$\mathscr{K}$-Convergence of Finite Volume Solutions of the Euler Equations

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Abstract We review our recent results on the convergence of invariant domain-preserving finite volume solutions to the Euler equations of gas dynamics. If the classical solution exists we obtain strong convergence of numerical solutions to the classical one applying the weak-strong uniqueness principle. On the other hand, if the classical solution does not exist we adapt the well-known Prokhorov compactness theorem to space-time probability measures that are generated by the sequences of finite volume solutions and show how to obtain the strong convergence in space and time of observable quantities. This can be achieved even in the case of ill-posed Euler equations having possibly many oscillatory solutions.

Keywords Convergence analysis · Finite volume methods · Euler equations · Ill-posedness · Dissipative measure–valued solutions · $\mathscr{K}$-convergence

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1 Introduction

Hyperbolic conservation laws are fundamental for most of physical, biological and mechanical processes. The iconic example of this class of partial differential equations are the Euler equations of gas dynamics. Being published in 1757 by Leonhard Euler in Mémoires de l'Académie des Sciences de Berlin in his article "Principes généraux du mouvement des fluides" the Euler equations are one of the first written partial differential equations at all. Recently, multidimensional Euler equations have achieved renewed interest in mathematical community. Indeed, it is a well-known fact that the classical (i.e., continuously differentiable) solution exists in general only for a short time since discontinuities (shocks) may develop. A suitable gener-

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alization is to consider weak solutions, which moreover satisfy the second law of thermodynamics.

As shown by De Lellis and Székelyhidi [8] and by Chiodaroli et al. [6] infinitely many weak entropy solutions can be constructed for the multidimensional compressible Euler equations. Their ill-posedness is related to the lack of compactness of a set of weak entropy solutions. Such a failure of well-posedness (i.e., uniqueness) of the multidimensional Euler equations in the class of weak entropy solutions is connected to the turbulence effects which are apparently not appropriately described by the concept of distributional solutions.

On the other hand, we can find in literature a large variety of powerful numerical schemes, typically finite volume or discontinuous Galerkin methods, that are successfully used in order to approximate multidimensional hyperbolic conservation laws and the Euler equations, in particular. We refer to monographs [11, 12, 18, 22, 28, 30, 33] and the references therein. Despite the popularity of the finite volume and discontinuous Galerkin methods for practical applications their theoretical convergence analysis for multidimensional hyperbolic conservation laws is still incomplete. We should mention for example the convergence and error analysis obtained in [26] for the Cauchy problem of a general multidimensional hyperbolic conservation law. Under the assumption of the existence of the classical solution the authors applied the stability result due to Dafermos [7] and DiPerna's method [9] in order to derive the error estimates for the explicit finite volume schemes satisfying the discrete entropy inequality. Consequently, they proved the strong convergence of the entropy stable finite volume schemes.

In view of the facts that the classical solution may not exist and the weak entropy solutions are non-unique new probabilistic concepts have been developed. In [9, 10, 29, 31] the so-called measure–valued solutions to hyperbolic conservation laws are studied. The latter are represented by the Young measures, which are space-time parametrized probability measures acting on a (solution) phase space. Measure–valued solutions have been also successfully used in [19, 21] in order to show convergence of the entropy stable finite volume schemes for general hyperbolic conservation laws under additional assumptions on the boundedness of numerical solutions or a growth condition on the flux function. Another interesting contribution to the convergence analysis of the Euler equations was presented in [29], where the limit of higher order viscous regularization to the Euler equations was identified with a measure–valued solution that exists globally in time.

Clearly, the set of (entropy) measure–valued solutions is larger than that of (entropy) weak solutions and thus the question of uniqueness remains still open. However, a recently introduced concept of dissipative measure–valued (DMV) solutions [5, 25] allows to show the DMV-strong uniqueness principle. It means that DMV solutions coincide with the strong solution as long as the latter exists.

The aim of the present paper is to review our recent results on the convergence analysis of some finite volume methods. It turned out that some invariant domain-preserving properties, such as the entropy stability, preservation of positivity of density and internal energy and minimum entropy principle are important in order to obtain convergence of a numerical scheme without any additional nonphysical assumptions [15]. We also report on the recently established concept of $\mathcal{K}$-convergence which allows to compute observable quantities of possibly strongly oscillating dissipative measure–valued solutions [14, 17]. We wish to give a clear overview of main convergence results without going into deep theoretical justifications. In this way we hope to attract the attention of more experimentally oriented computational scientists and to encourage them to apply $\mathcal{K}$-convergence to other well-known finite volume and discontinuous Galerkin methods. A reader interested in further theoretical details and proofs is referred to [14–17] and the references therein.

In what follows we firstly introduce the dissipative measure–valued and dissipative weak solutions of the Euler equations and describe a suitable invariant domain-preserving finite volume method. Consequently, we present the strong convergence results for single realizations under the assumption that the classical solution to the Euler equation exists and the strong convergence result of observable quantities in a general case.

2 Euler Equations and Dissipative Solutions

The gas dynamics of inviscid compressible flows is governed by the Euler equations

$$\begin{aligned} \partial_t \rho + \mathrm{div}\boldsymbol{m} &= 0, \\ \partial_t \boldsymbol{m} + \mathrm{div}(\boldsymbol{m} \otimes \boldsymbol{u}) + \nabla p &= 0, \\ \partial_t E + \mathrm{div}((E + p)\boldsymbol{u}) &= 0, \end{aligned} \tag{1}$$

where ρ, $\boldsymbol{m}$ and E represent the conservative variables, the density, momentum and the total energy, respectively. Further, p and $\boldsymbol{u} = \boldsymbol{m}/\rho$ stand for the pressure and velocity. The total energy $E = \frac{1}{2}\frac{\boldsymbol{m}^2}{\rho} + \rho e$ consists of the kinetic energy and the internal energy e.

System (1) is closed by the standard pressure law for a perfect gas $p(\rho, \vartheta) = R\rho\vartheta$, ϑ is the temperature and R the gas constant. We assume without loss of generality that $R = 1$. We denote by γ the adiabatic coefficient and by c_V the specific heat at constant volume, $c_V = \frac{1}{\gamma-1}$. In what follows we will assume that $1 < \gamma < 2$ and note that this covers the physically reasonable range for gases $1 < \gamma \leq 5/3$. Denoting s the specific physical entropy and S the total entropy we have

$$s(\rho, \vartheta) = \log\left(\frac{\vartheta^{c_V}}{\rho}\right) = \frac{1}{\gamma - 1}\log\left(\frac{p}{\rho^{\gamma}}\right), \quad S = \rho s \quad \text{and} \quad e(\rho, \vartheta) = c_V\vartheta.$$

On a space-time cylinder $\Omega \times (0, T)$, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, $T > 0$, the system of the Euler equations is accompanied by appropriate boundary and initial conditions. Here we assume the periodic or the no flux boundary conditions

$$\boldsymbol{u}|_{\partial\Omega}\cdot\boldsymbol{n}=0,\ \frac{\partial\vartheta}{\partial\boldsymbol{n}}=0$$

and set

$$\rho(t=0)=\rho_0,\quad \boldsymbol{m}(t=0)=\boldsymbol{m}_0,\quad E(t=0)=E_0.$$

In [16] it has been proved that numerical solutions obtained by suitable numerical schemes (such as invariant domain-preserving finite volume methods) either converge strongly in suitable Bochner spaces or their (weak) limit is not a weak entropy solution. Clearly, such a result calls for a new concept of generalized solutions to the Euler equations. Following [2, 5] we introduce the dissipative measure–valued solutions and dissipative weak solutions. The latter can be seen as the statistical mean values with respect to the corresponding Young measures.

Definition 1 (*Dissipative measure–valued solution*) [5, 16] A parametrized probability measure $\{\mathscr{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$,

$$\mathscr{V}\in L^\infty((0,T)\times\Omega;\mathscr{P}(\mathbb{R}^{d+2})),\ \mathbb{R}^{d+2}=\left\{[\tilde\rho,\tilde{\boldsymbol{m}},\widetilde{S}]\ \middle|\ \tilde\rho\in\mathbb{R},\ \tilde{\boldsymbol{m}}\in\mathbb{R}^d,\ \widetilde{S}\in\mathbb{R}\right\},$$

is called *dissipative measure valued (DMV) solution* of the Euler system (1) if the following holds:

- **lower bound on density and entropy**
  there exists $\underline{s}\in\mathbb{R}$ such that

$$\mathscr{V}_{t,x}\left[\left\{\rho\ge 0,\ S\equiv s\rho\ge\underline{s}\rho\right\}\right]=1 \text{ for a.a. } (t,x); \tag{2}$$

- **integral energy inequality**[1]

$$\int_\Omega\left\langle\mathscr{V}_{\tau,x};\frac{1}{2}\frac{|\tilde{\boldsymbol{m}}|^2}{\tilde\rho}+\tilde\rho e(\tilde\rho,\widetilde{S})\right\rangle\,\mathrm{d}x+\int_{\overline{\Omega}}\mathrm{d}\mathfrak{E}_{cd}(\tau)\le\int_\Omega\left[\frac{1}{2}\frac{|\boldsymbol{m}_0|^2}{\rho_0}+\rho_0 e(\rho_0,S_0)\right]\,\mathrm{d}x \tag{3}$$

  holds for a.a. $0\le\tau\le T$, with the energy concentration defect

$$\mathfrak{E}_{cd}\in L^\infty(0,T;M^+(\overline{\Omega})),$$

  where $M^+(\overline{\Omega})$ denotes the space of positive Radon measures on $\overline{\Omega}$;
- **equation of continuity**

$$\left[\int_\Omega\langle\mathscr{V};\tilde\rho\rangle\,\mathrm{d}x\right]_{t=0}^{t=\tau}=\int_0^\tau\int_\Omega\left[\langle\mathscr{V};\tilde\rho\rangle\,\partial_t\varphi+\langle\mathscr{V};\tilde{\boldsymbol{m}}\rangle\cdot\nabla\varphi\right]\,\mathrm{d}x\,\mathrm{d}t \tag{4}$$

[1]Here the mean value $\left\langle\mathscr{V}_{t,x};b\left(\tilde{\boldsymbol{U}}\right)\right\rangle\equiv\int_{\mathbb{R}^{d+2}}b\left(\tilde{\boldsymbol{U}}\right)d\mathscr{V}_{t,x}(\tilde{\boldsymbol{U}})$ for $\boldsymbol{U}\in\mathbb{R}^{d+2}$ and b bounded continuous function.

for any $0 \le \tau \le T$, and any $\varphi \in W^{1,\infty}((0,T)\times\Omega)$;

- **momentum equation**

$$\left[\int_\Omega \langle \mathscr{V}; \tilde{\boldsymbol{m}}\rangle \cdot \boldsymbol{\varphi}\ \mathrm{d}x\right]_{t=0}^{t=\tau}$$
$$= \int_0^\tau \int_\Omega \left[\langle \mathscr{V}; \tilde{\boldsymbol{m}}\rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle \mathscr{V}; 1_{\tilde{\rho}>0} \frac{\tilde{\boldsymbol{m}} \otimes \tilde{\boldsymbol{m}}}{\tilde{\rho}} \right\rangle : \nabla \boldsymbol{\varphi} + \left\langle \mathscr{V}; 1_{\tilde{\rho}>0} p(\tilde{\rho}, \widetilde{S})\right\rangle \mathrm{div}\boldsymbol{\varphi}\right]\ \mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^\tau \int_{\overline{\Omega}} \nabla\boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}_{cd}(t)\,\mathrm{d}t \tag{5}$$

for any $0 \le \tau \le T$, and any $\boldsymbol{\varphi} \in C^m([0,T]\times\overline{\Omega}; R^d)$, $\boldsymbol{\varphi}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$, $m \ge 1$, with the Reynolds concentration defect

$$\mathfrak{R}_{cd} \in L^\infty(0,T; M^+(\overline{\Omega}; R^{d\times d}_{\mathrm{sym}}))$$

satisfying

$$\underline{d}\ \mathfrak{E}_{cd} \le \mathrm{tr}[\mathfrak{R}_{cd}] \le \overline{d}\ \mathfrak{E}_{cd} \text{ for some constants } 0 < \underline{d} \le \overline{d}; \tag{6}$$

- **entropy inequality**

$$\left[\int_\Omega \left\langle \mathscr{V}; \widetilde{S}\right\rangle \varphi\ \mathrm{d}x\right]_{t=\tau_1-}^{t=\tau_2+} \ge \int_{\tau_1}^{\tau_2} \int_\Omega \left[\left\langle \mathscr{V}; \widetilde{S}\right\rangle \partial_t\varphi + \left\langle \mathscr{V}; 1_{\tilde{\rho}>0}\left(\widetilde{S}\tilde{\boldsymbol{u}}\right)\right\rangle \cdot \nabla\varphi\right]\ \mathrm{d}x\,\mathrm{d}t \tag{7}$$

for any $0 \le \tau_1 \le \tau_2 < T$, and any $\varphi \in W^{1,\infty}((0,T)\times\Omega)$, $\varphi \ge 0$.

The DMV solution is a very general concept that allows to show the convergence of invariant domain-preserving schemes in an elegant way. Despite its generality it still satisfies the DMV-strong uniqueness principle [5] and thus the DMV solutions coincide with the classical solution as long as the latter exists. To prove the latter the crucial properties are the energy dissipation (3) and (6) controlling the Reynolds defect in the momentum equation by the energy concentration defect. It is to be pointed out that the Reynolds concentration defect brings an additional freedom to model turbulent flow behaviour.

To simplify the viewpoint on this generalized solutions it often suffices to consider only the mean values of DMV solutions, which are the below-mentioned dissipative solutions.

Definition 2 (*dissipative weak solution*) [16] A triple $[\rho, \boldsymbol{m}, S]$ is *dissipative weak (DW) solution* of the full Euler system (1) if the following holds:

- **weak continuity** in time

$$\begin{aligned} &\rho \in C_{\mathrm{weak}}([0,T]; L^\gamma(\Omega)), \ (\gamma \text{ being the adiabatic constant}) \\ &\boldsymbol{m} \in C_{\mathrm{weak}}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d)), \\ &S \in L^\infty(0,T; L^\gamma(\Omega)) \cap BV_{\mathrm{weak}}([0,T]; L^\gamma(\Omega)); \end{aligned} \tag{8}$$

- **energy inequality**: there exists a measure

$$\mathfrak{E} \in L^\infty(0,T; M^+(\overline{\Omega})),$$

such that the inequality

$$\int_\Omega \left[\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\rho} + \rho e(\rho, S)\right](\tau,\cdot)\ \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(\tau) \leq \int_\Omega \left[\frac{1}{2}\frac{|\boldsymbol{m}_0|^2}{\rho_0} + \rho_0 e(\rho_0, S_0)\right]\ \mathrm{d}x \tag{9}$$

holds for a.a. $0 \leq \tau \leq T$;
- **equation of continuity**

$$\left[\int_\Omega \rho\varphi\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\rho\partial_t\varphi + \boldsymbol{m}\cdot\nabla\varphi]\ \mathrm{d}x\,\mathrm{d}t \tag{10}$$

holds for any $0 \leq \tau \leq T$;
- **momentum equation**

$$\left[\int_\Omega \boldsymbol{m}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\boldsymbol{m}\cdot\partial_t\boldsymbol{\varphi} + 1_{\rho>0}\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\rho} : \nabla\boldsymbol{\varphi} + 1_{\rho>0}p(\rho,S)\mathrm{div}\boldsymbol{\varphi}\right]\ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \nabla\boldsymbol{\varphi} : \mathrm{d}\mathfrak{R} \tag{11}$$

for any $0 \leq \tau \leq T$, any test function $\boldsymbol{\varphi} \in C^m([0,T]\times\overline{\Omega};\mathbb{R}^d)$, $\boldsymbol{\varphi}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$, and a defect measure

$$\mathfrak{R} \in L^\infty(0,T; M^+(\overline{\Omega};\mathbb{R}^d));$$

- **entropy inequality**

$$\left[\int_\Omega S\varphi\ \mathrm{d}x\right]_{t=\tau_1-}^{t=\tau_2+} \geq \int_{\tau_1}^{\tau_2} \int_\Omega \left[S\partial_t\varphi + \left\langle\mathcal{V}; 1_{\tilde{\rho}>0}\left(\widetilde{S}\tilde{\boldsymbol{u}}\right)\right\rangle\cdot\nabla\varphi\right]\ \mathrm{d}x\,\mathrm{d}t,\ S(0-,\cdot) = S_0, \tag{12}$$

for any $0 \leq \tau_1 \leq \tau_2 < T$, any $\varphi \in W^{1,\infty}((0,T)\times\Omega)$, $\varphi \geq 0$, where $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ is the aforementioned DMV solution
- **defect compatibility conditions**

$$\underline{d}\ \mathfrak{E} \leq \mathrm{tr}\,[\mathfrak{R}] \leq \overline{d}\ \mathfrak{E} \text{ for some constants } 0 \leq \underline{d} \leq \overline{d}, \tag{13}$$

and

$$\mathfrak{E} \geq \left\langle\mathcal{V}; \frac{1}{2}\frac{|\tilde{\boldsymbol{m}}|^2}{\tilde{\rho}} + \tilde{\rho}e(\tilde{\rho},\widetilde{S})\right\rangle - \left(\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\rho} + \rho e(\rho,S)\right). \tag{14}$$

The existence of DMV or DW solutions can be shown by the convergence of suitable invariant domain-preserving finite volume schemes. In what follows we will present such a finite volume method and review its convergence results for multidimensional

Euler equations. We mention in passing that in [13] the convergence of the standard Lax-Friedrichs finite volume method has been shown in an analogous way as presented below.

3 A Finite Volume Method Based on the Brenner Model

In [15] the two-velocity model for compressible flows by Brenner [3, 4] was revisited and a new invariant domain-preserving finite volume method, denoted here by the FLM method, has been proposed and analysed. To fix the notation we start by introducing a suitable discrete space and a finite volume mesh.

The finite volume grid $\mathbb{T}_h$ consists of finite volumes, denoted by K, that can be triangles, rectangles or polygons and cover the physical domain Ω

$$\overline{\Omega} = \bigcup_{K \in \mathbb{T}_h} K.$$

The parameter $h \in (0, 1)$ is the maximum element size, i.e., the size of the mesh $\mathbb{T}_h$. We assume that $\mathbb{T}_h$ is regular and quasi-uniform. The set of all faces is denoted by Σ, while the set of faces on the boundary is denoted by Σ_{ext}, and the set of interior faces by $\Sigma_{int} = \Sigma \backslash \Sigma_{ext}$. For periodic boundary conditions we set $\Sigma_{ext} = \emptyset$ and $\Sigma_{int} = \Sigma$. Further, we associate each face with its outer normal vector $\boldsymbol{n}$.

We denote by Q_h the set of piecewise constant functions on $\mathbb{T}_h$ and define for any $v \in Q_h$, $x \in \sigma \in \Sigma_{int}$

$$v^{\text{out}}(x) = \lim_{\delta \to 0+} v(x + \delta \boldsymbol{n}), \qquad v^{\text{in}}(x) = \lim_{\delta \to 0+} v(x - \delta \boldsymbol{n}),$$
$$\overline{v}(x) = \frac{v^{\text{in}}(x) + v^{\text{out}}(x)}{2}, \qquad [\![v]\!] = v^{\text{out}}(x) - v^{\text{in}}(x).$$

A numerical flux function in our finite volume method is based on the so-called *dissipative upwinding*. Let a velocity $\boldsymbol{u}_h \in Q_h$ and a function $r_h \in Q_h$, then the (classical) upwinding reads

$$\begin{aligned} Up[r_h, \boldsymbol{u}_h] &= r_h^{\text{up}} \boldsymbol{u}_h \cdot \boldsymbol{n} = r_h^{\text{in}} [\overline{\boldsymbol{u}_h} \cdot \boldsymbol{n}]^+ + r_h^{\text{out}} [\overline{\boldsymbol{u}_h} \cdot \boldsymbol{n}]^- \\ &= \overline{r_h}\, \overline{\boldsymbol{u}_h} \cdot \boldsymbol{n} - \frac{1}{2} |\overline{\boldsymbol{u}_h} \cdot \boldsymbol{n}|\, [\![r_h]\!] , \end{aligned}$$

where

$$[f]^{\pm} = \frac{f \pm |f|}{2} \quad \text{and} \quad r^{\text{up}} = \begin{cases} r^{\text{in}} & \text{if } \overline{\boldsymbol{u}_h} \cdot \boldsymbol{n} \geq 0, \\ r^{\text{out}} & \text{if } \overline{\boldsymbol{u}_h} \cdot \boldsymbol{n} < 0. \end{cases}$$

The numerical flux function is defined in the following way

$$F_h(r_h, \boldsymbol{u}_h) = Up[r_h, \boldsymbol{u}_h] - h^\beta [\![r_h]\!], \quad 0 < \beta < 1.$$

Note that the term $h^\beta [\![r_h]\!]$ leads to an additional vanishing viscosity term in the approximation of the Euler equations. Now we proceed to the formulation of a semi-discrete finite volume method for the Euler system (1).

Definition 3 (*FLM method*) Given the initial values $(\rho_{0,h}, \boldsymbol{m}_{0,h}, E_{0,h}) \in Q_h \times Q_h \times Q_h$, we seek a piecewise constant approximation $(\rho_h, \boldsymbol{m}_h, E_h) \in Q_h \times Q_h \times Q_h$ which solves at any time $t \in (0, T]$ the following equations:

$$\begin{aligned} &D_t \rho_h\Big|_K + \sum_{\sigma\in\partial K} \frac{|\sigma|}{|K|} F_h(\rho_h, \boldsymbol{u}_h) = 0, \\ &D_t \boldsymbol{m}_h\Big|_K + \sum_{\sigma\in\partial K} \frac{|\sigma|}{|K|} (\mathbf{F}_h(\boldsymbol{m}_h, \boldsymbol{u}_h) + \overline{p_h}\boldsymbol{n}) = h^{\alpha-1} \sum_{\sigma\in\partial K} \frac{|\sigma|}{|K|} [\![\boldsymbol{u}_h]\!], \\ &D_t E_h\Big|_K + \sum_{\sigma\in\partial K} \frac{|\sigma|}{|K|} \Big(F_h(E_h, \boldsymbol{u}_h) + (\overline{p_h}\, [\![\boldsymbol{u}_h]\!] + [\![p_h]\!]\, \overline{\boldsymbol{u}_h}) \cdot \boldsymbol{n} \Big) = \frac{h^{\alpha-1}}{2} \sum_{\sigma\in\partial K} \frac{|\sigma|}{|K|} [\![\boldsymbol{u}_h^2]\!], \end{aligned} \tag{15}$$

for any $K \in \mathbb{T}_h$.

By D_t we have denoted (continuous) time derivative; in practical implementation one can use any suitable ODE solver in order to approximate (15). In our recent work [15] we have shown that the FLM method (15) satisfies the following *invariant domain-preserving properties*, see [23, 24] where this notion was firstly introduced.

- **Positivity of the discrete density, pressure and internal energy**.
  For any fixed $h > 0$ the approximate density, pressure and internal energy remain strictly positive on any finite time interval. We refer the reader to [15, Sects. 4.3 and 4.4] for more details.
- **Discrete entropy inequality**.
  The discrete (renormalized) entropy inequality in the sense of Tadmor is satisfied, cf. [32]. More precisely, it holds that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}_h} \rho_h \chi(s_h) \Phi_h \,\mathrm{d}x \geq \sum_{\sigma\in\Sigma_{int}} \int_\sigma Up[\rho_h \chi(s_h), \boldsymbol{u}_h][[\Phi_h]] \mathrm{d}S_x + \\ + \sum_{\sigma\in\Sigma_{int}} \int_\sigma h^\beta \left(\overline{\nabla_\rho(\rho_h \chi(s_h))}[[\rho_h]] + \overline{\nabla_p(\rho_h \chi(s_h))}[[p_h]] \right) [[\Phi_h]] \mathrm{d}S_x, \end{aligned}$$

  where χ is a non-decreasing, concave, twice continuously differentiable function on $\mathbb{R}$ that is bounded from above. For the derivation and proof see [15, Sect. 3.2].
- **Minimum entropy principle**
  The discrete physical entropy $s_h = \log\left(\vartheta_h^{c_v}/\rho_h\right)$ attains its minimum at the initial time, i.e.,

$$s_h(t) \geq \underline{s}, \ t \geq 0, \quad \text{where} \quad -\infty < \underline{s} < \min s_h(0).$$

The entropy is either constant or produced over time, cf. [15, Sect. 4.2].

The above invariant domain-preserving properties are crucial in order to show that the approximate solutions obtained by the FLM method yield a consistent approximation to the Euler equations (1). Moreover, the discrete mass and energy conservation and some standard estimates, cf. [15], imply the stability of the FLM method, i.e., we have uniformly w.r.t. $h \to 0$

$$\|\rho_h\|_{L^\infty(0,T;L^\gamma(\Omega))} \lesssim 1, \quad \|\boldsymbol{m}_h\|_{L^\infty(0,T;L^{2\gamma/(\gamma+1)}(\Omega))} \lesssim 1, \quad \|E_h\|_{L^\infty(0,T;L^1(\Omega))} \lesssim 1.$$

In [15] the following type of nonlinear generalization of the *Lax-equivalence theorem* has been proven: Having consistent FLM method (15) for the Euler system (1), the stability of the FLM method is equivalent to its convergence. More precisely, we have shown the following results.

Theorem 1 (Existence of a DMV solution) *Let the initial data* $(\rho_{0,h}, \boldsymbol{m}_{0,h}, E_{0,h})$ *satisfy*

$$\rho_{0,h} \geq \underline{\rho} > 0, \quad E_{0,h} - \frac{1}{2}\frac{|\boldsymbol{m}_{0,h}|^2}{\rho_{0,h}} > 0.$$

Let $(\rho_h, \boldsymbol{m}_h, E_h) \in Q_h \times Q_h \times Q_h$ *be the solution of the FLM scheme* (15) *with*

$$0 < \beta < 1, \; 0 < \alpha < \frac{4}{3},$$

and there exist $\underline{\rho}, \overline{\vartheta} \in \mathbb{R}$, *such that the numerical solutions stay in a non-degenerate gas region*

$$0 < \underline{\rho} \leq \rho_h(t), \;\; \vartheta_h(t) \leq \overline{\vartheta} \textit{ for all } t \in [0,T] \textit{ uniformly for } h \to 0.$$

Then the family of approximate solutions $\{\rho_h, \boldsymbol{m}_h, E_h\}_{h\searrow 0}$ *generates a dissipative measure–valued (DMV) solution of the complete Euler system* (1) *in the sense of Definition 1.*

Further, taking into account the DMV–strong uniqueness principle proved in [5, Theorem 3.3] we obtain the desired strong convergence result.

Theorem 2 (Strong convergence of the FLM method) *In addition to the hypotheses of Theorem 1, suppose that the Euler system* (1) *admits the strong (Lipschitz–continuous) solution* $(\rho, \boldsymbol{m}, E)$ *defined on* $[0, T]$.

Then for $h \longrightarrow 0$ *it holds*

$$\rho_h \to \rho, \; \boldsymbol{m}_h \to \boldsymbol{m}, \; E_h \to E \textit{ (strongly) in } L^1((0,T) \times \Omega_h).$$

4 $\mathcal{K}$-Convergence

As demonstrated by numerical experiments, cf. [17, 19, 21], the finite volume approximations may not converge strongly. A typical example is the Kelvin-Helmholtz problem, where new and new small vortex substructures arise by refining the mesh. On the other hand, one can consider coarse-grained quantities, such as the mean or variance, averaged over different meshes. In our recent work [17] we have studied the question of strong convergence for these observable quantities. The aim of this section is to give an overview of our main results on the strong convergence without going deep into the theory of Young measures. Moreover, we would like to point out some connections to well-known and recent probabilistic concepts.

To start we recall a beautiful result of Komlós [27] on the pointwise convergence of the so-called Cèsaro averages.

Any sequence $\{F_n\}_{n=1}^{\infty}$ of uniformly L^1-bounded real valued functions on a set $Q \subset R^K$ admits a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$, such that the arithmetic averages (Cèsaro averages)

$$\frac{1}{N}\sum_{k=1}^{N} F_{n_k} \text{ converge a.e. to a function } F \in L^1(Q).$$

Moreover, any subsequence of $\{F_{n_k}\}_{k=1}^{\infty}$ enjoys the same property.

We note that analogous result holds also for sequences in the reflexive L^p spaces, $1 < p < \infty$, due to the Banach-Sachs theorem. Komlós theorem has been adapted by Balder [1] who introduced the concept of $\mathcal{K}$ (Komlós)-convergence for sequences of Young measures. Applying the Young measure adapted variant of the celebrated Prokhorov theorem for random processes one obtains compactness of the empirical measures and the strong convergence in space and time of mean values and variances, see [14, 16, 17].

Theorem 3 ($\mathcal{K}$-convergence of the FLM method) *Let* $\{\rho_{h_n}, \boldsymbol{m}_{h_n}, S_{h_n}\}_{n=1}^{\infty}$ *be a sequence of finite volume solutions obtained by the FLM method* (15) *with* $0 < \beta < 1,\ 0 < \alpha < \frac{4}{3}$. *Further, assume that the FLM solutions remain in a non-degenerate gas region, i.e., there exist* $\underline{\rho}, \overline{\vartheta} \in \mathbb{R}$, *such that*

$$0 < \underline{\rho} \leq \rho_{h_n}(t), \quad \vartheta_{h_n}(t) \leq \overline{\vartheta} \textit{ for all } t \in [0, T] \textit{ uniformly for } h_n \to 0.$$

Then there exists a subsequence of $\{\rho_{h_n}, \boldsymbol{m}_{h_n}, S_{h_n}\}_{n=1}^{\infty}$ *denoted by* $\{\rho_{n_k}, \boldsymbol{m}_{n_k}, S_{n_k}\}$, *for which we have*

- **strong convergence of Cesàro averages to a DW solution**

$$\frac{1}{N}\sum_{k=1}^{N} \rho_{n_k} \to \rho \textit{ as } N \to \infty \textit{ in } L^q(0, T; L^\gamma(\Omega)) \textit{ for any } 1 \leq q < \infty,$$

$$\frac{1}{N}\sum_{k=1}^{N}\boldsymbol{m}_{n_k} \to \boldsymbol{m} \text{ as } N \to \infty \text{ in } L^q(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)) \text{ for any } 1 \le q < \infty,$$

$$\frac{1}{N}\sum_{k=1}^{N} S_{n_k} \to S \text{ as } N \to \infty \text{ in } L^q(0,T; L^{\gamma}(\Omega)) \text{ for any } 1 \le q < \infty, \tag{16}$$

where $\rho, \boldsymbol{m}, S$ are the density, momentum and total entropy components of the DW solution in the sense of Definition 2.

- **strong convergence to a DMV solution in the Wasserstein metric**[2]

$$W_q\left[\frac{1}{N}\sum_{k=1}^{N}\delta_{[\rho_{n_k}(t,x),\boldsymbol{m}_{n_k}(t,x),S_{n_k}(t,x)]}; \mathscr{V}_{t,x}\right] \to 0 \text{ as } N \to \infty \text{ in } L^q((0,T)\times\Omega) \tag{17}$$

for any $1 \le q < \frac{2\gamma}{\gamma+1}$. Here δ denotes the Dirac measure acting on numerical solutions $[\rho_{n_k}, \boldsymbol{m}_{n_k}, S_{n_k}]$.

- **strong convergence of the variance**

Let $\widetilde{\boldsymbol{U}} = (\tilde{\rho}, \tilde{\boldsymbol{m}}, \widetilde{S})$ and $\boldsymbol{U}_{n_k} \equiv \left(\rho_{n_k}, \boldsymbol{m}_{n_k}, S_{n_k}\right)$, then

$$\left\| \frac{1}{N}\sum_{k=1}^{N}\left|\boldsymbol{U}_{n_k} - \frac{1}{N}\sum_{j=1}^{N}\boldsymbol{U}_{n_j}\right| - \left\langle \mathscr{V}_{t,x}; \left|\widetilde{\boldsymbol{U}} - \left\langle \mathscr{V}_{t,x}; \widetilde{\boldsymbol{U}}\right\rangle\right|\right\rangle \right\|_{L^1((0,T)\times\Omega)} \quad \text{as } N \to \infty. \tag{18}$$

Theorem 3 offers an elegant way how to compute DW solutions and the statistical moments of DMV solutions in the case that the strong solution does not exist. It indicates that we still have strong convergence to the observable quantities that can be approximated directly by averaging of numerical solutions over different meshes. We refer a reader to [17] where the numerical solutions obtained by the FLM method were presented for several tests. Depending on chosen numerical experiments it may happen that the mesh-convergence of single numerical solutions is not achieved. On the other hand, the strong convergence of empirical mean values and variances was clearly shown, see [17, Figs. 1–7]. In future it will be interesting to investigate the rate of $\mathscr{K}$-convergence.

In this context we should also mention an interesting work [20], where a different probabilistic concept of the so-called statistical solutions for general multidimensional hyperbolic conservation laws has been developed. Analogously to the DMV solutions the statistical solutions are probabilistic-type solutions. In fact, they are time-parametrized probability measures satisfying an infinite set of partial differential equations consistent with the underlying hyperbolic conservation laws. Thus,

[2] We recall that the Wasserstein metric of q-th order, $q \in [1,\infty)$, is defined in the following way $W_q(\mathscr{N},\mathscr{V}) := \left\{\inf_{\pi\in\Pi(\mathscr{N},\mathscr{V})}\int_{\mathbb{R}^{d+2}\times\mathbb{R}^{d+2}}|\zeta-\xi|^q d\pi(\zeta,\xi)\right\}^{1/q}$, where $\Pi(\mathscr{N},\mathscr{V})$ is the set of probability measures on $\mathbb{R}^{d+2}\times\mathbb{R}^{d+2}$ with marginals $\mathscr{N}$ and $\mathscr{V}$.

they are the measure–valued solutions augmented by information on multi-point spatial correlations. In order to obtain strong convergence of the entropy stable finite volume solutions (or more precisely, approximate statistical solutions) to a statistical solution one however needs to assume that a special condition on an approximate scaling of structure factors holds. The latter is related to the Kolmogorov compactness criterium. On the other hand, the concept of $\mathscr{K}$-convergence based on the averaging over different meshes naturally inherits compactness. Consequently, the empirical mean values (Cèsaro averages) converge strongly to a DW solution. In future it will be interesting to generalize the concept of DMV and DW solutions to general hyperbolic conservation laws.

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