On Certain Commuting Isometries, Joint Invariant Subspaces and C^*-Algebras

B. Krishna Das, Ramlal Debnath, and Jaydeb Sarkar

Dedicated to the memory of Ronald G. Douglas, our teacher, mentor and friend

Abstract In this paper, motivated by the Berger, Coburn and Lebow and Bercovici, Douglas and Foias theory for tuples of commuting isometries, we study analytic representations and joint invariant subspaces of a class of n tuples of commuting isometries and prove that the C^*-algebra generated by the n-tuple of multiplication operators by the coordinate functions restricted to an invariant subspace of finite codimension in $H^2(\mathbb{D}^n)$ is unitarily equivalent to the C^*-algebra generated by the n-tuple of multiplication operators by the coordinate functions on $H^2(\mathbb{D}^n)$.

Keywords Unilateral shift · Commuting isometries · Joint invariant subspaces · Hardy space over unit polydisc · C^*-algebras · Finite codimensional subspaces

Mathematics Subject Classification (2010) Primary 47A13, 47C15, 47L80; Secondary 47A20, 47A45, 47B35, 47A65, 46E22, 46E40, 47A05

B. K. Das
Department of Mathematics, Indian Institute of Technology Bombay, Mumbai, India
e-mail: dasb@math.iitb.ac.in

R. Debnath · J. Sarkar (✉)
Indian Statistical Institute, Statistics and Mathematics Unit, Bangalore, India
e-mail: ramlal_rs@isibang.ac.in; jay@isibang.ac.in

R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_8

Notations

$\mathcal{H}, \mathcal{E}, \mathcal{E}_*$	Hilbert spaces
$\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$	The space of all bounded linear operators from $\mathcal{E}$ to $\mathcal{E}_*$
$\mathcal{B}(\mathcal{E})$	The space of all bounded linear operators on $\mathcal{E}$
$\mathbb{D}^n$	Open unit polydisc in $\mathbb{C}^n$
$H^2(\mathbb{D}^n)$	Hardy space on $\mathbb{D}^n$
$H^2_{\mathcal{E}}(\mathbb{D}^n)$	$\mathcal{E}$-valued Hardy space on $\mathbb{D}^n$
$H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D}^n)$	Set of all $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$-valued bounded analytic functions on $\mathbb{D}^n$.
$(M_{z_1}, \dots, M_{z_n})$	n-tuple of multiplication operator by the coordinate functions on $H^2(\mathbb{D}^n)$

(1) All Hilbert spaces are assumed to be over the complex numbers.
(2) For a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, we denote by $P_{\mathcal{S}}$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$.
(3) For nested closed subspaces $\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{H}$, the orthogonal projection of $\mathcal{M}_2$ onto $\mathcal{M}_1$ is denoted by $P^{\mathcal{M}_2}_{\mathcal{M}_1}$.

1 Introduction

Tuples of commuting isometries on Hilbert spaces are cental objects of study in (multivariable) operator theory. This paper is concerned with the study of analytic representations, joint invariant subspaces and C^*-algebras of a certain class of tuples of commuting isometries.

To be precise, let $\mathcal{H}$ be a Hilbert space, and let $(V_1, \dots, V_n)$ be an n-tuple of commuting isometries on $\mathcal{H}$. In what follows, we always assume that $n \geq 2$. Set

$$V = \prod_{i=1}^{n} V_i.$$

We say that $(V_1, \dots, V_n)$ is a *pure n-isometry* if V is a unilateral shift. A closed subspace $\mathcal{S} \subseteq H^2(\mathbb{D}^n)$ is said to be an *invariant subspace* of $H^2(\mathbb{D}^n)$ if $M_{z_i}\mathcal{S} \subseteq \mathcal{S}$ for all $i = 1, \dots, n$ where M_{z_i} is the multiplication operator by the coordinate function z_i on $H^2(\mathbb{D}^n)$. Simpler (but complex enough) examples of pure n-isometry can be obtained by taking restrictions of the n-tuple of multiplication operators by coordinate functions $(M_{z_1}, \dots, M_{z_n})$ on $H^2(\mathbb{D}^n)$ to invariant subspaces of $H^2(\mathbb{D}^n)$ as follows. Given an invariant subspace $\mathcal{S}$ of $H^2(\mathbb{D}^n)$, we let

$$R_{z_i} = M_{z_i}|_{\mathcal{S}} \in \mathcal{B}(\mathcal{S}) \qquad (i = 1, \dots, n).$$

Then it is easy to see that $(R_{z_1}, \ldots, R_{z_n})$ is a pure n-isometry. We denote by $\mathcal{T}(\mathcal{S})$ the C^*-algebra generated by the commuting isometries $\{R_{z_1}, \ldots, R_{z_n}\}$. We simply say that $\mathcal{T}(\mathcal{S})$ is the *C^*-algebra corresponding to the invariant subspace $\mathcal{S}$*.

In this paper we aim to address three basic issues of pure n-isometries: (i) analytic and canonical models for pure n-isometries, (ii) an abstract classification of joint invariant subspaces for pure n-isometries, and (iii) the nature of C^*-algebra $\mathcal{T}(\mathcal{S})$ where $\mathcal{S}$ is a finite codimensional invariant subspace in $H^2(\mathbb{D}^n)$. To that aim, for (i) and (ii), we consider the initial approach by Berger et al. [6] from a more modern point of view (due to Bercovici et al. [5]) along with the technique of [20]. For (iii), we will examine Seto's approach [26] more closely from "subspace" approximation point of view.

We now briefly outline the setting and the main contributions of this paper. Let $\mathcal{E}$ be a Hilbert space, and let $\varphi \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$. We say that φ is an inner function if $\varphi(e^{it})^*\varphi(e^{it}) = I_{\mathcal{E}}$ for almost every t (cf. page 196, [21]). Recall that two n-tuples of commuting operators $(A_1, \ldots, A_n)$ on $\mathcal{H}$ and $(B_1, \ldots, B_n)$ on $\mathcal{K}$ are said to be *unitarily equivalent* if there exists a unitary operator $U : \mathcal{H} \to \mathcal{K}$ such that $UA_i = B_iU$ for all $i = 1, \ldots, n$. In [5], motivated by Berger et al. [6], Bercovici, Douglas and Foias proved the following result: A pure n-isometry is unitarily equivalent to a model pure n-isometry. The model pure n-isometries are defined as follows [5]: Consider a Hilbert space $\mathcal{E}$, unitary operators $\{U_1, \ldots, U_n\}$ on $\mathcal{E}$ and orthogonal projections $\{P_1, \ldots, P_n\}$ on $\mathcal{E}$. Let $\{\Phi_1, \ldots, \Phi_n\} \subseteq H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ be bounded $\mathcal{B}(\mathcal{E})$-valued holomorphic functions (polynomials) on $\mathbb{D}$, where

$$\Phi_i(z) = U_i(P_i^\perp + zP_i) \qquad (z \in \mathbb{D}),$$

and $i = 1, \ldots, n$. Then the n-tuple of multiplication operators $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{E}}(\mathbb{D})$ is called a *model pure n-isometry* if the following conditions are satisfied:

(a) $U_iU_j = U_jU_i$ for all $i, j = 1, \ldots n$;
(b) $U_1 \cdots U_n = I_{\mathcal{E}}$;
(c) $P_i + U_i^* P_j U_i = P_j + U_j^* P_i U_j \leq I_{\mathcal{E}}$ for all $i \neq j$; and
(d) $P_1 + U_1^* P_2 U_1 + U_1^* U_2^* P_3 U_2 U_1 + \cdots + U_1^* U_2^* \cdots U_{n-1}^* P_n U_{n-1} \cdots U_2 U_1 = I_{\mathcal{E}}$.

It is easy to see that a model pure n-isometry is also a pure n-isometry (see page 643 in [5]).

We refer to Bercovici et al. [3–5] and also [8–10, 12, 14, 15, 17, 19, 22, 26] and [27, 28] for more on pure n-isometries, $n \geq 2$, and related topics.

Our first main result, Theorem 2.1, states that a pure n-isometry is unitarily equivalent to an explicit (and canonical) model pure n-isometry. In other words, given a pure n-isometry $(V_1, \ldots, V_n)$ on $\mathcal{H}$, we explicitly solve the above conditions (a)–(d) for some Hilbert space $\mathcal{E}$, unitary operators $\{U_1, \ldots, U_n\}$ on $\mathcal{E}$ and orthogonal projections $\{P_1, \ldots, P_n\}$ on $\mathcal{E}$ so that the corresponding model pure n-isometry $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ is unitarily equivalent to $(V_1, \ldots, V_n)$. This also gives a new proof of Bercovici, Douglas and Foias theorem. On the one hand, our model pure n-isometry is explicit and canonical. On the other hand, our proof is perhaps more computational than the one in [5]. Another advantage of our approach is the proof of a list of useful equalities related to commuting isometries, which can be useful in other contexts.

Our second main result concerns a characterization of joint invariant subspaces of model pure n-isometries. To be precise, let $\mathcal{W}$ be a Hilbert space, and let $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ be a model pure n-isometry on $H^2_{\mathcal{W}}(\mathbb{D})$. Let $\mathcal{S}$ be a closed subspace of $H^2_{\mathcal{W}}(\mathbb{D})$. In Theorem 3.1, we prove that $\mathcal{S}$ is invariant for $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ if and only if there exist a Hilbert space $\mathcal{W}_*$, an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}_*, \mathcal{W})}(\mathbb{D})$ and a model pure n-isometry $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ on $H^2_{\mathcal{W}_*}(\mathbb{D})$ such that

$$\mathcal{S} = \Theta H^2_{\mathcal{W}_*}(\mathbb{D}),$$

and

$$\Phi_i \Theta = \Theta \Psi_i,$$

for all $i = 1, \ldots, n$. Moreover, the above representation is unique in an appropriate sense (see the remark following Theorem 3.1).

The third and final result concerns C^*-algebras corresponding to finite codimensional invariant subspaces in $H^2(\mathbb{D}^n)$. To be more specific, recall that if $n = 1$ and $\mathcal{S}$ and $\mathcal{S}'$ are invariant subspaces of $H^2(\mathbb{D})$, then $U\mathcal{T}(\mathcal{S})U^* = \mathcal{T}(\mathcal{S}')$ for some unitary $U : \mathcal{S} \to \mathcal{S}'$. Indeed, since $\mathcal{S} = \theta H^2(\mathbb{D})$ for some inner function $\theta \in H^\infty(\mathbb{D})$, it follows, by Beurling theorem, that $U := M_\theta : H^2(\mathbb{D}) \to \mathcal{S}$ is a unitary and hence $U^*\mathcal{T}(\mathcal{S})U = \mathcal{T}(H^2(\mathbb{D}))$. Clearly, the general case follows from this special case. For invariant subspaces $\mathcal{S}$ and $\mathcal{S}'$ of $H^2(\mathbb{D}^n)$, we say that $\mathcal{T}(\mathcal{S})$ and $\mathcal{T}(\mathcal{S}')$ are *isomorphic* as C^*-algebras if $U\mathcal{T}(\mathcal{S})U^* = \mathcal{T}(\mathcal{S}')$ holds for some unitary $U : \mathcal{S} \to \mathcal{S}'$. It is then natural to ask: If $n > 1$ and $\mathcal{S}$ and $\mathcal{S}'$ are invariant subspaces of $H^2(\mathbb{D}^n)$, are $\mathcal{T}(\mathcal{S})$ and $\mathcal{T}(\mathcal{S}')$ isomorphic as C^*-algebras?

In the same paper [6], Berger, Coburn and Lebow asked whether $\mathcal{T}(\mathcal{S})$ is isomorphic to $\mathcal{T}(H^2(\mathbb{D}^2))$ for every finite codimensional invariant subspaces $\mathcal{S}$ in $H^2(\mathbb{D}^2)$. This question was recently answered positively by Seto in [26]. Here we extend Seto's answer from $H^2(\mathbb{D}^2)$ to the general case $H^2(\mathbb{D}^n)$, $n \geq 2$.

The rest of this paper is organized as follows. In Sect. 2 we study and review the analytic construction of pure n-isometries. We also examine a (canonical) model pure n-isometry. A characterization of invariant subspaces is given in Sect. 3. Finally, in Sect. 4, we prove that $\mathcal{T}(\mathcal{S})$ is isomorphic to $\mathcal{T}(H^2(\mathbb{D}^n))$ where $\mathcal{S}$ is a finite codimensional invariant subspaces in $H^2(\mathbb{D}^n)$.

2 Pure n-Isometries and Model Pure n-Isometries

In this section, we first derive an explicit analytic representation of a pure n-isometry. Then we propose a canonical model for pure n-isometries.

For motivation, let us recall that if X on $\mathcal{H}$ is a bounded linear operator, then X is a unilateral shift operator if and only if X and M_z on $H^2_{\mathcal{W}(X)}(\mathbb{D})$ are unitarily equivalent. Here

$$\mathcal{W}(X) = \ker X^* = \mathcal{H} \ominus X\mathcal{H},$$

is the *wandering subspace* for X (see Halmos [16]) and M_z denotes the multiplication operator by the coordinate function z on $H^2_{\mathcal{W}(X)}(\mathbb{D})$, that is, $(M_z f)(w) = wf(w)$ for all $f \in H^2_{\mathcal{W}(X)}(\mathbb{D})$ and $w \in \mathbb{D}$. Explicitly, if X is a unilateral shift on $\mathcal{H}$, then

$$\mathcal{H} = \bigoplus_{m=0}^{\infty} X^m \mathcal{W}(X).$$

Hence the natural map $\Pi_X : \mathcal{H} \to H^2_{\mathcal{W}(X)}(\mathbb{D})$ defined by

$$\Pi_X(X^m \eta) = z^m \eta,$$

for all $m \geq 0$ and $\eta \in \mathcal{W}(X)$, is a unitary operator and

$$\Pi_X X = M_z \Pi_X.$$

We call Π_X the *Wold-von Neumann decomposition* of the shift X.

Now let $\mathcal{H}$ be a Hilbert space, and let $(V_1, \ldots, V_n)$ be a pure n-isometry on $\mathcal{H}$. Throughout this paper, we shall use the following notation:

$$\tilde{V}_i = \prod_{j \neq i} V_j,$$

for all $i = 1, \ldots, n$. For simplicity, we also use the notation

$$\mathcal{W} = \mathcal{W}(V),$$

and

$$\mathcal{W}_i = \mathcal{W}(V_i) \ \text{ and } \tilde{\mathcal{W}}_i = \mathcal{W}(\tilde{V}_i),$$

for all $i = 1, \ldots, n$. Since $V = \Pi_{i=1}^n V_i$ and $\tilde{V}_i = V_i^* V$ for all $i = 1, \ldots, n$, it is easy to see that

$$\mathcal{W}_i, \tilde{\mathcal{W}}_i \subseteq \mathcal{W},$$

for all $i = 1, \ldots, n$. We denote by $P_{\mathcal{W}_i}$ and $P_{\tilde{\mathcal{W}}_i}$ the orthogonal projections of $\mathcal{W}$ onto the subspaces $\mathcal{W}_i$ and $\tilde{\mathcal{W}}_i$, respectively.

Theorem 2.1 *Let $(V_1, \ldots, V_n)$ be a pure n-isometry on a Hilbert space $\mathcal{H}$, $V = \Pi_{i=1}^n V_i$, and let $\mathcal{W} = \mathcal{W}(V)$. Let $\Pi_V : \mathcal{H} \to H^2_{\mathcal{W}}(\mathbb{D})$ be the Wold-von Neumann decomposition of V. If $\tilde{V}_i = V_i^* V$ and $\tilde{\mathcal{W}}_i = \mathcal{W}(\tilde{V}_i)$, then*

$$\Pi_V V_i = M_{\Phi_i} \Pi_V,$$

where

$$\Phi_i(z) = U_i(P_{\tilde{\mathcal{W}}_i} + z P^{\perp}_{\tilde{\mathcal{W}}_i}),$$

for all $z \in \mathbb{D}$, and

$$U_i = (P_{\mathcal{W}} V_i + \tilde{V_i}^*)|_{\mathcal{W}},$$

is a unitary operator on $\mathcal{W}$ and $i = 1, \ldots, n$. In particular, $(V_1, \ldots, V_n)$ on $\mathcal{H}$ and $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent.

Proof Let $\Pi_V : \mathcal{H} \to H^2_{\mathcal{W}}(\mathbb{D})$ be the Wold-von Neumann decomposition of V. Then

$$\Pi_V V_i \Pi_V^* \in \{M_z\}',$$

and hence there exists $\Phi_i \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ [16, 21] such that $\Pi_V V_i \Pi_V^* = M_{\Phi_i}$ or, equivalently,

$$\Pi_V V_i = M_{\Phi_i} \Pi_V,$$

for all $i = 1, \ldots, n$. Note that M_{Φ_i} on $H^2_{\mathcal{W}}(\mathbb{D})$ is defined by

$$(M_{\Phi_i} f)(z) = \Phi_i(z) f(z), \tag{2.1}$$

for all $f \in H^2_{\mathcal{W}}(\mathbb{D})$, $z \in \mathbb{D}$ and $i = 1, \ldots, n$. We now proceed to compute the bounded analytic functions $\{\Phi_i\}_{i=1}^n$. Our method follows the construction in [20]. In fact, a close variant of Theorem 2.1 below follows from Theorems 3.4 and 3.5 of [20]. We will only sketch the construction, highlighting the essential ingredients for our present purpose. Let $i \in \{1, \ldots, n\}$, $z \in \mathbb{D}$ and $\eta \in \mathcal{W}$. By an abuse of notation, we will also denote the constant function η in $H^2_{\mathcal{W}}(\mathbb{D})$ corresponding to the vector $\eta \in \mathcal{W}$ by η itself. Then from (2.1), we have that

$$\Phi_i(z)\eta = (M_{\Phi_i}\eta)(z) = (\Pi_V V_i \Pi_V^* \eta)(z).$$

Now it follows from the definition of Π_V that $\Pi_V^*\eta = \eta$, and hence $\Phi_i(z)\eta = (\Pi_V V_i\eta)(z)$. But $I_{\mathcal{W}} = P_{\tilde{\mathcal{W}}_i} + \tilde{V}_i\tilde{V}_i^*|_{\mathcal{W}}$ yields that $V_i\eta = V_iP_{\tilde{\mathcal{W}}_i}\eta + V\tilde{V}_i^*\eta$ and thus

$$\begin{aligned}\Pi_V V_i\eta &= \Pi_V(V_iP_{\tilde{\mathcal{W}}_i}\eta + V\tilde{V}_i^*\eta)\\ &= \Pi_V(V_iP_{\tilde{\mathcal{W}}_i}\eta) + \Pi_V(V\tilde{V}_i^*\eta)\\ &= \Pi_V(V_iP_{\tilde{\mathcal{W}}_i}\eta) + M_z\Pi_V(\tilde{V}_i^*\eta),\end{aligned}$$

as $\Pi_V V = M_z\Pi_V$. Now, since $V^*(V_i(I - \tilde{V}_i\tilde{V}_i^*)V_i^*) = 0$ and $V^*(\tilde{V}_i^*\eta) = 0$, it follows that $V_iP_{\tilde{\mathcal{W}}_i}\eta \in \mathcal{W}$ and $\tilde{V}_i^*\eta \in \mathcal{W}$. This implies that

$$\Pi_V V_i\eta = V_iP_{\tilde{\mathcal{W}}_i}\eta + M_z\tilde{V}_i^*\eta,$$

and so $\Phi_i(z)\eta = V_iP_{\tilde{\mathcal{W}}_i}\eta + z\tilde{V}_i^*\eta$. It follows that $\Phi_i(z) = V_i|_{\tilde{\mathcal{W}}_i} + z\tilde{V}_i^*|_{\tilde{V}_i\mathcal{W}_i}$ as $\mathcal{W} = \tilde{V}_i\mathcal{W}_i \oplus \tilde{\mathcal{W}}_i$. Finally, $\mathcal{W} = \mathcal{W}_i \oplus V_i\tilde{\mathcal{W}}_i$ implies that

$$U_i = \begin{bmatrix}\tilde{V}_i^*|_{\tilde{V}_i\mathcal{W}_i} & 0\\ 0 & V_i|_{\tilde{\mathcal{W}}_i}\end{bmatrix} : \begin{matrix}\tilde{V}_i\mathcal{W}_i\\ \oplus\\ \tilde{\mathcal{W}}_i\end{matrix} \to \begin{matrix}\mathcal{W}_i\\ \oplus\\ V_i\tilde{\mathcal{W}}_i\end{matrix},$$

is a unitary operator on $\mathcal{W}$. Therefore

$$\Phi_i(z) = U_i(P_{\tilde{\mathcal{W}}_i} + zP_{\tilde{\mathcal{W}}_i}^{\perp}),$$

for all $z \in \mathbb{D}$. By definition of U_i, it follows that $U_i = (V_iP_{\tilde{\mathcal{W}}_i} + \tilde{V}_i{}^*)|_{\mathcal{W}}$. This and

$$V_iP_{\tilde{\mathcal{W}}_i} = P_{\mathcal{W}}V_i, \tag{2.2}$$

yields $U_i = (P_{\mathcal{W}}V_i + \tilde{V}_i{}^*)|_{\mathcal{W}}$. ∎

We now study the coefficients of the one-variable polynomials in Theorem 2.1 more closely and prove that the corresponding pure n-isometry $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ is a model pure n-isometry (see Sect. 1 for the definition of model pure n-isometries).

Let $(V_1, \ldots, V_n)$ be a pure n-isometry on a Hilbert space $\mathcal{H}$. Consider the analytic representation $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ of $(V_1, \ldots, V_n)$ as in Theorem 2.1. First we prove that $\{U_j\}_{j=1}^n$ is a commutative family. Let $p, q \in \{1, \ldots, n\}$ and $p \neq q$. As $\mathcal{W} = \ker V^*$, it follows that

$$\tilde{V}_p^*\tilde{V}_q^*|_{\mathcal{W}} = 0.$$

Then using (2.2) we obtain

$$\begin{aligned}
U_pU_q &= (P_{\mathcal{W}}V_p + \tilde{V}_p^*)(P_{\mathcal{W}}V_q + \tilde{V}_q^*)|_{\mathcal{W}}\\
&= (P_{\mathcal{W}}V_pP_{\mathcal{W}}V_q + \tilde{V}_p^*P_{\mathcal{W}}V_q + P_{\mathcal{W}}V_p\tilde{V}_q^*)|_{\mathcal{W}}\\
&= (P_{\mathcal{W}}V_pV_q + \prod_{i\neq p,q} V_i^*P_{\tilde{\mathcal{W}}_q} + V_pP_{\tilde{\mathcal{W}}_p}\tilde{V}_q^*)|_{\mathcal{W}}\\
&= (P_{\mathcal{W}}V_pV_q + (\prod_{i\neq p,q} V_i^*)(P_{\tilde{\mathcal{W}}_q} + \tilde{V}_qP_{\tilde{\mathcal{W}}_p}\tilde{V}_q^*))|_{\mathcal{W}}\\
&= (P_{\mathcal{W}}V_pV_q + (\prod_{i\neq p,q} V_i^*))|_{\mathcal{W}},
\end{aligned}$$

as $(P_{\tilde{\mathcal{W}}_q} + \tilde{V}_qP_{\tilde{\mathcal{W}}_p}\tilde{V}_q^*)|_{\mathcal{W}} = I_{\mathcal{W}}$, and hence

$$U_pU_q = U_qU_p,$$

follows by symmetry. Now if $I \subseteq \{1, \ldots, n\}$, then the same line of arguments as above yields

$$\prod_{i\in I} U_i = (P_{\mathcal{W}}(\prod_{i\in I} V_i) + (\prod_{i\in I^c} V_i^*))|_{\mathcal{W}}. \tag{2.3}$$

In particular, since $P_{\mathcal{W}}V|_{\mathcal{W}} = 0$, we have that

$$\prod_{i=1}^n U_i = I_{\mathcal{W}}.$$

The following lemma will be crucial in what follow.

Lemma 2.2 *Fix* $1 \le j \le n$*. Let* $I \subseteq \{1, \ldots, n\}$*, and let* $j \notin I$*. Then*

$$(\prod_{i\in I} U_i^*)P_{\tilde{\mathcal{W}}_j}^{\perp}(\prod_{i\in I} U_i) = (\prod_{i\in I^c\setminus\{j\}} V_i)(\prod_{i\in I^c\setminus\{j\}} V_i^*)|_{\mathcal{W}} - (\prod_{i\in I^c} V_i)(\prod_{i\in I^c} V_i^*)|_{\mathcal{W}}.$$

Proof Since $P_{\tilde{\mathcal{W}}_j} = I_{\mathcal{W}} - P_{\mathcal{W}}\tilde{V}_j\tilde{V}_j^*|_{\mathcal{W}}$, we have $P_{\tilde{\mathcal{W}}_j}^{\perp} = P_{\mathcal{W}}\tilde{V}_j\tilde{V}_j^*|_{\mathcal{W}} = \tilde{V}_j\tilde{V}_j^*|_{\mathcal{W}}$. By once again using the fact that $V^*|_{\mathcal{W}} = P_{\mathcal{W}}V|_{\mathcal{W}} = 0$, and by (2.3), one sees that

$$\begin{aligned}
(\prod_{i\in I} U_i^*)P_{\tilde{\mathcal{W}}_j}^{\perp}(\prod_{i\in I} U_i) &= [(\prod_{i\in I} V_i^*) + P_{\mathcal{W}}(\prod_{i\in I^c} V_i)]\tilde{V}_j\tilde{V}_j^*[P_{\mathcal{W}}(\prod_{i\in I} V_i) + (\prod_{i\in I^c} V_i^*)]|_{\mathcal{W}}\\
&= (\prod_{i\in I^c\setminus\{j\}} V_i)\tilde{V}_j^*P_{\mathcal{W}}(\prod_{i\in I} V_i)|_{\mathcal{W}}\\
&= (\prod_{i\in I^c\setminus\{j\}} V_i)\tilde{V}_j^*(I - VV^*)(\prod_{i\in I} V_i)|_{\mathcal{W}}\\
&= (\prod_{i\in I^c\setminus\{j\}} V_i)(\prod_{i\in I^c\setminus\{j\}} V_i^*)|_{\mathcal{W}} - (\prod_{i\in I^c} V_i)(\prod_{i\in I^c} V_i^*)|_{\mathcal{W}}
\end{aligned}$$

This completes the proof of the lemma. ■

Theorem 2.3 *If $(V_1, \ldots, V_n)$ be an n-isometry on a Hilbert space $\mathcal{H}$, and let $U_1, \ldots, U_n$ be unitary operators as in Theorem 2.1. Then*

(a) $U_pU_q = U_qU_p$ *for* $p, q = 1, \ldots n$,
(b) $\prod_{p=1}^{n} U_p = I_{\mathcal{W}}$,
(c) $(P^{\perp}_{\tilde{\mathcal{W}}_i} + U_i^* P^{\perp}_{\tilde{\mathcal{W}}_j} U_i) = (P^{\perp}_{\tilde{\mathcal{W}}_j} + U_j^* P^{\perp}_{\tilde{\mathcal{W}}_i} U_j) \leq I_{\mathcal{W}}$ $(1 \leq i < j \leq n)$,
(d) $P^{\perp}_{\tilde{\mathcal{W}}_1} + U_1^* P^{\perp}_{\tilde{\mathcal{W}}_2} U_1 + U_1^* U_2^* P^{\perp}_{\tilde{\mathcal{W}}_2} U_2 U_1 + \cdots + (\Pi_{i=1}^{n-1} U_i^*) P^{\perp}_{\tilde{\mathcal{W}}_n} (\Pi_{i=1}^{n-1} U_i) = I_{\mathcal{W}}$.

Proof By Lemma 2.2 applied to $I = \{p\}$ and $j = q$, where $p, q \in \{1, \ldots, n\}$ and $p \neq q$, we have

$$U_p^* P^{\perp}_{\tilde{\mathcal{W}}_q} U_p = (\prod_{i \neq p,q} V_i)(\prod_{i \neq p,q} V_i^*)|_{\mathcal{W}} - \tilde{V}_p \tilde{V}_p^{\,*}|_{\mathcal{W}},$$

hence

$$\begin{aligned}(P^{\perp}_{\tilde{\mathcal{W}}_p} + U_p^* P^{\perp}_{\tilde{\mathcal{W}}_q} U_p) &= P_{\mathcal{W}} \tilde{V}_p \tilde{V}_p^{\,*}|_{\mathcal{W}} + (\prod_{i \neq p,q} V_i)(\prod_{i \neq p,q} V_i^*)|_{\mathcal{W}} - P_{\mathcal{W}} \tilde{V}_p \tilde{V}_p^{\,*}|_{\mathcal{W}} \\ &= (\prod_{i \neq p,q} V_i)(\prod_{i \neq p,q} V_i^*)|_{\mathcal{W}} \\ &\leq I_{\mathcal{W}}.\end{aligned}$$

Therefore by symmetry, we have

$$(P^{\perp}_{\tilde{\mathcal{W}}_p} + U_p^* P^{\perp}_{\tilde{\mathcal{W}}_q} U_p) = (P^{\perp}_{\tilde{\mathcal{W}}_q} + U_q^* P^{\perp}_{\tilde{\mathcal{W}}_p} U_q) \leq I_{\mathcal{W}}.$$

Finally, we let $I_j = \{1, \ldots, j-1\}$ for all $1 < j \leq n$ and $I_{n+1} = \{1, \ldots, n\}$. Then Lemma 2.2 implies that for $1 < j \leq n$,

$$(\prod_{i \in I_j} U_i) P^{\perp}_{\tilde{\mathcal{W}}_j} (\prod_{i \in I_j} U_i^*) = [(\prod_{i \in I^c_{j+1}} V_i)(\prod_{i \in I^c_{j+1}} V_i^*) - (\prod_{i \in I^c_j} V_i)(\prod_{i \in I^c_j} V_i^*)]|_{\mathcal{W}}.$$

This and $P^{\perp}_{\tilde{\mathcal{W}}_1} = \tilde{V}_1 \tilde{V}_1^*|_{\mathcal{W}}$ imply that

$$P^{\perp}_{\tilde{\mathcal{W}}_1} + U_1^* P^{\perp}_{\tilde{\mathcal{W}}_2} U_1 + U_1^* U_2^* P^{\perp}_{\tilde{\mathcal{W}}_3} U_2 U_1 + \cdots + (\prod_{i=1}^{n-1} U_i^*) P^{\perp}_{\tilde{\mathcal{W}}_n} (\prod_{i=1}^{n-1} U_i) = I_{\mathcal{W}}.$$

This completes the proof of the theorem. ∎

As a corollary, we have:

Corollary 2.4 *Let $\mathcal{H}$ be a Hilbert space and $(V_1, \ldots, V_n)$ be a pure n-isometry on $\mathcal{H}$. Let $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ be the pure n-isometry as constructed in Theorem 2.1, and let $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ on $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$, for some Hilbert space $\tilde{\mathcal{W}}$, unitary operators*

$\{\tilde{U}_i\}_{i=1}^n$ and orthogonal projections $\{P_i\}_{i=1}^n$ on $\tilde{\mathcal{W}}$, be a model pure n-isometry. Then:

(a) *$(M_{\Phi_1}, \ldots, M_{\Phi_n})$ is a model pure n-isometry.*
(b) *$(V_1, \ldots, V_n)$ and $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ are unitarily equivalent.*
(c) *$(V_1, \ldots, V_n)$ and $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ are unitarily equivalent if and only if there exists a unitary operator $W : \mathcal{W} \to \tilde{\mathcal{W}}$ such that $WU_i = \tilde{U}_i W$ and $WP_i = \tilde{P}_i W$ for all $i = 1, \ldots, n$.*

Proof Parts (a) and (b) follows directly from the previous theorem. The third part is easy and readily follows from Theorem 4.1 in [20] or Theorem 2.9 in [5]. ■

Combining Corollary 2.4 with Theorem 2.3, we have the following characterization of commutative isometric factors of shift operators.

Corollary 2.5 *Let $\mathcal{E}$ be a Hilbert space, and let $\{\Phi_i\}_{i=1}^n \subseteq H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ be a commutative family of isometric multipliers. Then*

$$M_z = \prod_{i=1}^n M_{\Phi_j},$$

or, equivalently

$$\prod_{i=1}^n \Phi_j(z) = zI_{\mathcal{E}}, \quad (z \in \mathbb{D})$$

if and only if, up to unitary equivalence, $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ is a model pure n-isometry.

In other words, $zI_{\mathcal{E}}$ factors as n commuting isometric multipliers $\{\Phi_i\}_{i=1}^n$ in $H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ if and only if there exist unitary operators $\{U_i\}_{i=1}^n$ on $\mathcal{E}$ and orthogonal projections $\{P_i\}_{i=1}^n$ on $\mathcal{E}$ satisfying the properties (a)–(d) in Theorem 2.3 such that $\Phi_i(z) = U_i(P_i^\perp + zP_i)$ for all $i = 1, \ldots, n$.

3 Joint Invariant Subspaces

Let $\mathcal{W}$ be a Hilbert space. Let $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ be a model pure n-isometry on $H^2_{\mathcal{W}}(\mathbb{D})$, and let $\mathcal{S}$ be a closed invariant subspace for $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$, that is

$$M_{\Phi_i}\mathcal{S} \subseteq \mathcal{S},$$

for all $i = 1, \ldots, n$. Then $(M_{\Phi_1}|_{\mathcal{S}}, \ldots, M_{\Phi_n}|_{\mathcal{S}})$ is an n-tuple of commuting isometries on $\mathcal{S}$. Clearly

$$\prod_{i=1}^n (M_{\Phi_i}|_{\mathcal{S}}) = (\prod_{i=1}^n M_{\Phi_i})|_{\mathcal{S}},$$

and since

$$\prod_{j=1}^{n} M_{\Phi_j} = M_z,$$

it follows that

$$(\prod_{i=1}^{n} M_{\Phi_i})|_{\mathcal{S}} = M_z|_{\mathcal{S}}, \tag{3.1}$$

that is, $\mathcal{S}$ is a invariant subspace for M_z on $H^2_{\mathcal{W}}(\mathbb{D})$. Moreover, since $M_z|_{\mathcal{S}}$ is a unilateral shift on $\mathcal{S}$, the tuple $(M_{\Phi_1}|_{\mathcal{S}}, \ldots, M_{\Phi_n}|_{\mathcal{S}})$ is a pure n-isometry on $\mathcal{S}$. Then by Corollary 2.4 there is a model pure n-isometry $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ on $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$, for some Hilbert space $\tilde{\mathcal{W}}$, such that $(M_{\Phi_1}|_{\mathcal{S}}, \ldots, M_{\Phi_n}|_{\mathcal{S}})$ and $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ are unitarily equivalent. The main purpose of this section is to describe the invariant subspaces $\mathcal{S}$ in terms of the model pure n-isometry $(M_{\Psi_1}, \ldots, M_{\Psi_n})$.

As a motivational example, consider the classical $n = 1$ case. Here the model pure 1-isometry is the multiplication operator M_z on $H^2_{\mathcal{W}}(\mathbb{D})$ for some Hilbert space $\mathcal{W}$. Let $\mathcal{S}$ be a closed subspace of $H^2_{\mathcal{W}}(\mathbb{D})$. Then by the Beurling [7], Lax [18] and Halmos [16] theorem (or see page 239, Theorem 2.1 in [13]), $\mathcal{S}$ is invariant for M_z if and only if there exist a Hilbert space $\mathcal{W}_*$ and an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}_*, \mathcal{W})}(\mathbb{D})$ such that

$$\mathcal{S} = \Theta H^2_{\mathcal{W}_*}(\mathbb{D}).$$

Moreover, in this case, if we set

$$V = M_z|_{\mathcal{S}},$$

then $\mathcal{W}_* = \mathcal{S} \ominus z\mathcal{S}$ and V on $\mathcal{S}$ and M_z on $H^2_{\mathcal{W}_*}(\mathbb{D})$ are unitarily equivalent. This follows directly from the above representation of $\mathcal{S}$. Indeed, it follows that $X = M_\Theta : H^2_{\mathcal{W}_*}(\mathbb{D}) \to \text{ran} M_\Theta = \mathcal{S}$ is a unitary operator and

$$XM_z = VX.$$

Now, we proceed with the general case.

Theorem 3.1 *Let $n > 1$. Let $\mathcal{W}$ be a Hilbert space, $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ be a model pure n-isometry on $H^2_{\mathcal{W}}(\mathbb{D})$, and let $\mathcal{S}$ be a closed subspace of $H^2_{\mathcal{W}}(\mathbb{D})$. Then $\mathcal{S}$ is invariant for $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ if and only if there exist a Hilbert space $\mathcal{W}_*$, an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}_*, \mathcal{W})}(\mathbb{D})$ and a model pure n-isometry $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ on $H^2_{\mathcal{W}_*}(\mathbb{D})$ such that*

$$\mathcal{S} = \Theta H^2_{\mathcal{W}_*}(\mathbb{D}),$$

and

$$\Phi_j \Theta = \Theta \Psi_j,$$

for all $j = 1, \ldots, n$.

Proof Let $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ be a model pure n-isometry on $H^2_{\mathcal{W}}(\mathbb{D})$, and let $\mathcal{S}$ be a closed invariant subspace for $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$. Let

$$\mathcal{W}_* = \mathcal{S} \ominus z\mathcal{S}.$$

Since $\mathcal{S}$ is an invariant subspace for M_z on $H^2_{\mathcal{W}}(\mathbb{D})$ (see Eq. (3.1)), by Beurling, Lax and Halmos theorem, there exists an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}_*, \mathcal{W})}(\mathbb{D})$ such that $\mathcal{S}$ can be represented as

$$\mathcal{S} = \Theta H^2_{\mathcal{W}_*}(\mathbb{D}),$$

If $1 \leq j \leq n$, then

$$\Phi_j \mathcal{S} \subseteq \mathcal{S},$$

implies that ran $(M_{\Phi_j} M_\Theta) \subseteq$ ran M_Θ, and so by Douglas's range and inclusion theorem [11]

$$M_{\Phi_j} M_\Theta = M_\Theta M_{\Psi_j},$$

for some $\Psi_j \in H^\infty_{\mathcal{B}(\mathcal{W}_*)}(\mathbb{D})$. Note that $M_{\Phi_j} M_\Theta$ is an isometry and $\|\Theta \Psi_j f\| = \|\Psi_j f\|$ for each $f \in H^2_{\mathcal{W}_*}(\mathbb{D})$. But then $\|M_{\Psi_j} f\| = \|f\|$ implies that M_{Ψ_j} is an isometry, that is, Ψ_j is an inner function, and hence

$$M_{\Psi_j} = M^*_\Theta M_{\Phi_j} M_\Theta,$$

for all $j = 1, \ldots, n$. So

$$\prod_{i=1}^{n} M_{\Psi_i} = (M^*_\Theta M_{\Phi_1} M_\Theta) \cdots (M^*_\Theta M_{\Phi_n} M_\Theta).$$

Now $P_{\text{ran}\, M_\Theta} = M_\Theta M^*_\Theta$ and $\Phi_j \Theta H^2_{\mathcal{W}_*}(\mathbb{D}) \subseteq \Theta H^2_{\mathcal{W}_*}(\mathbb{D})$ implies that

$$M_\Theta M^*_\Theta M_{\Phi_j} M_\Theta = M_{\Phi_j} M_\Theta,$$

for all $j = 1, \ldots, n$. Consequently

$$\prod_{j=1}^{n} M_{\Psi_j} = M^*_\Theta (\prod_{j=1}^{n} M_{\Phi_j}) M^*_\Theta = M^*_\Theta M_z M_\Theta = M^*_\Theta M_\Theta M_z = M_z,$$

that is, $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ is a pure n-isometry on $H^2_{\mathcal{W}_*}(\mathbb{D})$. In view of Corollary 2.5, this also implies that the tuple $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ is a model pure n-isometry. This completes the proof of the theorem. ■

The representation of $\mathcal{S}$ is unique in the following sense: if there exist a Hilbert space $\hat{\mathcal{W}}$, an inner multiplier $\hat{\Theta} \in H^\infty_{\mathcal{B}(\hat{\mathcal{W}},\mathcal{W})}(\mathbb{D})$ and a model pure n-isometry $(M_{\hat{\Psi}_1}, \ldots, M_{\hat{\Psi}_n})$ on $H^2_{\hat{\mathcal{W}}}(\mathbb{D})$ such that $\mathcal{S} = \hat{\Theta} H^2_{\hat{\mathcal{W}}}(\mathbb{D})$ and $\Phi_i \hat{\Theta} = \hat{\Theta}\hat{\Psi}_i$ for all $i = 1, \ldots, n$, then there exists a unitary $\tau : \mathcal{W}_* \to \hat{\mathcal{W}}$ such that

$$\Theta = \hat{\Theta}\tau,$$

and

$$\hat{\Psi}_j \tau = \tau \Psi_j \qquad (j = 1, \ldots, n).$$

In other words, the model pure n-isometries $(M_{\hat{\Psi}_1}, \ldots, M_{\hat{\Psi}_n})$ on $H^2_{\hat{\mathcal{W}}}(\mathbb{D})$ and $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ on $H^2_{\mathcal{W}_*}(\mathbb{D})$ are unitary equivalent (under the same unitary τ). Indeed, the existence of the unitary τ along with the first equality follows from the uniqueness of the Beurling, Lax and Halmos theorem (cf. page 239, Theorem 2.1 in [13]). For the second equality, observe that (see the uniqueness part in [19])

$$\hat{\Theta}\tau\Psi_i = \Theta\Psi_i = \Phi_i\Theta = \Phi_i\hat{\Theta}\tau,$$

that is $\hat{\Theta}\tau\Psi_i = \hat{\Theta}\hat{\Psi}_i\tau$, and so

$$\tau\Psi_i = \hat{\Psi}_i\tau,$$

for all $i = 1, \ldots, n$.

It is curious to note that the content of Theorem 3.1 is related to the question [1] and its answer [24] on the classifications of invariant subspaces of Γ-isometries. A similar result also holds for invariant subspaces for the multiplication operator tuple on the Hardy space over the unit polydisc in $\mathbb{C}^n$ (see [19]).

Our approach to pure n-isometries has other applications to n-tuples, $n \geq 2$, of commuting contractions (cf. see [9]) that we will explore in a future paper.

4 C^*-Algebras Generated by Commuting Isometries

In this section, we extend Seto's result [26] on isomorphic C^*-algebras of invariant subspaces of finite codimension in $H^2(\mathbb{D}^2)$ to that in $H^2(\mathbb{D}^n)$, $n \geq 2$. Given a Hilbert space $\mathcal{H}$, the set of all compact operators from $\mathcal{H}$ to itself is denoted by $K(\mathcal{H})$. Recall that, for a closed subspace $\mathcal{S} \subseteq H^2(\mathbb{D}^n)$, we say that $\mathcal{S}$ is an invariant subspace of $H^2(\mathbb{D}^n)$ if $M_{z_i}\mathcal{S} \subseteq \mathcal{S}$ for all $i = 1, \ldots, n$. Also recall that in the case of an invariant subspace $\mathcal{S}$ of $H^2(\mathbb{D}^n)$, $(R_{z_1}, \dots, R_{z_n})$ is an n-isometry on $\mathcal{S}$ where

$$R_{z_i} = M_{z_i}|_{\mathcal{S}} \in \mathcal{B}(\mathcal{S}) \qquad (i = 1, \dots, n).$$

Lemma 4.1 *If $\mathcal{S}$ is an invariant subspace of finite codimension in $H^2(\mathbb{D}^n)$, then $K(\mathcal{S}) \subseteq \mathcal{T}(\mathcal{S})$.*

Proof Since $\mathcal{T}(\mathcal{S})$ is an irreducible C^*-algebra (cf. [26, Proposition 2.2]), it is enough to prove that $\mathcal{T}(\mathcal{S})$ contains a non-zero compact operator. As

$$\prod_{i=1}^{n} (I_{H^2(\mathbb{D}^n)} - M_{z_i} M_{z_i}^*) = P_{\mathbb{C}} \in \mathcal{T}(H^2(\mathbb{D}^n)),$$

we are done when $\mathcal{S} = H^2(\mathbb{D}^n)$. Let us now suppose that $\mathcal{S}$ is a proper subspace of $H^2(\mathbb{D}^n)$. For arbitrary $1 \le i < j \le n$, we have

$$[R_{z_i}^*, R_{z_j}] = P_{\mathcal{S}} M_{z_i}^* M_{z_j}|_{\mathcal{S}} - P_{\mathcal{S}} M_{z_j} P_{\mathcal{S}} M_{z_i}^*|_{\mathcal{S}} = P_{\mathcal{S}} M_{z_j} P_{\mathcal{S}^\perp} M_{z_i}^*|_{\mathcal{S}} \in K(\mathcal{S}),$$

as $\mathcal{S}^\perp$ is finite dimensional. It remains for us to prove that $[R_{z_i}^*, R_{z_j}] \neq 0$ for some $1 \le i < j \le n$. If not, then $\mathcal{S}$ is a proper doubly commuting invariant subspace with finite codimension. As a result, we would have $\mathcal{S} = \varphi H^2(\mathbb{D}^n)$ for some inner function $\varphi \in H^\infty(\mathbb{D}^n)$ ([25]) and hence $\mathcal{S}$ has infinite codimension (see the corollary in page 969, [2]), a contradiction. ■

In what follows, a finite rank operator on a Hilbert space will be denoted by F (without referring to the ambient Hilbert space). Also, if $\mathcal{M}$ is an invariant subspaces of $H^2(\mathbb{D}^n)$, then we set

$$R_{z_i}^{\mathcal{M}} = M_{z_i}|_{\mathcal{M}} \in \mathcal{B}(\mathcal{M}),$$

and simply write R_{z_i}, $i = 1, \dots, n$, when $\mathcal{M}$ is clear from the context.

Lemma 4.2 *Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are invariant subspaces of $H^2(\mathbb{D}^n)$, $\mathcal{M}_1 \subseteq \mathcal{M}_2$ and $\dim(\mathcal{M}_2 \ominus \mathcal{M}_1) < \infty$. Then $\mathcal{T}(\mathcal{M}_1) = \{P_{\mathcal{M}_1} T|_{\mathcal{M}_1} : T \in \mathcal{T}(\mathcal{M}_2)\}$. Moreover, if $\mathcal{L}$ is a closed subspace of $\mathcal{M}_1$ and $P_{\mathcal{L}}^{\mathcal{M}_2} \in \mathcal{T}(\mathcal{M}_2)$, then $P_{\mathcal{L}}^{\mathcal{M}_1} \in \mathcal{T}(\mathcal{M}_1)$.*

Proof Note that $R_{z_i}^{\mathcal{M}_2}|_{\mathcal{M}_1} = R_{z_i}^{\mathcal{M}_1}$ and so, by taking adjoint, we have

$$P_{\mathcal{M}_1} (R_{z_i}^{\mathcal{M}_2})^*|_{\mathcal{M}_1} = (R_{z_i}^{\mathcal{M}_1})^*,$$

for all $i = 1, \dots, n$. Then $R_{z_i}^{\mathcal{M}_1} (R_{z_j}^{\mathcal{M}_1})^* = P_{\mathcal{M}_1} R_{z_i}^{\mathcal{M}_2} P_{\mathcal{M}_1}^{\mathcal{M}_2} (R_{z_j}^{\mathcal{M}_2})^*|_{\mathcal{M}_1}$, $i = 1, \dots, n$. This yields

$$\begin{aligned} R_{z_i}^{\mathcal{M}_1} (R_{z_j}^{\mathcal{M}_1})^* &= P_{\mathcal{M}_1} R_{z_i}^{\mathcal{M}_2} I_{\mathcal{M}_2} (R_{z_j}^{\mathcal{M}_2})^*|_{\mathcal{M}_1} - P_{\mathcal{M}_1} R_{z_i}^{\mathcal{M}_2} P_{\mathcal{M}_2 \ominus \mathcal{M}_1}^{\mathcal{M}_2} (R_{z_i}^{\mathcal{M}_2})^*|_{\mathcal{M}_1} \\ &= P_{\mathcal{M}_1} R_{z_i}^{\mathcal{M}_2} (R_{z_j}^{\mathcal{M}_2})^*|_{\mathcal{M}_1} + F, \end{aligned}$$

for all $i, j = 1, \ldots, n$, as $\dim(\mathcal{M}_2 \ominus \mathcal{M}_1) < \infty$. Similarly $(R_{z_j}^{\mathcal{M}_1})^* R_{z_i}^{\mathcal{M}_1} = P_{\mathcal{M}_1}(R_{z_j}^{\mathcal{M}_2})^* R_{z_i}^{\mathcal{M}_2}|_{\mathcal{M}_1} + F$ for all $i, j = 1, \ldots, n$. Now let $T_1 \in \mathcal{T}(\mathcal{M}_1)$ be a finite word formed from the symbols

$$\{R_{z_i}^{\mathcal{M}_1}, (R_{z_i}^{\mathcal{M}_1})^* : i = 1, \ldots, n\},$$

and let $T_2 \in \mathcal{T}(\mathcal{M}_2)$ be the same word but formed from the corresponding symbols in

$$\{R_{z_i}^{\mathcal{M}_2}, (R_{z_i}^{\mathcal{M}_2})^* : i = 1, \ldots, n\}.$$

Then $T_1 = P_{\mathcal{M}_1} T_2|_{\mathcal{M}_1} + F$. Since both $\mathcal{T}(\mathcal{M}_1)$ and $\{P_{\mathcal{M}_1} T|_{\mathcal{M}_1} : T \in \mathcal{T}(\mathcal{M}_2)\}$ are closed subspaces of $\mathcal{B}(\mathcal{M}_1)$ and both contain all the compact operators in $\mathcal{B}(\mathcal{M}_1)$, it follows that $\mathcal{T}(\mathcal{M}_1) = \{P_{\mathcal{M}_1} T|_{\mathcal{M}_1} : T \in \mathcal{T}(\mathcal{M}_2)\}$. The second assertion now clearly follows from the first one. ■

A thorough understanding of co-doubly commuting invariant subspaces of finite codimension is important to analyze C^*-algebras of invariant subspaces of finite codimension in $H^2(\mathbb{D}^n)$. If $\mathcal{S}$ is a closed invariant subspace of $H^2(\mathbb{D})$, then we know that $\mathcal{S} = \theta H^2(\mathbb{D})$ for some inner function $\theta \in H^\infty(\mathbb{D})$. To simplify notations, for a given inner function $\theta \in H^\infty(\mathbb{D})$, we denote

$$\mathcal{S}_\theta = \theta H^2(\mathbb{D}), \quad \text{and} \quad \mathcal{Q}_\theta = H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D}).$$

Also, given an inner function $\theta_i \in H^\infty(\mathbb{D})$, $1 \leq i \leq n$, denote by M_{θ_i} the multiplication operator

$$(M_{\theta_i} f)(z_1, \ldots, z_n) = \theta_i(z_i) f(z_1, \ldots, z_n)$$

for all $f \in H^2(\mathbb{D}^n)$ and $(z_1, \ldots, z_n) \in \mathbb{D}^n$. Recall now that an invariant subspace $\mathcal{S}$ of $H^2(\mathbb{D}^n)$ is said to be *co-doubly commuting* [23] if $\mathcal{S} = \mathcal{S}_\Phi$ where

$$\mathcal{S}_\Phi = (\mathcal{Q}_{\varphi_1} \otimes \cdots \otimes \mathcal{Q}_{\varphi_n})^\perp, \tag{4.1}$$

and φ_i, $i = 1, \ldots, n$, is either inner or the zero function. We warn the reader that the suffix Φ in $\mathcal{S}_\Phi$ refers to the finite Blaschke products $\{\varphi_i\}_{i=1}^n$. Here, in view of (4.1) (or see [23]), we have

$$(M_{\varphi_p} M_{\varphi_p}^*)(M_{\varphi_q} M_{\varphi_q}^*) = (M_{\varphi_q} M_{\varphi_q}^*)(M_{\varphi_p} M_{\varphi_p}^*),$$

for all $p, q = 1, \ldots, n$, and

$$P_{\mathcal{S}_\Phi} = I_{H^2(\mathbb{D}^n)} - \prod_{i=1}^n (I_{H^2(\mathbb{D}^n)} - M_{\varphi_i} M_{\varphi_i}^*). \tag{4.2}$$

It also follows that

$$\mathcal{S}_\Phi = M_{\varphi_1} H^2(\mathbb{D}^n) + \cdots + M_{\varphi_n} H^2(\mathbb{D}^n).$$

Therefore, $\mathcal{S}_\Phi$ has finite codimension if and only if φ_i is a finite Blashcke product for all $i = 1, \ldots, n$. Moreover, it can be proved following the same line of argument as Lemma 3.1 in [26] that if $\mathcal{S}$ is an invariant subspace of $H^2(\mathbb{D}^n)$ then $\mathcal{S}$ is of finite codimension if and only if there exist finite Blaschke products $\varphi_1, \ldots, \varphi_n$ such that

$$\mathcal{S}_\Phi \subseteq \mathcal{S}.$$

Given $\mathcal{S}_\Phi$ as in (4.1) and $1 \le i < j \le n$, we define $\mathcal{Q}_\Phi[i, j]$ by

$$\mathcal{Q}_\Phi[i, j] = \mathcal{Q}_{\varphi_i} \otimes \mathcal{Q}_{\varphi_{i+1}} \otimes \cdots \otimes \mathcal{Q}_{\varphi_j} \subseteq H^2(\mathbb{D}^{j-i+1}).$$

Lemma 4.3 *Let $\{\varphi_i\}_{i=1}^n$ be finite Blaschke products. If*

$$\mathcal{L}_1 = \mathcal{Q}_\Phi[1, n-1]^\perp \otimes H^2(\mathbb{D}),\ \mathcal{L}_2 = \mathcal{Q}_\Phi[1, n-1] \otimes \mathcal{S}_{\varphi_n},$$

$$\mathcal{L}_3 = \mathcal{Q}_\Phi[1, n-1] \otimes H^2(\mathbb{D}), \mathcal{L}_2' = \mathcal{Q}_\Phi[1, n-1] \otimes \varphi_n \mathcal{S}_{\varphi_n}$$

and

$$\mathcal{L}_2'' = \mathcal{Q}_\Phi[1, n-1] \otimes \varphi_n \mathcal{Q}_{\varphi_n},$$

then $P_{\mathcal{L}_1}, P_{\mathcal{L}_2}, P_{\mathcal{L}_2'}$ and $P_{\mathcal{L}_2''}$ are in $\mathcal{T}(H^2(\mathbb{D}^n))$ and $P_{\mathcal{L}_1}^{\mathcal{S}_\Phi}, P_{\mathcal{L}_2}^{\mathcal{S}_\Phi}, P_{\mathcal{L}_2'}^{\mathcal{S}_\Phi}$ and $P_{\mathcal{L}_2''}^{\mathcal{S}_\Phi}$ are in $\mathcal{T}(\mathcal{S}_\Phi)$.

Proof Clearly $\mathcal{S}_\Phi = \mathcal{L}_1 \oplus \mathcal{L}_2$, $H^2(\mathbb{D}^n) = \mathcal{L}_1 \oplus \mathcal{L}_3$ and $\mathcal{L}_2 = \mathcal{L}_2' \oplus \mathcal{L}_2''$. By virtue of Lemma 4.2, we only prove the lemma for $H^2(\mathbb{D}^n)$. Since $\mathcal{L}_2''$ is finite-dimensional, it follows, by Lemma 4.1, that $P_{\mathcal{L}_2''} \in \mathcal{T}(H^2(\mathbb{D}^n))$. Since $\varphi_i \in H^\infty(\mathbb{D})$ is a finite Blaschke product, it follows that φ_i is holomorphic in an open set containing the closure of the disc, and hence $M_{\varphi_i} = \varphi_i(M_{z_i}) \in \mathcal{T}(H^2(\mathbb{D}^n))$ for all $i = 1, \ldots, n$. Then, by (4.2), $P_{\mathcal{S}_\Phi} \in \mathcal{T}(H^2(\mathbb{D}^n))$. In view of $\mathcal{S}_\Phi = \mathcal{L}_1 \oplus \mathcal{L}_2$, it is then enough to prove only that $P_{\mathcal{L}_2} \in \mathcal{T}(H^2(\mathbb{D}^n))$. This readily follows from the equality

$$P_{\mathcal{L}_2} = \Big(\prod_{i=1}^{n-1} (I_{H^2(\mathbb{D}^n)} - M_{\varphi_i} M_{\varphi_i}^*) \Big) M_{\varphi_n} M_{\varphi_n}^*.$$

This completes the proof of the lemma. ∎

In particular, $\mathcal{T}(\mathcal{S}_\Phi)$ contains a wealth of orthogonal projections. This leads to some further observations concerning the C^*-algebra $\mathcal{T}(\mathcal{S}_\Phi)$. First, given $\mathcal{S}_\Phi$ as in (4.1), we consider the unitary operator $U : H^2(\mathbb{D}^n) \to \mathcal{S}_\Phi$ defined by

$$U = \begin{bmatrix} I_{\mathcal{L}_1} & 0 \\ 0 & M_{\varphi_n} \end{bmatrix} : \begin{matrix} \mathcal{L}_1 \\ \oplus \\ \mathcal{L}_3 \end{matrix} \to \begin{matrix} \mathcal{L}_1 \\ \oplus \\ \mathcal{L}_2 \end{matrix}.$$

Then $U = P_{\mathcal{L}_1} + M_{\varphi_n} P_{\mathcal{L}_3}$ and $U^* = P_{\mathcal{L}_1}^{\mathcal{S}_\Phi} + M_{\varphi_n}^* P_{\mathcal{L}_2}^{\mathcal{S}_\Phi}$. We have the following result:

Theorem 4.4 *If $\{\varphi_i\}_{i=1}^n$ are finite Blaschke products, then*

$$U^* \mathcal{T}(\mathcal{S}_\Phi) U = \mathcal{T}(H^2(\mathbb{D}^n)).$$

In particular, $\mathcal{T}(\mathcal{S}_\Phi)$ and $\mathcal{T}(H^2(\mathbb{D}^n))$ are unitarily equivalent.

Proof A simple computation first confirms that

$$U^* R_{z_n} U = M_{z_n} \in \mathcal{T}(H^2(\mathbb{D}^n)),$$

that is

$$M_{z_n} \in U^* \mathcal{T}(\mathcal{S}_\Phi) U \quad \text{and} \quad R_{z_n} \in U \mathcal{T}(H^2(\mathbb{D}^n)) U^*.$$

Next, let $i = 1, \ldots, n-1$. Then

$$R_{z_i} U = M_{z_i} P_{\mathcal{L}_1} + R_{z_i} M_{\varphi_n} P_{\mathcal{L}_3} = M_{z_i} P_{\mathcal{L}_1} + M_{z_i} M_{\varphi_n} P_{\mathcal{L}_3},$$

as $M_{\varphi_n} \mathcal{L}_3 = \mathcal{L}_2 \subseteq \mathcal{S}_\Phi$, and so

$$\begin{aligned} U^* R_{z_i} U &= (P_{\mathcal{L}_1}^{\mathcal{S}_\Phi} + M_{\varphi_n}^* P_{\mathcal{L}_2}^{\mathcal{S}_\Phi})(M_{z_i} P_{\mathcal{L}_1} + M_{z_i} M_{\varphi_n} P_{\mathcal{L}_3}) \\ &= M_{z_i} P_{\mathcal{L}_1} + P_{\mathcal{L}_1} M_{z_i} M_{\varphi_n} P_{\mathcal{L}_3} + M_{\varphi_n}^* P_{\mathcal{L}_2} M_{z_i} M_{\varphi_n} P_{\mathcal{L}_3}, \end{aligned}$$

as $M_{z_i} \mathcal{L}_1 \subseteq \mathcal{L}_1$ and $M_{z_i} M_{\varphi_n} \mathcal{L}_3 = M_{z_i} \mathcal{L}_2 \subseteq \mathcal{S}_\Phi$. Then $U^* R_{z_i} U \in \mathcal{T}(H^2(\mathbb{D}^n))$ for al $i = 1, \ldots, n$, by Lemma 4.3. In particular

$$U^* \mathcal{T}(\mathcal{S}_\Phi) U \subseteq \mathcal{T}(H^2(\mathbb{D}^n)).$$

On the other hand, since $\mathcal{L}_2 = \mathcal{L}_2' \oplus \mathcal{L}_2''$ and $\mathcal{L}_2''$ is finite dimensional, it follows that $P_{\mathcal{L}_2} = P_{\mathcal{L}_2'} + F$, and thus $U^* = U^*|_{\mathcal{L}_1} + U^*|_{\mathcal{L}_2'} + F$. Now $U M_{z_i} U^*|_{\mathcal{L}_1} = U M_{z_i}|_{\mathcal{L}_1} = M_{z_i}|_{\mathcal{L}_1}$ as $z_i \mathcal{L}_1 \subseteq \mathcal{L}_1$ and hence

$$U M_{z_i} U^*|_{\mathcal{L}_1} = R_{z_i}|_{\mathcal{L}_1},$$

and on the other hand

$$UM_{z_i}U^*|_{\mathcal{L}_2'} = U(M_{z_i}M_{\varphi_n}^*|_{\mathcal{L}_2'}) = U(M_{z_i}P_{\mathcal{S}_\Phi}M_{\varphi_n}^*)|_{\mathcal{L}_2'} = U(R_{z_i}R_{\varphi_n}^*)|_{\mathcal{L}_2'},$$

where $R_{\varphi_n} = M_{\varphi_n}|_{\mathcal{S}_\Phi}$. Moreover, since $\mathcal{L}_3 = \mathcal{L}_2 \oplus \mathcal{S}_\Phi^\perp$ and $\mathcal{S}_\Phi^\perp$ is finite dimensional, it follows that $P_{\mathcal{L}_3} = P_{\mathcal{L}_2} + F$, and thus

$$\begin{aligned} UM_{z_i}U^*|_{\mathcal{L}_2'} &= P_{\mathcal{L}_1}R_{z_i}R_{\varphi_n}^*|_{\mathcal{L}_2'} + M_{\varphi_n}P_{\mathcal{L}_3}R_{z_i}R_{\varphi_n}^*|_{\mathcal{L}_2'} \\ &= P_{\mathcal{L}_1}R_{z_i}R_{\varphi_n}^*|_{\mathcal{L}_2'} + M_{\varphi_n}P_{\mathcal{L}_2}R_{z_i}R_{\varphi_n}^*|_{\mathcal{L}_2'} + F \\ &= P_{\mathcal{L}_1}^{\mathcal{S}_\Phi}R_{z_i}R_{\varphi_n}^*|_{\mathcal{L}_2'} + R_{\varphi_n}P_{\mathcal{L}_2}^{\mathcal{S}_\Phi}R_{z_i}R_{\varphi_n}^*|_{\mathcal{L}_2'} + F, \end{aligned}$$

and hence

$$UM_{z_i}U^* = R_{z_i}P_{\mathcal{L}_1}^{\mathcal{S}_\Phi} + P_{\mathcal{L}_1}^{\mathcal{S}_\Phi}R_{z_i}R_{\varphi_n}^*P_{\mathcal{L}_2'}^{\mathcal{S}_\Phi} + R_{\varphi_n}P_{\mathcal{L}_2}^{\mathcal{S}_\Phi}R_{z_i}R_{\varphi_n}^*P_{\mathcal{L}_2'}^{\mathcal{S}_\Phi} + F.$$

By Lemma 4.3, it follows then that $UM_{z_i}U^* \in \mathcal{T}(\mathcal{S}_\Phi)$ and so

$$U\mathcal{T}(H^2(\mathbb{D}^n))U^* \subseteq \mathcal{T}(\mathcal{S}_\Phi).$$

Therefore, the conclusion follows from the fact that $U^*R_{z_n}U = M_{z_n} \in \mathcal{T}(H^2(\mathbb{D}^n))$. ■

Now let $\mathcal{S}$ be an invariant subspace of finite codimension, and let $\mathcal{S}_\Phi \subseteq \mathcal{S}$, as in (4.1), for some finite Blashcke products $\{\varphi_i\}_{i=1}^n$. We proceed to prove that $\mathcal{T}(\mathcal{S})$ is unitarily equivalent to $\mathcal{T}(\mathcal{S}_\Phi)$. Let

$$m := \dim(\mathcal{S} \ominus \mathcal{S}_\Phi).$$

Observe that

$$P_{\mathcal{S}_\Phi} = M_{\varphi_1}M_{\varphi_1}^* + (I_{H^2(\mathbb{D}^n)} - M_{\varphi_1}M_{\varphi_1}^*)\Big(I_{H^2(\mathbb{D}^n)} - \prod_{i=2}^n (I_{H^2(\mathbb{D}^n)} - M_{\varphi_i}M_{\varphi_i}^*)\Big),$$

and so

$$\mathcal{S}_\Phi = \Big(\mathcal{S}_{\varphi_1} \otimes H^2(\mathbb{D}^{n-1})\Big) \oplus \Big(\mathcal{Q}_{\varphi_1} \otimes \mathcal{Q}_\Phi[2,n]^\perp\Big).$$

Lemma 4.5 $P^{\mathcal{S}}_{\mathcal{S}_{\varphi_1}\otimes H^2(\mathbb{D}^{n-1})}, P^{\mathcal{S}}_{\mathcal{Q}_{\varphi_1}\otimes \mathcal{Q}_\Phi[2,n]^\perp} \in \mathcal{T}(\mathcal{S})$ *and*

$$P^{\mathcal{S}_\Phi}_{\mathcal{S}_{\varphi_1}\otimes H^2(\mathbb{D}^{n-1})}, P^{\mathcal{S}_\Phi}_{\mathcal{Q}_{\varphi_1}\otimes \mathcal{Q}_\Phi[2,n]^\perp} \in \mathcal{T}(\mathcal{S}_\Phi).$$

Proof First one observes that, by virtue of Lemma 4.2, it is enough to prove the result for $\mathcal{S}$. Note that $M_{\varphi_1}\mathcal{S} \subseteq \mathcal{S}$. Define $R_{\varphi_1} \in \mathcal{B}(\mathcal{S})$ by $R_{\varphi_1} = M_{\varphi_1}|_{\mathcal{S}}$. Then $R_{\varphi_1} = \varphi_1(M_{z_1})|_{\mathcal{S}} \in \mathcal{T}(\mathcal{S})$ and

$$P_{M_{\varphi_1}\mathcal{S}} = R_{\varphi_1}R_{\varphi_1}^* \in \mathcal{T}(\mathcal{S}).$$

Now on the one hand

$$\mathcal{S}_{\varphi_1} \otimes H^2(\mathbb{D}^{n-1}) = M_{\varphi_1}H^2(\mathbb{D}^n) = M_{\varphi_1}\mathcal{S} \oplus \Big(M_{\varphi_1}H^2(\mathbb{D}^n) \ominus M_{\varphi_1}\mathcal{S}\Big),$$

also, $M_{\varphi_1}H^2(\mathbb{D}^n) \ominus M_{\varphi_1}\mathcal{S} = M_{\varphi_1}(H^2(\mathbb{D}^n) \ominus \mathcal{S})$ is finite dimensional, and hence we conclude $P_{\mathcal{S}_{\varphi_1}\otimes H^2(\mathbb{D}^{n-1})} \in \mathcal{T}(\mathcal{S})$. This along with $\dim\,(\mathcal{S} \ominus \mathcal{S}_\Phi) < \infty$ and the decomposition

$$\mathcal{S} = (\mathcal{S}_{\varphi_1} \otimes H^2(\mathbb{D}^{n-1})) \oplus (\mathcal{Q}_{\varphi_1} \otimes \mathcal{Q}_\Phi[2,n]^\perp) \oplus (\mathcal{S} \ominus \mathcal{S}_\Phi),$$

implies that $P_{\mathcal{Q}_{\varphi_1}\otimes\mathcal{Q}_\Phi[2,n]^\perp} \in \mathcal{T}(\mathcal{S})$. This completes the proof of the lemma. ■

For simplicity, let us introduce some more notation. Given $q \in \mathbb{N}$, let us denote

$$\mathbb{C}^{\otimes q} = \mathbb{C} \otimes \cdots \otimes \mathbb{C} \subseteq H^2(\mathbb{D}^q).$$

Note that $\mathbb{C}^{\otimes q}$ is the one-dimensional subspace consisting of the constant functions in $H^2(\mathbb{D}^q)$. Recalling $\dim(\mathcal{S} \ominus \mathcal{S}_\Phi) = m(< \infty)$, we consider the orthogonal decomposition of $\mathcal{S}_{\varphi_1} \otimes H^2(\mathbb{D}^{n-1})$ as:

$$\mathcal{S}_{\varphi_1} \otimes H^2(\mathbb{D}^{n-1}) = \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{S}_3,$$

where

$$\begin{cases} \mathcal{S}_1 = (\varphi_1 \mathcal{Q}_{z^m}) \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D}) \\ \mathcal{S}_2 = \mathcal{S}_{z^m\varphi_1} \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D}) \\ \mathcal{S}_3 = \mathcal{S}_{\varphi_1} \otimes (\mathbb{C}^{\otimes(n-2)})^\perp \otimes H^2(\mathbb{D}). \end{cases}$$

Finally, we define

$$\mathcal{L} = \mathcal{S}_2 \oplus \mathcal{S}_3 \oplus \Big(\mathcal{Q}_{\varphi_1} \otimes \mathcal{Q}_\Phi[2,n]^\perp\Big).$$

With this notation we have

$$\mathcal{S}_\Phi = \mathcal{S}_1 \oplus \mathcal{L},$$

and

$$\mathcal{S} = (\mathcal{S} \ominus \mathcal{S}_\Phi) \oplus \mathcal{S}_1 \oplus \mathcal{L}.$$

Lemma 4.6 $P^{\mathcal{S}}_{\mathcal{S}_i} \in \mathcal{T}(\mathcal{S})$ *and* $P^{\mathcal{S}_\Phi}_{\mathcal{S}_i} \in \mathcal{T}(\mathcal{S}_\Phi)$ *for all* $i = 1, 2, 3$.

Proof In view of Lemma 4.2, it is enough to prove that $P^{\mathcal{S}}_{\mathcal{S}_i} \in \mathcal{T}(\mathcal{S})$, $i = 1, 2, 3$. Note that $P_{\mathcal{S}_{\varphi_1} \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D})} \in \mathcal{T}(\mathcal{S})$ as

$$P_{\mathcal{S}_{\varphi_1} \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D})} = P_{\mathcal{S}_{\varphi_1} \otimes H^2(\mathbb{D}^{n-1})}(I_{\mathcal{S}} - X) P_{\mathcal{S}_{\varphi_1} \otimes H^2(\mathbb{D}^{n-1})},$$

where

$$X = \sum_{2 \le i_1 < \cdots < i_k \le n-1} (-1)^{k+1} R_{z_{i_1}} \cdots R_{z_{i_k}} R^*_{z_{i_1}} \cdots R^*_{z_{i_k}}.$$

Therefore

$$P_{\mathcal{S}_3} = P_{\mathcal{S}_{\varphi_1} \otimes H^2(\mathbb{D}^{n-1})} - P_{\mathcal{S}_{\varphi_1} \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D})} \in \mathcal{T}(\mathcal{S}).$$

Finally, since $P_{\mathcal{S}_2} = R^m_{z_1} P_{\mathcal{S}_{\varphi_1} \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D})} R^{*m}_{z_1}$ and $\mathcal{S}_1 \oplus \mathcal{S}_2 = \mathcal{S}_{\varphi_1} \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D})$, it follows that $P_{\mathcal{S}_1}$ and $P_{\mathcal{S}_2}$ are in $\mathcal{T}(\mathcal{S})$. ■

Before we proceed to the unitary equivalence of the C^*-algebras $\mathcal{T}(\mathcal{S})$ and $\mathcal{T}(\mathcal{S}_\Phi)$ we note that

$$\varphi_1 \mathcal{Q}_{z^m} = \text{span}\,\{\varphi_1, \varphi_1 z, \ldots, \varphi_1 z^{m-1}\}.$$

Theorem 4.7 *If* $\mathcal{S}$ *is a finite co-dimensional invariant subspace of* $H^2(\mathbb{D}^n)$ *and* $\mathcal{S}_\Phi \subseteq \mathcal{S}$ *for some finite Blaschke products* $\{\varphi_i\}_{i=1}^n$, *then* $\mathcal{T}(\mathcal{S})$ *and* $\mathcal{T}(\mathcal{S}_\Phi)$ *are unitarily equivalent.*

Proof By noting that $H^2(\mathbb{D}) = \mathbb{C} \oplus \mathcal{S}_z$, we decompose $\mathcal{S}_1$ as $\mathcal{S}_1 = \mathcal{F}_1 \oplus \mathcal{M}_1$ where

$$\mathcal{F}_1 = (\varphi_1 \mathcal{Q}_{z^m}) \otimes \mathbb{C}^{\otimes(n-1)}, \quad \text{and} \quad \mathcal{M}_1 = (\varphi_1 \mathcal{Q}_{z^m}) \otimes \mathbb{C}^{\otimes(n-2)} \otimes \mathcal{S}_z.$$

Taking into consideration $\dim \mathcal{F}_1 = \dim(\mathcal{S} \ominus \mathcal{S}_\Phi)$, we have a unitary $V : \mathcal{F}_1 \to \mathcal{S} \ominus \mathcal{S}_\Phi$, and then, using the decompositions

$$\mathcal{S}_\Phi = \mathcal{F}_1 \oplus \mathcal{M}_1 \oplus \mathcal{L}.$$

and

$$\mathcal{S} = (\mathcal{S} \ominus \mathcal{S}_\Phi) \oplus \mathcal{S}_1 \oplus \mathcal{L},$$

we see that

$$U = \begin{bmatrix} V & 0 & 0 \\ 0 & M_{z_n}^* & 0 \\ 0 & 0 & I_{\mathcal{L}} \end{bmatrix} : \mathcal{F}_1 \oplus \mathcal{M}_1 \oplus \mathcal{L} \to (\mathcal{S} \ominus \mathcal{S}_\Phi) \oplus \mathcal{S}_1 \oplus \mathcal{L},$$

defines a unitary from $\mathcal{S}_\Phi$ to $\mathcal{S}$. We claim that $U^*\mathcal{T}(\mathcal{S})U = \mathcal{T}(\mathcal{S}_\Phi)$. First we prove that $U^*\mathcal{T}(\mathcal{S})U \subseteq \mathcal{T}(\mathcal{S}_\Phi)$. Since $\dim \mathcal{F}_1 < \infty$, it suffices to prove that $U^* R_{z_i}^{\mathcal{S}} U|_{\mathcal{M}_1 \oplus \mathcal{L}} \in \mathcal{T}(\mathcal{S}_\Phi)$ for all $i = 1, \cdots, n$. Observe first that $U\mathcal{M}_1 = M_{z_n}^* \mathcal{M}_1 = \mathcal{S}_1 \subseteq \mathcal{S}_\Phi$, $M_{z_n}\mathcal{S}_1 \subseteq \mathcal{S}_1$ and $M_{z_n}\mathcal{L} \subseteq \mathcal{L}$. Since

$$U^* R_{z_n}^{\mathcal{S}} U|_{\mathcal{M}_1 \oplus \mathcal{L}} = U^* M_{z_n} M_{z_n}^*|_{\mathcal{M}_1} + M_{z_n}|_{\mathcal{L}},$$

and $U^* M_{z_n} M_{z_n}^*|_{\mathcal{M}_1} = M_{z_n}^2 M_{z_n}^*|_{\mathcal{M}_1} = M_{z_n}^2 P_{\mathcal{S}_\Phi} M_{z_n}^*|_{\mathcal{M}_1}$, it follows that

$$U^* R_{z_n}^{\mathcal{S}} U|_{\mathcal{M}_1 \oplus \mathcal{L}} = (R_{z_n}^{\mathcal{S}_\Phi})^2 (R_{z_n}^{\mathcal{S}_\Phi})^* P_{\mathcal{M}_1}^{\mathcal{S}_\Phi} + R_{z_n}^{\mathcal{S}_\Phi} P_{\mathcal{L}}^{\mathcal{S}_\Phi} \in \mathcal{T}(\mathcal{S}_\Phi).$$

Now for $1 < i < n$, we have

$$U^* R_{z_i}^{\mathcal{S}} U|_{\mathcal{M}_1 \oplus \mathcal{L}} = U^* M_{z_i} M_{z_n}^*|_{\mathcal{M}_1} + U^* M_{z_i}|_{\mathcal{L}},$$

where $U^* M_{z_i} M_{z_n}^*|_{\mathcal{M}_1} = M_{z_i} M_{z_n}^*|_{\mathcal{M}_1}$ as $z_i \mathcal{S}_1 \subseteq \mathcal{S}_3 \subseteq \mathcal{L}$. On the other hand, since $z_i \mathcal{S}_2 \subseteq \mathcal{S}_3$ we have $z_i \mathcal{L} \subseteq \mathcal{L}$ and hence $U^* M_{z_i}|_{\mathcal{L}} = M_{z_i}|_{\mathcal{L}}$, whence

$$U^* R_{z_i}^{\mathcal{S}} U|_{\mathcal{M}_1 \oplus \mathcal{L}} = R_{z_i}^{\mathcal{S}_\Phi} (R_{z_n}^{\mathcal{S}_\Phi})^* P_{\mathcal{M}_1}^{\mathcal{S}_\Phi} + R_{z_i}^{\mathcal{S}_\Phi} P_{\mathcal{L}}^{\mathcal{S}_\Phi} \in \mathcal{T}(\mathcal{S}_\Phi).$$

Now we decompose $\mathcal{M}_1$ as $\mathcal{M}_1 = \mathcal{K}_1 \oplus \tilde{\mathcal{K}}_1$ where

$$\mathcal{K}_1 = (\varphi_1 \mathcal{Q}_{z^{m-1}}) \otimes \mathbb{C}^{\otimes(n-2)} \otimes \mathcal{S}_z \quad \text{and} \quad \tilde{\mathcal{K}}_1 = (\varphi_1 z^{m-1}\mathbb{C}) \otimes \mathbb{C}^{\otimes(n-2)} \otimes \mathcal{S}_z.$$

Then

$$U^* R_{z_1}^{\mathcal{S}} U|_{\mathcal{M}_1} = U^* M_{z_1} M_{z_n}^*|_{\mathcal{K}_1} + U^* M_{z_1} M_{z_n}^*|_{\tilde{\mathcal{K}}_1} = M_{z_n} M_{z_1} M_{z_n}^*|_{\mathcal{K}_1} + M_{z_1} M_{z_n}^*|_{\tilde{\mathcal{K}}_1},$$

as $M_{z_1} M_{z_n}^* \mathcal{K}_1 \subseteq \mathcal{S}_1$ and $M_{z_1} M_{z_n}^* \tilde{\mathcal{K}}_1 \subseteq \mathcal{S}_2$. On the other hand, $U^* R_{z_1}^{\mathcal{S}} U|_{\mathcal{S}_2 \oplus \mathcal{S}_3} = M_{z_1}|_{\mathcal{S}_2 \oplus \mathcal{S}_3}$ as $M_{z_1}(\mathcal{S}_2 \oplus \mathcal{S}_3) \subseteq \mathcal{S}_2 \oplus \mathcal{S}_3 \subseteq \mathcal{L}$, and finally, by denoting $\mathcal{N} = \mathcal{Q}_{\varphi_1} \otimes \mathcal{Q}_\Phi[2, n]^\perp$, we have

$$U^* R_{z_1}^{\mathcal{S}} U|_{\mathcal{N}} = U^* M_{z_1}|_{\mathcal{N}} = U^*(I_{\mathcal{S}} - P_{\mathcal{S}_1}^{\mathcal{S}}) M_{z_1}|_{\mathcal{N}} + U^* P_{\mathcal{S}_1}^{\mathcal{S}} M_{z_1}|_{\mathcal{N}}.$$

Then $\mathcal{S} \ominus \mathcal{S}_1 = (\mathcal{S} \ominus \mathcal{S}_\Phi) \oplus \mathcal{L}$ and $M_{z_1}\mathcal{N} \subseteq \mathcal{S}_\Phi$ implies that

$$U^* R_{z_1}^{\mathcal{S}} U|_{\mathcal{N}} = P_{\mathcal{L}}^{\mathcal{S}_\Phi} M_{z_1}|_{\mathcal{N}} + M_{z_n} P_{\mathcal{S}_1}^{\mathcal{S}_\Phi} M_{z_1}|_{\mathcal{N}},$$

and so

$$U^* R_{z_1}^{\mathcal{S}} U|_{\mathcal{M}_1 \oplus \mathcal{L}} = R_{z_n}^{\mathcal{S}_\Phi} R_{z_1}^{\mathcal{S}_\Phi} (R_{z_n}^{\mathcal{S}_\Phi})^* P_{\mathcal{K}_1}^{\mathcal{S}_\Phi} + R_{z_1}^{\mathcal{S}_\Phi} (R_{z_n}^{\mathcal{S}_\Phi})^* P_{\tilde{\mathcal{K}}_1}^{\mathcal{S}_\Phi} + R_{z_1}^{\mathcal{S}_\Phi} P_{\mathcal{S}_2 \oplus \mathcal{S}_3}^{\mathcal{S}_\Phi}$$
$$+ P_{\mathcal{L}}^{\mathcal{S}_\Phi} R_{z_1}^{\mathcal{S}_\Phi} P_{\mathcal{N}}^{\mathcal{S}_\Phi} + R_{z_n}^{\mathcal{S}_\Phi} P_{\mathcal{S}_1}^{\mathcal{S}_\Phi} R_{z_1}^{\mathcal{S}_\Phi} P_{\mathcal{N}}^{\mathcal{S}_\Phi} + F.$$

This implies that $U^* R_{z_1}^{\mathcal{S}} U \in \mathcal{T}(\mathcal{S}_\Phi)$, and therefore $U^* \mathcal{T}(\mathcal{S}) U \subseteq \mathcal{T}(\mathcal{S}_\Phi)$. We now proceed to prove the reverse inclusion $U \mathcal{T}(\mathcal{S}_\Phi) U^* \in \mathcal{T}(\mathcal{S})$. Since $\dim(\mathcal{S} \ominus \mathcal{S}_\Phi) < \infty$, it is enough to prove that $U R_{z_i}^{\mathcal{S}_\Phi} U^*|_{\mathcal{S}_1 \oplus \mathcal{L}} \in \mathcal{T}(\mathcal{S})$ for all $i = 1, \ldots, n$. Once again, note that $U^* \mathcal{S}_1 = \mathcal{M}_1 \subseteq \mathcal{S}_\Phi$, $z_n \mathcal{M}_1 \subseteq \mathcal{M}_1$, $z_n \mathcal{S}_1 \subseteq \mathcal{S}_1$ and $z_n \mathcal{L} \subseteq \mathcal{L}$. Hence

$$U R_{z_n}^{\mathcal{S}_\Phi} U^*|_{\mathcal{S}_1 \oplus \mathcal{L}} = U M_{z_n}^2|_{\mathcal{S}_1} + U M_{z_n}|_{\mathcal{L}} = M_{z_n}|_{\mathcal{S}_1} + M_{z_n}|_{\mathcal{L}},$$

that is

$$U R_{z_n}^{\mathcal{S}_\Phi} U^*|_{\mathcal{S}_1 \oplus \mathcal{L}} = R_{z_n}^{\mathcal{S}} P_{\mathcal{S}_1 \oplus \mathcal{L}}^{\mathcal{S}} \in \mathcal{T}(\mathcal{S}).$$

Now, for fixed $1 < i < n$, we have $z_i \mathcal{M}_1 \subseteq \mathcal{S}_3$ and $z_i \mathcal{L} \subseteq \mathcal{L}$. Then

$$\begin{aligned} U R_{z_i}^{\mathcal{S}_\Phi} U^*|_{\mathcal{S}_1 \oplus \mathcal{L}} &= U M_{z_i} M_{z_n}|_{\mathcal{S}_1} + U M_{z_i}|_{\mathcal{L}} \\ &= M_{z_i} M_{z_n}|_{\mathcal{S}_1} + M_{z_i}|_{\mathcal{L}} \\ &= R_{z_i}^{\mathcal{S}} R_{z_n}^{\mathcal{S}} P_{\mathcal{S}_1}^{\mathcal{S}} + R_{z_i}^{\mathcal{S}} P_{\mathcal{L}} \in \mathcal{T}(\mathcal{S}). \end{aligned}$$

Finally, we consider the decomposition $\mathcal{S}_1 = \mathcal{S}_1' \oplus \mathcal{S}_1''$ where

$$\mathcal{S}_1' = (\varphi_1 \mathcal{Q}_{z^{m-1}}) \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D}) \text{ and } \mathcal{S}_1'' = (\varphi_1 z^{m-1} \mathbb{C}) \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D}).$$

Then

$$\begin{aligned} U R_{z_1}^{\mathcal{S}_\Phi} U^*|_{\mathcal{S}_1} &= U M_{z_1} M_{z_n}|_{\mathcal{S}_1'} + U M_{z_1} M_{z_n}|_{\mathcal{S}_1''} \\ &= M_{z_n}^* M_{z_1} M_{z_n}|_{\mathcal{S}_1'} + M_{z_1} M_{z_n}|_{\mathcal{S}_1''} \\ &= M_{z_1}|_{\mathcal{S}_1'} + M_{z_1} M_{z_n}|_{\mathcal{S}_1''}, \end{aligned}$$

as $z_1 z_n \mathcal{S}_1' \subseteq \mathcal{M}_1$ and $z_1 z_n \mathcal{S}_1'' \subseteq \mathcal{S}_2$. Moreover

$$U R_{z_1}^{\mathcal{S}_\Phi} U^*|_{\mathcal{S}_2 \oplus \mathcal{S}_3} = U M_{z_1}|_{\mathcal{S}_2 \oplus \mathcal{S}_3} = M_{z_1}|_{\mathcal{S}_2 \oplus \mathcal{S}_3},$$

as $z_1(\mathcal{S}_2 \oplus \mathcal{S}_3) \subseteq \mathcal{S}_2 \oplus \mathcal{S}_3$. From the definition of $\mathcal{N}$, it follows that

$$U R_{z_1}^{\mathcal{S}_\Phi} U^*|_{\mathcal{N}} = U P_{\mathcal{M}_1}^{\mathcal{S}_\Phi} M_{z_1}|_{\mathcal{N}} + U (I_{\mathcal{S}_\Phi} - P_{\mathcal{M}_1}^{\mathcal{S}_\Phi}) M_{z_1}|_{\mathcal{N}},$$

this in turn implies that

$$U R_{z_1}^{\mathcal{S}_\Phi} U^*|_{\mathcal{N}} = M_{z_n}^* P_{\mathcal{M}_1}^{\mathcal{S}} M_{z_1}|_{\mathcal{N}} + P_{\mathcal{L}}^{\mathcal{S}} M_{z_1}|_{\mathcal{N}} + F,$$

as $\mathcal{S}_\Phi \ominus \mathcal{M}_1 = \mathcal{F}_1 \oplus \mathcal{L}$ and $\mathcal{F}_1$ is finite dimensional. Therefore

$$\begin{aligned} U R_{z_1}^{\mathcal{S}_\Phi} U^*|_{\mathcal{S}_1 \oplus \mathcal{L}} &= R_{z_1}^{\mathcal{S}} P_{\mathcal{S}_1'}^{\mathcal{S}} + R_{z_1}^{\mathcal{S}} R_{z_n}^{\mathcal{S}} P_{\mathcal{S}_1''}^{\mathcal{S}} + R_{z_1}^{\mathcal{S}} P_{\mathcal{S}_2 \oplus \mathcal{S}_3}^{\mathcal{S}} \\ &\quad + (R_{z_n}^{\mathcal{S}})^* P_{\mathcal{M}_1}^{\mathcal{S}} M_{z_1} P_{\mathcal{N}}^{\mathcal{S}} + P_{\mathcal{L}}^{\mathcal{S}} R_{z_1}^{\mathcal{S}} P_{\mathcal{N}}^{\mathcal{S}} + F \in \mathcal{T}(\mathcal{S}). \end{aligned}$$

This completes the proof of the theorem. ■

On combining Theorems 4.4 and 4.7, we have the following:

Theorem 4.8 *If $\mathcal{S}$ is a finite co-dimensional invariant subspace of $H^2(\mathbb{D}^n)$, then $\mathcal{T}(\mathcal{S})$ and $\mathcal{T}(H^2(\mathbb{D}^n))$ are unitarily equivalent.*

In the case $n = 2$, the proof of the above result is considerably simpler and direct than the one by Seto [26] (for instance, if $n = 2$, then $1 < i < n$ case does not appear in the proof of Theorem 4.7).

Acknowledgments The research of the first named author is supported by DST-INSPIRE Faculty Fellowship No. DST/INSPIRE/04/2015/001094. The research of the third named author is supported in part by NBHM (National Board of Higher Mathematics, India) grant NBHM/R.P.64/2014, and the Mathematical Research Impact Centric Support (MATRICS) grant, File No: MTR/2017/000522 and Core Research Grant, File No: CRG/2019/000908, by the Science and Engineering Research Board (SERB), Department of Science & Technology (DST), Government of India.

References

1. J. Agler and N. J. Young, *A model theory for Γ-contractions*, J. Operator Theory 49 (2003), 45–60.
2. P. R. Ahern and D.N. Clark, *Invariant subspaces and analytic continuation in several variables*, J. Math. Mech. 19 (1970), 963–969.
3. H. Bercovici, R. Douglas and C. Foias, *Canonical models for bi-isometries*, A panorama of modern operator theory and related topics, 177–205, Oper. Theory Adv. Appl., 218, Birkhauser/Springer Basel AG, Basel, 2012.
4. H. Bercovici, R. Douglas and C. Foias, *Bi-isometries and commutant lifting*, Characteristic functions, scattering functions and transfer functions, 51–76, Oper. Theory Adv. Appl., 197, Birkhauser Verlag, Basel, 2010.
5. H. Bercovici, R. Douglas and C. Foias, *On the classification of multi-isometries*, Acta Sci. Math. (Szeged) 72 (2006), 639–661.
6. C. Berger, L. Coburn and A. Lebow, *Representation and index theory for C^*-algebras generated by commuting isometries*, J. Funct. Anal. 27 (1978), 51–99.
7. A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. 81 (1949), 239–255.

8. Z. Burdak, M. Kosiek and M. Slocinski, *The canonical Wold decomposition of commuting isometries with finite dimensional wandering spaces*, Bull. Sci. Math. 137 (2013), 653–678.
9. B.K. Das, S. Sarkar and J. Sarkar, *Factorizations of contractions*, Advances in Mathematics 322 (2017), 186–200.
10. R. Douglas, *On the C^*-algebra of a one-parameter semigroup of isometries*, Acta Math. 128 (1972), 143–151.
11. R. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. 17 (1966), 413–415.
12. R. Douglas and R. Yang, *Operator theory in the Hardy space over the bidisk. I*, Integral Equations Operator Theory 38 (2000), 207–221.
13. C. Foias and A. Frazho, *The commutant lifting approach to interpolation problems*, Operator Theory: Advances and Applications, 44. Birkhäuser Verlag, Basel, 1990.
14. D. Gaspar and P. Gaspar, *Wold decompositions and the unitary model for bi-isometries*, Integral Equations Operator Theory 49 (2004), 419–433.
15. D. Gaspar and N. Suciu, *Wold decompositions for commutative families of isometries*, An. Univ. Timisoara Ser. Stint. Mat. 27 (1989), 31–38.
16. P. Halmos, *Shifts on Hilbert spaces*, J. Reine Angew. Math. 208 (1961), 102–112.
17. W. He, Y. Qin, and R. Yang, *Numerical invariants for commuting isometric pairs*, Indiana Univ. Math. J. 64 (2015), 1–19.
18. P. Lax, *Translation invariant spaces*, Acta Math. 101 (1959), 163–178.
19. A. Maji, A. Mundayadan, J. Sarkar and Sankar T. R, *Characterization of invariant subspaces in the polydisc*, J. Operator Theory 82 (2019), 445–468.
20. A. Maji, J. Sarkar and Sankar T. R, *Pairs of Commuting Isometries - I*, Studia Mathematica 248 (2019), 171–189.
21. B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North Holland, Amsterdam, 1970.
22. D. Popovici, *On the structure of c.n.u. bi-isometries*, Acta Sci. Math. (Szeged) 66 (2000), 719–729.
23. J. Sarkar, *Jordan Blocks of $H^2(\mathbb{D}^n)$*, J. Operator theory 72 (2014), 371–385.
24. J. Sarkar, *Operator theory on symmetrized bidisc*, Indiana Univ. Math. J. 64 (2015), 847–873.
25. J. Sarkar, A. Sasane and B. W. Wick, *Doubly commuting submodules of the Hardy module over polydiscs*, Studia Math. 217 (2013), 179–192.
26. M. Seto, *On the Berger-Coburn-Lebow problem for Hardy submodules*, Canad. Math. Bull. 47 (2004), 456–467.
27. R. Yang, *A note on classification of submodules in $H^2(\mathbb{D}^2)$*, Proc. Amer. Math. Soc. 137 (2009), 2655–2659.
28. R. Yang, *Hilbert-Schmidt submodules and issues of unitary equivalence*, J. Operator Theory 53 (2005), 169–184.