. Then we have

$$\mathscr{A}^{(N)}_{\mathrm{crys},\lambda}\Omega^q_{\mathcal{B}}(\overline{\mathcal{A}}) = \bigoplus_{J\subset\mathbb{N}\cap[1,e],\sharp J=q} \mathscr{A}^{J,(N)}_{\mathrm{crys},\mathcal{B},\lambda}(\overline{\mathcal{A}}) \otimes \omega_J,$$

where $\omega_J = \omega_{j_1} \wedge \cdots \wedge \omega_{j_q}$ for a subset $J = \{j_1 < \cdots < j_q\}$ of $\mathbb{N} \cap [1, e]$.

Lemma 197 *Let a be an element of $A_{\mathrm{inf}}(O_{\overline{K}})$, and assume that its image in $A_{\mathrm{inf}}(O_{\overline{K}})/p \cong R_{O_{\overline{K}}}$ is neither zero nor invertible and that $aA_{\mathrm{inf}}(O_{\overline{K}})$ is G_K-stable.*

Then the action of G_K on $a^{-1}A_{\mathrm{inf}}(O_{\overline{K}})$ is continuous with respect to the $(p, [\underline{p}])$-adic topology. (Note that the element a is regular by Lemma 1 (4)).

Proof For $x \in A_{\mathrm{inf}}(O_{\overline{K}})$ and $g \in G_K$, we have $g(a^{-1}x) - a^{-1}x = g(a)^{-1}(1 - a^{-1}g(a))g(x) + a^{-1}(g(x) - x)$. By Lemma 5, it suffices to prove the following: For any $m \in \mathbb{N}_{>0}$, there exists an open subgroup H_m of G such that $a^{-1}g(a) - 1 \in (p, [\underline{p}])^m$ for every $g \in H_m$. By Lemma 5 again, it suffices to prove that the multiplication by a induces a homeomorphism from $A_{\mathrm{inf}}(O_{\overline{K}})$ to $aA_{\mathrm{inf}}(O_{\overline{K}})$ endowed with the topology induced by that of $A_{\mathrm{inf}}(O_{\overline{K}})$. By Lemma 1 (3), we have $(p^m A_{\mathrm{inf}}(O_{\overline{K}}) + a^m A_{\mathrm{inf}}(O_{\overline{K}})) \cap aA_{\mathrm{inf}}(O_{\overline{K}}) = a(p^m A_{\mathrm{inf}}(O_{\overline{K}}) + a^{m-1}A_{\mathrm{inf}}(O_{\overline{K}}))$ for $m \in \mathbb{N}_{>0}$. Hence the claim follows from Lemma 1 (1). □

Lemma 198 *Let λ be one of π and $\varphi^{-1}(\pi)$.*

(1) *$\mathscr{A}^{(N)}_{\mathrm{crys},\lambda}\Omega^q_{\mathcal{B}}(\overline{\mathcal{A}})$ is the $\mathscr{A}^{(N)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$-submodule of $\lambda^{-e}\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega^q_{\mathcal{B}}$.*
(2) *$\mathscr{A}^{(N)}_{\mathrm{crys},\lambda}\Omega^q_{\mathcal{B}}(\overline{\mathcal{A}})$ does not depend on the choice of the compatible system of p^nth roots $s_{i,n}$ of the image of s_i in $\mathcal{A}$ used in the construction of $u_{i,m}$ and u_i.*
(3) *$\mathscr{A}^{(N)}_{\mathrm{crys},\lambda}\Omega^q_{\mathcal{B}}(\overline{\mathcal{A}})$ is stable under the action of $G_{\mathcal{A}}$, and the action of $G_{\mathcal{A}}$ is continuous with respect to the p-adic topology of $\mathscr{A}^{(N)}_{\mathrm{crys},\lambda}\Omega^q_{\mathcal{B}}(\overline{\mathcal{A}})$.*
(4) *The inclusion homomorphism $A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}}) \to \mathscr{A}^{(N)}_{\mathrm{crys},\lambda}\Omega^0_{\mathcal{B}}(\overline{\mathcal{A}})$ gives a resolution $A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})/p^m \to \mathscr{A}^{(N)}_{\mathrm{crys},\lambda}\Omega^\bullet_{\mathcal{B}}(\overline{\mathcal{A}})/p^m$ for every $m \in \mathbb{N}_{>0}$.*
(5) *We have $\mathscr{A}^{(N)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}} \subset \eta^+_\lambda(\mathscr{A}^{(N)}_{\mathrm{crys},\lambda}\Omega^\bullet_{\mathcal{B}}(\overline{\mathcal{A}}))$.*
(6) *We have $\mathscr{A}^{(N)}_{\mathrm{crys},\varphi^{-1}(\pi)}\Omega^q_{\mathcal{B}}(\overline{\mathcal{A}}) \subset \mathscr{A}^{(N)}_{\mathrm{crys},\pi}\Omega^q_{\mathcal{B}}(\overline{\mathcal{A}})$ and $\varphi(\mathscr{A}^{(N)}_{\mathrm{crys},\varphi^{-1}(\pi)}\Omega^q_{\mathcal{B}}(\overline{\mathcal{A}})) \subset \mathscr{A}^{(N)}_{\mathrm{crys},\pi}\Omega^q_{\mathcal{B}}(\overline{\mathcal{A}})$ for $q \in \mathbb{N}$.*

Proof The claim (1) follows from $\sharp((\mathrm{Supp}\,\underline{n}) \cup J) \le \sharp((\mathrm{Supp}\,\underline{n} + \underline{m}) \cup J)$ for $\underline{n}, \underline{m} \in \mathbb{N}^e$ and $J \subset \mathbb{N} \cap [1, e]$. By Lemma 182 and $\pi \in \varphi^{-1}(\pi)A_{\mathrm{inf}}(O_{\overline{K}})$, we have $\lambda^{-1}([\underline{\varepsilon}^a] - 1)^{[m]} \in A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$ for $a \in \mathbb{Z}_p$ and $m \in \mathbb{N}_{>0}$. Hence $\lambda^{-1}([\underline{\varepsilon}^a]u_i + [\underline{\varepsilon}^a] - 1)^{[n]}$ is contained in $\mathscr{A}^{i,0,(N)}_{\mathrm{crys},\lambda}(\overline{\mathcal{A}})$ ($\subset \mathscr{A}^{i,1,(N)}_{\mathrm{crys},\lambda}(\overline{\mathcal{A}})$) for $n \in \mathbb{N}_{>0}$. By (6) and the description of $u'_{i,m}$ in the proof of Lemma 126, we see that $\mathscr{A}^{i,0,(N)}_{\mathrm{crys},\mathcal{B},\lambda}(\overline{\mathcal{A}})$ and $\mathscr{A}^{i,1,(N)}_{\mathrm{crys},\mathcal{B},\lambda}(\overline{\mathcal{A}})\omega_i$ do not depend on the choice of $s_{i,n}$ and are $G_{\mathcal{A}}$-stable, and then the claim (2) and the $G_{\mathcal{A}}$-stability in (3) hold. For $a \in \mathbb{Z}_p$, we have $[\underline{\varepsilon}^a] \in 1 + \pi A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$ and $\pi^p \in p\pi A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$ (Proposition 147), from which we obtain $[\underline{\varepsilon}^a]^{p^n} \in 1 + p^n\pi A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}) \Leftrightarrow \pi^{-1}([\underline{\varepsilon}^{p^n a}] - 1) \in p^n A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$ ($n \in \mathbb{N}$) by induction on n. This together with Lemma 197, the continuity of $A_{\mathrm{inf}}(O_{\overline{K}}) \to A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})$, and Proposition 162 for $(\Lambda, \Lambda_0) = (\overline{\mathcal{A}}, \mathcal{A})$ implies that the actions of $G_{\mathcal{A}}$ on $\mathscr{A}^{i,0,(N)}_{\mathrm{crys},\mathcal{B},\lambda}(\overline{\mathcal{A}})$, $\mathscr{A}^{i,1,(N)}_{\mathrm{crys},\mathcal{B},\lambda}(\overline{\mathcal{A}})\omega_i$, and hence on $\mathscr{A}^{(N)}_{\mathrm{crys},\lambda}\Omega^q_{\mathcal{B}}(\overline{\mathcal{A}})$ are continuous. For the claim (4), we can construct an $A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})/p^m$-linear homotopy between the identity map and the zero map in the same way as Lemma 127 because $(\mathrm{Supp}\,\underline{n}) \cup J = (\mathrm{Supp}(\underline{n} + \underline{1}_i)) \cup (J\backslash\{i\})$ for $\underline{n} \in \mathbb{N}^e$, $J \subset \mathbb{N} \cap [1, e]$, and $i \in J$. The claim (5) and the first inclusion in (6) are obvious by definition. The second inclusion in (6) follows from $\varphi(u_i) = u_i(\sum_{\nu=1}^p \binom{p}{\nu} u_i^{\nu-1})$, $\varphi(\omega_i) = -p(u_i + 1)^{p-1}\omega_i$, and the claim (1). □

Remark 199 Let $\mathcal{B}'$ be $\varprojlim_m (\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B})/p^m$, and let $\mathscr{A}^{(N)}_{\mathrm{crys},\lambda}\Omega^\bullet_{\mathcal{B}'}(\overline{\mathcal{A}})$ be the complex associated to the product map $\mathcal{B}' \to \mathcal{A}$ and $s_i \otimes 1, 1 \otimes s_i$. Then the two homomorphisms $\mathcal{B} \rightrightarrows \mathcal{B}'; a \mapsto a \otimes 1, 1 \otimes a$ induce isomorphisms

$$\mathscr{A}^{(N)}_{\mathrm{crys},\lambda}\Omega^\bullet_{\mathcal{B}}(\overline{\mathcal{A}})/p^m \otimes_{A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})/p^m} \mathscr{A}^{(N)}_{\mathrm{crys},\lambda}\Omega^\bullet_{\mathcal{B}}(\overline{\mathcal{A}})/p^m \xrightarrow{\cong} \mathscr{A}^{(N)}_{\mathrm{crys},\lambda}\Omega^\bullet_{\mathcal{B}'}(\overline{\mathcal{A}})/p^m$$

for $m \in \mathbb{N}_{>0}$. It seems that this will allow us to study the compatibility of (192) with products.

To construct the morphism (182), we first need to put the isomorphisms in Proposition 190 together in the derived category of inductive systems with respect to N. For $\Lambda = \overline{\mathcal{A}}, O_{\overline{K}}$, we define $A^{(N)}_{\mathrm{crys},m}(\Lambda)$ ($m \in \mathbb{N}$) to be $A^{(N)}_{\mathrm{crys}}(\Lambda)/p^m$, and $A^{(N)}_{\mathrm{crys},\bullet}(\Lambda)$ to be the inverse system $(A^{(N)}_{\mathrm{crys},m}(\Lambda))_{m\in\mathbb{N}}$. Let $A^{(\bullet)}_{\mathrm{crys},\bullet}(\overline{\mathcal{A}})$ denote the ring object $(A^{(2)}_{\mathrm{crys},\bullet}(\overline{\mathcal{A}}) \to A^{(3)}_{\mathrm{crys},\bullet}(\overline{\mathcal{A}}) \to \cdots \to A^{(N)}_{\mathrm{crys},\bullet}(\overline{\mathcal{A}}) \to \cdots)$ of $(G_{\mathcal{A}}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ})^{\mathbb{N}}$. The index N starts with $N = 2$ because of (189). We define the ring object $A^{(\bullet)}_{\mathrm{crys},\bullet}(O_{\overline{K}})$ of $(G_K\text{-}\mathbf{Sets}^{\mathbb{N}^\circ})^{\mathbb{N}}$ in the same way. Let $A^{(\bullet)}_{\mathrm{crys}}(O_{\overline{K}})$ denote the ring object of $\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}}$ defined by the inductive system $(A^{(2)}_{\mathrm{crys}}(O_{\overline{K}}) \to A^{(3)}_{\mathrm{crys}}(O_{\overline{K}}) \to \cdots \to A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}) \to \cdots)$.

Then we have the following functors (cf. (111)).

$$\begin{aligned} &D^+((G_{\mathcal{A}}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ})^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys},\bullet}(\overline{\mathcal{A}})) \xrightarrow{R(\mathrm{inv}^{\mathbb{N}^\circ}_{\Delta_{\mathcal{A}}})^{\mathbb{N}}_*} D^+((G_K\text{-}\mathbf{Sets}^{\mathbb{N}^\circ})^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys},\bullet}(O_{\overline{K}})) \\ &\xrightarrow{(\iota^{\mathbb{N}^\circ})^{\mathbb{N}*}} D^+((\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ})^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys},\bullet}(O_{\overline{K}})) \xrightarrow{R\varprojlim^{\mathbb{N}}_*} D^+(\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys}}(O_{\overline{K}})) \\ &\xrightarrow{L\eta^+_\pi} D^+(\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys}}(O_{\overline{K}})). \end{aligned} \tag{185}$$

By [3, $\mathrm{V}^{\mathrm{bis}}$ Corollaire (1.3.12)] (resp. Lemma 102 (1) and [3, $\mathrm{V}^{\mathrm{bis}}$ Proposition (1.2.9)]), the evaluation of the first three functors (resp. the last functor) at the N-component of inductive systems is given by the corresponding functors appearing in the definition of $R\Gamma(\Delta_{\mathcal{A}}, -)$ (110) on $D^+(G_{\mathcal{A}}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ}, A^{(N)}_{\mathrm{crys},\bullet}(\overline{\mathcal{A}}))$ (resp. η^+_π on $D^+(\underline{G_K}\text{-}\mathbf{Sets}, A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}))$).

Let $TA^{(\bullet)}_{\mathrm{crys},\bullet}(M)$ denote the $A^{(\bullet)}_{\mathrm{crys},\bullet}(\overline{\mathcal{A}})$-module

$$(TA^{(2)}_{\mathrm{crys},\bullet}(M) \to TA^{(3)}_{\mathrm{crys},\bullet}(M) \to TA^{(4)}_{\mathrm{crys},\bullet}(M) \to \cdots \to TA^{(N)}_{\mathrm{crys},\bullet}(M) \to \cdots)$$

on $(G_{\mathcal{A}}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ})^{\mathbb{N}}$, where $TA^{(N)}_{\mathrm{crys},m}(M) = TA^{(N)}_{\mathrm{crys}}(M)/p^m$ for $m \in \mathbb{N}$. Then the image of $TA^{(\bullet)}_{\mathrm{crys},\bullet}(M)$ under the composition of the functors (185) may be regarded as the "inductive system" consisting of $L\eta^+_\pi R\Gamma(\Delta_{\mathcal{A}}, TA^{(N)}_{\mathrm{crys}}(M))$ ($N \in \mathbb{N}, N \geq 2$) and is denoted by $(L\eta^+_\pi R\Gamma(\Delta_{\mathcal{A}}, TA^{(N)}_{\mathrm{crys}}(M)))_N$ in the following.

One can define the constant inductive system $(L\eta^+_\pi R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M)))_N$ by taking the image of the constant inductive system $(TA_{\mathrm{inf},\bullet}(M))_N$ under the functor $D^+((G_{\mathcal{A}}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ})^{\mathbb{N}}, (A_{\mathrm{inf},\bullet}(\overline{\mathcal{A}}))_N) \to D^+(\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}}, (A_{\mathrm{inf}}(O_{\overline{K}}))_N)$ defined similarly as above. Here $A_{\mathrm{inf},m}(\overline{\mathcal{A}}) = A_{\mathrm{inf}}(\overline{\mathcal{A}})/(p^m, [\underline{p}]^{pm})$ and $TA_{\mathrm{inf},m}(M) = TA_{\mathrm{inf}}(M) \otimes_{A_{\mathrm{inf}}(\overline{\mathcal{A}})} A_{\mathrm{inf},m}(\overline{\mathcal{A}})$ for $m \in \mathbb{N}$.

We have an obvious variant of (115) for inductive systems, and it gives us a morphism

$$(A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}))_N \otimes^L_{(A_{\mathrm{inf}}(O_{\overline{K}}))_N} (L\eta^+_\pi R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M)))_N \longrightarrow (L\eta^+_\pi R\Gamma(\Delta_{\mathcal{A}}, TA^{(N)}_{\mathrm{crys}}(M)))_N. \quad (186)$$

in $D^+(\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys}}(O_{\overline{K}}))$. By taking $R\underset{\leftarrow}{l}{}^{\mathbb{N}}_* \circ L\underset{\leftarrow}{l}{}^{\mathbb{N}*}$ with respect to the morphism of topos $\underset{\leftarrow}{l}{}^{\mathbb{N}}\colon ((\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ})^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys},\bullet}(O_{\overline{K}})) \to (\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys}}(O_{\overline{K}}))$ (Sect. 14), we obtain isomorphisms in $D^+(\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys}}(O_{\overline{K}}))$

$$\begin{aligned} &(A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}))_N \widehat{\otimes}^L_{(A_{\mathrm{inf}}(O_{\overline{K}}))_N} (L\eta^+_\pi R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M)))_N \\ \xrightarrow{\cong}\; &(A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}))_N \widehat{\otimes}^L_{(A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}))_N} (L\eta^+_\pi R\Gamma(\Delta_{\mathcal{A}}, TA^{(N)}_{\mathrm{crys}}(M)))_N \\ \xleftarrow{\cong}\; &(L\eta^+_\pi R\Gamma(\Delta_{\mathcal{A}}, TA^{(N)}_{\mathrm{crys}}(M)))_N, \end{aligned} \quad (187)$$

where $(A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}))_N \widehat{\otimes}^L_{(A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}))_N}$ (resp. $(A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}))_N \widehat{\otimes}^L_{(A_{\mathrm{inf}}(O_{\overline{K}}))_N}$) denotes the functor $R\underset{\leftarrow}{l}{}^{\mathbb{N}}_* \circ L\underset{\leftarrow}{l}{}^{\mathbb{N}*}$ (resp. $R\underset{\leftarrow}{l}{}^{\mathbb{N}}_* \circ L\underset{\leftarrow}{l}{}^{\mathbb{N}*} \circ [(A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}))_N \otimes^L_{(A_{\mathrm{inf}}(O_{\overline{K}}))_N} -]$); we see that these are isomorphisms by looking at the evaluation on each N-component and using Proposition 190.

By Lemma 198 (3) and (4), we have a resolution

$$TA^{(\bullet)}_{\mathrm{crys},\bullet}(M) \longrightarrow TA^{(\bullet)}_{\mathrm{crys},\bullet}(M) \otimes_{A^{(\bullet)}_{\mathrm{crys}}(\overline{\mathcal{A}})} \mathscr{A}^{(\bullet)}_{\mathrm{crys},\pi} \Omega^\bullet_{\mathcal{B}}(\overline{\mathcal{A}})$$

in $\mathrm{Mod}((G_{\mathcal{A}}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ})^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys},\bullet}(\overline{\mathcal{A}}))$. By applying (185) to this resolution, and using Lemmas 100 (2) and 198 (5), we obtain a morphism

$$(\Gamma(\Delta_{\mathcal{A}}, TA^{(N)}_{\mathrm{crys}}(M) \otimes_{A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})} \mathscr{A}^{(N)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}}))_N \longrightarrow (L\eta^+_\pi R\Gamma(\Delta_{\mathcal{A}}, TA^{(N)}_{\mathrm{crys}}(M)))_N \quad (188)$$

in $D^+(\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys}}(O_{\overline{K}}))$. On the other hand, we have a morphism

$$(A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}) \widehat{\otimes}_{O_K} M_{\mathcal{P}^{(N-1)}} \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}})_N \longrightarrow (\Gamma(\Delta_{\mathcal{A}}, TA^{(N)}_{\mathrm{crys}}(M) \otimes_{A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})} \mathscr{A}^{(N)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}}))_N \quad (189)$$

in $C^+(\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys}}(O_{\overline{K}}))$ by Proposition 196.

We take the image of the composition

$$(A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K}M_{\mathcal{P}^{(N-1)}}\otimes_{\mathcal{B}}\Omega^{\bullet}_{\mathcal{B}})_N \longrightarrow (A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}))_N\widehat{\otimes}^{L}_{(A_{\mathrm{inf}}(O_{\overline{K}}))_N}(L\eta^{+}_{\pi}R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M)))_N \tag{190}$$

of (187), (188), and (189) under the functors

$$D^{+}(\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys}}(O_{\overline{K}})) \xrightarrow{L\underset{\leftarrow}{l}^{\mathbb{N}*}} D^{+}(\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}^{\circ}\times\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys},\bullet}(O_{\overline{K}}))$$
$$\xrightarrow{L\underset{\rightarrow}{l}^{\mathbb{N}^{\circ}*}} D^{+}(\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}^{\circ}}, A_{\mathrm{crys},\bullet}(O_{\overline{K}})) \xrightarrow{R\underset{\leftarrow}{l}_{*}} D^{+}(\underline{G_K}\text{-}\mathbf{Sets}, A_{\mathrm{crys}}(O_{\overline{K}})). \tag{191}$$

By the commutativity of the right square in Lemmas 103 (1) and 7, we see that $L\underset{\leftarrow}{l}^{\mathbb{N}*}(\mathrm{adj})\colon L\underset{\leftarrow}{l}^{\mathbb{N}*}\to L\underset{\leftarrow}{l}^{\mathbb{N}*}R\underset{\leftarrow}{l}^{\mathbb{N}}_{*}L\underset{\leftarrow}{l}^{\mathbb{N}*}$ is an isomorphism. Hence we obtain the desired morphism (182) as follows.

$$\begin{aligned}
&A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K}M_{\mathcal{P}}\otimes_{\mathcal{B}}\Omega^{\bullet}_{\mathcal{B}} \\
\xleftarrow{\cong}\ &\varprojlim_{m}\varinjlim_{N}(A^{(N)}_{\mathrm{crys},m}(O_{\overline{K}})\otimes_{O_K}M_{\mathcal{P}^{(N-1)}}\otimes_{\mathcal{B}}\Omega^{\bullet}_{\mathcal{B}}) \\
\longrightarrow\ &R\varprojlim_{m}\varinjlim_{N}((A^{(N)}_{\mathrm{crys},m}(O_{\overline{K}}))_{m,N}\otimes^{L}_{(A_{\mathrm{inf}}(O_{\overline{K}}))_N}(L\eta^{+}_{\pi}R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M)))_N) \\
\xrightarrow{\cong}\ &A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}^{L}_{A_{\mathrm{inf}}(O_{\overline{K}})}L\eta^{+}_{\pi}R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M))
\end{aligned} \tag{192}$$

Here we use (179) and $\varinjlim_N A^{(N)}_{\mathrm{crys},m}(O_{\overline{K}})\cong A_{\mathrm{crys},m}(O_{\overline{K}})$ for the first isomorphism.

Let us prove that (192) is compatible with the Frobenius φ. First note that we have an obvious variant of Lemma 105 for inductive systems. By applying it to the morphisms of ringed topos φ defined by the endomorphisms φ of the ring objects $A^{(\bullet)}_{\mathrm{crys},\bullet}(\overline{\mathcal{A}})$, $(A_{\mathrm{inf},\bullet}(\overline{\mathcal{A}}))_N$, $A^{(\bullet)}_{\mathrm{crys},\bullet}(O_{\overline{K}})$ etc. and the morphisms f, g, $\underline{g}$, and h of ringed topos defined by the morphisms of ring objects $(A_{\mathrm{inf},\bullet}(\overline{\mathcal{A}}))_N\to A^{(\bullet)}_{\mathrm{crys},\bullet}(\overline{\mathcal{A}})$, $(A_{\mathrm{inf},\bullet}(O_{\overline{K}}))_N\to A^{(\bullet)}_{\mathrm{crys},\bullet}(O_{\overline{K}})$ etc., we see that the morphism (186) is compatible with φ. Then, by Lemma 200 below applied to φ_* and the adjunction morphism for the morphism of ringed topos $\underset{\leftarrow}{l}^{\mathbb{N}}\colon((\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}^{\circ}})^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys},\bullet}(O_{\overline{K}}))\to(\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys}}(O_{\overline{K}}))$, we obtain the compatibility of (187) with φ.

Lemma 200 *For a commutative diagram of ringed topos*

$$\begin{array}{ccc}
(E', A') & \xrightarrow{h} & (E, A) \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
(F', B') & \xrightarrow{g} & (F, B),
\end{array}$$

the following diagram of functors from $D(F', B')$ to $D(F, B)$ is commutative, where the bottom horizontal arrow is the base change morphism.

$$\begin{array}{ccc} Rg_* & \xrightarrow{Rg_*(adj)} & Rg_*Rf'_*Lf'^* \\ {\scriptstyle adj\circ Rg_*}\downarrow & & \downarrow\cong \\ Rf_*Lf^*Rg_* & \longrightarrow & Rf_*Rh_*Lf'^* \end{array}$$

Proof Straightforward. □

Next let us verify the compatibility of (188) with φ. To simplify the notation, we put $T := TA^{(\bullet)}_{\mathrm{crys},\bullet}(M)$, $\pi' := \varphi^{-1}(\pi)$, $\mathrm{DR}_\lambda := TA^{(\bullet)}_{\mathrm{crys},\bullet}(M) \otimes_{A^{(\bullet)}_{\mathrm{crys}}(\overline{\mathcal{A}})} \mathscr{A}^{(\bullet)}_{\mathrm{crys},\lambda}\Omega^\bullet_{\mathcal{B}}(\overline{\mathcal{A}})$ $(\lambda = \pi, \pi')$, and $\mathrm{DR} := TA^{(\bullet)}_{\mathrm{crys}}(M) \otimes_{A^{(\bullet)}_{\mathrm{crys}}(\overline{\mathcal{A}})} \mathscr{A}^{(\bullet)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}}$. Then, by Lemma 198 (6), we have commutative diagrams

$$\begin{array}{ccc} T & \xrightarrow{\varphi} & \varphi_*T \\ {\scriptstyle \text{q.i.}}\downarrow & & \downarrow{\scriptstyle \text{q.i.}} \\ \mathrm{DR}_{\pi'} & \xrightarrow{\varphi} & \varphi_*\mathrm{DR}_\pi, \end{array} \qquad \begin{array}{ccc} T & = & T \\ {\scriptstyle \text{q.i.}}\downarrow & & \downarrow{\scriptstyle \text{q.i.}} \\ \mathrm{DR}_{\pi'} & \hookrightarrow & \mathrm{DR}_\pi \end{array} \tag{193}$$

in $C^+((G_{\mathcal{A}}\text{-}\mathbf{Sets})^{\mathbb{N}^\circ\times\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys},\bullet}(\overline{\mathcal{A}}))$, where q.i. means quasi-isomorphism. Let $R\Gamma^{\mathbb{N}}(\Delta_{\mathcal{A}}, -)$ denote the composition of the first three functors in (185). By using Lemmas 198 (5), 102 (2), and an analogue of (113) for inductive systems indexed by $\mathbb{N}$, we see that the left commutative diagram in (193) induces the following commutative diagram in $D(\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys}}(O_{\overline{K}}))$; we use Lemma 198 (5) (resp. Lemma 102 (2) and the analogue of (113)) for the morphisms (A) (resp. (B)).

$$\begin{array}{ccccc} L\eta^+_{\pi'}R\Gamma^{\mathbb{N}}(\Delta_{\mathcal{A}}, T) & \xrightarrow{\varphi} & L\eta^+_{\pi'}R\Gamma^{\mathbb{N}}(\Delta_{\mathcal{A}}, \varphi_*T) & \xrightarrow{(B)} & \varphi_*L\eta^+_{\pi}R\Gamma^{\mathbb{N}}(\Delta_{\mathcal{A}}, T) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ L\eta^+_{\pi'}R\Gamma^{\mathbb{N}}(\Delta_{\mathcal{A}}, \mathrm{DR}_{\pi'}) & \xrightarrow{\varphi} & L\eta^+_{\pi'}R\Gamma^{\mathbb{N}}(\Delta_{\mathcal{A}}, \varphi_*\mathrm{DR}_\pi) & \xrightarrow{(B)} & \varphi_*L\eta^+_{\pi}R\Gamma^{\mathbb{N}}(\Delta_{\mathcal{A}}, \mathrm{DR}_\pi) \\ {\scriptstyle (A)}\uparrow & & & & \uparrow{\scriptstyle (A)} \\ \Gamma(\Delta_{\mathcal{A}}, \mathrm{DR}) & & \xrightarrow{\qquad\qquad\varphi\qquad\qquad} & & \varphi_*\Gamma(\Delta_{\mathcal{A}}, \mathrm{DR}) \end{array} \tag{194}$$

The composition of the right vertical morphism in (194) is $\varphi_*(188)$. From the right commutative diagram of (193), we obtain a commutative diagram

$$\begin{array}{ccccc} L\eta^+_{\pi}R\Gamma^{\mathbb{N}}(\Delta_{\mathcal{A}}, T) & \longrightarrow & L\eta^+_{\pi'}R\Gamma^{\mathbb{N}}(\Delta_{\mathcal{A}}, T) & & \\ \downarrow\cong & & \downarrow\cong & \searrow\cong & \\ L\eta^+_{\pi}R\Gamma^{\mathbb{N}}(\Delta_{\mathcal{A}}, \mathrm{DR}_\pi) & \longrightarrow & L\eta^+_{\pi'}R\Gamma^{\mathbb{N}}(\Delta_{\mathcal{A}}, \mathrm{DR}_\pi) & \xleftarrow{\cong} & L\eta^+_{\pi'}R\Gamma(\Delta_{\mathcal{A}}, \mathrm{DR}_{\pi'}) \\ & \nwarrow & & \nearrow & \\ & & \Gamma(\Delta_{\mathcal{A}}, \mathrm{DR}) & & \end{array} \tag{195}$$

in $D(\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys}}(O_{\overline{K}}))$. This implies that the composition of the left vertical morphisms in (194) coincides with the composition of (188) with the natural morphism $L\eta_\pi^+ R\Gamma^{\mathbb{N}}(\Delta_{\mathcal{A}}, T) \to L\eta_{\pi'}^+ R\Gamma^{\mathbb{N}}(\Delta_{\mathcal{A}}, T)$. Thus we see that (188) is compatible with φ. The morphism (189) is obviously compatible with φ. By using Lemma 200 for φ_* and the adjunction morphism for the morphism of ringed topos $\underleftarrow{l}^{\mathbb{N}}\colon ((\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}^\circ})^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys},\bullet}(O_{\overline{K}})) \to (\underline{G_K}\text{-}\mathbf{Sets}^{\mathbb{N}}, A^{(\bullet)}_{\mathrm{crys}}(O_{\overline{K}}))$ again, we see that (192) is compatible with φ.

In the rest of this section, we show a functoriality of (192) in $(\mathcal{B}, s_1, \ldots, s_e)$ and prove that the morphism (192) does not depend on the choice of s_i.

Let $\mathcal{B} \to \mathcal{B}' \to \mathcal{A}$, $s'_1, \ldots, s'_{e'} \in \mathcal{B}'^{\times}$, $\Omega_{\mathcal{B}'}$, $\Omega^q_{\mathcal{B}'}$, $\mathcal{P}'_m$, $\mathcal{P}'$, $M_{\mathcal{P}'_m}$, and $M_{\mathcal{P}'}$ be the same as after Theorem 136. Recall that we have PD-homomorphisms $\mathcal{P}_m \to \mathcal{P}'_m$ and $\mathcal{P} \to \mathcal{P}'$ compatible with $\mathcal{B} \to \mathcal{B}'$, φ, and ∇. We also have homomorphisms of $A_{\mathrm{crys}}(\overline{\mathcal{A}})$-algebras $\mathscr{A}_{\mathrm{crys},\mathcal{B},m}(\overline{\mathcal{A}}) \to \mathscr{A}_{\mathrm{crys},\mathcal{B}',m}(\overline{\mathcal{A}})$ and $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \to \mathscr{A}_{\mathrm{crys},\mathcal{B}'}(\overline{\mathcal{A}})$ compatible with the homomorphisms $\mathcal{P}_m \to \mathcal{P}'_m$ and $\mathcal{P} \to \mathcal{P}'$, the $G_{\mathcal{A}}$-action, φ, and ∇.

Proposition 201 *Under the notation and the assumption as above, the following diagram is commutative, where the left vertical morphism is induced by (143).*

$$\begin{array}{ccc} A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M_{\mathcal{P}} \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}} & \xrightarrow{(192)} & A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}^L_{A_{\mathrm{inf}}(O_{\overline{K}})} L\eta_\pi^+ R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{inf}}(M)) \\ \downarrow & \nearrow_{(192)} & \\ A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M_{\mathcal{P}'} \otimes_{\mathcal{B}'} \Omega^\bullet_{\mathcal{B}'} & & \end{array}$$

Proof We define $\mathscr{A}^{(N)}_{\mathrm{crys},\mathcal{B}'}(\overline{\mathcal{A}})$ and $\mathscr{A}^{(N)}_{\mathrm{crys},\pi}\Omega^q_{\mathcal{B}'}(\overline{\mathcal{A}})$ for $N \in \mathbb{N}_{>0}$ and $q \in \mathbb{N}$ associated to $(\mathcal{B}', s_1, \ldots, s_e, s'_1, \ldots, s'_{e'})$ in the same way as those for $(\mathcal{B}, s_1, \ldots, s_e)$. Let $\mathrm{DR}^{(N)}_{\mathscr{A}_{\mathrm{crys}}}(\mathcal{C})$ ($\mathcal{C} = \mathcal{B}, \mathcal{B}'$) denote the complex $\mathscr{A}^{(N)}_{\mathrm{crys},\mathcal{C}}(\overline{\mathcal{A}}) \otimes_{\mathcal{C}} \Omega^\bullet_{\mathcal{C}}$, and let $T^{(N)}$ denote $TA^{(N)}_{\mathrm{crys}}(M)$. We see that the homomorphism $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \to \mathscr{A}_{\mathrm{crys},\mathcal{B}'}(\overline{\mathcal{A}})$ induces $G_{\mathcal{A}}$-equivariant morphisms of complexes $\mathrm{DR}^{(N)}_{\mathscr{A}_{\mathrm{crys}}}(\mathcal{B}) \to \mathrm{DR}^{(N)}_{\mathscr{A}_{\mathrm{crys}}}(\mathcal{B}')$ and $\mathscr{A}^{(N)}_{\mathrm{crys},\pi}\Omega^\bullet_{\mathcal{B}}(\overline{\mathcal{A}}) \to \mathscr{A}^{(N)}_{\mathrm{crys},\pi}\Omega^\bullet_{\mathcal{B}'}(\overline{\mathcal{A}})$ compatible with the morphisms from the former to the latter. Hence we have the following commutative diagram.

$$\begin{array}{ccc} (\Gamma(\Delta_{\mathcal{A}}, T^{(N)} \otimes_{A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})} \mathrm{DR}^{(N)}_{\mathscr{A}_{\mathrm{crys}}}(\mathcal{B}))_N & \xrightarrow{(188)} & (L\eta_\pi^+ R\Gamma(\Delta_{\mathcal{A}}, TA^{(N)}_{\mathrm{crys}}(M))_N \\ \downarrow & \nearrow_{(188)} & \\ (\Gamma(\Delta_{\mathcal{A}}, T^{(N)} \otimes_{A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})} \mathrm{DR}^{(N)}_{\mathscr{A}_{\mathrm{crys}}}(\mathcal{B}'))_N & & \end{array}$$

We define $\mathcal{P}'^{(N)} \subset \mathcal{P}'$ and $M_{\mathcal{P}'^{(N)}} \subset M_{\mathcal{P}'}$ in the same way as $\mathcal{P}^{(N)}$ and $M_{\mathcal{P}^{(N)}}$ by using $\mathcal{B}' \to \mathcal{A}$. Then we can verify that the homomorphisms $\mathcal{P} \to \mathcal{P}'$ and $M_{\mathcal{P}} \to M_{\mathcal{P}'}$ (143) induce $\mathcal{P}^{(N)} \to \mathcal{P}'^{(N)}$ and $M_{\mathcal{P}^{(N)}} \to M_{\mathcal{P}'^{(N)}}$ for $N \in \mathbb{N}_{>0}$. For the former, we simply note that the image of $\mathcal{J}_m = \mathrm{Ker}(\mathcal{B}_m \to \mathcal{A}_m)$ under $\mathcal{B}_m \to \mathcal{B}'_m$ is contained in $\mathcal{J}'_m = \mathrm{Ker}(\mathcal{B}'_m \to \mathcal{A}_m)$. For the latter, we choose $f \in \mathbb{N}_{>0}$ satisfying $\mathcal{J}_m^f \mathcal{P}_m = 0$ and $(\mathcal{J}'_m)^f \mathcal{P}'_m = 0$, and then $\overline{\delta}\colon \mathcal{A}_m \to \mathcal{B}_m/\mathcal{J}_m^f$ as before (176). By defining $M_{\mathcal{P}^{(1)}_m} \subset M_{\mathcal{P}_m}$ and $M_{\mathcal{P}'^{(1)}_m} \subset M_{\mathcal{P}'_m}$ by using $\overline{\delta}$ and the composition $\mathcal{A}_m \xrightarrow{\overline{\delta}}$

$\mathcal{B}_m/\mathcal{J}_m^f \to \mathcal{B}'_m/(\mathcal{J}'_m)^f$, we see that the image of $M_{\mathcal{P}_m^{(1)}}$ under $M_{\mathcal{P}_m} \to M_{\mathcal{P}'_m}$ is contained in $M_{\mathcal{P}'^{(1)}_m}$.

Since the isomorphisms (39) for $\mathcal{B}$ and $\mathcal{B}'$ are compatible with $M_{\mathcal{P}} \to M_{\mathcal{P}'}$ and $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \to \mathscr{A}_{\mathrm{crys},\mathcal{B}'}(\overline{\mathcal{A}})$, we see that the isomorphisms (180) for $\mathcal{B}$ and $\mathcal{B}'$ are compatible with $M_{\mathcal{P}^{(N)}} \to M_{\mathcal{P}^{(N')}}$ and $\mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}}) \to \mathscr{A}^{(N+)}_{\mathrm{crys},\mathcal{B}'}(\overline{\mathcal{A}})$. Hence we have the following commutative diagram.

$$\begin{array}{ccc} (A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}) \otimes_{O_K} M_{\mathcal{P}^{(N-1)}} \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}})_N & \xrightarrow{(189)} & (\Gamma(\Delta_{\mathcal{A}}, T^{(N)} \otimes_{A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})} \mathrm{DR}^{(N)}_{\mathscr{A}_{\mathrm{crys}}}(\mathcal{B}))_N \\ \downarrow & & \downarrow \\ (A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}) \otimes_{O_K} M_{\mathcal{P}'^{(N-1)}} \otimes_{\mathcal{B}'} \Omega^\bullet_{\mathcal{B}'})_N & \xrightarrow{(189)} & (\Gamma(\Delta_{\mathcal{A}}, T^{(N)} \otimes_{A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})} \mathrm{DR}^{(N)}_{\mathscr{A}_{\mathrm{crys}}}(\mathcal{B}'))_N \end{array}$$

Combining the above diagrams and (187), and applying the functors (191), we obtain the desired commutative diagram. □

Lemma 202 *Under the notation and the assumption as above, the morphism*

$$A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M_{\mathcal{P}} \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}} \to A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M_{\mathcal{P}'} \otimes_{\mathcal{B}'} \Omega^\bullet_{\mathcal{B}'}$$

induced by (143) is a quasi-isomorphism.

Proof For each $m \in \mathbb{N}_{>0}$ the morphism $M_{\mathcal{P}_m} \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}} \to M_{\mathcal{P}'_m} \otimes_{\mathcal{B}'} \Omega^\bullet_{\mathcal{B}'}$ is a quasi-isomorphism because both sides compute $R\Gamma((X_m/(\Sigma_m, \gamma))_{\mathrm{crys}}, \mathcal{F}_m)$. (See Sect. 5 for the definition of $\mathcal{F}_m$.) We obtain the desired quasi-isomorphism by taking $A_{\mathrm{crys}}(O_{\overline{K}})\otimes_{O_K}$ and then $R\varprojlim_m$. □

Theorem 203 *The morphism (192)*

$$A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M_{\mathcal{P}} \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}} \longrightarrow A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{A_{\inf}(O_{\overline{K}})} L\eta^+_\pi R\Gamma(\Delta_{\mathcal{A}}, TA_{\inf}(M))$$

associated to $\mathcal{B} \to \mathcal{A}$ and $s_i \in \mathcal{B}^\times$ does not depend on the choice of s_i.

Proof Let $s'_1, \dots, s'_e$ be another system of coordinates of $\mathcal{B}$ over O_K. Put $\mathcal{B}' := \varprojlim_m (\mathcal{B} \otimes_{O_K} \mathcal{B})/p^m$. Then, by Proposition 201, the morphism (192) associated to $\mathcal{B} \to \mathcal{A}$ and s_i (resp. s'_i) factors through the morphism (192) associated to the product map $\mathcal{B}' \to \mathcal{A}$ and $s_i \otimes 1, 1 \otimes s'_i$ via the morphism λ (resp. λ') $A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M_{\mathcal{P}} \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}} \to A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M_{\mathcal{P}'} \otimes_{\mathcal{B}'} \Omega^\bullet_{\mathcal{B}'}$ induced by $\mathcal{B} \to \mathcal{B}'$; $a \mapsto a \otimes 1$ (resp. $1 \otimes a$). The product map $\mathcal{B}' \to \mathcal{B}$ induces a G_K-equivariant $A_{\mathrm{crys}}(O_{\overline{K}})$-linear morphism $\mu\colon A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{A_{\inf}(O_{\overline{K}})} M_{\mathcal{P}'} \otimes_{\mathcal{B}'} \Omega^\bullet_{\mathcal{B}'} \to A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{A_{\inf}(O_{\overline{K}})} M_{\mathcal{P}} \otimes_{\mathcal{B}} \Omega^\bullet_{\mathcal{B}}$ such that $\mu \circ \lambda$ and $\mu \circ \lambda'$ are both the identity map. Since λ and λ' are quasi-isomorphisms by Lemma 202, this implies that λ and λ' coincide with each other in $D(\underline{G_K}\text{-}\mathbf{Sets}, A_{\mathrm{crys}}(O_{\overline{K}}))$. □

21 Comparison Theorem with de Rham Complex over A_{crys}

Theorem 204 *Let M be an object of $\mathrm{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(\mathcal{A},\Phi)$. Then the morphism (192)*

$$A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K}M_{\mathcal{P}}\otimes_{\mathcal{B}}\Omega^{\bullet}_{\mathcal{B}}\longrightarrow A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}^{L}_{A_{\mathrm{inf}}(O_{\overline{K}})}L\eta^{+}_{\pi}R\Gamma(\Delta_{\mathcal{A}},TA_{\mathrm{inf}}(M))$$

is an isomorphism in $D(\underline{G_K}\text{-}\mathbf{Sets},A_{\mathrm{crys}}(O_{\overline{K}}))$.

By applying Proposition 201 to $\mathcal{A},\mathcal{B}\to\varprojlim_m(\mathcal{A}\otimes_{O_K}\mathcal{B})/p^m$ and using Lemma 202, Theorem 204 is immediately reduced to the special case $\mathcal{B}=\mathcal{A}$. In this case, the image of $(A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K}M\otimes_{\mathcal{A}}\Omega^{\bullet}_{\mathcal{A}})_N$ under the functor (191) is isomorphic to $A_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K}M\otimes_{\mathcal{A}}\Omega^{\bullet}_{\mathcal{A}}$ because $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K}M\otimes_{\mathcal{A}}\Omega^{\bullet}_{\mathcal{A}}$ are p-torsion free. Hence, by the construction of (192), it suffices to prove that the N-component of the composition of (188) and (189) for $\mathcal{A}$ and $t_1,\dots,t_d$

$$A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K}M\otimes_{\mathcal{A}}\Omega^{\bullet}_{\mathcal{A}}\longrightarrow L\eta^{+}_{\pi}R\Gamma(\Delta_{\mathcal{A}},TA^{(N)}_{\mathrm{crys}}(M))\tag{196}$$

is an isomorphism in $D^{+}(\underline{G_K}\text{-}\mathbf{Sets},A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}))$ for every integer $N\geq 2$. We will construct a morphism (196) with $\Delta_{\mathcal{A}}$ and $TA^{(N)}_{\mathrm{crys}}(M)$ replaced by $\Gamma_{\mathcal{A}}$ and $TA^{\square,(N)}_{\mathrm{crys}}(M)$ by using $\mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})$ and $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})$ instead of $\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})$ and $A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})$. Then we will prove that it is an isomorphism by comparing it with variants of (119) and Proposition 123 for $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$ and $TA^{\square,(N)}_{\mathrm{crys}}(M)$, and obtain the desired isomorphism by using Corollary 193 and Proposition 194.

Similarly to $\mathscr{A}^{(N)}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$, we define $\mathscr{A}^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})$ to be the p-adic completion of $\oplus_{\underline{n}\in\mathbb{N}^d}A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})\prod_{1\leq i\leq d}v_i^{[n_i]}$ regarded as a subring of $\mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})$. Then the claims corresponding to Lemma 195 hold for $\mathscr{A}^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})$. We define the subcomplex $\mathscr{A}^{\square,(N)}_{\mathrm{crys},\pi}\Omega^{\bullet}_{\mathcal{A}}\subset\pi^{-d}\mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})\otimes_{\mathcal{A}}\Omega^{\bullet}_{\mathcal{A}}$ in the same way as $\mathscr{A}^{(N)}_{\mathrm{crys},\pi}\Omega^{\bullet}_{\mathcal{A}}(\overline{\mathcal{A}})$ by using $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})$, $\mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})$, and v_i instead of $A^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})$, $\mathscr{A}_{\mathrm{crys},\mathcal{B}}(\overline{\mathcal{A}})$, and u_i. Then the claims corresponding to Lemma 198 (1)–(5) hold for $\mathscr{A}^{\square,(N)}_{\mathrm{crys},\pi}\Omega^{\bullet}_{\mathcal{A}}$. The natural homomorphism $\mathscr{A}^{\square}_{\mathrm{crys}}(\mathcal{A})\to\mathscr{A}_{\mathrm{crys}}(\overline{\mathcal{A}})$ (85) induces $\mathscr{A}^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})\to\mathscr{A}^{(N)}_{\mathrm{crys}}(\overline{\mathcal{A}})$ and $\mathscr{A}^{\square,(N)}_{\mathrm{crys},\pi}\Omega^{\bullet}_{\mathcal{A}}\to\mathscr{A}^{(N)}_{\mathrm{crys},\pi}\Omega^{\bullet}_{\mathcal{A}}(\overline{\mathcal{A}})$.

We define $A^{\square,(N)}_{\mathrm{crys},m}(\mathcal{A})$ ($m\in\mathbb{N}$) to be $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})/p^m$, and let $A^{\square,(N)}_{\mathrm{crys},\bullet}(\mathcal{A})$ denote the ring object $(A^{\square,(N)}_{\mathrm{crys},m}(\mathcal{A}))_{m\in\mathbb{N}}$ of $(\widetilde{\Gamma}_{\mathcal{A}}\text{-}\mathbf{Sets})^{\mathbb{N}^{\circ}}$. Let $TA^{\square,(N)}_{\mathrm{crys},\bullet}(M)$ denote the $A^{\square,(N)}_{\mathrm{crys},\bullet}(\mathcal{A})$-module $(TA^{\square,(N)}_{\mathrm{crys}}(M)/p^m)_{m\in\mathbb{N}}$ on $(\widetilde{\Gamma}_{\mathcal{A}}\text{-}\mathbf{Sets})^{\mathbb{N}^{\circ}}$. Then we have a resolution

$$TA^{\square,(N)}_{\mathrm{crys},\bullet}(M)\longrightarrow TA^{\square,(N)}_{\mathrm{crys},\bullet}(M)\otimes_{A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})}\mathscr{A}^{\square,(N)}_{\mathrm{crys},\pi}\Omega^{\bullet}_{\mathcal{A}}\tag{197}$$

in $\mathrm{Mod}((\widetilde{\Gamma}_{\mathcal{A}}\text{-}\mathbf{Sets})^{\mathbb{N}^{\circ}},A^{\square,(N)}_{\mathrm{crys},\bullet}(\mathcal{A}))$. Applying the functor (see (109) and (110))

$$L\eta^{+}_{\pi}R\Gamma(\Gamma_{\mathcal{A}},-):D^{+}((\widetilde{\Gamma}_{\mathcal{A}}\text{-}\mathbf{Sets})^{\mathbb{N}^{\circ}},A^{\square,(N)}_{\mathrm{crys},\bullet}(\mathcal{A}))\longrightarrow D^{+}(\underline{G_K}\text{-}\mathbf{Sets},A^{(N)}_{\mathrm{crys}}(O_{\overline{K}}))$$

and using $\mathscr{A}_{\mathrm{crys}}^{\square,(N)}(\mathcal{A})\otimes_{\mathcal{A}}\Omega^\bullet_{\mathcal{A}}\subset\eta_\pi^+\mathscr{A}_{\mathrm{crys},\pi}^{\square,(N)}\Omega^\bullet_{\mathcal{A}}$, we obtain a morphism

$$\Gamma(\Gamma_{\mathcal{A}}, TA_{\mathrm{crys}}^{\square,(N)}(M)\otimes_{A_{\mathrm{crys}}^{\square,(N)}(\mathcal{A})}\mathscr{A}_{\mathrm{crys}}^{\square,(N)}(\mathcal{A})\otimes_{\mathcal{A}}\Omega^\bullet_{\mathcal{A}}) \longrightarrow L\eta_\pi^+R\Gamma(\Gamma_{\mathcal{A}}, TA_{\mathrm{crys}}^{\square,(N)}(M)) \quad (198)$$

in $D^+(\underline{G_K}\text{-}\mathbf{Sets}, A_{\mathrm{crys}}^{(N)}(O_{\overline{K}}))$. Assume $N\geq 2$. Then, by Proposition 189 and (167) (cf. Proposition 196), there is a natural G_K-equivariant morphism from $A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})\widehat{\otimes}_{O_K} M\otimes_{\mathcal{A}}\Omega^\bullet_{\mathcal{A}}$ to the source of (198). Thus we obtain a morphism

$$A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})\widehat{\otimes}_{O_K} M\otimes_{\mathcal{A}}\Omega^\bullet_{\mathcal{A}} \longrightarrow L\eta_\pi^+R\Gamma(\Gamma_{\mathcal{A}}, TA_{\mathrm{crys}}^{\square,(N)}(M)) \quad (199)$$

in $D^+(\underline{G_K}\text{-}\mathbf{Sets}, A_{\mathrm{crys}}^{(N)}(O_{\overline{K}}))$. Since the isomorphism (180) for $\mathcal{B}=\mathcal{A}$ and (181) are compatible with the natural homomorphisms $TA_{\mathrm{crys}}^{\square,(N+)}(M)\to TA_{\mathrm{crys}}^{(N+)}(M)$ and $\mathscr{A}_{\mathrm{crys}}^{\square,(N+)}(\mathcal{A})\to\mathscr{A}_{\mathrm{crys}}^{(N+)}(\overline{\mathcal{A}})$, the following diagram is commutative.

$$\begin{array}{ccc} A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})\widehat{\otimes}_{O_K} M\otimes_{\mathcal{A}}\Omega^\bullet_{\mathcal{A}} & \xrightarrow{(196)} & L\eta_\pi^+R\Gamma(\Delta_{\mathcal{A}}, TA_{\mathrm{crys}}^{(N)}(M)) \\ & \searrow^{(196)} & \uparrow \\ & & L\eta_\pi^+R\Gamma(\Gamma_{\mathcal{A}}, TA_{\mathrm{crys}}^{\square,(N)}(M)). \end{array} \quad (200)$$

The right vertical morphism is an isomorphism by Corollary 193 and Proposition 194, and therefore, it suffices to prove the following.

Proposition 205 *For any integer $N\geq 2$, the morphism (199) is an isomorphism.*

By Lemma 103, we may forget the action of $\underline{G_K}$ in the following. By the same argument as the proof of Lemma 117, we obtain an isomorphism in the derived category of $A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})$-modules

$$R\Gamma(\Gamma_{\mathcal{A}}, TA_{\mathrm{crys}}^{\square,(N)}(M))\cong K(TA_{\mathrm{crys}}^{\square,(N)}(M);\gamma_1-1,\dots,\gamma_d-1). \quad (201)$$

Since $(\gamma_i-1)(TA_{\mathrm{crys}}^{\square,(N)}(M))\subset\pi TA_{\mathrm{crys}}^{\square,(N)}(M)$ by Lemma 98 and Proposition 168 (3), the degree q-part of the complex $\eta_\pi^+K(TA_{\mathrm{crys}}^{\square,(N)}(M);\gamma_1-1,\dots,\gamma_d-1)$ is given by $\pi^qTA_{\mathrm{crys}}^{\square,(N)}(M)\otimes_{\mathbb{Z}}\wedge^qE$, where $E=\oplus_{1\leq i\leq d}\mathbb{Z}e_i$. By multiplying the degree q-part by π^{-q}, we obtain an isomorphism

$$\eta_\pi^+K(TA_{\mathrm{crys}}^{\square,(N)}(M);\gamma_1-1,\dots,\gamma_d-1)\cong K(TA_{\mathrm{crys}}^{\square,(N)}(M);\frac{\gamma_1-1}{\pi},\dots,\frac{\gamma_d-1}{\pi}). \quad (202)$$

We will prove that the right-hand side of (202) is isomorphic to the de Rham complex $A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})\widehat{\otimes}_{O_K} M\otimes_{\mathcal{A}}\Omega^\bullet_{\mathcal{A}}$ by an explicit computation (Proposition 209, (207)) similarly to Proposition 123, and then verify that the composition of it with the isomorphisms $L\eta_\pi^+(201)$ and (202) gives (199).

Put $\tau_i := 1 \otimes t_i - [\underline{t}_i] \otimes 1 \in \mathscr{A}_{\mathrm{crys}}^{\square}(\mathcal{A})$ for $i \in \mathbb{N} \cap [1, d]$. Then the $A_{\mathrm{crys}}^{\square}(\mathcal{A})$-algebra $\mathscr{A}_{\mathrm{crys}}^{\square}(\mathcal{A})$ is the p-adic completion of the PD-polynomial ring over $A_{\mathrm{crys}}^{\square}(\mathcal{A})$ with d-variables τ_i. We have

$$\nabla(\tau_i^{[n]}) = \tau_i^{[n-1]} \otimes dt_i \quad (n \in \mathbb{N}_{>0}), \tag{203}$$

$$\gamma_i(\tau_j) = \tau_j \text{ if } j \neq i, \tau_i - \pi[\underline{t}_i] \text{ if } j = i. \tag{204}$$

We have $(1+v_i)^{-1} - 1 = [\underline{t}_i]^{-1}\tau_i$, which implies $v_i = \sum_{n\geq 1}(-1)^n[\underline{t}_i]^{-n}\tau_i^n$, $\tau_i = [\underline{t}_i]\sum_{n\geq 1}(-1)^n v_i^n$, $\nabla(v_i) = (\sum_{n\geq 1}(-1)^n n[\underline{t}_i]^{-n}\tau_i^{n-1})\nabla(\tau_i)$, and $\nabla(\tau_i) = ([\underline{t}_i]\sum_{n\geq 1}(-1)^n n v_i^{n-1})\nabla(v_i)$. Using these formulae, we see that we obtain the same algebras if we replace v_i and $\nabla(v_i) = -(v_i + 1)d\log t_i$ by τ_i and $\nabla(\tau_i) = dt_i$ in the construction of $\mathscr{A}_{\mathrm{crys}}^{\square,(N)}(\mathcal{A})$ and $\mathscr{A}_{\mathrm{crys},\pi}^{\square,(N)}\Omega^{\bullet}_{\mathcal{A}}$. We use this alternative construction in the following.

By combining the isomorphisms in Propositions 186 and 189 with N replaced by $N-1$, and taking the scalar extension by $\mathscr{A}_{\mathrm{crys}}^{\square,((N-1)+)}(\mathcal{A}) \to \mathscr{A}_{\mathrm{crys}}^{\square,(N)}(\mathcal{A})$ (167), we obtain $\mathscr{A}_{\mathrm{crys}}^{\square,(N)}(\mathcal{A})$-linear isomorphisms

$$\begin{aligned} \delta_M \colon M \otimes_{\mathcal{A},\iota} \mathscr{A}_{\mathrm{crys}}^{\square,(N)}(\mathcal{A}) &\xrightarrow[\text{Prop. 189}]{\cong} TA_{\mathrm{crys}}^{\square,(N)}(M) \otimes_{A_{\mathrm{crys}}^{\square,(N)}(\mathcal{A})} \mathscr{A}_{\mathrm{crys}}^{\square,(N)}(\mathcal{A}) \\ &\xrightarrow[\text{Prop. 186}]{\cong} M \otimes_{\mathcal{A},\alpha} \mathscr{A}_{\mathrm{crys}}^{\square,(N)}(\mathcal{A}). \end{aligned} \tag{205}$$

The first isomorphism in (205) is $\widetilde{\Gamma}_{\mathcal{A}}$-equivariant, and the second one in (205) is compatible with $\mathrm{id} \otimes \nabla$. We see that the composition δ_M is induced by the inverse of (95) by its construction. Therefore, by (96) and Remark 18, we have

$$\delta_M(x \otimes 1) = \sum_{\underline{n}\in\mathbb{N}^d} \nabla^{\underline{n}}(x) \otimes \tau^{[\underline{n}]}, \quad \delta_M^{-1}(x \otimes 1) = \sum_{\underline{n}\in\mathbb{N}^d} \nabla^{\underline{n}}(x) \otimes (-\tau)^{[\underline{n}]} \tag{206}$$

for $x \in M$, where $\tau^{\underline{n}} = \prod_{1\leq i\leq d} \tau_i^{[n_i]}$, $(-\tau)^{[\underline{n}]} = \prod_{1\leq i\leq d}(-\tau_i)^{[n_i]}$, and $\nabla^{\underline{n}}(x) = \prod_{1\leq i\leq d} \nabla_i^{n_i}(x)$ for $\underline{n} = (n_i) \in \mathbb{N}^d$ and $x \in M$. The endomorphisms ∇_i $(i \in \mathbb{N} \cap [1, d])$ on M are defined by $\nabla(x) = \sum_{1\leq i\leq d} \nabla_i(x) \otimes dt_i$.

By Lemma 87 (2) and (164), we have an isomorphism $A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})\widehat{\otimes}_{O_K}\mathcal{A} \xrightarrow{\cong} A_{\mathrm{crys}}^{\square,(N)}(\mathcal{A})$. Hence by Proposition 186 and (167), we see that the second isomorphism in (205) induces the following isomorphism. Note $N \geq 2$.

$$TA_{\mathrm{crys}}^{\square,(N)}(M) \cong A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})\widehat{\otimes}_{O_K} M. \tag{207}$$

We define the integrable connection ∇ on $A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})\widehat{\otimes}_{O_K} M$ by $\mathrm{id} \otimes \nabla$, and the endomorphisms ∇_i $(i \in \mathbb{N} \cap [1, d])$ on $A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})\widehat{\otimes}_{O_K} M$ by $\nabla(x) = \sum_{1\leq i\leq d} \nabla_i(x) \otimes dt_i$. We equip $A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})\widehat{\otimes}_{O_K} M$ with the action of $\widetilde{\Gamma}_{\mathcal{A}}$ obtained from that on $TA_{\mathrm{crys}}^{\square,(N)}(M)$ via (207).

Lemma 206 (cf. Propositions 120 (3), 121) *For* $i \in \mathbb{N} \cap [1, d]$ *and* $x \in A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M$, *we have* $\gamma_i(x) = \sum_{n\in\mathbb{N}} \pi^{[n]} t_i^n \nabla_i^n(x)$.

Proof Since the action of $\Gamma_{\mathcal{A}}$ on $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M$ is $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$-linear, it suffices to prove the claim when $x \in M$. We can verify it by using (205) and (206) as follows. We abbreviate δ_M (205) to δ.

$$\begin{aligned}\delta\gamma_i\delta^{-1}(x\otimes 1) &= \sum_{\underline{n}\in\mathbb{N}^d} \delta\gamma_i(\nabla^{\underline{n}}(x)\otimes(-\tau)^{[\underline{n}]})\\ &= \delta(\sum_{\underline{n}\in\mathbb{N}^d} \nabla^{\underline{n}}(x)\otimes(-\tau)^{[\underline{n}-n_i\underline{1}_i]}(-\tau_i+\pi[\underline{t}_i])^{[n_i]})\\ &= \sum_{\underline{m}\in\mathbb{N}^d}\tau^{[\underline{m}]}\sum_{\underline{n}\in\mathbb{N}^d}\nabla^{\underline{m}+\underline{n}}(x)\otimes(-\tau)^{[\underline{n}-n_i\underline{1}_i]}(-\tau_i+\pi[\underline{t}_i])^{[n_i]}\\ &= \sum_{n\in\mathbb{N}}\nabla_i^n(x)\otimes(\pi[\underline{t}_i])^{[n]} = \sum_{n\in\mathbb{N}} t_i^n\nabla_i^n(x)\otimes\pi^{[n]}.\end{aligned}$$

The fourth equality follows from $\sum_{l=m+n}\tau_i^{[m]}(-\tau_i+\pi[\underline{t}_i])^{[n]} = (\pi[\underline{t}_i])^{[l]}$ and $\sum_{l=m+n}\tau_j^{[m]}(-\tau_j)^{[n]} = 0$ $(j \neq i)$ for $l \in \mathbb{N}_{>0}$. □

Lemma 207 *We have* $\pi^{[n]}\pi^{-1} \in I^{n-1}W^{\mathrm{PD},(1)}(R_{O_{\overline{K}}})$ *for* $n \in \mathbb{N}_{>0}$, *and it converges to* 0 *as* $n \to \infty$ *with respect to the* p*-adic topology of* $A^{(2)}_{\mathrm{crys}}(O_{\overline{K}})$.

Proof This follows from $\frac{\pi^{p-1}}{p} \in W^{\mathrm{PD},(1)}(R_{O_{\overline{K}}})$, $\frac{1}{p}(\frac{\pi^{p-1}}{p})^p \in W^{\mathrm{PD},(2)}(R_{O_{\overline{K}}})$ (Proposition 147), $v_p(n!) \leq \frac{n-1}{p-1}$, and $I^rW^{\mathrm{PD},(1)}(R_{O_{\overline{K}}}) = W^{\mathrm{PD},(1)}(R_{O_{\overline{K}}}) \cap \pi^r\cdot W(R_{O_{\overline{K}}})[\frac{1}{p}]$. □

By Lemma 207, we can define $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$-linear endomorphisms F_i $(i \in \mathbb{N} \cap [1, d])$ of $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M$ by

$$F_i(x) = \sum_{n\in\mathbb{N}_{>0}} \pi^{[n]}\pi^{-1}t_i^n\nabla_i^{n-1}(x). \tag{208}$$

Lemma 208 (cf. Lemma 122)

(1) F_i *is an isomorphism.*
(2) *We have* $F_i \circ \nabla_i = \pi^{-1}(\gamma_i - 1)$ *on* $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M$.
(3) *For* $i, j \in \mathbb{N} \cap [1, d]$ *such that* $i \neq j$, *we have* $F_iF_j = F_jF_i$, $F_i\nabla_j = \nabla_jF_i$ *and* $F_i\gamma_j = \gamma_jF_i$.

Proof (1) By the proof of Lemma 160 (2), $I^1W^{\mathrm{PD},(1)}(R_{O_{\overline{K}}})A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})/p$ is a nilpotent ideal of $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})/p$. As $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$ is p-adically complete and separated, it suffices to prove the claim after taking the reduction mod $pA^{(N)}_{\mathrm{crys}}(O_{\overline{K}}) + I^1W^{\mathrm{PD},(1)}(R_{O_{\overline{K}}})\cdot A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$. Then F_i becomes the multiplication by t_i, which is an isomorphism because $t_i \in A^\times$.

(2) This follows from Lemma 206.

(3) One can verify the first two equalities by explicit computation using $\nabla_i \circ \nabla_j = \nabla_j \circ \nabla_i$ and $\nabla_i \circ t_j \mathrm{id} = t_j \mathrm{id} \circ \nabla_i$ $(i \neq j)$. The last one follows from the first two and (2). □

For $I = \{i_1 < \cdots < i_q\} \subset \mathbb{N} \cap [1, d]$, we define $dt_I \in \Omega^q_{\mathcal{A}}$ and $e_I \in \wedge^q E$ to be $dt_{i_1} \wedge \ldots \wedge dt_{i_q}$ and $e_{i_1} \wedge \ldots \wedge e_{i_q}$. Let F_I denote the composition $F_{i_q} \circ F_{i_{q-1}} \circ \cdots \circ F_{i_1}$, which is an $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$-linear automorphism of $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M$. We define the isomorphism $F^q \colon A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M \otimes_{\mathcal{A}} \Omega^q_{\mathcal{A}} \xrightarrow{\cong} A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M \otimes_{\mathbb{Z}} \wedge^q E$ by $F^q(x \otimes dt_I) = F_I(x) \otimes e_I$ for $x \in A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M$ and $I \subset \mathbb{N} \cap [1, d]$ with $\sharp I = q$.

Proposition 209 *The isomorphisms F^q $(q \in \mathbb{N})$ define an isomorphism of complexes between the de Rham complex of $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M$ and the Koszul complex of $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M$ with respect to $\pi^{-1}(\gamma_i - 1)$:*

$$F \colon A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M \otimes_{\mathcal{A}} \Omega^\bullet_{\mathcal{A}} \xrightarrow{\cong} K(A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M; \frac{\gamma_1 - 1}{\pi}, \ldots, \frac{\gamma_d - 1}{\pi}). \tag{209}$$

Proof For $x \in A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M$ and $I \subset \mathbb{N} \cap [1, d]$, we have

$$F \circ \nabla(x dt_I) = F(\textstyle\sum_{i \in I^c} \nabla_i(x) dt_i \wedge dt_I) = \sum_{i \in I^c} F_{I \cup \{i\}} \circ \nabla_i(x) \otimes e_i \wedge e_I$$
$$d \circ F(x dt_I) = d(F_I(x) e_I) = \textstyle\sum_{i \in I^c} \pi^{-1}(\gamma_i - 1) \circ F_I(x) \otimes e_i \wedge e_I,$$

where $I^c = (\mathbb{N} \cap [1, d]) \backslash I$. Lemma 208 implies that these two elements coincide. □

Now it remains to prove the following proposition.

Proposition 210 *The isomorphism*

$$L\eta^+_\pi R\Gamma(\Gamma_{\mathcal{A}}, TA^{\square,(N)}_{\mathrm{crys}}(M)) \cong A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M \otimes_{\mathcal{A}} \Omega^\bullet_{\mathcal{A}}$$

obtained from (201), (202), (207), and (209) coincides with the morphism (199).

We prove Proposition 210 in the rest of this section.

For a $\Gamma_{\mathcal{A}}$-module T, let $K_\gamma(T)$ denote the Koszul complex $K(T; \gamma_1 - 1, \ldots \gamma_d - 1)$. If the action of $\Gamma_{\mathcal{A}}$ on $T/\pi T$ is trivial, we define $K_{\pi^{-1}\gamma}(T)$ to be the Koszul complex $K(T; \pi^{-1}(\gamma_1 - 1), \ldots, \pi^{-1}(\gamma_d - 1))$. To simplify the notation, we abbreviate $A^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})$ and $\mathscr{A}^{\square,(N)}_{\mathrm{crys}}(\mathcal{A})$ to $A^{\square,(N)}_{\mathrm{crys}}$ and $\mathscr{A}^{\square,(N)}_{\mathrm{crys}}$, and put $M^{\square,(N)}_{A_{\mathrm{crys}}} := A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})\widehat{\otimes}_{O_K} M$,

$$T\mathscr{A}^{\square,(N)}_{\mathrm{crys},\pi}\Omega^\bullet_{\mathcal{A}}(M) := TA^{\square,(N)}_{\mathrm{crys}}(M) \otimes_{A^{\square,(N)}_{\mathrm{crys}}} \mathscr{A}^{\square,(N)}_{\mathrm{crys},\pi}\Omega^\bullet_{\mathcal{A}},$$
$$T\mathscr{A}^{\square,(N)}_{\mathrm{crys}}\Omega^\bullet_{\mathcal{A}}(M) := TA^{\square,(N)}_{\mathrm{crys}}(M) \otimes_{A^{\square,(N)}_{\mathrm{crys}}} \mathscr{A}^{\square,(N)}_{\mathrm{crys}} \otimes_{\mathcal{A}} \Omega^\bullet_{\mathcal{A}}.$$

We have

$$\pi^{-q} T\mathscr{A}_{\mathrm{crys}}^{\square,(N)}\Omega_{\mathcal{A}}^{q}(M) \subset T\mathscr{A}_{\mathrm{crys},\pi}^{\square,(N)}\Omega_{\mathcal{A}}^{q}(M). \tag{210}$$

We identify $TA_{\mathrm{crys}}^{\square,(N)}(M)$ with $M_{A_{\mathrm{crys}}}^{\square,(N)}$ via the isomorphism (207).

Let

$$G_{(0)}\colon M_{A_{\mathrm{crys}}}^{\square,(N)} \otimes_{\mathcal{A}} \Omega_{\mathcal{A}}^{\bullet} \longrightarrow \eta_{\pi}^{+}K_{\gamma}(T\mathscr{A}_{\mathrm{crys},\pi}^{\square,(N)}\Omega_{\mathcal{A}}^{\bullet}(M))$$

be the composition of

$$\begin{aligned} A_{\mathrm{crys}}^{(N)}(O_{\overline{K}})\widehat{\otimes}_{O_K} M \otimes_{\mathcal{A}} \Omega_{\mathcal{A}}^{\bullet} &\longrightarrow \Gamma(\Gamma_{\mathcal{A}}, (\mathscr{A}_{\mathrm{crys}}^{\square,(N)} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{A}} \Omega_{\mathcal{A}}^{\bullet}) \\ &\xrightarrow{\cong} \Gamma(\Gamma_{\mathcal{A}}, T\mathscr{A}_{\mathrm{crys}}^{\square,(N)}\Omega_{\mathcal{A}}^{\bullet}(M)) \longrightarrow \eta_{\pi}^{+}\Gamma(\Gamma_{\mathcal{A}}, T\mathscr{A}_{\mathrm{crys},\pi}^{\square,(N)}\Omega_{\mathcal{A}}^{\bullet}(M)) \\ &\hookrightarrow \eta_{\pi}^{+}K_{\gamma}(T\mathscr{A}_{\mathrm{crys},\pi}^{\square,(N)}\Omega_{\mathcal{A}}^{\bullet}(M)). \end{aligned} \tag{211}$$

Here the first morphism is induced by the structure homomorphism $A_{\mathrm{crys}}^{(N)}(O_{\overline{K}}) \to \mathscr{A}_{\mathrm{crys}}^{\square,(N)}$ and id_M, and the second one is induced by the first isomorphism in (205). Let

$$G_{(d)}\colon K_{\pi^{-1}\gamma}(M_{A_{\mathrm{crys}}}^{\square,(N)}) \longrightarrow \eta_{\pi}^{+}K_{\gamma}(T\mathscr{A}_{\mathrm{crys},\pi}^{\square,(N)}\Omega_{\mathcal{A}}^{\bullet}(M))$$

be the composition of the inverse of the isomorphism (202) with the quasi-isomorphism $\eta_{\pi}^{+}K_{\gamma}(TA_{\mathrm{crys}}^{\square,(N)}(M)) \to \eta_{\pi}^{+}K_{\gamma}(T\mathscr{A}_{\mathrm{crys},\pi}^{\square,(N)}\Omega_{\mathcal{A}}^{\bullet}(M))$. It suffices to show that $G_{(d)} \circ F$ and $G_{(0)}$ are homotopic for the isomorphism F in Proposition 209.

By replacing ∇_i with $\pi^{-1}(\gamma_i - 1)$ one by one, we construct a decomposition of F into the composition $F_{(d-1)} \circ \cdots \circ F_{(0)}$ of d isomorphisms $F_{(r)}\colon C_{(r)}^{\bullet} \xrightarrow{\cong} C_{(r+1)}^{\bullet}$ $(r \in \mathbb{N} \cap [0, d-1])$ of complexes. Then we construct morphisms of complexes $G_{(r)}\colon C_{(r)}^{\bullet} \to \eta_{\pi}^{+}K_{\gamma}(T\mathscr{A}_{\mathrm{crys},\pi}^{\square,(N)}\Omega_{\mathcal{A}}^{\bullet}(M))$ $(r \in \mathbb{N} \cap [0, d])$ such that $G_{(0)}$ and $G_{(d)}$ are as above, and show that $G_{(r+1)} \circ F_{(r)}$ and $G_{(r)}$ are homotopic for each $r \in \mathbb{N} \cap [0, d-1]$.

Lemma 211 *For $i, j \in \mathbb{N} \cap [1, d]$ with $i \neq j$, ∇_i and $\pi^{-1}(\gamma_j - 1)$ on $M_{A_{\mathrm{crys}}}^{\square,(N)}$ are commutative.*

Proof This follows from Lemma 206, $\nabla_i \circ \nabla_j = \nabla_j \circ \nabla_i$ and $\nabla_i \circ t_j\mathrm{id} = t_j\mathrm{id} \circ \nabla_i$. □

Let $r \in \mathbb{N} \cap [0, d]$. We first construct the complex $C_{(r)}^{\bullet}$. For $i \in \mathbb{N} \cap [1, d]$, we define the endomorphism $\partial_i^{(r)}$ of $M_{A_{\mathrm{crys}}}^{\square,(N)}$ to be $\pi^{-1}(\gamma_i - 1)$ if $i \in [1, r]$ and ∇_i if $i \in [r+1, d]$. For $i \in \mathbb{N} \cap [1, d]$, we define $\omega_i^{(r)}$ to be $e_i \in E$ if $i \in [1, r]$, and $dt_i \in \Omega_{\mathcal{A}}$ if $i \in [r+1, d]$. Let $E_{(r)}$ be the free $\mathbb{Z}$-module $\oplus_{i \in \mathbb{N} \cap [1,d]} \mathbb{Z}\omega_i^{(r)}$. For $I = \{i_1 < \cdots < i_q\} \subset \mathbb{N} \cap [1, d]$, let $\omega_I^{(r)}$ denote the element $\omega_{i_1}^{(r)} \wedge \ldots \wedge \omega_{i_q}^{(r)}$ of $\wedge^q E_{(r)}$.

We put $C_{(r)}^{q} := M_{A_{\mathrm{crys}}}^{\square,(N)} \otimes_{\mathbb{Z}} \wedge^q E_{(r)}$ $(q \in \mathbb{N})$ and define the homomorphism $d_{(r)}^{q}\colon C_{(r)}^{q} \to C_{(r)}^{q+1}$ by $d_{(r)}^{q}(x \otimes \omega_I) = \sum_{1 \le i \le d} \partial_i^{(r)}(x) \otimes \omega_i^{(r)} \wedge \omega_I^{(r)}$ for $x \in M_{A_{\mathrm{crys}}}^{\square,(N)}$ and $I \subset \mathbb{N} \cap [1, d]$ with $\sharp I = q$. By Lemma 211, $\partial_i^{(r)}$ $(i \in \mathbb{N} \cap [1, d])$ are mutually commutative, and therefore $d^{q+1}_{(r)} \circ d^{q}_{(r)} = 0$ for every $q \in \mathbb{N}$. The complex $C^{\bullet}_{(0)}$ is identified with $M^{\square,(N)}_{A_{\mathrm{crys}}} \otimes_{\mathcal{A}} \Omega^{\bullet}_{\mathcal{A}}$ via the canonical isomorphisms $\Omega^{q}_{\mathcal{A}} \cong \mathcal{A} \otimes_{\mathbb{Z}} \wedge^{q} E^{(0)}$ ($q \in \mathbb{N}$), and we have $C^{\bullet}_{(d)} = K_{\pi^{-1}\gamma}(M^{\square,(N)}_{A_{\mathrm{crys}}})$.

Let $r \in \mathbb{N} \cap [0, d-1]$. For $q \in \mathbb{N}$, we define the isomorphism $F^{q}_{(r)} \colon C^{q}_{(r)} \xrightarrow{\cong} C^{q+1}_{(r)}$ by $F^{q}_{(r)}(x \otimes \omega^{(r)}_{I}) = x \otimes \omega^{(r+1)}_{I}$ if $r+1 \notin I$, $F_{r+1}(x) \otimes \omega^{(r+1)}_{I}$ if $r+1 \in I$ for $x \in M^{\square,(N)}_{A_{\mathrm{crys}}}$ and $I \subset \mathbb{N} \cap [1, d]$ with $\sharp I = q$. See (208) for the definition of F_{r+1}. We have $F_{r+1} \circ \partial^{(r)}_{i} = \partial^{(r+1)}_{i} \circ F_{r+1}$ for $i \neq r+1$ and $\partial^{(r+1)}_{r+1} = F_{r+1} \circ \partial^{(r)}_{r+1}$ by Lemma 208 (2) and (3). This shows that $F^{q}_{(r)}$ ($q \in \mathbb{N}$) define an isomorphism of complexes $F_{(r)} \colon C^{\bullet}_{(r)} \xrightarrow{\cong} C^{\bullet}_{(r+1)}$. It is obvious that the composition $F_{(d-1)} \circ \cdots \circ F_{(0)}$ coincides with the isomorphism $F \colon C^{\bullet}_{(0)} \xrightarrow{\cong} C^{\bullet}_{(d)}$ (209).

Let $r \in \mathbb{N} \cap [0, d]$. Let us construct a morphism of complexes $G_{(r)} \colon C^{\bullet}_{(r)} \to \eta^{+}_{\pi} K_{\gamma}(T\mathscr{A}^{\square,(N)}_{\mathrm{crys},\pi} \Omega^{\bullet}_{\mathcal{A}}(M))$. We first define an $A^{(N)}_{\mathrm{crys}}(O_{\overline{K}})$-linear homomorphism

$$G_r \colon M^{\square,(N)}_{A_{\mathrm{crys}}} \longrightarrow \mathscr{A}^{\square,(N)}_{\mathrm{crys}} \otimes_{A^{\square,(N)}_{\mathrm{crys}}} M^{\square,(N)}_{A_{\mathrm{crys}}} = \mathscr{A}^{\square,(N)}_{\mathrm{crys}} \otimes_{A^{\square,(N)}_{\mathrm{crys}}} TA^{\square,(N)}_{\mathrm{crys}}(M)$$

by

$$G_r(x) = \sum_{\underline{n} \in \{0\}^r \times \mathbb{N}^{d-r}} \tau^{[\underline{n}]} \otimes \nabla^{\underline{n}}(x), \quad x \in M^{\square,(N)}_{A_{\mathrm{crys}}}.$$

Lemma 212 (1) *For $i \in \mathbb{N} \cap [1, r]$, we have $(\gamma_i - 1) \circ G_r = G_r \circ (\gamma_i - 1)$.*
(2) *For $i \in \mathbb{N} \cap [r+1, d]$, we have $(\nabla_i \otimes \mathrm{id}) \circ G_r = G_r \circ \nabla_i$.*
(3) *For $j \in \mathbb{N} \cap [r+1, d]$, we have $(\gamma_j - 1) \circ G_r = 0$.*
(4) *For $j \in \mathbb{N} \cap [1, r]$, we have $(\nabla_j \otimes \mathrm{id}) \circ G_r = 0$.*

Proof We can verify the claim by using Lemma 206, (203), (204), and the equalities $\nabla_i \circ \nabla_j = \nabla_j \circ \nabla_i$ and $\nabla_i \circ t_j\mathrm{id} = t_j\mathrm{id} \circ \nabla_i$ ($i \neq j$) on $M^{\square,(N)}_{A_{\mathrm{crys}}}$ as follows. Let Λ_r denote $\{0\}^r \times \mathbb{N}^{d-r}$. The last equality for $\gamma_j \circ G_r(x)$ follows from $\sum_{l=n+m}(\tau_j - \pi[\underline{t}_j])^{[n]}(\pi[\underline{t}_j])^{[m]} = \tau_j^{[l]}$ for $l \in \mathbb{N}$.

$$\begin{aligned}
\gamma_i \circ G_r(x) &= \sum_{\underline{n} \in \Lambda_r} \tau^{[\underline{n}]} \otimes \left(\sum_{n \in \mathbb{N}} \pi^{[n]} t_i^n \nabla_i^n (\nabla^{\underline{n}}(x)) \right) \\
&= \sum_{\underline{n} \in \Lambda_r} \tau^{[\underline{n}]} \otimes \nabla^{\underline{n}} \left(\sum_{n \in \mathbb{N}} \pi^{[n]} t_i^n \nabla_i^n(x) \right) = G_r \circ \gamma_i(x), \\
(\nabla_i \otimes \mathrm{id}) \circ G_r(x) &= \sum_{\underline{n} \in \Lambda_r, \underline{n} \geq \underline{1}_i} \tau^{[\underline{n}-\underline{1}_i]} \otimes \nabla^{\underline{n}}(x) \\
&= \sum_{\underline{n} \in \Lambda_r} \tau^{[\underline{n}]} \otimes \nabla^{\underline{n}}(\nabla_i(x)) = G_r \circ \nabla_i(x),
\end{aligned}$$

$$\begin{aligned}\gamma_j \circ G_r(x) &= \sum_{\underline{n}\in\Lambda_r} \tau^{[\underline{n}-n_j\underline{1}]}(\tau_j - \pi[\underline{t}_j])^{[n_j]} \otimes \left(\sum_{m\in\mathbb{N}} \pi^{[m]} t_j^m \nabla_j^m(\nabla^{\underline{n}}(x)\right) \\ &= \sum_{\underline{n}\in\Lambda_r}\sum_{m\in\mathbb{N}} \tau^{[\underline{n}-n_j\underline{1}]}(\tau_j - \pi[\underline{t}_j])^{[n_j]}(\pi[\underline{t}_j])^{[m]} \otimes (\nabla_j^{n_j+m}(\nabla^{\underline{n}-n_j\underline{1}}(x)) \\ &= \sum_{\underline{n}\in\Lambda_r} \tau^{[\underline{n}]} \otimes \nabla^{\underline{n}}(x) = G_r(x),\end{aligned}$$

$$(\nabla_j \otimes \mathrm{id}) \circ G_r(x) = (\nabla_j \otimes \mathrm{id}) \left(\sum_{\underline{n}\in\Lambda_r} \tau^{[\underline{n}]} \otimes \nabla^{\underline{n}}(x)\right) = 0.$$

□

We define the homomorphism $G^q_{(r)}\colon C^q_{(r)} \to K^q_\gamma(T\mathscr{A}^{\square,(N)}_{\mathrm{crys},\pi}\Omega^\bullet_{\mathcal{A}}(M))$ by

$$G^q_{(r)}(x \otimes \omega_I^{(r)}) = \pi^{\sharp(I\cap[1,r])} G_r(x) \otimes \omega_I^{(r)}$$

for $x \in M^{\square,(N)}_{A_{\mathrm{crys}}}$ and $I \subset \mathbb{N}\cap[1,d]$ with $\sharp I = q$. By using Lemma 212, we see that this defines a morphism of complexes

$$G_{(r)}\colon C^\bullet_{(r)} \longrightarrow K_\gamma(T\mathscr{A}^{\square,(N)}_{\mathrm{crys},\pi}\Omega^\bullet_{\mathcal{A}}(M)). \tag{212}$$

For x and I as above, the image of $x\otimes\omega_I^{(r)}$ under the homomorphism $G^q_{(r)}$ is contained $\pi^{q_2}T\mathscr{A}^{\square,(N)}_{\mathrm{crys}}\Omega^{q_1}_{\mathcal{A}}(M)\otimes_{\mathbb{Z}}\wedge^{q_2}E \subset \pi^q T\mathscr{A}^{\square,(N)}_{\mathrm{crys},\pi}\Omega^{q_1}_{\mathcal{A}}(M)\otimes_{\mathbb{Z}}\wedge^{q_2}E$, where $q_1 = \sharp(I\cap[r+1,d])$ and $q_2 = \sharp(I\cap[1,r])$. Hence the morphism of complexes $G_{(r)}$ (212) factors through the subcomplex $\eta^+_\pi K_\gamma(T\mathscr{A}^{\square,(N)}_{\mathrm{crys},\pi}\Omega^\bullet_{\mathcal{A}}(M))$. By the construction of (207) and (202), and the description of δ_M in (206), we see that $G_{(0)}$ and $G_{(d)}$ coincide with the homomorphisms defined before Lemma 211.

It remains to construct a homotopy between $G_{(r+1)}\circ F_{(r)}$ and $G_{(r)}$ for $r\in\mathbb{N}\cap[0,d-1]$. We use the "integration homomorphism" with respect to the variable τ_{r+1} defined as follows.

$$I_{r+1}\colon \mathscr{A}^{\square,(N)}_{\mathrm{crys}} \otimes_{A^{\square,(N)}_{\mathrm{crys}}} TA^{\square,(N)}_{\mathrm{crys}}(M) \longrightarrow \mathscr{A}^{\square,(N)}_{\mathrm{crys}} \otimes_{A^{\square,(N)}_{\mathrm{crys}}} TA^{\square,(N)}_{\mathrm{crys}}(M),$$
$$\textstyle\sum_{\underline{n}\in\mathbb{N}^d} \tau^{[\underline{n}]}\otimes x_{\underline{n}} \longmapsto \sum_{\underline{n}\in\mathbb{N}^d}\tau^{[\underline{n}+\underline{1}_{r+1}]}\otimes x_{\underline{n}}.$$

Lemma 213 (1) *We have* $I_{r+1}\circ G_r\circ\nabla_{r+1} = G_r - G_{r+1}$.
(2) *For* $i\in\mathbb{N}\cap[1,d]$, *we have*

$$(\nabla_i\otimes\mathrm{id})\circ I_{r+1}\circ G_r = \begin{cases} 0 & \text{if } i\in[1,r],\\ G_r & \text{if } i = r+1,\\ I_{r+1}\circ G_r\circ\nabla_i & \text{if } i\in[r+2,d].\end{cases}$$

(3) *For* $i\in\mathbb{N}\cap[1,d]$, *we have*

$$(\gamma_i - 1) \circ I_{r+1} \circ G_r = \begin{cases} I_{r+1} \circ G_r \circ (\gamma_i - 1) & \textit{if } i \in [1, r], \\ -\pi G_{r+1} \circ F_{r+1} & \textit{if } i = r+1, \\ 0 & \textit{if } i \in [r+2, d]. \end{cases}$$

Proof (1) For $x \in M^{\square,(N)}_{A_{\mathrm{crys}}}$, we have

$$\begin{aligned} I_{r+1} \circ G_r \circ \nabla_{r+1}(x) =& I_{r+1}\left(\sum_{\underline{n}\in\{0\}^r\times\mathbb{N}^{d-r}} \tau^{[\underline{n}]} \otimes \nabla^{\underline{n}+\underline{1}_{r+1}}(x)\right) \\ =& \sum_{\underline{m}\in\{0\}^r\times\mathbb{N}^{d-r}, \underline{m}\geq\underline{1}_{r+1}} \tau^{[\underline{m}]} \otimes \nabla^{\underline{m}}(x) = G_r(x) - G_{r+1}(x). \end{aligned}$$

(2) For $i \in \mathbb{N} \cap [1, d]$, we see that $(\nabla_i \otimes \mathrm{id}) \circ I_{r+1} = I_{r+1} \circ (\nabla_i \otimes \mathrm{id})$ if $i \neq r+1$ and $(\nabla_i \otimes \mathrm{id}) \circ I_{r+1} = \mathrm{id}$ if $i = r+1$ by using (203). Hence the claim follows from Lemma 212 (2) and (4).

(3) For $i \in \mathbb{N} \cap [1, d]$ different from $r+1$ and $x = \sum_{\underline{n}\in\mathbb{N}^d} \tau^{[\underline{n}]} \otimes x_{\underline{n}} \in \mathscr{A}^{\square,(N)}_{\mathrm{crys}} \otimes_{A^{\square,(N)}_{\mathrm{crys}}} TA^{\square,(N)}_{\mathrm{crys}}(M)$ ($x_{\underline{n}} \in TA^{\square,(N)}_{\mathrm{crys}}(M)$), we have

$$I_{r+1} \circ \gamma_i(x) = \gamma_i \circ I_{r+1}(x) = \sum_{\underline{n}\in\mathbb{N}^d} \tau^{[\underline{n}+\underline{1}_{r+1}-n_i\underline{1}_i]}(\tau_i - \pi[\underline{t}_i])^{[n_i]} \otimes \gamma_i(x_{\underline{n}})$$

by (204). Hence the equality for $i \neq r+1$ follows from Lemma 212 (1) and (3). The equality for $i = r+1$ is verified as follows. Let $x \in M^{\square,(N)}_{A_{\mathrm{crys}}}$. For $s \in \{r, r+1\}$, put $\Lambda_s := \{0\}^s \times \mathbb{N}^{d-s}$.

$$\begin{aligned} &\gamma_{r+1} \circ I_{r+1} \circ G_r(x) \\ &= \gamma_{r+1}\left(\sum_{\underline{n}\in\Lambda_r} \tau^{[\underline{n}+\underline{1}_{r+1}]} \otimes \nabla^{\underline{n}}(x)\right) \\ &= \sum_{\underline{n}\in\Lambda_r} \tau^{[\underline{n}-n_{r+1}\underline{1}_{r+1}]}(\tau_{r+1} - \pi[\underline{t}_{r+1}])^{[n_{r+1}+1]} \otimes \left(\sum_{n\in\mathbb{N}} \pi^{[n]} t^n_{r+1} \nabla^n_{r+1}(\nabla^{\underline{n}}(x))\right) \\ &= \sum_{\underline{l}\in\Lambda_{r+1}} \tau^{[\underline{l}]}\left(\sum_{(m,n)\in\mathbb{N}^2} (\tau_{r+1} - \pi[\underline{t}_{r+1}])^{[m+1]}(\pi[\underline{t}_{r+1}])^{[n]} \otimes \nabla^{m+n}_{r+1}(\nabla^{\underline{l}}(x))\right) \\ &= \sum_{\underline{l}\in\Lambda_{r+1}} \tau^{[\underline{l}]} \sum_{l\in\mathbb{N}} (\tau^{[l+1]}_{r+1} - (\pi[\underline{t}_{r+1}])^{[l+1]}) \otimes \nabla^l_{r+1}(\nabla^{\underline{l}}(x)) \\ &= I_{r+1} \circ G_r(x) - \sum_{\underline{l}\in\Lambda_{r+1}} \tau^{[\underline{l}]} \otimes \nabla^{\underline{l}}\left(\sum_{l\in\mathbb{N}} \pi^{[l+1]} t^{l+1}_{r+1} \nabla^l_{r+1}(x)\right) \end{aligned}$$

$$=I_{r+1} \circ G_r(x) - \pi G_{r+1} \circ F_{r+1}(x).$$

□

We define the homomorphism $k^q_{(r)} \colon C^q_{(r)} \to K^{q-1}_\gamma(T\mathscr{A}^{\Box,(N)}_{\mathrm{crys},\pi}\Omega^\bullet_{\mathcal{A}}(M))$ for $q \in \mathbb{N}_{>0}$ by

$$k^q_{(r)}(x \otimes \omega^{(r)}_I) = \begin{cases} 0 & \text{if } r+1 \notin I, \\ \varepsilon_{I,r+1}\pi^{\sharp(I\cap[1,r])} I_{r+1} \circ G_r(x) \otimes \omega^{(r)}_{I\backslash\{r+1\}} & \text{if } r+1 \in I, \end{cases}$$

where $\varepsilon_{I,r+1} \in \{\pm 1\}$ is defined by $\omega^{(r)}_I = \varepsilon_{I,r+1} dt_{r+1} \wedge \omega^{(r)}_{I\backslash\{r+1\}}$. We set $k^0_{(r)} = 0$. The proof of Proposition 210 is completed by the following lemma.

Lemma 214 (1) *For* $q \in \mathbb{N}$, *we have* $d^{q-1} \circ k^q_{(r)} + k^{q+1}_{(r)} \circ d^q_{(r)} = G^q_{(r)} - G^q_{(r+1)} \circ F^q_{(r)}$.

(2) *For* $q \in \mathbb{N}_{>0}$, *the image of* $k^q_{(r)}$ *is contained in the degree* $(q-1)$*-part of* $\eta^+_\pi K_\gamma(T\mathscr{A}^{\Box,(N)}_{\mathrm{crys},\pi}\Omega^\bullet_{\mathcal{A}}(M))$.

Proof (1) We prove the equality for the images of $x \otimes \omega^{(r)}_I$ for $x \in M^{\Box,(N)}_{A_{\mathrm{crys}}}$ and $I \subset \mathbb{N} \cap [1,d]$ with $\sharp I = q$. In the case $r+1 \notin I$, we have

$$(d^{q-1} \circ k^q_{(r)} + k^{q+1}_{(r)} \circ d^q_{(r)})(x \otimes \omega^{(r)}_I) = k^{q+1}_{(r)}\left(\sum_{i\in\mathbb{N}\cap[1,d]} \partial^{(r)}_i(x) \otimes \omega^{(r)}_i \wedge \omega^{(r)}_I\right)$$
$$= \pi^{\sharp(I\cap[1,r])} I_{r+1} \circ G_r \circ \nabla_{r+1}(x) \otimes \omega^{(r)}_I.$$

The last term equals to $(G^q_{(r)} - G^q_{(r+1)} \circ F^q_{(r)})(x \otimes \omega^{(r)}_I)$ by Lemma 213 (1). Suppose that $r+1 \in I$. Put $J = \backslash\{r+1\}$ and $y = \varepsilon_{I,r+1}x$. Then we have $x \otimes \omega^{(r)}_I = y \otimes (\omega^{(r)}_{r+1} \wedge \omega^{(r)}_J)$. Its image under $k^q_{(r)}$ is $\pi^{\sharp(J\cap[1,r])} I_{r+1} \circ G_r(y) \otimes \omega^{(r)}_J$ by definition. Put $J^c = \mathbb{N}\cap[1,d]\backslash J$, $J^c_0 := J^c \cup (\mathbb{N}\cap[1,r])$, and $J^c_1 := J^c \cup (\mathbb{N}\cap[r+1,d])$. By Lemma 213 (2) and (3), we obtain

$$\begin{aligned} &d^{q-1} \circ k^q_{(r)}(x \otimes \omega^{(r)}_I) \\ &= \pi^{\sharp(J\cap[1,r])}\Bigg\{\sum_{i\in J^c_0} (\nabla_i \otimes \mathrm{id}) \circ I_{r+1} \circ G_r(y) \otimes (dt_i \wedge \omega^{(r)}_J) \\ &\qquad + \sum_{i\in J^c_1} (\gamma_i - 1) \circ I_{r+1} \circ G_r(y) \otimes (e_i \wedge \omega^{(r)}_J)\Bigg\} \\ &= \pi^{\sharp(J\cap[1,r])}\Bigg\{G_r(y) \otimes (dt_{r+1} \wedge \omega^{(r)}_J) \\ &\qquad + \sum_{i\in J^c\cap[r+2,d]} I_{r+1} \circ G_r \circ \nabla_i(y) \otimes (dt_i \wedge \omega^{(r)}_J) \end{aligned}$$

$$+\pi \sum_{i\in J^c\cap[1,r]} I_{r+1}\circ G_r\circ\pi^{-1}(\gamma_i-1)(y)\otimes(e_i\wedge\omega_J^{(r)})$$
$$-\pi G_{r+1}\circ F_{r+1}(y)\otimes(e_{r+1}\wedge\omega_J^{(r)})\Big\}.$$

On the other hand, we have

$$\begin{aligned}
&k_{(r)}^{q+1}\circ d_{(r)}^q(x\otimes\omega_I^{(r)})\\
&=k_{(r)}^{q+1}\left(\sum_{i\in J^c\setminus\{r+1\}}\partial_i^{(r)}(y)\otimes(\omega_i^{(r)}\wedge\omega_{r+1}^{(r)}\wedge\omega_J^{(r)})\right)\\
&=-\sum_{i\in J^c\cap[1,r]}\pi^{\sharp(J\cap[1,r])+1}I_{r+1}\circ G_r\circ\pi^{-1}(\gamma_i-1)(y)\otimes(e_i\wedge\omega_J^{(r)})\\
&\quad-\sum_{i\in J^c\cap[r+2,d]}\pi^{\sharp(J\cap[1,r])}I_{r+1}\circ G_r\circ\nabla_i(y)\otimes(dt_i\wedge\omega_J^{(r)}).
\end{aligned}$$

By taking the sum of the two, we obtain

$$\begin{aligned}
&(d^{q-1}\circ k_{(r)}^q+k_{(r)}^{q+1}\circ d_{(r)}^q)(x\otimes\omega_I^{(r)})\\
&=\pi^{\sharp(J\cap[1,r])}\{G_r(y)\otimes(\omega_{r+1}^{(r)}\wedge\omega_J^{(r)})-\pi G_{r+1}\circ F_{r+1}(y)\otimes(\omega_{r+1}^{(r+1)}\wedge\omega_J^{(r+1)})\}\\
&=G_{(r)}^q(y\otimes(\omega_{r+1}^{(r)}\wedge\omega_J^{(r)}))-G_{(r+1)}^q(F_{r+1}(y)\otimes(\omega_{r+1}^{(r+1)}\wedge\omega_J^{(r+1)}))\\
&=(G_{(r)}^q-G_{(r+1)}^q\circ F_{(r)}^q)(x\otimes\omega_I^{(r)}).
\end{aligned}$$

(2) Let $x\otimes\omega_I^{(r)}$ be the same as above. If $r+1\notin I$, we have $k_{(r)}^q(x\otimes\omega_I^{(r)})=0$. Suppose that $r+1\in I$. Under the notation in the proof of (1), we have $\omega_J^{(r)}\in\Omega_{\mathcal{A}}^{\sharp(J\cap[r+1,d])}\otimes_{\mathbb{Z}}\wedge^{\sharp(J\cap[1,r])}E$. By the computation of $z:=k_{(r)}^q(x\otimes\omega_I^{(r)})$ and $d^{q-1}(z)$ above and (210), we obtain $z\in\pi^{q-1}K_\gamma^{q-1}(T\mathscr{A}_{\mathrm{crys},\pi}^{\square,(N)}\Omega_{\mathcal{A}}^\bullet(M))$ and $d^{q-1}(z)\in\pi^qK_\gamma^q(T\mathscr{A}_{\mathrm{crys},\pi}^{\square,(N)}\Omega_{\mathcal{A}}^\bullet(M))$. □

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