Idempotence and Divisoriality in Prüfer-Like Domains

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Abstract Let D be a Prüfer $\star$-multiplication domain, where $\star$ is a semistar operation on D. We show that certain ideal-theoretic properties related to idempotence and divisoriality hold in Prüfer domains, and we use the associated semistar Nagata ring of D to show that the natural counterparts of these properties also hold in D.

Keywords Idempotent ideal · Semistar operation · Prüfer $\star$-multiplication domain · Nagata ring · Divisorial ideal

1 Introduction and Preliminaries

Throughout this work, D will denote an integral domain, and K will denote its quotient field. Recall that Arnold [1] proved that D is a Prüfer domain if and only if its associated Nagata ring $D[X]_N$, where N is the set of polynomials in $D[X]$ whose coefficients generate the unit ideal, is a Prüfer domain. This was generalized

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to Prüfer v-multiplication domains (PvMDs) by Zafrullah [16] and Kang [14] and to Prüfer $\star$-multiplication domains (P$\star$MDs) by Fontana, Jara, and Santos [8].

Our goal in this paper is to show that certain ideal-theoretic properties that hold in Prüfer domains transfer in a natural way to P$\star$MDs. For example, we show that an ideal I of a Prüfer domain is idempotent if and only if it is a radical ideal each of whose minimal primes is idempotent (Theorem 2.9), and we use a Nagata ring transfer "machine" to transfer a natural counterpart of this characterization to P$\star$MDs. For another example, in Theorem 3.5 we show that an ideal in a Prüfer domain of finite character is idempotent if and only if it is a product of idempotent prime ideals and, perhaps more interestingly, we characterize ideals that are simultaneously idempotent and divisorial as (unique) products of incomparable divisorial idempotent primes; and we then extend this to P$\star$MDs.

Let us review the terminology and notation. Denote by $\overline{\boldsymbol{F}}(D)$ the set of all nonzero D–submodules of K, and by $\boldsymbol{F}(D)$ the set of all nonzero fractional ideals of D, i.e., $E \in \boldsymbol{F}(D)$ if $E \in \overline{\boldsymbol{F}}(D)$ and there exists a nonzero $d \in D$ with $dE \subseteq D$. Let $\boldsymbol{f}(D)$ be the set of all nonzero finitely generated D–submodules of K. Then, obviously, $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D)$.

Following Okabe-Matsuda [15], a *semistar operation* on D is a map $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, $E \mapsto E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\boldsymbol{F}}(D)$, the following properties hold:

($\star_1$) $(xE)^\star = xE^\star$;
($\star_2$) $E \subseteq F$ implies $E^\star \subseteq F^\star$;
($\star_3$) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Of course, semistar operations are natural generalizations of star operations–see the discussion following Corollary 2.5 below.

The semistar operation $\star$ is said to have *finite type* if $E^\star = \bigcup\{F^\star \mid F \in \boldsymbol{f}(D),\ F \subseteq E\}$ for each $E \in \overline{\boldsymbol{F}}(D)$. To any semistar operation $\star$ we can associate a finite-type semistar operation $\star_{_f}$ given by

$$E^{\star_{_f}} := \bigcup\{F^\star \mid F \in \boldsymbol{f}(D), F \subseteq E\}.$$

We say that a nonzero ideal I of D is a *quasi-$\star$-ideal* if $I = I^\star \cap D$, a *quasi-$\star$-prime ideal* if it is a prime quasi-$\star$-ideal, and a *quasi-$\star$-maximal ideal* if it is maximal in the set of all proper quasi-$\star$-ideals. A quasi-$\star$-maximal ideal is a prime ideal. We will denote by $\mathrm{QMax}^\star(D)$ ($\mathrm{QSpec}^\star(D)$) the set of all quasi-$\star$-maximal ideals (quasi-$\star$-prime ideals) of D. While quasi-$\star$-maximal ideals may not exist, quasi-$\star_{_f}$-maximal ideals are plentiful in the sense that each proper quasi-$\star_{_f}$-ideal is contained in a quasi-$\star_{_f}$-maximal ideal. (See [9] for details.) Now we can associate to $\star$ yet another semistar operation: for $E \in \overline{\boldsymbol{F}}(D)$, set

$$E^{\widetilde{\star}} := \bigcap\{ED_Q \mid Q \in \mathrm{QMax}^{\star_{_f}}(D)\}.$$

Then $\widetilde{\star}$ is also a finite-type semistar operation, and we have $E^{\widetilde{\star}} \subseteq E^{\star_f} \subseteq E^{\star}$ for all $E \in \overline{\boldsymbol{F}}(D)$.

Let $\star$ be a semistar operation on D. Set $N(\star) = \{g \in D[X] \mid c(g)^{\star} = D^{\star}\}$, where $c(g)$ is the *content* of the polynomial g, i.e., the ideal of D generated by the coefficients of g. Then $N(\star)$ is a saturated multiplicatively closed subset of $D[X]$, and we call the ring $\mathrm{Na}(D, \star) := D[X]_{N(\star)}$ the *semistar Nagata ring of* D *with respect to* $\star$. The domain D is called a *Prüfer* $\star$*- multiplication domain* (P$\star$MD) if $(FF^{-1})^{\star_f} = D^{\star_f}$ $(= D^{\star})$ for each $F \in \boldsymbol{f}(D)$ (i.e., each such F is $\star_f$-invertible). (Recall that $F^{-1} = (D : F) = \{u \in K \mid uF \subseteq D\}$.)

In the following two lemmas, we assemble the facts we need about Nagata rings and P$\star$MDs. Most of the proofs can be found in [6, 9] or [5].

Lemma 1.1. *Let $\star$ be a semistar operation on D. Then:*

(1) $D^{\star} = D^{\star_f}$.
(2) $\mathrm{QMax}^{\star_f}(D) = \mathrm{QMax}^{\widetilde{\star}}(D)$.
(3) *The map* $\mathrm{QMax}^{\star_f}(D) \to \mathrm{Max}(\mathrm{Na}(D, \star))$*,* $P \mapsto P\mathrm{Na}(D, \star)$*, is a bijection with inverse map* $M \mapsto M \cap D$.
(4) $P \mapsto P\mathrm{Na}(D, \star)$ *defines an injective map* $\mathrm{QSpec}^{\widetilde{\star}}(D) \to \mathrm{Spec}(\mathrm{Na}(D, \star))$.
(5) $N(\star) = N(\star_f) = N(\widetilde{\star})$ *and (hence)* $\mathrm{Na}(D, \star) = \mathrm{Na}(D, \star_f) = \mathrm{Na}(D, \widetilde{\star})$.
(6) *For each* $E \in \overline{\boldsymbol{F}}(D)$*,* $E^{\widetilde{\star}} = E\mathrm{Na}(D, \star) \cap K$*, and* $E^{\widetilde{\star}}\mathrm{Na}(D, \star) = E\mathrm{Na}(D, \star)$.
(7) *A nonzero ideal I of D is a quasi-$\widetilde{\star}$-ideal if and only if* $I = I\mathrm{Na}(D, \star) \cap D$.

Lemma 1.2. *Let $\star$ be a semistar operation on D.*

(1) *The following statements are equivalent.*

  (a) *D is a P$\star$MD.*
  (b) $\mathrm{Na}(D, \star)$ *is a Prüfer domain.*
  (c) *The ideals of* $\mathrm{Na}(D, \star)$ *are extended from ideals of D.*
  (d) *D_P is a valuation domain for each* $P \in \mathrm{QMax}^{\star_f}(D)$.

(2) *Assume that D is a P$\star$MD. Then:*

  (a) $\widetilde{\star} = \star_f$ *and (hence)* $D^{\star} = D^{\widetilde{\star}}$.
  (b) *The map* $\mathrm{QSpec}^{\star_f}(D) \to \mathrm{Spec}(\mathrm{Na}(D, \star))$*,* $P \mapsto P\mathrm{Na}(D, \star)$*, is a bijection with inverse map* $Q \mapsto Q \cap D$.
  (c) *Finitely generated ideals of* $\mathrm{Na}(D, \star)$ *are extended from finitely generated ideals of D.*

2 Idempotence

We begin with our basic definition.

Definition 2.1. *Let $\star$ be a semistar operation on D. An element* $E \in \overline{\boldsymbol{F}}(D)$ *is said to be $\star$-idempotent if* $E^{\star} = (E^2)^{\star}$.

Our primary interest will be in (nonzero) $\star$-idempotent *ideals* of D. Let $\star$ be a semistar operation on D, and let I be a nonzero ideal of D. It is well known that $I^\star \cap D$ is a quasi-$\star$-ideal of D. (This is easy to see: we have

$$(I^\star \cap D)^\star \subseteq I^{\star\star} = I^\star = (I \cap D)^\star \subseteq (I^\star \cap D)^\star,$$

and hence $I^\star = (I^\star \cap D)^\star$; it follows that $I^\star \cap D = (I^\star \cap D)^\star \cap D$.) It, therefore, seems natural to call $I^\star \cap D$ the *quasi-$\star$-closure* of I. If we also call I *$\star$-proper* when $I^\star \subsetneq D^\star$, then it is easy to see that I is $\star$-proper if and only if its quasi-$\star$-closure is a proper quasi-$\star$-ideal. Now suppose that I is $\star$-idempotent. Then

$$(I^\star \cap D)^\star = I^\star = (I^2)^\star = ((I^\star)^2)^\star = (((I^\star \cap D)^\star)^2)^\star = ((I^\star \cap D)^2)^\star,$$

whence $I^\star \cap D$ is a $\star$-idempotent quasi-$\star$-ideal of D. A similar argument gives the converse. Thus a ($\star$-proper) nonzero ideal is $\star$-idempotent if and only if its quasi-$\star$-closure is a (proper) $\star$-idempotent quasi-$\star$-ideal.

Our study of idempotence in Prüfer domains and P$\star$MDs involves the notions of sharpness and branchedness. We recall some notation and terminology.

Given an integral domain D and a prime ideal $P \in \mathrm{Spec}(D)$, set

$$\nabla(P) := \{M \in \mathrm{Max}(D) \mid M \not\supseteq P\} \text{ and}$$
$$\Theta(P) := \bigcap\{D_M \mid M \in \nabla(P)\}.$$

We say that P is *sharp* if $\Theta(P) \nsubseteq D_P$ (see [11, Lemma 1] and [3, Section 1 and Proposition 2.2]). The domain D itself is *sharp* (*doublesharp*) if every maximal (prime) ideal of D is sharp. (Note that a Prüfer domain D is doublesharp if and only if each overring of D is sharp [7, Theorem 4.1.7].) Now let $\star$ be a semistar operation on D. Given a prime ideal $P \in \mathrm{QSpec}^{\star_f}(D)$, set

$$\nabla^{\star_f}(P) := \{M \in \mathrm{QMax}^{\star_f}(D) \mid M \not\supseteq P\} \text{ and}$$
$$\Theta^{\star_f}(P) := \bigcap\{D_M \mid M \in \nabla^{\star_f}(P)\}.$$

Call P *$\star_f$-sharp* if $\Theta^{\star_f}(P) \nsubseteq D_P$. For example, if $\star = d$ is the identity, then the $\star_f$-sharp property coincides with the sharp property. We then say that D is $\star_f$-(double)sharp if each quasi-$\star_f$-maximal (quasi-$\star_f$-prime) ideal of D is $\star_f$-sharp. (For more on sharpness, see [10, 11, 13], [7, page 62], [3], [4, Chapter 2, Section 3] and [5].)

Recall that a prime ideal P of a ring is said to be *branched* if there is a P-primary ideal distinct from P. Also, recall that the domain D has *finite character* if each nonzero ideal of D is contained in only finitely many maximal ideals of D.

We now prove a lemma that discusses the transfer of ideal-theoretic properties between D (on which a semistar operation $\star$ has been defined) and its associated Nagata ring.

Lemma 2.2. *Let $\star$ be a semistar operation on D.*

(1) *Let $E \in \overline{\boldsymbol{F}}(D)$. Then E is $\widetilde{\star}$-idempotent if and only if $E\mathrm{Na}(D, \star)$ is idempotent. In particular, if D is a P$\star$MD, then E is $\star_f$-idempotent if and only if $E\mathrm{Na}(D, \star)$ is idempotent.*

(2) *Let P be a quasi-$\widetilde{\star}$-prime of D and I a nonzero ideal of D. Then:*

  (a) *I is P-primary in D if and only if I is a quasi-$\widetilde{\star}$-ideal of D and $I\mathrm{Na}(D, \star)$ is $P\mathrm{Na}(D, \star)$-primary in $\mathrm{Na}(D, \star)$.*

  (b) *P is branched in D if and only if $P\mathrm{Na}(D, \star)$ is branched in $\mathrm{Na}(D, \star)$.*

(3) *D has $\star_f$-finite character (i.e., each nonzero element of D belongs to only finitely many (possibly zero) $M \in \mathrm{QMax}^{\star_f}(D)$) if and only if $\mathrm{Na}(D, \star)$ has finite character.*

(4) *Let I be a quasi-$\widetilde{\star}$-ideal of D. Then I is a radical ideal if and only if $I\mathrm{Na}(D, \star)$ is a radical ideal of $\mathrm{Na}(D, \star)$.*

(5) *Assume that D is a P$\star$MD. Then:*

  (a) If $P \in \mathrm{QSpec}^{\star_f}(D)$, then P is $\star_f$-sharp if and only if $P\mathrm{Na}(D, \star)$ is sharp in $\mathrm{Na}(D, \star)$.

  (b) D is $\star_f$-(double)sharp if and only if $\mathrm{Na}(D, \star)$ is (double)sharp.

Proof. (1) We use Lemma 1.1(6). If $E\mathrm{Na}(D, \star)$ is idempotent, then $E^{\widetilde{\star}} = E\mathrm{Na}(D, \star) \cap K = E^2\mathrm{Na}(D, \star) \cap K = (E^2)^{\widetilde{\star}}$. Conversely, if E is $\widetilde{\star}$-idempotent, then $(E\mathrm{Na}(D, \star))^2 = E^2\mathrm{Na}(D, \star) = (E^2)^{\widetilde{\star}}\mathrm{Na}(D, \star) = E^{\widetilde{\star}}\mathrm{Na}(D, \star) = E\mathrm{Na}(D, \star)$. The "in particular" statement follows because $\star_f = \widetilde{\star}$ in a P$\star$MD (Lemma 1.2(2a)).

(2) (a) Suppose that I is P-primary. Then $ID[X]$ is $PD[X]$-primary. Since P is a quasi-$\widetilde{\star}$-prime of D, $P\mathrm{Na}(D, \star)$ is a prime ideal of $\mathrm{Na}(D, \star)$ (Lemma 1.1(4)), and then, since $\mathrm{Na}(D, \star)$ is a quotient ring of $D[X]$, $I\mathrm{Na}(D, \star)$ is $P\mathrm{Na}(D, \star)$-primary in $\mathrm{Na}(D, \star)$. Also, again using the fact that $ID[X]$ is $PD[X]$-primary (along with Lemma 1.1(6)), we have

$$I^{\widetilde{\star}} \cap D = I\mathrm{Na}(D, \star) \cap D \subseteq ID[X]_{PD[X]} \cap D[X] \cap D = ID[X] \cap D = I,$$

whence I is a quasi-$\widetilde{\star}$-ideal of D. Conversely, assume that I is a quasi-$\widetilde{\star}$-ideal of D and that $I\mathrm{Na}(D, \star)$ is $P\mathrm{Na}(D, \star)$-primary. Then for $a \in P$, there is a positive integer n for which $a^n \in I\mathrm{Na}(D, \star) \cap D = I^{\widetilde{\star}} \cap D = I$. Hence $P = \mathrm{rad}(I)$. It now follows easily that I is P-primary. (b) Suppose that P is branched in D. Then there is a P-primary ideal I of D distinct from P, and $I\mathrm{Na}(D, \star)$ is $P\mathrm{Na}(D, \star)$-primary by (a). Also by (a), I is a quasi-$\widetilde{\star}$-ideal, from which it follows that $I\mathrm{Na}(D, \star) \neq P\mathrm{Na}(D, \star)$. Now suppose that $P\mathrm{Na}(D, \star)$ is branched and that J is a $P\mathrm{Na}(D, \star)$-primary ideal of $\mathrm{Na}(D, \star)$ distinct from $P\mathrm{Na}(D, \star)$. Then it is straightforward to show that $J \cap D$ is distinct from P and is P-primary.

(3) Let ψ be a nonzero element of $\mathrm{Na}(D, \star)$, and let N be a maximal ideal with $\psi \in N$. Then $\psi\mathrm{Na}(D, \star) = f\mathrm{Na}(D, \star)$ for some $f \in D[X]$, and writing $N = M\mathrm{Na}(D, \star)$ for some $M \in \mathrm{QMax}^{\star_f}(D)$ (Lemma 1.1(3)), we must have $f \in MD[X]$

and hence $c(f) \subseteq M$. Therefore, if D has finite $\star_f$-character, then $\text{Na}(D, \star)$ has finite character. The converse is even more straightforward.

(4) Suppose that I is a radical ideal of D, and let $\psi^n \in I\text{Na}(D, \star)$ for some $\psi \in \text{Na}(D, \star)$ and positive integer n. Then there is an element $g \in N(\star)$ with ($g\psi^n$ and hence) $(g\psi)^n \in ID[X]$. Since $ID[X]$ is a radical ideal of $D[X]$, $g\psi \in ID[X]$ and we must have $\psi \in I\text{Na}(D, \star)$. Therefore, $I\text{Na}(D, \star)$ is a radical ideal of $\text{Na}(D, \star)$. The converse follows easily from the fact that $I\text{Na}(D, \star) \cap D = I^{\tilde{\star}} \cap D = I$ (Lemma 1.1(7)).

(5) (a) This is part of [5, Proposition 3.5], but we give here a proof more in the spirit of this paper. Let $P \in \text{QSpec}^{\star_f}(D)$. If P is $\star_f$-sharp, then by [5, Proposition 3.1] P contains a finitely generated ideal I with $I \nsubseteq M$ for all $M \in \nabla^{\star_f}(P)$, and, using the description of $\text{Max}(\text{Na}(D, \star))$ given in Lemma 1.1(3), $I\text{Na}(D, \star)$ is a finitely generated ideal of $\text{Na}(D, \star)$ contained in $P\text{Na}(D, \star)$ but in no element of $\nabla(P\text{Na}(D, \star))$. Therefore, $P\text{Na}(D, \star)$ is sharp in the Prüfer domain $\text{Na}(D, \star)$. For the converse, assume that $P\text{Na}(D, \star)$ is sharp in $\text{Na}(D, \star)$. Then $P\text{Na}(D, \star)$ contains a finitely generated ideal J with $J \subseteq P\text{Na}(D, \star)$ but $J \nsubseteq N$ for $N \in \nabla(P\text{Na}(D, \star))$ [13, Corollary 2]. Then $J = I\text{Na}(D, \star)$ for some finitely generated ideal I of D (necessarily) contained in P by Lemma 1.2(2c), and it is easy to see that $I \nsubseteq M$ for $M \in \nabla^{\star_f}(D)$. Then by [5, Proposition 3.1], P is $\star_f$-sharp. Statement (b) follows easily from (a) (using Lemma 1.2). □

Let D be an almost Dedekind domain with a non-finitely generated maximal ideal M. Then $M^{-1} = D$, but M is not idempotent (since MD_M is not idempotent in the Noetherian valuation domain D_M). Our next result shows that this cannot happen in a sharp Prüfer domain.

Theorem 2.3. *Let D be a Prüfer domain. If D is (d-)sharp and I is a nonzero ideal of D with $I^{-1} = D$, then I is idempotent.*

Proof. Assume that D is sharp. Proceeding contrapositively, suppose that I is a nonzero, non-idempotent ideal of D. Then, for some maximal ideal M of D, ID_M is not idempotent in D_M. Since D is a sharp domain, we may choose a finitely generated ideal J of D with $J \subseteq M$ but $J \nsubseteq N$ for all maximal ideals $N \neq M$. Since ID_M is a non-idempotent ideal in the valuation domain D_M, there is an element $a \in I$ for which $I^2 D_M \subsetneq aD_M$. Let $B := J + Da$. Then $I^2 D_M \subseteq BD_M$ and, for $N \in \text{Max}(D) \setminus \{M\}$, $I^2 D_N \subseteq D_N = BD_N$. Hence $I^2 \subseteq B$. Since B is a proper finitely generated ideal, we then have $(I^2)^{-1} \supseteq B^{-1} \supsetneq D$. Hence $(I^2)^{-1} \neq D$, from which it follows that $I^{-1} \neq D$, as desired. □

Kang [14, Proposition 2.2] proves that, for a nonzero ideal I of D, we always have $I^{-1}\text{Na}(D, v) = (\text{Na}(D, v)) : I)$. This cannot be extended to general semistar Nagata rings; for example, if D is an almost Dedekind domain with non-invertible maximal ideal M and we define a semistar operation $\star$ by $E^\star = ED_M$ for $E \in \overline{\boldsymbol{F}}(D)$, then $(D : M) = D$ and hence $(D : M)\text{Na}(D, \star) = \text{Na}(D, \star) = D[X]_{M[X]} = D_M(X) \subsetneq (D_M : MD_M)D_M(X) = (\text{Na}(D, \star) : M\text{Na}(D, \star))$ (where the proper inclusion holds because MD_M is principal in D_M). At any rate, what we really require is the equality

$(D^\star : E)\mathrm{Na}(D, \star) = (\mathrm{Na}(D, \star) : E)$ for $E \in \overline{\boldsymbol{F}}(D)$. In the next lemma, we show that this holds in a P⋆MD but not in general. The proof of part (1) of the next lemma is a relatively straightforward translation of the proof of [14, Proposition 2.2] to the semistar setting. In carrying this out, however, we discovered a minor flaw in the proof of [14, Proposition 2.2]. The flaw involves a reference to [12, Proposition 34.8], but this result requires that the domain D be integrally closed. To ensure complete transparency, we give the proof in full detail.

Lemma 2.4. *Let $\star$ be a semistar operation on D. Then:*

(1) $(D^\star : E)\mathrm{Na}(D, \star) \supseteq (\mathrm{Na}(D, \star) : E)$ *for each* $E \in \overline{\boldsymbol{F}}(D)$.
(2) *The following statements are equivalent:*
   (a) $(D^\star : E)\mathrm{Na}(D, \star) = (\mathrm{Na}(D, \star) : E)$ *for each* $E \in \overline{\boldsymbol{F}}(D)$.
   (b) $D^\star = D^{\tilde{\star}}$.
   (c) $D^\star \subseteq \mathrm{Na}(D, \star)$.
(3) $(D^{\tilde{\star}} : E)\mathrm{Na}(D, \star) = (\mathrm{Na}(D, \star) : E)$ *for each* $E \in \overline{\boldsymbol{F}}(D)$.
(4) *If D is a P⋆MD, then the equivalent conditions in (2) hold.*

Proof. (1) Let $E \in \overline{\boldsymbol{F}}(D)$, and let $\psi \in (\mathrm{Na}(D, \star) : E)$. For $a \in E$, $a \neq 0$, we may find $g \in N(\star)$ such that $\psi a g \in D[X]$. This yields $\psi g \in a^{-1}D[X] \subseteq K[X]$, and hence $\psi = f/g$ for some $f \in K[X]$. We claim that $c(f) \subseteq (D^\star : E)$. Granting this, we have $f \in (D^\star : E)D[X]$, from which it follows that $\psi = f/g \in (D^\star : E)\mathrm{Na}(D, \star)$, as desired. To prove the claim, take $b \in E$, and note that $fb \in \mathrm{Na}(D, \star)$. Hence $fbh \in D[X]$ for some $h \in N(\star)$, and so $c(fh)b \subseteq D$. By the content formula [12, Theorem 28.1], there is an integer m for which $c(f)c(h)^{m+1} = c(fh)c(h)^m$. Using the fact that $c(h)^\star = D^\star$, we obtain $c(f)^\star = c(fh)^\star$ and hence that $c(f)b \subseteq c(fh)^\star b \subseteq D^\star$. Therefore, $c(f) \subseteq (D^\star : E)$, as claimed.

(2) Under the assumption in (c), $D^\star \subseteq \mathrm{Na}(D, \star) \cap K = D^{\tilde{\star}}$ (Lemma 1.1(6)). Hence (c) ⇒ (b). Now assume that $D^\star = D^{\tilde{\star}}$. Then for $E \in \overline{\boldsymbol{F}}(D)$, we have $(D^\star : E)E \subseteq D^\star = D^{\tilde{\star}} \subseteq \mathrm{Na}(D, \star)$; using (1), the implication (b) ⇒ (a) follows. That (a) ⇒ (c) follows upon taking $E = D$ in (a).

(3) This follows easily from (2), because $\mathrm{Na}(D, \star) = \mathrm{Na}(D, \tilde{\star})$ by Lemma 1.1(5).

(4) This follows from (2), since if D is a P⋆MD, then $D^\star = D^{\tilde{\star}}$ by Lemma 1.2(2a). □

The conditions in Lemma 2.4(2) need not hold: Let $F \subsetneq k$ be fields, $V = k[[x]]$ the power series ring over V in one variable, and $D = F + M$, where $M = xk[[x]]$. Define a (finite-type) semistar operation $\star$ on D by $A^\star = AV$ for $A \in \overline{\boldsymbol{F}}(D)$. Then $D^\star = V \supsetneq D = D_M = D^{\tilde{\star}}$.

We can now extend Theorem 2.3 to P⋆MDs.

Corollary 2.5. *Let $\star$ be a semistar operation on D such that D is a $\star_f$-sharp P⋆MD, and let I be a nonzero ideal of D with $(D^\star : I) = D^\star$. Then I is $\star_f$-idempotent.*

Proof. By Lemma 2.4(3), we have

$$(\mathrm{Na}(D,\star) : I\mathrm{Na}(D,\star)) = (D^\star : I)\mathrm{Na}(D,\star) = D^\star\mathrm{Na}(D,\star) = \mathrm{Na}(D,\star).$$

Hence $I\mathrm{Na}(D,\star)$ is idempotent in the Prüfer domain $\mathrm{Na}(D,\star)$ by Theorem 2.3. Lemma 2.2(1) then yields that I is $\star_f$-idempotent. ☐

Many semistar counterparts of ideal-theoretic properties in domains result in equations that are "external" to D, since for a semistar operation $\star$ on D and a nonzero ideal I of D, it is possible that $I^\star \nsubseteq D$. Of course, $\star$-idempotence is one such property. Often, one can obtain a "cleaner" counterpart by specializing from P$\star$MDs to "ordinary" PvMDs. We recall some terminology. Semistar operations are generalizations of *star* operations, first considered by Krull and repopularized by Gilmer [12, Sections 32, 34]. Roughly, a star operation is a semistar operation restricted to the set $\boldsymbol{F}(D)$ of nonzero fractional ideals of D with the added requirement that one has $D^\star = D$. The most important star operation (aside from the d-, or trivial, star operation) is the *v-operation*: For $E \in \boldsymbol{F}(D)$, put $E^{-1} = \{x \in K \mid xE \subseteq D\}$ and $E^v = (E^{-1})^{-1}$. Then v_f (restricted to $\boldsymbol{F}(D)$) is the *t-operation* and $\widetilde{v}$ is the w-operation. Thus a *PvMD* is a domain in which each nonzero finitely generated ideal is t-invertible. Corollary 2.5 then has the following restricted interpretation (which has the advantage of being *internal* to D).

Corollary 2.6. *If D is a t-sharp PvMD and I is a nonzero ideal of D for which $I^{-1} = D$, then I is t-idempotent.*

Our next result is a partial converse to Theorem 2.3.

Proposition 2.7. *Let D be a Prüfer domain such that I is idempotent whenever I is a nonzero ideal of D with $I^{-1} = D$. Then, every branched maximal ideal of D is sharp.*

Proof. Let M be a branched maximal ideal of D. Then $MD_M = \mathrm{rad}(aD_M)$ for some nonzero element $a \in M$ [12, Theorem 17.3]. Let $I := aD_M \cap D$. Then I is M-primary, and since $ID_M = aD_M$, (ID_M and hence) I is not idempotent. By hypothesis, we may choose $u \in I^{-1} \setminus D$. Since $Iu \subseteq D$ and $ID_N = D_N$ for $N \in \mathrm{Max}(D) \setminus \{M\}$, then $u \in \bigcap\{D_N \mid N \in \mathrm{Max}(D), N \neq M\}$. On the other hand, since $u \notin D$, $u \notin D_M$. It follows that M is sharp. ☐

Now we extend Proposition 2.7 to P$\star$MDs.

Corollary 2.8. *Let $\star$ be a semistar operation on D, and assume that D is a P$\star$MD such that I is $\star_f$-idempotent whenever I is a nonzero ideal of D with $(D^\star : I) = D^\star$. Then, each branched quasi-$\star_f$-maximal ideal of D is $\star_f$-sharp.* (*In particular if D is a PvMD in which I is t-idempotent whenever I is a nonzero ideal of D with $I^{-1} = D$, then each branched maximal t-ideal of D is t-sharp.*)

Proof. Let J be a a nonzero ideal of the Prüfer domain $\text{Na}(D, \star)$ with $(\text{Na}(D, \star) : J) = \text{Na}(D, \star)$. By Lemma 1.2(1c), $J = I\text{Na}(D, \star)$ for some ideal I of D. Applying Lemma 2.4(3) and Lemma 1.1(6), we obtain $(D^\star : I) = D^\star$. Hence, by hypothesis, I is $\star_f$-idempotent, and this yields that $J = I\text{Na}(D, \star)$ is idempotent in the Prüfer domain $\text{Na}(D, \star)$ (Lemma 2.2(1)). Now, let M be a branched quasi-$\star_f$-maximal ideal of D. Then, by Lemma 2.2(2), $M\text{Na}(D, \star)$ is a branched maximal ideal of $\text{Na}(D, \star)$. We may now apply Proposition 2.7 to conclude that $M\text{Na}(D, \star)$ is sharp. Therefore, M is $\star_f$-sharp in D by Lemma 2.2(5). □

If P is a prime ideal of a Prüfer domain D, then powers of P are P-primary by [12, Theorem 23.3(b)]; it follows that P is idempotent if and only if PD_P is idempotent. We use this fact in the next result.

It is well known that a proper idempotent ideal of a valuation domain must be prime [12, Theorem 17.1(3)]. In fact, according to [12, Exercise 3, p. 284], a proper idempotent ideal in a Prüfer domain must be a radical ideal. We (re-)prove and extend this fact and add a converse.

Theorem 2.9. *Let D be a Prüfer domain, and let I be an ideal of D. Then I is idempotent if and only if I is a radical ideal each of whose minimal primes is idempotent.*

Proof. The result is trivial for $I = (0)$ and vacuously true for $I = D$. Suppose that I is a proper nonzero idempotent ideal of D, and let P be a prime minimal over I. Then ID_P is idempotent, and we must have $ID_P = PD_P$ [12, Theorem 17.1(3)]. Hence PD_P is idempotent, and therefore, by the comment above, so is P. Now let M be a maximal ideal containing I. Then ID_M is idempotent, hence prime (hence radical). It follows (checking locally) that I is a radical ideal.

Conversely, let I be a radical ideal each of whose minimal primes is idempotent. If M is a maximal ideal containing I and P is a minimal prime of I contained in M, then $ID_M = PD_M$. Since P is idempotent, this yields $ID_M = I^2D_M$. It follows that I is idempotent. □

We next extend Theorem 2.9 to P$\star$MDs.

Corollary 2.10. *Let D be a P$\star$MD, where $\star$ is a semistar operation on D, and let I be a quasi-$\star_f$-ideal of D. Then I is $\star_f$-idempotent if and only if I is a radical ideal each of whose minimal primes is $\star_f$-idempotent.* (*In particular, if D is a PvMD and I is a t-ideal of D, then I is t-idempotent if and only if I is a radical ideal each of whose minimal primes is t-idempotent.*)

Proof. Suppose that I is $\star_f$-idempotent. Then $I\text{Na}(D, \star)$ is an idempotent ideal in $\text{Na}(D, \star)$ by Lemma 2.2(1). By Theorem 2.9, $I\text{Na}(D, \star)$ is a radical ideal of $\text{Na}(D, \star)$, and hence, by Lemma 2.2(4), I is a radical ideal of D. Now let P be a minimal prime of I in D. Then P is a quasi-$\star_f$-prime of D. By Lemma 1.2(2b) $P\text{Na}(D, \star)$ is minimal over $I\text{Na}(D, \star)$, whence $P\text{Na}(D, \star)$ is idempotent, again by Theorem 2.9. The $\star_f$-idempotence of P now follows from Lemma 2.2(1).

The converse follows by similar applications of Theorem 2.9 and Lemma 2.2. □

Recall that a Prüfer domain is said to be *strongly discrete* (*discrete*) if it has no nonzero (branched) idempotent prime ideals. Since unbranched primes in a Prüfer domain must be idempotent [12, Theorem 23.3(b)], a Prüfer domain is strongly discrete if and only if it is discrete and has no unbranched prime ideals. We have the following straightforward application of Theorem 2.9.

Corollary 2.11. *Let D be a Prüfer domain.*

(1) *If D is discrete, then an ideal I of D is idempotent if and only if I is a radical ideal each of whose minimal primes is unbranched.*
(2) *If D is strongly discrete, then D has no proper nonzero idempotent ideals.*

Let us call a P$\star$MD $\star_f$- *strongly discrete* ($\star_f$- *discrete*) if it has no (branched) $\star_f$-idempotent quasi-$\star_f$-prime ideals. From Lemma 2.2(1, 2), we have the usual connection between a property of a P$\star$MD and the corresponding property of its $\star$-Nagata ring.

Proposition 2.12. *Let $\star$ be a semistar operation on D. Then D is $\star_f$-*(*strongly*) *discrete P$\star$MD if and only if* $\mathrm{Na}(D, \star)$ *is a* (*strongly*) *discrete Prüfer domain.*

Applying Corollary 2.10 and Lemma 2.2(1, 2), we have the following extension of Corollary 2.11.

Corollary 2.13. *Let D be a domain.*

(1) *Assume that D is a P$\star$MD for some semistar operation $\star$ on D.*
   (a) *If D is $\star_f$-discrete, then a nonzero quasi-$\star_f$-ideal I of D is $\star_f$-idempotent if and only if I is a radical ideal each of whose minimal primes is unbranched.*
   (b) *If D is $\star_f$-strongly discrete, then D has no $\star_f$-proper $\star_f$-idempotent ideals.*
(2) *Assume that D is a PvMD.*
   (a) *If D is t-discrete, then a t-ideal I of D is t-idempotent if and only if I is a radical ideal each of whose minimal primes is unbranched.*
   (b) *If D is t-strongly discrete, then D has no t-proper t-idempotent ideals.*

3 Divisoriality

According to [7, Corollary 4.1.14], if D is a doublesharp Prüfer domain and P is a nonzero, nonmaximal ideal of D, then P is divisorial. The natural question arises: If D is a $\star_f$-doublesharp P$\star$MD and $P \in \mathrm{QSpec}^{\star_f}(D) \setminus \mathrm{QMax}^{\star_f}(D)$, is P necessarily divisorial? Since $\star$ is an arbitrary semistar operation and divisoriality specifically involves the v-operation, one might expect the answer to be negative. Indeed, we give a counterexample in Example 3.4 below. However, in Theorem 3.2 we prove a general result, a corollary of which does yield divisoriality in the "ordinary" PvMD case. First, we need a lemma, the first part of which may be regarded as an extension of [14, Proposition 2.2(2)].

Lemma 3.1. *Let $\star$ be a semistar operation on D. Then*

(1) $(D^{\widetilde{\star}} : (D^{\widetilde{\star}} : E))\text{Na}(D, \star) = (\text{Na}(D, \star) : (\text{Na}(D, \star) : E))$ *for each* $E \in \overline{\boldsymbol{F}}(D)$, *and*

(2) *if I is a nonzero ideal of D, then $I^{\widetilde{\star}}$ is a divisorial ideal of $D^{\widetilde{\star}}$ if and only if* $I\text{Na}(D, \star)$ *is a divisorial ideal of* $\text{Na}(D, \star)$.

In particular, if D is a P$\star$MD, then $(D^{\star} : (D^{\star} : E))\text{Na}(D, \star) = (\text{Na}(D, \star) : (\text{Na}(D, \star) : E))$ *for each $E \in \overline{\boldsymbol{F}}(D)$; and, for a nonzero ideal I of D, $I^{\star_f}$ is divisorial in $D^{\star}$ if and only if $I\text{Na}(D, \star)$ is divisorial in* $\text{Na}(D, \star)$.

Proof. Set $\mathcal{N} = \text{Na}(D, \star)$. For (1), applying Lemma 2.4, we have

$$(D^{\widetilde{\star}} : (D^{\widetilde{\star}} : E))\mathcal{N} = (\mathcal{N} : (D^{\widetilde{\star}} : E)) = (\mathcal{N} : (\mathcal{N} : E)).$$

(2) Assume that I is a nonzero ideal of D. If $I^{\widetilde{\star}}$ is divisorial in $D^{\widetilde{\star}}$, then (using (1))

$$(\mathcal{N} : (\mathcal{N} : I\mathcal{N})) = (D^{\widetilde{\star}} : (D^{\widetilde{\star}} : I^{\widetilde{\star}}))\mathcal{N} = I^{\widetilde{\star}}\mathcal{N} = I\mathcal{N}.$$

Now suppose that $I\mathcal{N}$ is divisorial. Then

$$(D^{\widetilde{\star}} : (D^{\widetilde{\star}} : I^{\widetilde{\star}}))\mathcal{N} = (\mathcal{N} : (\mathcal{N} : I)) = I\mathcal{N},$$

whence

$$(D^{\widetilde{\star}} : (D^{\widetilde{\star}} : I^{\widetilde{\star}})) \subseteq I\mathcal{N} \cap K = I^{\widetilde{\star}}.$$

The "in particular" statement follows from standard considerations. □

Theorem 3.2. *Let $\star$ be a semistar operation on D such that D is a $\star_f$-doublesharp P$\star$MD, and let $P \in \text{QSpec}^{\star_f}(D) \setminus \text{QMax}^{\star_f}(D)$. Then $P^{\star_f}$ is a divisorial ideal of $D^{\star}$.*

Proof. Since $\text{Na}(D, \star)$ is a doublesharp Prüfer domain (Lemma 2.2(5)), $P\text{Na}(D, \star)$ is divisorial by [7, Corollary 4.1.14]. Hence $P^{\star_f}$ is divisorial in $D^{\star}$ by Lemma 3.1. □

Corollary 3.3. *If D is a t-doublesharp PvMD, and P is a non-t-maximal t-prime of D, then P is divisorial.*

Proof. Take $\star = v$ in Theorem 3.2. (More precisely, take $\star$ to be any extension of the star operation v on D to a semistar operation on D, so that $\star_f$ (restricted to D) is the t-operation on D.) Then $P = P^t = P^{\star_f}$ is divisorial by Theorem 3.2. □

Example 3.4. *Let p be a prime integer and let $D := \text{Int}(\mathbb{Z}_{(p)})$. Then D is a two-dimensional Prüfer domain by [2, Lemma VI.1.4 and Proposition V.1.8]. Choose a height 2 maximal ideal M of D, and let P be a height 1 prime ideal of D contained in M. Then $P = q\mathbb{Q}[X] \cap D$ for some irreducible polynomial $q \in \mathbb{Q}[X]$ [2, Proposition V.2.3]. By [2, Theorems VIII.5.3 and VIII.5.15], P is not a divisorial ideal of D. Set $E^{\star} = ED_M$ for $E \in \overline{\boldsymbol{F}}(D)$. Then $\star$ is a finite-type semistar operation on D.*

Clearly, M is the only quasi-$\star$-maximal ideal of D, and, since D_M is a valuation domain, D is a P$\star$MD by Lemma 1.2. Moreover, $\mathrm{Na}(D, \star) = D_M(X)$ *is also a valuation domain and hence a doublesharp Prüfer domain, which yields that D is a $\star_f$-doublesharp P$\star$MD* (*Lemma* 2.2). *Finally, since $P = PD_M \cap D = P^\star \cap D$, P is a non-$\star_f$-maximal quasi-$\star_f$-prime of D.* □

In the remainder of the paper, we impose on Prüfer domains (P$\star$MDs) the finite character (finite $\star_f$-character) condition. As we shall see, this allows improved versions of Theorem 2.9 and Corollary 2.10. It also allows a type of unique factorization for (quasi-$\star_f$-)ideals that are simultaneously ($\star_f$-)idempotent and ($\star_f$-)divisorial.

Theorem 3.5. *Let D be a Prüfer domain with finite character, and let I be a nonzero ideal of D. Then:*

(1) *I is idempotent if and only if I is a product of idempotent prime ideals.*
(2) *The following statements are equivalent.*
   (a) *I is idempotent and divisorial.*
   (b) *I is a product of non-maximal idempotent prime ideals.*
   (c) *I is a product of divisorial idempotent prime ideals.*
   (d) *I has a unique representation as the product of incomparable divisorial idempotent primes.*

Proof. (1) Suppose that I is idempotent. By Theorem 2.9, I is the intersection of its minimal primes, each of which is idempotent. Since D has finite character, I is contained in only finitely many maximal ideals, and, since no two distinct minimal primes of I can be contained in a single maximal ideal, I has only finitely many minimal primes and they are comaximal. Hence I is the product of its minimal primes (and each is idempotent). The converse is trivial.

(2) (a) $\Rightarrow$ (b): Assume that I is idempotent and divisorial. By (1) and its proof, $I = P_1 \cdots P_n = P_1 \cap \cdots \cap P_n$, where the P_i are the minimal primes of I. We claim that each P_i is divisorial. To see this, observe that

$$(P_1)^v P_2 \cdots P_n \subseteq (P_1 \cdots P_n)^v = I^v = I \subseteq P_1.$$

Since the P_i are incomparable, this gives $(P_1)^v \subseteq P_1$, that is, P_1 is divisorial. By symmetry each P_i is divisorial. It is well known that in a Prüfer domain, a maximal ideal cannot be both idempotent and divisorial. Hence the P_i are non-maximal.

(b) $\Rightarrow$ (c): Since D has finite character, it is a (d)-doublesharp Prüfer domain [13, Theorem 5], whence nonmaximal primes are automatically divisorial by [7, Corollary 4.1.14].

(c) $\Rightarrow$ (a): Write $I = Q_1 \cdots Q_m$, where each Q_j is a divisorial idempotent prime. Since I is idempotent (by (1)), we may also write $I = P_1 \cdots P_n$, where the P_i are the minimal primes of I. For each i, we have $Q_1 \cdots Q_m = I \subseteq P_i$, from which it follows that $Q_j \subseteq P_i$ for some j. By minimality, we must then have $Q_j = P_i$. Thus each P_i is divisorial, whence $I = P_1 \cap \cdots \cap P_n$ is divisorial.

Finally, we show that (d) follows from the other statements. We use the notation in the proof of (c) $\Rightarrow$ (a). In the expression $I = P_1 \cdots P_n$, the P_i are (divisorial, idempotent, and) incomparable, and it is clear that no P_i can be omitted. To see that this is the only such expression, consider a representation $I = Q_1 \cdots Q_m$, where the Q_i are divisorial, idempotent, and incomparable. Fix a Q_k. Then $P_1 \cdots P_n = I \subseteq Q_k$, and we have $P_i \subseteq Q_k$ for some i. However, as above, $Q_j \subseteq P_i$ for some j, whence, by incomparability, $Q_k = P_i$. The conclusion now follows easily. □

We note that incomparability is necessary for uniqueness above, for example, if D is a valuation domain and $P \subsetneq Q$ are non-maximal (necessarily divisorial) primes, then $P = PQ$.

We close by extending Theorem 3.5 to P⋆MDs and then to "ordinary" PvMDs. We omit the (by now) straightforward proofs.

Corollary 3.6. *Let $\star$ be a semistar operation on D such that D is a P⋆MD with finite $\star_f$-character, and let I be a quasi-$\star_f$-ideal of D. Then:*

(1) *I is $\star_f$-idempotent if and only if $I^{\star_f}$ is a $\star_f$-product of $\star_f$-idempotent quasi-$\star_f$-prime ideals in D, that is, $I^{\star_f} = (P_1 \cdots P_n)^{\star_f}$, where the P_i are $\star_f$-idempotent quasi-$\star_f$-primes of D.*

(2) *The following statements are equivalent.*

  (a) *I is $\star_f$-idempotent and $\star_f$-divisorial ($I^{\star_f}$ is divisorial in $D^\star$).*
  (b) *I is a $\star_f$-product of non-quasi-$\star_f$-maximal idempotent quasi-$\star_f$-prime ideals.*
  (c) *I is a $\star_f$-product of $\star_f$-divisorial $\star_f$-idempotent prime ideals.*
  (d) *I has a unique representation as a $\star_f$-product of incomparable $\star_f$-divisorial $\star_f$-idempotent primes.*

Corollary 3.7. *Let D be a PvMD with finite t-character, and let I be a nonzero t-ideal of D. Then:*

(1) *I is t-idempotent if and only if I is a t-product of t-idempotent t-prime ideals in D.*

(2) *The following statements are equivalent.*

  (a) *I is t-idempotent and divisorial.*
  (b) *I is a t-product of non-t-maximal t-idempotent t-primes.*
  (c) *I is a t-product of divisorial t-idempotent t-primes.*
  (d) *I has a unique representation as a t-product of incomparable divisorial t-idempotent t-primes.*

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