A Survey on the Local Invertibility of Ideals in Commutative Rings

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Abstract Let D be an integral domain. We give an overview on connections between the (t)-finite character property of D (i.e., each nonzero element of D is contained in finitely many (t)-maximal ideals) and problems of local invertibility of ideals.

Keywords Finite character · Invertible ideal · Star operation

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1 Introduction

A well-known characterization of invertible ideals in integral domains states that "a nonzero ideal is invertible if and only if it is finitely generated and locally principal" [8, II §5, Theorem 4].

The condition that the ideal is finitely generated can be dropped down, for instance, if the domain has the finite character on maximal ideals, i.e., each nonzero element is contained in finitely many maximal ideals (see, for instance, the argument provided in the proof of [3, Chapter 7, Exercise 9]. Here the fact that finite character allows to characterize Noetherian domains among locally Noetherian domains is put in light).

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An interesting problem considered, for instance, by S. Glaz and W. Vasconcelos in [22, 23], asks for conditions on a domain D in order to have that flat ideals of D are invertible. This question can be specialized by asking when faithfully flat ideals are invertible.

We recall that for ideals in a domain, projective is equivalent to invertible and faithfully flat is equivalent to locally principal ([2]). Thus the condition "faithfully flat ideals are projective" is exactly "locally principal ideals are invertible".

In [22], the authors conjecture the equivalence between faithfully flat and projective ideals in H-domains (i.e., a domain in which t-maximal ideals are divisorial). This conjecture has been disproved by G. Picozza and F. T. in [32, Example 1.10], but the problem that S. Glaz and W. Vasconcelos posed has also been considered for other classes of domains like the Prüfer ones (i.e., domains whose localization at a prime ideal is a valuation domain).

Bazzoni [6] conjectured that:

In a Prüfer domain D "locally principal ideals are invertible" if and only if D has the finite character on maximal ideals.

Bazzoni's conjecture was proved at the same time by W.C. Holland, J. Martinez, W. Wm. McGovern, M. Tesemma in [27] by using methods (of independent interest) of the theory of lattice-ordered groups and by F. Halter-Koch using the theory of r-Prüfer monoids [25], where r is an ideal system.

After the publication of these papers, a growing interest in this question (in more general contexts) came up.

It was considered a kind of t-version of Bazzoni's conjecture which replaces the finite character with the t-finite character and the local invertibility with the t-local invertibility. In this context, the original conjecture has been generalized to Prüfer v-multiplication domains and to even larger classes of integral domains (cfr. [18, 25, 35]). Section 3 of the paper is completely dedicated to a discussion of the t-local invertibility.

Finally, in Section 4, we present some recent results about the local invertibility and its connections with the finite character on maximal ideals in commutative (not necessarily integral) rings. As seen in Theorem 4.5, Bazzoni's conjecture can be extended to general rings with zero-divisors. In this context, the authors use the concept of Manis valuations and Prüfer extensions in place of Prüfer domains (see [28]).

2 The Prüfer Case

In this section, we will provide a deeper insight into the so-called Bazzoni's conjecture, which states that a Prüfer domain D has finite character if and only if every locally principal of D is invertible (i.e., finitely generated). This conjecture, stated in [4] and [6], was first solved by Holland, Martinez, Mc Govern and Tesemma in [27] and their result was then generalized to several other classes of rings (see, for instance, [18, 25, 35]). We are going to present the main steps of the first proof of Bazzoni's conjecture. It is based on an argument involving some basic tools on lattice-ordered abelian groups. Thus we will recall now some preliminaries for the reader's convenience.

As usual, for any ring R, $\mathrm{Spec}(R)$ denotes the set of all prime ideals of R and, if S is any subset of R, we set

$$V(S) := \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq S\}.$$

If $(X, \leq)$ is a partially ordered set and $x_1, \ldots, x_n \in X$, then $\sup(x_1, \ldots, x_n)$ (resp., $\inf(x_1, \ldots, x_n)$) will denote the supremum (resp., the infimum) of $\{x_1, \ldots, x_n\}$ in X, if it exists. Recall that a nonempty and proper subset F of a partially ordered set $(X, \leq)$ is a *filter* (see [12, Definition 14.1]) if it satisfies the following properties:

- given $x, y \in F$, there exists $\inf(x, y)$ and $\sup(x, y) \in F$;
- if $f \in F, x \in X$ and $f \leq x$, then $x \in F$.

Let $\mathcal{F}(X)$ denote the set of all filters on X. If $(X, \leq)$ is a lattice, for any $a \in X$, the set $\{x \in X : x \geq a\}$ is clearly a filter and it is called *a principal filter*.

Now, let $(G, \cdot, \leq)$ be a lattice-ordered abelian group (for short, a ℓ-group). Recall that an ℓ-subgroup H of G is *convex* if, given elements $h, k \in H$, $g \in G$ such that $h \leq g \leq k$, then $g \in H$. Clearly, the intersection of any nonempty collection of convex ℓ-subgroups of G is still convex, and thus for any subset S of G there exists the smallest convex ℓ-subgroup $\mathrm{Conv}(S)$ of G containing S, and it is called *the convex envelop of* S. If e is the identity element of G, let

$$G^+ := \{g \in G : g \geq e\}$$

be the positive cone of G. If $S \subseteq G^+$, then, by [12, Proposition 7.11(b)],

$$\mathrm{Conv}(S) = \{g \in G : |g| \leq s_1 \cdots s_n, \text{ for some } s_1, \ldots, s_n \in S, n \in \mathbb{N}^+\},$$

where $|g| := \sup(g, e) \cdot \sup(g^{-1}, e)$. A convex ℓ-subgroup P of G is *prime* if, whenever $g, h \in G$ and $\inf(g, h) = e$, then either $g \in P$ or $h \in P$. A straightforward application of Zorn's Lemma shows that G admits *minimal prime subgroups* (i.e., prime subgroups which are minimal under inclusion) and every prime subgroup of G contains some minimal prime subgroup [12, Theorem 9.6]. Let $\mathcal{P}(G)$ (resp., $\mathcal{M}(G)$) denote the set of all prime (resp., minimal prime) subgroups of G. For every $g \in G$, let $U(g) := \{P \in \mathcal{M}(G) : g \notin P\}$.

The bridge which links Bazzoni's conjecture and the theory of ℓ-groups is the ideal structure of a Prüfer domain. Let D be an integral domain and let $\mathrm{Inv}(D)$ be the multiplicative group consisting of all invertible fractional ideals of D, endowed with partial order given by the opposite inclusion $\supseteq$. Since D is the identity of $\mathrm{Inv}(D)$, the positive cone $\mathrm{Inv}(D)^+$ of $\mathrm{Inv}(D)$ is just the set of all integral invertible ideals.

Theorem 2.1 ([9, Theorem 2]). *If D is a Prüfer domain, then* $\mathrm{Inv}(D)$ *is an ℓ-group.*

More precisely, given two fractional ideals $I, J \in \mathrm{Inv}(D)$, then they are finitely generated, in particular. Thus $I + J$ is finitely generated too and, since D is Prüfer, $I + J \in \mathrm{Inv}(D)$, proving that $I + J = \inf(I, J)$. Moreover, D is a coherent domain (meaning that the intersection of finitely many finitely generated fractional ideals is finitely generated too), being it Prüfer, by [21, Proposition (25.4)(1)], and thus $I \cap J \in \mathrm{Inv}(D)$, proving that $I \cap J = \sup(I, J)$. Let $\mathcal{I}^\bullet(D)$ denote the set of all nonzero (integral) ideals of D.

Lemma 2.2 (see [27, Lemma 1]). *Let D be a Prüfer domain. The following properties hold.*

(1) The map $\varphi : \mathcal{I}^\bullet(D) \longrightarrow \mathcal{F}(\mathrm{Inv}(D)^+)$ defined by setting

$$\varphi(\mathfrak{i}) := \{\mathfrak{a} \in \mathrm{Inv}(D)^+ : \mathfrak{a} \subseteq \mathfrak{i}\}$$

is a bijection.

(2) For every $\mathfrak{i} \in \mathcal{I}^\bullet(D)$, $\varphi(\mathfrak{i})$ is a principal filter if and only if $\mathfrak{i}$ is invertible (i.e., finitely generated).

Proof. (1). The fact that φ is well defined and injective is trivial. Now, let F be a filter on $\mathrm{Inv}(D)^+$ and set $\mathfrak{i} := \sum_{\mathfrak{a} \in F} \mathfrak{a}$. Thus, by definition, $F \subseteq \varphi(\mathfrak{i})$. Conversely, take an ideal $\mathfrak{b} \in \varphi(\mathfrak{i})$, i.e., $\mathfrak{b}$ is invertible and $\mathfrak{b} \subseteq \mathfrak{i}$. Since $\mathfrak{b}$ is, in particular, finitely generated, there exist ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \in F$ such that $\mathfrak{b} \subseteq \mathfrak{a}_1 + \ldots + \mathfrak{a}_n$, that is, $\mathfrak{b} \geq \inf(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$. Keeping in in mind that F is a filter it follows $\inf(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) \in F$ and finally $\mathfrak{b} \in F$.

(2) is clear, for the definitions. □

Remark 2.3. Let D be a Prüfer domain and let $\mathrm{Spec}(D)^\bullet$ denote the set of all nonzero prime ideals of D. For any ideal $\mathfrak{p} \in \mathrm{Spec}(D)^\bullet$, set $X_\mathfrak{p} := \{\mathfrak{a} \in \mathrm{Inv}(D)^+ : \mathfrak{a} \nsubseteq \mathfrak{p}\}$ and $\overline{\mathfrak{p}} := \mathrm{Conv}(X_\mathfrak{p})$.

(1) $\overline{\mathfrak{p}}$ is a prime subgroup of $\mathrm{Inv}(D)$. Take invertible ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ of D such that $\inf(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) := \mathfrak{a}_1 + \ldots + \mathfrak{a}_n = D$. It follows $\mathfrak{a}_i \nsubseteq \mathfrak{p}$, for some i, i.e., $\mathfrak{a}_i \in X_\mathfrak{p} \subseteq \overline{\mathfrak{p}}$. Furthermore, by [12, Proposition 7.11(b)] and keeping in mind that $X_\mathfrak{p}$ is closed under multiplication, we infer

$$\overline{\mathfrak{p}} = \{I \in \mathrm{Inv}(D) : |I| \supseteq \mathfrak{a}, \text{ for some } \mathfrak{a} \in X_\mathfrak{p}\}.$$

(2) Since $|\mathfrak{a}| = \mathfrak{a}$, for any $\mathfrak{a} \in \mathrm{Inv}(D)^+$, it follows that $\overline{\mathfrak{p}} \cap \mathrm{Inv}(D)^+ = X_\mathfrak{p}$.
(3) Consider $G = \mathrm{Inv}(D)$ and the map $\psi : \mathrm{Spec}(D)^\bullet \longrightarrow \mathcal{P}(G)$ defined by setting, $\psi(\mathfrak{p}) := \overline{\mathfrak{p}}$, for every $\mathfrak{p} \in \mathrm{Spec}(D)^\bullet$. Keeping in mind part (2) of the present remark, it easily follows that, for every $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(D)^\bullet$, $\mathfrak{p} \subseteq \mathfrak{q}$ if and only if $\overline{\mathfrak{q}} \subseteq \overline{\mathfrak{p}}$. In particular, ψ is injective and order reversing.
(4) ψ restricts to a bijection of $\mathrm{Max}(D)$ onto $\mathcal{M}(\mathrm{Inv}(D))$. As a matter of fact, let P be a minimal prime subgroup of $\mathrm{Inv}(D)$. If, for every maximal ideal $\mathfrak{m}$ of

D, $X_{\mathfrak{m}} \nsubseteq P$, there is an invertible integral ideal $\mathfrak{a}_{\mathfrak{m}}$ of D such that $\mathfrak{a}_{\mathfrak{m}} \nsubseteq \mathfrak{m}$ and $\mathfrak{a}_{\mathfrak{m}} \notin P$. It follows $\sum_{\mathfrak{m}\in \mathrm{Max}(D)} \mathfrak{a}_{\mathfrak{m}} = D$ and thus (since every ring has the identity element) there are maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ of D such that $\mathfrak{a}_{\mathfrak{m}_1} + \ldots + \mathfrak{a}_{\mathfrak{m}_r} = D$. Since P is a prime subgroup, there exists $i \in \{1, \ldots, r\}$ such that $\mathfrak{a}_{\mathfrak{m}_i} \in P$, a contradiction. It follows that there is a maximal ideal $\mathfrak{m}$ of D such that $X_{\mathfrak{m}} \subseteq P$ and, since P is a minimal prime subgroup (and, in particular, it is convex), we deduce $\overline{\mathfrak{m}} := \mathrm{Conv}(X_{\mathfrak{m}}) = P$. Conversely, if $\mathfrak{m}$ any maximal ideal of D and P is a minimal prime subgroup of $\mathrm{Inv}(D)$ such that $\overline{\mathfrak{m}} \supseteq P$, take a maximal ideal $\mathfrak{n}$ of D such that $P = \overline{\mathfrak{n}}$ (in view of what we have just proved). By part (3) it follows $\mathfrak{m} = \mathfrak{n}$ and thus $\overline{\mathfrak{m}}$ is minimal.

(5) By the previous parts, for every integral invertible ideal $\mathfrak{a}$ of D, we have

$$\psi(\mathrm{Max}(D) \cap V(\mathfrak{a})) = U(\mathfrak{a}).$$

The following result immediately follows from parts (4, 5) of the previous remark.

Lemma 2.4 ([27, Theorem 2]). *A Prüfer domain D has a finite character if and only if, for every integral invertible ideal* $\mathfrak{a}$ *of D, the set* $U(\mathfrak{a})$ *is finite.*

Thus the previous lemma provides the translation of the finite character of a Prüfer domain into a statement in the language of ℓ-groups. The next goal is to provide the translation of the LPI property.

Definition 2.5 ([27, Definition 3]). Let G be an ℓ-group and let F be a filter on G^+. Then F is said to be a *cold filter* provided that, for every $P \in \mathcal{M}(G)$, there exists some element $f \in F$ such that $f + P \leq g + P$, for all $g \in F$ (where $\leq$ is the canonical total order induced by the order of G into the factor group G/P).

Lemma 2.6 ([27, Proposition 5]). *Let D be a Prüfer domain. Then a nonzero ideal* $\mathfrak{i}$ *of D is locally principal if and only if the filter* $\varphi(\mathfrak{i})$ *(see Lemma 2.2) is a cold filter.*

Combining Lemmas 2.2 and 2.6, it is clear that for a Prüfer domain D the following conditions are equivalent.

(1) Every nonzero locally principal ideal of D is invertible.
(2) Every cold filter on $\mathrm{Inv}(D)^+$ is principal.

This provides the complete translation of Bazzoni's conjecture into a conjecture regarding ℓ-groups. The key step to show it is to observe that, if G is an ℓ-group such that every cold filter on G^+ is principal, then every element of G^+ is greater than only finitely many mutually disjoint elements ([27, Proposition 7]). Now several relevant results of Conrad [11] are very helpful, together with the Finite Basis Theorem ([12, Theorem 46.12]). These tools lead to the main result.

Theorem 2.7 ([27, Theorem 9]). *Let G be an* ℓ*-group such that every cold filter on* G^+ *is principal. Then, for every* $g \in G^+$*, the set* $U(g)$ *is finite.*

Thus finally, keeping in mind the previous theorem and Lemmas 2.4, 2.6, the desired conclusion follows.

Corollary 2.8. *A Prüfer domain D has finite character if and only if every nonzero locally principal ideal of D is invertible.*

3 Generalization to Non-Prüfer Domains

We start this section by recalling some basic facts about star operations.

Let D be an integral domain with quotient field K. The set $\mathbf{F}(D)$ denotes the nonzero fractional ideals of D and $\mathbf{f}(D)$ the nonzero finitely generated fractional ideals of D.

A map $\star : \mathbf{F}(D) \to \mathbf{F}(D)$, $I \mapsto I^\star$ is called a *star operation* if the following conditions hold for all $x \in K \setminus \{0\}$ and $I, J \in \mathbf{F}(D)$:

($\star_1$) $(xD)^\star = xD$;
($\star_2$) $I \subseteq J \Rightarrow I^\star \subseteq J^\star$;
($\star_3$) $I \subseteq I^\star$ and $I^{\star\star} := (I^\star)^\star = I^\star$.

Given a star operation $\star$, a nonzero ideal I of D such that $I = I^\star$ is called a $\star$–*ideal*.

Examples of star operations are the d-operation, the v-operation, and the t-operation:

- The *d-operation* is the identity map $I \mapsto I$.
- The *v-operation* is the map:

$$I \mapsto I^v := (D\colon (D\colon I)), \quad \text{where} \quad (D\colon I) := I^{-1} = \{x \in K \mid xI \subseteq D\}.$$

- the *t-operation* is the map:

$$I \mapsto I^t := \bigcup_{J \in \mathbf{f}(D),\, J \subseteq I} J^v.$$

A star operation $\star$ on D is *of finite type* if for all $I \in F(D)$,

$$I^\star = \bigcup \{J^\star : J \subseteq I,\, J \in \mathbf{f}(D)\}.$$

From the definition, it follows that the t-operation is of finite type.

Definition 3.1. Given a star operation $\star$ on a domain D and $I \in \mathbf{F}(D)$,

I is *$\star$-invertible* if there exists $J \in \mathbf{F}(D)$ such that $(IJ)^\star = D$ (it is easy to see that, in this case, $J = I^{-1}$).

Thus, taking $\star = d$, we find the usual definition of invertible ideal, that is $II^{-1} = D$.

An ideal I is $\star$-finite if there exists a finitely generated ideal J such that $J^\star = I^\star$. If $\star$ is of finite type, J can always be taken inside I.

If $\star$ is a star operation of finite type, the set of $\star$-ideals has (proper) maximal elements called $\star$-maximal ideals (this set is denoted by $\star - \text{Max}(D)$). A $\star$-maximal ideal is a prime ideal and every integral $\star$-ideal is contained in a $\star$-maximal ideal.

A domain D has the $\star$-*finite character* if each $\star$-ideal (equivalently, each nonzero element) of D is contained in finitely many $\star$-maximal ideals.

It is well-known that a nonzero ideal of a domain D is invertible if and only if I is finitely generated and locally principal (see [8, II Section 5, Theorem 4]).

A similar characterization holds for $\star$-invertible ideals.

In fact, an ideal I is $\star$-invertible if and only it is $\star$-finite and $\star$-locally principal (that is, ID_M is principal for each $M \in \star - \text{Max}(D)$) ([26, page 137]).

In the characterization of invertible ideals given above, the hypothesis that I is finitely generated can be dropped down in some classes of domains, called LPI domains, introduced by D. D. Anderson and M. Zafrullah in [1].

A domain D is LPI if every nonzero locally principal ideal is invertible, or equivalently, if every faithfully flat ideal is finitely generated. Thus, LPI domains are exactly the domains in which faithfully flat ideals are projective.

Mori domains, and therefore Noetherian domains are LPI.

The finite character condition on a domain D is sufficient to have that D is LPI. In fact, it is a straightforward exercise to prove that the finite character implies that a locally principal ideal is finitely generated. Nevertheless, this condition is not necessary for the LPI property of D.

For instance, Noetherian domains are LPI but they do not always have the finite character (see $\mathbb{Z}[X]$).

Definition 3.2. A domain D has the t-finite character if each nonzero element $x \in D$ is contained in only finitely many t-maximal ideals.

Noetherian domains have the t-finite character (see [5, Proposition 2.2(b)]) and the t-finite character is a sufficient condition for a general domain D to be LPI ([32, Lemma 1.12]).

Remark 3.3. In Prüfer domains the t-operation is the identity ([21, Theorem 22.1 (3)]), that is, each ideal is a t-ideal. Thus, for this class of domains, the t-finite character coincides with the finite character, which is exactly the property required for Prüfer domains to be LPI in the conjecture by S. Bazzoni.

The above remark brings to consider the t-version of Prüfer domain, the Prüfer v-multiplication domains (PvMD).

We recall that a domain D is a PvMD if each t-finite ideal is t-invertible. Equivalently, if and only if D_P is a valuation domain for each t-prime (or t-maximal) ideal P of D ([29, Theorem 4.3]).

Thus, by replacing the finite character with the t-finite character and the invertibility property for ideals with the t-invertibility property it is possible to generalize S. Bazzoni's conjecture using the t-operation.

A first step in this direction is the definition of t-LPI domains.

Definition 3.4. A domain D is t-LPI if each nonzero t-locally principal t-ideal is t-invertible.

The t-version of Bazzoni's conjecture is then:

"A PvMD is t-LPI if and only if it has the t-finite character."

M. Zafrullah and F. Halter-Koch proved this result almost at the same time using different techniques (see [35, Proposition 5] and [25]). F. Halter-Koch considered the problem in a more general setting involving a general $\star$-operation.

Now, we consider the t-version of Bazzoni's conjecture outside the natural context of PvMD and present contributions that prove the equivalence ($\diamond$)

"t-finite character $\Leftrightarrow$ each nonzero t-locally principal ideal is t-invertible"

for more general classes of domains.

We have seen that the t-finite character is a sufficient condition for a general domain D in order to have that D is t-LPI ([32, Lemma 1.12]).

Conversely, by [18, Example 2.3], if D is not a PvMD, the t-finite character is not a necessary condition to have that D is t-LPI.

Thus, a question that was investigated, for instance, by T. Dumitrescu and M. Zafrullah in [13] and, independently, by C.A. F. - G. Picozza and F.T. in [18], concerns the characterization of classes of domains strictly larger than PvMD verifying condition ($\diamond$) given above.

T. Dumitrescu and M. Zafrullah considered the case of t-Schreier domains, that we define below.

Given a domain D, $\mathrm{Inv}_t(D)$ is the set of the t-invertible t-ideals of D.

A domain D is t-Schreier if $\mathrm{Inv}_t(D)$ is a Riesz group, that is: if every finite intersection of nonzero principal ideals is a direct union of t-invertible t-ideals. For instance, PvMD's are t-Schreier (see [15, Lemma 1.8]). More precisely, an integral domain is a PvMD if and only if it is t-Schreier and v-coherent, by [14, Corollary 6(a)].

Theorem 3.5. [14, Proposition 17] *If D is t-Schreier, then D is t-LPI if and only if D has the t-finite character.*

Thus t-Schreier domains enlarge the class of domains verifying the t-version of Bazzoni's conjecture.

In this direction, we also find the results by C.A. F. - G. Picozza and F.T. about v-coherent domains.

In their paper [18] the authors consider the more general case of a $\star$-operation of finite type, thus including the t-operation.

The general question is then to find a characterization of domains for which the $\star$-finite character is equivalent to $\star$-LPI (that is, every $\star$-locally principal $\star$-ideal is $\star$-invertible).

We have seen as Noetherian domains suggest to use the t-finite character condition in the study of (t)-local invertibility for ideals.

Again, Noetherian domains bring to consider another interesting condition involving (t)-comaximal ideals.

In a Noetherian domain every nonzero nonunit element belongs to only a finite number of *mutually comaximal* proper invertible ideals.

Definition 3.6. Two proper ideals $I, J \subset D$ are t-comaximal if $(I + J)_t = D$. In particular, I, J are t-comaximal if $(I + J)_t = D$, which means that I and J are not contained in a common t-maximal ideal.

Theorem 3.7. [18, Proposition 1.6] *Let D be an integral domain. Then the following conditions are equivalent.*

(i) D has the $t-$finite character.
(ii) Every family of mutually t-comaximal t-finite t-ideals of D with nonzero intersection is finite.

Thus, in order to prove that t-LPI is equivalent to the t-finite character, it would be interesting to see whether the condition (ii) of the above Theorem has connections with the t-LPI property.

Since in Prüfer domains the t-operation is the identity, we can restate Theorem 3.7 as follows:

Corollary 3.8. *A Prüfer domain D has the finite character if and only if each invertible integral ideal of D is contained in at most a finite number of mutually comaximal invertible ideals.*

Corollary 3.8 can be easily extended to PvMD's by replacing comaximality with t-comaximality.

Corollary 3.9. *A PvMD has the t-finite character if and only if each integral t-invertible t-ideal is contained in at most a finite number of mutually t-comaximal t-invertible t-ideals.*

Remark 3.10. Consider the following t-invertibility like conditions for ideals in a domain D:

(1) t-locally t-finite (i.e. I_M is t-finite for each $M \in t-\mathrm{Max}(D)$, with respect to the t-operation of D_M) t-ideals are t-finite;
(2) t-locally principal (i.e., I_M is principal for each $M \in t-\mathrm{Max}(D)$) t-ideals are t-invertible (t-LPI);

We observe that

(a) (1) $\Rightarrow$ (2);
(b) conditions (1)–(2) are equivalent to LPI in the case of Prüfer domains and to t-LPI for PvMD's;
(c) the t-finite character implies conditions (1)–(2).

We recall that a domain D is *v-coherent* if for any nonzero finitely generated ideal I of D, I^{-1} is v-finite (see, for instance, [16, Proposition 3.6] and [30]).

A domain D is *t-locally v-coherent* if D_M is v-coherent, for each $M \in t-\mathrm{Max}(D)$.

Important classes of v-coherent domains are Noetherian domains, Mori domains, Prüfer domains, PvMD's, finite conductor domains (i.e., $(x) \cap (y)$ is finitely generated for each $x, y \in A$), coherent domains (i.e., the intersection of two finitely generated ideals is finitely generated).

Using pullback constructions it is possible to give examples of t-locally v-coherent domains which are not v-coherent (cfr. [20]).

Since both Prüfer domains and PvMD's are t-locally v-coherent, a first step in the direction of generalizing Bazzoni's conjecture to any domain is the following theorem.

Theorem 3.11. [18, Theorem 1.11] *Let D be an integral domain which is t-locally v-coherent. Then the following conditions are equivalent.*

(i) D has the t-finite character;
(ii) every family of t-finite, t-comaximal, t-ideals over a nonzero element $a \in D$ is finite;
(iii) every nonzero t-locally t-finite t-ideal is t-finite.

Remark 3.12. (a) A t-locally v-coherent domain is not necessarily a PvMD. In fact any Noetherian domain is t-locally v-coherent (and it is not always a PvMD). Thus, the class of domains considered in Theorem 3.11 is larger than one of the PvMD's.

(b) Condition (iii) of Theorem 3.11 is exactly point (1) of Remark 3.10. Thus it implies the t-LPI property (point (2) of the same Remark) and it is equivalent to t-LPI in PvMD's. In general, we don't know whether these two conditions are equivalent.
Anyway, Theorem 3.11 suggests that a natural statement to generalize to any domain of the t-version of Bazzoni's conjecture should claim the equivalence between the t-finite character and condition (1) of Remark 3.10.

(c) In general, Theorem 3.11 cannot be extended to any finite type star operation. For instance, it fails if we take the identity operation d. In fact, a Noetherian domain does not need to have the finite character on maximal ideals, but each locally finitely generated ideal is finitely generated.

So far, the t-operation seems to be the only star operation of finite type that has an interesting role in the generalization of Bazzoni's conjecture.

In fact there is not an analogue of Theorem 3.11 for a generic $\star$-operation. Here below we give two partial results in this direction.

Independently, the authors in [13, Corollary 3] and in [18, Proposition 1.6] put in connection some families of mutually $\star$-comaximal ideals of D with the $\star$-finite character of D.

The following theorem generalizes Theorem 3.7 to any star operation of finite type.

Theorem 3.13. *Let D be an integral domain and $\star$ a finite type star operation on D. Then the following conditions are equivalent.*

(i) D has the $\star$-finite character.
(ii) Every nonzero element $a \in D$ is contained in at most finitely many proper $\star$-finite, mutually $\star$-comaximal $\star$-ideals of D.

There is not a proven connection between condition (ii) of the theorem above and generalizations of $\star$-LPI condition as it happens for the t-operation by Theorem 3.11.

The following result states that if things "work well" for the t-operation, then there are positive cascade results for finite type star operations.

Proposition 3.14. [18, Proposition 2.2] *Let D be a domain in which each t-locally principal t-ideal is t-finite. Then, for any star operation of finite type, each $\star$-locally principal $\star$-ideal is $\star$-finite. In particular, a locally principal ideal is finitely generated (and so, invertible).*

In Proposition 3.14 the t-operation cannot be replaced by any finite type star operation. For instance, [18, Example 2.3] shows that it does not hold when $\star = d$.

Another interesting class of domains recently studied in this context are the *finitely stable* domains.

Definition 3.15. An ideal I of a domain D is *finitely stable* if I is invertible (or projective) in its endomorphism ring

$$End(I) = (I : I) = \{x \in K \mid xI \subseteq I\}.$$

A domain D is *stable* if each nonzero ideal is stable and D is *finitely stable* if each finitely generated ideal is stable. Obviously, stable domains are finitely stable.

An important result proven by B. Olberding in [31] states that stable domains have the finite character. Moreover, integrally closed stable domains are Prüfer, thus they are LPI.

It is also well-known that integrally closed, finitely stable domains are exactly Prüfer domains, hence they generalize the Prüfer ones.

Moreover, finitely stable domains are a distinct class from v-coherent domains. In fact, $D = K[[X^2, X^3]]$ (where K is any field) is Mori, hence v-coherent, but it is not finitely stable because its maximal ideal is not stable (see [7, Example 1]).

On the other hand, if we take a PvMD that is not Prüfer (e.g., $\mathbb{Z}[X]$), then this is not finitely stable but it is v-coherent.

S. Bazzoni in [7] before, and S. Xing and F. Wang in [34] after, study conditions on finitely stable domains in order to verify the LPI property.

A domain has the *local stability property* if each nonzero ideal that is locally stable is stable.

In [7, Theorem 4.5] a characterization of finitely stable domains with finite character is given.

Theorem 3.16. *Let D be a finitely stable domain. Then D has the finite character if and only if it has the local stability property.*

[7, Lemma 3.2] shows that if D is a finitely stable domain that has the local stability property then it is LPI.

As we know, the question whether a finitely stable LPI domain has the local stability property is still open ([7, Question 4.6]).

In view of Theorem 3.16, if the answer to this question is positive, then this would imply that also for (non integrally closed) finitely stable domain the finite character is equivalent to LPI, as it happens in the Prüfer case.

Other interesting results about the interplay between the finite character and the LPI property for finitely stable domains are given in [34].

First of all, in this paper the authors show that LPI is not a local property. In fact they give an example of a domain D that is LPI, but D_S is not LPI for a suitable multiplicatively closed subset $S \subseteq D$ ([34, Example 2.4]). This fact has no connections with the finite character question, but it is interesting by itself.

Anyway, the main result of [34] gives a characterization of LPI finitely stable involving the finite character.

The authors denote by $\mathcal{T}(D)$ the set of maximal ideals m of D for which there exists a finitely generated ideal I such that m is the only maximal ideal containing I. For each ideal I, $\Omega(I)$ is the set of maximal ideals of D containing I. Thus, the finite character is equivalent to ask that $\Omega(I)$ is finite for each nonzero ideal $I \subseteq D$.

Then, the following results are proven:

Theorem 3.17. [34, Theorem 2.6] *Let D be a finitely stable LPI domain. Then every nonzero element of D is contained in, at most, finitely many ideals of $\mathcal{T}(D)$.*

Thus we have that LPI on finitely stable domains implies the finite character on the subset $\mathcal{T}(D)$ of $\mathrm{Max}(D)$.

From Theorem 3.17 it follows the next corollary:

Corollary 3.18. [34, Corollary 2.7] *Let D be a finitely stable LPI domain. Then, the following conditions are equivalent:*

(i) *D ha the finite character;*
(ii) *$\mathcal{T}(D) \cap \Omega(I) \neq \emptyset$, for each nonzero, finitely generated ideal I of D.*

We observe that Corollary 3.18 does not hold without the LPI hypothesis.

In fact, the following example shows that there exists a Prüfer domain D without the finite character and such that $\mathcal{T}(D) = \emptyset$. In this case D is finitely stable (since it is Prüfer) and point (ii) of Corollary 3.18 trivially holds since $\mathcal{T}(D) = \emptyset$, but the domain has not the finite character.

Example 3.19. Consider the integer valued polynomial ring

$$\mathrm{Int}(\mathbb{Z}) = \{f(X) \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}.$$

It is well-known that Int($\mathbb{Z}$) is a Prüfer domain ([10, Theorem VI.1.7]) and it has not the finite character. In fact, each prime $p \in \mathbb{Z}$ is contained in infinitely many maximal ideals of the type $M_{p,\alpha}$ described in [10, Theorem V.2.7].

We easily see that $\mathcal{T}(\text{Int}(\mathbb{Z})) = \emptyset$. By [10, Theorem V.2.7], the maximal ideals of Int($\mathbb{Z}$) are the $M_{p,\alpha}$, $p \in \mathbb{Z}$, $\alpha \in \widehat{\mathbb{Z}_p}$ (the p-adic completion of $\mathbb{Z}$). Suppose that a maximal ideal $M_{p,\alpha}$ belongs to $\mathcal{T}(\text{Int}(\mathbb{Z}))$ and let I be a finitely generated ideal such that $M_{p,\alpha}$ is the only maximal ideal containing I. Since Int($\mathbb{Z}$) is a Prüfer domain, I is invertible. But

$$I^{-1} \subseteq \bigcap_{\mathcal{M} \in \text{Max}(\text{Int}(\mathbb{Z})),\, I \nsubseteq \mathcal{M}} \text{Int}(\mathbb{Z})_{\mathcal{M}},$$

where second term is an overring of Int($\mathbb{Z}$) defined by Kaplansky and so-called the *Kaplansky transform* of the ideal I [17, Theorem 3.2.2]. Using the same argument of [33, Proposition 2.2] to see that $\mathfrak{P}^{-1} = \text{Int}(D)$, we can show that $I^{-1} = \text{Int}(\mathbb{Z})$ against the hypothesis that I is a proper ideal.

4 Generalization to Rings with Zero-Divisors

In the present section we will present a generalization of Bazzoni's conjecture to (commutative) rings with zero-divisors. We start with recalling some terminology and preliminaries and we will follow [28]. Let $A \subseteq B$ be a ring extension and let X be an A-submodule of B. We will say that X is *B-regular* if $XB = B$. Furthermore, X is said to be *B-invertible* if $XY = A$, for some A-submodule Y of B. It is worth noting that, in case B is the total ring of fractions $\mathrm{T}(A)$ of A, then X is B-regular if and only if it is regular (i.e., it contains a regular element of A) and X is B-invertible if and only if it is invertible in the sense of Griffin (see [24]). Several known facts about fractional ideals of integral domains extend naturally in this setting: for instance, X is B-invertible if and only if X is B-regular, finitely generated and locally principal [28, Section 2, Proposition 2.3].

Definition 4.1. Let $A \subseteq B$ be a ring extension. We say that $A \subseteq B$ *has finite character* if every B-regular ideal of A is contained only in finitely many maximal ideals of A.

Clearly, if $B = \mathrm{T}(A)$, then $A \subseteq B$ has finite character if and only if every regular ideal of A is contained only in finitely many maximal ideal. The following result extends Theorem 3.13 (in case $\star = d$) to every ring extension.

Proposition 4.2. [19, Corollary 3.3] *For a ring extension $A \subseteq B$ the following conditions are equivalent.*

(1) $A \subseteq B$ has finite character.
(2) For any finitely generated and B-regular ideal $\mathfrak{a}$ of A, every collection of mutually comaximal finitely generated (and B-regular) ideals of A containing $\mathfrak{a}$ is finite.

Finite character ring extensions allows to test relevant properties of ideals locally, as the following result shows.

Proposition 4.3. ([19, Proposition 3.4 and Corollary 3.5]) *Let $A \subseteq B$ be a ring extension with finite character and let $\mathfrak{a}$ be a B-regular ideal of A. If $\mathfrak{a}$ is locally finitely generated, then $\mathfrak{a}$ is finitely generated. In particular, $\mathfrak{a}$ is B-invertible if and only if it is locally principal.*

According to [28, Chapter 2, Theorem 2.1], we say that a ring extension $A \subseteq B$ is a *Prüfer extension* if $A \subseteq B$ is a flat epimorphism (in the category of rings) and every finitely generated B-regular ideal of A is B-invertible. In case $B = \mathrm{T}(A)$, then the previous definition extends the notion of Prüfer ring given by Griffin (i.e., every regular finitely generated ideal is invertible). In what follows it will suffice to work with the following weaker notion.

Definition 4.4. A ring extension $A \subseteq B$ is said to be an *almost Prüfer* extension if every finitely generated B-regular ideal of A is B-invertible.

Now we are in condition to state Bazzoni's conjecture for rings with zero-divisors.

Theorem 4.5. [19, Theorem 4.5] *For an almost Prüfer extension $A \subseteq B$ the following conditions are equivalent.*

(1) $A \subseteq B$ has finite character.
(2) Every B-regular locally principal ideal of A is B-invertible.

Keeping in mind the remarks made at the beginning of the present section, the following corollary is now clear.

Corollary 4.6. [19, Corollary 4.6] *Let A be a Prüfer ring. Then every regular locally principal ideal of A is invertible if and only if every regular element of A is contained in only finitely many maximal ideal.*

Finally we list some problems and questions that can motivate further investigation about this topic.

Question 1. Is there a class of integral domains, larger than that of t-locally v-coherent domains, for which the equivalent conditions of Theorem 3.11 hold?

Question 2. Does the statement of Theorem 3.16 admit some generalization for rings with zero-divisors?

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