A Bazzoni-Type Theorem for Multiplicative Lattices

Tiberiu Dumitrescu

Abstract We prove a Bazzoni-type theorem for multiplicative lattices thus unifying several ring/monoid theoretic results of this type.

Keywords Prüfer domain · Multiplicative lattice

2010 Mathematics Subject Classification Primary 06F05 · Secondary 13F15

1 Introduction

Let D be an integral domain. Consider the following two assertions:

(i) If I is an ideal of D whose localizations at the maximal ideals are finitely generated, then I is finitely generated.

(ii) Every $x \in D - \{0\}$ belongs to only finitely many maximal ideals of D.

While $(ii) \Rightarrow (i)$ is well-known and easy to prove, Bazzoni [3, p. 630] conjectured that the converse is true for Prüfer domains. Recall that D is a *Prüfer domain* if every finitely generated ideal I of D is locally principal.

Holland et al. [10, Theorem 10] proved Bazzoni's conjecture for Prüfer domains using techniques from lattice-ordered groups theory and McGovern [14, Theorem 11] proved the same result using a direct ring theoretic approach. Halter-Koch [9, Theorem 6.11] proved Bazzoni's conjecture for r-Prüfer monoids (see Section 4). Zafrullah [16, Proposition 5] proved Bazzoni's conjecture for Prüfer v-multiplication domains. Finocchiaro and Tartarone [6, Theorem 4.5] proved Bazzoni's conjecture for almost Prüfer ring extensions (see Section 3). Recently, Chang and Hamdi [4, Theorem 2.4] proved Bazzoni's conjecture for almost Prüfer v-multiplication domains (see Section 4).

T. Dumitrescu (✉)
Facultatea de Matematica si Informatica, University of Bucharest,
14 Academiei Street, 010014 Bucharest, RO, Romania
e-mail: tiberiu@fmi.unibuc.ro

A. Facchini et al. (eds.), *Advances in Rings, Modules and Factorizations*,
Springer Proceedings in Mathematics & Statistics 321,
https://doi.org/10.1007/978-3-030-43416-8_6

The purpose of this paper is to prove a Bazzoni-type theorem for multiplicative lattices (see Section 2), thus unifying the results mentioned above (see Sections 3 and 4). Our standard references are [1, 7, 8].

2 Main Result

We use an abstract ideal theory approach, so we work with multiplicative lattices.

Definition 1. A *multiplicative lattice* is a complete lattice $(L, \leq)$ (with bottom element 0 and top element 1) which is also a multiplicative commutative monoid with identity 1 (the top element) and satisfies $a(\bigvee b_\alpha) = \bigvee ab_\alpha$ for each $a, b_\alpha \in L$.

Let L be a multiplicative lattice. The elements in $L - \{1\}$ are said to be *proper*. Denote by $Max(L)$ the set of maximal elements of L. For $x, y \in L$, set $(y : x) = \bigvee\{a \in L;\ ax \leq y\}$.

We recall some standard terminology.

Definition 2. Let L be a multiplicative lattice and let $x, p \in L$.

(1) p is *prime* if $p \neq 1$ and for all $a, b \in L$, $ab \leq p$ implies $a \leq p$ or $b \leq p$. It follows easily that every maximal element is prime.

(2) x is *compact* if whenever $x \leq \bigvee_{y \in S} y$ with $S \subseteq L$, we have $x \leq \bigvee_{y \in T} y$ for some finite subset T of S.

(3) L is a C*-lattice* if the set L^* of compact elements of L is closed under multiplication, $1 \in L^*$ and every element in L is a join of compact elements.

(4) x is *meet-principal* if $y \wedge zx = ((y : x) \wedge z)x$ for all $y, z \in L$ (in particular $(y : x)x = x \wedge y$).

(5) x is *join-principal* if $y \vee (z : x) = ((yx \vee z) : x)$ for all $y, z \in L$ (in particular $(xy : x) = y \vee (0 : x)$).

(6) x is *cancellative* if for all $y, z \in L$, $xy = xz$ implies $y = z$.

(7) x is *CMP* (ad hoc name) if x is cancellative and meet-principal.

(8) L is a *lattice domain* if $(0 : a) = 0$ for all $a \in L - \{0\}$.

In the sequel, we work with C-lattices and their localization theory. Let L be a C-lattice. For $p \in L$ a prime element and $x \in L$, we set

$$x_p = \bigvee\{a \in L^*;\ ab \leq x \text{ for some } b \in L^* \text{ with } b \not\leq p\}.$$

Then $L_p := \{x_p;\ x \in L\}$ is again a lattice with multiplication $(x_p, y_p) \mapsto (xy)_p = (x_p y_p)_p$, join $\{(b_\alpha)_p\} \mapsto (\bigvee(b_\alpha)_p)_p = (\bigvee b_\alpha)_p$ and meet $\{(b_\alpha)_p\} \mapsto (\bigwedge(b_\alpha)_p)_p$. The next lemma collects several basic properties.

Lemma 3. *Let L be a C-lattice, let $x, y \in L$ and let $p \in L$ be a prime element.*

(1) $x_p = 1$ *if and only if* $x \not\leq p$.

(2) $(x \wedge y)_p = x_p \wedge y_p$.

(3) *If x is compact, then* $(y : x)_p = (y_p : x_p)$.

(4) $x = y$ *if and only if* $x_m = y_m$ *for each* $m \in Max(L)$.

(5) *A cancellative element x is CMP if and only if* $(y : x)x = x \wedge y$ *for all* $y \in L$.

(6) *If x is compact, then x_p is compact in L_p. Conversely, if x_p is compact in L_p, then $x_p = c_p$ for some compact element $c \leq x$.*

(7) *If x and y are CMP elements, then so is xy.*

(8) *If x is compact, y is CMP and $x \leq y$, then $(x : y)$ is compact.*

Proof. For (1–4) see [11, pp. 201–202], for (5) see [15, Lemma 2.10], while (6–7) follow easily from definitions. We prove (8). Note that $(x : y)y = x \wedge y = x$. Suppose that $(x : y) \leq \bigvee_{i \in A} z_i$. Then $x = (x : y)y \leq \bigvee_{i \in A} z_i y$, so $(x : y)y \leq \bigvee_{i \in B} z_i y$ for some finite subset B of A. Cancel y to get $(x : y) \leq \bigvee_{i \in B} z_i$. □

Say that x and $y \in L$ are *comaximal* if $x \vee y = 1$. Clearly, $x \vee yz = 1$ if and only if $x \vee y = 1$ and $x \vee z = 1$. When $t \leq u$ we say that t is *below* u or that u is *above* t.

Lemma 4. *Let L be a C-lattice and $z \in L - \{1\}$ a compact element such that $\{m \in Max(L);\ z \leq m\}$ is infinite. There exists an infinite set $\{a_n;\ n \geq 1\}$ of pairwise comaximal proper compact elements such that $z \leq a_n$ for each n.*

Proof. We may clearly assume that $z = 0$ (just change L by $\{x \in L;\ x \geq z\}$). Say that a proper compact element h is *big* (ad hoc name) if h is below only one maximal element $M(h)$. We separate in two cases.

Case (1): *Every proper compact element is below some big compact element*. We proceed by induction. Suppose that $n \geq 1$ and we already have big compacts $a_1,...,a_n$ such that $M(a_1),...,M(a_n)$ are distinct maximal elements (for $n = 1$ just pick an arbitrary big compact a_1). Let p be a maximal element other than $M(a_1),...,M(a_n)$. There exists a compact element $c \leq p$ such that $c \not\leq M(a_i)$ for $1 \leq i \leq n$ (take $c = c_1 \vee \cdots \vee c_n$ where each $c_i \in L^*$ satisfies $c_i \leq p$ and $c_i \not\leq M(a_i)$). Then take a big compact element $a_{n+1} \geq c$. This way we construct an infinite set $\{a_n;\ n \geq 1\}$ of big compacts such that all $M(a_n)$'s are distinct. Hence the a_n's are pairwise comaximal.

Case (2): *There exists a proper compact element a_0 which is not below any big compact element*. Clearly every proper compact above a_0 inherits this property. Pick two distinct maximal elements p and q above a_0. As L is a C-lattice, there exist two comaximal compacts $a_1 \leq p$ and $b_1 \leq q$ (note that $p \vee q = 1$, express p and q as joins of compact elements and use the fact that 1 is compact). Repeating this argument for a_1, there exist two comaximal proper compacts $a_2 \geq a_1$ and $b_2 \geq a_1$. Note that b_2 and b_1 are comaximal. Thus we construct inductively an infinite set $\{b_n;\ n \geq 1\}$ of pairwise comaximal proper compact elements. □

In a C-lattice L, we say that an element x is *locally compact* if x_m is compact in L_m for each $m \in Max(L)$. We state our main result which is a Bazzoni-type theorem for C-lattices.

Theorem 5. *Let L be a C-lattice domain satisfying the following two conditions:*

(a) *every nonzero element is above some cancellative compact element, and*

(b) *every compact element $x \neq 1$ has some power x^n below some proper CMP element.*

Then the following conditions are equivalent:

(i) *Every locally compact element of L is compact.*

(ii) *Every nonzero element is below at most finitely many maximal elements.*

Proof. $(ii) \Rightarrow (i)$. Although this part is well-known and easy, we include a proof for the reader's convenience. Let x be a nonzero locally compact element of L and let $a \leq x$ be a nonzero compact element. By (ii), there are only finitely many maximal elements above a, say $m_1,...,m_k$. For each i between 1 and k, pick a compact element $c_i \leq x$ such that $x_{m_i} = (c_i)_{m_i}$. A local check shows that $x = a \vee c_1 \vee \cdots \vee c_k$, so x is compact. Note that this part works for any C-lattice.

$(i) \Rightarrow (ii)$. Deny, so suppose that some nonzero element c is below infinitely many maximal elements. By hypothesis (a), we may assume that c is a cancellative compact element. By Lemma 4 and hypothesis (b), there exist an infinite set $\{b_n;\ n \geq 1\}$ of proper pairwise comaximal CMP elements and integers $k_n \geq 1$ such that $c^{k_n} \leq b_n$ for $n \geq 1$ (k_n minimal with this property). Restricting to a subsequence, we may assume that $k_n \leq k_{n+1}$ for all n. We then have $c^{k_n} \leq b_1 \wedge \cdots \wedge b_n = b_1 \cdots b_n$ for all n.

Claim $(*)$: *The element $a := \bigvee_{n\geq 1}(c^{k_n} : b_1 \cdots b_n)$ is locally compact.*
Pick $m \in \mathrm{Max}(L)$. Since the b_n's are pairwise comaximal, m is above at most one of them. Assume first that $m \geq b_s$. Since each product $b_1 \cdots b_n$ is compact, we get

$$a_m = (\bigvee_{n\geq 1}((c^{k_n})_m : (b_1 \cdots b_n)_m))_m = (c^{k_1} \vee (c^{k_s} : b_s))_m$$

which is compact in L_m, cf. Lemma 3. Similarly, when m is above no b_n, we get $a_m = (c^{k_1})_m$, so a_m is compact in L_m, hence Claim $(*)$ is proved. By (i), a is compact. So $a = \bigvee_{n=1}^{q}(c^{k_n} : b_1 \cdots b_n)$ for some $q \geq 1$. We get

$$(c^{k_{q+1}} : b_1 \cdots b_{q+1}) \leq (c^{k_1} : b_1 \cdots b_q)$$

so multiplying by $b_1 \cdots b_{q+1}$ (which is a CMP element) and taking into account that $c^{k_{q+1}} \leq b_1 \cdots b_{q+1}$, we get

$$c^{k_{q+1}} \leq (c^{k_1} : b_1 \cdots b_q)b_1 \cdots b_{q+1} \leq c^{k_1} b_{q+1}.$$

Since $k_{q+1} \geq k_1$ and c^{k_1} is cancellative, we get $c^{k_{q+1}-k_1} \leq b_{q+1}$, which is a contradiction since k_{q+1} was minimal with $c^{k_{q+1}} \leq b_{q+1}$. □

Recall that a C-lattice domain is a *Prüfer lattice* if every compact element is principal (i.e., meet-principal and join-principal). In a C-lattice domain, every nonzero join-principal element x is cancellative (because $(yx : x) = y \vee (0 : x) = y$ for each y). So in a Prüfer lattice domain every nonzero compact element is CMP.

Corollary 6. *Let L be a C-lattice domain in which every nonzero compact element is CMP (e.g., a Prüfer lattice domain). Then conditions* (i) *and* (ii) *of Theorem 5 are equivalent.*

Bazzoni's conjecture for Prüfer domains [10, Theorem 10] (see Introduction) follows from Corollary 6 since the ideal lattice of a Prüfer domain is clearly a Prüfer lattice.

3 Almost Prüfer Extensions

We recall several definitions from [6, 12]. Let $A \subseteq B$ be a commutative ring extension and I an ideal of A. Then I is called *B-regular* if $IB = B$ and I is called *B-invertible* if $IJ = A$ for some A-submodule J of B. Every B-invertible ideal is B-regular, since $A = IJ \subseteq IB$ implies $IB = B$. We say that $A \subseteq B$ is an *almost Prüfer extension* if every finitely generated B-regular ideal of A is B-invertible.

Finocchiaro and Tartarone [6, Theorem 4.5] proved Bazzoni's conjecture for almost Prüfer ring extensions. We state their result and derive it from Corollary 6.

Theorem 7. (Finocchiaro and Tartarone) *If $A \subseteq B$ is an almost Prüfer extension, the following are equivalent:*

(i) *Every B-regular locally principal ideal of A is B-invertible.*

(ii) *Every B-regular ideal of A is contained in only finitely many maximal ideals of A.*

Proof. It is well-known and easy to prove that (ii) implies (i), see [6, Corollary 3.5]. We prove the converse. Let L be the set of all B-regular ideals of A together with the zero ideal and order L by inclusion. As shown in [15, Lemma 7.1], L is a C-lattice domain under usual ideal multiplication, where the join is the ideal sum and the meet is the ideal intersection except the case when we get a non-B-regular ideal when we put $\bigwedge = 0$. By [15, Lemma 7.1], the set L^* of compact elements in L is exactly the set of (B-regular) finitely generated ideals of A together with the zero ideal. After this preparation it becomes clear that $[(i) \Rightarrow (ii)]$ follows from Corollary 6 provided we prove the two claims below. Write $x \in L$ as $\widehat{x}$ when considered as an ideal of A.

Claim 1: *Every nonzero compact element of L is a CMP element.*

Let c be a nonzero compact element of L. As $A \subseteq B$ is almost Prüfer, $\widehat{c}$ is a B-invertible ideal, so $\widehat{c}J = A$ for some A-submodule J of B. Then c is clearly cancellative. By Lemma 3, it suffices to show that $(x : c)c = x \wedge c$ for each $x \in L$. Changing x by $x \wedge c$, we may assume that $x \leq c$. We have $\widehat{x} = \widehat{x}J\widehat{c}$, so $x = yc$ where $y \in L$ is such that $\widehat{y} = \widehat{x}J$ (note that $\widehat{x}J \subseteq A$). From $x = yc$ we get $y \leq (x : c)$, so $x = yc \leq (x : c)c \leq x$, thus $(x : c)c = x$.

Claim 2: *Every locally compact element of L is compact.*

Suppose that c is a nonzero locally compact element of L. Let m be a maximal element of L, that is, $\widehat{m}$ is a B-regular maximal ideal of A. So $c_m = \bigvee\{y \in L^*;\ ys \leq c$ for some $s \in L^*, s \not\leq m\}$ is compact in the lattice $L_m = \{x_m;\ x \in L\}$. Then $c_m = h_m$ for some $h \in L^*$. Extending these ideals in $A_{\widehat{m}}$, we get $\widehat{c}A_{\widehat{m}} = \widehat{c_m}A_{\widehat{m}} = \widehat{h_m}A_{\widehat{m}} = \widehat{h}A_{\widehat{m}}$. Since $A \subseteq B$ is almost Prüfer, $\widehat{h}$ is B-invertible. By [12, Proposition 2.3], $\widehat{h}A_{\widehat{m}} = \widehat{c}A_{\widehat{m}}$ is a principal ideal of $A_{\widehat{m}}$. Thus $\widehat{c}$ is a locally principal ideal of A. By (i), $\widehat{c}$ is B-invertible, so $\widehat{c}$ is finitely generated, cf. [12, Proposition 2.3]. Thus c is compact in L. □

4 Ideal Systems on Monoids and Integral Domains

Let H be a commutative multiplicative monoid (with zero element 0 and unit element 1) such that every nonzero element of H is cancellative and let $\mathcal{P}(H)$ be the power set of H. A map $r : \mathcal{P}(H) \to \mathcal{P}(H)$, $X \mapsto X_r$, is an *ideal system* on H if the following conditions hold for all $X, Y \in \mathcal{P}(H)$ and $c \in H$:

$$(1)\ cX_r = (cX)_r,\ (2)\ X \subseteq X_r,\ (3)\ X \subseteq Y \text{ implies } X_r \subseteq Y_r,\ (4)\ (X_r)_r = X_r.$$

Then X_r is called the *r-closure* of X and a set of the form X_r is called an r-ideal. An r-ideal I is *r-finite* if $I = Y_r$ for some finite subset Y of I. The ideal system r is called *finitary* if $X_r = \bigcup\{Y_r;\ Y \subseteq X \text{ finite}\}$.

Assume that r is finitary. A proper r-ideal P is *prime* if, for $x, y \in H$, $xy \in P$ implies $x \in P$ or $y \in P$. The *localization of H at P* is the fraction monoid $H_P = \{x/t;\ x \in H, t \in H - P\}$ which comes together with the finitary ideal system r_P defined by $\{a_1/s_1, ..., a_n/s_n\}_{r_P} = (\{a_1, ..., a_n\}_r)_P$ for all $a_1, ..., a_n \in H$ and $s_1, ..., s_n \in H - P$. A *maximal r-ideal* is a maximal element of the set of proper r-ideals of H. Any proper r-ideal is contained in a maximal one and a maximal r-ideal is prime. A nonzero r-ideal I is *r-invertible* if I is r-finite and *r-locally principal* (i.e., $I_M = \{x/t;\ x \in I, t \in H - M\} = yH_M$ with $y \in I_M$ (depending on M) for all maximal r-ideals M). Next H is an *r-Prüfer monoid* if every nonzero r-finite r-ideal of H is r-invertible. For complete details we refer to [8].

Halter-Koch [9, Theorem 6.11] proved Bazzoni's conjecture for r-Prüfer monoids. We state his result and derive it from Corollary 6.

Theorem 8. (Halter-Koch) *If H is an r-Prüfer monoid for some finitary ideal system r on H, the following are equivalent.*

(i) *Every r-locally principal r-ideal of H is r-finite.*

(ii) *Every nonzero r-ideal of H is contained in only finitely many maximal r-ideals.*

Proof. It is well-known and easy to prove that (ii) implies (i). We prove the converse. Let L be the set of all r-ideals of H ordered by inclusion. As shown in [8, Chapter 8], L is a C-lattice domain under r-ideal multiplication $(I, J) \mapsto (IJ)_r$, join $\bigvee\{J_\alpha\} := (\bigcup J_\alpha)_r$ and meet $\bigwedge\{J_\alpha\} := \bigcap J_\alpha$. By [15, Lemma 8.1], the set L^* of compact elements in L is exactly the set of r-finite r-ideals of H. After this preparation it becomes clear that $[(i) \Rightarrow (ii)]$ follows from Corollary 6 provided we prove the two claims below. Write $x \in L$ as $\widehat{x}$ when considered as an r-ideal of H.

Claim 1: *Every nonzero compact element of L is a CMP element.*
Let $c \in L$ be a nonzero compact, in other words $\widehat{c}$ is nonzero r-finite r-ideal. Since H is r-Prüfer, $\widehat{c}$ is r-invertible. So c is a CMP element of L, cf. [15, Lemma 8.2].

Claim 2: *Every locally compact element of L is compact.*
Suppose that c is a locally compact element of L. Let m be a maximal element of L, that is, $\widehat{m}$ is a maximal r-ideal of H. So $c_m = \bigvee\{y \in L^*;\ ys \leq c$ for some $s \in L^*, s \nleq m\}$ is compact in the lattice $L_m = \{x_m;\ x \in L\}$. Then $c_m = h_m$ for some $h \in L^*$. Switching to H, we have $(\widehat{c})_{\widehat{m}} = (\widehat{c_m})_{\widehat{m}} = (\widehat{h_m})_{\widehat{m}} = \widehat{h}_{\widehat{m}}$. Since H is an r-Prüfer monoid, $\widehat{h}_{\widehat{m}}$ is a principal $r_{\widehat{m}}$-ideal of $H_{\widehat{m}}$, cf. [8, Theorem 12.3]. Thus $\widehat{c}$ is an r-locally principal r-ideal. By (i), $\widehat{c}$ is r-finite, thus c is compact in L. □

Next we present an application for integral domains. Let D be an integral domain. The t ideal system on D is defined by $X_t = \bigcup\{Y_v;\ Y \subseteq X$ finite$\}$ for all $X \subseteq D$, where $Y_v = \bigcap\{(aD :_D b);\ a, b \in D, bY \subseteq aD\}$. Clearly t is finitary, so the set $Max_t(D)$ of maximal t-ideals is nonempty, see [8, Chapter 11].

The w ideal system on D is defined by $X_w = \bigcap\{XD_M;\ M \in Max_t(D)\}$ for all $X \subseteq D$, where XD_M is the ideal generated by X in D_M. So a w-ideal is an ideal of the ring D. For $X \subseteq D$, we have $X_w = (XD)_w$ and $X_w D_M = XD_M$ for each $M \in Max_t(D)$. Moreover, w is finitary and the set of maximal w-ideals is exactly $Max_t(D)$. A w-finite ideal has the form $((a_1, ..., a_n)D)_w$ for some a_i's in D. And a nonzero w-ideal is *w-invertible* if it is w-finite and t-locally principal (i.e., ID_M is a principal ideal of D_M for each $M \in Max_t(D)$). For details on the w ideal system we refer to [2, 5].

According to [13], D is an *almost Prüfer v-multiplication domain* (in short *APVMD*) if for every $a_1, ..., a_n \in D - \{0\}$, the ideal $((a_1^k, ..., a_n^k)D)_w$ is w-invertible for some $k \geq 1$. Say that a w-ideal I of D is *t-locally finitely generated*, if ID_M is a finitely generated ideal of D_M for each $M \in Max_t(D)$.

Chang and Hamdi [4, Theorem 2.4] proved Bazzoni's conjecture for APVMDs. We state their result and derive it from Theorem 5.

Theorem 9. (Chang and Hamdi) *For an APVMD D, the following statements are equivalent:*

(i) *Each nonzero t-locally finitely generated w-ideal of D is w-finite.*

(ii) *Every nonzero ideal of D is contained in only finitely many maximal t-ideals.*

Proof. It is well-known and easy to prove that (ii) implies (i), see for instance the proof of (3) ⇒ (1) in [4, Theorem 2.4]. We prove that (i) implies (ii). Let L be the set of all w-ideals of D ordered by inclusion. As shown in [8, Chapter 8], L is a C-lattice domain under w-ideal multiplication $(I, J) \mapsto (IJ)_w$, join $\bigvee\{J_\alpha\} := (\bigcup J_\alpha)_w$ and meet $\bigwedge\{J_\alpha\} := \bigcap J_\alpha$. By [15, Lemma 8.1], the set L^* of compact elements in L is exactly the set of w-finite ideals of D. Condition (a) of Theorem 5 holds clearly for L (any nonzero ideal contains a nonzero principal ideal). After this preparation it becomes clear that $[(i) \Rightarrow (ii)]$ follows from Theorem 5 provided we prove the two claims below. Write $x \in L$ as $\widehat{x}$ when considered as a w-ideal of D.

Claim 1: *Every* $d \in L^* - \{1\}$ *has some power* d^n *below some proper CMP element.*
Let $d \in L^* - \{1\}$. Then $\widehat{d} = (a_1, ..., a_k)_w$ for some elements $a_i \in \widehat{d}$. Since D is an APVMD, $(a_1^s, ..., a_k^s)_w$ is w-invertible for some $s \geq 1$. If $(a_1^s, ..., a_k^s)_w = \widehat{f}$ with $f \in L$, then f is a proper CMP element of L, cf. [15, Lemma 8.2]. Moreover $d^{sk} \leq f$, so Claim 1 is proved.

Claim 2: *Every locally compact element of L is compact.*
Suppose that c is a locally compact element of L. Let m be a maximal element of L, that is, $\widehat{m}$ is a maximal w-ideal of D. So $c_m = \bigvee\{y \in L^*;\ ys \leq c$ for some $s \in L^*, s \not\leq m\}$ is compact in the lattice $L_m = \{x_m;\ x \in L\}$. Then $c_m = h_m$ for some $h \in L^*$. Hence $\widehat{h} = (b_1, ..., b_n)_w$ for some elements $b_i \in \widehat{h}$. Switching to D, we have

$$\widehat{c}D_{\widehat{m}} = \widehat{c_m}D_{\widehat{m}} = (\widehat{h_m})D_{\widehat{m}} = \widehat{h}D_{\widehat{m}} = (b_1, ..., b_n)_w D_{\widehat{m}} = (b_1, ..., b_n)D_{\widehat{m}}.$$

Thus $\widehat{c}$ is a t-locally finitely generated ideal of D. By (i), $\widehat{c}$ is w-finite, thus c is compact in L. □

Acknowledgements I thank the referee for the close reading of the manuscript and for the detailed report which heavily improved the quality of this paper.

References

1. Anderson, D.D.: Abstract commutative ideal theory without chain conditions. Algebra Univ. **6**, 131–145 (1976)
2. Anderson, D.D., Cook, S.J.: Two star-operations and their induced lattices. Comm. Algebra **28**, 2461–2475 (2000)
3. Bazzoni, S.: Class semigroups of Prüfer domains. J. Algebra **184**, 613–631 (1996)
4. Chang, G.W. Hamdi, H.: Bazzoni's conjecture and almost Prüfer domains, to appear in Comm. Algebra
5. Fanggui, W., McCasland, R.L.: On w-modules over strong Mori domains. Comm. Algebra **25**, 1285–1306 (1997)
6. Finnochiaro, C., Tartarone, F.: Invertibility of ideals in Prüfer extensions. Comm. Algebra **45**, 4521–4527 (2017)
7. Gilmer, R.: Multiplicative Ideal Theory. Marcel Dekker, New York (1972)
8. Halter-Koch, F.: Ideal System. An Introduction to Multiplicative Ideal Theory, Marcel Dekker, New York (1998)
9. Halter-Koch, F.: Clifford semigroups of ideals in monoids and domains. Forum Math. **21**, 1001–1020 (2009)
10. Holland, W.C., Martinez, J., McGovern, W.W., Tesemma, M.: Bazzoni's conjecture. J. Algebra **320**, 1764–1768 (2008)
11. Jayaram, C., Johnson, E.G.: S-prime elements in multiplicative lattices. Periodica Math. Hungarica **31**, 201–208 (1995)
12. Knebusch, M., Zhang, D.: Manis Valuations and Prüfer Extensions I: A New Chapter in Commutative Algebra. Springer, Berlin, Heidelberg (2002)
13. Lewin, R.L.: Generalizations of GCD-domains and related topics, Ph.D. Thesis, University of Iowa (1991)

14. McGovern, W.W.: Prüfer domains with Clifford class semigroup. J. Comm. Algebra **3**, 551–559 (2011)
15. Olberding, B., Reinhart, A.: Radical factorization in commutative rings, monoids and multiplicative lattices, Math. (2018). arXiv: 1811.00242v1, math.AC
16. Zafrullah, M.: t-invertibility and Bazzoni-like statements. J. Pure Appl. Algebra **214**, 654–657 (2010)