The Multiplicative Ideal Theory of Leavitt Path Algebras of Directed Graphs—A Survey

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Abstract Let L be the Leavitt path algebra of an arbitrary directed graph E over a field K. This survey article describes how this highly non-commutative ring L shares a number of the characterizing properties of a Dedekind domain or a Prüfer domain expressed in terms of their ideal lattices. Special types of ideals such as the prime, the primary, the irreducible, and the radical ideals of L are described in terms of the graphical properties of E. The existence and the uniqueness of the factorization of a non-zero ideal of L as an irredundant product of prime or primary or irreducible ideals are established. Such factorization always exists for every ideal in L if the graph E is finite or if L is two-sided Artinian or two-sided Noetherian. In all these factorizations, the graded ideals of L seem to play an important role. Necessary and sufficient conditions are given under which L is a generalized ZPI ring, that is, when every ideal of L is a product of prime ideals. Intersections of various special types of ideals are investigated and an analogue of Krull's theorem on the intersection of powers of an ideal in L is established.

Keywords Leavitt path algebras · Multiplicative ideal theory · Factorization of ideals

1 Introduction

Leavitt path algebras of directed graphs are algebraic analogues of graph C*-algebras and, ever since they were introduced in 2004, have become an active area of research [1]. Every Leavitt path algebra $L := L_K(E)$ of a directed graph E over a field K is equipped with three mutually compatible structures: L is an associative K-algebra, L is a $\mathbb{Z}$-graded algebra, and L is an algebra with an involution *. Further, L possesses a large supply of idempotents, but it is highly non-commutative. Indeed, in most of the cases, the center of this K-algebra is trivial, being just the field K. In spite

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of this, it is somewhat intriguing and certainly interesting that the ideals of such a non-commutative algebra L exhibit the behavior of the ideals of a Prüfer domain and sometimes that of a Dedekind domain, thus making the multiplicative ideal theory of these algebras L worth investigating. The purpose of this survey is to give a detailed account of some of these properties of L and the resulting factorizations of its ideals. To start with, the ideal multiplication in L is commutative: $AB = BA$ for any two ideals A, B of L. As we shall see, the Prüfer-domain-like properties of L lead to satisfactory factorizations of ideals of L as products of prime, primary, or irreducible ideals. The graded ideals of L seem to possess interesting properties such as coinciding with their own radical, being realizable as Leavitt path algebras of suitable graphs, possessing local units and many others. They play an important role in the factorization of non-graded ideals of L. As noted in ([1], Theorem 2.8.10 and in [19]), the two-sided ideal structure of L can be described completely in terms of the hereditary saturated subsets and breaking vertices and cycles without exits in the graph E and irreducible polynomials in $K[x, x^{-1}]$, and the association preserves the lattice structures. This fact facilitates the description of various factorization properties of the two-sided ideals in L.

This paper is organized as follows. After the preliminaries, Section 3 describes the various properties of the graded ideals of L which are foundational to the study of non-graded ideals and in the factorization of ideals in L. In Section 4, L is shown to be an arithmetical ring, that is, its ideal lattice is distributive and, as a consequence, the Chinese Remainder Theorem holds in L. In addition, L is shown to be a multiplication ring. The ideal version of the number-theoretic theorem $\gcd(m, n) \cdot \operatorname{lcm}(m, n) = mn$ for positive integers m, n holds in L, namely, for any two ideals M, N in L, $(M \cap N)(M + N) = MN$, again a characterizing property of Prüfer domains. In the next section, the prime, the primary, the irreducible, and the radical ideals of L are described in terms of the graph properties of E. It is interesting to note that for a graded ideal I of L the first three of these properties coincide and that I is always a radical ideal. In Section 6, we consider the existence and the uniqueness of factorizations of a non-zero ideal I as a product of prime, primary, or irreducible ideals of L. It is shown that if E is a finite graph or more generally, if L is two-sided Noetherian or Artinian, then every ideal of L is a product of prime ideals. This leads to a complete characterization of L as a generalized ZPI ring, that is, a ring in which every ideal of L is a product of prime ideals. Finally, an analogue of the Krull's theorem on powers of an ideal is proved for Leavitt path algebras. The results of this paper indicate the potential for successful utilization of the ideas and results from the ideal theory of commutative rings in the deeper study of the ideal theory of Leavitt path algebras (of course using different techniques, as L is non-commutative, and using the graphical properties of E and the nature of the graded ideals of L).

2 Preliminaries

For the general notation, terminology and results in Leavitt path algebras, we refer to [1, 18, 22] and for those in graded rings, we refer to [14, 17]. We refer to [8–13, 16] for results in commutative rings. Below we give an outline of some of the needed basic concepts and results.

A (directed) graph $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 together with maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. For each $e \in E^1$, say,

$$\bullet_{s(e)} \xrightarrow{e} \bullet_{r(e)}$$

$s(e)$ is called the **source** of e and $r(e)$ the **range** of e. If $\overset{u}{\bullet} \xrightarrow{e} \overset{v}{\bullet}$ is an edge, then $\overset{u}{\bullet} \xleftarrow{e^*} \overset{v}{\bullet}$ denotes the **ghost edge** e^* with $s(e^*) = v$ and $r(e^*) = u$.

A vertex v is called a **sink** if it emits no edges and a vertex v is called a **regular vertex** if it emits a non-empty finite set of edges. An **infinite emitter** is a vertex which emits infinitely many edges.

A **path** μ of length n is a sequences of edges $\mu = e_1 \dots e_n$ where $r(e_i) = s(e_{i+1})$ for all $i = 1, \cdots, n-1$. $|\mu|$ denotes the length of μ. The path $\mu = e_1 \dots e_n$ in E is **closed** if $r(e_n) = s(e_1)$, in which case μ is said to be *based at the vertex* $s(e_1)$. A closed path μ as above is called **simple** provided it does not pass through its base more than once, i.e., $s(e_i) \neq s(e_1)$ for all $i = 2, \dots, n$. The closed path μ is called a **cycle** if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$.

An *exit* for a path $\mu = e_1 \dots e_n$ is an edge e such that $s(e) = s(e_i)$ for some i and $e \neq e_i$.

If there is a path from vertex u to a vertex v, we write $u \geq v$. A subset D of vertices is said to be **downward directed** if for any $u, v \in D$, there exists a $w \in D$ such that $u \geq w$ and $v \geq w$. A subset H of E^0 is called **hereditary** if, whenever $v \in H$ and $w \in E^0$ satisfy $v \geq w$, then $w \in H$. A hereditary set is **saturated** if, for any regular vertex v, $r(s^{-1}(v)) \subseteq H$ implies $v \in H$.

Definition 1. *Given an arbitrary graph E and a field K, the Leavitt path algebra $L_K(E)$ is defined to be the K-algebra generated by a set $\{v : v \in E^0\}$ of pair-wise orthogonal idempotents, together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:*

(1) $s(e)e = e = er(e)$ for all $e \in E^1$.

(2) $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.

(3) (The "CK-1 relations") For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if $e \neq f$.

(4) (The "CK-2 relations") For every regular vertex $v \in E^0$,

$$v = \sum_{e \in E^1, s(e) = v} ee^*.$$

Note that L need not have an identity. Indeed, L will have the identity 1 exactly when the vertex set E^0 is finite and in that case $1 = \sum_{v \in E^0} v$. However, L possesses **local units**, namely, given any finite set of elements $a_1, \ldots, a_n \in L$, there is an idempotent u such that $ua_i = a_i = a_i u$ for all $i = 1, \ldots, n$. Every element $a \in L := L_K(E)$ can be written as $a = \sum_{i=1}^{n} k_i \alpha_i \beta_i^*$ where α_i, β_i are paths and $k_i \in K$. Here $r(\alpha_i) = s(\beta_i^*) = r(\beta_i)$. From this, it is easy to see that $L = \bigoplus_{u \in E^0} Lu$.

Many well-known examples of rings occur as Leavitt path algebras.

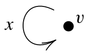
Example 1. The Leavitt path algebra of the straight line graph E :

$$\bullet_{v_1} \xrightarrow{e_1} \bullet_{v_2} \xrightarrow{e_2} \cdots \xrightarrow{e_{n-1}} \bullet_{v_n}$$

is isomorphic to the matrix ring $M_n(K)$.

(Indeed, if $p_1 = e_1 \cdots e_{n-1}$, $p_2 = e_2 \ldots e_{n-1}$, ..., $p_{n-1} = e_{n-1}$, $p_n = v_n$, then $\{\epsilon_{ij} = p_i p_j^* : 1 \leq i, j \leq n\}$ is a set of matrix units, that is, $\epsilon_{ii}^2 = \epsilon_{ii}$ and $\epsilon_{ij}\epsilon_{jk} = \epsilon_{ik}$. Then $\epsilon_{ij} \longmapsto E_{ij}$ induces the isomorphism, where E_{ij} is the $n \times n$ matrix with 1 at (i, j) position and 0 everywhere else.)

Example 2. If E is the graph with a single vertex and a single loop



then $L_K(E) \cong K[x, x^{-1}]$, the Laurent polynomial ring, induced by the map $v \mapsto 1$, $x \mapsto x$, $x^* \mapsto x^{-1}$.

The defining relations of a Leavitt path algebra $L_K(E)$ show that it is a non-commutative ring. Indeed if e is an edge in E, say, $\overset{u}{\bullet} \overset{e}{\longrightarrow} \overset{v}{\bullet}$ where $u \neq v$, then by defining relation (1), $ue = e$, but $eu = evu = e(vu) = 0$. The following proposition describes when $L_K(E)$ becomes a commutative ring.

Proposition 1. *Let E be a connected graph. Then the Leavitt path algebra $L_K(E)$ is commutative if and only if either E consists of just a single vertex $\{\bullet\}$ or E is the graph with a single vertex and a single loop as in Example 2. In this case $L_K(E) \cong K$ or $K[x, x^{-1}]$.*

Every Leavitt path algebra $L_K(E)$ is a $\mathbb{Z}$ **-graded algebra**, namely, $L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_n$ induced by defining, for all $v \in E^0$ and $e \in E^1$, $\deg(v) = 0$, $\deg(e) = 1$, $\deg(e^*) = -1$. Here the L_n, called **homogeneous components**, are abelian subgroups satisfying $L_m L_n \subseteq L_{m+n}$ for all $m, n \in \mathbb{Z}$. Further, for each $n \in \mathbb{Z}$, the subgroup L_n is given by

$$L_n = \{\sum k_i \alpha_i \beta_i^* \in L : |\alpha_i| - |\beta_i| = n\}.$$

An ideal I of $L_K(E)$ is said to be a **graded ideal** if $I = \bigoplus_{n\in\mathbb{Z}}(I \cap L_n)$. If I is a non-graded ideal, then $\bigoplus_{n\in\mathbb{Z}}(I \cap L_n)$ is the largest graded ideal contained in I and is called the **graded part** of I, denoted by $gr(A)$.

We will also be using the fact that the Jacobson radical (and in particular, the prime/Baer radical) of $L_K(E)$ is always zero (see [1]).

Let Λ be an arbitrary non-empty (possibly, infinite) index set. For any ring R, we denote by $M_\Lambda(R)$ the ring of matrices over R whose entries are indexed by $\Lambda \times \Lambda$ and whose entries, except for possibly a finite number, are all zero. It follows from the works in [4] that $M_\Lambda(R)$ is Morita equivalent to R.

Throughout this paper L will denote the Leavitt path algebra $L_K(E)$ of an arbitrary directed graph E over a field K.

3 Graded Ideals of a Leavitt Path Algebra

In this section, we shall describe some of the salient properties of the graded ideals of a Leavitt path algebra L. As we shall see in a later section, these properties impact the factorization of ideals of L. Every ideal of L, whether graded or not, is shown to possess an orthogonal set of generators. As a consequence, we get the interesting property that every finitely generated ideal of L is a principal ideal. It is interesting to note that if I is a graded ideal of L, then both I and L/I can be realized as Leavitt path algebras of suitable graphs.

Suppose H is a hereditary saturated subset of vertices. A **breaking vertex** of H is an infinite emitter $w \in E^0\backslash H$ with the property that $0 < |s^{-1}(w) \cap r^{-1}(E^0\backslash H)| < \infty$. The set of all breaking vertices of H is denoted by B_H. For any $v \in B_H$, v^H denotes the element $v - \sum_{s(e)=v, r(e)\notin H} ee^*$. The following theorem of Tomforde describes graded ideals of L by means of their generators.

Theorem 1. *([22]) Suppose H is a hereditary saturated set of vertices and S is a subset of B_H. Then the ideal $I(H, S)$ generated by the set of idempotents $H \cup \{v^H : v \in S\}$ is a graded ideal of L, and conversely every graded ideal I of L is of the form $I(H, S)$ where $H = I \cap E^0$ and $S = \{u \in B_H : u^H \in I\}$.*

Given a pair (H, S) where H is a hereditary saturated set of vertices in the graph E and S is a subset of B_H, one could construct the **Quotient graph** $E\backslash(H, S)$ given by $(E\backslash(H, S))^0 = E^0\backslash H \cup \{u' : u \in B_H\backslash S\}$, $(E\backslash(H, S))^1 = \{e \in E^1 : r(e) \notin H\} \cup \{e' : e \in E^1$ with $r(e) \in B_H\backslash S\}$ and r, s are extended to $(E\backslash(H, S))^0$ by setting $s(e') = s(e)$ and $r(e') = r(e)'$.

The next theorem describes a generating set Y for a not necessarily graded non-zero ideal of L. This set Y is actually an orthogonal set of generators.

Theorem 2. *([19]) Let E be an arbitrary graph and let I be an arbitrary non-zero ideal of $L = L_K(E)$ with $H = I \cap E^0$ and $S = \{u \in B_H : u^H \in I\}$. Then I is generated by the set*

$$Y = H \cup \{v^H : v \in S\} \cup \{f_t(c_t) : t \in T\},$$

where T is some index set (which may be empty), for each $t \in T$, c_t is a cycle without exits in $E\backslash(H, S)$, no v in S is on any cycle c_t, and $f_t(x) \in K[x]$ is a polynomial with a non-zero constant term and is of the smallest degree such that $f_t(c_t) \in I$. Any two elements $x \neq y$ in Y are orthogonal, that is, $xy = 0 = yx$.

If I is a finitely generated ideal, then the orthogonal set Y of generators mentioned in the above theorem can be shown to be finite and, in that case, the single element $a = \sum_{y \in Y} y$ will be a generator for the ideal I. Consequently, we obtain the following interesting result.

Theorem 3. *([19]) Every finitely generated ideal in a Leavitt path algebra is a principal ideal, i.e., of the form LaL for some $a \in L$.*

Remark 1. In [3], the above theorem has been extended by showing that every finitely generated one-sided ideal of L is a principal ideal, that is, L is a Bêzout ring.

An important property of graded ideals is the following.

Theorem 4. *([21]) Every graded ideal $I(H, S)$ of L can be realized as a Leavitt path algebra $L_K(F)$ of some graph F and further the corresponding quotient ring $L/I(H, S)$ is also a Leavitt path algebra, being isomorphic to the Leavitt path algebra $L_K(E\backslash(H, S))$ of the quotient graph $E\backslash(H, S)$.*

Since Leavitt path algebras possess local units, we conclude that the graded ideals I of L possess local units. Using this, we obtain some interesting properties of graded ideals.

Proposition 2. *([20]) (i) Let A be a graded ideal of L. Then*
(a) for any ideal B of L, $AB = A \cap B$, $BA = B \cap A$ and, in particular, $A^2 = A$;
(b) $AB = BA$ for all ideals B;
(c) If $A = A_1 \cdots A_m$ is a product of ideals, then $A = \bigcap_{i=1}^{m} gr(A_i) = \prod_{i=1}^{m} gr(A_i)$.

Similarly, if $A = A_1 \cap \cdots \cap A_m$ is an intersection of ideals A_i, then $A = \bigcap_{i=1}^{m} gr(A_i) = \prod_{i=1}^{m} gr(A_i)$.

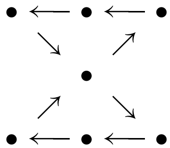
(ii) If $A_1, \ldots, A_m$ are graded ideals of L, then $\prod_{i=1}^{m} A_i = \bigcap_{i=1}^{m} A_i$.

Proof. We shall point out the easy proof of (i)(a). We need only to prove $A \cap B \subseteq AB$. Let $x \in A \cap B$. Since the graded ideal A has local units, there is an idempotent $u \in A$ such that $ua = a = au$. Clearly then $a = ua \in AB$. So $A \cap B = AB$. Similarly, $B \cap A = BA$. Hence $AB = BA$. In particular, $A^2 = A \cap A = A$.

A natural question is when every ideal of L will be a graded ideal. This can happen when E satisfies the following graph property.

Definition 2. *A graph E satisfies **Condition (K)** if whenever a vertex v lies on a simple closed path α, v also lies on another simple closed path β distinct from α.*

Here is a simple graph satisfying Condition (K), where every vertex satisfies the required property.



Theorem 5. *([18, 22]) The following conditions are equivalent for $L := L_K(E)$:*
(a) Every ideal of L is graded;
(b) Every prime ideal of L is graded;
(c) The graph E satisfies Condition (K).

4 The Lattice of Ideals of a Leavitt Path Algebra

This section describes how the ideals of a Leavitt path algebra L share lattice-theoretic properties and module-theoretic properties of the ideals of a Dedekind domain or a Prüfer domain. We start with noting that, in this non-commutative ring L, the multiplication of ideals is commutative. Moreover, L is left/right hereditary, that is, every left/right or two-sided ideal of L is projective as a left or a right ideal. The ideal lattice of L is distributive and multiplicative. It is also shown how many of the characterizing properties of a Prüfer domain stated in terms of its ideals hold in L.

Using a deep theorem of George Bergman, Ara and Goodearl proved the following result that every Leavitt path algebra is a left/right hereditary ring, a property shared by Dedekind domains.

Theorem 6. *(Theorem 3.7, [5]) Every ideal (including any one-sided ideal) of a Leavitt path algebra L is projective as a left/right L-module.*

In Section 3, we noted that if A is a graded ideal of L, then $AB = BA$ for any ideal B of L. What happens if A is not a graded ideal? With an analysis of the "non-graded parts" of A and B, it was shown in [1, 20] that even though L is, in general, non-commutative, the multiplication of its ideals is commutative as noted next.

Theorem 7. *([1, 20]) For any two arbitrary ideals A, B of a Leavitt path algebra L, $AB = BA$.*

The next result shows that every Leavitt path algebra L is an arithmetical ring, that is, the ideal lattice of L is distributive, a property that characterizes Prüfer domains.

Theorem 8. *([20]) For any three ideals A, B, C of the Leavitt path algebra L, we have*

$$A \cap (B + C) = (A \cap B) + (A \cap C).$$

Remark 2. A well-known result in commutative rings (see, e.g., Theorem 18, Chapter V, [23]) states that if the ideal lattice of a commutative ring R is distributive (such as when R is a Dedekind domain), then the Chinese Remainder Theorem holds in R: This means that the simultaneous congruences $x \equiv x_i \pmod{A_i}$ $(i = 1, \cdots, n)$ where the A_i are ideals and the elements $x_i \in R$, admits a solution for x in R provided the compatibility condition $x_i + x_j \equiv 0 \pmod{A_i + A_j}$ holds for all $i \neq j$. The proof of this theorem does not require R to be commutative and nor does it require the existence of a multiplicative identity in R. So, as a consequence of Theorem 8, one can show that the Chinese Remainder Theorem holds in Leavitt path algebras. (Thus Leavitt path algebras satisfy another property of Dedekind domains.)

We next use Theorem 8 to show that every Leavitt path algebra is a multiplication ring, a useful property in the multiplicative ideal theory of Leavitt path algebras.

Theorem 9. *([20]) The Leavitt path algebra $L = L_K(E)$ of an arbitrary graph E is a multiplication ring, that is, for any two ideals A, B of L with $A \subseteq B$, there is an ideal C of L, such that $A = BC = CB$. Moreover, if A is a prime ideal, then $AB = A = BA$.*

A well-known property of a Dedekind domain R is that if there are only finitely many prime ideals in R, then R is a principal ideal domain (see Theorem 16, Chapter V in [23]). Interestingly, as the next theorem shows, a Leavitt path algebra possesses this property.

Theorem 10. *([6]) Let $L := L_K(E)$ be the Leavitt path algebra of an arbitrary graph E. If L has only a finite number of prime ideals, then every ideal of L is a principal ideal, i.e., of the form LaL for some $a \in L$.*

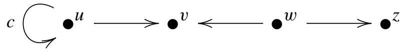
Recently, it was shown (see [7]) that the ideals of a Leavitt path algebra satisfy two more characterizing properties of Prüfer domains.

Theorem 11. *([7]) Let A, B, C be any three ideals of a Leavitt path algebra L. Then*
(i) $A(B \cap C) = AB \cap AC$;
(ii) $(A \cap B)(A + B) = AB$.

Note that the statement (ii) in the preceding theorem is the ideal version of a well-known theorem in elementary number theory that, for any two positive integers a, b, $\gcd(a, b) \cdot \mathrm{lcm}(a, b) = ab$.

However not all characterizing properties of a Prüfer domain hold in a Leavitt path algebra. For instance, a domain R is a Prüfer domain if and only if finitely generated ideals of R are cancellative, that is, if A is a non-zero finitely generated ideal, then for any two ideals B, C of R, $AB = AC$ implies $B = C$. This property may not hold in a Leavitt path algebra as the next example shows.

Example 3. Consider the graph E



Here $H = \{v\}$ is a hereditary saturated subset. Let $A = \langle H \rangle$, the ideal generated by H. Clearly the cycle c has no exits in $E \backslash H$. Let B be the non-graded ideal $A + \langle p(c) \rangle$, where $p(x) = 1 + x \in K[x]$. Clearly $gr(B) = A$. Since A is a graded ideal, we apply Proposition 2 (a), to conclude that $AB = A \cap B = A = A^2 = AA$. But $A \neq B$.

5 Prime, Radical, Primary, and Irreducible Ideals of a Leavitt Path Algebra

In this section, we describe special types of ideals in L such as the prime, the irreducible, the primary, and the radical (= semiprime) ideals using graphical properties. While these concepts are independent for ideals in a commutative ring, we show that the first three properties of ideals coincide for graded ideals in the Leavitt path algebra L. We also show that a non-graded ideal I of L is irreducible if and only if I is a primary ideal if and only if $I = P^n$, a power of a prime ideal P. This is useful in the factorization of ideals in the next section. We also characterize the radical ideals of L. It may be some interest to note that every graded ideal of L is a radical ideal.

The following description of prime ideals of L was given in [18].

Theorem 12. *(Theorem 3.2, [18]) An ideal P of $L := L_K(E)$ with $P \cap E^0 = H$ is a prime ideal if and only if P satisfies one of the following properties:*

(i) $P = I(H, B_H)$ and $E^0 \backslash H$ is downward directed;

(ii) $P = I(H, B_H \backslash \{u\})$, $v \geq u$ for all $v \in E^0 \backslash H$ and the vertex u' that corresponds to u in $E \backslash (H, B_H \backslash \{u\})$ is a sink;

(iii) P is a non-graded ideal of the form $P = I(H, B_H) + \langle p(c) \rangle$, where c is a cycle without exits based at a vertex u in $E \backslash (H, B_H)$, $v \geq u$ for all $v \in E^0 \backslash H$ and $p(x)$ is an irreducible polynomial in $K[x, x^{-1}]$ such that $p(c) \in P$.

Recall, an ideal I of a ring R is called an **irreducible ideal** if, for ideals A, B of R, $I = A \cap B$ implies that either $I = A$ or $I = B$. Given an ideal I, the **radical of the ideal** I, denoted by $Rad(I)$ or $\sqrt{I}$, is the intersection of all prime ideals containing I. A useful property is that if $a \in Rad(I)$, then $a^n \in I$ for some integer $n \geq 0$. (The proof of this property is given in Theorem 10.7 of [15] for non-commutative rings with identity, but the proof also works for rings without identity but with local units.)

If $Rad(I) = I$ for an ideal I, then I is called a **radical ideal** or a **semiprime ideal**. An ideal I of R is said to be a **primary ideal** if, for any two ideals A, B, if $AB \subseteq I$ and $A \nsubseteq I$, then $B \subseteq Rad(I)$.

Remark 3. We note in passing that for any graded ideal I of L, say $I = I(H, S)$, $Rad(I) = I$. Because, $Rad(I)/I$ is a nil ideal in L/I and L/I, being isomorphic to the Leavitt path algebra $L_K(E\backslash(H, S))$, has no non-zero nil ideals.

We now point out an interesting property of graded ideals of L.

Theorem 13. *([20]) Suppose I is a graded ideal of L. Then the following are equivalent:*
(i) I is a primary ideal;
(ii) I is a prime ideal;
(iii) I is an irreducible ideal.

The next theorem extends the above result to arbitrary ideals of L.

Theorem 14. *([20]) Suppose I is a non-graded ideal of L. Then the following are equivalent:*
(i) I is a primary ideal;
(ii) $I = P^n$*, a power of a prime ideal P for some* $n \geq 1$*;*
(iii) I is an irreducible ideal.

The final result of this section describes the radical (also known as semiprime) ideals of L.

Theorem 15. *([2]) Let A be an arbitrary ideal of L with* $A \cap E^0 = H$ *and* $S = \{v \in B_H : v^H \in A\}$*. Then the following properties are equivalent:*
(i) A is a radical ideal of L;
(ii) $A = I(H, S) + \sum_{i \in Y} \langle f_i(c_i) \rangle$*, where Y is an index set which may be empty, for each* $i \in Y$*,* c_i *is a cycle without exits based at a vertex* v_i *in* $E\backslash(H, S)$ *and* $f_i(x)$ *is a polynomial with its constant term non-zero which is a product of distinct irreducible polynomials in* $K[x, x^{-1}]$.

6 Factorization of Ideals in L

As noted in the introduction, ideals in an arithmetical ring admit interesting representations as products of special types of ideals ([10–12]). In this section, we explore the existence and the uniqueness of factorizations of an arbitrary ideal in a Leavitt path algebra L as a product of prime ideals and as a product of irreducible/primary ideals. The prime factorization of graded ideals of L seems to influence that of the non-graded ideals in L. Indeed, an ideal I is a product of prime ideals in L if and only its graded part $gr(I)$ has the same property and, moreover, $I/gr(I)$ is finitely generated with a generating set of cardinality no more than the number of distinct prime ideals in an irredundant factorization of $gr(I)$. It is interesting to note that if I is a graded ideal and if $I = P_1 \cdots P_n$ is an irredundant product of prime ideals, then necessarily each of the ideals P_j must be graded. We also show that I is an intersection of irreducible ideals if and only if I is an intersection of prime ideals. If L is the Leavitt path algebra of a finite graph or, more generally, if L is two-sided Noetherian or two-sided Artinian, then every ideal of L is shown to be a product of prime ideals. We also give necessary and sufficient conditions under which every non-zero ideal of L is a product of prime ideals, that is, when L is a generalized ZPI ring. We end this section by proving for L an analogue of the Krull's theorem on the intersection of powers of an ideal.

We begin with the following useful proposition.

Proposition 3. *([20]) Suppose I is a non-graded ideal of L. If $gr(I)$ is a prime ideal, then I is a product of prime ideals.*

Using this, we obtain the following main factorization theorem.

Theorem 16. *([20]) Let E be an arbitrary graph. For a non-graded ideal I of $L := L_K(E)$, the following are equivalent:*

(i) I is a product of prime ideals;

(ii) I is a product of primary ideals;

(iii) I is a product of irreducible ideals;

(iv) $gr(I)$ is a product of (graded) prime ideals;

(v) $gr(I) = P_1 \cap \cdots \cap P_m$ is an irredundant intersection of m graded prime ideals P_j and $I/gr(I)$ is generated by at most m elements and is of the form $I/gr(I) = \bigoplus_{r=1}^{k} \langle f_r(c_r)\rangle$ where $k \leq m$ and, for each $r = 1 \cdots k$, c_r is a cycle without exits in $E^0 \backslash I$ and $f_r(x) \in K[x]$ is a polynomial with non-zero constant term of smallest degree such that $f_r(c_r) \in I$.

As a consequence of Theorem 16, we obtain a number of corollaries.

Corollary 1. *([20]) Let E be a finite graph, or more generally, let E^0 be finite. Then every non-zero ideal of $L = L_K(E)$ is a product of prime ideals.*

Using a minimal or maximal argument, the above corollary can be extended to the case when the ideals of L satisfy the DCC or ACC as noted below.

Corollary 2. *([20]) Suppose L is two-sided Artinian or two-sided Noetherian. Then every non-zero ideal of L is a product of prime ideals.*

We now give the necessary and sufficient conditions under which L is a generalized ZPI ring, that is, when every ideal of L is a product of prime ideals.

Theorem 17. *([20]) Let E be an arbitrary graph and let $L := L_K(E)$. Then every proper ideal of L is a product of prime ideals if and only if every homomorphic image of L is either a prime ring or contains only finitely many minimal prime ideals.*

The next theorem states that an irredundant factorization of an ideal A as a product of prime ideals in L is unique up to a permutation of the factors. It also points out the interesting fact that if A is a graded ideal, then every factor in this irredundant factorization must also be a graded ideal.

Recall that $A = P_1 \cdots P_n$ is an **irredundant product** of the ideals P_i, if A is not the product of a proper subset of the set $\{P_1, \ldots, P_n\}$.

Theorem 18. *([6]) (a) Suppose A is an arbitrary ideal of L and $A = P_1 \cdots P_m = Q_1 \cdots Q_n$ are two representations of A as irredundant products of prime ideals P_i and Q_j. Then $m = n$ and $\{P_1, \ldots, P_m\} = \{Q_1, \ldots, Q_n\}$;*

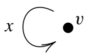
(b) If A is a graded ideal of L and if $A = P_1 \cdots P_m$ is an irredundant product of prime ideals P_j, then the ideals are all graded and $A = P_1 \cap \cdots \cap P_m$.

From Proposition 2(c) and the equivalence of conditions (i) and (iv) of Theorem 16, we derive following proposition.

Proposition 4. *If an ideal I of L is an intersection of finitely many prime ideals, then I is a product of (finitely many) prime ideals.*

But a product of prime ideals in L need not be an intersection of prime ideals as the next example shows.

Example 4. If E is the graph with a single vertex and a single loop



then $L_K(E) \cong K[x, x^{-1}]$, the Laurent polynomial ring, induced by the map $v \mapsto 1$, $x \mapsto x$, $x^* \mapsto x^{-1}$. So it is enough to find a ideal A in $K[x, x^{-1}]$ with the desired property. Consider the prime ideal $A = \langle p(x) \rangle$ in $K[x, x^{-1}]$, where $p(x)$ is an irreducible polynomial. We claim that $B = A^2$ is not an intersection of prime ideals in $K[x, x^{-1}]$. Suppose, on the contrary, $B = \bigcap_{\lambda \in \Lambda} M_\lambda$ where Λ is some (finite or infinite) index set and each M_λ is a (non-zero) prime ideal of $K[x, x^{-1}]$ and hence a maximal ideal of the principal ideal domain $K[x, x^{-1}]$. Now there is a homomorphism $\phi : R \longrightarrow \prod_{\lambda \in \Lambda} R/M_\lambda$ given by $r \mapsto (\ldots, r + M_\lambda, \ldots)$ with $\ker(\phi) = B$. Then $\bar{A} = \phi(A) \cong A/B \neq 0$ satisfies $(\bar{A})^2 = 0$ and this is impossible since $\prod_{\lambda \in \Lambda} R/M_\lambda$, being a direct product of fields, does not contain any non-zero nilpotent ideals.

The next proposition is new and gives necessary and sufficient conditions under which a product of prime ideals in a Leavitt path algebra is also an intersection of prime ideals. This happens exactly when every ideal of L is a radical ideal.

Proposition 5. *Let E be an arbitrary graph and let $L := L_K(E)$. Then the following properties are equivalent:*

(i) Every product of prime ideals in L is an intersection of prime ideals;

(ii) The graph E satisfies Condition (K);

(iii) Every ideal of L is a radical ideal;

(iv) Every ideal of L is a graded ideal.

Proof. Assume (i). Assume, by way of contradiction, that the graph E does not satisfy Condition (K). Then, for some admissible pair (H, S), the quotient graph $E\backslash(H, S)$ does not satisfy Condition (L) (see [1]) and thus there is a cycle c without exits in $E\backslash(H, S)$. By [1, Lemma 2.7.1], the ideal M of $L_K(E\backslash(H, S))$ generated by $\{c^0\}$ is isomorphic to the matrix ring $M_\Lambda(K[x, x^{-1}])$ where Λ is some index set. Then [7, Proposition 1] and Example 4 imply that, for any prime ideal P of M, P^2 is not an intersection of prime ideals of M. Since the graded ideal M is a ring with local units ([1, Corollary 2.5.23]), every ideal (prime ideal) of M is an ideal (prime ideal) of $L_K(E\backslash(H, S))$ and, for any prime ideal Q of $L_K(E\backslash(H, S))$, $M \cap Q$ is a prime ideal of M. Consequently, P^2 cannot be an intersection of prime ideals of $L_K(E\backslash(H, S))$. This is a contradiction, since $L_K(E\backslash(H, S))$, being isomorphic to the quotient ring $L/I(H, S)$, satisfies (i). Consequently, the graph E must satisfy Condition (K), thus proving (ii).

Assume (ii). By [1, Proposition 2.9.9], every ideal of L is graded. On the other hand if $I = I(H, S)$ is a graded ideal, then L/I is isomorphic to the Leavitt path algebra $L_K(E\backslash(H, S)$ and since the prime radical (the intersection of all prime ideals of $L_K(E\backslash(H, S))$ is zero, I is the intersection of all the prime ideals containing I and hence is a radical ideal. This proves (iii).

Assume (iii). We claim that every ideal of L must be a graded ideal. Suppose, by way of contradiction, there is a non-graded ideal I in L, say, $I = I(H, S) + \sum_{i \in Y} \langle f_i(c_i) \rangle$, where Y is an index set and, for each $i \in Y$, $f_i(x) \in K[x]$ and c_i is a cycle without exits in $E\backslash(H, S)$. Now for a fixed $i \in Y$ and an irreducible polynomial $p(x) \in K[x, x^{-1}]$, $P = I(H, S) + \langle p(c_i) \rangle$ is a prime ideal and $\bar{P} = P/I(H, S) = \langle p(c_i) \rangle \subsetneqq M = \langle \{c_i^0\} \rangle$. As noted in the proof of (i) $\implies$ (ii), $\bar{P}^2$ is not a radical ideal of $L/I(H, S)$ and hence P^2 is not a radical ideal in L, a contradiction. Hence every ideal of L is a graded ideal. This proves (iv).

Now (iv) $\implies$ (i), by Proposition 2(c).

We end this section by considering the powers of an ideal in L. From Proposition 2, it is clear that if A is a graded ideal of L, then $A = A^2$ and so $A = A^n$ for all $n \geq 1$. What happens if A is a non-graded ideal? The next proposition implies that, for such an A, $A \neq A^n$ for any $n > 1$.

Proposition 6. *([6]) If A is a non-graded ideal in L, then $\bigcap_{n=1}^{\infty} A^n$ is a graded ideal, being equal to $gr(A)$.*

As a corollary, we obtain

Corollary 3. *An ideal A of L is a graded ideal if and only if $A = A^n$ for all $n \geq 1$.*

W. Krull showed that if A is an ideal of a commutative Noetherian ring with identity 1, then $\bigcap_{n=1}^{\infty} A^n = 0$ if and only if $1 - x$ is not a zero divisor for all $x \in A$ (see Theorem 12, Section 7 in [23]). As a consequence of Proposition 6, we obtain an analogue of Krull's theorem for Leavitt path algebras.

Corollary 4. *([6]) Let A be an arbitrary ideal of L. Then $\bigcap_{n=1}^{\infty} A^n = 0$ if and only if A contains no vertices of the graph E.*

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