Average Failure Rate and Its Applications of Preventive Replacement Policies

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Abstract When a mission arrives at a random time and lasts for an interval, it becomes an important constraint to plan preventive replacement policies, as the unit should provide reliability and no maintenance can be done during the mission interval. From this viewpoint, this chapter firstly gives a definition of an average failure rate, which is based on the conditional failure probability and the mean time to failure, given that the unit is still survival at the mission arrival time. Next, age replacement models are discussed analytically to show that how the average failure rate function appears in the models. In addition, periodic replacement models with minimal repairs are discussed in similar ways. Numerical examples are given when the mission arrival time follows a gamma distribution and the failure time of the unit has a Weibull distribution.

Keywords Age replacement · Minimal repair · Failure rate · Mission interval · Reliability

1 Introduction

Preventive replacement policies have been studied extensively in literatures [1–7]. Barlow and Proschan [1] have firstly given an age replacement model for a finite operating time span, where the unit operates from installation to a fixed interval caused by external factors, and it is replaced at the end of the interval even if no

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failure has occurred. When the finite time span becomes a random interval with working cycles, during which, it is impossible to perform maintenance policies, optimal replacement policies with random works have been discussed [3, 6, 8, 9].

When the working cycles are taken into account for planning replacement polices, Zhao and Nakagawa [10] proposed the policies of replacement first and replacement last, that would become alternatives in points of cost rate, reliability and maintainability. Replacement first means the unit is replaced preventively at events such as operating time, number of repairs, or mission numbers, etc, whichever takes place first, while replacement last means the unit is replaced preventively at the above events, whichever takes place last. It has been shown that replacement last could let the unit operate working cycles as longer as possible while replacement first are more easier to save total maintenance cost [10]. More recent models of replacement first and replacement last can be found in [11–15].

In this chapter, the above working cycle is reconsidered as mission interval, and we suppose that the arrival time of a mission is a random variable rather than it begins from installation and lasts for an interval, during which, the unit should provide reliability and no maintenance can be done. The typical example is maintaining a hot spare for a key unit in a working system, in which, the spare unit should be active at the time when the key unit fails and provide system reliability for an interval when the key unit is unavailable. From this viewpoint, this chapter discusses preventive replacement policies for random arrival of missions. For this, an average failure rate is firstly given based on the conditional failure probability and the mean time to failure, given that the unit is still survival at time t. We next formulate and optimize the models of age replacement policies and the periodic policies with minimal repairs in analytical ways. Numerical examples are given when the mission arrival time follows a gamma distribution and the failure time of the unit has a Weibull distribution.

2 Average Failure Rate

It is assumed that a unit has a general failure distribution $F(t) \equiv \Pr\{X \leq t\}$ with a density function $f(t) \equiv \mathrm{d}F(t)/\mathrm{d}t$ and a finite mean $\mu \equiv \int_0^\infty \overline{F}(t)\mathrm{d}t$. The conditional failure probability is given by [2]:

$$\lambda(t; x) \equiv \frac{F(t+x) - F(t)}{\overline{F}(t)} \quad (0 < x < \infty), \tag{1}$$

which represents the probability that the unit fails in interval $[t, t+x]$, given that it is still survival at time t. Note that $0 \leq \lambda(t; x) \leq 1$. When $x \to 0$, $\lambda(t; x)/x$ becomes an instant failure rate:

$$h(t) \equiv \frac{f(t)}{\overline{F}(t)} = -\frac{1}{\overline{F}(t)} \frac{\mathrm{d}\overline{F}(t)}{\mathrm{d}t}. \tag{2}$$

We usually suppose, in modeling maintenance policies, that $h(t)$ increases with t from $h(0) = 0$ to $h(\infty) \equiv \lim_{t\to\infty} h(t)$ that might be infinity, i.e., $\lambda(t; x)$ increases with t from $F(x)$ to 1.

We next define:

$$F(t; x) = \frac{\int_t^{t+x} \overline{F}(u)\mathrm{d}u}{\overline{F}(t)}, \tag{3}$$

which means the mean time to failure, given that the unit is still survival at time t. Obviously, when $t \to 0$, $F(t; x)$ becomes $\int_0^x \overline{F}(u)\mathrm{d}u$, that represents the mean time to replacement when the unit is replaced preventively at time x or correctively at failure, whichever takes place first. When $t \to \infty$,

$$\lim_{t\to\infty} \frac{\int_t^{t+x} \overline{F}(u)\mathrm{d}u}{\overline{F}(t)} = \lim_{t\to\infty} \frac{F(t+x) - F(t)}{f(t)} = \lim_{t\to\infty} \frac{\lambda(t; x)}{h(t)} = \frac{1}{h(\infty)}.$$

Differentiating $\int_t^{t+x} \overline{F}(u)\mathrm{d}u/\overline{F}(t)$ with t, and noting that

$$\begin{aligned} &h(t) \int_t^{t+x} \overline{F}(t)\mathrm{d}t - [F(t+x) - F(t)] \\ &\le h(t) \int_t^{t+x} \left[\frac{f(u)}{h(t)}\right] \mathrm{d}u - [F(t+x) - F(t)] = 0. \end{aligned}$$

which shows that $F(t; x)$ decreases with t from $\int_0^x \overline{F}(u)\mathrm{d}u$ to $1/h(\infty)$.

Using $\lambda(t; x)$ and $F(t; x)$, we define:

$$\Lambda(t; x) \equiv \frac{F(t+x) - F(t)}{\int_t^{t+x} \overline{F}(u)\mathrm{d}u} \quad (0 < x < \infty), \tag{4}$$

which means the average failure rate, given that the unit is still survival at time t. It can be easily proved that $\Lambda(t; x)$ increases with t from $F(x)/\int_0^x \overline{F}(u)\mathrm{d}u$ to $h(\infty)$, and $h(t) \le \Lambda(t; x) \le h(t+x)$.

3 Age Replacement

In this section, we apply the above average failure rate into age replacement policies with random arrival of missions. That is, the unit begins to operate after installation, and its failure time X $(0 < X < \infty)$ has a general distribution $F(t) \equiv \Pr\{X \le t\}$ with finite mean $\mu \equiv \int_0^\infty \overline{F}(t)\mathrm{d}t$. In addition, the unit should be active at time T_o $(0 < T_o < \infty)$ for an interval $[T_o, T_o + t_x]$ $(0 \le t_x < \infty)$ to provide reliability. In this case, t_x can be considered as a mission interval during which the unit provides reliability in [2].

3.1 Constant T_o

We plan the unit is replaced preventively at time $T_o + t_x$ $(0 \le t_x \le \infty)$ when it is still survival at time T_o $(0 \le T_o < \infty)$, or it is replaced correctively at failure time X during $(0, T_o + t_x]$, whichever takes place first.

The probability that the unit is replaced at $T_o + t_x$ is

$$\Pr\{X > T_o + t_x\} = \overline{F}(T_o + t_x), \tag{5}$$

and the probability that it is replaced at failure is

$$\Pr\{X \le T_o + t_x\} = F(T_o + t_x). \tag{6}$$

The mean time from installation to replacement is

$$(T_o + t_x)\overline{F}(T_o + t_x) + \int_0^{T_o+t_x} t\mathrm{d}F(t) = \int_0^{T_o+t_x} \overline{F}(t)\mathrm{d}t \tag{7}$$

Thus, the expected replacement cost rate is

$$C_s(t_x; T_o) = \frac{c_p + (c_f - c_p)F(T_o + t_x)}{\int_0^{T_o+t_x} \overline{F}(t)\mathrm{d}t}, \tag{8}$$

where c_f and c_p $(c_p < c_f)$ are the costs of replacement policies done at failure and at $T_o + t_x$, respectively.

We find optimum t_x^* to minimize $C_s(t_x; T_o)$ in (8). Differentiating $C_s(t_x; T_o)$ with respect to t_x and setting it equal to zero,

$$h(T_o + t_x) \int_0^{T_o+t_x} \overline{F}(t)\mathrm{d}t - F(T_o + t_x) = \frac{c_p}{c_f - c_p}, \tag{9}$$

whose left-hand side increases with t_x from

$$h(T_o) \int_0^{T_o} \overline{F}(t)\mathrm{d}t - F(T_o)$$

to $h(\infty)/\mu - 1$. Thus, if $h(t)$ increases strictly with t to $h(\infty) = \infty$, then there exists a finite and unique t_x^* $(0 \le t_x^* < \infty)$ which satisfies (9), and the resulting cost rate is

$$C_s(t_x^*; T_o) = (c_f - c_p)h(T_o + t_x^*). \tag{10}$$

Noting that the left-hand side of (9) increases with T_o, t_x^* decreases with T_o from T^* to 0, where T^* is an optimum age replacement time that satisfies

$$h(T)\int_0^T \overline{F}(t)\mathrm{d}t - F(T) = \frac{c_p}{c_f - c_p}. \tag{11}$$

3.2 Random T_o

When T_o is a random variable and has a general distribution $Y(t) \equiv \Pr\{T_o \le t\}$ with a density function $y(t) \equiv \mathrm{d}Y(t)/\mathrm{d}t$ and a finite mean $\gamma = \int_0^\infty \overline{Y}(t)\mathrm{d}t$, we plan that the unit is replaced preventively at time $T_o + t_x$ $(0 \le t_x \le \infty)$ when it is still survival at a random time T_o $(0 \le T_o < \infty)$, or it is replaced correctively at failure time X during $(0, T_o + t_x]$, whichever takes place first.

The probability that the unit is replaced at $T_o + t_x$ is

$$\Pr\{X > T_o + t_x\} = \int_0^\infty \overline{F}(t + t_x)\mathrm{d}Y(t), \tag{12}$$

and the probability that it is replaced at failure is

$$\Pr\{X \le T_o + t_x\} = \int_0^\infty F(t + t_x)\mathrm{d}Y(t). \tag{13}$$

The mean time from installation to replacement is

$$\begin{aligned} &\int_0^\infty (t + t_x)\overline{F}(t + t_x)\mathrm{d}Y(t) + \int_0^\infty \left[\int_0^{t+t_x} u\mathrm{d}F(u)\right]\mathrm{d}Y(t) \\ &= \int_0^\infty \left[\int_0^{t+t_x} \overline{F}(u)\mathrm{d}u\right]\mathrm{d}Y(t). \end{aligned} \tag{14}$$

Thus, the expected replacement cost rate is

$$C_s(t_x; Y) = \frac{c_p + (c_f - c_p)\int_0^\infty F(t + t_x)\mathrm{d}Y(t)}{\int_0^\infty [\int_0^{t+t_x} \overline{F}(u)\mathrm{d}u]\mathrm{d}Y(t)}, \tag{15}$$

where c_f and c_p $(c_p < c_f)$ are the costs of replacement policies done at failure and at $T_o + t_x$, respectively.

Clearly,

$$\lim_{t_x\to\infty} C_s(t_x;Y) = \frac{c_f}{\mu},$$

$$\lim_{t_x\to 0} C_s(t_x;Y) = \frac{c_p + (c_f - c_p)\int_0^\infty F(t)\mathrm{d}Y(t)}{\int_0^\infty \overline{F}(t)\overline{Y}(t)\mathrm{d}t},$$

which agrees with random replacement model in [3].

We find optimum t_x^* to minimize $C_s(t_x;Y)$ in (15). Differentiating $C_s(t_x;Y)$ with respect to t_x and setting it equal to zero,

$$h_s(t_x)\int_0^\infty \left[\int_0^{t+t_x} \overline{F}(u)\mathrm{d}u\right]\mathrm{d}Y(t) - \int_0^\infty F(t+t_x)\mathrm{d}Y(t) = \frac{c_p}{c_f - c_p}, \tag{16}$$

where

$$h_s(t_x) \equiv \frac{\int_0^\infty f(t+t_x)\mathrm{d}Y(t)}{\int_0^\infty \overline{F}(t+t_x)\mathrm{d}Y(t)}.$$

When $Y(t) = 1 - \mathrm{e}^{-\theta t}$,

$$h_s(t_x) \equiv \lim_{T\to\infty} h_f(T;t_x) \equiv \lim_{T\to\infty} \frac{\int_0^T f(t+t_x)\mathrm{d}Y(t)}{\int_0^T \overline{F}(t+t_x)\mathrm{d}Y(t)},$$

and it increases with t_x from $h_s(0) = \int_0^\infty f(t)\mathrm{e}^{-\theta t}\mathrm{d}t / \int_0^\infty \overline{F}(t)\mathrm{e}^{-\theta t}\mathrm{d}t$ to $h(\infty)$. Then, the left-hand side of (16) increases with t_x to ∞ as $h(\infty) \to \infty$. In this case, there exists a finite and unique t_x^* $(0 \le t_x^* < \infty)$ which satisfies (16), and the resulting cost rate is

$$C_s(t_x^*;Y) = (c_f - c_p)h_s(t_x^*). \tag{17}$$

When T_o has a gamma distribution with a density function $y(t) = \theta^k t^{k-1} \mathrm{e}^{-\theta t}/(k-1)!$ $(k = 1, 2, \ldots)$, and the failure time X has a Weibull distribution $F(t) = 1 - \mathrm{e}^{-(\alpha t)^\beta}$ $(\alpha > 0, \beta > 1)$, Table 1 presents optimum t_x^* and its cost rate $C_s(t_x^*;Y)$ for k and c_p when $\theta = 1.0$, $\alpha = 0.1$, $\beta = 2.0$, and $c_f = 100.0$. Table 1 shows that optimum interval $[T_o, T_o + t_x^*]$ decreases with k and increases with c_p. This means that if k becomes large, then the failure rate increases with T_o and t_x^* becomes small. On the other hand, if c_p $(<c_f)$ becomes large, then it is unnecessary to replace the unit at a early time and t_o^* becomes large.

Table 1 Optimum t_x^* and its cost rate $C_s(t_x^*; Y)$ when $\theta = 1.0$, $\alpha = 0.1$, $\beta = 2.0$, and $c_f = 100.0$

c_p	$k = 1$		$k = 2$		$k = 5$	
	t_x^*	$C_s(t_x^*; Y)$	t_x^*	$C_s(t_x^*; Y)$	t_x^*	$C_s(t_x^*; Y)$
10.0	2.564	6.269	1.756	6.466	$t_x^* \to 0$	7.048
15.0	3.446	7.397	2.625	7.537	0.147	7.911
20.0	4.282	8.279	3.457	8.385	0.968	8.662
25.0	5.114	8.989	4.286	9.067	1.797	9.276
30.0	5.966	9.565	5.141	9.624	2.659	9.780
35.0	6.863	10.031	6.043	10.076	3.575	10.191
40.0	7.832	10.406	7.018	10.439	4.569	10.521
45.0	8.901	10.700	8.096	10.723	5.671	10.779
50.0	10.111	10.921	9.316	10.937	6.924	10.974

3.3 Replace at T and $T_o + t_x$

In order to prevent early or late arrivals of time T_o, we plan that the unit is replaced preventively at time T $(0 < T \le \infty)$ or at time $T_o + t_x$ $(0 \le t_x \le \infty)$, whichever takes place first. However, no replacement can be done preventively during the interval $[T_o, T_o + t_x]$. In this policy, t_x is constantly given and T_o is a random variable with a general distribution $Y(t)$.

The probability that the unit is replaced at T is

$$\Pr\{X > T, T_o > T\} = \overline{F}(T)\overline{Y}(T), \tag{18}$$

the probability that it is replaced at $T_o + t_x$ is

$$\Pr\{X > T_o + t_x, T_o \le T\} = \int_0^T \overline{F}(t + t_x)\mathrm{d}Y(t), \tag{19}$$

and the probability that it is replaced at failure is

$$\Pr\{X \le T \text{ and } T_o \ge T, X \le T_o + t_x \text{ and } T_o < T\} = F(T)\overline{Y}(T) + \int_0^T F(t + t_x)\mathrm{d}Y(t), \tag{20}$$

where note that (18) + (19) + (20) = 1.

The mean time from installation to replacement is

$$
\begin{aligned}
&T\overline{F}(T)\overline{Y}(T) + \int_0^T (t+t_x)\overline{F}(t+t_x)\mathrm{d}Y(t) + \overline{Y}(T)\int_0^T t\mathrm{d}F(t) \\
&+ \int_0^T \left[\int_0^{t+t_x} u\mathrm{d}F(u)\right]\mathrm{d}Y(t) \\
&= \overline{Y}(T)\int_0^T \overline{F}(t)\mathrm{d}t + \int_0^T \left[\int_0^{t+t_x} \overline{F}(u)\mathrm{d}u\right]\mathrm{d}Y(t). \qquad (21)
\end{aligned}
$$

Thus, the expected replacement cost rate is

$$
C_f(T;t_x) = \frac{c_p + (c_f - c_p)[F(T)\overline{Y}(T) + \int_0^T F(t+t_x)\mathrm{d}Y(t)]}{\overline{Y}(T)\int_0^T \overline{F}(t)\mathrm{d}t + \int_0^T [\int_0^{t+t_x} \overline{F}(u)\mathrm{d}u]\mathrm{d}Y(t)}, \qquad (22)
$$

Note that when $t_x \to \infty$, $\lim_{t_x\to\infty} C_f(T;t_x)$ becomes age replacement model in [2], when $t_x \to 0$, $\lim_{t_x\to 0} C_f(T;t_x)$ becomes random replacement model in [3], when $T \to \infty$, $\lim_{T\to\infty} C_f(T;t_x) = C_s(t_x;Y)$ in (15), and when $T \to 0$, $\lim_{T\to 0} C_f(T;t_x) = \infty$.

We find optimum T_f^* and t_{xf}^* to minimize $C_f(T;t_x)$ in (22) for given t_x. Differentiating $C_f(T;t_x)$ with respect to T and setting it equal to zero,

$$
\begin{aligned}
&q_f(T;t_x)\left\{\overline{Y}(T)\int_0^T \overline{F}(t)\mathrm{d}t + \int_0^T \left[\int_0^{t+t_x} \overline{F}(u)\mathrm{d}u\right]\mathrm{d}Y(t)\right\} \\
&-\left[F(T)\overline{Y}(T) + \int_0^T F(t+t_x)\mathrm{d}Y(t)\right] = \frac{c_p}{c_f - c_p}, \qquad (23)
\end{aligned}
$$

where

$$
q_f(T;t_x) \equiv \frac{r(T)\lambda(T;t_x) + h(T)}{r(T)\dfrac{\lambda(T;t_x)}{\Lambda(T;t_x)} + 1} \quad \text{and} \quad r(T) \equiv \frac{y(T)}{\overline{Y}(T)},
$$

and the instant failure rate $h(T)$, the conditional failure probability $\lambda(T;t_x)$ and the average failure rate $\Lambda(T;t_x)$ are included in $q_f(T;t_x)$.

When $Y(t) = 1 - \mathrm{e}^{-\theta t}$, $r(T) = \theta$ and

$$
q_f(T;t_x) = \frac{\theta[F(T+t_x) - F(T)] + f(T)}{\theta\int_T^{T+t_x} \overline{F}(t)\mathrm{d}t + \overline{F}(T)}.
$$

Note that

$$
h(T) < \frac{F(T+t_x) - F(T)}{\int_T^{T+t_x} \overline{F}(t)\mathrm{d}t} < h(T+t_x),
$$

then $q_f(T; t_x)$ increases strictly with T to ∞ as $h(\infty) \to \infty$, and also increases strictly with t_x to $q_f(T; \infty)$. Thus, the left-hand side of (23) increases with T from 0 to ∞ as $h(\infty) \to \infty$. In this case, there exists a finite and unique T_f^* $(0 < T_f^* < \infty)$ which satisfies (23), and the resulting cost rate is

$$C_f(T_f^*; t_x) = (c_f - c_p)q_f(T_f^*; t_x). \tag{24}$$

In addition, the left-hand side of (23) increases with t_x, then T_f^* decreases with t_x from T^* which satisfies the following random replacement model [3],

$$h(T)\int_0^T \mathrm{e}^{-\theta t}\overline{F}(t)\mathrm{d}t - \int_0^T \mathrm{e}^{-\theta t}\mathrm{d}F(t) = \frac{c_p}{c_f - c_p}.$$

Nest, we find optimum t_{xf}^* for given T. Differentiating $C_f(T; t_x)$ with respect to t_x for given T and setting it equal to zero,

$$h_f(T; t_x)\left\{\overline{Y}(T)\int_0^T \overline{F}(t)\mathrm{d}t + \int_0^T \left[\int_0^{t+t_x}\overline{F}(u)\mathrm{d}u\right]\mathrm{d}Y(t)\right\}$$
$$-\left[F(T)\overline{Y}(T) + \int_0^T F(t+t_x)\mathrm{d}Y(t)\right] = \frac{c_p}{c_f - c_p}, \tag{25}$$

where

$$h_f(T; t_x) \equiv \frac{\int_0^T f(t+t_x)\mathrm{d}Y(t)}{\int_0^T \overline{F}(t+t_x)\mathrm{d}Y(t)} < h(T+t_x).$$

When $Y(t) = 1 - \mathrm{e}^{-\theta t}$, $h_f(T; t_x)$ increases with t_x to $h(\infty)$. Then, the left-hand side of (25) increases strictly with t_x from 0 to ∞ as $h(\infty) \to \infty$. In this case, there exists a finite and unique t_{xf}^* $(0 < t_{xf}^* < \infty)$ which satisfies (25), and the resulting cost rate is

$$C_f(T; t_{xf}^*) = (c_f - c_p)h_f(T; t_{xf}^*). \tag{26}$$

Note that t_{xf}^* decreases with T to t_x^* given in (16), as the left-hand side of (22) increases with T to that of (16).

When $y(t) = \theta^k t^{k-1}\mathrm{e}^{-\theta t}/(k-1)!$ $(k = 1, 2, \ldots)$ and $F(t) = 1 - \mathrm{e}^{-(\alpha t)^\beta}$, $(\alpha > 0, \beta \geq 1)$, Table 2 presents optimum T_f^* and its cost rate $C_f(T_f^*; t_x)$ for t_x and c_p when $\theta = 1.0, k = 2, \alpha = 0.1, \beta = 2.0$, and $c_f = 100.0$, and Table 3 presents optimum t_{xf}^* and its cost rate $C_f(T; t_{xf}^*)$ for T and c_p when $\theta = 1.0, k = 2, \alpha = 0.1, \beta = 2.0$, and $c_f = 100.0$. Table 3 shows that T_f^* increases with c_p and decreases with t_x and t_{xf}^* increases with c_p and decreases with T, as shown in the above analytical discussions.

Table 2 Optimum T_f^* and its cost rate $C_f(T_f^*; t_x)$ when $\theta = 1.0$, $k = 2$, $\alpha = 0.1$, $\beta = 2.0$, and $c_f = 100.0$

c_p	$t_x = 1.0$		$t_x = 2.0$		$t_x = 5.0$	
	T_f^*	$C_f(T_f^*; t_x)$	T_f^*	$C_f(T_f^*; t_x)$	T_f^*	$C_f(T_f^*; t_x)$
10.0	3.368	6.442	2.864	6.174	2.167	6.977
15.0	4.577	8.148	3.834	7.493	2.871	7.786
20.0	5.888	9.769	4.864	8.702	3.587	8.453
25.0	7.354	11.360	6.001	9.861	4.357	9.048
30.0	9.027	12.943	7.291	11.002	5.212	9.602
35.0	10.961	14.524	8.776	12.136	6.185	10.135
40.0	13.260	16.105	10.513	13.268	7.315	10.659
45.0	16.050	17.686	12.612	14.400	8.650	11.179
50.0	19.265	19.266	15.380	15.532	10.257	11.698

Table 3 Optimum t_{xf}^* and its cost rate $C_f(T; t_{xf}^*)$ when $\theta = 1.0$, $k = 2$, $\alpha = 0.1$, $\beta = 2.0$, and $c_f = 100.0$

c_p	$T = 1.0$		$T = 2.0$		$T = 5.0$	
	t_{xf}^*	$C_f(T; t_{xf}^*)$	t_{xf}^*	$C_f(T; t_{xf}^*)$	t_{xf}^*	$C_f(T; t_{xf}^*)$
10.0	3.918	8.137	2.446	6.330	1.800	6.350
15.0	5.541	10.440	3.512	7.781	2.673	7.447
20.0	7.135	12.374	4.543	8.964	3.510	8.317
25.0	8.776	14.058	5.581	9.953	4.345	9.022
30.0	10.523	15.564	6.659	10.791	5.206	9.598
35.0	12.442	16.943	7.811	11.510	6.116	10.070
40.0	14.606	18.234	9.069	12.126	7.101	10.452
45.0	17.113	19.468	10.479	12.659	8.190	10.754
50.0	20.088	20.670	12.101	13.122	9.424	10.985

4 Minimal Repair

It is assumed that the unit undergoes minimal repairs at failures and begins to operate again after repairs, where the time for repairs are negligible and the failure rate remains undisturbed by repairs. In this case, we define

$$\Lambda(t; x) = \frac{1}{x}\int_t^{t+x} h(u)\mathrm{d}u, \tag{27}$$

which means the average failure rate for an interval $[t, t + x]$. It is obviously to show that $\Lambda(t; x)$ increases with t from $H(x)/x$ to $h(\infty)$ and increases with x from $h(0)$ to $h(\infty)$, and $h(t) \le \Lambda(t; x) \le h(t + x)$.

4.1 Constant T_o

In order to prevent an increasing repair cost, we plan that the unit is replaced at time $T_o + t_x$ $(0 < T_o \leq \infty, 0 \leq t_x < \infty)$. Noting that the expected number of failures during $(0, T_o + t_x]$ is $H(T_o + t_x)$, the expected cost rate is

$$C_s(t_x; T_o) = \frac{c_m H(T_o + t_x) + c_p}{T_o + t_x}, \tag{28}$$

where c_m is minimal repair cost at failure, and c_p is given in (8).

We find optimum t_x^* to minimize $C_s(t_x; T_o)$ for given T_o. Differentiating $C_s(t_x; T_o)$ with respect to t_x and setting it equal to zero,

$$h(T_o + t_x)(T_o + t_x) - H(T_o + t_x) = \frac{c_p}{c_m}, \tag{29}$$

whose left-hand side increases with t_x from $h(T_o)T_o - H(T_o)$ to $\int_0^\infty [h(\infty) - h(t)]\mathrm{d}t$. Thus, if the failure rate $h(t)$ increases strictly with t to $h(\infty) = \infty$, then there exists a finite and unique t_x^* $(0 \leq t_x^* < \infty)$ which satisfies (29), and the resulting cost rate is

$$C_s(t_x^*; T_o) = c_m h(T_o + t_x^*), \tag{30}$$

Noting that the left-hand side of (29) increases with T_o, t_x^* decreases with T_o from T^* to 0, where T^* is an optimum periodic replacement time that satisfies

$$h(T)T - H(T) = \frac{c_p}{c_m}. \tag{31}$$

4.2 Random T_o

We plan that the unit is replaced at time $T_o + t_x$ $(0 \leq t_x < \infty)$, where T_o is a random variable with distribution $Y(t)$. Then, the expected cost rate is

$$C_s(t_x; Y) = \frac{c_m \int_0^\infty H(t + t_x)\mathrm{d}Y(t) + c_p}{\int_0^\infty (t + t_x)\mathrm{d}Y(t)}, \tag{32}$$

where c_m is minimal repair cost at failure, and c_p is given in (15).

Clearly, $\lim_{t_x \to \infty} C_s(t_x; Y) \to \infty$ and

$$\lim_{t_x \to 0} C_s(t_x; Y) = \frac{c_m \int_0^\infty H(t)\mathrm{d}Y(t) + c_p}{\int_0^\infty t\mathrm{d}Y(t)},$$

which agrees with random replacement model [3].

If there exists an optimum t_x^* to minimize $C_s(t_x; Y)$ in (32), it satisfies

$$\int_0^{\infty}(t+t_x)\mathrm{d}Y(t)\int_0^{\infty}h(t+t_x)\mathrm{d}Y(t)-\int_0^{\infty}H(t+t_x)\mathrm{d}Y(t)=\frac{c_p}{c_m}, \tag{33}$$

whose left-hand side increases with t_x to ∞ as $h(\infty)\to\infty$. In this case, the resulting cost rate is

$$C_s(t_x^*; Y)=c_m\int_0^{\infty}h(t+t_x^*)\mathrm{d}Y(t). \tag{34}$$

When $y(t)=\theta^k t^{k-1}\mathrm{e}^{-\theta t}/\Gamma(k)$ and $F(t)=1-\mathrm{e}^{-(\alpha t)^{\beta}}$, Table 4 presents optimum t_x^* and its cost rate $C_s(t_x^*; Y)$ for k and c_m when $\theta=1.0$, $\alpha=1.0$, $\beta=2.0$, and $c_p=100.0$. Table 4 shows that optimum interval $[T_o, T_o+t_x^*]$ decreases when c_m increases and T_o arrives at a late time due to the total increasing repair cost. Note that when $k=5$, $t_x^*\to 0$ for all of c_m.

4.3 Replace at T and T_o+t_x

We plan that the unit is replaced at time T $(0<T\le\infty)$ or at time T_o+t_x $(0\le t_x\le\infty)$, whichever takes place first; however, only minimal repairs can be done during the interval $[T_o, T_o+t_x]$. Then, the expected number of repairs between replacement policies is

$$H(T)\overline{Y}(T)+\int_0^{T}H(t+t_x)\mathrm{d}Y(t), \tag{35}$$

Table 4 Optimum t_x^* and its cost rate $C_s(t_x^*; Y)$ when $\theta=1.0$, $\alpha=1.0$, $\beta=2.0$, and $c_p=100.0$

c_m	$k=1$		$k=2$		$k=5$	
	t_x^*	$C_s(t_x^*; Y)$	t_x^*	$C_s(t_x^*; Y)$	t_x^*	$C_s(t_x^*; Y)$
10.0	2.317	66.324	1.465	69.178	$t_x^*\to 0$	77.383
15.0	1.769	83.324	0.944	88.133	$t_x^*\to 0$	105.355
20.0	1.449	97.953	0.644	105.529	$t_x^*\to 0$	133.327
25.0	1.236	111.782	0.447	122.048	$t_x^*\to 0$	161.300
30.0	1.081	124.855	0.307	138.057	$t_x^*\to 0$	189.272
35.0	0.963	137.392	0.201	153.651	$t_x^*\to 0$	217.244
40.0	0.871	149.613	0.118	168.977	$t_x^*\to 0$	245.217
45.0	0.794	161.407	0.051	184.072	$t_x^*\to 0$	273.189
50.0	0.732	173.128	$t_x^*\to 0$	199.001	$t_x^*\to 0$	301.161

and the mean time from installation to replacement is

$$T\overline{Y}(T) + \int_0^T (t + t_x)\mathrm{d}Y(t) = t_x Y(T) + \int_0^T \overline{Y}(t)\mathrm{d}t. \tag{36}$$

Thus, the expected replacement cost rate is

$$C_f(T; t_x) = \frac{c_m[H(T)\overline{Y}(T) + \int_0^T H(t + t_x)\mathrm{d}Y(t)] + c_p}{t_x Y(T) + \int_0^T \overline{Y}(t)\mathrm{d}t}. \tag{37}$$

Differentiating $C_f(T; t_x)$ with respect to T and setting it equal to zero,

$$q_f(T; t_x)\left[t_x Y(T) + \int_0^T \overline{Y}(t)\mathrm{d}t\right] - \left[H(T)\overline{Y}(T) + \int_0^T H(t + t_x)\mathrm{d}Y(t)\right] = \frac{c_p}{c_m}, \tag{38}$$

where

$$q_f(T; t_x) \equiv \frac{r(T)\Lambda(T; t_x) + h(T)/t_x}{r(T) + 1/t_x}. \tag{39}$$

When $Y(t) = 1 - \mathrm{e}^{-\theta t}$, $q_f(T; t_x)$ increases with T to $h(\infty)/(\theta t_x + 1)$. Then, the left-hand side of (38) increases with T from 0 to ∞ as $h(\infty) \to \infty$. Therefore, there exits a finite and unique T_f^* $(0 < T_f^* < \infty)$ which satisfies (38), and the resulting cost rate is

$$C_f(T_f^*; t_x) = c_m \frac{\theta\Lambda(T_f^*; t_x) + h(T_f^*)/t_x}{\theta + 1/t_x}. \tag{40}$$

Next, differentiating $C_f(T; t_x)$ with respect to t_x and setting it equal to zero,

$$\frac{\int_0^T h(t + t_x)\mathrm{d}Y(t)}{Y(T)}\left[t_x Y(T) + \int_0^T \overline{Y}(t)\mathrm{d}t\right] - \left[H(T)\overline{Y}(T) + \int_0^T H(t + t_x)\mathrm{d}Y(t)\right] = \frac{c_p}{c_m}, \tag{41}$$

whose left-hand side increases with t_x to ∞ as $h(\infty) \to \infty$. Therefore, there exists a finite and unique t_{xf}^* $(0 \le t_{xf}^* < \infty)$ which satisfies (40), and the resulting cost rate is

$$C_f(T; t_{xf}^*) = c_m \frac{\int_0^T h(t + t_{xf}^*)\mathrm{d}Y(t)}{Y(T)}. \tag{42}$$

Table 5 Optimum T_f^* and its cost rate $C_f(T_f^*; t_x)$ when $\theta = 1.0$, $\alpha = 1.0$, $\beta = 2.0$, $t_x = 1.0$, and $c_p = 100.0$

c_m	$k = 1$		$k = 2$		$k = 3$	
	T_f^*	$C_f(T_f^*; t_x)$	T_f^*	$C_f(T_f^*; t_x)$	T_f^*	$C_f(T_f^*; t_x)$
10.0	3.467	74.336	3.115	66.613	3.057	64.619
15.0	2.588	85.137	2.451	79.764	2.451	78.299
20.0	2.139	95.547	2.080	91.265	2.119	90.667
25.0	1.846	104.785	1.846	102.121	1.885	101.321
30.0	1.631	112.852	1.670	111.739	1.709	110.321
35.0	1.494	122.090	1.533	120.521	1.592	120.077
40.0	1.377	130.156	1.416	128.062	1.475	127.323
45.0	1.279	137.637	1.338	136.789	1.396	135.778
50.0	1.201	145.117	1.260	143.873	1.318	142.548

Table 6 Optimum t_{xf}^* and its cost rate $C_f(T; t_{xf}^*)$ when $\theta = 1.0$, $\alpha = 1.0$, $\beta = 2.0$, $T = 1.0$ and $c_p = 100.0$

c_m	$k = 1$		$k = 2$		$k = 3$	
	t_{xf}^*	$C_f(T; t_{xf}^*)$	t_{xf}^*	$C_f(T; t_{xf}^*)$	t_{xf}^*	$C_f(T; t_{xf}^*)$
10.0	3.096	70.275	3.564	83.445	4.189	97.977
15.0	2.393	84.318	2.588	95.870	2.861	107.121
20.0	1.982	96.018	2.041	105.952	2.158	114.704
25.0	1.709	106.350	1.689	114.862	1.709	120.919
30.0	1.514	115.902	1.436	122.600	1.416	127.524
35.0	1.357	124.281	1.240	129.362	1.182	132.372
40.0	1.221	131.098	1.084	135.342	1.025	138.782
45.0	1.123	138.696	0.967	141.713	0.889	143.825
50.0	1.045	146.294	0.869	147.693	0.791	150.040

When $y(t) = \theta^k t^{k-1} e^{-\theta t} / \Gamma(k)$ and $F(t) = 1 - e^{-(\alpha t)^\beta}$, Table 5 presents optimum T_f^* and its cost rate $C_f(T_f^*; t_x)$ for t_x and c_m when $\theta = 1.0$, $\alpha = 1.0$, $\beta = 2.0$, $t_x = 1.0$, and $c_p = 100.0$, and Table 6 presents optimum t_{xf}^* and its cost rate $C_f(T; t_{xf}^*)$ for k and c_m when $\theta = 1.0$, $\alpha = 1.0$, $\beta = 2.0$, $T = 1.0$, and $c_p = 100.0$.

5 Conclusions

We have firstly obtained a definition of average failure rate, i.e., $\Lambda(t; x)$, that is based on the conditional failure probability and the mean time to failure given that the unit is still survival at time t. The mathematical monotonicity of $\Lambda(t; x)$ has been proved analytically. Next, the average failure rate has been applied into preventive replacement policies when the arrival time of a mission is a random variable and lasts for an interval, during which, the unit provides reliability and no maintenance can be done. Optimum replacement time and mission interval have been discussed respectively for the models of age replacement and periodic replacement. Numerical examples have been illustrated when the mission arrival time follows a gamma distribution and the failure time of the unit has a Weibull distribution.

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