Tilting Modules and Tilting Torsion Pairs

Filtrations Induced by Tilting Modules

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Abstract Tilting modules, generalising the notion of progenerator, furnish equivalences between pieces of module categories. This paper is dedicated to study how much these pieces say about the whole category. We will survey the existing results in the literature, introducing also some new insights.

Keywords Tilting modules · Torsion pairs · t-structures · t-tree

1 Introduction

In 1958, Morita characterised equivalences between the entire categories of left (or right) modules over two rings. Let A be an arbitrary associative ring with $1 \neq 0$. A left A-module ${}_AP$ is a *progenerator* if it is projective, finitely generated and generates the category A-Mod of left A-modules. Set $B := \mathrm{End}({}_AP)$, the covariant functor $\mathrm{Hom}_A(P, ?)$ gives an equivalence between A-Mod and B-Mod; moreover, any equivalence between modules categories is of this type.

The notion of tilting module has been axiomatised in 1979 by Brenner and Butler [BB], generalising that of progenerator for modules of projective dimension 1. The various forms of generalisations to higher projective dimensions considered until today continue to follow their approach.

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A tilting module T of projective dimension n naturally gives rise to $n+1$ corresponding classes of modules in A-Mod and B-Mod, the Miyashita classes, with $n+1$ equivalences between them. These classes are

$$\mathrm{KE}_e(T) = \{M \in A\text{-Mod} : \mathrm{Ext}^i_A(T, M) = 0\ \forall i \neq e\}$$

$$\mathrm{KT}^e(T) = \{N \in B\text{-Mod} : \mathrm{Tor}^B_i(T, M) = 0\ \forall i \neq e\}, \quad e = 0, 1, ..., n$$

and the $n+1$ equivalences are

$$KE_e(T) \underset{\mathrm{Tor}^B_e(T,?)}{\overset{\mathrm{Ext}^e_A(T,?)}{\rightleftarrows}} KT^e(T)\ , \quad e = 0, 1, ..., n.$$

In the $n = 0$ case (progenerator), there is only one class on each side, and so every module is subject to the equivalence of categories (that of Morita); for $n = 1$, on each side, the two Miyashita classes form torsion pairs, so every module in both A-Mod and B-Mod can be decomposed in terms of modules in the Miyashita classes: precisely every module admits a composition series of length 2 with composition factors in the Miyashita classes.

For $n > 1$, the Miyashita classes fail to decompose every module; the way to recover a similar decomposition is the subject of this paper.

In Section 2, we define classical n-tilting modules and Miyashita classes; we show that they give a torsion pair for $n = 1$, and hence they can be used to decompose every module; we give an example showing that a similar decomposition does not exist for $n > 1$, and characterise those modules which can be decomposed.

In Section 3, we present some previous attempts to recover the decomposition for $n > 1$ as well, by extending the Miyashita classes, due to Jensen, Madsen, Su [11] and to Lo [13]. A useful tool in our analysis will be a characterisation of modules in $\bigcap_{i>e} \mathrm{Ker}\,\mathrm{Ext}^i_A(T, ?)$, $0 \leq e \leq n$ (see Lemma 1), which generalises the characterisation of modules in $\bigcap_{i>0} \mathrm{Ker}\,\mathrm{Ext}^i_A(T, ?)$ given by Bazzoni [3, Lemma 3.2]. These extensions deform in an irreversible way the Miyashita classes, weakening their role.

In Section 4, we recall some introductory notions about the derived category of an abelian category and about t-structures.

In Section 5, we drop the finiteness assumptions on the tilting modules, recalling the definition of non classical n-tilting modules [2]. In this setting, we recall the definition of the t-structure associated to such a module; we then study its interaction with the natural t-structure of the derived category.

In Section 6, we exploit the results of Section 5 to construct in the derived category the t-tree of a module with respect to a tilting module. This procedure, discovered in the classical tilting case by Fiorot, the first and the third author in [8], solves satisfactorily the decomposition problem for $n > 1$: the classes used for the decomposition intersect the module category exactly in the Miyashita classes. As a result of the work of the previous section, we prove that this construction can be reproduced also in the non classical case.

Throughout the paper, the concrete case considered in Example 1 introduced in Section 2 will be used to illustrate the various attempts to solve the decomposition problem (see Examples 2, 3).

2 Classical *n*-tilting Modules

In 1986, Miyashita [14] and Cline, Parshall and Scott [7] gave similar definitions of a *tilting module of projective dimension n*.

Definition 1 (Miyashita [14]) A left A-module T is a classical n-tilting module, for some integer $n \geq 0$, if

p_n) T has a finitely generated projective resolution of length n, i.e. a projective resolution

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow T \longrightarrow 0$$

with the P_i finitely generated;
e_n) T is rigid, i.e. $\mathrm{Ext}^i_A(T, T) = 0$ for every $0 < i \leq n$;
g_n) the ring A admits a coresolution of length n

$$0 \longrightarrow A \longrightarrow T_0 \longrightarrow \cdots \longrightarrow T_n \longrightarrow 0$$

with the T_i finitely generated direct summands of arbitrary coproducts of copies of T.

In the case when $n = 0$, p_0) says that the module is a finitely generated projective, e_0) is empty and g_0) says that it is a generator: this is then the definition of a progenerator module. As such, a classical 0-tilting module T induces a Morita equivalence of categories of modules, as follows. Let $B = \mathrm{End}_A(T)$ be its ring of endomorphisms, which acts on the right on T, and consider the category B-Mod of left B-modules. There are functors

$$\begin{aligned} \mathrm{Hom}_A(T, ?) :&\quad A\text{-Mod} \to B\text{-Mod} \\ T \otimes_B ? :&\quad B\text{-Mod} \to A\text{-Mod} \end{aligned}$$

which are category equivalences, with the unit and counit morphisms being those of the adjunction. This is the motivating example for the definition of tilting modules, along with the next case.

In the case when $n = 1$, we find what was originally (see Brenner and Butler [6]) defined as a *tilting module*; we will give a brief and incomplete overview of what is known about them.

Let T be a classical 1-tilting left A-module, and let as before $B = \mathrm{End}_A(T)$ be its ring of endomorphisms. In this case, the pair $(\mathrm{Hom}_A(T, ?), T \otimes_B ?)$ does not induce an equivalence of A-Mod and B-Mod anymore; however, a little less can be proved, as follows.

Define the following pairs of full subcategories of A-Mod and B-Mod, respectively,

$$\begin{aligned} KE_0(T) &= \left\{X \in A\text{-Mod} : \ \mathrm{Ext}^1_A(T, X) = 0\right\} \\ KE_1(T) &= \{X \in A\text{-Mod} : \ \mathrm{Hom}_A(T, X) = 0\} \\ KT^0(T) &= \left\{Y \in B\text{-Mod} : \ \mathrm{Tor}^B_1(T, Y) = 0\right\} \\ KT^1(T) &= \{Y \in B\text{-Mod} : \ T \otimes_B Y = 0\}\,. \end{aligned}$$

Then we have the following results.

Theorem 1 (Brenner and Butler [6]) *Let A be a ring, T a classical 1-tilting left A-module, $B = \mathrm{End}_A(T)$.*

i) The pairs $(KE_0(T), KE_1(T))$ and $(KT^1(T), KT^0(T))$ defined above are torsion pairs, respectively, in A-Mod and B-Mod.
ii) There are equivalences of (sub)categories

$$KE_0(T) \underset{T\otimes_B ?}{\overset{\mathrm{Hom}_A(T,?)}{\rightleftarrows}} KT^0(T)$$

$$KE_1(T) \underset{\mathrm{Tor}^B_1(T,?)}{\overset{\mathrm{Ext}^1_A(T,?)}{\rightleftarrows}} KT^1(T)\ .$$

This theorem shows that the 1-tilting case is slightly more complex than the 0-tilting one. Instead of having an equivalence of the whole categories A-Mod and B-Mod, we have two pairs of equivalent subcategories, giving a functorial decomposition of every module in its torsion and torsion free parts.

For an arbitrary $n \geq 0$, following Miyashita, we find that every classical n-tilting module T gives rise to two sets of $n + 1$ full subcategories of A-Mod and B-Mod, respectively, defined as follows for $e = 0, \ldots, n$:

$$\begin{aligned} KE_e(T) &= \left\{X \in A\text{-Mod} : \ \mathrm{Ext}^i_A(T, X) = 0 \text{ for every } i \neq e\right\} \ \subset A\text{-Mod} \\ KT^e(T) &= \left\{Y \in B\text{-Mod} : \ \mathrm{Tor}^B_i(T, Y) = 0 \text{ for every } i \neq e\right\} \ \subset B\text{-Mod} \end{aligned}$$

where conventionally $\mathrm{Ext}^0_A(T, X) = \mathrm{Hom}_A(T, X)$ and $\mathrm{Tor}^B_0(T, Y) = T \otimes_B Y$. As a generalisation of point (ii) of Theorem 1, we may state the following result.

Theorem 2 (Miyashita [14, Theorem 1.16]) *In the setting above, there are equivalences of (sub)categories, for every $e = 0, \dots, n$:*

$$KE_e(T) \underset{\mathrm{Tor}_e^B(T,?)}{\overset{\mathrm{Ext}_A^e(T,?)}{\rightleftarrows}} KT^e(T)\ .$$

For $n \geq 2$, however, the Miyashita classes do not provide a decomposition of every module, as it used to happen for $n = 1$. This is proved by the existence of simple modules (which can have only a trivial decomposition in the module category) not belonging to any class.

Example 1 ([20, Example 2.1]) Let k be an algebraically closed field. Let A be the k-algebra associated to the quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$ with the relation $b \circ a = 0$. The indecomposable projectives are $\begin{smallmatrix}1\\2\end{smallmatrix}$, $\begin{smallmatrix}2\\3\end{smallmatrix}$, $\begin{smallmatrix}3\end{smallmatrix}$, while the indecomposable injectives are $\begin{smallmatrix}1\end{smallmatrix}$, $\begin{smallmatrix}1\\2\end{smallmatrix}$, $\begin{smallmatrix}2\\3\end{smallmatrix}$. It follows that the module $T = \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\end{smallmatrix}$ is a classical 2-tilting module: a p_2) resolution is

$$P^\bullet \to T \to 0: \qquad 0 \to 0 \oplus 0 \oplus \begin{smallmatrix}3\end{smallmatrix} \to 0 \oplus 0 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \to \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\end{smallmatrix} \to 0\,;$$

T is a direct sum of injectives, so it is rigid; lastly, $A = \begin{smallmatrix}3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix}$ and so a g_2) co-resolution can be easily found. We shall show that the simple module $\begin{smallmatrix}2\end{smallmatrix}$ does not belong to any of the Miyashita classes.

In order to compute the $\mathrm{Ext}_A^i(T, \begin{smallmatrix}2\end{smallmatrix})$, we apply the contravariant functor $\mathrm{Hom}_A(?, \begin{smallmatrix}2\end{smallmatrix})$ to the resolution $P^\bullet$, obtaining

$$0 \to \mathrm{Hom}_A(\begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}2\end{smallmatrix}) \to \mathrm{Hom}_A(0 \oplus 0 \oplus \begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}2\end{smallmatrix}) \to \mathrm{Hom}_A(0 \oplus 0 \oplus \begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}2\end{smallmatrix}) \to 0$$

which is isomorphic to

$$0 \longrightarrow \mathrm{Hom}_A(\begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}2\end{smallmatrix}) \xrightarrow{0} \mathrm{Hom}_A(\begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}2\end{smallmatrix}) \xrightarrow{0} 0 \longrightarrow 0\ .$$

Hence, $\mathrm{Hom}_A(T, \begin{smallmatrix}2\end{smallmatrix}) \simeq \mathrm{Ext}_A^1(T, \begin{smallmatrix}2\end{smallmatrix}) \simeq \mathrm{Hom}_A(\begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}2\end{smallmatrix}) \neq 0$ as abelian groups.

Indeed, those modules for which the $KE_i(T)$ (resp. the $KT^i(T)$) provide a decomposition can be characterised in the following way.

Definition 2 A left A-module M (resp. a left B-module N) is *sequentially static* (resp. *costatic*) if for every $i \neq j \geq 0$,

$$\mathrm{Tor}_i^B(T, \mathrm{Ext}_A^j(T, M)) = 0 \quad (\text{resp. } \mathrm{Ext}_B^i(T, \mathrm{Tor}_j^A(T, N)) = 0).$$

Notice that for an A-module M (resp. a B-module N) to be sequentially static (resp. costatic) means that for every $e = 0, \dots, n$, we have that $\mathrm{Ext}_A^e(T, M)$ belongs to KT^e (resp. $\mathrm{Tor}_e^B(T, N)$ belongs to KE_e).

Proposition 1 ([20, Theorem 2.3]) *A left A-module M is sequentially static if and only if there exists a filtration*

$$M = M_n \geq M_{n-1} \geq M_{n-2} \geq \cdots \geq M_0 \geq M_{-1} = 0$$

such that for every $i = 0, \ldots, n$, the quotient M_i/M_{i-1} belongs to $KE_i(T)$. In this case, for every such i, we have that $M_i/M_{i-1} \simeq \mathrm{Tor}_i^B(T, \mathrm{Ext}_A^i(T, M))$.

Dually, a left B-module M is sequentially costatic if and only if there exists a filtration

$$N = N_{-1} \geq N_0 \geq N_1 \geq \cdots \geq N_{n-1} \geq N_n = 0$$

such that for every $i = 0, \ldots, n$, the quotient N_{i-1}/N_i belongs to $KT^i(T)$. In this case, for every such i, we have that $N_{i-1}/N_i \simeq \mathrm{Ext}_A^i(T, \mathrm{Tor}_i^B(T, N))$.

Remark 1 In Example 1, the module 2 was not sequentially static. Let us check that

$$\mathrm{Tor}_2^B(T, \mathrm{Hom}_A(T, 2)) \neq 0.$$

The ring $B = \mathrm{End}_A(T)$ (with multiplication the composition left to right) is the k-algebra associated to the quiver $4 \xrightarrow{c} 5 \xrightarrow{d} 6$ with the relation $d \circ c = 0$. In detail, the idempotents are the endomorphisms of T induced by the identities of its direct summands, e_4 of 1, e_5 of $\begin{smallmatrix}1\\2\end{smallmatrix}$ and e_6 of $\begin{smallmatrix}2\\3\end{smallmatrix}$, respectively; and c and d are the endomorphisms of T induced by the morphisms $\begin{smallmatrix}1\\2\end{smallmatrix} \to 1$ and $\begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix}$, respectively.

In order to compute the right B-module structure of T, we notice first that as a k-vector space T is generated by five elements: $x \in \begin{smallmatrix}2\\3\end{smallmatrix} \setminus 3$ and $y = bx \in 3$, $v \in \begin{smallmatrix}1\\2\end{smallmatrix} \setminus 2$ and $w = av \in 2$, and $z \in 1$. If we look at how B acts on the right on these elements, we see that T as a right B-module is isomorphic to $\begin{smallmatrix}5\\4\end{smallmatrix} \oplus \begin{smallmatrix}6\\5\end{smallmatrix} \oplus 6 = \begin{smallmatrix}v\\z\end{smallmatrix} \oplus \begin{smallmatrix}x\\w\end{smallmatrix} \oplus y$.

To compute $\mathrm{Ext}_A^1(T, 2)$, we consider the injective coresolution of 2 in A-Mod

$$0 \longrightarrow 2 \longrightarrow \begin{smallmatrix}1\\2\end{smallmatrix} \longrightarrow 1 \longrightarrow 0$$

and compute coker $\left[\mathrm{Hom}_A(T, \begin{smallmatrix}1\\2\end{smallmatrix}) \to \mathrm{Hom}_A(T, 1)\right]$ as left B-modules.

The left B-module $\mathrm{Hom}_A\left(T, \begin{smallmatrix}1\\2\end{smallmatrix}\right)$ is generated as a k-vector space by (the morphisms induced on T by) two morphisms $\begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix}$ and $\begin{smallmatrix}1\\2\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix}$. When we look at how B acts on the left on these elements, we see that the module is isomorphic to $_B\left(\begin{smallmatrix}5\\6\end{smallmatrix}\right)$. Similarly, it can be seen that $\mathrm{Hom}_A(T, 1)$ as a left B-module is isomorphic to $\begin{smallmatrix}4\\5\end{smallmatrix}$, hence the cokernel we are interested in is the simple 4. To compute $\mathrm{Tor}_2^B(T, 4)$, we now consider the presentation

$$0 \longrightarrow 5 \longrightarrow \begin{smallmatrix}4\\5\end{smallmatrix} \longrightarrow 4 \longrightarrow 0$$

where $\begin{smallmatrix}4\\5\end{smallmatrix}$ is a projective left B-module. It can be easily seen that $\mathrm{Tor}_2^B(T, 4) \simeq \mathrm{Tor}_1^B(T, 5)$. Take the injective coresolution of ${}_B 5$

$$0 \longrightarrow 6 \longrightarrow \begin{smallmatrix}5\\6\end{smallmatrix} \longrightarrow 5 \longrightarrow 0\ ;$$

similarly to what we did to compute $\mathrm{Ext}_A^1(T, 2)$, we can compute $\mathrm{Tor}_1^B(T, 5)$ as the kernel of $T \otimes_B 6 \to T \otimes_B \begin{smallmatrix}5\\6\end{smallmatrix}$ as a morphism of left A-modules.

If we call t a generator of 6, with the previous notation for the generators of T_B, as a k-vector space $T \otimes_B 6$ is generated by $v \otimes t, z \otimes t, x \otimes t, w \otimes t, y \otimes t$. Since however $e_6 t = t$, the only generators of these which are not zero are $x \otimes t = xe_6 \otimes t$ and $y \otimes t = ye_6 \otimes t$. If we look at the action of A on the left of these elements, we deduce that $T \otimes_B 6$ is isomorphic to $\begin{smallmatrix}2\\3\end{smallmatrix}$ as a left A-module. Similarly, $T \otimes_B \begin{smallmatrix}5\\6\end{smallmatrix}$ turns out to be isomorphic to 2, so in the end

$$\mathrm{Tor}_2^B(T, \mathrm{Hom}_A(T, 2)) \simeq 3 \neq 0.$$

3 First Attempts to Recover the Decomposition

In order to recover a decomposition of every module induced by a classical n-tilting module, different strategies have been proposed.

In [11], Jensen, Madsen and Su suggested a solution for the $n = 2$ case by enlarging the subcategories KE_0, KE_1, KE_2 in the following way. Let $\mathcal{K}_0$ be the full subcategory of cokernels of monomorphisms from objects in KE_2 to objects in KE_0; let $\mathcal{K}_1$ be KE_1; let $\mathcal{K}_2$ be the full subcategory of kernels of epimorphisms from objects in KE_2 to objects in KE_0:

$$\begin{aligned}
\mathcal{K}_0 &= \left\{ \operatorname{coker} f :\ X_2 \overset{f}{\hookrightarrow} X_0, \quad X_2 \in KE_2, X_0 \in KE_0 \right\} \\
\mathcal{K}_1 &= KE_1 \\
\mathcal{K}_2 &= \left\{ \ker g :\ X_2 \overset{g}{\twoheadrightarrow} X_0, \quad X_2 \in KE_2, X_0 \in KE_0 \right\}.
\end{aligned}$$

By considering the morphisms $f :\ 0 \hookrightarrow X_0$ and $g :\ X_2 \twoheadrightarrow 0$, we can see that $KE_i \subset \mathcal{K}_i$ for every $i = 0, 1, 2$, so this is indeed an enlargement.

Now, for $i = 0, 1, 2$, let $\mathcal{E}_i$ be the extension closure of $\mathcal{K}_i$, i.e. the smallest subcategory containing $\mathcal{K}_i$ and closed under extensions.

Proposition 2 (*[11, Corollary 15, Theorem 19, Lemma 24]*) *For any left A-module X, there exists a unique filtration*

$$0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 = X$$

with the quotients $X_{i+1}/X_i \in \mathcal{E}_i$ for every $i = 0, 1, 2$. Moreover, such a filtration is functorial.

Example 2 Let us apply this construction to find a decomposition of the simple module 2 considered in the Example 1. In a way similar to that used to study the $\mathrm{Ext}^i_A(T, 2), i = 0, 1, 2$, we may prove that $\frac{2}{3}$ belongs to KE_0 and 3 belongs to KE_2. Then, 2 belongs to $\mathcal{K}_0 \subseteq \mathcal{E}_0$, being the cokernel of the monomorphism $3 \to \frac{2}{3}$. Therefore, the trivial filtration $0 \leq 2$ has its only filtration factor in the new class $\mathcal{E}_0$.

In [13], Lo generalised this filtration to the $n > 2$ case as well. After giving a different proof of Proposition 2, he introduced the following subcategories. For a class of objects $\mathcal{S}$, denote by $[\mathcal{S}]$ the extension closure of the full subcategory of quotients of objects of $\mathcal{S}$:

$$[\mathcal{S}] = \langle\{X : \exists(S \twoheadrightarrow X) \text{ for some } S \in \mathcal{S}\}\rangle_{\text{ext}}.$$

This subcategory is closed under quotients ([13, Lemma 5.1]). Then set, for $i = 0, \ldots, n$:

$$\mathcal{T}_i = \left[\mathrm{Ker}\,\mathrm{Ext}^i_A(T, ?) \cap \cdots \cap \mathrm{Ker}\,\mathrm{Ext}^n_A(T, ?)\right]$$
$$\mathcal{F}_i = \mathrm{Ker}\,\mathrm{Hom}_A(\mathcal{T}_i, ?) = \{X : \mathrm{Hom}_A(\mathcal{T}_i, X) = 0\}$$

with our usual convention that $\mathrm{Ext}^0_A = \mathrm{Hom}_A$. Define also $\mathcal{T}_{n+1} = A$-Mod and $\mathcal{F}_{n+1} = 0$.

This provides pairs $(\mathcal{T}_i, \mathcal{F}_i)$ of full subcategories, which are torsion pairs since the $\mathcal{T}_i$'s are closed under extensions and quotients (see Polishchuk [16]). The following easy proposition can then be applied to these torsion pairs.

Proposition 3 ([13, Theorem 5.3]) *Let $(\mathcal{T}_i, \mathcal{F}_i)$ be torsion pairs in A-Mod, for $i = 0, \ldots, n+1$, such that*

$$0 = \mathcal{T}_0 \subseteq \mathcal{T}_1 \subseteq \cdots \subseteq \mathcal{T}_{n+1} = A\text{-}Mod.$$

Then for every left A-module X, there exists a functorial filtration

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n+1} = X$$

such that $X_i \in \mathcal{T}_i$ for $i = 0, \ldots, n+1$ and $X_i/X_{i-1} \in \mathcal{T}_i \cap \mathcal{F}_{i-1}$ for $i=1, \ldots, n+1$. Moreover, the $\mathcal{T}_i \cap \mathcal{F}_{i-1}$ have pairwise trivial intersection.

We now prove that the subcategories $\mathcal{T}_i \cap \mathcal{F}_{i-1}$ introduced by Lo are indeed enlargements of the Miyashita classes using the following generalisation of [3, Lemma 3.2], which we find of independent interest.

Lemma 1 *Let X be a module belonging to $\bigcap_{i>e} \operatorname{Ker}\operatorname{Ext}^i_A(T, X)$ for some $0 \leq e \leq n$. Then, there exists a sequence of direct summands of coproducts of copies of T,*

$$\cdots \longrightarrow T_{-1} \xrightarrow{d_{-1}} T_0 \xrightarrow{d_0} \cdots \longrightarrow T_e \longrightarrow 0$$

which is exactly everywhere except for degree 0, and having $\ker d_0/\operatorname{im} d_{-1} \simeq X$. *In particular, for $e = n$, $\bigcap_{i>n} \operatorname{Ker}\operatorname{Ext}^i_A(T, X) = A$-Mod and hence X may be any module.*

Proof Set $T^{\perp_\infty} := \bigcap_{i>0} \operatorname{Ker}\operatorname{Ext}^i(T, ?)$ and, for a family of modules $\mathcal{S}$, ${}^{\perp}\mathcal{S} := \operatorname{Ker}\operatorname{Ext}^1(?, \mathcal{S})$. It is well known (see [9], after Definition 5.1.1) that the pair of subcategories $({}^{\perp}(T^{\perp_\infty}), T^{\perp_\infty})$ is a complete hereditary cotorsion pair. This means (see [9, Lemma 2.2.6]) that X (as any other module) admits a special ${}^{\perp}(T^{\perp_\infty})$-precover

$$0 \longrightarrow J \longrightarrow K \longrightarrow X \longrightarrow 0\,.$$

In particular, J belongs to $({}^{\perp}(T^{\perp_\infty}))^{\perp}$, which equals $T^{\perp_\infty}$ by definition of cotorsion pair. Now we can apply [3, Lemma 3.2] to J and [9, Proposition 5.1.9] to K in order to construct a sequence of direct summands of coproducts of copies of T

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & T_{-2} & \longrightarrow & T_{-1} & \overset{d_{-1}}{\dashrightarrow} & T_0 & \xrightarrow{d_0} & \cdots & \longrightarrow & T_n & \longrightarrow 0 \\ & & & \downarrow & & \uparrow & & & & & \\ & & & J & \hookrightarrow & K & \overset{\pi}{\twoheadrightarrow} & X & & & \end{array} \qquad (*)$$

By construction, the first row is a sequence which is exact everywhere except for degree 0, where $\ker d_0/\operatorname{im} d_{-1} \simeq K/J \simeq X$.

This concludes the proof for the case where $e = n$. Otherwise, it can be easily proved that since by hypothesis $\operatorname{Ext}^i_A(T, X) = 0$ for $i > e$, then for these indices $T_i = 0$: let us show it for $i = n$, then the other cases follow similarly. First, notice that since $\operatorname{Ext}^j_A(T, J) = 0$, one gets $\operatorname{Ext}^j_A(T, K) = \operatorname{Ext}^j_A(T, X)$ for every $j > 0$. Then, if we call $K_j = \ker d_j$ for $j \geq 0$, we have

$$\operatorname{Ext}^1_A(T, K_{n-1}) \cong \operatorname{Ext}^n_A(T, K_0) = \operatorname{Ext}^n_A(T, X) = 0;$$

applying the functor $\operatorname{Hom}_A(T, ?)$ to the short exact sequence

$$0 \to K_{n-1} \to T_{n-1} \to K_n = T_n \to 0$$

we get that $\operatorname{Hom}_A(T, T_{n-1}) \to \operatorname{Hom}_A(T, T_n)$ is an epimorphism and hence all morphisms $T \to T_n$ factorise through T_{n-1}. Using the universal property of the coproduct of which T_n is a direct summand, it is easy to prove that this implies that

$0 \to K_{n-1} \to T_{n-1} \to T_n \to 0$ splits. Thus K_{n-1} is a direct summand of a coproduct of copies of T. Therefore, we may truncate the sequence $(*)$ as

$$\cdots \longrightarrow T_{n-3} \longrightarrow T_{n-2} \longrightarrow K_{n-1} \longrightarrow 0\ .$$

Notice that this lemma generalises [3, Lemma 3.2], which is the case where $e = 0$.

Remark 2 We shall prove that $KE_e \subseteq \mathcal{T}_{e+1} \cap \mathcal{F}_e$ for every $e = 0, \ldots, n$. Indeed, it is obvious that $KE_e \subseteq \mathcal{T}_{e+1}$. To see that any $M \in KE_e$ belongs to $\mathcal{F}_e$ as well, we will proceed in subsequent steps.

First, we prove that for every $X \in \bigcap_{i>e-1} \operatorname{Ker} \operatorname{Ext}^i_A(T, ?) \subseteq \mathcal{T}_e$, there are no non zero morphisms $X \to M$. Indeed, if $e = 0$, then $X = 0$; if $e > 0$, consider the sequence

$$T^\bullet := \ \cdots \longrightarrow T_{-1} \xrightarrow{d_{-1}} T_0 \xrightarrow{d_0} \cdots \longrightarrow T_{e-1} \longrightarrow 0$$

given by Lemma 1 applied to X. Set $K_j = \ker d_j$ for $j \geq 0$ (and so $K_0 = K$), applying the functor $\operatorname{Hom}(-, M)$ to the epimorphism $K_0 \to X$, one gets

$$\operatorname{Hom}_A(X, M) \hookrightarrow \operatorname{Hom}_A(K_0, M) \cong \operatorname{Ext}^1_A(K_1, M) \cong \cdots$$

$$\cdots \cong \operatorname{Ext}^{e-1}_A(K_{e-1}, M) = \operatorname{Ext}^{e-1}_A(T_{e-1}, M) = 0,$$

and hence $\operatorname{Hom}_A(X, M) = 0$.

Now, if X' is the epimorphic image of some $X \in \bigcap_{i>e-1} \operatorname{Ker} \operatorname{Ext}^i_A(T, ?)$, we have $\operatorname{Hom}_A(X', M) \hookrightarrow \operatorname{Hom}_A(X, M) = 0$ so $\operatorname{Hom}_A(X', M) = 0$ as well. Lastly, if X'' is an extension of such epimorphic images, we still find that $\operatorname{Hom}_A(X'', M) = 0$.

This proves the claim that M has no non zero morphisms from objects of $\mathcal{T}_e$, and therefore it belongs to $\mathcal{F}_e$.

The last result of [13] is the proof that for $n = 2$, the filtration procedure of Proposition 3 reduces to that provided by Jensen, Madsen, and Su.

It should be noted that these results, while providing a way to generalise the decomposition of every module found in the $n = 1$ case, do so by introducing enlargements of the Miyashita classes KE_i which are not very natural, at the point that the connection to the tilting object they originate from seems a bit weak.

The rest of the article is devoted to the description of an alternative approach to this enlarging strategy, introduced in [8], which takes place in the derived category $\mathcal{D}(A)$ of A-Mod. In the following section, we recall some basic facts about derived categories and t-structures.

4 Introducing Derived Categories and *t*-structures

Given an abelian category $\mathcal{A}$, one may construct its derived category $\mathcal{D}(\mathcal{A})$ defining objects and morphisms in the following way. As objects, one takes the cochain complexes with terms in $\mathcal{A}$:

$$\cdots \longrightarrow X^n \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} X^{n+2} \longrightarrow \cdots$$

In order to define morphisms, one first takes the quotient of morphisms of complexes modulo those satisfying the *nullohomotopy* condition; the category having these equivalence classes as morphisms is called the *homotopy category*. The step from this to the derived category is performed by an argument of *localisation*; in this way, morphisms of complexes which induce isomorphisms on the cohomologies get an inverse in the derived category.

The category $\mathcal{D}(\mathcal{A})$ so obtained is not abelian anymore, but it is a *triangulated category*. This means that it is equipped with the following structure. First, there is an autoequivalence, whose action on the complex $X^\bullet$ is denoted as $X^\bullet[1]$ and is defined as follows:

$$(X^\bullet[1])^n = X^{n+1} \qquad d^n_{X[1]} = -d_X^{n+1}.$$

This functor is called the *suspension functor*; its natural definition on chain morphisms induces a good definition on morphisms in $\mathcal{D}(\mathcal{A})$. We will sometimes denote this functor also as Σ; its inverse as Σ^{-1} or $?[-1]$; their powers as Σ^i or $?[i]$ for $i \in \mathbb{Z}$.

Given this autoequivalence, one calls *triangles* the diagrams of the form

$$X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} X[1]$$

such that $v \circ u = 0 = w \circ v$; in $\mathcal{D}(\mathcal{A})$, a particular role is played by the triangles isomorphic (as diagrams) to those of the form

$$X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow \operatorname{Cone} f \longrightarrow X[1]$$

where $\operatorname{Cone} f$ is defined as the complex having terms $(\operatorname{Cone} f)^i = X^{i+1} \oplus Y^i$ and differentials $d^i_{\operatorname{Cone} f} = \begin{bmatrix} -d_X^{i+1} & 0 \\ f^{i+1} & d_Y^i \end{bmatrix}$. These triangles are called *distinguished triangles* and are the analogous of short exact sequences in abelian categories.

In a triangulated category, hence also in $\mathcal{D}(\mathcal{A})$, products and coproducts of distinguished triangles, when they exist, are distinguished (see [15, Proposition 1.2.1, and its dual]). In particular, if $\mathcal{A}$ has arbitrary products or coproducts, $\mathcal{D}(\mathcal{A})$ has them as well: they are constructed degree-wise using those of $\mathcal{A}$.

Once we have set our context, we now define the main object which we will work with.

Definition 3 Let $\mathcal{S} = (\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 0})$ be a pair of full, strict (i.e. closed under isomorphisms) subcategories of $\mathcal{D}(\mathcal{A})$, and denote $\mathcal{S}^{\leq i} = \mathcal{S}^{\leq 0}[-i]$ and $\mathcal{S}^{\geq i} = \mathcal{S}^{\geq 0}[-i]$, for every $i \in \mathbb{Z}$.

The pair $\mathcal{S}$ is a *t-structure* if it satisfies the following properties:

T1) $\mathcal{S}^{\leq 0} \subseteq \mathcal{S}^{\leq 1}$ and $\mathcal{S}^{\geq 0} \supseteq \mathcal{S}^{\geq 1}$;
T2) $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 1}) = 0$;
T3) For any complex $X^\bullet$ in $\mathcal{D}(\mathcal{A})$, there exist complexes $A^\bullet \in \mathcal{S}^{\leq 0}$ and $B^\bullet \in \mathcal{S}^{\geq 1}$ and morphisms such that

$$A^\bullet \longrightarrow X^\bullet \longrightarrow B^\bullet \longrightarrow A^\bullet[1]$$

is a distinguished triangle in $\mathcal{D}(\mathcal{A})$. This is called an *approximating triangle* of $X^\bullet$.

In this case, $\mathcal{S}^{\leq 0}$ is called an *aisle*, $\mathcal{S}^{\geq 0}$ a *coaisle*. The t-structure $\mathcal{S}$ is called *non degenerate* if $\bigcap_{i \in \mathbb{Z}} \mathcal{S}^{\leq i} = 0$ (or equivalently $\bigcap_{i \in \mathbb{Z}} \mathcal{S}^{\geq i} = 0$). The full subcategory $\mathcal{H}_\mathcal{S} = \mathcal{S}^{\leq 0} \cap \mathcal{S}^{\geq 0}$ is called the *heart* of $\mathcal{S}$.

This definition immediately resembles that of a torsion pair in an abelian category. As it holds for torsion pairs, the approximating triangle of a complex with respect to a t-structure is functorial, as we are going to state.

Given a t-structure $\mathcal{S}$ in $\mathcal{D}(\mathcal{A})$, it can be proved that the embeddings of subcategories $\mathcal{S}^{\leq 0} \subseteq \mathcal{D}(\mathcal{A})$ and $\mathcal{S}^{\geq 0} \subseteq \mathcal{D}(\mathcal{A})$ have a right adjoint $\sigma^{\leq 0} : \mathcal{D}(\mathcal{A}) \to \mathcal{S}^{\leq 0}$ and a left adjoint $\sigma^{\geq 0} : \mathcal{D}(\mathcal{A}) \to \mathcal{S}^{\geq 0}$, respectively.

For $i \in \mathbb{Z}$, let us write $\sigma^{\leq i} = \Sigma^{-i} \circ \sigma^{\leq 0} \circ \Sigma^i : \mathcal{D}(\mathcal{A}) \to \mathcal{S}^{\leq i}$ and similarly $\sigma^{\geq i} = \Sigma^{-i} \circ \sigma^{\geq 0} \circ \Sigma^i : \mathcal{D}(\mathcal{A}) \to \mathcal{S}^{\geq i}$; $\sigma^{\leq i}$ and $\sigma^{\geq i}$ will be called, respectively, the left and the right *truncation functors at* i with respect to $\mathcal{S}$, for $i \in \mathbb{Z}$.

It can be proved that for every $X^\bullet$ in $\mathcal{D}(\mathcal{A})$, the approximation triangle for $X^\bullet$ provided by the definition of the t-structure $\mathcal{S}$ is precisely (isomorphic to):

$$\sigma^{\leq 0}(X^\bullet) \longrightarrow X^\bullet \longrightarrow \sigma^{\geq 1}(X^\bullet) \longrightarrow (\sigma^{\leq 0}(X^\bullet))[1] .$$

The truncation functors of $\mathcal{S}$ can be used to define the *i-th cohomology* with respect to $\mathcal{S}$. It can be proved that for every $i, j \in \mathbb{Z}$, there is a canonical natural isomorphism $\sigma^{\leq i}\sigma^{\geq j} \simeq \sigma^{\geq j}\sigma^{\leq i}$. Then, for every $i \in \mathbb{Z}$, the functor $H^i_\mathcal{S} = \Sigma^i \sigma^{\leq i}\sigma^{\geq i} \simeq \Sigma^i \sigma^{\geq i}\sigma^{\leq i} : \mathcal{D}(\mathcal{A}) \to \mathcal{H}_\mathcal{S}$ is called the i-th *cohomology functor* with respect to the t-structure $\mathcal{S}$ (or simply $\mathcal{S}$*-cohomology*).

We introduce now the first t-structure in $\mathcal{D}(\mathcal{A})$ we are going to use.

Definition 4 The *natural* t-structure $\mathcal{D}$ of $\mathcal{D}(\mathcal{A})$ has aisle and coaisle:

$$\mathcal{D}^{\leq 0} = \left\{X^\bullet \in \mathcal{D}(\mathcal{A}) : \; H^i(X^\bullet) = 0 \text{ for every } i > 0\right\}$$
$$\mathcal{D}^{\geq 0} = \left\{X^\bullet \in \mathcal{D}(\mathcal{A}) : \; H^i(X^\bullet) = 0 \text{ for every } i < 0\right\} .$$

Notice that by construction, the ith $\mathcal{D}$-cohomology of $X^\bullet$ is a complex having zero cohomology everywhere except for degree 0, where it has $H^i(X^\bullet)$, the usual i-th cohomology of $X^\bullet$: i.e., $H^i_{\mathcal{D}}(X^\bullet) = H^i(X^\bullet)[0]$.

The original proof that this is indeed a t-structure can be found in [5].

We now state the following fundamental theorem about t-structures. One may read it with our example $\mathcal{D}$ in mind.

Theorem 3 *Let $\mathcal{S}$ be a non degenerate t-structure in $\mathcal{D}(\mathcal{A})$. Then*

1. *The heart $\mathcal{H}_\mathcal{S}$ is an abelian category; moreover, a short sequence*

$$0 \longrightarrow X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow 0$$

   in $\mathcal{H}_\mathcal{S}$ is exact if and only if there exists a morphism $Z \to X[1]$ in $\mathcal{D}(\mathcal{A})$ such that the triangle

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow X^\bullet[1]$$

   is distinguished.
2. *Given any distinguished triangle*

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow X^\bullet[1]$$

   in $\mathcal{D}(\mathcal{A})$, there is a long exact sequence in $\mathcal{H}_\mathcal{S}$

$$\cdots \longrightarrow H^{i-1}_\mathcal{S} Z^\bullet \longrightarrow H^i_\mathcal{S} X^\bullet \longrightarrow H^i_\mathcal{S} Y^\bullet \longrightarrow H^i_\mathcal{S} Z^\bullet \longrightarrow H^{i+1}_\mathcal{S} \longrightarrow \cdots$$

As can be easily seen, the heart $\mathcal{H}_\mathcal{D}$ of the natural t-structure of $\mathcal{D}(\mathcal{A})$ is (equivalent to) $\mathcal{A}$ itself via the embedding $\mathcal{A} \to \mathcal{D}(\mathcal{A})$ defined by

$$X \mapsto X[0] = (\cdots \longrightarrow 0 \longrightarrow X \longrightarrow 0 \longrightarrow \cdots)$$

whose quasi-inverse is H^0, the usual 0th-cohomology functor.

As it happens for torsion pairs, the aisle or the coaisle of a t-structure is sufficient to characterise the whole t-structure. Indeed, we give the following lemma by Keller and Vossieck [12].

Lemma 2 *Let $\mathcal{R} = (\mathcal{R}^{\leq 0}, \mathcal{R}^{\geq 0})$ be a t-structure in $\mathcal{D}(\mathcal{A})$. Then*

$$\mathcal{R}^{\leq 0} = \left\{X^\bullet \in \mathcal{D}(\mathcal{A}) : \operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(X^\bullet, Y^\bullet) = 0 \text{ for all } Y^\bullet \in \mathcal{R}^{\geq 1}\right\}$$
$$\mathcal{R}^{\geq 0} = \left\{Y^\bullet \in \mathcal{D}(\mathcal{A}) : \operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(X^\bullet, Y^\bullet) = 0 \text{ for all } X^\bullet \in \mathcal{R}^{\leq -1}\right\}.$$

We now give the following proposition, which gives a very useful way to construct t-structures in the derived category of a Grothendieck category $\mathcal{G}$.

Proposition 4 (*[1, Lemma 3.1, Theorem 3.4]*) *Let $\mathcal{G}$ be a Grothendieck category. Given any complex E in $\mathcal{D}(\mathcal{G})$, let $\mathcal{U}$ be the smallest* cocomplete pre-aisle *contain-*

ing E, that is, the smallest full, strict subcategory of $\mathcal{D}(\mathcal{G})$ closed under positive shifts, extensions and coproducts. Then, $\mathcal{U}$ is an aisle and the corresponding coaisle is

$$\begin{aligned}\mathcal{U}^{\perp} &= \left\{Y^{\bullet} \in \mathcal{D}(\mathcal{G}) :\ \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(X^{\bullet}, Y^{\bullet}) = 0 \textit{ for every } X^{\bullet} \in \mathcal{U}[1]\right\} \\ &= \left\{Y^{\bullet} \in \mathcal{D}(\mathcal{G}) :\ \mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(E, Y^{\bullet}[i]) = 0 \textit{ for every } i < 0\right\}.\end{aligned}$$

Remark 3 As a first application of this proposition, it is easy to see that if $\mathcal{G}$ has a projective generator E, then the natural t-structure of $\mathcal{D}(\mathcal{G})$ will be that generated by E (in the sense of the proposition). This will be the case when we will consider $\mathcal{G} = A$-Mod, with $E = A$.

Remark 4 In the case where the object E is in fact a module, that is, a complex concentrated in degree zero, we shall give a characterisation of the aisle $\mathcal{U}$ generated by E.

First, $\mathcal{U}$ contains E; and it is closed under positive shifts, hence it contains $E[i]$ for every $i > 0$. $\mathcal{U}$ is closed under arbitrary coproducts; let then $J = \cup_{i>0} J_i$ be a set of indices, and let $E_j = E[i]$ for every $j \in J_i$. Then the coproduct $\coprod_{j\in J} E_j = \coprod_{i>0} E^{(J_i)}[i]$ belongs to $\mathcal{U}$ as well. If $\mathcal{V}$ is the full subcategory of all objects isomorphic to these coproducts, this means that $\mathcal{V} \subseteq \mathcal{U}$. Since $\mathcal{U}$ is also closed under extensions, if we call $\mathcal{V}'$ the extension closure of $\mathcal{V}$, we have $\mathcal{V}' \subseteq \mathcal{U}$ as well. Moreover, since coproducts of distinguished triangles are distinguished, from the fact that $\mathcal{V}$ is closed under arbitrary coproducts follows easily that $\mathcal{V}'$ is as well. Hence, $\mathcal{V}'$ is a cocomplete pre-aisle, and by definition $\mathcal{U} \subseteq \mathcal{V}'$.

In conclusion, objects of $\mathcal{U}$ are isomorphic to complexes having zero terms in positive degrees and coproducts of E in nonpositive degrees.

5 *n*-Tilting Objects and Associated *t*-structures

In the following, we are going to work with a generalisation of classical n-tilting modules, introduced by Angeleri Hügel and Coelho [2]; the equivalent definition we give is more oriented towards the derived category $\mathcal{D}(A)$ of A-Mod, which will be our setting.

Definition 5 A left A-module T is (non necessarily classical) *n-tilting* if it satisfies the following properties:

P_n) T has projective dimensions at most n, i.e. there exists an exact sequence

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow T \longrightarrow 0$$

in A-Mod with the P_i projectives;

E_n) T is rigid, i.e. $\mathrm{Ext}^i_A(T, T^{(\Lambda)}) = 0$ for every index $0 < i \leq n$ and set Λ;

G_n) T is a *generator* in $\mathcal{D}(A)$, meaning that if for a complex $X^{\bullet}$ we have $\mathrm{Hom}_{\mathcal{D}(A)}(T, X[i]) = 0$ for every $i \in \mathbb{Z}$, then $X^{\bullet} = 0$ in $\mathcal{D}(A)$.

Notice that a classical n-tilting module is indeed n-tilting: in particular, p_n) implies P_n), p_n) and e_n) imply E_n) (see the Stacks Project [19, Proposition 15.72.3] and g_n) implies G_n) (see Positselski and Stovicek [17, Corollary 2.6]).

The discussion about t-structures in the previous section is justified by the following construction. Let T be a n-tilting left A-module and consider the pair $\mathcal{T} = (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of subcategories of $\mathcal{D}(A)$

$$\mathcal{T}^{\leq 0} = \left\{X^\bullet \in \mathcal{D}(A) : \operatorname{Hom}_{\mathcal{D}(A)}(T, X^\bullet[i]) = 0 \text{ for every } i > 0\right\}$$
$$\mathcal{T}^{\geq 0} = \left\{X^\bullet \in \mathcal{D}(A) : \operatorname{Hom}_{\mathcal{D}(A)}(T, X^\bullet[i]) = 0 \text{ for every } i < 0\right\}.$$

Remark 5 This is the t-structure generated by T in the sense of Proposition 4, as proved in [4, Lemma 3.4], which in turn follows [18, Lemma 4.4]. We provide here another proof.

Let $\mathcal{G} = (\mathcal{G}^{\leq 0}, \mathcal{G}^{\geq 0})$ be the generated t-structure. We have $\mathcal{T}^{\geq 0} = \mathcal{G}^{\geq 0}$. For the aisle, notice that $\mathcal{T}^{\leq 0}$ contains T by (E_n); and it is clearly closed under positive shifts, hence it contains any $T[i]$ for $i > 0$. Now, we show that it is closed under arbitrary coproducts of such complexes $T[i]$. Let $J = \cup_{i>0} J_i$ be a set, let $T_j = T[i]$ for every $j \in J_i$, and consider the coproduct $\coprod_{j\in J} T_j = \coprod_{i>0} T^{(J_i)}[i]$. Notice that since by (P_n) T has projective dimension n, we have

$$\operatorname{Hom}_{\mathcal{D}(A)}\left(T, \coprod_{j\in J} T_j\right) = \operatorname{Hom}_{\mathcal{D}(A)}\left(T, \coprod_{i>0} T^{(J_i)}[i]\right) = \operatorname{Hom}_{\mathcal{D}(A)}\left(T, \coprod_{1\leq i\leq n} T^{(J_i)}[i]\right).$$

Now, since $\mathcal{D}(A)$ is an additive category, this is itself isomorphic to

$$\operatorname{Hom}_{\mathcal{D}(A)}\left(T, \prod_{1\leq i\leq n} T^{(J_i)}[i]\right) \simeq \prod_{1\leq i\leq n} \operatorname{Hom}_{\mathcal{D}(A)}\left(T, T^{(J_i)}[i]\right) = 0$$

which is zero by property (E_n). Lastly, $\mathcal{T}^{\leq 0}$ is clearly closed under extensions, and so by Remark 4, it contains $\mathcal{G}^{\leq 0}$.

For the inclusion $\mathcal{T}^{\leq 0} \subseteq \mathcal{G}^{\leq 0}$, take an object $X^\bullet \in \mathcal{T}^{\leq 0}$ and consider its approximation triangle with respect to $\mathcal{G}$,

$$A^\bullet \longrightarrow X^\bullet \longrightarrow B^\bullet \xrightarrow{+1} .$$

We have $A^\bullet \in \mathcal{G}^{\leq 0} \subseteq \mathcal{T}^{\leq 0}$; and since $\mathcal{T}^{\leq 0}$ is clearly closed under cones, $B^\bullet \in \mathcal{T}^{\leq 0}$ as well. So in the end $B^\bullet \in \mathcal{T}^{\leq 0} \cap \mathcal{G}^{\geq 1} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 1}$ which is 0 by G_3).

As a side note, observe that if T is classical n-tilting, it induces a triangulated equivalence $R\operatorname{Hom}_A(T, ?) : \mathcal{D}(A) \to \mathcal{D}(B)$ (see [7]); then, by the fact that

$$\operatorname{Hom}_{\mathcal{D}(A)}(T, X^\bullet[i]) = H^i R\operatorname{Hom}_A(T, X^\bullet)$$

we may recognise in $\mathcal{T} := (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ the "pullback" of the natural t-structure of $\mathcal{D}(B)$ along $R\operatorname{Hom}_A(T, ?)$.

Remark 6 Without further study, the t-structure $\mathcal{T}$ can be immediately used to review some previous results.

First, we can greatly simplify the proof of Remark 2. In the notation used there, to prove that there are no non zero morphisms $X \to M$, for X in $\cap_{i>e-1} \operatorname{Ker} \operatorname{Ext}^i_A(T, ?)$ and M in KE_e, we may just recognise that we have $X = X[0] \in \mathcal{T}^{\leq e-1}$ and $M \in \mathcal{T}^{\geq e}$, and use axiom (T2) of t-structures.

Second, we may read our Lemma 1 under a different light: given the characterisation of objects in $\mathcal{T}^{\leq 0}$ as in Remark 4, the lemma can be seen to be the equality

$$\cap_{i>e} \operatorname{Ker} \operatorname{Ext}^i_A(T, ?) = A\text{-Mod} \cap \mathcal{T}^{\leq e}.$$

In the following, T will be a n-tilting module; $\mathcal{T}$ will be the associated t-structure, as defined above. The solution that we are going to give to our decomposition problem originates from the interaction of the t-structure $\mathcal{T}$ with the natural t-structure $\mathcal{D}$ of $\mathcal{D}(A)$ (see Definition 4). First, we make an easy observation.

Proposition 5 *The following inclusions of aisles and coaisles hold:*

$$\mathcal{D}^{\leq -n} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0} \quad \textit{and} \quad \mathcal{D}^{\geq 0} \subseteq \mathcal{T}^{\geq 0} \subseteq \mathcal{D}^{\geq -n}.$$

Proof Some of the inclusions are easy to prove: if $X^\bullet \in \mathcal{D}^{\leq -n}$, then for every $i > 0$ we will have $X^\bullet[i] \in \mathcal{D}^{\leq -n-i} \subseteq \mathcal{D}^{\leq -n-1}$, hence $\operatorname{Hom}_{\mathcal{D}(A)}(T, X^\bullet[i]) = 0$ since T has projective dimension n. On the other hand, if $X^\bullet \in \mathcal{D}^{\geq 0}$, then for every $i < 0$ we will have $X^\bullet[i] \in \mathcal{D}^{\geq 0-i} \subseteq \mathcal{D}^{\geq 1}$, hence again $\operatorname{Hom}_{\mathcal{D}(A)}(T, X^\bullet[i]) = 0$ since $T \in \mathcal{D}^{\leq 0}$. The other two inclusions can be easily proved from these using Lemma 2.

Remark 7 With Proposition 5, we are ready to notice an important fact, which will be key later. Take a module X in KE_e, for some $e = 0, \ldots, n$; in particular, being a module, it belongs to $A\text{-Mod} \simeq \mathcal{H}_{\mathcal{D}} \subseteq \mathcal{D}^{\geq 0} \subseteq \mathcal{T}^{\geq 0}$. Moreover, by definition, for every $i = 0, \ldots, e-1$, we have $0 = \operatorname{Ext}^i_A(T, X) \simeq \operatorname{Hom}_{\mathcal{D}(A)}(T, X[i])$, hence X belongs in fact to $\mathcal{T}^{\geq e}$. Lastly, again by definition, for every $i > e$, we have $0 = \operatorname{Ext}^i_A(T, X) \simeq \operatorname{Hom}_{\mathcal{D}(A)}(T, X[i])$, hence X belongs to $\mathcal{T}^{\leq e}$ as well.

This proves that, after identifying $A\text{-Mod} \simeq \mathcal{H}_{\mathcal{D}}$, for every $e = 0, \ldots, n$ the e-th Miyashita class is

$$KE_e = A\text{-Mod} \cap \mathcal{H}_{\mathcal{T}}[-e].$$

Let us now look at Proposition 5 in the $n = 1$ case. Its proof suggests that we may focus on the inclusions between the aisles (those between the coaisles being their "dual" in the sense of Lemma 2). If T is a 1-tilting module, we will then have

$$\mathcal{D}^{\leq -1} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}. \tag{$*$}$$

In other words, complexes in $\mathcal{T}^{\leq 0}$ are allowed to have any cohomology (with respect to $\mathcal{D}$, which means the usual complex cohomology H^i) in degrees ≤ -1 and some kind of cohomology in degree 0, while they must have 0 cohomology in higher degrees.

Remark 8 In this situation, with T a 1-tilting module, we may try to characterise $H^0(X^\bullet)$ for $X^\bullet \in \mathcal{T}^{\leq 0}$. Notice that $X^\bullet$ sits in the approximation triangle with respect to $\mathcal{D}$

$$\delta^{\leq -1}(X^\bullet) \longrightarrow X^\bullet \longrightarrow H^0(X^\bullet)[0] \xrightarrow{+1}$$

where

$H^0(X^\bullet)[0] = H^0_{\mathcal{D}}(X^\bullet) = \delta^{\geq 0}\delta^{\leq 0}(X^\bullet) \simeq \delta^{\geq 0}(X^\bullet)$ since $X^\bullet \in \mathcal{D}^{\leq 0}$. If we apply the homological functor $\mathrm{Hom}_{\mathcal{D}(A)}(T, ?)$ to it, we get the long exact sequence of abelian groups

$$\cdots \to \mathrm{Hom}_{\mathcal{D}(A)}(T, X^\bullet[1]) \to \mathrm{Hom}_{\mathcal{D}(A)}(T, H^0(X^\bullet)[1]) \to \mathrm{Hom}_{\mathcal{D}(A)}(T, \delta^{\leq -1}(X^\bullet)[2]) \to \cdots$$

The last term is 0 because $\delta^{\leq -1}(X^\bullet) \in \mathcal{D}^{\leq -1} \subseteq \mathcal{T}^{\leq 0}$; similarly, the first is 0 because $X^\bullet \in \mathcal{T}^{\leq 0}$. This means that

$$\mathrm{Ext}^1_A(T, H^0(X^\bullet)) \simeq \mathrm{Hom}_{\mathcal{D}(A)}(T, H^0(X^\bullet)[1]) = 0$$

as well, i.e. that $H^0(X^\bullet) \in KE_0$. □

The inclusions (∗) are precisely the hypothesis of the following proposition by Polishchuk.

Proposition 6 (*[16, Lemma 1.1.2]*) *Let $\mathcal{R}, \mathcal{S}$ be two t-structures in a triangulated category $\mathcal{C}$ such that*

$$\mathcal{R}^{\leq -1} \subseteq \mathcal{S}^{\leq 0} \subseteq \mathcal{R}^{\leq 0} \qquad \textit{(or equivalently } \mathcal{R}^{\geq 0} \subseteq \mathcal{S}^{\geq 0} \subseteq \mathcal{R}^{\geq -1}\textit{).}$$

Then the classes:

$$\mathcal{X} = \mathcal{H}_{\mathcal{R}} \cap \mathcal{S}^{\leq 0} = \mathcal{R}^{\geq 0} \cap \mathcal{S}^{\leq 0}, \qquad \mathcal{Y} = \mathcal{H}_{\mathcal{R}} \cap \mathcal{S}^{\geq 1} = \mathcal{R}^{\leq 0} \cap \mathcal{S}^{\geq 1}$$

form a torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{H}_{\mathcal{R}}$. $\mathcal{S}$ can be reconstructed from $\mathcal{R}$ and $(\mathcal{X}, \mathcal{Y})$ as

$$\mathcal{S}^{\leq 0} = \left\{X^\bullet \in \mathcal{R}^{\leq 0} : \ H^0_{\mathcal{R}}(X^\bullet) \in \mathcal{X}\right\}$$
$$\mathcal{S}^{\geq 0} = \left\{X^\bullet \in \mathcal{R}^{\geq -1} : \ H^{-1}_{\mathcal{R}}(X^\bullet) \in \mathcal{Y}\right\}.$$

This procedure to recover $\mathcal{S}$ is called *tilting* of the t-structure $\mathcal{R}$ with respect to the torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{H}_{\mathcal{R}}$. It was introduced by Happel et al. [10], and it is a central tool in the construction we are going to present.

Remark 9 It can be proved without too much effort that in our case the torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{H}_{\mathcal{D}} \simeq A$-Mod so identified is exactly the pair (KE_0, KE_1) induced by the 1-tilting module T; this confirms Remark 8.

We would like to use a procedure analogous to the Happel-Reiten-Smalø tilting of Proposition 6 in order to link $\mathcal{D}$ and $\mathcal{T}$ in the $n > 1$ case. Notice that if we repeat this tilting operation n times, the first and last of the produced t-structures will be related by the inclusions of Proposition 5. Indeed, let $\mathcal{R}_0, \dots, \mathcal{R}_n$ be t-structures such that $\mathcal{R}_i$ is obtained by tilting $\mathcal{R}_{i-1}$ with respect to some torsion pair on $\mathcal{H}_{\mathcal{R}_{i-1}}$, for every $i = 1, \dots, n$. Then, we have by construction

$$\mathcal{R}_0^{\leq -1} \subseteq \mathcal{R}_1^{\leq 0} \subseteq \mathcal{R}_0^{\leq 0} \quad \text{and} \quad \mathcal{R}_1^{\leq -1} \subseteq \mathcal{R}_2^{\leq 0} \subseteq \mathcal{R}_1^{\leq 0}$$

which combined give

$$\mathcal{R}_0^{\leq -2} \subseteq \mathcal{R}_2^{\leq 0} \subseteq \mathcal{R}_0^{\leq 0}.$$

One can then clearly prove by induction that

$$\mathcal{R}_0^{\leq -n} \subseteq \mathcal{R}_n^{\leq 0} \subseteq \mathcal{R}_0^{\leq 0}.$$

If T is an n-tilting A-module, we shall show that the associated t-structure $\mathcal{T}$ in $\mathcal{D}(A)$ can indeed be constructed from the natural t-structure $\mathcal{D}$ with this iterated procedure. To do so, we are going to construct the "intermediate" t-structures produced after each tilting.

For $i = 0, \dots, n$, consider the strict full subcategories $\mathcal{D}_i^{\geq} = \mathcal{D}^{\geq -i} \cap \mathcal{T}^{\geq 0}$ (notice that we are working with the coaisles). We have as wanted that

$$\mathcal{D}^{\geq 0} = \mathcal{D}_0^{\geq} \subseteq \mathcal{D}_1^{\geq} \subseteq \cdots \subseteq \mathcal{D}_n^{\geq} = \mathcal{T}^{\geq 0}$$

and $\mathcal{D}_{i-1}^{\geq} \subseteq \mathcal{D}_i^{\geq} \subseteq \mathcal{D}_{i-1}^{\geq}[1]$ for $i = 1, \dots, n$. The only thing needed to proceed with an iterated application of Proposition 6 is to prove that these $\mathcal{D}_i^{\geq}$ are indeed the coaisles of some t-structures, for $i = 1, \dots, n-1$.

Lemma 3 *The* $\mathcal{D}_i^{\geq} = \mathcal{D}^{\geq -i} \cap \mathcal{T}^{\geq 0}$ *are coaisles of* t*-structures.*

Proof As we noticed before (see Remark 3 and the definition of $\mathcal{T}$), we have

$$\begin{aligned} \mathcal{D}^{\geq -i} &= \left\{ Y^\bullet \in \mathcal{D}(A) : \operatorname{Hom}_{\mathcal{D}(A)}(A[i], Y^\bullet[j]) = 0 \text{ for every } j < 0 \right\} \\ \mathcal{T}^{\geq 0} &= \left\{ Y^\bullet \in \mathcal{D}(A) : \operatorname{Hom}_{\mathcal{D}(A)}(T, Y^\bullet[j]) = 0 \text{ for every } j < 0 \right\}. \end{aligned}$$

Hence, we have

$$\mathcal{D}^{\geq -i} \cap \mathcal{T}^{\geq 0} = \left\{ Y^\bullet \in \mathcal{D}(A) : \operatorname{Hom}_{\mathcal{D}(A)}(T \oplus A[i]), Y^\bullet[j]) = 0 \text{ for every } j < 0 \right\}$$

which is the coaisle of the t-structure generated by $T \oplus A[i]$ in the sense of Proposition 4.

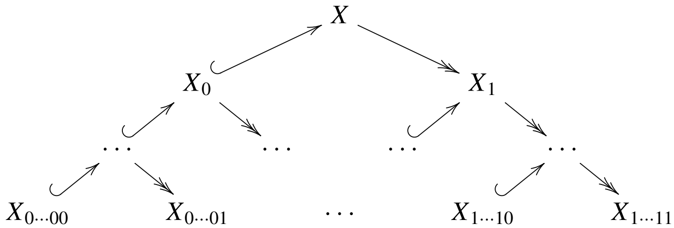
This concludes our previous discussion, making sure that $\mathcal{T}$ can be constructed from $\mathcal{D}$ with (at most) n applications of the procedure of tilting a t-structure with respect to a torsion pair on its heart.

6 The *t*-tree

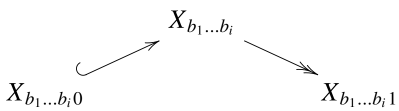
We are now going to exploit this fact to solve our decomposition problem.

First, we characterise the torsion pairs involved. According to Proposition 6, at the i-th step, the t-structure $\mathcal{D}_i$ (having coaisle $\mathcal{D}_i^{\geq 0} = \mathcal{D}_i^{\geq} = \mathcal{D}^{\geq -i} \cap \mathcal{T}^{\geq 0}$) is tilted with respect to the torsion pair $(\mathcal{X}_i, \mathcal{Y}_i) = (\mathcal{D}_i^{\geq 0} \cap \mathcal{D}_{i+1}^{\leq 0}, \mathcal{D}_i^{\leq 0} \cap \mathcal{D}_{i+1}^{\geq 1})$ in the heart $\mathcal{H}_i$ of $\mathcal{D}_i$, $i = 0, \ldots, n-1$, thus producing the t-structure $\mathcal{D}_{i+1}$.

Theorem 4 *Let T be a n-tilting left A-module. We can associate to each left A-module X a tree (we call it the t-tree of X with respect to the t-structure induced by the tilting module T)*



with $n+1$ rows, where

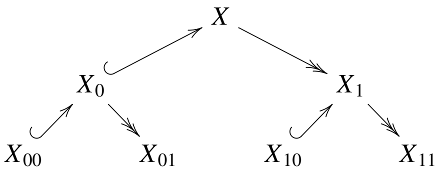


is the short exact sequence obtained decomposing $X_{b_1\ldots b_i}$ with respect to the torsion pair $(\mathcal{X}_i[-(b_1+\cdots+b_i)], \mathcal{Y}_i[-(b_1+\cdots+b_i)])$ in $\mathcal{H}_i[-(b_1+\cdots+b_i)]$.

Proof We may regard the left A-module X as a complex concentrated in degree 0, $X[0]$ in the heart $\mathcal{H}_{\mathcal{D}} = \mathcal{H}_0$. The first torsion pair $(\mathcal{X}_0, \mathcal{Y}_0)$ provides then a decomposition

$$X_0 \hookrightarrow X \twoheadrightarrow X_1 \quad \text{in } \mathcal{H}_0$$

with $X_0 \in \mathcal{X}_0$, $X_1 \in \mathcal{Y}_0$. Notice that by construction $\mathcal{X}_0 \subseteq \mathcal{H}_1$ and $\mathcal{Y}_0 \subseteq \mathcal{H}_1[-1]$ (see Proposition 6); this means that we can use $(\mathcal{X}_1, \mathcal{Y}_1)$ and $(\mathcal{X}_1[-1], \mathcal{Y}_1[-1])$ to further decompose X_0 and X_1, respectively, obtaining



with the exact sequences in the respective abelian categories:

$$X_0 \hookrightarrow X \twoheadrightarrow X_1 \quad \text{in } \mathcal{H}_0$$

$$X_{00} \hookrightarrow X_0 \twoheadrightarrow X_{01} \quad \text{in } \mathcal{H}_1 \qquad\qquad X_{10} \hookrightarrow X_1 \twoheadrightarrow X_{11} \quad \text{in } \mathcal{H}_1[-1]$$

Now, notice again that since $\mathcal{X}_1 \subseteq \mathcal{H}_2$ and $\mathcal{Y}_1 \subseteq \mathcal{H}_2[-1]$, we have that $X_{00} \in \mathcal{H}_2$, $X_{01}, X_{10} \in \mathcal{H}_2[-(0+1)] = \mathcal{H}_2[-(1+0)]$ and $X_{11} \in \mathcal{H}_2[-(1+1)]$.

Inductively, by decomposing each $X_{b_1 \ldots b_i}$ with respect to the torsion pair $(\mathcal{X}_i[-(b_1 + \cdots + b_i)], \mathcal{Y}_i[-(b_1 + \cdots + b_i)])$ in $\mathcal{H}_i[-(b_1 + \cdots + b_i)]$ we obtain objects $X_{b_1 \ldots b_i 0} \in \mathcal{H}_{i+1}[-(b_1 + \cdots + b_i)]$ and $X_{b_1 \ldots b_i 1} \in \mathcal{H}_{i+1}[-(b_1 + \cdots + b_i + 1)]$.

After n steps, we obtain the complete diagram.

We claim that Theorem 4 solves our decomposition problem. Indeed, by construction each object $X_{b_1 \cdots b_n}$ in the last row (called a *t-leaf*) belongs to $\mathcal{H}_n[-(b_1 + \cdots + b_n)] = \mathcal{H}_{\mathcal{T}}[-(b_1 + \cdots + b_n)]$: as noted in Remark 7, these shifted hearts are extensions of the Miyashita classes: $KE_{b_1 + \cdots + b_n} = A\text{-Mod} \cap \mathcal{H}_{\mathcal{T}}[-(b_1 + \cdots + b_n)]$. Moreover, these shifted hearts are obtained by adding only non-module objects (i.e., objects of $\mathcal{D}(A)$ outside of $\mathcal{H}_{\mathcal{D}}$) to the corresponding Miyashita class; for this reason, they are less artificial than other enlargments, and instead shed a new light on the Miyashita classes. The latter can indeed be regarded as the piece of the shifted hearts of $\mathcal{T}$ visible in the category of modules.

Example 3 We recall one last time the situation considered in Example 1 to show an application of the construction of the t-tree; we will do it for the simple module 2 again.

First, a computation shows that the indecomposable complexes in $\mathcal{D}(A)$ are (shifts of)

$$\left\{ 1,\ 2,\ 3,\ \begin{smallmatrix}1\\2\end{smallmatrix},\ \begin{smallmatrix}2\\3\end{smallmatrix},\ \begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix} \right\}.$$

Since we know that $\mathcal{D}^{\geq 0} \subseteq \mathcal{T}^{\geq 0}$, any bounded below complex will belong to $\mathcal{T}^{\geq 0}$, up to shifiting it enough to the right. We can then check for each of the indecomposable complexes what is their leftmost shift which still belongs to $\mathcal{T}^{\geq 0}$; with an easy computation, the following is the result:

$$\mathcal{T}^{\geq 0} = \left\langle 1,\ 2,\ 3[2],\ \begin{smallmatrix}1\\2\end{smallmatrix},\ \begin{smallmatrix}2\\3\end{smallmatrix},\ \begin{smallmatrix}2\\3\end{smallmatrix} \to \overset{\bullet}{\begin{smallmatrix}1\\2\end{smallmatrix}} \right\rangle$$

where the dot over a complex indicates its degree 0. The angle brackets will be used to denote the closure under direct sums and negative shifts.

Following the construction, we can compute the intermediate coaisles:

$$\mathcal{D}_0^{\geq 0} = \left\langle 1\,,\ 2\,,\ 3\,,\ \substack{1\\2},\ \substack{2\\3},\ \substack{\bullet\\2\\3} \rightarrow \substack{1\\2} \right\rangle = \mathcal{D}^{\geq 0}$$

$$\mathcal{D}_1^{\geq 0} = \left\langle 1\,,\ 2\,,\ 3\,[1],\ \substack{1\\2},\ \substack{2\\3},\ \substack{2\\3} \rightarrow \substack{\bullet\\1\\2} \right\rangle$$

$$\mathcal{D}_2^{\geq 0} = \left\langle 1\,,\ 2\,,\ 3\,[2],\ \substack{1\\2},\ \substack{2\\3},\ \substack{2\\3} \rightarrow \substack{\bullet\\1\\2} \right\rangle = \mathcal{T}^{\geq 0}.$$

Now we compute the hearts of the respective t-structures: to do this, we use Lemma 2. An object X of $\mathcal{D}_i^{\geq 0}$ will belong to $\mathcal{H}_i$ if and only if $\mathrm{Hom}_{\mathcal{D}(A)}(X, Y) = 0$ for every $Y \in \mathcal{D}_i^{\geq 1} = \mathcal{D}_i^{\geq 0}[-1]$. In particular, it is easy to see that we must look for objects of the heart only among the "leftmost shifts" we have listed. The resulting computation gives (only indecomposable objects are listed)

$$\mathcal{H}_0 = \left\{ 1\,,\ 2\,,\ 3\,,\ \substack{1\\2},\ \substack{2\\3} \right\} = \mathcal{H}_{\mathcal{D}} = A\text{-Mod}$$

$$\mathcal{H}_1 = \left\{ 1\,,\ 2\,,\ 3\,[1],\ \substack{1\\2},\ \substack{2\\3},\ \substack{2\\3} \rightarrow \substack{\bullet\\1\\2} \right\}$$

$$\mathcal{H}_2 = \left\{ 1\,,\ 3\,[2],\ \substack{1\\2},\ \substack{2\\3},\ \substack{2\\3} \rightarrow \substack{\bullet\\1\\2} \right\} = \mathcal{H}_{\mathcal{T}}.$$

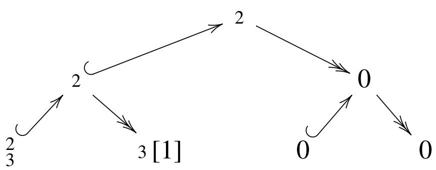
Notice that neither 2 nor its shifts belong to $\mathcal{H}_{\mathcal{T}}$, which means exactly that it does not belong to any Miyashita class.

Lastly, we can compute the torsion pairs $(\mathcal{X}_i, \mathcal{Y}_i)$ in $\mathcal{H}_i$, for $i = 0, 1$. We have

$$\mathcal{X}_0 = \mathcal{H}_0 \cap \mathcal{H}_1 = \left\{ 1\,,\ 2\,,\ \substack{1\\2},\ \substack{2\\3} \right\}, \qquad \mathcal{Y}_0 = \mathcal{H}_0 \cap \mathcal{H}_1[-1] = \{\,3\,\}$$

$$\mathcal{X}_1 = \mathcal{H}_1 \cap \mathcal{H}_2 = \left\{ 1\,,\ \substack{1\\2},\ \substack{2\\3},\ \substack{2\\3} \rightarrow \substack{\bullet\\1\\2} \right\}, \qquad \mathcal{Y}_1 = \mathcal{H}_1 \cap \mathcal{H}_2[-1] = \{\,3\,[1]\}\,.$$

The t-tree for the module 2 is then



where the bottom left exact sequence is that associated to the distinguished triangle

$$\substack{2\\3} \longrightarrow 2 \longrightarrow 3\,[1] \xrightarrow{+1} \,.$$

Notice that this triangle can be shifted to become $3 \longrightarrow {}^{2}_{3} \longrightarrow 2 \xrightarrow{+1}$, which can be read as a short exact sequence of modules. This says that 2 is realised as the cokernel of the monomorphism $3 \to {}^{2}_{3}$, which is what was found following Jensen et al. in Example 2.

We conclude with a remark about the construction presented above, giving a possible direction for future developments.

Remark 10 Notice that while our motivation comes from an n-tilting A-module, the construction of the t-tree only relies on the existence of some t-structures $\mathcal{D}_i$, for $i = 0, \ldots, n$, having the property that $\mathcal{D}_i^{\geq 0} \subseteq \mathcal{D}_{i+1}^{\geq 0} \subseteq \mathcal{D}_i^{\geq -1}$. Therefore, it can be replicated in an arbitrary triangulated category $\mathcal{C}$, given such t-structures: to any object in the heart $\mathcal{H}_0$ of $\mathcal{D}_0$, it is possible to associate a tree-like diagram with $n+1$ rows having leaves in the shifts $\mathcal{H}_n[-e]$ of the heart $\mathcal{H}_n$ of $\mathcal{D}_n$, for $e = 0, \ldots, n$. It is of natural interest to investigate other situations in which such t-structures may appear.

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