Operator Theory
Advances and Applications
278

Raul E. Curto, William Helton,
Huaxin Lin, Xiang Tang,
Rongwei Yang, Guoliang Yu,
Editors

Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology

Ronald G. Douglas Memorial Volume

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Operator Theory: Advances and Applications

Volume 278

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ISSN 0255-0156 ISSN 2296-4878 (electronic)
Operator Theory: Advances and Applications
ISBN 978-3-030-43379-6 ISBN 978-3-030-43380-2 (eBook)
https://doi.org/10.1007/978-3-030-43380-2

Mathematics Subject Classification: 47-XX

This book is published under the imprint Birkhäuser, www.birkhauser-science.com, by the registered company Springer Nature Switzerland AG.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland


International Workshop on Operator Theory and its Applications
July 23-27, 2018 on the campus of East China Normal University in Shanghai

Preface

The IWOTA 2018 was held at East China Normal University, Shanghai from July 23–27. To honor Ronald G. Douglas' expansive and profound contributions to mathematics, in particular his monumental contribution to operator theory research in China for the past 30 years, the organizers had planned to take this occasion to celebrate his 80th birthday. Sadly, Ron passed away at Brazos Valley Hospice in Bryan, Texas on February 27. The organizers thus decided to make this Proceeding of IWOTA 2018 a special memorial volume. In addition to papers pertinent to the themes of the conference, this volume collected papers from some of his collaborators and former students. Included also is an article by physicist Michael R. Douglas which gives a personal account of his father's influence.

Ron was born on December 10, 1938 in Osgood, Indiana. He earned his doctorate at Louisiana State University in 1962. His first paper was in measure theory and it was published in 1964 by Michigan Mathematical Journal. Ron's early career research centered mostly around classical operator theory topics such as invariant subspaces, Toeplitz operators, operator model theory, and C^*-algebras, and he soon emerged as one of the leaders in these fields. But in Ron's view, there is indeed no fence within mathematics. His work on operator theory extended naturally to complex geometry. The definition of Cowen–Douglas operator was announced at a symposium at Williamstown, Massachusetts in 1975. This notion made it possible to use geometric tools such as holomorphic bundle and curvature to study unitary equivalence of operators. The foundation of Brown–Douglas–Fillmore theory was laid in a 1977 joint paper. Although its original intention was to use topological methods to classify essentially normal operators, it in fact gave a simple analytic version of K-homology. This work turned out to be fundamental to noncommutative geometry and topology. His later work focused mainly on multivariable operator theory and in particular its analytic framework Hilbert modules in function spaces. The notion of Hilbert module was announced in a conference at Timişoara and Herculane, Romania in 1984. In response to Shunhua Sun's invitation, Ron gave a series of lectures on this topic at Sichuan University, China in 1985. A more extensive treatment was later carried out in a book coauthored with V. I. Paulsen in 1989. This framework greatly propelled the development of multivariable operator theory. A difficult problem in this field is the Arveson–Douglas conjecture which connects essentially normal Hilbert modules with algebraic geometry, differential equations, index theory, and K-homology.

Each of Ron's aforementioned visionary work has nurtured a large community of scholars. This volume contains a number of papers and surveys in these fields. In this regard, it serves as a testimony that Ron's mathematical ideas are still very much alive. In closing, we would like to thank the contributors to this volume and the many unnamed reviewers of the articles. Special thanks go to Huaxin Lin, Guoliang Yu, the local organizers Xiaoman Chen, Kunyu Guo, Qin Wang, Yi-Jun Yao, and the many volunteer helpers from East China Normal University and Fudan University, without whom this grand scale conference would not have been possible. This IWOTA 2018 is also indebted to East China Normal University, Fudan University, and the U.S. National Science Foundation (award no. 1800780) for their financial support.

St. Louis, MO, USA — Xiang Tang
Albany, NY, USA — Rongwei Yang
January 28, 2020

Contents

Following in the Footsteps of Ronald G. Douglas

Michael R. Douglas

Abstract Many of you knew my father Ronald Douglas as a mentor, a collaborator or a colleague. While I never wrote a paper with him, he was a powerful influence on my own career. Some of my most important works, such as those on Dirichlet branes and noncommutative geometry, turned out to have strong connections with his work. In this talk I will reminisce a bit and describe a few of these works and connections.

As I reflect on my father's life, I realize in how many ways I followed in his footsteps. I was one of his three children, growing up in Ann Arbor and then Stony Brook, and our father showed us how attractive the academic life could be—bringing back gifts and photos from conferences in exotic countries, hosting dinner parties for visiting friends and colleagues from around the world. Just as appealing were the simple things: his home office filled with books, some of which he had written, or his freedom to come home early from work when we needed him, say to help with a difficult project for school.

We made several long family trips which had a huge influence on me: especially a sabbatical semester in Newcastle-upon-Tyne in 1973, and a summer in France in 1970 which included a month in Les Houches. There our mother (his first wife Nancy) would take us walking in the mountains, while our father attended the well known summer school which that year was on statistical mechanics and quantum field theory. Arthur Jaffe has some nice reminiscences of that meeting in [1]. Later as a young string theorist and mathematical physicist, when I would meet older colleagues, I was often told that it was not for the first time, they remembered me from when I was little, many from that meeting.

This trip was also the seed of what would become a lifelong relationship with France. A year in 1990 visiting Volodya Kazakov and Edouard Brezin at the

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_1

Laboratoire de Physique Theorique of the ENS in Paris, my collaboration with Alain Connes and Albert Schwarz at the IHES, my many visits there between 1999 and 2008 as the Louis Michel chair, and my current role as chairman of the Friends of IHES, all were in some way fulfilling that early attraction to French and European culture, to physics and to mathematics.

Although I was fascinated by mathematics, I chose to major in theoretical physics in college, in part just to avoid following too closely in my father's footsteps. Operators were something I used in quantum mechanics, but my mathematical education did not include operator algebras. In grad school I did take a course in mathematical methods with Barry Simon, but that focused on topology as used to study solitons and instantons. I had by then heard the initials BDF, but I have to admit that when I first tried to read the paper, and for many years after that, I did not understand any of it. Still, later on my own research would turn out to have many points of contact with his, both by choice and by chance.

The most direct influence came in the early 1990s. In 1988, my first year as a postdoc at Chicago, I started working with Steve Shenker on random matrix theory. As many of you know, the usual starting point for this theory is the discussion of matrix ensembles such as the Gaussian unitary ensemble, defined by the following integral over $N \times N$ hermitian matrices,

$$\int \prod_{1 \leq i,j \leq N} dM_{i,j} e^{-N \operatorname{tr} M^2},$$

where the measure is independent and uniform for each of the matrix elements. We were particularly interested in generalizations such as

$$Z[N,\lambda] \equiv \int \prod_{1 \leq i,j \leq N} dM_{i,j} e^{-N\left(\frac{1}{2} \operatorname{tr} M^2 + \frac{\lambda}{3} \operatorname{tr} M^3\right)}$$

where λ is a real parameter.

Our interest in this was not because M was an operator, or any other property of the matrix M. Rather, it was because of a combinatorial interpretation of the integral, first pointed out in the physics literature by Gerard 't Hooft. It is a generating function for the number of planar triangulations of a genus g Riemann surface with F faces, call this $Z_{g,F}$,

$$Z[N,\lambda] = \sum_{g,F} N^{2-2g} \lambda^F Z_{g,F}.$$

This is explained in many references such as [3], and very recently in section 4 of [4], so I will not repeat it here.

One can go on, as proposed by Migdal and collaborators in the mid-1980s, to regard the terms in this expansion at fixed g as defining a discrete approximation to two-dimensional quantum gravity. The idea is that a planar triangulation can be thought of as defining a Riemannian metric on the surface, with curvature concentrated at each of the vertices. This is a very special class of metrics, but if one only looks at the total curvature for regions containing many triangles, since in two dimensions the only invariant of a metric is the curvature scalar, one can argue that a general metric can be approximated this way. Thus, a sum over all triangulations is some sort of approximation to an integral over all Riemannian metrics, which is what physicists mean by the term quantum gravity. The specific results will of course depend on the choice of triangles versus squares or some other class of diagrams, but if we just look at the asymptotics for F large, it is plausible that some aspects of the results are universal, and thus can be thought of as properties of a random Riemannian metric. This turns out to be true, and has even been shown rigorously in some cases [5].

Now, two-dimensional quantum gravity can be thought of as a simplified "toy" model of the quantum version of Einstein's theory of general relativity, but it can also be thought of as a simplified model of the two-dimensional world-sheet of a string as defined in superstring theory. Our goal was to understand the latter and in particular to find a model in which one could compute results for all genus g, and perhaps resum them to get a "nonperturbative string theory." To this end, we developed what we called the double scaling limit of taking $N \to \infty$ and $\lambda \to \lambda_c$ (the location of the singularity controlling the large F asymptotics) holding an appropriate combination fixed. In this limit, the universal quantities could be computed exactly in terms of a solution of the Painlevé I equation, related to the integrable KdV hierarchy as first argued in [6].

This work was quite influential, to the point where my father started hearing about it from other physicists. This brings me to the story of the second time I went to visit Chen Ning Yang at Stony Brook. My father knew him well of course and brought me to visit him when I was first deciding where to go to graduate school and what to study. Yang explained that although particle physics might look attractive, one had to keep in mind that ultimately it was based on experiments done at colliders, and that the progress in such experiments was becoming more and more difficult. In fact, he counseled me against going into the field. So I took his advice, and decided to go to Caltech to work with John Hopfield on his new theory of neural networks. This was fall 1983, but in the summer of 1984 came the famous paper of Green and Schwarz on anomaly cancellation in ten-dimensional superstrings. Soon most of my fellow graduate students were working on string theory, and I was caught up in the excitement as well. But I had not forgotten Yang's advice, and was rather worried about what he would say when I returned to visit him in 1990. But he had heard of my work too, and told me that perhaps I had been right not to follow his advice.

After many discussions on random matrix theory with my father, he suggested I talk to Dan Voiculescu. As you all know, Voiculescu had developed a framework called free probability theory, which axiomatized the key properties of random matrix integrals in the large N limit. This theory has been very influential in mathematics, but what was more attractive about it for a physicist was that it led to very simple and intuitive calculational tools, such as the R-transform and S-transform for additive and multiplicative free convolutions. Inspired by this, several physicists including Rajesh Gopakumar and David Gross, Matthias Staudacher, as well as Miao Li and myself, used free probability both to simplify the existing random matrix works and to solve new problems, most notably a problem raised by Is Singer of characterizing the master field for two-dimensional Yang–Mills theory.

Dan Voiculescu also invited me to a workshop he organized at the Fields Institute in March 1995, which I attended along with Tony Zee, a theoretical physicist who had moved from quantum field theory into statistical mechanics and condensed matter theory. This is a good point for me to comment on the difficulties of communication between physicists and mathematicians. Mathematicians often complain that they can't understand physicists because they never define what they are talking about, but it is just as difficult to follow a talk based on precise definitions which one is seeing for the first time, or even worse which are not spelled out in the talk. Now as a string theorist and as my father's son, I had some experience in interpreting mathematics, but Tony found the talks impenetrable and I remember having to give him many translations. Still this was time well spent, as Tony went on to write many papers using these ideas with Edouard Brezin and others, and free probability theory is now a well established tool in condensed matter physics.

While I first learned about the connection between random matrix theory and free probability theory from my father, the most direct connection between the topics of our research came a bit later and as a surprise to both of us. Now already by the time of the workshop I just recalled, string theorists including myself were moving on to a new topic, duality in supersymmetric field theory, epitomized by the famous Seiberg–Witten solution of $N = 2$ super Yang–Mills theory. That summer came the 1995 Strings conference held at USC, at which Chris Hull and Paul Townsend proposed their unification of superstring dualities, and Edward Witten gave the first talk on M theory. This was the beginning of the second superstring revolution, the most exciting part of my scientific career. For the next three years, almost every month there would be a new discovery which would force us to completely rethink our concept of string theory and our research directions.

Arguably the most important of these discoveries was the central role of the Dirichlet brane, explained at the end of 1995 by the late Joe Polchinski [7]. Now from the beginning of string theory, people had studied both open strings, maps from an oriented interval into space-time, and closed strings, maps from the circle into space-time. And during the 1970s it was realized that quantizing the open strings produced Yang–Mills theory, while quantizing the closed strings produced general relativity. One sign that this made sense was that whereas a closed string is in some sense unique, an open string can have "charged quarks" at its ends which couple to a background $U(N)$ Yang–Mills connection. In plainer mathematical terms, let V be the defining representation of the Yang–Mills gauge group $U(N)$, and let $U(P)$ be the holonomy for a path P of the Yang–Mills connection acting on V. Then the "coupling" means that we redefine the operator expressing the motion of the string from an initial interval I_i to a final interval I_f, by tensoring it with $U(P_0) \otimes U^\dagger(P_1)$, where P_0 is a path from the "left" end of I_i to the left end of I_f, and P_1 is a path from the "right" end of I_i to the right end of I_f. In the limit that the length of the interval goes to zero, we have $P_0 = P_1$ and this amounts to taking the holonomy in the adjoint representation. But in general it is different.

In the physical applications of string theory, the strings are very small and this difference is rather subtle. But one can vary the construction to make it much more evident. For example, one could take the two ends of the open string to couple to two different Yang–Mills connections. Next, although the original definition of the open string allowed its ends to move anywhere in space-time, it is also consistent to constrain the end to a single point, or to an affine subspace of Minkowski space-time. In the terms of the path integral formalism this amounts to putting Dirichlet boundary conditions on the coordinates of the embedding map and thus the nomenclature. And once one considers a more general metric on space-time, one generalizes the constraint from an affine subspace to an arbitrary submanifold. Thus, the full definition of an open string requires maing a choice for each of the two ends $i = 0, 1$ of the string, of data (Σ_i, A_i) where Σ_i is a submanifold of space-time and A_i is a connection on Σ_i.

All this had been pointed out by Dai, Leigh and Polchinski in 1989 [8], but what Polchinski showed in 1995 was that the Dirichlet brane could also be interpreted in closed string theory, as the natural object carrying Ramond–Ramond charge. Without going deeply into the physics, each string theory (and M theory) has a finite list of gauge fields. The philosophy of superstring duality then states that each of these gauge fields is associated to two fundamental objects, one carrying electric charge under the field and the other carrying magnetic charge. These fundamental objects were then the key to understanding the strong coupling behavior of the theory. Using arguments from supersymmetry, one could compute the mass of every fundamental object as a function of parameters, and then whichever was the lightest object would be "the" fundamental object in that regime. As an example, in the closed superstring theories, one of the gauge fields is the so-called "Neveu–Schwarz two-form field," for which the closed string is the fundamental object. And consistent with the philosophy, one finds that if the string coupling is weak, all of the other candidate fundamental objects have large mass. Now these other masses are proportional to an inverse power of the string coupling, so for strong coupling a different object will be the lightest. Which one depends on the theory. In the IIa superstring, one finds that the lightest object at strong coupling is a particle which is electrically charged under the "Ramond–Ramond one-form field," and treating it as fundamental leads to the identification of the strong coupling limit of IIa theory as M theory. But when this argument was made, there was no understanding of what this special particle might actually be in string theory terms. Polchinski showed that it is in fact the Dirichlet brane constrained to live at a point in space and move along a world-line in time, the so-called D-particle.

Hopefully the reader will not need to follow the details to see that the discovery of such a simple and intuitive idea led to another revolution in our understanding. My own contributions to these developments largely focused on the geometric interpretation of Dirichlet branes. It turned out that by just knowing how to calculate with Dirichlet branes, and following one's nose, one could rederive and extend many important mathematical results relating noncommutative algebra and geometry. This line of work began with Witten's "Small instantons in string theory" [9] and my [10], which rederived the ADHM construction of instanton moduli spaces this way, and with my joint work [11] with Greg Moore, which rederived and generalized the Kronheimer–Nakajima construction of four-dimensional self-dual metrics and their instanton moduli spaces.

Why do Dirichlet branes lead to noncommutative geometry? There should be a purely conceptual explanation of this point, but let me give the original argument in terms of the algebra of coordinates on space-time. In the physics of string theory, rather than work directly with the strings, one often proceeds through an intermediate step of "effective field theory," in which one identifies the subset of all of the degrees of freedom which are needed to describe the problem at hand. This is closely related to the idea of separating fast and slow variables in the analysis of ODE's and PDE's, and to the renormalization group. In particular, the relation I described earlier between open strings and Yang–Mills theory is an example; the effective theory of open strings is Yang–Mills theory. Following the physical arguments which lead to a Yang–Mills connection in the case of open strings moving in all of space-time, and modifying them to the case of a Dirichlet brane associated with the submanifold Σ of space-time, we find that the counterpart of the Yang–Mills connection is a map from Σ to its normal bundle $N\Sigma$, describing deformations of the embedding of Σ. Combining this with the "coupling" argument we gave earlier, this map to the normal bundle is tensored with an adjoint action of a gauge group for a Yang–Mills connection on Σ, becoming a map

$$X : \Sigma \to N\Sigma \otimes \mathrm{End}(V),$$

an intrinsically noncommutative object. While a general map of this type would contain far more data than a deformation of an embedding, the Yang–Mills equations also generalize to a flatness condition on the deformation,

$$[\delta X, \delta X] \sim 0.$$

In the simplest cases, say of Dirichlet branes in Minkowski space-time, this is zero and we conclude that the deformation lives in a diagonal subgroup of $\mathrm{End}(V)$, in other words it is like a direct sum of N independent deformations. This is the case that reduces to ordinary commutative geometry. But in more general problems the right hand side is more interesting, and one finds that the Dirichlet branes realize a noncommutative geometry.

Now I had lectured at the 1995 Les Houches lectures organized by Alain Connes and Krzysztof Gawedzki, though not about Dirichlet branes as this was the summer and these developments were still to come. But our writeups were not due until 1996, so after the Dirichlet brane revolution I decided that this was a much more interesting subject and I devoted part of my Les Houches write-up to it [12]. I had been intrigued by Connes' lectures there about his work relating noncommutative gauge theory and the Standard Model, and in my writeup I had a passage comparing and contrasting the two pictures, that of Connes and that from Dirichlet branes, for how noncommutative geometry related to physics. When Alain saw this, I think he was happy to see a string theorist giving his work the attention it deserved, and perhaps this entered into the discussions which led the IHES to offer me a permanent position late in 1996. By then I was convinced that noncommutative geometry had a deep relationship to Dirichlet branes, and I happily accepted the position on a trial basis. This led to a visit in the fall of 1997 and my collaboration with Alain and Albert Schwarz in which we explained how M theory could be compactified on the noncommutative torus. This was hugely influential and is still my most cited paper.

Although for family reasons I did not take up the permanent position, I continued to visit the IHES frequently and during these visits I enjoyed discussions with Maxim Kontsevich and his many visitors. Maxim had a somewhat different concept of noncommutative geometry, based on algebraic geometry and concepts such as the derived category of coherent sheaves. This was extremely difficult for a physicist to get any purchase on, but as I continued to develop the geometry of Dirichlet branes I found myself learning more and more of this mathematics, including quiver algebras, tilting equivalences, and deformation theory. One point where Maxim's intuition was of immediate guidance was the role of the superpotential, which in his terms was a reduction of the holomorphic Chern–Simons action. But the real prize in the story was the role of the derived category of coherent sheaves, which Maxim had brought in to formulate his homological mirror symmetry conjecture. Since the Dirichlet brane theory was in some sense a generalization of Yang–Mills, I and other physicists had brought in all of the successful approaches to Yang–Mills, including the Donaldson–Uhlenbeck–Yau theorem and the necessary prerequisite of stability of holomorphic bundles. Gradually we realized that the right approach to understanding Dirichlet branes on Calabi–Yau manifolds was to generalize the concepts entering this theorem. Thus holomorphic bundles became coherent sheaves and then the derived category of coherent sheaves, and we were able to see how to get all of these generalizations out of the physical constructions. On the other hand there was no counterpart in the derived category of the stability condition of DUY. During 1999–2000 I asked many mathematicians about this, and the universal opinion was that it did not and could not exist, because there was no concept of subobject in a derived category. Still, the physics said it had to exist.

The resolution of this contradiction involved a great deal of additional input from string theory, which led to the formulation of Π-stability [14], a definition of stability which made sense for a derived category. With Paul Aspinwall we showed that this formulation passed several nontrivial consistency checks, but at this point the development was becoming too difficult for our physics techniques. Happily we were able to the explain the ideas to mathematicians, as in my ICM lecture [15] and most importantly at the M theory workshop we organized at the Newton Institute in the winter of 2002. There Tom Bridgeland took up the mantle and was able to turn these ideas into rigorous mathematics, now generally referred to as a Bridgeland stability condition [16]. Much of this story is explained in our book [17].

While the rich structure of algebraic geometry allowed us to go very far, so far this has only been for a small subset of the space-times possible in string theory, the Calabi–Yau manifolds. For more general spaces one needs to start with more general foundations, and the most general mathematical context in which one study the Dirichlet brane is K theory, as pointed out by Edward Witten and especially by Greg Moore. This brings me to what is probably the most direct connection between my father's and my own respective bodies of research.

In 1982, working with Paul Baum, my father published "Index Theory, Bordism and K-homology." [2]. The introduction states that it was completed during a visit to the IHES, I believe the family visit we made that summer. One of the stories we still tell about that visit is about Bastille Day, when we went into Paris to watch the fireworks. We had taken public transit (the RER), and as some of you will know, this stops running around 1 in the morning, so we took care to leave a bit early to catch our train. Unfortunately, so many other spectators had similar constraints that the Metro was jam-packed, and we only made it in time for the last train. And the last train did not go all the way to Bures, it stopped in Massy-Palaiseau, several miles away. By the time we realized where the taxi stop was, they were all taken. So we had to make the long hike home, under the moonlight. Still we made the best of it, singing and playing word games, until my sister sprained her ankle, and we had to carry her the rest of the way. The sun was just coming up as we arrived at the Ormaille.

Despite having to watch us, evidently my father found time to do some work, and the resulting paper is (I am told) a classic in K theory. I will not get any farther than the first definition, however, which is that for a cycle in K-homology [2]:

Definition 1 A cycle for $K_0(X)$ is a triple (σ_0, σ_1, T) where σ_0 and σ_1 are $*$-representations of the algebra of complex continuous bounded functions on X, and T is a bounded intertwining operator.

Amazingly enough, in [13] Jeff Harvey and Greg Moore showed that one can derive this definition from Dirichlet branes as well. This uses the physical idea of "tachyon condensation," and since the map T in the definition corresponds physically to a tachyon, Harvey and Moore even used the same notation for it. This is (more or less) the reduction to K theory of the derived category constructions I was working out at the same time.

So I was destined to walk in my father's footsteps after all. Fortunately he had chosen some very fruitful directions to walk in, and I am eternally grateful for that and for all that we shared together.

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Functional Models for Commuting Hilbert-Space Contractions

Joseph A. Ball and Haripada Sau

Dedicated to the memory of Ron Douglas, a leader and dedicated mentor for the field

Abstract We develop a Sz.-Nagy–Foias-type functional model for a commutative contractive operator tuple $\underline{T} = (T_1, \ldots, T_d)$ having $T = T_1 \cdots T_d$ equal to a completely nonunitary contraction. We identify additional invariants $\mathbb{G}_\sharp, \mathbb{W}_\sharp$ in addition to the Sz.-Nagy–Foias characteristic function Θ_T for the product operator T so that the combined triple $(\mathbb{G}_\sharp, \mathbb{W}_\sharp, \Theta_T)$ becomes a complete unitary invariant for the original operator tuple $\underline{T}$. For the case $d \geq 3$ in general there is no commutative isometric lift of $\underline{T}$; however there is a (not necessarily commutative) isometric lift having some additional structure so that, when compressed to the minimal isometric-lift space for the product operator T, generates a special kind of lift of $\underline{T}$, herein called a *pseudo-commutative contractive lift* of $\underline{T}$, which in turn leads to the functional model for $\underline{T}$. This work has many parallels with recently developed model theories for symmetrized-bidisk contractions (commutative operator pairs (S, P) having the symmetrized bidisk Γ as a spectral set) and for tetrablock contractions (commutative operator triples (A, B, P) having the tetrablock domain $\mathbb{E}$ as a spectral set).

The research of the second named author was supported by SERB Indo-US Postdoctoral Research Fellowship, 2017.

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_2

Keywords Commutative contractive operator-tuples · Functional model · Unitary dilation · Isometric lift · Spectral set · Pseudo-commutative contractive lift

Mathematics Subject Classification (2010) Primary: 47A13; Secondary: 47A20, 47A25, 47A56, 47A68, 30H10

1 Introduction

A major development in the theory of nonnormal operator theory was the Sz.-Nagy dilation theorem (*any Hilbert-space contraction operator T can be represented as the compression of a unitary operator to the orthogonal difference of two invariant subspaces*) and the concomitant Sz.-Nagy–Foias functional model for a completely nonunitary contraction operator (we refer to [42] for a complete treatment). Since then there have been many forays into extensions of the formalism to more general settings. Perhaps the earliest was that of Andô [9] who showed that any pair of commuting contractions can be dilated to a pair of commuting unitary operators, but the construction had no functional form like that of the Sz.-Nagy–Foias model for the single-operator case and did not lead to a functional model for a commutative contractive pair. Around the same time the Commutant Lifting Theorem due to Sz.-Nagy et al. [42] appeared, with a seminal special case due to Sarason [49]. It was soon realized that there is a close connection between the Andô Dilation Theorem and Commutant Lifting (see [47, Section 3]). However in the same paper of Parrott it was shown that Andô's result fails for d commuting contractions as soon as $d \geq 3$. Arveson [11] gave a general operator-algebraic/function-algebraic formulation of the general problem which also revealed the key role of the property of complete contractivity as opposed to mere contractivity for representations of operator algebras.

Since the appearance of [21], much work has focused on the d-tuple of coordinate multipliers $M_{z_1}, \dots, M_{z_d}$ on the Hardy space over the polydisk $H^2_{\mathbb{D}^d}$ as well as the coordinate multipliers $M_{\zeta_1}, \dots, M_{\zeta_d}$ on the Lebesgue space over the torus $L^2_{\mathbb{T}^d}$ and variations thereof as models for commuting isometries, and the quest for Wold decompositions related to variations of these two simple examples. While the most definitive results are for the doubly-commuting case (see [40, 50, 52]), there has been additional progress developing models to handle more general classes of commuting isometries [29, 30, 56, 57]. One can then study examples of commutative contractive tuples by studying compressions of such commutative isometric tuples to jointly coinvariant subspaces (see e.g. the book of Douglas and Paulsen [35] for an abstract approach and work of Yang [61]). This work has led to a wealth of distinct new types of examples with special features, including strong rigidity results (see e.g. [36]). In case the commutative contractive tuple itself is doubly commuting, one can get a rather complete functional analogue of the Schäffer construction of the minimal unitary dilation (see [24, 58]).

More recent work of Agler and Young along with collaborators [1, 6, 7], inspired by earlier work of Bercovici et al. [18] having motivation from the notion of structured singular-value in Robust Control Theory (see [18, 38]), explored more general domains on which to explore the Arveson program: a broad overview of this direction is given in Sect. 2.2 below. Followup work by Bhattacharyya and collaborators (including the second author of the present manuscript) [22, 23, 25–28] as well as of Sarkar [51] found analogues of the Sz.-Nagy–Foias defect operator $D_T = (I - T^*T)^{\frac{1}{2}}$ and a more functional form for the dilation and model theory results established for these more general domains (specifically, the symmetrized bidisk Γ and tetrablock domain $\mathbb{E}$ to be discussed below).

The goal of the present paper is to adapt these recent advances in the theory of Γ- and $\mathbb{E}$-function-theoretic operator theory to the original Andô-Parrott setting where the domain is the polydisk $\mathbb{D}^d$ and the associated operator-theoretic object is a operator-tuple $\underline{T} = (T_1, \ldots, T_d)$ of commuting contraction operators on a Hilbert space $\mathcal{H}$. Specifically, we adapt the definition of *Fundamental Operators*, originally introduced in [25] for Γ-contractions and then adapted to $\mathbb{E}$-contractions in [22], to arrive at a definition of *Fundamental Operators* $\{F_{j1}, F_{j2} \colon j = 1, \ldots, d\}$ for a commutative contractive operator tuple $\underline{T} = (T_1, \ldots, T_d)$. We then show that the set of Fundamental Operators can be jointly Halmos-dilated to another geometric object which we call an *Andô tuple* as it appears implicitly as a key piece in Andô's construction of a joint unitary dilation in [9] for the pair case. While the set of Fundamental Operators is uniquely determined by $\underline{T}$, there is some freedom in the choice of Andô tuple associated with $\underline{T}$. With the aid of an Andô tuple, we are then able to construct a (not necessarily commutative) isometric lift for $\underline{T}$ which has the form of a Berger–Coburn–Lebow (BCL) model (as in [21]) for a commutative isometric operator-tuple. While any commutative isometric operator-tuple can be modeled as a BCL-model, there is no tractable characterization as to which BCL-models are commutative, except in the $d = 2$ case. For the $d = 2$ case it can be shown that there is an appropriate choice of the Andô tuple which leads to a commutative BCL-model—thereby giving a more succinct proof of Andô's original result. For the general case where $d \geq 3$, we next show how the noncommutative isometric lift constructed from the Andô tuple can be cut down to the minimal isometric-lift space for the single product operator $T = T_1 \cdots T_d$ to produce an analogue of the single-variable lift for the commutative tuple situation which we call a *pseudo-commutative contractive lift* of $\underline{T}$. For the case where $T = T_1 \cdots T_d$ is completely nonunitary, we model the minimal isometric-lift space for T as the Sz.-Nagy–Foias functional-model space $\begin{bmatrix} \frac{H^2(\mathcal{D}_{T^*})}{\Delta_{\Theta_T} L^2(\mathcal{D}_T)} \end{bmatrix}$ for T based on the Sz.-Nagy–Foias characteristic function Θ_T for T and we arrive at a functional model for the whole commutative tuple $\underline{T} = (T_1, \ldots, T_d)$ consistent with the standard Sz.-Nagy–Foias model for the product operator $T = T_1 \cdots T_d$. Let us mention that the basic ingredients of this model already appear in the work of Das et al. [31] for the pure-pair case ($d = 2$ and $T = T_1 T_2$ has the property that $T^{*n} \to 0$ strongly as $n \to \infty$). This leads to the identification of additional unitary invariants (in addition to the characteristic function Θ_T) so that the whole collection $\{\mathbb{G}, \mathbb{W}, \Theta_T\}$

(which we call a *characteristic triple* for the commutative contractive tuple $\underline{T}$) is a complete unitary invariant for $\underline{T}$ for the case where $T = T_1 \cdots T_d$ is completely nonunitary. Here $\mathbb{G} = \{G_{j1}, G_{j2} \colon j = 1, \ldots, d\}$ consists of the Fundamental Operators for the adjoint tuple $\underline{T}^* = (T_1^*, \ldots, T_d^*)$ and $\mathbb{W} = \{W_{\sharp 1}, \ldots, W_{\sharp d}\}$ consists of a canonically constructed commutative unitary tuple of multiplication operators on the Sz.-Nagy–Foias defect model space $\overline{\Delta_{\Theta_T} \cdot L^2(\mathcal{D}_{\Theta_T})}$ with product equal to multiplication by the coordinate M_ζ on $\overline{\Delta_{\Theta_T} \cdot L^2(\mathcal{D}_{\Theta_T})}$, all of which is vacuous for the case where $T = T_1 \cdots T_d$ is pure. Let us also mention that we obtain an analogue of the Sz.-Nagy–Foias canonical decomposition for a contraction operator, i.e.: any commutative contractive operator tuple $\underline{T}$ splits as an orthogonal direct sum $\underline{T} = \underline{T}_u \oplus \underline{T}_c$ where $\underline{T}_u$ is a commutative unitary operator-tuple and $\underline{T}_c$ is a commutative contractive operator tuple with $T = T_1 \cdots T_d$ completely nonunitary. As the unitary classification problem for commutative unitary tuples can be handled by the spectral theory for commuting normal operators (see [10, 34]), the results for the case where $T = T_1 \cdots T_d$ is completely nonunitary combined with the spectral theory for the commutative unitary case leads to a model theory and unitary classification theory for the general class of commutative contractive operator-tuples.

Let us mention that Bercovici et al. [14–16] have also recently obtained a wealth of structural information concerning commutative contractive tuples. This work also builds off the BCL-model for the commutative isometric case, but also derives additional insight concerning the BCL-model itself. There also appears the notion of *characteristic function* for a commutative contractive operato-tuple, but this is quite different from our notion of characteristic function (simply the Sz.-Nagy–Foias characteristic function of the single operator equal to the product $T = T_1 \cdots T_d$).

The paper is organized as follows. After the present Introduction, Sect. 2 on preliminaries provides (1) a reference for some standard notations to be used throughout, (2) a review of the rational dilation problem, especially in the context of the specific domains Γ (symmetrized bidisk) and $\mathbb{E}$ (tetrablock domain), including some discussion on how these domains arise from specific examples of the structured singular value arising in Robust Control theory, (3) some background on Fundamental Operators in the setting of the symmetrized bidisk, along with some additional information (4) concerning Berger–Coburn–Lebow models for commutative isometric-tuples [21] and (5) concerning the Douglas approach [32] to the Sz.-Nagy–Foias model theory which will be needed in the sequel. Let us also mention that the present manuscript is closely related to our companion paper [13] where the results of the present paper are developed directly for the pair case ($\underline{T} = (T_1, T_2)$ is a commutative contractive operator-pair) from a more general point of view where additional details are developed. Finally this manuscript and [13] subsume the preliminary report [54] posted on arXiv.

Acknowledgements Finally let us mention that this paper is dedicated to the memory of Ron Douglas, a role model and inspiring mentor for us. Indeed it is his approach to the Sz.-Nagy–Foias model theory in [32] which was a key intermediate

step in our development of the multivariable version appearing here. In addition his recent work with Bercovici and Foias [14–16] has informed our work as well.

2 Preliminaries

2.1 Notation

We here provide a reference for a core of common notation to be used throughout the paper.

Given an operator A on a Hilbert space $\mathcal{X}$, we write

- $\nu(A)$ = *numerical radius* of A= $\sup\{|\langle Ax, x\rangle_{\mathcal{X}}|: x \in \mathcal{X}$ with $\|x\| = 1\}$.
- $\rho_{\rm spec}(A)$ = *spectral radius* of A = $\sup\{|\lambda|: \lambda \in \mathbb{C}$ and $\lambda I - A$ not invertible$\}$.
- If $T \in \mathcal{L}(\mathcal{X})$ with $\|T\| \le 1$, then D_T denotes the *defect operator* of T defined as $D_T = (I - T^*T)^{\frac{1}{2}}$ and $\mathcal{D}_T = \overline{\text{Ran}}\ D_T$.
- Given the set of d indices $\{j: 1 \le j \le d\}$, (j) denotes the tuple of $d-1$ indices $(1, \ldots, j-1, j+1, \ldots, d)$.
- For a d-tuple $(T_1, T_2, \ldots, T_d)$ of operators and an index j such that $1 \le j \le d$, $T_{(j)}$ denotes the operator $T_1 \cdots T_{j-1} T_{j+1} \cdots T_d$.

2.2 Domains with Motivation from Control: The Symmetrized Bidisk $\mathbb{G}$ and the Tetrablock $\mathbb{E}$

The symmetrized bidisk $\mathbb{G}$ is the domain in $\mathbb{C}^2$ defined as

$$\mathbb{G} = \{(s, p) \in \mathbb{C}^2: \exists\, (\lambda_1, \lambda_2) \in \mathbb{D}^2 \text{ such that } s = \lambda_1 + \lambda_2 \text{ and } p = \lambda_1\lambda_2\}. \tag{2.1}$$

The study of this domain from a function-theoretic and operator-theoretic point of view was initiated in a series of papers by Agler and Young starting in the late 1990s (see [3–8]) with original motivation from Robust Control Theory (see [38] and the papers of Bercovici et al. [17–20]). The control motivation can be explained as follows.

A key role is played by the notion of structured singular value introduced in the control literature by Packer and Doyle [45]. The *structured singular value* $\mu_{\mathbf{\Delta}}(A)$ of a $N \times N$ matrix over $\mathbb{C}$ with respect to an *uncertainty set* $\mathbf{\Delta}$ (to be thought of as the admissible range for an additional unknown variable Δ which is used to parametrize the set of possible true plants around the chosen nominal (oversimplified) model plant) is defined to be

$$\mu_{\mathbf{\Delta}}(A) = [\sup\{r \in \mathbb{R}_+: I - \Delta A \text{ invertible for } \Delta \in \mathbf{\Delta} \text{ with } \|\Delta\| \le r\}]^{-1}$$

After appropriate normalizations, it suffices to test whether $\mu_{\mathbf{\Delta}}(A) < 1$;

$$\mu_{\mathbf{\Delta}}(A) < 1 \Leftrightarrow I - A\Delta \text{ invertible for all } \Delta \in \mathbf{\Delta} \text{ with } \|\Delta\| \leq 1.$$

In the control theory context, this appears as the test for internal stability not only for the nominal plant but for all other possible true plants as modeled by the uncertainty set $\mathbf{\Delta}$. In practice the uncertainty set is taken to be the set of all matrices having a prescribed block diagonal structure.

For the case of 2×2 matrices, there are three possible block-diagonal structures:

$$\mathbf{\Delta}_{\text{full}} = \left\{ \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} : z_{ij} \in \mathbb{C} \right\} = \text{ all } 2 \times 2 \text{ matrices.}$$

$$\mathbf{\Delta}_{\text{scalar}} = \left\{ \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} : z \in \mathbb{C} \right\} = \text{ all scalar } 2 \times 2 \text{ matrices.}$$

$$\mathbf{\Delta}_{\text{diag}} = \left\{ \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} : z_1, z_2 \in \mathbb{C} \right\} = \text{ all diagonal matrices.}$$

An easy exercise using the theory of singular-value decompositions is to show that

$$\mu_{\mathbf{\Delta}_{\text{full}}}(A) = \|A\|.$$

To compute $\mu_{\mathbf{\Delta}_{\text{scalar}}}(A)$, one can proceed as follows. Given $A = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$, from the definitions we see that

$$\begin{aligned} \mu_{\mathbf{\Delta}_{\text{scalar}}}(A) < 1 &\Leftrightarrow \det \left(\begin{bmatrix} 1 - za_{11} & -za_{12} \\ -za_{21} & 1 - za_{22} \end{bmatrix} \right) \neq 0 \text{ for all } z \text{ with } |z| \leq 1 \\ &\Leftrightarrow 1 - (\operatorname{tr} A)z + (\det A)z^2 \neq 0 \text{ for all } z \text{ with } |z| \leq 1. \end{aligned} \tag{2.2}$$

Thus the decision as to whether $\mu_{\mathbf{\Delta}_{\text{scalar}}}(A) < 1$ depends only on $\operatorname{tr} A$ and $\det A$, i.e., on $\operatorname{tr} A = \lambda_1 + \lambda_2$ and $\det A = \lambda_1 \lambda_2$ where λ_1, λ_2 are the eigenvalues of A. This suggests that we define a map $\pi_{\mathbb{G}} : \mathbb{C}^{2\times 2} \to \mathbb{C}^2$ by

$$\pi_{\mathbb{G}}(A) = (\operatorname{tr} A, \det A)$$

and introduce the domain

$$\begin{aligned} \mathbb{G}' = \{ & x = (x_1, x_2) \in \mathbb{C}^2 : \exists\, A = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right] \in \mathbb{C}^{2\times 2} \\ & \text{with } \pi_{\mathbb{G}}(A) = x \text{ and } \mu_{\mathbf{\Delta}_{\text{scalar}}}(A) < 1 \}. \end{aligned} \tag{2.3}$$

Note next that the first form of the criterion (2.2) for $\mu_{\mathbf{\Delta}_{\text{scalar}}}(A) < 1$ can also be interpreted as saying that A has no inverse-eigenvalues inside the closed unit disk, i.e., all eigenvalues of A are in the open unit disk, meaning that $\rho_{\text{spec}}(A) < 1$. In this way we see that the symmetrized bidisk $\mathbb{G}$ (2.1) is exactly the same as the domain $\mathbb{G}'$ given by (2.3). This equivalence gives the connection between the symmetrized bidisk and the structured singular value $A \mapsto \mu_{\boldsymbol{\Delta}_{\text{scalar}}}(A)$.

Noting that similarity transformations

$$A \mapsto A' = SAS^{-1} \text{ for some invertible } S$$

preserve eigenvalues and using the fact that $\rho_{\text{spec}}(A) < 1$ if and only if A is similar to a strict contraction (known as Rota's Theorem [48] among mathematicians whereas engineers think in terms of $X = S^*S \succ 0$ being a solution of the Linear Matrix Inequality $A^*XA - X \prec 0$—see e.g. [38, Theorem 11.1 (i)]), we see that yet another characterization of the domain $\mathbb{G}$ is

$$\mathbb{G} = \{x = (s, p) \in \mathbb{C}^2 \colon \exists\, A \in \mathbb{C}^{2\times 2} \text{ with } \pi_{\mathbb{G}}(A) = x \text{ and } \|A\| < 1\}. \tag{2.4}$$

The fact that one can always write down a companion matrix A whose characteristic polynomial $\det(zI - A)$ is equal to a given polynomial $1 - sz + ps^2$ leads us to one more equivalent definition of $\mathbb{G}$:

$$\mathbb{G} = \{(s, p) \in \mathbb{C}^2 \colon 1 - sz + pz^2 \neq 0 \text{ for } |z| \leq 1\}. \tag{2.5}$$

The closure of $\mathbb{G}$ is denoted by Γ.

A similar story holds for the tetrablock domain $\mathbb{E}$ defined as

$$\mathbb{E} = \{x = (x_1, x_2, x_3) \in \mathbb{C}^3 \colon 1 - x_1 z - x_2 w + x_3 zw \neq 0 \text{ whenever } |z| \leq 1, |w| \leq 1\} \tag{2.6}$$

(the analogue of definition (2.5) for the symmetrized bidisk $\mathbb{G}$) and its connection with the structured singular value $A \mapsto \mu_{\boldsymbol{\Delta}_{\text{diag}}}(A)$. From the definitions we see that, for $A = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$,

$$\begin{aligned} \mu_{\boldsymbol{\Delta}_{\text{diag}}}(A) < 1 &\Leftrightarrow \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} z & 0 \\ 0 & w \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) \neq 0 \text{ for } |z| \leq 1,\ |w| \leq 1 \\ &\Leftrightarrow 1 - za_{11} - wa_{22} + zw \cdot \det A \neq 0 \text{ whenever } |z| \leq 1,\ |w| \leq 1. \end{aligned}$$

This suggests that we define a mapping $\pi_{\mathbb{E}} \colon \mathbb{C}^{2\times 2} \to \mathbb{C}^3$ by

$$\pi_{\mathbb{E}}\left(\left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]\right) = (a_{11}, a_{22}, a_{11}a_{22} - a_{12}a_{21})$$

and we define a domain $\mathbb{E}$ by

$$\mathbb{E} = \{x = (x_1, x_2, x_3) \in \mathbb{C}^3 \colon \exists\, A \in \mathbb{C}^{2\times 2} \text{ with } \pi_{\mathbb{E}}(A) = x \text{ and } \mu_{\boldsymbol{\Delta}_{\text{diag}}}(A) < 1\}. \tag{2.7}$$

If $x = (x_1, x_2, x_3)$ belongs to $\mathbb{E}$ as defined in (2.6) above, we can always take $A = \begin{bmatrix} x_1 & x_1x_2 - x_3 \\ 1 & x_2 \end{bmatrix}$ to produce a matrix A with $\pi_{\mathbb{E}}(A) = (x_1, x_2, x_3)$ and then this A has the property that $\mu_{\mathbf{\Delta}_{\rm diag}}(A) < 1$. Thus definitions (2.6) and (2.7) are equivalent. Among the many equivalent definitions of $\mathbb{E}$ (see [1, Theorem 2.2]), one of the more remarkable ones is the following variation of definition (2.7):

$$\mathbb{E} = \{x = (x_1, x_2, x_3) \in \mathbb{C}^3 \colon \exists\, A \in \mathbb{C}^{2\times 2} \text{ with } \pi_{\mathbb{E}}(A) = x \text{ and } \|A\| < 1\}. \tag{2.8}$$

That (2.7) and (2.8) are equivalent can be seen as a consequence of the $2s + f$ theorem in the control literature (with $s = 0$, $f = 2$ so that $2s + f = 2 \le 3$)—see [38, Theorem 8.27], but is also proved in [1] directly.

While the original motivation was the control theory connections, most of the ensuing research concerning the domains $\mathbb{G}$ and $\mathbb{E}$ focused on their role as new concrete domains to explore operator- and function-theoretic questions concerning general domains in $\mathbb{C}^d$. One such question is the *rational dilation problem* formulated by Arveson [11]. Let us assume that K is a compact set in $\mathbb{C}^d$ (as is the case for K equal to $\Gamma = \overline{\mathbb{G}}$ or $\overline{\mathbb{E}}$). Suppose that we are given a commutative tuple $\underline{T} = (T_1, \dots, T_d)$ of Hilbert space operators with Taylor joint spectrum contained in K (if the Hilbert space $\mathcal{H}$ is finite-dimensional, one can take Taylor joint spectrum to mean the set of joint eigenvalues). If r is any function holomorphic in a neighborhood of K (if K is polynomially convex, one can take r to be polynomial) any reasonable functional calculus can be used to define $r(\underline{T})$. We say that $\underline{T}$ *is a K-contraction* (sometimes also phrased as K *is a spectral set for* $\underline{T}$), if for all $r \in \operatorname{Rat}(K)$ (rational functions holomorphic in a neighborhood of K) it is the case that the following *von Neumann inequality* holds:

$$\|r(\underline{T})\|_{\mathcal{B}(\mathcal{H})} \le \|r\|_{\infty, K} = \sup_{z \in K}\{|r(z)|\}$$

where $\mathcal{B}(\mathcal{H})$ is the Banach algebra of bounded linear operators on $\mathcal{H}$ with the operator norm. Let us say that operator tuple $\underline{U} = (U_1, \dots, U_d)$ is K*-unitary* if $\underline{U}$ is a commutative tuple of normal operators with joint spectrum contained in the distinguished boundary $\partial_e K$ of K. We say that $\underline{T}$ has a *K-unitary dilation* if there is a K-unitary operator-tuple $\underline{U}$ on a larger Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that $r(\underline{T}) = P_{\mathcal{H}} r(\underline{U})|_{\mathcal{H}}$ for all $r \in \operatorname{Rat} K$. If $\underline{T}$ has a K-unitary dilation $\underline{U}$, it follows that

$$\|r(\underline{T})\| = \|P_{\mathcal{H}} r(\underline{U})|_{\mathcal{H}}\| \le \|r(\underline{U})\| = \sup_{z \in \partial_e K} |r(z)|$$

(by the functional calculus for commutative normal operators)

$$= \sup_{z \in K} |r(z)| \text{ (by the definition of the distinguished boundary)}$$

and it follows that $\underline{T}$ has K as a spectral set. The *rational dilation question* asks: for a given compact set K, when is it the case that the converse direction holds, i.e., that $\underline{T}$ being a K-contraction implies that $\underline{T}$ has a K-unitary dilation $\underline{U}$? For the case of K equal to the closed polydisk $\overline{\mathbb{D}}^d$, the rational dilation question is known to have an affirmative answer in case $d = 1$ (by the Sz.-Nagy dilation theorem [43]) as well as $d = 2$ (by the Andô dilation theorem [9]) but has a negative answer for $d \geq 3$ by the result of Parrott [47]. For the case of $K = \Gamma$ it is known that the rational dilation question has an affirmative answer ([7, 25]) while the case of $K = \overline{\mathbb{E}}$ was initially thought to be settled in the negative [46] but now appears to be still undecided [12].

It is known that existence of a K-unitary dilation for $\underline{T}$ is equivalent to the existence of a *K-isometric lift* for $\underline{T}$. Here a commutative operator-tuple $\underline{V} = (V_1, \ldots, V_d)$ defined on a Hilbert space $\mathcal{K}_+$ is said to be a *K-isometry* if there is a K-unitary d-tuple $\underline{U} = (U_1, \ldots, U_d)$ on a Hilbert space $\mathcal{K}$ containing $\mathcal{K}_+$ such that $\mathcal{K}_+$ is invariant for $\underline{U}$ and $\underline{U}$ restricted to $\mathcal{K}_+$ is equal to $\underline{V}$, i.e.,

$$U_j \mathcal{K}_+ \subset \mathcal{K}_+ \text{ and } U_j|_{\mathcal{K}_+} = V_j \text{ for } j = 1, \ldots, d.$$

Then we say that the K-contraction $\underline{T}$ has a K-isometric lift if there is a K-isometric operator-tuple $\underline{V}$ on a Hilbert space $\mathcal{K}_+$ containing $\mathcal{H}$ such that $\underline{V}$ is a lift of $\underline{T}$, i.e., for each $j = 1, \ldots, d$,

$$V_j^* \mathcal{H} \subset \mathcal{H} \text{ and } V_j^*|_{\mathcal{H}} = T_j^*.$$

It is known that a K-contraction $\underline{T}$ has a K-unitary dilation if and only if $\underline{T}$ has a K-isometric lift. In practice K-isometric lifts are easier to work with, so in the sequel we shall only deal with K-isometric lifts. This point has been made in a number of places (see e.g. the introduction in [12]).

We define a couple of terminologies here. To add flexibility to the construction of such lifts, we often drop the requirement that $\mathcal{H}$ be a subspace of $\mathcal{K}_+$ but instead require only an isometric identification map $\Pi \colon \mathcal{H} \to \mathcal{K}_+$. We summarize the precise language which we shall be using in the following definition.

Definition 2.1 We say that $(\Pi, \mathcal{K}_+, \underline{S} = (S_1, \ldots, S_d))$ is a *lift* of $\underline{T} = (T_1, \ldots, T_d)$ on $\mathcal{H}$ if

- $\Pi \colon \mathcal{H} \to \mathcal{K}_+$ is isometric, and
- $S_j^* \Pi h = \Pi T_j^* h$ for all $h \in \mathcal{H}$ and $j = 0, 1, 2, \ldots, d$.

A lift $(\Pi, \mathcal{K}_+, \underline{S})$ of $\underline{T}$ is said to be *minimal* if

$$\mathcal{K}_+ = \overline{\text{span}}\{S_1^{m_1} S_2^{m_2} \cdots S_d^{m_d} h \colon h \in \mathcal{H}, m_j \geq 0\}.$$

Two lifts $(\Pi, \mathcal{K}_+, \underline{S})$ and $(\Pi,' \mathcal{K}'_+, \underline{S}')$ of the same $(T_1, \ldots, T_d)$ are said to be *unitarily equivalent* if there is a unitary operator $\tau : \mathcal{K}_+ \to \mathcal{K}'_+$ so that

$$\tau S_j = S'_j \tau \text{ for each } j = 1, \ldots, d, \quad \text{and} \quad \tau \Pi = \Pi'.$$

It is known (see Chapter I of [42]) that when $K = \overline{\mathbb{D}}$, any two minimal isometric lifts of a given contraction are unitarily equivalent. However, minimality in several variables does not imply uniqueness, in general. For example, two minimal $\overline{\mathbb{D}}^2$-isometric lifts need not be unique (see [41]).

Instead of $\overline{\mathbb{E}}$-contraction, the terminology *tetrablock contraction* was used in [22]. We follow this terminology.

2.3 Fundamental Operators

For our study of commutative contractive tuples $\underline{T} = (T_1, \dots, T_d)$, we shall have use for the following theorem concerning Γ-contractions. We refer back to Sect. 2.1 for other notational conventions.

Theorem 2.2 *Let (S, T) be a Γ-contraction on a Hilbert space $\mathcal{H}$. Then*

1. (See [25, Theorem 4.2].) *There exists a unique operator $F \in \mathcal{B}(\mathcal{D}_T)$ with $\nu(F) \leq 1$ such that*

$$S - S^*T = D_T F D_T .$$

2. (See [22, Lemma 4.1].) *The operator F in part (1) above is the unique solution $X = F$ of the operator equation*

$$D_T S = X D_T + X^* D_T T.$$

This theorem has been a major influence on further developments in the theory of both Γ-contractions [26, 27] and tetrablock contractions [22, 28, 53]. The unique operator F in Theorem 2.2 is called the *fundamental operator* of the Γ-contraction (S, T).

2.4 Models for Commutative Isometric Tuples

The following result of Berger, Coburn and Lebow for commutative-tuples of isometries is a fundamental stepping stone for our study of commutative-tuples of contractions.

Theorem 2.3 *Let $d \geq 2$ and $(V_1, V_2, \dots, V_d)$ be a d-tuple of commutative isometries acting on $\mathcal{K}$. Then there exit Hilbert spaces $\mathcal{F}$ and $\mathcal{K}_u$, unitary operators $U_1, \dots, U_d$ and projection operators $P_1, \dots, P_d$ on $\mathcal{F}$, commutative unitary operators $W_1, \dots, W_d$ on $\mathcal{K}_u$, such that $\mathcal{K}$ can be decomposed as*

$$\mathcal{K} = H^2(\mathcal{F}) \oplus \mathcal{K}_u \tag{2.9}$$

and with respect to this decomposition

$$V_j = M_{U_j P_j^\perp + zU_j P_j} \oplus W_j,\ \ V_{(j)} = M_{P_j U_j^* + zP_j^\perp U_j^*} \oplus W_{(j)} \textit{ for } 1 \le j \le d, \tag{2.10}$$

$$\textit{and } V = V_1 V_2 \cdots V_d = M_z \oplus W_1 W_2 \cdots W_d.$$

Proof See Theorem 3.1 in [21] as well as [14, Section 2] for a different perspective. □

Definition 2.4 Given two Hilbert spaces $\mathcal{F}$, $\mathcal{E}$, d projections $P_1, P_2, \ldots, P_d$ in $\mathcal{B}(\mathcal{F})$, d unitaries $U_1, U_2, \ldots, U_d$ in $\mathcal{B}(\mathcal{F})$, and d commuting unitaries $W_1, W_2, \ldots, W_d$ in $\mathcal{B}(\mathcal{E})$, the tuple

$$(\mathcal{F}, \mathcal{E}, P_j, U_j, W_j)_{j=1}^d$$

will be referred to as a *BCL tuple*. We shall call the tuple of isometries acting on $H^2(\mathcal{F}) \oplus \mathcal{E}$ given by

$$(M_{U_1 P_1^\perp + zU_1 P_1} \oplus W_1, M_{U_2 P_2^\perp + zU_2 P_2} \oplus W_2, \ldots, M_{U_d P_d^\perp + zU_d P_d} \oplus W_d) \tag{2.11}$$

the *BCL model* associated with the BCL tuple $(\mathcal{F}, \mathcal{E}, P_j, U_j, W_j)_{j=1}^d$.

Remark 2.5 If P and U are a projection and a unitary acting on a Hilbert space $\mathcal{F}$, then one can check that the multiplication operator $M_{U(P^\perp + zP)}$ acting on $H^2(\mathcal{F})$ is an isometry. It should however be noted that given d projections $P_1, P_2, \ldots, P_d$ and unitaries $U_1, U_2, \ldots, U_d$ on $\mathcal{F}$, the tuple of isometries

$$(M_{U_1 P_1^\perp + zU_1 P_1}, M_{U_2 P_2^\perp + zU_2 P_2}, \ldots, M_{U_d P_d^\perp + zU_d P_d})$$

need not be commutative, in general. Necessary conditions for such a tuple of isometries to be commuting are given in Theorem 3.2 of [21]:

$$\begin{aligned} &U_1 U_2 \cdots U_d = I_{\mathcal{F}}, \quad U_i U_j = U_j U_i \text{ for } 1 \le i, j \le d, \\ &P_{j_1} + U_{j_1}^* P_{j_2} U_{j_1} + U_{j_2}^* U_{j_1}^* P_{j_3} U_{j_1} U_{j_2} + \cdots + U_{j_{d-1}}^* \cdots U_{j_1}^* P_{j_n} U_{j_1} \cdots U_{j_{d-1}} = I_{\mathcal{F}} \\ &\quad \text{for } (j_1, j_2, \ldots, j_d) \in S_d \end{aligned} \tag{2.12}$$

where S_d is the permutation group on d indices $\{1, 2, \ldots, d\}$. When $d = 2$, these necessary conditions (2.12) simplify to

$$U_2 = U_1^* \text{ and } P_2 = I_{\mathcal{F}} - P_1. \tag{2.13}$$

which turn out to be sufficient as well for the $d = 2$ case.

It is well known that an arbitrary family of commutative isometries has a commutative unitary extension (see [42, Proposition I.6.2]). The Berger–Coburn–Lebow model for commutative isometries gives some additional information regarding such extensions.

Lemma 2.6 *Let $\underline{V} = (V_1, V_2, \ldots, V_d)$ be a d-tuple of commutative isometries on a Hilbert space $\mathcal{H}$ and $V = V_1 V_2 \cdots V_d$. Then $\underline{V}$ has a commutative unitary extension $\underline{Y} = (Y_1, Y_2, \ldots, Y_d)$ such that $Y = Y_1 Y_2 \cdots Y_d$ is the minimal unitary extension of V.*

Proof See [21, Theorem 3.6]. □

2.5 Canonical Commutative Unitary Tuple Associated with a Commutative Tuple of Contractions

Let $(T_1, T_2, \cdots, T_d)$ be a d-tuple of commutative contractions on a Hilbert space $\mathcal{H}$ and $T = T_1 T_2 \cdots T_d$. Since T is a contraction, the sequence $T^n T^{*n}$ converges in the strong operator topology. Let Q be the positive semidefinite square root of the limit operator, so

$$Q^2 := \text{SOT-}\lim T^n T^{*n}. \tag{2.14}$$

Then the operator $X^* : \overline{\text{Ran}}\ Q \to \overline{\text{Ran}}\ Q$ defined densely by

$$X^* Q = Q T^*, \tag{2.15}$$

is an isometry because for all $h \in \mathcal{H}$,

$$\langle Q^2 h, h\rangle = \lim_{n\to\infty} \langle T^n T^{*n} T^* h, T^* h\rangle = \langle Q T^* h, Q T^* h\rangle. \tag{2.16}$$

Let W_D^* on $\mathcal{R}_D \supseteq \overline{\text{Ran}}\ Q$ be the minimal unitary extension of X^*. Define the operator $\widehat{\mathcal{O}}_{D_{T^*},T^*} : \mathcal{H} \to H^2(\mathcal{D}_{T^*})$ as

$$\widehat{\mathcal{O}}_{D_{T^*},T^*}(z)h = \sum_{n=0}^{\infty} z^n D_{T^*} T^{*n} h, \text{ for every } h \in \mathcal{H}. \tag{2.17}$$

Then the operator $\Pi_D : \mathcal{H} \to H^2(\mathcal{D}_{T^*}) \oplus \mathcal{R}_D$ defined by

$$\Pi_D(h) = \widehat{\mathcal{O}}_{D_{T^*},T^*}(z)h \oplus Q(h) = \sum_{n=0}^{\infty} z^n D_{T^*} T^{*n} h \oplus Qh \tag{2.18}$$

is an isometry and satisfies the intertwining property

$$\Pi_D T^* = (M_z \oplus W_D)^* \Pi_D \tag{2.19}$$

(see e.g. [32, Section 4]. We conclude that with the isometry V_D defined on $\mathcal{K}_D := H^2(\mathcal{D}_{T^*}) \oplus \mathcal{R}_D$ as

$$V_D := M_z \oplus W_D, \tag{2.20}$$

$(\Pi_D, \mathcal{K}_D, V_D)$ is an isometric lift of T. One can furthermore show that this lift is minimal as well (see [32, Lemma 1]).

If we now recall that $T = T_1, T_2, \dots, T_d$, we see that for all $h \in \mathcal{H}$ and $i = 1, 2, \dots, d$,

$$\langle T_i Q^2 T_i^* h, h\rangle = \lim\langle T^n (T_i T_i^*) T^{*n} h, h\rangle \leq \lim\langle T^n T^{*n} h, h\rangle = \langle Q^2 h, h\rangle.$$

By the Douglas Lemma [33], this implies that there exists a contraction X_i^* such that

$$X_i^* Q = Q T_i^*. \tag{2.21}$$

Since $\underline{T} = (T_1, T_2, \dots, T_d)$ is commutative, it is evident that $(X_1, X_2, \dots, X_d)$ is a commutative tuple of contractions and that

$$X_1^* \cdots X_d^* = X^*,$$

where X^* is as in (2.15). Since X^* is an isometry, so also is each X_i. By Lemma 2.6 we have a commutative unitary extension $(W_{\partial 1}^*, W_{\partial 2}^*, \dots, W_{\partial d}^*)$ of $(X_1^*, X_2^*, \dots, X_d^*)$ on the same space $\mathcal{R}_D \supseteq \overline{\operatorname{Ran}}\, Q$, where the minimal unitary extension W_D^* of X^* acts and

$$W_D = W_{\partial 1} W_{\partial 2} \cdots W_{\partial d}. \tag{2.22}$$

Note that this means

$$\mathcal{R}_D = \overline{\operatorname{span}}\{W_D^n x : x \in \overline{\operatorname{Ran}}\, Q \text{ and } n \geq 0\}. \tag{2.23}$$

The tuple

$$\underline{W}_\partial := (W_{\partial 1}, W_{\partial 2}, \dots, W_{\partial d}) \tag{2.24}$$

will be referred to as the *canonical commutative unitary tuple* associated with $(T_1, T_2, \dots, T_d)$. We next show that the canonical tuple of commutative unitary operators is uniquely determined by the tuple $(T_1, T_2, \dots, T_d)$.

Lemma 2.7 *Let $\underline{T} = (T_1, T_2, \ldots, T_d)$ on $\mathcal{H}$ and $\underline{T}' = (T_1', T_2', \ldots, T_d')$ on $\mathcal{H}'$ be tuples of commutative contractions. Let $\underline{W}_\partial = (W_{\partial 1}, W_{\partial 2}, \ldots, W_{\partial d})$ on $\mathcal{R}_D$ and $\underline{W}_\partial' = (W_{\partial 1}', W_{\partial 2}', \ldots, W_{\partial d}')$ on $\mathcal{R}_D'$ be the $\underline{W}_\partial' = (W_{\partial 1}', W_{\partial 2}', \ldots, W_{\partial d}')$ on $\mathcal{R}_D'$ be the respective commutative tuples of unitaries obtained from $\underline{T}$ and $\underline{T}'$ as above, respectively. If $\underline{T}$ is unitarily equivalent to $\underline{T}'$ via the unitary similarity $\phi\colon \mathcal{H} \to \mathcal{H}'$, then so are $\underline{W}_\partial$ and $\underline{W}_\partial'$ via the induced unitary transformation $\tau_\phi\colon \mathcal{R}_D \to \mathcal{R}_D'$ determined by $\tau_\phi\colon W_D^n Qh \to W_D'^n Q'\phi h$. In particular, if $\underline{T} = \underline{T}'$, then $\underline{W}_\partial = \underline{W}_\partial'$.*

Proof That the tuples $\underline{W}_\partial$ and $\underline{W}_\partial'$ are obtained from $\underline{T}$ and $\underline{T}'$ respectively means that

$$\begin{aligned} &W_{\partial j}^* Q = Q T_j^*, \;\; W_{\partial j}'^* Q' = Q' T_j'^* \text{ for each } j = 1, 2, \ldots, d, \\ &W_D = \prod_{j=1}^{\infty} W_{\partial j}, \quad W_D' = \prod_{j=1}^{\infty} W_{\partial j}', \end{aligned} \tag{2.25}$$

where $Q^2 = \text{SOT-lim}_{n\to\infty} T^n T^{*n}$ and $Q'^2 = \text{SOT-lim}_{n\to\infty} T'^n T'^{*n}$ with $T = T_1 T_2 \cdots T_d$ and $T' = T_1' T_2' \cdots T_d'$. We shall show that set of Eqs. (2.25) is all that is needed to prove the lemma.

So suppose that the tuples $\underline{T}$ and $\underline{T}'$ are unitarily equivalent via the unitary similarity $\phi\colon \mathcal{H} \to \mathcal{H}'$. By definitions of Q and Q', it is easy to see that ϕ intertwines Q and Q' also and hence ϕ takes $\overline{\text{Ran}}\, Q$ onto $\overline{\text{Ran}}\, Q'$. By (2.25) it follows that ϕ intertwines $W_{\partial j}^*|_{\overline{\text{Ran}}\, Q}$ and $W_{\partial j}'^*|_{\overline{\text{Ran}}\, Q'}$ for each $j = 1, 2, \ldots, d$. Now remembering the formula (2.23) for the spaces $\mathcal{R}_D$ and $\mathcal{R}_D'$, we define $\tau_\phi : \mathcal{R}_D \to \mathcal{R}_D'$ by

$$\tau_\phi : W_D^n x \mapsto W_D'^n \phi x, \;\text{ for every } x \in \overline{\text{Ran}}\, Q \text{ and } n \geq 0$$

and extend linearly and continuously. It is evident that τ_ϕ is unitary and intertwines W_D and W_D'. For a non-negative integer n, $j = 1, 2, \ldots, d$ and x in $\overline{\text{Ran}}\, Q$, we compute

$$\begin{aligned} \tau_\phi W_{\partial j}(W_D^n x) &= \tau_\phi W_D^{n+1} \prod_{j\neq i=1}^{d} W_{\partial i}^* x = W_D'^{n+1} \phi \left(\prod_{j\neq i=1}^{d} W_{\partial i}^* x \right) \\ &= W_D'^{n+1} \prod_{j\neq i=1}^{d} W_{\partial i}'^* \phi x = W_{\partial 1}' W_D'^n \phi x = W_{\partial j}' \tau_\phi (W_D^n x). \end{aligned}$$

and the lemma follows. □

3 Fundamental Operators for a Tuple of Commutative Contractions

The following result reduces the study of commutative contractive d-tuples to the study of a family of Γ-contractions. This enables us to apply the substantial body of existing results concerning Γ-contractions to the study of commutative contractive operator-tuples.

Proposition 3.1 *Let $d \geq 2$ and $\underline{T} = (T_1, T_2, \ldots, T_d)$ be a d-tuple of commutative contractions on a Hilbert space $\mathcal{H}$ and let $T = T_1 \cdots T_d$. Then for each $j = 1, 2, \ldots, d$ and $w \in \overline{\mathbb{D}}$, the pair*

$$(S_j(w), T(w)) := (T_j + wT_{(j)}, wT) \tag{3.1}$$

is a Γ-contraction, where $T_{(j)} := T_1 \ldots T_{j-1}T_{j+1} \ldots T_d$.

Proof Note that for each $j = 1, 2, \ldots, d$ and $w \in \overline{\mathbb{D}}$, the pair $(S_j(w), T(w))$ is actually the symmetrization of two commutative contractions, viz., T_j and $wT_{(j)}$. Since every such pair is a Γ-contraction, the result follows. □

Proposition 3.1 allows us to apply the Γ-contraction theory to obtain fundamental operators associated with a d-tuple of commutative contractions. This is the main result of this section.

Theorem 3.2 *Let $d \geq 2$ and $\underline{T} = (T_1, T_2, \ldots, T_d)$ be a d-tuple of commutative contractions on a Hilbert space $\mathcal{H}$ and let $T = T_1T_2 \cdots T_d$. Then*

1. *For each $i = 1, 2, \ldots, d$, there exist unique bounded operators $F_{i1}, F_{i2} \in \mathcal{B}(\mathcal{D}_T)$ with $\nu(F_{i1} + wF_{i2}) \leq 1$ for all $w \in \overline{\mathbb{D}}$ such that*

$$\begin{aligned} T_i - T_{(i)}^* T &= D_T F_{i1} D_T, \\ T_{(i)} - T_i^* T &= D_T F_{i2} D_T. \end{aligned} \tag{3.2}$$

2. *For each $i = 1, 2, \ldots, d$, the pair (F_{i1}, F_{i2}) as in part (1) is the unique solution $(X_{i1}, X_{i2}) = (F_{i1}, F_{i2})$ of the system of operator equations*

$$\begin{aligned} D_T T_i &= X_{i1} D_T + X_{i2}^* D_T T, \\ D_T T_{(i)} &= X_{i2} D_T + X_{i1}^* D_T T. \end{aligned} \tag{3.3}$$

Proof Note first that Proposition 3.1 ensures us that for all $w \in \mathbb{T}$, the pairs $(S_i(w), T(w)) := (T_i + wT_{(i)}, wT)$ are Γ-contractions. Hence by Theorem 2.2 there exist operators $F_i(w) \in \mathcal{B}(\mathcal{D}_T)$ such that

$$S_i(w) - S_i(w)^* T(w) = D_T F_i(w) D_T$$

which in turn simplifies to

$$(T_i - T_{(i)}^* T) + w(T_{(i)} - T_i^* T) = D_T F_i(w) D_T. \tag{3.4}$$

Let us introduce the notation

$$L_0 = T_i - T_{(i)}^* T, \quad L_1 = T_{(i)} - T_i^* T, \quad L(w) = L_0 + wL_1$$

so that we can write (3.4) more compactly as

$$L(w) = D_T F_i(w) D_T. \tag{3.5}$$

Our goal is to show that then necessarily $F_i(w)$ has the pencil form

$$F_i(w) = F_{i1} + wF_{i2} \tag{3.6}$$

for some uniquely determined operators F_{i1} and F_{i2} in $\mathcal{B}(\mathcal{D}_T)$. Note that we recover L_0 and L_1 from $L(w)$ via the formulas

$$L_0 = L(0), \quad L_1 = \frac{L(w) - L(0)}{w} \text{ for any } w \in \mathbb{D} \setminus \{0\}.$$

Since $L(0) = D_T F_i(0) D_T$, it is natural to set

$$F_{i1} = F_i(0). \tag{3.7}$$

Similarly, since we recover L_1 from $L(w)$ via the formula

$$L_1 = \frac{L(w) - L(0)}{w} \text{ for any } w \in \mathbb{D} \setminus \{0\},$$

it is natural to set

$$F_{i2} = \frac{F_i(w) - F_i(0)}{w} \text{ for } w \in \mathbb{D} \setminus \{0\}. \tag{3.8}$$

To see that the right-hand side of (3.8) is independent of w, we note that

$$L_1 = \frac{L(w) - L(0)}{w} = D_T \frac{F_i(w) - F_i(0)}{w} D_T.$$

Since L_1 is independent of w and $(F_i(w) - F_i(0))/w \in \mathcal{B}(\mathcal{D}_T)$, it follows that, for any two points $w, w' \in \mathbb{D} \setminus \{0\}$, we have

$$D_T \left(\frac{F_i(w) - F_i(0)}{w} - \frac{F_i(w') - F_i(0)}{w'} \right) D_T = 0.$$

From the general fact

$$X \in \mathcal{L}(\mathcal{D}_T),\ D_T X D_T = 0 \Rightarrow X = 0, \tag{3.9}$$

it follows that $\frac{F_i(w)-F_i(0)}{w} = \frac{F_i(w')-F_i(0)}{w'}$ and hence F_{i2} is well-defined by (3.8). From the definitions we see that $L_0 + wL_1 = D_T(F_{i1} + wF_{i2})D_T$ and hence, again by the uniqueness statement (3.9), we have established that $F_i(w)$ has the pencil form (3.6) as wanted.

Finally Eqs. (3.2) now follow by equations coefficients in the pencil identity $L(w) = D_T T(w) D_T$.

To prove part (2), we see by part (2) of Theorem 2.2 that for each $i = 1, 2, \ldots, d$ and $w \in \mathbb{T}$, the operator $F_i(w)$ is the unique operator that satisfies

$$D_T S_i(w) = F_i(w) D_T + F_i(w)^* D_T T(w).$$

Hence it follows that for all $w \in \mathbb{T}$ we have

$$D_T(T_i + wT_{(i)}) = (F_{i1} + wF_{i2})D_T + w(F_{i1} + wF_{i2})^* D_T T.$$

A comparison of the constant terms and the coefficients of w gives the equations in (3.3).

The uniqueness part follows from that of the function $F_i(w)$ as follows. If F'_{i1} and F'_{i2} are operators on $\mathcal{D}_T$ that satisfy (3.3), then setting $F_i(w)' := F'_{i1} + wF'_{i2}$ gives

$$D_T S_i(w) = F_i(w)' D_T + F_i(w)'^* D_T T(w).$$

By the uniqueness in part (2) of Theorem 2.2, we conclude $F_i(w) = F_i(w)'$ proving $F_{i1} = F'_{i1}$ and $F_{i2} = F'_{i2}$ for all $i = 1, 2, \ldots, d$. □

Definition 3.3 For a d-tuple $\underline{T} = (T_1, T_2, \ldots, T_d)$ of commutative contractions on a Hilbert space $\mathcal{H}$, the unique operators $\{F_{i1}, F_{i2} : i = 1, 2, \ldots, d\}$ obtained in Theorem 3.2 are called the *fundamental operators* of $\underline{T}$. The fundamental operators of the adjoint tuple $\underline{T}^* = (T_1^*, T_2^*, \ldots, T_d^*)$ will be denoted by $\{G_{i1}, G_{i2} : i = 1, 2, \ldots, d\}$.

The following is a straightforward consequence of Theorem 3.2.

Corollary 3.4 *Let $\underline{T} = (T_1, T_2, \ldots, T_d)$ be a d-tuple of commutative contractions on a Hilbert space $\mathcal{H}$. Then the fundamental operators $\{G_{i1}, G_{i2} : i = 1, \ldots, d\}$ of the adjoint tuple $\underline{T}^* = (T_1^*, T_2^*, \ldots, T_d^*)$ are the unique operators satisfying the systems of equations*

$$\begin{cases} T_i^* - T_{(i)} T^* = D_{T^*} G_{i1} D_{T^*} \text{ and} \\ T_{(i)}^* - T_i T^* = D_{T^*} G_{i2} D_{T^*} \end{cases} \tag{3.10}$$

and

$$\begin{cases} D_{T^*}T_i^* = G_{i1}D_{T^*} + G_{i2}^*D_{T^*}T^* \text{ and} \\ D_{T^*}T_{(i)}^* = G_{i2}D_{T^*} + G_{i1}^*D_{T^*}T^* \end{cases} \tag{3.11}$$

for each $i = 1, 2, \ldots, d$*:*

We next note some additional properties of the fundamental operators. These properties will not be used in this paper but are of interest in their own right.

Proposition 3.5 *Let* $\underline{T} = (T_1, T_2, \ldots, T_d)$ *be a* d*-tuple of commutative contractions on a Hilbert space* $\mathcal{H}$ *and* $T = T_1T_2\cdots T_d$*. Let* $\{F_{j1}, F_{j2} : j = 1, \ldots, d\}$ *and* $\{G_{j1}, G_{j2} : j = 1, \ldots, d\}$ *be the fundamental operators of* $\underline{T}$ *and* $\underline{T}^*$*, respectively. Then for each* $j = 1, 2, \ldots, d$*,*

1. $TF_{j1} = G_{j1}^*T|_{\mathcal{D}_T}$*;*
2. $D_TF_{j1} = (T_jD_T - D_{T^*}G_{j2}T)|\mathcal{D}_T$*,* $D_TF_{j2} = (T_{(j)}D_T - D_{T^*}G_{j1}T)|\mathcal{D}_T$*;*
3. $(F_{j1}^*D_TD_{T^*} - F_{j2}T^*)|_{\mathcal{D}_{T^*}} = D_TD_{T^*}G_{j1} - T^*G_{j2}^*$ *and* $(F_{j2}^*D_TD_{T^*} - F_{j1}T^*)|_{\mathcal{D}_{T^*}} = D_TD_{T^*}G_{j2} - T^*G_{j1}^*$*.*

Proof Let (T_1, T_2) be a commutative pair of contractions on a Hilbert space $\mathcal{H}$. We claim that the triple (T_1, T_2, T_1T_2) is a tetrablock contraction or equivalently the closure of the tetrablock domain $\mathbb{E}$ as in (2.6) is a spectral set for (T_1, T_2, T_1T_2). Let $\pi_{\mathbb{D}^2,\mathbb{E}} : \mathbb{D}^2 \to \mathbb{E}$ be the map defined by

$$\pi_{\mathbb{D}^2,\mathbb{E}} \colon (z_1, z_2) \mapsto (z_1, z_2, z_1z_2) \tag{3.12}$$

and let f be any polynomial in three variables. Then by Andô's theorem

$$\|f(T_1, T_2, T_1T_2)\| = \|f \circ \pi_{\mathbb{D}^2,\mathbb{E}}(T_1, T_2)\| \le \|f \circ \pi_{\mathbb{D}^2,\mathbb{E}}\|_{\infty,\mathbb{D}^2} \le \|f\|_{\infty,\mathbb{E}}.$$

Therefore the triple (T_1, T_2, T_1T_2) is a tetrablock contraction whenever (T_1, T_2) is a commutative pair of contractions.

Thus, given a d-tuple $\underline{T} = (T_1, T_2, \ldots, T_d)$ of commutative contractions on a Hilbert space $\mathcal{H}$, there is an associated family of tetrablock contractions, viz.,

$$(T_j, T_{(j)}, T), \quad j = 1, 2, \ldots, d. \tag{3.13}$$

Hence parts (1), (2), and (3) of Proposition 3.5 follow immediately from Lemmas 8, 9 and 10 of [28], respectively. □

Remark 3.6 We note that a result parallel to Theorem 3.2 appears in the theory of tetrablock contractions (see [22, Theorem 3.4], namely: *if* (A, B, T) *is a tetrablock contraction, then* [22] *that there exists two bounded operators* F_1, F_2 *acting on* $\mathcal{D}_T$ *with* $\nu(F_1 + zF_2) \le 1$*, for every* $z \in \overline{\mathbb{D}}$ *such that*

$$A - B^*T = D_TF_1D_T \text{ and } B - A^*T = D_TF_2D_T.$$

Moreover, Corollary 4.2 in [22] shows that F_1, F_2 *are the unique operators* (X_1, X_2) *such that*

$$D_T A = X_1 D_T + X_2^* D_T T, \quad D_T B = X_2 D_T + X_1^* D_T T.$$

These unique operators F_1, F_2 are called the *fundamental operators* of the tetrablock contraction (A, B, T).

Furthermore, it is possible to arrive at the result of Theorem 3.2 via applying these results to the special tetrablock contractions (3.13). Our proof of Theorem 3.2 instead relies only on the properties of fundamental operators for Γ-contractions.

4 Joint Halmos Dilation of Fundamental Operators

The following notion of dilation for the case $d = 1$ goes back to a 1950 paper of Halmos [39], hence our term *joint Halmos dilation*.

Definition 4.1 For a tuple $\underline{A} = (A_1, A_2, \ldots, A_d)$ of operators on a Hilbert space $\mathcal{H}$, a tuple $\underline{B} = (B_1, B_2, \ldots, B_d)$ of operators acting on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ is called a *joint Halmos dilation* of $\underline{A}$, if there exists an isometry $\Lambda : \mathcal{H} \to \mathcal{K}$ such that $A_i = \Lambda^* B_i \Lambda$ for each $i = 1, 2, \ldots, d$.

Lemma 4.2 *Let $d \geq 2$ and $(T_1, T_2, \ldots, T_d)$ be a d-tuple of commutative contractions on $\mathcal{H}$ and $T = T_1 T_2 \cdots T_d$.*

1. *Let $\alpha = (j_1, \ldots, j_k)$ be a k-tuple such that $1 \leq j_1 < j_2 < \cdots < j_k \leq d$. Consider the k-tuple $(T_{j_1}, T_{j_2}, \ldots, T_{j_k})$ and define $T_\alpha = T_{j_1} \cdots T_{j_k}$. Let $\Delta_\alpha : \mathcal{D}_{T_\alpha} \to \mathcal{D}_{T_{j_1}} \oplus \mathcal{D}_{T_{j_2}} \oplus \cdots \oplus \mathcal{D}_{T_{j_k}}$ be the operator defined by*

$$\Delta_\alpha : D_{T_\alpha} h \mapsto D_{T_{j_1}} T_{j_2} \cdots T_{j_k} h \oplus D_{T_{j_2}} T_{j_3} \cdots T_{j_k} h \oplus \cdots \oplus D_{T_{j_{k-1}}} T_{j_k} h \oplus D_{T_{j_k}} h \tag{4.1}$$

   for all $h \in \mathcal{H}$. Then Δ_α is an isometry.
2. *For each $j = 1, 2, \ldots, d$, the operator*

$$\Lambda_j : \mathcal{D}_T \to \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2} \oplus \cdots \oplus \mathcal{D}_{T_d}$$

   given by

$$\Lambda_j : D_T h \mapsto D_{T_j} T_{(j)} h \oplus \Delta_{(j)} D_{T_{(j)}} h, \text{ for all } h \in \mathcal{H} \tag{4.2}$$

   is an isometry, where $\Delta_{(j)}$ for the tuple $(j) = (1, \ldots, j-1, j+1, \ldots, d)$ is as in (4.1).

3. *For each* $j = 1, 2 \ldots, d$*, the operator* $U_j^* \colon \operatorname{Ran} \Lambda_j \to \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2} \oplus \cdots \oplus \mathcal{D}_{T_d}$ *defined by*

$$U_j^* \colon D_{T_j} T_{(j)} h \oplus \Delta_{(j)} D_{T_{(j)}} h \mapsto D_{T_j} h \oplus \Delta_{(j)} D_{T_{(j)}} T_j h \text{ for all } h \in \mathcal{H} \tag{4.3}$$

is an isometry.

4. *After possibly enlarging the Hilbert space* $\mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2} \oplus \cdots \oplus \mathcal{D}_{T_d}$ *to a larger Hilbert space*

$$\mathcal{F} := \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2} \oplus \cdots \oplus \mathcal{D}_{T_d} \oplus \mathcal{E}$$

for some auxiliary Hilbert space $\mathcal{E}$*,*

(a) *the isometries* U_j^* *in part (3) can be extended to be unitary operators on* $\mathcal{F}$ *(still denoted as* U_j^**).*
(b) *for each* $j = 1, 2 \ldots, d$*, there exists a unitary operator* τ_j *on* $\mathcal{F}$ *such that*

$$\tau_j \Lambda_j = \Lambda_1, \tag{4.4}$$

where $\tau_1 = I_{\mathcal{F}}$*.*

Proof of Part (1) Note that the norm of the vector on the RHS of (4.1) is

$$\|D_{T_{j_1}} T_{j_2} \cdots T_{j_k} h\|^2 + \|D_{T_{j_2}} T_{j_3} \cdots T_{j_k} h\|^2 + \cdots + \|D_{T_{j_{k-1}}} T_{j_k} h\|^2 + \|D_{T_{j_k}} h\|^2 \tag{4.5}$$

Making use of the general fact that if T is a contraction on a Hilbert space $\mathcal{H}$, then $\|D_T h\|^2 = \|h\|^2 - \|Th\|^2$ for every $h \in \mathcal{H}$, we can convert (4.5) to the telescoping sum

$$\begin{aligned}(\|T_{j_2} \cdots T_{j_k} h\|^2 - \|T_{j_1} T_{j_2} \cdots T_{j_k} h\|^2) + (\|T_{j_3} \cdots T_{j_k} h\|^2 - \|T_{j_2} T_{j_3} \cdots T_{j_k} h\|^2) \\ + \cdots + (\|T_{j_k} h\|^2 - \|T_{j_{k-1}} T_{j_k} h\|^2) + (\|h\|^2 - \|T_{j_k} h\|^2). \\ = \|h\|^2 - \|T_{j_1} T_{j_2} \cdots T_{j_k} h\|^2 = \|h\|^2 - \|T_\alpha h\|^2 = \|D_{T_\alpha} h\|^2.\end{aligned}$$

This shows that Δ_α is an isometry.

Proof of Part (2) Use the fact that $\Delta_{(j)}$ is an isometry by Part (1) to get

$$\begin{aligned}\|D_{T_j} T_{(j)} h\|^2 + \|\Delta_{(j)} D_{T_{(j)}} h\|^2 &= \|D_{T_j} T_{(j)} h\|^2 + \|D_{T_{(j)}} h\|^2 \\ &= (\|T_{(j)} h\|^2 - \|Th\|^2) + (\|h\|^2 - \|T_{(j)} h\|^2) = \|D_T h\|^2.\end{aligned}$$

Proof of Part (3) By a similar computation as done in Part (2), one can show that the norms of the vectors $D_{T_j} T_{(j)} h \oplus \Delta_{(j)} D_{T_{(j)}} h$ and $D_{T_j} h \oplus \Delta_{(j)} D_{T_{(j)}} T_j h$ are the same for every h in $\mathcal{H}$.

Proof of Part (4) Denote by $\mathcal{D}$ the Hilbert space $\mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2} \oplus \cdots \oplus \mathcal{D}_{T_d}$. If for each $j = 1, 2, \ldots, d$,

$$\begin{aligned} &\dim(\mathcal{D} \ominus \overline{\{D_{T_j} T_{(j)} h \oplus \Delta_{(j)} D_{T_{(j)}} h : h \in \mathcal{H}\}}) \\ &= \dim(\mathcal{D} \ominus \overline{\{D_{T_j} h \oplus \Delta_{(j)} D_{T_{(j)}} T_j h : h \in \mathcal{H}\}}), \end{aligned} \tag{4.6}$$

then clearly the isometric operators U_j^* defined as in (4.3) extend to unitary operators on $\mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2} \oplus \cdots \oplus \mathcal{D}_{T_d}$. Then we may define unitary operators τ_j on $\mathcal{D}$ by

$$\tau_j = U_1 U_j^*. \tag{4.7}$$

(so in particular $\tau_1 = I_{\mathcal{D}}$). Note next that

$$\tau : \Lambda_j D_T h = D_{T_j} T_{(j)} h \oplus \Delta_{(j)} D_{T_{(j)}} h \mapsto D_{T_1} T_{(1)} h \oplus \Delta_{(1)} D_{T_{(1)}} h = \Lambda_1 D_T h$$

Hence τ_j is a well-defined unitary operator on all of $\mathcal{D}$ satisfying the intertwining relation (4.4). If any of the equalities in (4.6) does not hold, then we add an infinite dimensional Hilbert space $\mathcal{E}$ to $\mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2} \oplus \cdots \oplus \mathcal{D}_{T_d}$ so that (4.6) does hold with $\mathcal{D}$ replaced by

$$\mathcal{F} := \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2} \oplus \cdots \oplus \mathcal{D}_{T_d} \oplus \mathcal{E}.$$

This proves (4). □

Notation For the adjoint tuple $(T_1^*, T_2^*, \ldots, T_d^*)$, the symbols $\mathcal{F}, \Lambda_j, U_j, \tau_j$ introduced in Lemma 4.2 will be changed to $\mathcal{F}_{j*}, \Lambda_{j*}, U_{j*}$ and τ_{j*}, respectively. In addition to this, we denote by P_j and P_{j*} the projections of $\mathcal{F}$ and $\mathcal{F}_*$ onto $\mathcal{D}_{T_j}$ and $\mathcal{D}_{T_{j*}}$, respectively.

Definition 4.3 Let $d \geq 2$ and $\underline{T} = (T_1, T_2, \ldots, T_d)$ be a d-tuple of commutative contractions. Let $\mathcal{F}, \Lambda_j, U_j$ be as in Lemma 4.2, and for $j = 1, 2, \ldots, d$ let P_j denote the projection of $\mathcal{F}$ onto $\mathcal{D}_{T_j}$. Then we say that the tuple $(\mathcal{F}, \Lambda_j, P_j, U_j)_{j=1}^d$ is an *Andô tuple* for $\underline{T}$.

Theorem 4.4 *Let $d \geq 2$, $\underline{T} = (T_1, T_2, \ldots, T_d)$ be a d-tuple of commutative contractions on a Hilbert space $\mathcal{H}$, $(\mathcal{F}, \Lambda_j, P_j, U_j)_{j=1}^d$ be an Andô tuple for $\underline{T}$ and $\{F_{j1}, F_{j2} : j = 1, 2, \ldots, d\}$ be the fundamental operators for $\underline{T}$. Then*

1. *For each $j = 1, 2 \ldots, d$, the pair (F_{j1}, F_{j2}) pair of partial isometries, viz.,*

$$(F_{j1}, F_{j2}) = \Lambda_j^* (P_j^{\perp} U_j^*, U_j P_j) \Lambda_j. \tag{4.8}$$

2. *With the unitaries τ_j as obtained in part (5) of Lemma 4.2, there is a joint Halmos dilation for the set $\{F_{j1}, F_{j2} : j = 1, 2, \ldots, d\}$, viz.,*

$$(F_{j1}, F_{j2}) = \Lambda_1^*(\tau_j P_j^\perp U_j^* \tau_j^*, \tau_j U_j P_j \tau_j^*)\Lambda_1, \text{ for each } j = 1, 2, \ldots, d. \tag{4.9}$$

Proof of (1) The proof of this part uses the uniqueness of the fundamental operators. For $h, h' \in \mathcal{H}$ we have

$$\begin{aligned}
&\langle D_T \Lambda_j^* P_j^\perp U_j^* \Lambda_j D_T h, h' \rangle \\
&\quad = \langle P_j^\perp U_j^* (D_{T_j} T_{(j)} h \oplus \Delta_{(j)} D_{T_{(j)}} h), D_{T_j} T_{(j)} h' \oplus \Delta_{(j)} D_{T_{(j)}} h' \rangle \\
&\quad = \langle 0 \oplus \Delta_{(j)} D_{T_{(j)}} T_j h, D_{T_j} T_{(j)} h' \oplus \Delta_{(j)} D_{T_{(j)}} h' \rangle \\
&\quad = \langle D_{T_{(j)}} T_j h, D_{T_{(j)}} h' \rangle = \langle (T_j - T_{(j)}^* T) h, h' \rangle.
\end{aligned}$$

Therefore $D_T \Lambda_j^* P_j^\perp U_j^* \Lambda_j D_T = T_j - T_{(j)}^* T$. By a similar computation one can show that $D_T \Lambda_j^* U_j P_j \Lambda_j D_T = T_{(j)} - T_j^* T$. By Theorem 3.2, the fundamental operators are the unique operators satisfying these equations. Hence (4.8) follows.

Proof of (2) This follows from the property (4.4) of the unitaries τ_j: $\tau_j \Lambda_j = \Lambda_1$, for each $j = 1, 2, \ldots, d$. Using this in (4.8), we obtain (4.9). □

5 Non-commutative Isometric Lift of Tuples of Commutative Contractions

As was mentioned in connection with the rational dilation problem in Sect. 2.2, it can happen that a $\overline{\mathbb{D}}^d$-contraction fails to have a $\overline{\mathbb{D}}^d$-isometric lift once $d \geq 3$, unlike the case of $d = 1$ and $d = 2$. Note that a $\overline{\mathbb{D}}^d$-contraction consists of a commutative d-tuple $\underline{T} = (T_1, \ldots, T_d)$ of contraction operators, while a $\overline{\mathbb{D}}^d$-isometry consists of a commutative d-tuple of isometries. Here we show that, even for the case of a general $d \geq 3$, a $\overline{\mathbb{D}}^d$-contraction always has a (in general noncommutative) isometric lift $\underline{V} = (V_1, \ldots, V_d)$.

Let T be a contraction on a Hilbert space $\mathcal{H}$. Schäffer [55] showed that the following 2×2 block operator matrix

$$V^S = \begin{bmatrix} T & 0 \\ \mathbf{e}_0^* D_T & M_z \end{bmatrix} : \mathcal{H} \oplus H^2(\mathcal{H}) \to \mathcal{H} \oplus H^2(\mathcal{H}) \tag{5.1}$$

is an isometry and hence a lift of the contraction T. Here, $\mathbf{e}_0 \colon H^2(\mathcal{H}) \to \mathcal{H}$ is the "evaluation at zero" map: $\mathbf{e}_0 \colon g \mapsto g(0)$. For a given d-tuple $\underline{T} = (T_1, \ldots, T_d)$

of contraction operators, let V_j^S be the Schäffer isometric lift of T_j, for each $1 \le j \le d$. Then the tuple $\underline{V}^S := (V_1^S, V_2^S, \ldots, V_d^S)$ is an (in general noncommutative) isometric lift of $\underline{T}$.

It is of interest to develop other constructions for such possibly noncommutative isometric lifts which have more structure and provide additional insight. Our next goal is to provide one such construction where the (possibly noncommutative) isometric lift of the given commutative contractive d-tuple $\underline{T} = (T_1, \ldots, T_d)$ has the form of a (possibly noncommutative) BCL model. The starting point for the construction is an Andô tuple $(\mathcal{F}_*, \Lambda_{j*}, P_{j*}, U_{j*})_{j=1}^d$ of $\underline{T}^*$, with the BCL model for the lift then having the form

$$(M_{\tau_{j*}(U_{j*}P_{j*}^\perp + zU_{j*}P_{j*})\tau_{j*}^*} \oplus W_{\partial j}^*)_{j=1}^d,$$

where $\underline{W}_\partial := (W_{\partial 1}, W_{\partial 2}, \ldots, W_{\partial d})$ is the canonical commutative unitary tuple associated with the commutative contractive $\underline{T}$ as in (2.24), and where τ_j ($j = 1, \ldots, d$) are unitaries acting on $\mathcal{F}$ as in Part (5) of Lemma 4.2. We first need a preliminary lemma.

Lemma 5.1 *Let $d \ge 2$, $(T_1, T_2, \ldots, T_d)$ be a d-tuple of commutative contractions and $(\mathcal{F}, \Lambda_j, P_j, U_j)_{j=1}^d$ be an Andô tuple for $(T_1, T_2, \ldots, T_d)$. Then the operator identities*

$$\begin{aligned} P_j^\perp U_j^* \Lambda_j D_T + P_j U_j^* \Lambda_j D_T T &= \Lambda_j D_T T_j, \\ U_j P_j \Lambda_j D_T + U_j P_j^\perp \Lambda_j D_T T &= \Lambda_j D_T T_{(j)} \end{aligned} \tag{5.2}$$

hold for $j = 1, \ldots, d$.

Proof Let j be some integer between 1 and d, and h be in $\mathcal{H}$. Then

$$\begin{aligned} &P_j^\perp U_j^* \Lambda_j D_T h + P_j U_j^* \Lambda_j D_T T h \\ &= P_j^\perp U_j^* (D_{T_j} T_{(j)} h \oplus \Delta_{(j)} D_{T_{(j)}} h) + P_j U_j^* (D_{T_j} T_{(j)} T h \oplus \Delta_{(j)} D_{T_{(j)}} T h) \\ &= P_j^\perp (D_{T_j} h \oplus \Delta_{(j)} D_{T_{(j)}} T_j h) + P_j (D_{T_j} T h \oplus \Delta_{(j)} D_{T_{(j)}} T_j T h) \\ &= (0 \oplus \Delta_{(j)} D_{T_{(j)}} T_j h) + (D_{T_j} T h \oplus 0) = D_{T_j} T_{(j)} T_j h \oplus \Delta_{(j)} D_{T_{(j)}} T_j h \\ &= \Lambda_j D_T T_j h. \end{aligned}$$

The proof of the second equality in (5.2) is similar to that of the first one. □

Theorem 5.2 *Let $\underline{T} = (T_1, T_2, \ldots, T_d)$ be a d-tuple of commutative contractions, $\underline{W}_\partial := (W_{\partial 1}^*, W_{\partial 2}^*, \ldots, W_{\partial d}^*)$ be the canonical commutative unitary tuple associated with $\underline{T}$ as in (2.24), let $(\mathcal{F}_*, \Lambda_{j*}, P_{j*}, U_{j*})_{j=1}^d$ be an Andô tuple for $\underline{T}^*$, and let $T = T_1 T_2 \cdots T_d$. Then*

1. *For each* $1 \le j \le d$, *define the isometries* $\Pi_{j*} : \mathcal{H} \to H^2(\mathcal{F}_*) \oplus \mathcal{R}_D$ *as*

$$\Pi_{j*}h = (I_{H^2} \otimes \Lambda_{j*})\widehat{\mathcal{O}}_{D_{T^*},T^*}h \oplus Qh. \tag{5.3}$$

Then the identities

$$\begin{aligned}\Pi_{j*}T_j^* &= (M^*_{U_{j*}P_{j*}^\perp + zU_{j*}P_{j*}} \oplus W^*_{\partial j})\Pi_{j*} \\ \Pi_{j*}T_{(j)}^* &= (M^*_{P_{j*}U_{j*}^* + zP_{j*}^\perp U_{j*}^*} \oplus W^*_{(\partial j)})\Pi_{j*}\end{aligned} \tag{5.4}$$

hold for $1 \le j \le d$, *i.e. for each* $j = 1, 2, \dots, d$,

$$\big(\Pi_{j*}, M_{U_{j*}P_{j*}^\perp + zU_{j*}P_{j*}} \oplus W_{\partial j}, M_{P_{j*}U_{j*}^* + zP_{j*}^\perp U_{j*}^*} \oplus W_{(\partial j)}, M_z \oplus W_D\big) \tag{5.5}$$

is an isometric lift of $(T_j, T_{(j)}, T)$.

2. *With the unitaries* τ_{j*} *obtained as in part (4) of Lemma 4.2 applied to* $(T_1^*, T_2^*, \dots, T_d^*)$, *we have for each* $j = 1, 2, \dots, d$,

$$\begin{cases}\Pi_{1*}T_j^* = (M^*_{\tau_{j*}(U_{j*}P_{j*}^\perp + zU_{j*}P_{j*})\tau_{j*}^*} \oplus W^*_{\partial j})\Pi_{1*} \\ \Pi_{1*}T_{(j)}^* = (M^*_{\tau_{j*}(P_{j*}U_{j*}^* + zP_{j*}^\perp U_{j*}^*)\tau_{j*}^*} \oplus W^*_{(\partial j)})\Pi_{1*},\end{cases} \tag{5.6}$$

i.e. if we denote the projections $\tau_{j*}P_{j*}\tau_{j*}^*$ *and unitaries* $\tau_{j*}U_{j*}\tau_{j*}^*$ *by* P'_{j*} *and* U'_{j*}, *respectively, then the* d*-tuple of (in general non-commutative) isometries*

$$(M_{(U'_{j*}P'^\perp_{j*} + zU'_{j*}P'_{j*})} \oplus W_{\partial j})_{j=1}^d \tag{5.7}$$

is a lift of $(T_1, T_2, \dots, T_d)$ *via the embedding* $\Pi_{1*} : \mathcal{H} \to H^2(\mathcal{F}_*) \oplus \mathcal{R}_D$.

Proof of Part (1) For every $h \in \mathcal{H}$, we have for each $j = 1, 2, \dots, d$,

$$\begin{aligned}&(M^*_{U_{j*}P_{j*}^\perp + zU_{j*}P_{j*}} \oplus W^*_{\partial j})\Pi_{j*}h \\ &= \sum_{n\ge 0} z^n P_{j*}^\perp U_{j*}^*\Lambda_{j*}D_{T^*}T^{*n}h + \sum_{n\ge 0} z^n P_{j*}U_{j*}^*\Lambda_{j*}D_{T^*}T^{*n+1}h \oplus W^*_{\partial j}Qh \\ &= \sum_{n\ge 0} z^n (P_{j*}^\perp U_{j*}^*\Lambda_{j*}D_{T^*} + P_{j*}U_{j*}^*\Lambda_{j*}D_{T^*}T^*)T^{*n}h \oplus QT_j^*h \\ &= \sum_{n\ge 0} z^n \Lambda_{j*}D_{T^*}T_j^*T^{*n}h \oplus QT_j^*h = \Pi_{j*}T_j^*h\end{aligned}$$

where we use the first equation in (5.2)) for the last step.

A similar computation using the second equation in (5.2) leads to the second equation in (5.4).

Proof of Part (2) It follows from property (4.4) of τ_j and the definition (5.3) of Π_{j*} that

$$((I_{H^2} \otimes \tau_{j*}) \oplus I_{\mathcal{R}})\Pi_{j*} = \Pi_{1*}, \text{ for each } j = 1, 2, \ldots, d.$$

Using this in (5.4) one obtains (5.6). □

Remark 5.3 Note that for a given d-tuple $\underline{T} = (T_1, T_2, \ldots, T_d)$ of commutative contractions, if there exists an Andô tuple of $\underline{T}^*$ such that the d-tuple of isometries given in (5.7) is commutative, then there exists a commutative isometric lift of $\underline{T}$. Therefore a priori, we have a sufficient condition for dilation in $\overline{\mathbb{D}}^d$.

Remark 5.4 Note that in the terminology of Definition 2.1, the context of part (2) of Theorem 5.2 the collection of objects $(\Pi_D, H^2(\mathcal{F}_*) \oplus \mathcal{R}_D, \underline{V})$, where we set $\underline{V} = (V_1, \ldots, V_d)$ with

$$V_j = M_{U'_{j*}P'^{\perp}_{j*} + zU_{j*}P'_{j*}} \oplus W_{\partial j} \text{ for } j = 1, \ldots, d \tag{5.8}$$

is a (not necessarily commutative) isometric lift for of $(T_1, \ldots, T_d)$, where the construction involves only an Andô tuple $(\mathcal{F}_*, \Lambda_{j*}, P_{j*}, U_{j*})_{j=1}^d$ for $\underline{T} = (T_1, \ldots, T_d)$. Note also that the presentation (5.8) shows that $(V_1, \ldots, V_d)$ is just the (not necessarily commutative) BCL-model associated with the BCL-tuple $(\mathcal{F}_*, \mathcal{R}_D, P_{*j}, U_{*j}, W_{\partial j})_{j=1}^d$ as in Definition 2.4.

6 Pseudo-Commutative Contractive Lifts and Models for Tuples of Commutative Contractions

One disadvantage of dilation theory in $\overline{\mathbb{D}}^d$ ($d \geq 2$) is that there is no uniqueness of minimal isometric lifts when such exist, even in the case $d = 2$ (where at least we know such exist)—unlike the classical case $d = 1$. We next identify an alternative generalization of the notion of isometric lift for the $d = 1$ case, which, as we shall see, always exists and has good uniqueness properties.

Definition 6.1 For a given d-tuple $\underline{T} = (T_1, T_2, \ldots, T_d)$ of commutative contractions acting on a Hilbert space $\mathcal{H}$, we say that $(\Pi, \mathcal{K}, \underline{S}, V)$ is a *pseudo-commutative contractive lift* of $(T_1, T_2, \ldots, T_d, T)$, where $\underline{S} = (S_1, S_2, \ldots, S_d)$ and $T = T_1 T_2 \cdots T_d$, if

1. $\Pi : \mathcal{H} \to \mathcal{K}$ is an isometry such that $(\Pi, \mathcal{K}, V)$ is the minimal isometric lift of the single operator T, and

2. with $S'_j = S_j^* V$ for each $1 \le j \le d$, the pairs (S_j, V), (S'_j, V) are commutative and

$$(S_j^*, S_j'^*)\Pi = \Pi(T_j^*, T_{(j)}^*).$$

Remark 6.2 Note that we do not assume that the tuples $\underline{S} = (S_1, S_2, \dots, S_d)$ and $\underline{S'} = (S'_1, S'_2, \dots, S'_d)$ be commutative but we do require that each of the pairs (S_j, V) and (S'_j, V) be commutative. Also one can show that the validity of the equation $S'_j = S_j^* V$ implies the validity of the equation $S_j = S_j'^* V$ for each $j = 1, 2, \dots, d$ as follows:

$$S'_j = S_j^* V \Rightarrow S_j'^* = V^* S_j \Rightarrow S_j'^* V = V^* S_j V = V^* V S_j = S_j.$$

Suppose that $(V_1, V_2, \dots, V_d)$ on $\mathcal{K}$ is a commutative isometric lift of a given d tuple of commutative contractions $\underline{T} = (T_1, T_2, \dots, T_d)$ on $\mathcal{H}$ via an isometric embedding $\Pi : \mathcal{H} \to \mathcal{K}$. Let us denote by V the isometry $V_1 V_2 \cdots V_d$. Then with $V'_j := V_1 \cdots V_{j-1} V_{j+1} \cdots V_d$ for $j = 1, 2, \dots, d$, we see that part (2) in Definition 6.1 is satisfied. However, the lift $(\Pi, \mathcal{K}, V)$ of $T = T_1 T_2 \cdots, T_d$ need not be minimal, i.e. condition (1) in Definition 6.1 may not hold.

The next theorem shows that for a given tuple $\underline{T}$ of commutative contractions, a pseudo-commutative contractive lift exists and any two such lifts are unitarily equivalent in a sense explained in the theorem.

Theorem 6.3 *Let $\underline{T} = (T_1, T_2, \dots, T_d)$ be a d-tuple of commutative contractions acting on a Hilbert space $\mathcal{H}$ and let $T = T_1 T_2 \cdots T_d$. Then there exists a pseudo-commutative contractive lift of $(T_1, T_2, \dots, T_d, T)$. Moreover, if $(\Pi_1, \mathcal{K}_1, \underline{S}, V_1)$ and $(\Pi_2, \mathcal{K}_2, \underline{R}, V_2)$ are two pseudo-commutative contractive lifts of $(T_1, T_2, \dots, T_d, T)$, where $\underline{S} = (S_1, S_2, \dots, S_d)$ and $\underline{R} = (R_1, R_2, \dots, R_d)$, then $(\Pi_1, \mathcal{K}_1, \underline{S}, V_1)$ and $(\Pi_2, \mathcal{K}_2, \underline{R}, V_2)$ are unitarily equivalent in the sense of Definition 2.1.*

Proof of Existence Roughly the idea is that a pseudo-commutative contractive lift of $\underline{T} = (T_1, \dots, T_d)$ arises as the compression of the Andô-tuple-based noncommutative isometric lift constructed in Theorem 5.2 to the minimal lift space for the single contraction operator $T = T_1 \cdots T_d$. Precise details are as follows.

We use the Douglas model for the minimal isometric lift of the single contraction operator $T = T_1 \cdots, T_d$ as described in Sect. 2.5, namely

$$(H^2(\mathcal{D}_{T^*}) \oplus \mathcal{R}_D, \Pi_D, V_D = M_z \oplus W_D)$$

as in (2.18), (2.19), and (2.20). We let $(W_{\partial 1}, \dots, W_{\partial d})$ be the canonical unitary tuple associated with $(T_1, \dots, T_d)$ as in (2.24), and let $\{G_{i1}, G_{i2} \colon i = 1, \dots d\}$ be the fundamental operators associated with $\underline{T}^* = (T_1^*, \dots, T_d^*)$ as in (3.10). Set

$$\begin{aligned}\underline{S}^D &:= (S_1^D, S_2^D, \dots, S_d^D)\\ &:= (M_{G_{11}^*+zG_{12}} \oplus W_{\partial 1}, M_{G_{21}^*+zG_{22}} \oplus W_{\partial 2}, \dots, M_{G_{d1}^*+zG_{d2}} \oplus W_{\partial d})\end{aligned} \tag{6.1}$$

and

$$\begin{aligned}\underline{S'}^D &:= (S_1'^D, S_2'^D, \dots, S_d'^D)\\ &:= (M_{G_{12}^*+zG_{11}} \oplus W_{(\partial 1)}, M_{G_{22}^*+zG_{21}} \oplus W_{(\partial 2)} \dots, M_{G_{d2}^*+zG_{d1}} \oplus W_{(\partial d)}).\end{aligned} \tag{6.2}$$

We shall show that

$$(\Pi_D, \mathcal{K}_D = H^2(\mathcal{D}_{T^*}) \oplus \mathcal{R}_D, \underline{S}^D, M_z \oplus W_D) \tag{6.3}$$

is a pseudo-commutative contractive lift of $\underline{T} = (T_1, \dots, T_d)$.

Toward this goal let us first note that part (1) of Theorem 4.4 for the d tuple $\underline{T}^*$, we get

$$(G_{j1}, G_{j2}) = \Lambda_{j*}^*(P_{j*}^\perp U_{j*}^*, U_{j*}P_{j*})\Lambda_{j*} \text{ for each } j = 1, 2, \dots, d, \tag{6.4}$$

where $(\mathcal{F}_*, \Lambda_{j*}, P_{j*}, U_{j*})_{j=1}^d$ is an Andô tuple for $\underline{T}^*$. We now recall the construction of a noncommutative isometric lift described in Theorem 5.2. Notice that the isometries $\Pi_{j*} : \mathcal{H} \to H^2(\mathcal{F}_*) \oplus \mathcal{R}_D$ as in (5.3) can be factored as

$$\Pi_{j*}h = \begin{bmatrix}(I_{H^2} \otimes \Lambda_{j*}) & 0\\ 0 & I_{\mathcal{R}_D}\end{bmatrix}\begin{bmatrix}\widehat{\mathcal{O}}_{D_{T^*},T^*}(z)h\\ Qh\end{bmatrix} = \begin{bmatrix}(I_{H^2} \otimes \Lambda_{j*}) & 0\\ 0 & I_{\mathcal{R}_D}\end{bmatrix}\Pi_D h. \tag{6.5}$$

Therefore from the first equation in (5.4) we get for each $j = 1, 2, \dots, d$,

$$\begin{aligned}\Pi_D T_j^* h &= \begin{bmatrix}(I_{H^2} \otimes \Lambda_{j*}^*) & 0\\ 0 & I_{\mathcal{R}_D}\end{bmatrix}\begin{bmatrix}M^*_{U_{j*}P_{j*}^\perp+zU_{j*}P_{j*}} & 0\\ 0 & W_{\partial j}^*\end{bmatrix}\begin{bmatrix}(I_{H^2} \otimes \Lambda_{j*}) & 0\\ 0 & I_{\mathcal{R}_D}\end{bmatrix}\Pi_D h\\ &= \begin{bmatrix}M^*_{\Lambda_{j*}^*U_{j*}P_{j*}^\perp\Lambda_{j*}+z\Lambda_{j*}^*U_{j*}P_{j*}\Lambda_{j*}} & 0\\ 0 & W_{\partial j}^*\end{bmatrix}\Pi_D h\\ &= \begin{bmatrix}M^*_{G_{j1}^*+zG_{j2}} & 0\\ 0 & W_{\partial j}^*\end{bmatrix}\Pi_D h.\end{aligned}$$

Consequently, we have for each $j = 1, 2, \ldots, d$,

$$\Pi_D T_j^* = (M^*_{G^*_{j1}+zG_{j2}} \oplus W^*_{\partial j})\Pi_D = S_j^{D*}\Pi_D h \tag{6.6}$$

Similarly starting with the second equation in (5.4) and proceeding as above we obtain

$$\Pi_D T^*_{(i)} = (M^*_{G^*_{i2}+zG_{i1}} \oplus W^*_{(\partial i)})\Pi_D = S_i'^{D*}\Pi_D. \tag{6.7}$$

Then with $V_D = M_z \oplus W_D$, the Douglas isometric lift of $T = T_1 T_2 \cdots T_d$ as discussed in Sect. 2.5, it follows from the equality (see (2.22))

$$W_D = W_{\partial 1} W_{\partial 2} \cdots W_{\partial d}$$

that $S_j'^D = S_j^{D*} V_D$ for each $j = 1, 2, \ldots, d$. As we have already noted, $(\Pi, \mathcal{K}_D, V_D)$ is a minimal isometric lift of T. Therefore part (1) of Definition 6.1 is satisfied. Also it follows from definitions (6.1) and (6.2) that the pairs (S_j, V_D) and (S'_j, V_D) are commutative for each $j = 1, 2, \ldots, d$. And finally from Eqs. (6.6) and (6.7) we see that part (2) of Definition 6.1 is also satisfied. Consequently, (6.3) is a pseudo-commutative contractive lift of $(T_1, T_2, \ldots, T_d, T)$. □

Proof of Uniqueness in Theorem 6.3 The strategy is to show that any pseudo-commutative contractive lift $(\Pi, \mathcal{K}, \underline{S}, V)$ is unitarily equivalent to the canonical-model pseudo-commutative contractive lift $(\Pi_D, \mathcal{K}_D, \underline{S}^D, V_D)$, as constructed in the existence part of the proof, Since (Π, V) and (Π_D, V_D) are two minimal isometric dilations of $T = T_1 T_2 \cdots T_d$, there exists a unitary $\tau : \mathcal{K} \to \mathcal{K}_D$ such that $\tau V = V_D \tau$ and $\tau \Pi = \Pi_D$. We show that this unitary does the rest of the job.

Without loss of generality we may assume that $(\Pi, V) = (\Pi_D, V_D)$. Due to this reduction all we have to show is that $\underline{S} = \underline{S}^D$ and $\underline{S}' = \underline{S}'^D$. First let us suppose

$$S_j = \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix} \text{ and } S'_j = \begin{bmatrix} A'_j & B'_j \\ C'_j & D'_j \end{bmatrix} \tag{6.8}$$

for each $j = 1, 2, \ldots, d$ with respect to the decomposition $\mathcal{K}_D = H^2(\mathcal{D}_{T^*}) \oplus \mathcal{R}_D$. Since each S_j commutes with $V_D = M_z \oplus W_D$, we have

$$\begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix}\begin{bmatrix} M_z & 0 \\ 0 & W_D \end{bmatrix} = \begin{bmatrix} M_z & 0 \\ 0 & W_D \end{bmatrix}\begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix}$$
$$\Leftrightarrow \begin{bmatrix} A_j M_z & B_j W_D \\ C_j M_z & D_j W_D \end{bmatrix} = \begin{bmatrix} M_z A_j & M_z B_j \\ W_D C_j & W_D D_j \end{bmatrix}. \tag{6.9}$$

It is well-known that any operator that intertwines a unitary and a pure isometry is zero (see e.g. [44, page 227] or [13, Chapter 3]), hence $B_j = 0$ for $j = 1, 2, \ldots, d$.

Since each S'_j commutes with V also, by a similar computation with S'_j, we have $B'_j = 0$ for each $j = 1, 2, \dots, d$. From the identity of (1,1)-entries in (6.9) we see that

$$A_j = M_{\varphi_j} \text{ and } A'_j = M_{\varphi'_j}, \text{ for some } \varphi_j, \varphi'_j \in H^\infty(\mathcal{B}(\mathcal{D}_{T^*})). \tag{6.10}$$

Hence S_j and S'_j have the form

$$S_j = \begin{bmatrix} M_{\varphi_j} & 0 \\ C_j & D_j \end{bmatrix}, \quad S'_j = \begin{bmatrix} M_{\varphi'_j} & 0 \\ C'_j & D'_j \end{bmatrix}. \tag{6.11}$$

Since $S'_j = S^*_j V$ and hence also by Remark 6.2 $S_j = S'^*_j V$, we then have

$$\begin{bmatrix} M_{\varphi'_j} & 0 \\ 0 & D'_j \end{bmatrix} = \begin{bmatrix} M^*_{\varphi_j} & C^*_j \\ 0 & D^*_j \end{bmatrix} \begin{bmatrix} M_z & 0 \\ 0 & W_D \end{bmatrix} = \begin{bmatrix} M^*_{\varphi_j} M_z & C^*_j W_D \\ 0 & D^*_j W_D \end{bmatrix},$$
$$\begin{bmatrix} M_{\varphi_j} & 0 \\ 0 & D_j \end{bmatrix} = \begin{bmatrix} M^*_{\varphi'_j} & C'^*_j \\ 0 & D'^*_j \end{bmatrix} \begin{bmatrix} M_z & 0 \\ 0 & W_D \end{bmatrix} = \begin{bmatrix} M^*_{\varphi'_j} M_z & C'^*_j W_D \\ 0 & D'^*_j W_D \end{bmatrix}. \tag{6.12}$$

From equality of the (1, 2) entries we see that

$$0 = C^*_j W_D, \quad 0 = C'^*_j W_D.$$

As W_D is unitary, in particular W_D is surjective and we may conclude that in fact $C_j = 0$, $C'^*_j = 0$ and the form (6.11) for S_j and S'_j collapses to

$$S_j = \begin{bmatrix} M_{\varphi_j} & 0 \\ 0 & D_j \end{bmatrix}, \quad S'_j = \begin{bmatrix} M_{\varphi'_j} & 0 \\ 0 & D'_j \end{bmatrix}. \tag{6.13}$$

Looking next at the identities $M_{\varphi'_j} = M^*_{\varphi_j} M_z$ and $M_{\varphi_j} = M^*_{\varphi'_j} M_z$ for each $j = 1, 2, \dots, d$ in terms of power series expansions of φ_j and φ'_j then leads to

$$\varphi_j(z) = \widetilde{G}^*_{j1} + z\widetilde{G}_{j2} \text{ and } \varphi'_j(z) = \widetilde{G}^*_{j2} + z\widetilde{G}_{j1} \text{ for } j = 1, 2, \dots, d \tag{6.14}$$

for some $\widetilde{G}_{j1}, \widetilde{G}_{j2} \in \mathcal{B}(\mathcal{D}_{T^*})$. We shall eventually see that $\{\widetilde{G}_{j1}, \widetilde{G}_{j2} \colon j = 1, 2, \dots, d\}$ is exactly the set of fundamental operators $\{G_{j1}, G_{j2} \colon j = 1, 2, \dots, d\}$ for $\underline{T}^*$.

Let us now analyze the second components in (6.13) involving the operator tuples $(D_1, D_2, \dots, D_d)$ and $(D'_1, D'_2, \dots, D'_d)$. From the relations

$$(S^*_j, S'^*_j)\Pi_D = \Pi_D(T^*_j, T^*_{(j)})$$

we have for all $h \in \mathcal{H}$,

$$D_j^* Qh = QT_j^* h \text{ and } D_j'^* Qh = QT_{(j)}^* h,$$

which by (2.21) implies that $D_j^*|_{\overline{\operatorname{Ran} Q}} = X_j^*$, for each $j = 1, 2, \ldots, d$. Since each S_j and S_j' commute with V_D, D_j and D_j' commute with W_D and since W_D is a unitary, D_j^* and $D_j'^*$ also commute with W_D. Using this we have for every $\xi \in \overline{\operatorname{Ran} Q}$ and $n \geq 0$,

$$D_j^*(W_D^n)\xi = W_D^n D_j^* \xi = W_D^n X_j^* \xi = W_D^n W_j^* \xi = W_j^* W_d^n \xi$$
$$D_j'^*(W_D^n)\xi = W_D^n D_j'^* \xi = W_D^n X_{(j)}^* \xi = W_D^n W_{(j)}^* \xi = W_{(j)}^* W_d^n \xi.$$

As the set of elements of the form $W_D^n \xi$ is dense in $\mathcal{R}_D$, we conclude that

$$D_j = W_j \text{ and } D_j' = W_{(j)} \text{ for each } j = 1, 2, \ldots, d. \tag{6.15}$$

To show that $G_{j1} = \widetilde{G}_{j1}$, by the uniqueness result in part (2) of Theorem 3.2 it suffices to show that

$$D_{T^*} G_{j1} D_{T^*} = D_{T^*} \widetilde{G}_{j1} D_{T^*}. \tag{6.16}$$

for $j = 1, \ldots, d$. The fundamental operator G_{j1} is characterized as the unique solution of

$$D_{T^*} G_{j1} D_{T^*} = T_j^* - T_{(j)} T^* \tag{6.17}$$

As $\underline{S}$ is a pseudo-commutative contractive lift of $\underline{T}$ with embedding operator Π_D, we have by Definition 6.1 the intertwining conditions

$$S_j^* \Pi_D = \Pi_D T_j^*, \quad S_j'^* \Pi_D = \Pi_D T_{(j)}^*, \quad W_D^* \Pi_D = \Pi_D T^*$$

from which we also deduce that

$$T_{(j)} = T_{(j)} \Pi_D^* \Pi_D = \Pi_D^* S_j' \Pi_D.$$

We may then compute

$$\begin{aligned}
T_j^* - T_{(j)} T^* &= \Pi_D^* \Pi_D (T_j^* - T_{(j)} T^*) = \Pi_D^* S_j^* \Pi_D - \Pi_D^* \Pi_D \Pi_D^* S_j' \Pi_D T^* \\
&= \Pi_D^* (S_j^* - S_j' W_D^*) \Pi_D = \Pi_D^* (S_j^* - S_j^* W_D W_D^*) = \Pi_D^* S_j^* (I - W_D W_D^*) \\
&= \Pi_D^* \begin{bmatrix} M_{\varphi_j}^* (I - M_z M_z^*) & 0 \\ 0 & 0 \end{bmatrix} = \Pi_D^* \begin{bmatrix} (I - M_z M_z^*) \otimes \widetilde{G}_{j1} & 0 \\ 0 & 0 \end{bmatrix} \\
&= \widehat{\mathcal{O}}_{D_{T^*}, T^*}^* \left((I - M_z M_z^*) \otimes \widetilde{G}_{j1} \right).
\end{aligned} \tag{6.18}$$

From the general formula

$$\widehat{\mathcal{O}}^*_{D_{T^*},T^*}\colon \sum_{n=0}^{\infty} h_n z^n \mapsto \sum_{n=0}^{\infty} T^n D_{T^*} h_n$$

for the action of the adjoint observability operator $\mathcal{O}^*_{D_{T^*},T^*}$ and combining (6.17) and (6.18), we finally arrive at

$$D_{T^*} G_{j1} D_{T^*} = \widehat{\mathcal{O}}^*_{D_{T^*},T^*} \left((I - M_z M_z^*) \otimes \widehat{G}_{j1}\right) D_{T^*} = D_{T^*} \widetilde{G}_{j1} D_{T^*}$$

and (6.16) follows as wanted.

A similar computation shows that

$$D_T^* G_{j2} D_{T^*} = T^*_{(i)} - T_i T^* = \cdots = D_{T^*} \widetilde{G}_{j2} D_{T^*}$$

for $j = 1, \ldots, d$ from which it follows that $\widetilde{G}_{j2} = G_{j2}$ as well. This completes the proof of uniqueness in Theorem 6.3. □

Remark 6.4 The proof of the existence part of Theorem 6.3 actually gives a canonical model (6.1)–(6.2) for an arbitrary pseudo-commutative contractive lift of a given commutative contractive operator-tuple $\underline{T} = (T_1, \ldots, T_d)$. By compressing the operators $\underline{S}^D$ to the subspace

$$\mathcal{H}_D = (H^2(\mathcal{D}_{T^*}) \oplus \mathcal{R}_D) \ominus \operatorname{Ran} \Pi_D,$$

we arrive at a Douglas-type functional model for the original commutative contractive operator tuple. The precise statement is: *Let $\underline{T} = (T_1, T_2, \ldots, T_d)$ be a d-tuple of commutative contractions on a Hilbert space $\mathcal{H}$ and $T = T_1 T_2 \cdots T_d$. Let $\{G_{i1}, G_{i2} : i = 1, \ldots, d\}$ be the fundamental operators of the adjoint tuple $\underline{T}^* = (T_1^*, T_2^*, \ldots, T_d^*)$ and $\underline{W}_\partial = (W_{\partial 1}, W_{\partial 2}, \ldots, W_{\partial d})$ be the canonical commutative unitary-operator tuple associated with $\underline{T}$ as in (2.24). Then the tuple $(T_1, \ldots, T_d, T)$ is unitarily equivalent to*

$$P_{\mathcal{H}_D}(M_{G_{11}^* + zG_{12}} \oplus W_{\partial 1}, \ldots, M_{G_{d1}^* + zG_{d2}} \oplus W_{(\partial d)}, M_z \oplus W_D)|_{\mathcal{H}_D}, \tag{6.19}$$

and $(T_{(1)}, \ldots, T_{(d)}, T)$ is unitarily equivalent to

$$P_{\mathcal{H}_D}(M_{G_{12}^* + zG_{11}} \oplus W_{(1)}, \ldots, M_{G_{d2}^* + zG_{d1}} \oplus W_{(d)}, M_z \oplus W_D)|_{\mathcal{H}_D}, \tag{6.20}$$

where $\mathcal{H}_D = (H^2(\mathcal{D}_{T^*}) \oplus \mathcal{R}_D) \ominus \operatorname{Ran} \Pi_D$.

We saw in the above proof of uniqueness that the unitary involved in two pseudo-commutative contractive lifts $(\Pi_1, \mathcal{K}_1, \underline{S}, V_1)$ and $(\Pi_2, \mathcal{K}_2, \underline{R}, V_2)$ is the same unitary that is involved in the unitary equivalence of the two minimal isometric lifts $(\Pi_1, \mathcal{K}_1, V_1)$ and $(\Pi_2, \mathcal{K}_2, V_2)$ of T. Since such a unitary is unique (see the proof of Theorem I.4.1 in [42]), we have the following consequence of Theorem 6.3.

Corollary 6.5 *Let $\underline{T} = (T_1, T_2, \dots, T_d)$ be a d-tuple of commutative contractions acting on a Hilbert space and $T = T_1 T_2 \cdots T_d$. If $(\Pi_1, \mathcal{K}_1, \underline{S}, V_1)$ and $(\Pi_2, \mathcal{K}_2, \underline{R}, V_2)$ be two pseudo-commutative contractive lifts of $(T_1, T_2, \dots, T_d, T)$ such that $(\Pi_1, \mathcal{K}_1, V_1) = (\Pi_2, \mathcal{K}_2, V_2)$, then $\underline{S} = \underline{R}$.*

We end this section with another model for tuples of commutative contractive operator-tuples. This model will be used crucially in the next section where we analyze characteristic tuples for a given commutative contractive operator-tuple.

Sz.-Nagy and Foias gave a concrete functional model for the minimal isometric dilation for the case of a completely nonunitary (c.n.u.) contraction (see [42] for a comprehensive treatment). In their construction of this functional model appears what they called the *characteristic function* for a contraction operator T on a Hilbert space $\mathcal{H}$, a contractive analytic function on the unit disk $\mathbb{D}$ defined explicitly in terms of T via the formula

$$\Theta_T(z) := [-T + z D_{T^*}(I_{\mathcal{H}} - zT^*)^{-1} D_T]|_{\mathcal{D}_T} : \mathcal{D}_T \to \mathcal{D}_{T^*} \text{ for } z \in \mathbb{D}. \tag{6.21}$$

Also key to their analysis is the so-called *defect* of the characteristic function Δ_T defined a.e. on the unit circle $\mathbb{T}$ as

$$\Delta_T(\zeta) := (I - \Theta_T(\zeta)^* \Theta_T(\zeta))^{1/2}, \tag{6.22}$$

where $\Theta_T(\zeta)$ is the radial limit of the characteristic function. There it is shown that (Π_{NF}, V_{NF}) is a minimal isometric dilation of T, where V_{NF} is the isometry

$$V_{NF} := M_z \oplus M_\zeta|_{\overline{\Delta_T L^2(\mathcal{D}_T)}} \text{ on } \mathcal{K}_{NF} := H^2(\mathcal{D}_{T^*}) \oplus \overline{\Delta_T L^2(\mathcal{D}_T)} \tag{6.23}$$

and $\Pi_{NF} : \mathcal{H} \to \mathcal{K}_{NF}$ is some isometry with

$$\mathcal{H}_{NF} := \operatorname{Ran} \Pi_{NF} = \begin{bmatrix} H^2(\mathcal{D}_{T^*}) \\ \overline{\Delta_T L^2(\mathcal{D}_T)} \end{bmatrix} \ominus \begin{bmatrix} \Theta_T \\ \Delta_T \end{bmatrix} \cdot H^2(\mathcal{D}_T). \tag{6.24}$$

It is shown in [13] that, in case T is completely nonunitary, the isometric embedding Π_{NF} has the explicit formula

$$\Pi_{NF} = (I_{H^2 \otimes \mathcal{D}_{T^*}} \oplus u_{\min}) \Pi_D, \tag{6.25}$$

where $u_{\min} : \mathcal{R}_D \to \overline{\Delta_T L^2(\mathcal{D}_T)}$ is a unitary that intertwines W_D and $M_\zeta|_{\overline{\Delta_T L^2(\mathcal{D}_T)}}$. Let us introduce the notation

$$W_{\sharp j} := u_{\min} W_{\partial j} u_{\min}^*, \quad U_{\min} := ((I_{H^2} \otimes \mathcal{D}_{T^*}) \oplus u_{\min} \tag{6.26}$$

for unitary operators $W_{\sharp j}$ on $\overline{\Delta_T L^2(\mathcal{D}_T)}$ for $j = 1, \dots, d$ and a unitary operator $U_{\min}\colon \mathcal{R}_D \to \overline{\Delta_T L^2(\mathcal{D}_T)}$. Then we have

$$U_{\min} V_D = V_{NF} U_{\min} \text{ and } U_{\min} \Pi_D = \Pi_{NF}.$$

Using this relation between Π_D and Π_{NF} we have the following intertwining relations that follow from (6.6) and (6.7), respectively.

$$\Pi_{NF} T_i^* = (M^*_{G^*_{i1}+zG_{i2}} \oplus W^*_{\sharp i})\Pi_{NF}, \quad \Pi_{NF} T_{(i)}^* = (M^*_{G^*_{i2}+zG_{i1}} \oplus W^*_{(\sharp i)})\Pi_{NF}. \tag{6.27}$$

Equations (6.27) then provide us a Sz.-Nagy–Foias type functional model for tuples of commutative contractions.

Theorem 6.6 *Let $\underline{T} = (T_1, T_2, \dots, T_d)$ be a d-tuple of commutative contractions on a Hilbert space $\mathcal{H}$ such that the contraction operator $T = T_1 T_2 \cdots T_d$ is c.n.u. Let $\{G_{i1}, G_{i2} : i = 1, \dots, d\}$ be the fundamental operators of the adjoint tuple $\underline{T}^* = (T_1^*, T_2^*, \dots, T_d^*)$ and let the model space $\mathcal{H}_{NF}$ be as in (6.24). Then $(T_1, \dots, T_d, T)$ is unitarily equivalent to*

$$P_{\mathcal{H}_{NF}}(M_{G^*_{11}+zG_{12}} \oplus W_{\sharp 1}, \dots, M_{G^*_{d1}+zG_{d2}} \oplus W_{\sharp d}, M_z \oplus M_\zeta|_{\overline{\Delta_T L^2(\mathcal{D}_T)}})|_{\mathcal{H}_{NF}}, \tag{6.28}$$

and $(T_{(1)}, \dots, T_{(d)}, T)$ is unitarily equivalent to

$$P_{\mathcal{H}_{NF}}(M_{G^*_{12}+zG_{11}} \oplus W_{(\sharp 1)}, \dots, M_{G^*_{d2}+zG_{d1}} \oplus W_{(\sharp d)}, M_z \oplus M_\zeta|_{\overline{\Delta_T L^2(\mathcal{D}_T)}})|_{\mathcal{H}_{NF}}. \tag{6.29}$$

Remark 6.7 Equations (6.27) also provide us another model for pseudo-commutative contractive lifts, at least for the case where the $T = T_1 \cdots T_d$ is c.n.u. Indeed, let $\underline{T} = (T_1, T_2, \dots, T_d)$ be a d-tuple of commutative contractions on a Hilbert space $\mathcal{H}$, $T = T_1 T_2 \cdots T_d$ and let $\{G_{i1}, G_{i2} : i = 1, \dots, d\}$ be the set of fundamental operators of the adjoint tuple $\underline{T}^* = (T_1^*, T_2^*, \dots, T_d^*)$. Let us set

$$\begin{aligned} \underline{S}^{NF} &:= (S_1^{NF}, S_2^{NF}, \dots, S_d^{NF}) \\ &:= (M_{G^*_{11}+zG_{12}} \oplus W_{\sharp 1}, M_{G^*_{21}+zG_{22}} \oplus W_{\sharp 2}, \dots, M_{G^*_{d1}+zG_{d2}} \oplus W_{\sharp d}) \qquad (6.30) \\ \underline{S'}^{NF} &:= (S_1'^{NF}, S_2'^{NF}, \dots, S_d'^{NF}), \\ &:= (M_{G^*_{12}+zG_{11}} \oplus W_{(\sharp 1)}, M_{G^*_{22}+zG_{21}} \oplus W_{(\sharp 2)} \dots, M_{G^*_{d2}+zG_{d1}} \oplus W_{(\sharp d)}). \qquad (6.31) \end{aligned}$$

Then it follows from the definition and from $M_\zeta|_{\overline{\Delta_T L^2(\mathcal{D}_T)}} = W_{\sharp 1} W_{\sharp 2} \cdots W_{\sharp d}$ that $S_j'^{NF} = S_j^{NF*} V_{NF}$ for each $j = 1, 2, \ldots, d$, where V_{NF} is the minimal isometric lift of T as defined in (6.23). Hence by Eqs. (6.27) it follows that *the tuple* $(\Pi_{NF}, \mathcal{K}_{NF}, \underline{S}^{NF}, V_{NF})$ *is a pseudo-commutative contractive lift of* $(T_1, T_2, \ldots, T_d, T)$.

7 Characteristic Triple for a Tuple of Commutative Contractions

Let $\underline{T} = (T_1, T_2, \ldots, T_d)$ be a d-tuple of commutative contractions on a Hilbert space $\mathcal{H}$ and $(\mathcal{F}_*, \Lambda_{j*}, P_{j*}, U_{j*})_{j=1}^d$ be an Andô tuple for $\underline{T}^* = (T_1^*, T_2^*, \ldots, T_d^*)$. Let $\{G_{j1}, G_{j2} : j = 1, 2, \ldots, d\}$ be the set of fundamental operators of $\underline{T}^*$. Then note that by part (2) of Theorem 4.4 we have

$$(G_{j1}, G_{j2}) = \Lambda_{1*}^*(\tau_{j*} P_{j*}^\perp U_{j*}^* \tau_{j*}^*, \tau_{j*} U_{j*} P_{j*} \tau_{j*}^*)\Lambda_{1*} \text{ for } j = 1, 2, \ldots, d. \tag{7.1}$$

Definition 7.1 Let $\underline{T} = (T_1, T_2, \ldots, T_d)$ be a d-tuple of commutative contractions on a Hilbert space, $\mathbb{G}_\sharp := \{G_{j1}, G_{j2} : j = 1, 2, \ldots, d\}$ be the set of fundamental operators of $\underline{T}^*$ and $\mathbb{W}_\sharp := (W_{\sharp 1}, W_{\sharp 2}, \ldots, W_{\sharp d})$ be the tuple of commutative unitaries as in (6.26). The triple $(\mathbb{G}_\sharp, \mathbb{W}_\sharp, \Theta_T)$ is called the characteristic triple for $\underline{T}$, where Θ_T is the characteristic function for the contraction $T = T_1 T_2 \cdots T_d$.

Note that the expression (7.1) of the fundamental operators of $\underline{T}^*$ indicates the dependence of the characteristic triple on a choice of an Andô tuple for $\underline{T}^*$. However, the uniqueness part of Theorem 3.2 says that the fundamental operators are uniquely determined by $\underline{T}$. Consequently, the characteristic triple, despite its apparent dependence on a choice of an Andô tuple, turns out to be uniquely determined already by the d-tuple $\underline{T}$ of commutative contractions. In fact, as Theorem 7.3 below explains, the characteristic triple (up to the natural notion of equivalence to be defined next) is a complete unitary invariant for tuples of commutative contractions.

Definition 7.2 Let $(\mathcal{D}, \mathcal{D}_*, \Theta)$, $(\mathcal{D}', \mathcal{D}'_*, \Theta')$ be two purely contractive analytic functions. Let $\mathbb{G} = \{G_{j1}, G_{j2} : j = 1, 2, \ldots, d\}$ on $\mathcal{D}_*$, $\mathbb{G}' = \{G'_{j1}, G'_{j2} : j = 1, 2, \ldots, d\}$ on $\mathcal{D}'_*$ be two sets of contraction operators and $\mathbb{W} = (W_1, W_2, \ldots, W_d)$ on $\overline{\Delta_\Theta L^2(\mathcal{D})}$, $\mathbb{W}' = (W'_1, W'_2, \ldots, W'_d)$ on $\overline{\Delta_{\Theta'} L^2(\mathcal{D}')}$ be two tuples of commutative unitaries such that their product is M_ζ on the respective spaces. We say that the two triples $(\mathbb{G}, \mathbb{W}, \Theta)$ and $(\mathbb{G}', \mathbb{W}', \Theta')$ *coincide* if:

(i) $(\mathcal{D}, \mathcal{D}_*, \Theta)$ and $(\mathcal{D}', \mathcal{D}'_*, \Theta')$ *coincide*, i.e., there exist unitary operators $u : \mathcal{D} \to \mathcal{D}$ and $u_* : \mathcal{D}_* \to \mathcal{D}_*$ such that the diagram

$$\begin{array}{ccc} \mathcal{D}_T & \xrightarrow{\Theta(z)} & \mathcal{D}_{T^*} \\ u\big\downarrow & & \big\downarrow u_* \\ \mathcal{D}_{T'} & \xrightarrow[\Theta'(z)]{} & \mathcal{D}_{T'^*} \end{array} \tag{7.2}$$

commutes for each $z \in \mathbb{D}$.

(ii) The same unitary operators u, u_* as in part (i) satisfy the additional intertwining conditions:

$$\mathbb{G}' = (G'_1, G'_2) = u_* \mathbb{G} u_*^* = (u_* G_1 u_*^*, u_* G_2 u_*^*, \ldots, u_* G_d u_*^*),$$
$$\mathbb{W}' = (W'_1, W'_2) = \omega_u \mathbb{W} \omega_u^* = (\omega_u W_1 \omega_u^*, \omega_u W_2 \omega_u^*, \ldots, \omega_u W_d \omega_u^*),$$

where $\omega_u : \overline{\Delta_\Theta L^2(\mathcal{D})} \to \overline{\Delta_{\Theta'} L^2(\mathcal{D})}$ is the unitary map induced by u according to the formula

$$\omega_u := (I_{L^2} \otimes u)|_{\overline{\Delta_\Theta L^2(\mathcal{D})}}. \tag{7.3}$$

Theorem 7.3 *Let $\underline{T} = (T_1, T_2, \ldots, T_d)$ on $\mathcal{H}$ and $\underline{T}' = (T'_1, T'_2, \ldots, T'_d)$ on $\mathcal{H}'$ be two tuples of commutative contractions. Let $(\mathbb{G}_\sharp, \mathbb{W}_\sharp, \Theta_T)$ and $(\mathbb{G}'_\sharp, \mathbb{W}'_\sharp, \Theta_{T'})$ be the characteristic triples of $\underline{T}$ and $\underline{T}'$, respectively, where $T = T_1 T_2 \ldots, T_d$ and $T' = T'_1 T'_2 \cdots, T'_d$. If $\underline{T}$ and $\underline{T}'$ are unitarily equivalent, then $(\mathbb{G}_\sharp, \mathbb{W}_\sharp, \Theta_T)$ and $(\mathbb{G}'_\sharp, \mathbb{W}'_\sharp, \Theta_{T'})$ coincide.*

Conversely, suppose in addition that T and T' are c.n.u. with characteristic triples $(\mathbb{G}_\sharp, \mathbb{W}_\sharp, \Theta_T)$ and $(\mathbb{G}'_\sharp, \mathbb{W}'_\sharp, \Theta_{T'})$ coinciding. Then $\underline{T}$ and $\underline{T}'$ are unitarily equivalent.

Proof First let us suppose that $\underline{T}$ and $\underline{T}'$ be unitarily equivalent via a unitary similarity $U : \mathcal{H} \to \mathcal{H}'$. Then

$$U(I - T^*T) = (I - T'^*T')U \text{ and } U(I - TT^*) = (I - T'T'^*)U \tag{7.4}$$

and the functional calculus for positive operators implies that U induces two unitary operators

$$u := U|_{\mathcal{D}_T} : \mathcal{D}_T \to \mathcal{D}_{T'} \text{ and } u_* := U|_{\mathcal{D}_{T^*}} : \mathcal{D}_{T^*} \to \mathcal{D}_{T'^*}. \tag{7.5}$$

A consequence of the Sz.-Nagy–Foias theory [42] is that $u_* \Theta_T u^* = \Theta_{T'}$ showing Θ_T and $\Theta_{T'}$ coincide, i.e., condition (i) holds.

As for condition (ii), note that since the fundamental operators satisfy the fundamental equations (3.10), one can easily deduce using (7.4) that

$$u_*(G_{j1}, G_{j2}) = (G'_{j1}, G'_{j2})u_* \text{ for each } j = 1, 2, \dots, d, \tag{7.6}$$

where $\mathbb{G}_\sharp = \{G_{j1}, G_{j2} : j = 1, 2, \dots, d\}$ and $\mathbb{G}'_\sharp = \{G'_{j1}, G'_{j2} : j = 1, 2, \dots, d\}$. It remains to establish the unitary equivalence of $\mathbb{W}_\sharp$ and $\mathbb{W}'_\sharp$ via $\omega_u = (I_{L^2} \otimes u)|_{\overline{\Delta_T L^2(\mathcal{D}_T)}}$. To this end, we consider the tuple $(\widetilde{\Pi}, \widetilde{\mathcal{K}}, \underline{\widetilde{S}}, \underline{\widetilde{S}'}, \widetilde{V})$, where $\widetilde{\mathcal{K}} = \mathcal{K}_{NF}$ and $\widetilde{V} = V_{NF}$ as in (6.23) and where

$$\begin{aligned}
\widetilde{\Pi} :=& ((I_{H^2} \otimes u_*^*) \oplus \omega_u^*)\Pi'_{NF} U : \mathcal{H} \to H^2(\mathcal{D}_{T^*}) \oplus \overline{\Delta_T L^2(\mathcal{D}_T)} \\
\underline{\widetilde{S}} :=& (\widetilde{S}_1, \widetilde{S}_2, \dots, \widetilde{S}_d) := \\
& (M_{G_{11}^* + zG_{12}} \oplus W''_1, M_{G_{21}^* + zG_{22}} \oplus W''_2, \dots, M_{G_{d1}^* + zG_{d2}} \oplus W''_d) \\
\underline{\widetilde{S}'} :=& (\widetilde{S}'_1, \widetilde{S}'_2, \dots, \widetilde{S}'_d) := \\
& (M_{G_{12}^* + zG_{11}} \oplus W''_{(1)}, M_{G_{22}^* + zG_{21}} \oplus W''_{(2)} \dots, M_{G_{d2}^* + zG_{d1}} \oplus W''_{(d)})
\end{aligned} \tag{7.7}$$

with

$$\mathbb{W}'' := (W''_1, W''_2, \dots, W''_d) := \omega_u \mathbb{W}_\sharp \omega_u^* = (\omega_u W_{\sharp 1}\omega_u^*, \omega_u W_{\sharp 2}\omega_u^*, \dots, \omega_u W_{\sharp d}\omega_u^*).$$

By tracing through the intertwining properties of the unitary identification maps U and ω_u, one can see that actually $\widetilde{\Pi} = \Pi_{NF}$. A further consequence of these intertwining properties is that the tuple

$$(\Pi_{NF}, \mathcal{K}_{NF}, \underline{S}^{NF}, \underline{S}'^{NF}, V_{NF})$$

as in (6.30) and (6.30) being a pseudo-commutative contractive lift of $\underline{T}$ implies that $(\Pi, \mathcal{K}_{NF}, \underline{\widetilde{S}}, \underline{\widetilde{S}}', V_{NF})$ is a pseudo-commutative contractive lift of $\underline{T}$ as well. A direct application of Corollary 6.5 then tells us that $\mathbb{W}'' = \mathbb{W}_\sharp$, i.e.,

$$\omega_u W_{\sharp j}\omega_u^* = W_{\sharp j} \text{ for } j = 1, \dots, d$$

and condition (ii) in Definition 7.2 is now verified as wanted.

Conversely, assume that the product operators T and T' are c.n.u. and $\underline{T}$ and $\underline{T}'$ have characteristic triples which coincide. By Theorem 6.6 each of $\underline{T}$ and $\underline{T}'$ is unitarily equivalent to its respective Sz.-Nagy–Foias functional model. It is now a straightforward exercise to see that the unitary identification maps u and u_* in the definition of the coincidence of the characteristic triples leads to a unitary identification of the model spaces $\mathcal{H}_{NF}$ and $\mathcal{H}'_{NF}$ which also implements a unitary similarity of the respective model operator tuples (6.28) and (6.29) associated with $\underline{T}$ and $\underline{T}'$. This completes the proof of Theorem 7.3. □

We next introduce the notion of *admissible triple* for a collection $\{\mathbb{G}, \mathbb{W}, \Theta\}$ of the same sort as appearing in Definition 7.2 but which satisfies some additional conditions; the additional conditions correspond to what is needed to conclude that the triple arises as the characteristic triple for some contractive commutative tuple $\underline{T}$.

Definition 7.4 Suppose that $(\mathcal{D}, \mathcal{D}_*, \Theta)$ is a purely contractive analytic function, $\mathbb{G} = \{G_{j1}, G_{j2} \colon j = 1, \dots, d\}$ is a collections of operators on $\mathcal{D}_*$, $\mathbb{W} = \{W_1, \dots, W_d\}$ is a commutative d-tuple of unitary operators on $\overline{\Delta_\Theta L^2(\mathcal{D})}$ such that:

1. Each $M_{G_{j1}^* + zG_{j2}}$ is a contraction operator on $H^2(\mathcal{D}_*)$.
2. $W_1 \cdots W_d = M_\zeta|_{\overline{\Delta_\Theta L^2(\mathcal{D})}}$.
3. The space $\mathcal{Q}_\Theta := \begin{bmatrix} \Theta \\ \Delta_\Theta \end{bmatrix} H^2(\mathcal{D}) \subset \begin{bmatrix} H^2(\mathcal{D}_*) \\ \overline{\Delta_\Theta L^2(\mathcal{D})} \end{bmatrix}$ is jointly invariant for the operator tuple $\left\{ \begin{bmatrix} M_{G_{j1}^* + zG_{j2}} & 0 \\ 0 & W_j \end{bmatrix} : j = 1, \dots, d \right\}$.
4. With $\mathcal{K}_\Theta = \begin{bmatrix} H^2(\mathcal{D}_*) \\ \overline{\Delta_\Theta L^2(\mathcal{D})} \end{bmatrix}$ and $\mathcal{H}_\Theta = \mathcal{K}_\Theta \ominus \mathcal{Q}_\Theta$ and with operators $\mathbf{T}_j$ on $\mathcal{H}_\Theta$ defined by

$$\mathbf{T}_j = P_{\mathcal{H}(\Theta)} \begin{bmatrix} M_{G_{j1}^* + zG_{j2}} & 0 \\ 0 & W_j \end{bmatrix} |_{\mathcal{H}(\Theta)} \text{ for } j = 1, \dots, d, \tag{7.8}$$

   the operator-tuple $(\mathbf{T}_1, \dots, \mathbf{T}_d)$ is commutative with product (in any order) then given by

$$\mathbf{T}_1 \cdots \mathbf{T}_d = P_{\mathcal{H}(\Theta)} \begin{bmatrix} M_z & 0 \\ 0 & M_\zeta \end{bmatrix} |_{\mathcal{H}(\Theta)}.$$

Then we shall say that the collection $\{\mathbb{G}, \mathbb{W}, \Theta\}$ is an *admissible triple* and that the commutative contractive operator-tuple $\underline{\mathbf{T}} = (\mathbf{T}_1, \dots, \mathbf{T}_d)$ acting on the space $\mathcal{H}(\Theta)$ (7.8) is the *functional model* associated with the admissible triple $\{\mathbb{G}, \mathbb{W}, \Theta\}$.

Let us note that the functional model associated with an admissible triple $\{\mathbb{G}, \mathbb{W}, \Theta\}$ also displays a pseudo-commutative contractive lift for its functional-model commutative, contractive operator tuple $\mathbf{T}$, namely:

$$\mathbf{S} := \left\{ \begin{bmatrix} M_{G_{j1}^* + zG_{j2}} & 0 \\ 0 & W_j \end{bmatrix} : j = 1, \dots, d \right\}, \quad V = \begin{bmatrix} M_z & 0 \\ 0 & M_\zeta \end{bmatrix}$$

$$\mathbf{S}' := \left\{ \begin{bmatrix} M_{G_{j2}^* + zG_{j1}} & 0 \\ 0 & W_{(j)} \end{bmatrix} : j = 1, \dots, d \right\}.$$

Note also that it easily follows from the definitions that the characteristic triple for a commutative contractive d-tuple $\underline{T}$ is an admissible triple. Furthermore the functional model associated with the characteristic triple $(\mathbb{G}_\sharp, \mathbb{W}_\sharp, \Theta_T)$ is the same as the functional model obtained by considering $(\mathbb{G}_\sharp, \mathbb{W}_\sharp, \Theta_T)$ as an admissible triple. The content of Theorem 6.6 is that any commutative contractive tuple $\underline{T}$ is unitarily equivalent to its associated functional model $\underline{\mathbf{T}}$.

Our next goal is to indicate the reverse path: how to go from an admissible triple to a characteristic triple for some commutative contractive pair $\underline{T}$. We state the result without proof.

Theorem 7.5 *If* $(\mathbb{G}, \mathbb{W}, \Theta)$ *is an admissible triple, then* $(\mathbb{G}, \mathbb{W}, \Theta)$ *is a characteristic triple for some contractive operator tuple. More precisely, the admissible triple* $(\mathbb{G}, \mathbb{W}, \Theta)$ *coincides with the characteristic triple* $(\mathbb{G}_\sharp, \mathbb{W}_\sharp, \Theta_{\mathbf{T}})$ *of its functional model.*

Since model theory and unitary classification for commuting tuples of unitary operators can be handled by spectral theory, the importance of the next result is that the c.n.u. restriction on $T = T_1 \cdots T_d$ appearing in Theorem 6.6 and Theorem 7.3 is not essential. This result for the case $d = 1$ goes back to Sz.-Nagy et al. [42].

Theorem 7.6 *Let* $\underline{T} = (T_1, T_2, \dots, T_d)$ *be a commutative contractive operator-tuple acting on a Hilbert space* $\mathcal{H}$*. Then there corresponds a decomposition of* $\mathcal{H}$ *into the orthogonal sum of two subspaces reducing each* T_j*,* $j = 1, 2, \dots, d$*, say* $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_c$*, such that with*

$$\begin{aligned} (T_{1u}, T_{2u}, \dots, T_{du}) &= (T_1, T_2, \dots, T_d)|_{\mathcal{H}_u}, \\ (T_{1c}, T_{2c}, \dots, T_{dc}) &= (T_1, T_2, \dots, T_d)|_{\mathcal{H}_c}, \end{aligned} \tag{7.9}$$

$T_u = T_{1u} T_{2u} \cdots T_{du}$ *is a unitary and* $T_c = T_{1c} T_{2c} \cdots T_{dc}$ *is a completely nonunitary contraction. Moreover, then* $T_u \oplus T_c$ *with respect to* $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_c$ *is the Sz.-Nagy–Foias canonical decomposition for the contraction operator* $T = T_1 T_2 \cdots T_d$*.*

Proof Let $\{F_{j1}, F_{j2} : j = 1, 2, \dots, d\}$ be the set of fundamental operators of $\underline{T}$. Then by Theorem 3.2, for each $j = 1, 2, \dots, d$, we have

$$T_j - T_{(j)}^* T = D_T F_{j1} D_T \text{ and } T_{(j)} - T_j^* T = D_T F_{j2} D_T. \tag{7.10}$$

By part (1) of Theorem 4.4 each of F_{j1} and F_{j2} are contractions. Consequently, we have for every ω and ζ in $\mathbb{T}$

$$I_{\mathcal{D}_T} - \mathrm{Re}(\omega F_{j1}) \geq 0 \text{ and } I_{\mathcal{D}_T} - \mathrm{Re}(\zeta F_{j2}) \geq 0. \tag{7.11}$$

Adding together the two inequalities (7.11) then gives

$$2 I_{\mathcal{D}_T} - \mathrm{Re}(\omega F_{j1} + \zeta F_{j2}) \geq 0 \text{ for all } \omega, \zeta \in \mathbb{T}. \tag{7.12}$$

Recall that the fundamental operators act on $\mathcal{D}_T = \overline{\text{Ran}}\ D_T$. Therefore inequality (7.12) is equivalent to

$$2D_T^2 - \text{Re}(\omega D_T F_{j1} D_T + \zeta D_T F_{j2} D_T) \geq 0, \text{ for all } \omega, \zeta \in \mathbb{T},$$

By (7.10) we see that this in turn is the same as

$$2D_T^2 - \text{Re}(\omega(T_j - T_{(j)}^* T)) - \text{Re}(\zeta(T_{(j)} - T_j^* T)) \geq 0, \text{ for all } \omega, \zeta \in \mathbb{T}. \tag{7.13}$$

Let

$$T = \begin{bmatrix} T_u & 0 \\ 0 & T_c \end{bmatrix} : \mathcal{H}_u \oplus \mathcal{H}_c \to \mathcal{H}_u \oplus \mathcal{H}_c \tag{7.14}$$

be the canonical decomposition of T into unitary piece T_u and completely nonunitary piece T_c. We show below that each T_j is block diagonal with respect to the decomposition $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_c$. Toward this end, we first suppose that with respect to the decomposition $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_c$ for each $j = 1, 2, \ldots d$ we have

$$T_j = \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix} \text{ and } T_{(j)} = \begin{bmatrix} E_j & K_j \\ L_j & H_j \end{bmatrix}. \tag{7.15}$$

Apply (7.13) to obtain that for each $j = 1, 2, \ldots, d$,

$$\begin{bmatrix} 0 & 0 \\ 0 & 2D_{T_c}^2 \end{bmatrix} - \text{Re}\left(\omega \begin{bmatrix} A_j - E_j^* T_u & B_j - L_j^* T_c \\ C_j - K_j^* T_u & D_j - H_j^* T_c \end{bmatrix}\right) - \text{Re}\left(\zeta \begin{bmatrix} E_j - A_j^* T_u & K_j - C_j^* T_c \\ L_j - B_j^* T_u & H_j - D_j^* T_c \end{bmatrix}\right) \geq 0, \text{ for all } \omega, \zeta \in \mathbb{T}. \tag{7.16}$$

In particular, the (1, 1)-entry in this inequality must satisfy

$$\mathbb{P}_{11}^j(\omega, \zeta) := \text{Re}(\omega(A_j - E_j^* T_u)) + \text{Re}(\zeta(E_j - A_j^* T_u)) \leq 0, \text{ for all } \omega, \zeta \in \mathbb{T}, \tag{7.17}$$

which implies that

$$\mathbb{P}_{11}^j(\omega, 1) + \mathbb{P}_{11}^j(\omega, -1) = 2\,\text{Re}(\omega(A_j - E_j^* T_u)) \leq 0 \text{ and}$$
$$\mathbb{P}_{11}^j(1, \zeta) + \mathbb{P}_{11}^j(-1, \zeta) = 2\,\text{Re}(\zeta(E_j - A_j^* T_u)) \leq 0.$$

It is an elementary exercise to show that, if a bounded operator X such that $Re(\zeta X) \leq 0$ for all $\zeta \in \mathbb{T}$, then $X = 0$ (see e.g. Lemma 2.4 in [25]). We apply this fact to conclude that

$$A_j = E_j^* T_u \text{ and } E_j = A_j^* T_u \text{ for each } j = 1, 2, \ldots, d. \tag{7.18}$$

This shows that the (1, 1)-entry of the matrix on the left-hand side of (7.16) is zero. Since the matrix is positive semi-definite, the (1, 2)-entry (and hence also the (2, 1)-entry) is also zero, i.e., for all $\omega, \zeta \in \mathbb{T}$

$$\mathbb{P}_{12}^j(\omega, \zeta) := \omega(B_j - L_j^* T_c) + \bar{\omega}(C_j^* - T_u^* K_j) + \zeta(K_j - C_j^* T_c) + \bar{\zeta}(L_j^* - T_u^* B_j) = 0,$$

which in particular implies that

$$\mathbb{P}^j(\omega) := \mathbb{P}_{12}^j(\omega, 1) + \mathbb{P}_{12}^j(\omega, -1) = 2\omega(B_j - L_j^* T_c) + 2\bar{\omega}(C_j^* - T_u^* K_j) = 0$$

for every $\omega \in \mathbb{T}$. This implies the first two of the following equations while the last two are obtained similarly:

$$B_j = L_j^* T_c, \quad C_j^* = T_u^* K_j, \quad K_j = C_j^* T_c \quad \text{and} \quad L_j^* = T_u^* B_j. \tag{7.19}$$

Commutativity of each T_j with T gives

$$A_j T_u = T_u A_j, \quad B_j T_c = T_u B_j, \quad C_j T_u = T_c C_j \quad \text{and} \quad T_c D_j = D_j T_c, \tag{7.20}$$

while commutativity of $T_{(j)}$ with T implies

$$E_j T_u = T_u E_j, \quad K_j T_c = T_u K_j, \quad L_j T_u = T_c L_j \quad \text{and} \quad T_c H_j = H_j T_c. \tag{7.21}$$

Using the last equation in (7.19) and the third equation in (7.21) we get

$$B_j^* T_u^2 = L_j T_u = T_c L_j = T_c B_j^* T_u.$$

As T_u is unitary, this leads to

$$B_j^* T_u = T_c B_j^*. \tag{7.22}$$

Using the second equality in (7.20) together with (7.22) leads to

$$T_c T_c^* B_j^* = T_c B_j^* T_u^* = B_j^* = B_j^* T_u^* T_u = T_c^* B_j^* T_u = T_c^* T_c B_j^*,$$

which implies that T_c is unitary on $\overline{\text{Ran}\, B_j^*}$ for $j = 1, 2, \ldots, d$. Since T_c is completely nonunitary, each B_j must be zero. By similar arguments one can show that $C_j = 0$, for each $j = 1, 2, \ldots, d$. This completes the proof. □

Remark 7.7 We note that a proof of Theorem 7.6 is given in [13] for the pair case ($d = 2$). It is of interest to note that the general case can be reduced to the pair case simply by applying the result for the pair case to the special pair $(T_j, T_{(j)})$ for each $j = 1, \dots, d$. Our proof on the other hand is a direct multivariable proof.

Remark 7.8 (Examples and Special Cases) If we consider the special case with $\underline{T} = \underline{V}$ is a commutative tuple of isometries $\underline{V} = (V_1, \dots, V_d)$ with product operator $V = V_1 \cdots V_d$ c.n.u. (i.e., V is a pure isometry or shift operator), then the associated characteristic function Θ_V is zero, and the model theory presented here amounts to the BCL-model for commuting isometries as in Theorem 2.3. In detail, the characteristic triple collapses to the first component $\mathbb{G}$ which has the additional structure of the form

$$G_{j1} = P_j^{\perp} U_j^*, \quad G_{j2} = U_j P_j$$

for a collection of projection operators P_j and unitary operators U_j on a space $\mathcal{F}$ ($j = 1, \dots, d$) forming a BCL-tuple (Definition 2.4) for which the associated isometric operator-tuple V_j (with W_j trivial for $j = 1, \dots, d$) is commutative. The difficulty in writing down examples is that there is no explicit way to write down such operator tuples $\mathbb{G}$ so that the associated isometric-tuple $\underline{V}$ is commutative.

As we have seen in Sect. 2.4, given such a collection of operators forming a BCL-tuple as in Definition 2.4 (with $\mathbb{W}$ taken to be trivial for simplicity), the operators $V_j = M_{G_{j1}^* + zG_{j2}}$ ($j = 1, \dots, d$) form an isometric tuple but there are no explicit criteria for deciding when it is the case that this is a commutative isometric tuple, unless $d = 1, 2$.

Similarly from Theorem 7.5, to construct examples of commutative contractive tuples, it suffices to construct examples of admissible triples. At its core, according to Definition 7.4, an admissible triple consists of a pure contractive operator function $(\mathcal{D}, \mathcal{D}_*, \Theta)$ together with a collection of operators $\mathbb{G} = \{G_{j1}, G_{j2} \colon 1 \le j \le d\}$ on $\mathcal{D}_*$, and a commutative unitary tuple $\mathbb{W} = \{W_j \colon 1 \le j \le d\}$ acting on $\overline{\Delta_\Theta L^2(\mathcal{D})}$ satisfying auxiliary conditions (1)–(5). While conditions (1) and (2) are not so difficult to analyze, the joint-invariance property in condition (3) and the joint-commutativity property in condition (4) are mysterious: for a general Θ there is no apparent way to write down interesting explicit examples of potential admissible triples $(\mathbb{G}, \mathbb{W}, \Theta)$ which satisfy these additional properties, even for the $d = 2$ case. In case Θ is inner, the $\mathbb{W}$-component becomes trivial, condition (1) is just the requirement that the operator pencil $G_j(z) = G_{j1}^* + zG_{j2}$ have H^∞-norm at most 1 but one is still left with the nontrivial requirement (3) that $M_{G_{j1}^* + zG_{j2}}$ leave the subspace $M_\Theta H^2(\mathcal{D})$ invariant. Unlike the case for the BCL-model for commutative isometric tuples, this flaw happens even in the $d = 2$ case.

An example which may be tractable is the case where the commutative contractive tuple $\underline{T} = (T_1, \dots, T_d)$ acts on a finite-dimensional Hilbert space $\mathcal{X}$ and has a basis of joint eigenvectors. Similar examples are discussed in [2, 16].

More detail on all these issues will appear in forthcoming work of the authors [13].

Acknowledgments This work was done when the second named author was visiting Virginia Tech as an SERB Indo-US Postdoctoral Research Fellow. He wishes to thank the Department of Mathematics, Virginia Tech for all the facilities provided to him.

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The Extended Aluthge Transform

Chafiq Benhida and Raul E. Curto

To the memory of Professor Ronald G. Douglas

Abstract Given a bounded linear operator T with canonical polar decomposition $T \equiv V|T|$, the Aluthge transform of T is the operator $\Delta(T) := \sqrt{|T|}V\sqrt{|T|}$. For P an arbitrary positive operator such that $VP = T$, we define the *extended* Aluthge transform of T associated with P by $\Delta_P(T) := \sqrt{P}V\sqrt{P}$. First, we establish some basic properties of Δ_P; second, we study the fixed points of the extended Aluthge transform; third, we consider the case when T is an idempotent; next, we discuss whether Δ_P leaves invariant the class of complex symmetric operators. We also study how Δ_P transforms the numerical radius and numerical range. As a key application, we prove that the spherical Aluthge transform of a commuting pair of operators corresponds to the extended Aluthge transform of a 2×2 operator matrix built from the pair; thus, the theory of extended Aluthge transforms yields results for spherical Aluthge transforms.

The first named author was partially supported by Labex CEMPI (ANR-11-LABX-0007-01). The second named author was partially supported by NSF Grant DMS-1302666 and by Labex CEMPI (ANR-11-LABX-0007-01).

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_3

Keywords Extended Aluthge transform · Polar decomposition · Extended quasinormal operator

Mathematics Subject Classification (2010) 2010

1 Introduction

The Aluthge transform for a bounded operator T acting on a Hilbert space $\mathcal{H}$ was introduced by A. Aluthge in [1]. If $T \equiv V|T|$ is the canonical polar decomposition of T, the Aluthge transform $\Delta(T)$ is given as $\Delta(T) := \sqrt{|T|}V\sqrt{|T|}$. One of Aluthge's motivations was to use this transform in the study of p-hyponormal and log-hyponormal operators. Roughly speaking, the idea was to convert an operator, T, into another operator, $\Delta(T)$, which shares with the first one many spectral properties, but which is closer to being a normal operator. Over the last two decades, substantial and significant results about $\Delta(T)$, and how it relates to T, have been obtained by a long list of mathematicians who devoted considerable attention to this topic (see, for instance [2, 4, 9, 12–15, 20–23, 25–27, 30–33]). Aluthge transforms have been generalized to the case of powers of $|T|$ different from $\frac{1}{2}$ [5, 7, 8, 28] and to the case of commuting pairs of operators [10, 11].

In this paper, we set out to extend the Aluthge transform in a different direction. Starting with the canonical polar decomposition $T \equiv V|T|$, we consider the class of positive operators P such that $VP = V|T|$, that is, all positive operators P that mimic the action of $|T|$ in the canonical polar decomposition. For each such P we then define the *extended* Aluthge transform as $\Delta_P(T) := \sqrt{P}V\sqrt{P}$. Naturally, the classical Aluthge transform is simply $\Delta_{|T|}(T)$.

We first study the basic properties of this new operator transform, and how it relates to the classical Aluthge transform. We do this in Sect. 2. We then study, in Sect. 3, the fixed points of the extended Aluthge transform, in an effort to see what is the correct generalization of quasinormality to this new environment. Third, in Sect. 4 we consider the case when T is an idempotent. Next, we discuss whether Δ_P leaves invariant the class of complex symmetric operators (Sect. 7).

We also study how Δ_P transforms the numerical radius and numerical range; we do this in Sect. 8. As a key application, we prove that the spherical Aluthge transform of a commuting pair of operators (introduced in [10] and further studied in [11]) corresponds to the extended Aluthge transform of a 2×2 operator matrix built from the pair; thus, the theory of extended Aluthge transforms is well positioned to yield new results for spherical Aluthge transforms.

Along the way, we strive to maintain contact with the classical Aluthge transform, in an effort to shed light on how this new extended Aluthge transform can help unravel the relative position of $|T|$ within the equation $V|T| = T$. For instance, we prove in Sect. 2 that $|T|$ is the smallest positive solution of the equation $VP = T$.

2 The Extended Aluthge Transform

Let $\mathcal{H}$ denote a (complex, separable) Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, let $T \equiv V|T|$ be the canonical polar decomposition of T; that is, $|T| := (T^*T)^{\frac{1}{2}}$, V is a partial isometry, and $\ker V = \ker |T| = \ker T$. The Aluthge transform of T is the operator $\Delta(T) := |T|^{\frac{1}{2}} V |T|^{\frac{1}{2}}$.

Consider now an arbitrary positive operator $P \in \mathcal{B}(\mathcal{H})$ such that $VP = T$. The *extended Aluthge transform* of T associated with P is the operator

$$\Delta_P(T) := P^{\frac{1}{2}} V P^{\frac{1}{2}}.$$

Lemma 2.1 *For P as above,*

$$|T| \le P.$$

Proof

$$|T|^2 = T^*T = PV^*VP \le P^2,$$

since V is a contraction. It follows that $|T| \le P$. □

Corollary 2.2 *For P as above,*

$$\ker P \subseteq \ker |T|.$$

Corollary 2.3 *For P as above,*

$$\overline{\operatorname{Ran}|T|} \subseteq \overline{\operatorname{Ran} P},$$

where $\overline{\mathcal{M}}$ denotes the closure of the linear space $\mathcal{M}$.

Lemma 2.4 *For P as above, $|T|$ commutes with P.*

Proof

$$V|T| = VP \Longrightarrow (|T| - P)\mathcal{H} \subseteq \ker V = \ker |T|.$$

It follows that

$$\begin{aligned} |T|(|T| - P) = 0 \Longrightarrow |T|^2 = |T|P \Longrightarrow |T|^2 = (|T|^2)^* \\ = (|T|P)^* = P|T| \Longrightarrow |T|P = P|T|. \end{aligned}$$ □

Lemma 2.5 *For P as above,*

$$P|_{\operatorname{Ran}|T|} = |T|\,|_{\operatorname{Ran}|T|}.$$

Proof By the Proof of Lemma 2.4, we have

$$P\,|T|\,x = |T|\,Px = |T|\,|T|\,x \quad (\text{all } x \in \mathcal{H})\,.$$

It follows that P and $|T|$ agree on Ran $|T|$. □

Lemma 2.6 *Write* $\mathcal{H} = \overline{\text{Ran}\,|T|} \oplus \ker T$. *Then*

$$|T| = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where $A := |T|\,|_{\overline{\text{Ran}|T|}}$ *and* $B := P|_{\ker T}$.

Proof By Lemma 2.5, P leaves $\overline{\text{Ran}\,|T|}$ invariant, so $\overline{\text{Ran}\,|T|}$ is a reducing subspace for P. □

Consider now the orthogonal decomposition

$$\mathcal{H} = \overline{\text{Ran}\,|T|} \oplus (\overline{\text{Ran}\,B} \oplus \ker P),$$

where the orthogonal sum in parentheses equals $\ker |T|$. Then

$$|T| = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} A & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Observe that $P = |T|$ if and only if $C = 0$. We wish to find the matrix for V. Recall that $\ker V = \ker |T|$. Therefore,

$$V = \begin{pmatrix} X & 0 & 0 \\ Y & 0 & 0 \\ Z & 0 & 0 \end{pmatrix}.$$

Since V^*V is the projection onto $(\ker V)^{\perp} = \overline{\text{Ran}\,|T|}$, we must have

$$X^*X + Y^*Y + Z^*Z = I_{\overline{\text{Ran}|T|}}.$$

Since $VP = V|T|$, it follows that

$$\begin{pmatrix} X & 0 & 0 \\ Y & 0 & 0 \\ Z & 0 & 0 \end{pmatrix} \begin{pmatrix} A & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} X & 0 & 0 \\ Y & 0 & 0 \\ Z & 0 & 0 \end{pmatrix} \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Also,

$$T = \begin{pmatrix} XA & 0 & 0 \\ YA & 0 & 0 \\ ZA & 0 & 0 \end{pmatrix}. \tag{2.1}$$

Then

$$\Delta(T) = |T|^{\frac{1}{2}} V |T|^{\frac{1}{2}} = \begin{pmatrix} A^{\frac{1}{2}} X A^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.2}$$

while

$$\Delta_P(T) = P^{\frac{1}{2}} V P^{\frac{1}{2}} = \begin{pmatrix} A^{\frac{1}{2}} & 0 & 0 \\ 0 & C^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X & 0 & 0 \\ Y & 0 & 0 \\ Z & 0 & 0 \end{pmatrix} \begin{pmatrix} A^{\frac{1}{2}} & 0 & 0 \\ 0 & C^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A^{\frac{1}{2}} X A^{\frac{1}{2}} & 0 & 0 \\ C^{\frac{1}{2}} Y A^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.3}$$

Therefore,

$$\Delta(T)^* \Delta(T) = \begin{pmatrix} A^{\frac{1}{2}} X^* A X A^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\Delta_P(T)^* \Delta_P(T) = \begin{pmatrix} A^{\frac{1}{2}} X^* A X A^{\frac{1}{2}} + A^{\frac{1}{2}} Y^* C Y A^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As a consequence,

$$|\Delta_P(T)| \geq |\Delta(T)|$$

and

$$\|\Delta_P(T)\| \geq \|\Delta(T)\|\,.$$

As is well known, the Aluthge transform is homogeneous, that is $\Delta(\lambda T) = \lambda\Delta(T)$ for every $\lambda \in \mathbb{C}$. The following result shows what form of homogeneity holds for the extended Aluthge transform.

Proposition 2.7 *Let $T \equiv V\,|T|$ be the canonical polar decomposition of T, and let P be a positive operator such that $T = VP$. For $\lambda \in \mathbb{C}$ we have*

$$\Delta_{|\lambda|P}(\lambda T) = \lambda\Delta_P(T).$$

Proof Without loss of generality, assume that $\lambda \neq 0$, and let $\lambda \equiv e^{i\theta}\,|\lambda|$ be its canonical polar decomposition. Then

$$\lambda T = e^{i\theta}V\,|\lambda|\,|T| = (e^{i\theta}V)(|\lambda|\,|T|)$$

is the canonical polar decomposition of λT. Moreover,

$$\lambda T = e^{i\theta}V\,|\lambda|\,P = (e^{i\theta}V)(|\lambda|\,P),$$

so that

$$\Delta_{|\lambda|P}(\lambda T) = (|\lambda|)^{1/2}P^{\frac{1}{2}}e^{i\theta}V(|\lambda|)^{1/2}P^{\frac{1}{2}} = \lambda P^{1/2}VP^{1/2} = \lambda\Delta_P(T).\Box$$

□

In an entirely similar way, one can establish the following result.

Proposition 2.8 *Let $T \equiv V\,|T|$ be the canonical polar decomposition of T, and let P be a positive operator such that $T = VP$. Let U be a unitary operator on $\mathcal{H}$. Then*

$$\Delta_{UPU^*}(UTU^*) = U\Delta_P(T)U^*.$$

We now discuss an extension of the so-called *-Aluthge transform, used by P.Y. Wu [30] and T. Yamazaki [32] to prove [30, Theorem 1] (cf. Theorem 8.4). This transform is defined as $\Delta(T)^{(*)} := \sqrt{|T^*|}V\sqrt{|T^*|}$. It is not difficult to prove that $\Delta(T)^{(*)} = (\Delta(T^*))^*$; thus, $\Delta(T)^{(*)} = V\Delta(T)V^*$.

We will now obtain the proper analog for the extended Aluthge transform. Let $T = V|T| = VP$ where $T = V\,|T|$ is the canonical polar decomposition of T. It is well known that $T^* = V^*|T^*|$ is the canonical polar decomposition of T^*. Now observe that, with the notation from Sect. 2, we have

$$PV^* = \begin{pmatrix} A & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X^* & Y^* & Z^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} AX^* & AY^* & AZ^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$V^*V = \begin{pmatrix} X^* & Y^* & Z^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X & 0 & 0 \\ Y & 0 & 0 \\ Z & 0 & 0 \end{pmatrix} = \begin{pmatrix} X^*X + Y^*Y + Z^*Z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that

$$(V^*V)PV^* = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} AX^* & AY^* & AZ^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} AX^* & AY^* & AZ^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = PV^*. \tag{2.4}$$

As a result, $V^*V\sqrt{P}V^* = \sqrt{P}V^*$.

To state the following result, we first recall that the canonical polar decomposition of T^* is $T^* = V^*|T^*|$. Since $T = VP$ we get $T^* = PV^*$, and using (2.4) we obtain $T^* = PV^* = (V^*V)PV^* = V^*(VPV^*)$. Moreover, VPV^* is a positive operator, so we may consider the extended Aluthge transform of T^* associated with VPV^*.

Proposition 2.9 *With T, V and P as above, we have*

$$\Delta_{VPV^*}(T^*) = V\Delta_P(T)^*V^*.$$

Proof

$$\begin{aligned} \Delta_{VPV^*}(T^*) &= \sqrt{VPV^*}V^*\sqrt{VPV^*} \\ &= (V\sqrt{P}V^*)V^*(V\sqrt{P}V^*) \\ &= V\sqrt{P}V^*V^*V\sqrt{P}V^* \\ &= V\sqrt{P}V^*(V^*V\sqrt{P}V^*) \\ &= V\sqrt{P}V^*\sqrt{P}V^* \\ &= V(\sqrt{P}V^*\sqrt{P})V^* \\ &= V\Delta_P(T)^*V^*. \end{aligned}$$

□

Corollary 2.10 *In the case when $P = |T|$ we have*

$$\Delta(T^*)^* = V\Delta(T)V^*.$$

Proof We first observe that $V\,|T|\,V^* = |T^*|$, which is established as follows: $T = V\,|T|$ and $T^* = V^*\,|T^*|$ imply that $V^*\,|T^*| = |T|\,V^*$, and therefore $V\,|T|\,V^* = VV^*\,|T^*| = |T^*|$. Next, we use Proposition 2.9 to conclude that $\Delta(T^*) = V\Delta(T)^*V^*$. Finally, we take adjoints to get $\Delta(T^*)^* = V\Delta(T)V^*$. □

We end this section with a result about orthogonal direct sums.

Proposition 2.11 *Let $T_i \equiv V_i\,|T_i|$ be the canonical polar decomposition of T_i $(i = 1, 2)$, and let P_i be a positive operator such that $T_i = V_iP_i$ $(i = 1, 2)$. Then*

$$\Delta_{P_1 \oplus P_2}(T_1 \oplus T_2) = \Delta_{P1}(T_1) \oplus \Delta_{P_2}(T_2).$$

3 Fixed Points of the Extended Aluthge Transform

It is well known that the fixed points of the classical Aluthge transform are the quasinormal operators, that is, those operators $T = V\,|T|$ such that V and $|T|$ commute. In this section we study the class of operators which are fixed points for the extended Aluthge transform. In what follows, we frequently use the matricial decompositions introduced in Sect. 2. From (2.1) and (2.3), we easily see that

$$\Delta_P(T) = T \Longleftrightarrow \begin{cases} A^{\frac{1}{2}}XA^{\frac{1}{2}} = XA \\ C^{\frac{1}{2}}YA^{\frac{1}{2}} = YA \\ \quad 0 \quad = ZA. \end{cases}$$

Recall that Ran A is dense in $\overline{\text{Ran}\,|T|}$. Thus, $ZA = 0 \Rightarrow Z = 0$. Also,

$$(A^{\frac{1}{2}}X - XA^{\frac{1}{2}})A^{\frac{1}{2}} = A^{\frac{1}{2}}XA^{\frac{1}{2}} - XA = 0 \Longrightarrow A^{\frac{1}{2}}X = XA^{\frac{1}{2}}$$

and therefore

$$AX = A^{\frac{1}{2}}A^{\frac{1}{2}}X = A^{\frac{1}{2}}XA^{\frac{1}{2}} = XA^{\frac{1}{2}}A^{\frac{1}{2}} = XA.$$

It follows that A and X commute. Finally,

$$(C^{\frac{1}{2}}Y - YA^{\frac{1}{2}})A^{\frac{1}{2}} = C^{\frac{1}{2}}YA^{\frac{1}{2}} - YA = 0 \Longrightarrow C^{\frac{1}{2}}Y = YA^{\frac{1}{2}}$$

so that

$$CY = C^{\frac{1}{2}}(C^{\frac{1}{2}}Y) = C^{\frac{1}{2}}YA^{\frac{1}{2}} = YA^{\frac{1}{2}}A^{\frac{1}{2}} = YA.$$

We then have:

$$\begin{cases} AX = XA \\ CY = YA \\ Z = 0, \end{cases}$$

which readily implies

$$V|T| = PV$$

and

$$VP = PV.$$

It follows that $\ker P$ reduces V.

We summarize the previous discussion in the following result.

Theorem 3.1 *Let $P \in \mathcal{B}(\mathcal{H})$ be a positive operator such that $VP = T$, and assume that $\Delta_P(T) = T$. Then T commutes with P, V commutes with P, and* $\ker P$ *reduces T and V.*

Corollary 3.2 *In Theorem 3.1, assume that $P = |T|$, so that $\Delta_P(T) = \Delta(T) = T$. Then T is quasinormal (i.e., $|T|$ commutes with V, or equivalently, $|T|$ commutes with T).*

4 The Case of T Idempotent

In this section we consider the case when T is an idempotent, that is, $T^2 = T$. From (2.1) it easily follows that

$$\begin{cases} XAXA = XA \\ YAXA = YA \\ ZAXA = ZA. \end{cases}$$

Since $\operatorname{Ran} A$ is dense in $\overline{\operatorname{Ran} |T|}$, we have

$$\begin{cases} XAX = X \\ YAX = Y \\ ZAX = Z, \end{cases}$$

and therefore

$$\begin{cases} X^*XAX = X^*X \\ Y^*YAX = Y^*Y \\ Z^*ZAX = Z^*Z. \end{cases}$$

Since $X^*X + Y^*Y + Z^*Z = I_{\overline{\text{Ran}|T|}}$, we readily obtain

$$AX = I_{\overline{\text{Ran}|T|}}.$$

As a result,

$$A^{\frac{1}{2}}XA^{\frac{1}{2}} = I_{\overline{\text{Ran}|T|}}.$$

(For, given $x \in \text{Ran}\, A^{\frac{1}{2}}$ one has

$$\left\langle (A^{\frac{1}{2}}XA^{\frac{1}{2}})A^{\frac{1}{2}}x, A^{\frac{1}{2}}x \right\rangle = \langle AXAx, x\rangle = \langle Ax, x\rangle = \left\langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}x \right\rangle,$$

and it follows that $A^{\frac{1}{2}}XA^{\frac{1}{2}} = I$ on $\overline{\text{Ran}\, A^{\frac{1}{2}}} = \overline{\text{Ran}\,|T|}$.)

Using (2.2) we readily see that $\Delta(T)$ is the projection from $\mathcal{H}$ onto $\overline{\text{Ran}\,|T|}$; using (2.3) we see that

$$\Delta_P(T) = \begin{pmatrix} I & 0\ 0 \\ C^{\frac{1}{2}}YA^{\frac{1}{2}} & 0\ 0 \\ 0 & 0\ 0 \end{pmatrix}.$$

Since $AX = I_{\overline{\text{Ran}|T|}}$, we know that A is right invertible on $\overline{\text{Ran}\,|T|}$, therefore invertible (as an operator on $\overline{\text{Ran}\,|T|}$). We summarize the previous discussion in the following result.

Theorem 4.1 *Let T be an idempotent. Then*

$$\Delta(T) = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and

$$\Delta_P(T) = \begin{pmatrix} I & 0\ 0 \\ C^{\frac{1}{2}}YA^{\frac{1}{2}} & 0\ 0 \\ 0 & 0\ 0 \end{pmatrix},$$

with A invertible.

The information we have gathered is somewhat optimal, as the following example shows.

Example 4.2 Given $a, b \in \mathbb{R}$, let

$$T := \begin{pmatrix} 1 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \in M_3(\mathbb{C}).$$

Let $\delta := \sqrt{1 + a^2 + b^2}$. It is straightforward to verify that $T^2 = T$, with canonical polar decomposition

$$T = \begin{pmatrix} 1 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \equiv V\,|T| = \begin{pmatrix} \frac{1}{\delta} & 0 & 0 \\ \frac{a}{\delta} & 0 & 0 \\ \frac{b}{\delta} & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $f > 0$ let

$$P \equiv P_{\delta, f} := \begin{pmatrix} \delta & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\Delta(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\Delta_P(T) = \begin{pmatrix} 1 & 0 & 0 \\ a\sqrt{\frac{f}{\delta}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Notice in particular that V does not commute with P, so T is not a fixed point for Δ_P. Moreover, $\Delta(\Delta_P(T)) \neq \Delta_P(T)$; however, $\Delta(\Delta(\Delta_P(T))) = \Delta(\Delta_P(T))$. Also, as expected, $\Delta(T)$ is a projection, while $\Delta_P(T)$ is again an idempotent. Therefore, it makes sense to repeat this construction (with a new δ and a new f) to obtain the iterate $\Delta_{P_2}(\Delta_P(T))$, which is again an idempotent. One can then study the asymptotic behavior of these iterates, in a manner resembling the results in [3, 12, 22] and [29] for the classical Aluthge transform. We plan to report on the behavior of the iterates of the extended Aluthge transform in a forthcoming paper.

□

5 Some Useful Identities

We devote this section to the proof of some identities involving T, its classical Aluthge transform $\Delta(T)$ and the extended Aluthge transform $\Delta_P(T)$. First, recall from (2.2) and (2.3) that

$$\Delta(T) = \begin{pmatrix} A^{\frac{1}{2}}XA^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \Delta_P(T) = \begin{pmatrix} A^{\frac{1}{2}}XA^{\frac{1}{2}} & 0 & 0 \\ C^{\frac{1}{2}}YA^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Direct matrix calculation shows that

$$\Delta(T)P = \Delta(T)\,|T|\,,$$

consistent with Lemma 2.4. Similarly, one obtains the following result.

Proposition 5.1 (Intertwining Property) *For T, P, $\Delta(T)$ and $\Delta_P(T)$ as in Sect. 2, we have:*

$$|T|^{\frac{1}{2}}\,\Delta_P(T)P^{\frac{1}{2}} = P^{\frac{1}{2}}\Delta(T)\,|T|^{\frac{1}{2}}\,.$$

We briefly pause to recall an important feature of the class $\mathcal{C}_2$ of Hilbert-Schmidt operators on $\mathcal{H}$. Recall that the inner product of two Hilbert-Schmidt operators S and T is given by

$$\langle S, T\rangle_{\mathcal{C}_2} := \mathrm{Tr}(T^*S).$$

The class $\mathcal{C}_2$ is a Hilbert space, with norm $\|S\|_2 := (\langle S, S\rangle_{\mathcal{C}_2})^{\frac{1}{2}} = (\mathrm{Tr}(S^*S))^{\frac{1}{2}}$. For E and F in the class $\mathcal{C}_2$, consider the operator matrix

$$\begin{pmatrix} E & 0 & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\left\| \begin{pmatrix} E & 0 & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\|_2^2 = \left\langle \begin{pmatrix} E & 0 & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} E & 0 & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle_{\mathcal{C}_2} = \mathrm{Tr}(E^*E + F^*F) = \|E\|_2^2 + \|F\|_2^2\,.$$

A direct consequence of this calculation is the following result.

Theorem 5.2 *Let T be a Hilbert-Schmidt operator. Then*

$$\|\Delta_P(T)\|_2^2 = \|\Delta(T)\|_2^2 + \left\| C^{\frac{1}{2}} Y A^{\frac{1}{2}} \right\|_2^2.$$

6 An Application: The Spherical Aluthge Transform

In this section we will show how the spherical Aluthge transform (introduced in [10] and [11]) can be obtained as a particular case of the extended Aluthge transform, for a suitable positive operator P. Given a commuting pair $\mathcal{T} \equiv (T_1, T_2)$ of operators acting on $\mathcal{H}$, let $Q := (T_1^* T_1 + T_2^* T_2)^{\frac{1}{2}}$. Clearly, $\ker Q = \ker T_1 \bigcap \ker T_2$. For $x \in \ker Q$, let $V_i x := 0$ $(i = 1, 2)$; for $y \in \operatorname{Ran} Q$, say $y = Qx$, let $V_i y := T_i x$ $(i = 1, 2)$. It is easy to see that V_1 and V_2 are well defined, and extend continuously to $\overline{\operatorname{Ran} Q}$. We then have

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} V_1 Q \\ V_2 Q \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} Q, \tag{6.1}$$

as operators from $\mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}$. Moreover, this is the canonical polar decomposition of $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$. It follows that $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ is a partial isometry from $(\ker Q)^\perp$ onto $\overline{\operatorname{Ran} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}}$.

The spherical Aluthge transform of $\mathcal{T}$ is $\widehat{\mathcal{T}} \equiv (\widehat{T_1}, \widehat{T_2})$, where

$$\widehat{T_i} := Q^{\frac{1}{2}} V_i Q^{\frac{1}{2}} \ (i = 1, 2) \ \text{(cf. [10, 11])}.$$

Lemma 6.1 (cf. [11]) *$\widehat{\mathcal{T}}$ is commutative.*

We now let

$$\Phi(\mathcal{T}) := \begin{pmatrix} T_1 & 0 \\ T_2 & 0 \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{H}).$$

It is clear that

$$|\Phi(\mathcal{T})| = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}.$$

We also let $\mathbf{V} := (V_1, V_2)$. (Notice that $\mathbf{V}$ is not necessarily commuting.) Finally, let

$$\Phi(\mathbf{V}) := \begin{pmatrix} V_1 & 0 \\ V_2 & 0 \end{pmatrix}.$$

Lemma 6.2 *With $\mathcal{T}$ and* $\mathbf{V}$ *as above,* $\Phi(\mathcal{T}) = \Phi(\mathbf{V})\,|\Phi(\mathcal{T})|$ *is the canonical polar decomposition of* $\Phi(\mathcal{T})$.

Proof This is straightforward from the fact that (6.1) is the canonical polar decomposition of $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$. □

Consider now the positive operator

$$P \equiv P(Q) := \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}.$$

Then

$$\Phi(\mathbf{V})P = \Phi(\mathbf{V})\,|\Phi(\mathcal{T})| = \Phi(\mathcal{T}).$$

We wish to study the extended Aluthge transform $\Delta_P(\Phi(\mathcal{T}))$.

Theorem 6.3 *With $\mathcal{T}$ and P as above,*

$$\Delta_P(\Phi(\mathcal{T})) = \Phi(\widehat{\mathcal{T}}).$$

Proof

$$\Phi(\widehat{\mathcal{T}}) = \begin{pmatrix} \widehat{T}_1 & 0 \\ \widehat{T}_2 & 0 \end{pmatrix} = \begin{pmatrix} Q^{\frac{1}{2}} V_1 Q^{\frac{1}{2}} & 0 \\ Q^{\frac{1}{2}} V_2 Q^{\frac{1}{2}} & 0 \end{pmatrix} = \begin{pmatrix} Q^{\frac{1}{2}} & 0 \\ 0 & Q^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ V_2 & 0 \end{pmatrix} \begin{pmatrix} Q^{\frac{1}{2}} & 0 \\ 0 & Q^{\frac{1}{2}} \end{pmatrix}$$
$$= P^{\frac{1}{2}} \Phi(\mathbf{V}) P^{\frac{1}{2}} = \Delta_P(\Phi(\mathcal{T})).$$

□

Remark 6.4 Proposition 6.3 shows that the spherical Aluthge transform can be expressed in terms of the extended Aluthge transform of the 2×2-operator matrix $\Phi(\mathcal{T})$. □

Observation 6.5 For the spherical Aluthge transform, the operator P is uniquely determined by the commuting pair $\mathcal{T}$; that is, the pair $\mathcal{T}$ determines Q, which in turn determines P. □

7 Extended Aluthge Transforms of Complex Symmetric Operators

Recall that a *conjugation* C on a Hilbert space $\mathcal{H}$ is an antilinear map satisfying: (1) $C^2 = I$; and (2) $\langle Cx, Cy \rangle = \langle y, x \rangle$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *complex symmetric* if there exists a conjugation C such that $T^* = CTC$.

If $T = U|T|$ is the canonical polar decomposition of T, one may use a generalization of a theorem of Godič and Lucenko to write $U = CJ$ where J is a partial conjugation supported on $\overline{\mathrm{Ran}(|T|)}$ such that $J|T| = |T|J$, where T is a C-complex symmetric operator (see [17, Theorem 2]); as a result, $T = CJ|T|$. (For additional results, see [6, 7, 16, 18] and [19].)

Of course, J can be extended to a conjugation $\tilde{J}$ (which, with minor abuse of notation, we will usually denote again by J (cf. [5])) acting on the whole space $\mathcal{H}$, without affecting the equation $T = CJ|T|$. In this case, it was proven in [15] that the Aluthge transform $\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is also complex symmetric, with the conjugation $\tilde{J}$.

Theorem 7.1 ([15]) *Let T be a complex symmetric operator. Then $\Delta(T)$ is complex symmetric.*

We will now establish that the extended Aluthge transform does not preserve the property of being complex symmetric. To this end, we will focus attention on a special class of finite rank operators. Let n be an integer and assume that $n \geq 2$. For a finite family of complex numbers $\lambda \equiv \lambda_1, \cdots, \lambda_{n-1}$ consider the operator

$$T(\lambda) := \sum_{i=1}^{n-1} \lambda_i e_{i+1} \otimes e_i,$$

where $e_1, \cdots, e_n, \cdots$ are elements of an orthonormal basis for $\mathcal{H}$, and for vectors $x, y \in \mathcal{H}$ we denote by $x \otimes y$ the rank-one operator $(x \otimes y)(z) := \langle z, y\rangle\, x \;\; (z \in \mathcal{H})$. Without loss of generality we may assume that $\lambda_i > 0$ for all $i = 1, \cdots, n-1$. The results, however, will be stated for complex λ_i's. It is easy to see that T admits the following matricial representation:

$$T(\lambda) = \begin{pmatrix} 0 & 0 & 0 & & \dots & 0 \\ \lambda_1 & 0 & 0 & \dots & \dots & 0 \\ 0 & \lambda_2 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \dots & \dots & \lambda_{n-1} & 0 \end{pmatrix}.$$

It is straightforward to see that

$$V(\lambda) = \begin{pmatrix} 0 & 0 & 0 & & \dots & 0 \\ 1 & 0 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \dots & \dots & 1 & 0 \end{pmatrix}$$

and

$$|T(\lambda)| = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_{n-1} & \\ & & & & 0 \end{pmatrix}$$

are the factors in the canonical polar decomposition of $T(\lambda)$. Let γ be a given positive real number, and consider a positive operator $P_\gamma(\lambda)$ of the form

$$P_\gamma(\lambda) := \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_{n-1} & \\ & & & & \gamma \end{pmatrix} \tag{7.1}$$

We now recall the following result, proved in [34, Theorem 3.1], appropriately adjusted to our situation.

Proposition 7.2 *Let* $T(\lambda) \equiv \sum_{i=1}^{n-1} \lambda_i e_{i+1} \otimes e_i$ *be as above. Then* T *is complex symmetric if and only if* $|\lambda_i| = |\lambda_{n-i}|$ $(1 \le i \le n-1)$.

For the class of operators $T(\lambda)$ we now determine which λ's and γ's give rise to a complex symmetric extended Aluthge transform $\Delta_{P_\gamma(\lambda)}$.

Theorem 7.3 *Let* $T(\lambda)$ *and* γ *be as above, and let* $\Delta_{P_\gamma(\lambda)}(T(\lambda))$ *be the associated extended Aluthge transform. Then* $\Delta_{P_\gamma(\lambda)}(T(\lambda))$ *is complex symmetric if and only if* $\gamma\,|\lambda_{n-1}| = |\lambda_1 \lambda_2|$ *and* $|\lambda_i \lambda_{i+1}| = |\lambda_{n-i} \lambda_{n-i+1}|$ *for every* $2 \le i \le n-2$.

Proof We have

$$\Delta_{P_\gamma(\lambda)}(T(\lambda)) = \sqrt{P_\gamma(\lambda)}V\sqrt{P_\gamma(\lambda)}$$
$$= \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \ddots & & \\ & & \sqrt{\lambda_{n-1}} & \\ & & & \sqrt{\gamma} \end{pmatrix} \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \ddots & & \\ & & \sqrt{\lambda_{n-1}} & \\ & & & \sqrt{\gamma} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & & \dots & 0 \\ \sqrt{\lambda_1\lambda_2} & 0 & 0 & \dots & \dots & 0 \\ 0 & \sqrt{\lambda_2\lambda_3} & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \sqrt{\lambda_{n-2}\lambda_{n-1}} & 0 & 0 \\ 0 & 0 & \dots & 0 & \sqrt{\gamma\lambda_{n-1}} & 0 \end{pmatrix}.$$

From Proposition 7.2 we see that for $\Delta_{P_\gamma(\lambda)}$ to be complex symmetric one needs

$$\begin{cases} |\lambda_1\lambda_2| = |\gamma\lambda_{n-1}| \\ |\lambda_i\lambda_{i+1}| = |\lambda_{n-i}\lambda_{n-i+1}| \;\; for 2 \le i \le n-2, \end{cases}$$

as desired. □

Corollary 7.4 *Let $T(\lambda)$ and γ be as above. Then $T(\lambda)$ and its extended Aluthge transform $\Delta_{P_\gamma(\lambda)}(T(\lambda))$ are both complex symmetric if and only if*

(i) *(when n is odd)* $\gamma = |\lambda_1| = |\lambda_2| = \cdots = |\lambda_{n-1}|$;
(ii) *(when n is even)*

$$\begin{cases} |\lambda_1| = |\lambda_3| = \cdots = |\lambda_{n-1}| \\ \gamma = |\lambda_2| = |\lambda_4| = \cdots = |\lambda_{n-2}| . \end{cases}$$

Remark 7.5

(i) For $t \in [0, 1]$, one may define the *generalized* extended Aluthge Transform as follows:

$$\Delta_{P_\gamma(\lambda)}(T(\lambda); t) := P_\gamma(\lambda)^t V P_\gamma(\lambda)^{1-t}.$$

As in the classical case, $\Delta_{P_\gamma(\lambda)}(T(\lambda); \frac{1}{2}) = \Delta_{P_\gamma(\lambda)}(T(\lambda))$ is the extended Aluthge transform, and $\Delta_{P_\gamma(\lambda)}(T(\lambda); 0) = T$; also, $\Delta_{P_\gamma(\lambda)}(T(\lambda); 1)$ is the analog of the so called Duggal transform.

(ii) As in the classical case, the generalized extended Aluthge transform of a complex symmetric operator may fail to be complex symmetric (except for the cases $t = 0$ and $t = \frac{1}{2}$ (see [34] and [5]). □

Let us consider the same class of operators $T(\lambda)$, this time looking at the generalized extended Aluthge transforms. As before, P_γ is given by (7.1).

$$\Delta_{P_\gamma(\lambda)}(T(\lambda); t) := \big(P_\gamma(\lambda)\big)^t V \big(P_\gamma(\lambda)\big)^{1-t}$$

$$= \begin{pmatrix} \lambda_1^t & & & & \\ & \lambda_2^t & & & \\ & & \ddots & & \\ & & & \lambda_{n-1}^t & \\ & & & & \gamma^t \end{pmatrix} \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1^{1-t} & & & & \\ & \lambda_2^{1-t} & & & \\ & & \ddots & & \\ & & & \lambda_{n-1}^{1-t} & \\ & & & & \gamma^{1-t} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & & \dots & 0 \\ \lambda_2^t\lambda_1^{1-t} & 0 & 0 & \dots & \dots & 0 \\ 0 & \lambda_3^t\lambda_2^{1-t} & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \lambda_{n-1}^t\lambda_{n-2}^{1-t} & 0 & 0 \\ 0 & 0 & \dots & \dots & \gamma^t\lambda_{n-1}^{1-t} & 0 \end{pmatrix}.$$

Using again Proposition 7.2, we obtain the following result.

Theorem 7.6 *Let $T(\lambda)$ and γ be as above. Then the generalized extended Aluthge transform $\Delta_{P_\gamma(\lambda)}(T(\lambda); t)$ is complex symmetric if and only if $\gamma^t |\lambda_{n-1}|^{1-t} = |\lambda_2|^t |\lambda_1|^{1-t}$ and $|\lambda_{i+1}|^t |\lambda_i|^{1-t} = |\lambda_{n-i+1}|^t |\lambda_{n-i}|^{1-t}$ for every $2 \le i \le n-2$.*

Rather surprisingly, for the case $\gamma > 0$, the generalized extended Aluthge transforms allows one to simplify the conditions describing complex symmetry, in the sense that if we start with $T(\lambda)$ complex symmetric, there is *one* condition that ensures that *all* $\Delta_{P_\gamma(\lambda)}(T(\lambda); t)$ are complex symmetric operators. Namely, we have the following result.

Corollary 7.7 *Let $T(\lambda)$ and γ be as above. Then $T(\lambda)$ and its generalized extended Aluthge transforms $\Delta_{P_\gamma(\lambda)}(T(\lambda); t)$ ($0 \le t \le 1$) are all complex symmetric if and only if*

(i) *(when n is odd)* $\gamma = |\lambda_1| = |\lambda_2| = \cdots = |\lambda_{n-1}|$;
(ii) *(when n is even)*

$$\begin{cases} |\lambda_1| = |\lambda_3| = \cdots = |\lambda_{n-1}| \\ \gamma = |\lambda_2| = |\lambda_4| = \cdots = |\lambda_{n-2}| . \end{cases}$$

8 The Numerical Range and the Extended Aluthge Transform

For an operator $A \in \mathcal{B}(\mathcal{H})$, recall that the numerical range $W(A)$ of A is defined as

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H} \text{ with } \|x\| = 1\}.$$

The following lemmas are interesting and useful; they appear in [30] and the references therein.

Lemma 8.1 *Let A and B be operators on $\mathcal{H}$ such that $A = X^*BX$ for some contraction X. Then $W(A) \subseteq co(W(B) \cup \{0\})^\wedge$. If, in addition, X is a coisometry, then we also have $W(B) \subseteq W(A)$.*

Lemma 8.2 (Heinz Inequality) *Let A, X and B be operators on $\mathcal{H}$, and assume that A and B are positive. Then the following inequalities hold:*

(i) $\|A^r XB^r\| \leq \|AXB\|^r \|X\|^{1-r}$ *for* $r \in [0, 1]$.
(ii) $\|A^r XB^r\| \geq \|AXB\|^r \|X\|^{1-r}$ *for* $r > 1$.

From the last result we can derive the following lemma (see [30]).

Lemma 8.3 *Let A and X be operators on $\mathcal{H}$, and assume that A is positive. Then*

$$\|A^r XA^{1-r} - zI\| \leq \|AX - zI\|^r \|XA - zI\|^{1-r}, \text{ for all } r \in [0, 1] \text{ and } z \in \mathbb{C}$$

The previous results were used to prove the following inclusion.

Theorem 8.4 ([30], Theorem 1) *Let T be an operator on $\mathcal{H}$. Then*

$$\overline{W(\Delta(T))} \subseteq \overline{W(T)}$$

We now turn our attention to the extended Aluthge transform.

8.1 Numerical Range for Extended Aluthge Transforms

We begin with a natural question.

Question 8.5 Is Theorem 8.4 still true for the extended Aluthge transform?

We'll show here that Theorem 8.4 is not true for all extended Aluthge transforms. However we have a relationship connecting the numerical ranges.
Recall that we have following decompositions

$$T = \begin{pmatrix} XA & 0 & 0 \\ YA & 0 & 0 \\ ZA & 0 & 0 \end{pmatrix} \quad \Delta(T) = \begin{pmatrix} \sqrt{A}X\sqrt{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Delta_P(T) = \begin{pmatrix} \sqrt{A}X\sqrt{A} & 0 & 0 \\ \sqrt{C}Y\sqrt{A} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then, we have

$$\overline{W(\Delta(T))} \subseteq \overline{W(\Delta_P(T))} \text{ and } \overline{W(\Delta(T))} \subseteq \overline{W(T)}.$$

It has been shown in [24, Theorem 4.1] that the numerical range of an upper triangular matrix of the form

$$A = \begin{pmatrix} p & 0 & 0 \\ x & p & 0 \\ y & z & p \end{pmatrix}$$

when $xyz = 0$ is the closed disc centered at p and with radius $\frac{1}{2}\sqrt{|x|^2 + |y|^2 + |z|^2}$.

So, if

$$T = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix},$$

and we recall that

$$P_\gamma = \begin{pmatrix} |\alpha| & 0 & 0 \\ 0 & |\beta| & 0 \\ 0 & 0 & \gamma \end{pmatrix},$$

it follows that

(i) $W(T) = \bar{\mathbf{D}}\big(0, \frac{1}{2}\sqrt{|\alpha|^2 + |\beta|^2}\big)$.
(ii) $W(\Delta(T)) = \bar{\mathbf{D}}\big(0, \frac{1}{2}\sqrt{|\alpha||\beta|}\big)$.
(iii) $W(\Delta_{P_\gamma}(T)) = \bar{\mathbf{D}}\Big(0, \frac{1}{2}\sqrt{|\alpha||\beta| + |\beta|\gamma}\Big)$.

As expected, we have

$$\begin{cases} W(\Delta(T)) \subseteq W(T) \\ W(\Delta(T)) \subseteq W(\Delta_{P_\gamma}(T)). \end{cases}$$

Remark 8.6

(i) We may choose α, β and γ in the previous example such that the inclusions above are strict; see Fig. 1.
(ii) $W(\Delta_{P_\gamma}(T))$ and $W(T)$ are not comparable in general, unless we impose restrictions on γ. For example, observe that if $\gamma \le |\alpha|$ in the previous discussion, then $W(\Delta_{P_\gamma}(T)) \subseteq W(T)$, with equality holding if and only if $\gamma = |\alpha| = |\beta|$. □

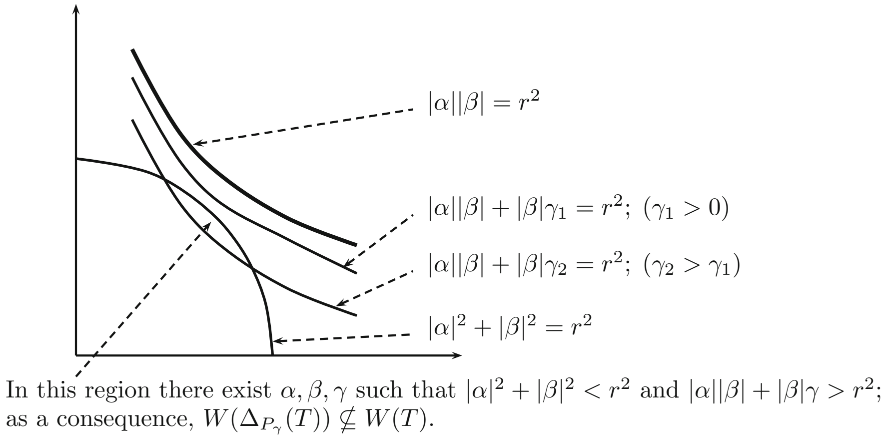

Fig. 1 Graphs of radii in Remark 8.6

Acknowledgments The authors are grateful to the referee for some helpful suggestions that improved the presentation.

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Open Problems in Wavelet Theory

Marcin Bownik and Ziemowit Rzeszotnik

Abstract We present a collection of easily stated open problems in wavelet theory and we survey the current status of answering them. This includes a problem of Larson ((2007) Unitary systems and Wavelet sets. In: Wavelet analysis and applications. Applied and Numerical Harmonic Analysis. Birkhäuser, Basel, pp 143–171) on minimally supported frequency wavelets. We show that it has an affirmative answer for MRA wavelets.

Keyword Wavelets

Mathematics Subject Classification (2010) Primary: 42C40, Secondary: 46C05

1 Introduction

The goal of this paper is twofold. The first goal is to present a collection of open problems on wavelets which have simple formulations. Many of these problems are well-known, such as connectivity of the set of wavelets. Others are less known, but nevertheless deserve a wider dissemination. At the same time we present the current state of knowledge about these problems. These include several results giving a partial progress, which indicate inherent difficulties in answering them. One of such problems was formulated by Larson [43] and asks about frequency supports

The first author was supported in part by the NSF grant DMS-1665056.

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_4

of orthonormal wavelets. Must they contain a wavelet set? The second goal of the paper is to give an affirmative answer to this problem for the class of MRA wavelets.

2 One Dimensional Wavelets

In this section we discuss problems in wavelet theory that remain unanswered even in the classical setting of one dimensional dyadic wavelets. Many of these problems have higher dimensional analogues which also remain open.

Definition 2.1 We say that $\psi \in L^2(\mathbb{R})$ is an o.n. wavelet if the collection of translates and dyadic dilates

$$\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k), \qquad j, k \in \mathbb{Z} \tag{2.1}$$

forms an o.n. basis of $L^2(\mathbb{R})$.

2.1 Connectivity of Wavelets

One of the fundamental areas in the theory of wavelets is the investigation of properties of the collection of all wavelets as a subset of $L^2(\mathbb{R})$. The most prominent problem in this area was formulated independently by D. Larson and G. Weiss around the year 1995.

Problem 2.1 Is the collection of all orthonormal wavelets (as a subset of the unit sphere in $L^2(\mathbb{R})$) path connected in $L^2(\mathbb{R})$ norm?

Despite several attempts and significant initial progress Problem 2.1 remains open. In addition, variants of Problem 2.1 for Parseval wavelets and Riesz wavelets are also open. A strong initial thrust toward answering this problem was given by a joint work by a group of authors from Texas A&M University and Washington University led by D. Larson and G. Weiss, respectively. The paper [60] written by the Wutam consortium gave a positive answer to Problem 2.1 for the class of MRA wavelets. A concept of a multiresolution analysis (MRA) is one of the most fundamental in the wavelet theory. It was introduced by Mallat and Meyer [46, 47].

Definition 2.2 A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ is called a *multiresolution analysis* (MRA) if

(M1) $V_j \subset V_{j+1}$,
(M2) $f(\cdot) \in V_j \iff f(2\cdot) \in V_{j+1}$,
(M3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,
(M4) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$,

(M5) There exists $\varphi \in V_0$ such that its integer translates $(\varphi(\cdot - k))_{k\in\mathbb{Z}}$ form an o.n. basis of V_0.

We say that an o.n. wavelet $\psi \in L^2(\mathbb{R})$ is associated with an MRA $\{V_j : j \in \mathbb{Z}\}$ if ψ belongs to the orthogonal complement $V_1 \ominus V_0$ of V_0 inside V_1.

A Fourier transform defined initially for $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is given by

$$\hat{\psi}(\xi) = \int_R \psi(x)e^{-2\pi i x\xi}dx \qquad \xi \in \mathbb{R}.$$

There is a simple characterization of MRA wavelets in terms of the wavelet dimension function, see [38, Theorem 7.3.2]. The notion of the wavelet dimension function was introduced by Auscher in [1] and studied in [3–5, 18, 52].

Theorem 2.1 *Let $\psi \in L^2(\mathbb{R})$ be an orthonormal wavelet. Then ψ is an MRA wavelet if and only if*

$$\mathcal{D}_\psi(\xi) := \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + k))|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}.$$

The main theorem of the Wutam consortium [60, Theorem 4] shows that the collection of all MRA wavelets is path connected.

Theorem 2.2 *Let ψ_0 and ψ_1 be two MRA wavelets which are not necessarily associated with the same MRA. Then, there exists a continuous map $\Psi : [0, 1] \to L^2(\mathbb{R})$ such that $\Psi(0) = \psi_0$, $\Psi(1) = \psi_1$, and $\Psi(t)$ is an MRA wavelet for all $t \in [0, 1]$.*

Another fundamental connectivity result for the class of minimally supported frequency (MSF) wavelets was obtained by Speegle [56].

Definition 2.3 Let $\psi \in L^2(\mathbb{R})$ be an o.n. wavelet. We say that ψ is an MSF wavelet if its frequency support

$$\operatorname{supp}\hat{\psi} = \{\xi \in \mathbb{R} : \hat{\psi}(\xi) \neq 0\}$$

has minimal Lebesgue measure (equal 1).

Equivalently, $\psi \in L^2(\mathbb{R})$ is an MSF wavelet if and only if $|\hat{\psi}| = \mathbf{1}_W$ for some measurable set $W \subset \mathbb{R}$, known as *wavelet set*, which satisfies simultaneous translation and dilation tiling of $\mathbb{R}$. That is,

- $\{W + k\}_{k\in\mathbb{Z}}$ is a partition of $\mathbb{R}$ modulo null sets, and
- $\{2^j W\}_{j\in\mathbb{Z}}$ is a partition of $\mathbb{R}$ modulo null sets.

Speegle [56, Theorem 2.5 and Corollary 2.6] has shown the following result.

Theorem 2.3 *The wavelet sets are path-connected in the symmetric difference metric. Consequently, the collection of MSF wavelets forms a path connected subset of* $L^2(\mathbb{R})$.

Besides the last two results, little is known about the connectivity problem for general o.n. wavelets. However, there is a partial evidence that the answer to Problem 2.1 is affirmative. The following result characterizing wavelet dimension function was shown in [18].

Theorem 2.4 *Let* $\psi \in L^2(\mathbb{R})$ *be an orthonormal wavelet. Then its wavelet dimension function*

$$\mathcal{D}(\xi) = \mathcal{D}_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + k))|^2 \qquad \xi \in \mathbb{R}, \tag{2.2}$$

satisfies the following 4 conditions:

(D1) $\mathcal{D} : \mathbb{R} \to \mathbb{N} \cup \{0\}$ *is a measurable 1-periodic function,*
(D2) $\mathcal{D}(\xi) + \mathcal{D}(\xi + 1/2) = \mathcal{D}(2\xi) + 1$ *for a.e.* $\xi \in \mathbb{R}$,
(D3) $\sum_{k\in\mathbb{Z}} \mathbf{1}_\Delta(\xi + k) \ge \mathcal{D}(\xi)$ *for a.e.* $\xi \in \mathbb{R}$, *where*

$$\Delta = \{\xi \in \mathbb{R} : \mathcal{D}(2^{-j}\xi) \ge 1 \text{ for } j \in \mathbb{N} \cup \{0\}\},$$

(D4) $\liminf_{j\to\infty} \mathcal{D}(2^{-j}\xi) \ge 1$ *for a.e.* $\xi \in \mathbb{R}$.

Conversely, for any function $\mathcal{D}$ *satisfying the above 4 conditions, there exists an orthonormal MSF wavelet* ψ *such that* (2.2) *holds for a.e.* $\xi \in \mathbb{R}$.

In light of Theorems 2.3 and 2.4 the affirmative answer to Problem 2.1 would follow from the following conjecture. Our joint work [17] was meant as an initial step toward this conjecture.

Conjecture 2.1 Let $\mathcal{D}$ be any wavelet dimension function, i.e., $\mathcal{D}$ satisfies (D1)–(D4). Then, the collection of o.n. wavelets with the same dimension function

$$\{\psi \in L^2(\mathbb{R}) : \psi \text{ is an o.n. wavelet and } \mathcal{D}_\psi = \mathcal{D}\}.$$

is a path connected subset of $L^2(\mathbb{R})$.

Variants of Problem 2.1 have been studied for other classes of wavelets such as Parseval wavelets. We say that $\psi \in L^2(\mathbb{R})$ is a Parseval wavelet if its wavelet system is a Parseval frame. That is,

$$\sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 = ||f||^2 \qquad \text{for all } f \in L^2(\mathbb{R}).$$

This problem also remains open in its full generality. Paluszyński, Šikić, Weiss, and Xiao showed the connectivity for the class of MRA Parseval wavelets [48, 49], which is an extension of Theorem 2.2. Moreover, Garrigós, Hernández, Šikić, Soria, Weiss, and Wilson showed that the class of Parseval wavelets satisfying very mild conditions on their spectrum is also connected [33, 34]. Likewise, a variant of Problem 2.1 for Riesz wavelets, which was posed by Larson [42, 43], is also open. However, the same problem for frame wavelets was solved by the first author [10].

A frame wavelet, or in short a framelet, is a function $\psi \in L^2(\mathbb{R})$ such that the wavelet system (3.1) forms a frame for $L^2(\mathbb{R})$. Hence, we require the existence of constants $0 < c \le d < \infty$ such that

$$c||f||^2 \le \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 \le d||f||^2 \qquad \text{for all } f \in L^2(\mathbb{R}). \tag{2.3}$$

We say that a wavelet system is Bessel if only the upper bound holds in (2.3). Then we have the following result [10, Theorem 3.1].

Theorem 2.5 *The collection of all framelets*

$$\mathcal{W}_f = \{\psi \in L^2(\mathbb{R}) : \psi \textit{ is a framelet}\}.$$

is path connected in $L^2(\mathbb{R})$.

2.2 Wavelets for $H^2(\mathbb{R})$

Auscher in his influential work [2] has solved two problems on wavelets. He has shown that all biorthogonal wavelets satisfying mild regularity conditions come from biorthogonal MRAs. In particular, we have the following result [2, Theorem 1.2].

Theorem 2.6 *Let* $\psi \in L^2(\mathbb{R})$ *be an o.n. wavelet such that:*

- $\hat{\psi}$ *is continuous on* $\mathbb{R}$,
- $|\hat{\psi}(\xi)| = O((1 + |\xi|)^{-\alpha-1/2})$ *as* $|\xi| \to \infty$ *for some* $\alpha > 0$.

Then, ψ *is an MRA wavelet.*

The original formulation in [2] has one more condition, $|\hat{\psi}(\xi)| = O(|\xi|^{\alpha})$ as $\xi \to 0$, which is not essential. The proof of Theorem 2.6 is actually not that difficult in light of Theorem 2.1. It suffices to observe that the regularity conditions imply that series defining the wavelet dimension function (2.2) is uniformly convergent on compact subsets of $\mathbb{R} \setminus \mathbb{Z}$. Since $\mathcal{D}$ is integer-valued and periodic, it must be a constant function (equal to 1).

The other problem solved by Auscher deals with the Hardy space

$$H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \quad \text{for } \xi \le 0\}.$$

Meyer [47] has shown the existence of o.n. wavelets in the Schwartz class. His famous construction produces a band-limited wavelet ψ such that $\hat{\psi} \in C^\infty$ has compact support. He has asked if it is possible to such nice wavelets also in the Hardy space $H^2(\mathbb{R})$. Auscher [2, Theorem 1.1] has shown that this is not possible, see also [38, Theorem 7.6.20].

Theorem 2.7 *There is no o.n. wavelet $\psi \in H^2(\mathbb{R})$ satisfying the regularity assumptions as in Theorem 2.6. In particular, there is no ψ in the Schwartz class such that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an o.n. basis of $H^2(\mathbb{R})$.*

This leaves open the problem of existence of Riesz wavelets which was posed by Seip [55]. We say that ψ is a Riesz wavelet for $\mathcal{H} = H^2(\mathbb{R})$ or $L^2(\mathbb{R})$ if the wavelet system is a Riesz basis of $\mathcal{H}$. A Riesz basis in a Hilbert space $\mathcal{H}$ can be defined as an image of an orthonormal basis under an invertible operator on $\mathcal{H}$. Every Riesz basis has a dual Riesz basis. However, the dual of Riesz wavelet system might not be a wavelet system. If it is, then we say that ψ is a biorthogonal (Riesz) wavelet.

Problem 2.2 Does there exist a Riesz wavelet ψ in $H^2(\mathbb{R})$ such that ψ belongs to the Schwartz class?

Auscher [2] has shown that the answer is negative for biorthogonal Riesz wavelets. However, Auscher's result does not preclude the existence of more general types of Riesz wavelets for which wavelet dimension techniques are not applicable.

2.3 Minimality of MSF Wavelets

Larson [43] has posed an interesting problem about frequency supports of wavelets. Must the support of the Fourier transform of a wavelet contain a wavelet set? This problem stems from the observation that there are two ways of describing minimality of frequency support. The first one is that $\operatorname{supp}\hat{\psi}$ has the smallest possible Lebesgue measure (equal to 1), which is used in the actual definition of an MSF wavelet. The second possibility is to insist that the support is minimal with respect to the inclusion partial order. It is not known whether these two natural definitions of minimality of frequency supports are the same. This is the essence of the following problem posed by Larson in late 1990s although its official formulation appeared only in [43].

Problem 2.3 Is it true that for any orthonormal wavelet $\psi \in L^2(\mathbb{R})$, there exists a wavelet set W such that such that $W \subset \operatorname{supp}\hat{\psi}$?

A positive answer to this problem was given by the second author [53] for the class of MRA wavelets. A special case of Theorem 2.8 for band-limited MRA wavelets was shown in [61].

Theorem 2.8 *Suppose that $\psi \in L^2(\mathbb{R})$ is an MRA wavelet. Then there exists a wavelet set W such that $W \subset \operatorname{supp}\hat{\psi}$.*

In Sect. 4 we give the proof of Theorem 2.8. Despite this initial progress, not much is known about frequency supports of non-MRA wavelets where Problem 2.3 remains wide open. The second author and Speegle [54] have investigated this problem using the concept of an interpolation pair of wavelet sets, which was introduced by Dai and Larson in [28].

2.4 Density of Riesz Wavelets

Another fundamental problem posed by Larson [43] asks about density of Riesz wavelets.

Problem 2.4 Is the collection of all Riesz wavelets dense in $L^2(\mathbb{R})$?

Larson in [43] gives several pieces of evidence why the answer to Problem 2.4 might be affirmative. For example, if ψ_0 and ψ_1 are o.n. wavelets, then their convex combination $(1-t)\psi_0 + t\psi_1$ is a Riesz wavelet for all $t \in \mathbb{R}$ possibly with the exception of $t = 1/2$. Hence, a line connecting any two o.n. wavelets is in the norm closure of the set of Riesz wavelets. In the case of frame wavelets the first author has shown the following positive result [10, Theorem 2.1]. A similar density result was independently obtained by Cabrelli and Molter [22].

Theorem 2.9 *The collection of all framelets*

$$\mathcal{W}_f = \{\psi \in L^2(\mathbb{R}) : \psi \text{ is a framelet}\}.$$

is dense in $L^2(\mathbb{R})$.

In addition, Han and Larson [36] has shown that any $f \in L^2(\mathbb{R})$ can be approximated in $L^2(\mathbb{R})$-norm by a sequence $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{W}_f$ of asymptotically tight frame wavelets. Namely, if $0 < c_k \le d_k < \infty$ denote the lower and the upper frame bounds of ψ_k, then $d_k/c_k \to 1$ as $k \to \infty$. However, the situation changes drastically if we restrict ourselves to the class of tight frame wavelets. These are functions $\psi \in L^2(\mathbb{R})$ satisfying (2.3) with equal bounds $c = d$. Then the answer becomes negative by Bownik [12, Corollary 2.1].

Theorem 2.10 *The collection of all tight frame wavelets*

$$\mathcal{W}_{tf} = \{\psi \in L^2(\mathbb{R}) : \psi \text{ is a tight framelet}\}.$$

is not dense in $L^2(\mathbb{R})$.

A partial positive result related to Problem 2.4 was obtained by Cabrelli and Molter [22], where the authors proved that any $f \in L^2(\mathbb{R}^n)$ can be approximated in $L^2(\mathbb{R}^n)$ norm by Riesz wavelets associated to expansive dilation matrices A and lattices of translates Γ; for definitions see Sect. 3. However, both dilations A and lattices Γ vary with the accuracy of approximation. Hence, Problem 2.4 remains open, since it asks about density of Riesz wavelets for a fixed (dyadic) dilation and a fixed lattice of translates (integers).

2.5 Intersection of Negative Dilates

Yet another fundamental problem in the theory of wavelets was posed by Baggett in 1999. Baggett's problem asks whether every Parseval wavelet ψ must necessarily come from a generalized multiresolution analysis (GMRA). A concept of GMRA was introduced by Baggett et al. [4] as a natural generalization of MRA.

Definition 2.4 A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ is called a *multiresolution analysis* (MRA) if (M1)–(M4) in Definition 2.2 hold and the space V_0 is shift-invariant

(M5') $\quad f(\cdot) \in V_0 \implies f(\cdot - k)$ for all $k \in \mathbb{Z}$.

To formulate Baggett's problem we also need a concept of space of negative dilates.

Definition 2.5 Let $\psi \in L^2(\mathbb{R})$ be a frame wavelet. A *space of negative dilates* of ψ is defined as

$$V(\psi) = \overline{\operatorname{span}}\{\psi_{j,k} : j < 0, k \in \mathbb{Z}\}. \tag{2.4}$$

We say that ψ is associated with a GMRA $\{V_j : j \in \mathbb{Z}\}$ if $V(\psi) = V_0$.

Suppose that $\psi \in L^2(\mathbb{R}^n)$ is a Parseval wavelet. Then, we can define spaces

$$V_j = D^j(V(\psi)) \qquad j \in \mathbb{Z},$$

where $Df(x) = \sqrt{2} f(x)$ is a dilation operator. Baggett has shown that a sequence $\{V_j : j \in \mathbb{Z}\}$ satisfies all properties of GMRA (M1), (M2), (M4), and (M5') possibly with the exception of (M3). Hence, it is natural to ask the following question.

Problem 2.5 Let ψ be a Parseval wavelet with the space of negative dilates $V = V(\psi)$. Is it true that

$$\bigcap_{j \in \mathbb{Z}} D^j(V(\psi)) = \{0\}?$$

Despite its simplicity Problem 2.5 is a difficult open problem and only partial results are known. The authors proved in [16] that if the dimension function (also called multiplicity function) of $V(\psi)$ is not identically ∞, then the answer to Problem 2.5 is affirmative. A generalization of this result was shown in [12]. Problem 2.5 is not only interesting for its own sake, but it also has several implications for other aspects of the wavelet theory. For example, it was shown in [16] that a positive answer would imply that all compactly supported Parseval wavelets come from a MRA, thus generalizing the well-known result of Lemarié-Rieusset [2, 44] for compactly supported (orthonormal) wavelets. However, there is some evidence that the answer to Problem 2.5 might be negative. The authors in [16] have shown examples of (non-tight) frame wavelet ψ such that its space of negative dilates is the largest possible $V(\psi) = L^2(\mathbb{R})$. In fact, the following theorem was shown in [11, Theorem 8.20].

Theorem 2.11 *For any $\delta > 0$, there exists a frame wavelet $\psi \in L^2(\mathbb{R})$ such that:*

(i) the frame bounds of a wavelet system $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ are 1 and $1 + \delta$,
(ii) the space V of negative dilates of ψ satisfies $V(\psi) = L^2(\mathbb{R})$,
(iii) $\hat{\psi}$ is C^∞ and all its derivatives have exponential decay,
(iv) ψ has a dual frame wavelet.

2.6 Extension of Wavelet Frames

A more recent problem was proposed by Christensen and his collaborators [24, 25].

Problem 2.6 Suppose ψ is Bessel wavelet with bound < 1. Does there exist ψ' such that the combined wavelet system

$$\{\psi_{j,k} : j, k \in \mathbb{Z}\} \cup \{\psi'_{j,k} : j, k \in \mathbb{Z}\}.$$

generated by ψ and ψ' is a Parseval frame?

The original formulation of Problem 2.6 asks for an extension of a pair of Bessel wavelets to a pair of dual frames. Hence, Problem 2.6 is a simplified version of a problem proposed in [24]. Despite partial progress in a subsequent work of Christensen et al. [25], either formulation of this problem remains open. It is worth adding that an analogue of Problem 2.6 for Gabor Bessel sequences has been proven in [24, Theorem 3.1].

2.7 A Simple Question that Nobody has Bothered to Answer

The last problem illustrates the difficulty of determining whether a function is a frame wavelet or not. The following problem was proposed by Weber and the first author [21].

Problem 2.7 For $0 < b < 1$ define $\psi_b \in L^2(\mathbb{R})$ by $\hat{\psi}_b = \mathbf{1}_{(-1,-b)\cup(b,1)}$. For what values of $1/8 < b \le 1/6$, is ψ_b a frame wavelet?

The above range of parameter b seems to be the hardest in determining a frame wavelet property of ψ_b. Outside of this range, the following table lists properties of

ψ_b which were shown in [21].

Range of b	Property of ψ_b	Dual frame wavelets of ψ_b
$b = 0$	not a frame wavelet	no duals exist
$0 < b \leq 1/8$	frame wavelet (not Riesz)	no duals exist
$1/6 < b < 1/3$	not a frame wavelet	no duals exist
$1/3 \leq b < 1/2$	biorthogonal Riesz wavelet	a unique dual exists
$b = 1/2$	orthonormal wavelet	a unique dual exists
$1/2 < b < 1$	not a frame wavelet	no duals exist

3 Higher Dimensional Wavelets

In this section we concentrate on problems involving higher dimensional wavelets. Most of the one dimensional problems discussed in Sect. 2 have higher dimensional analogues. Rather surprisingly, their higher dimensional analogues have definitive answers for certain classes of dilation matrices. Subsequently, we shall focus on problems which have been resolved in one or two dimensions, but remain open in higher dimensions.

We start by a higher dimensional analogue of Definition 2.1.

Definition 3.1 Let $A \in GL_n(\mathbb{R})$ be $n \times n$ invertible matrix. Let $\Gamma \subset \mathbb{R}^n$ be a full rank lattice. We say that $\psi \in L^2(\mathbb{R}^n)$ is an o.n. wavelet associated with a pair (A, Γ) if the collection of translates and dilates

$$\psi_{j,k}(x) := |\det A|^{j/2}\psi(A^j x - k), \qquad j \in \mathbb{Z}, k \in \Gamma, \tag{3.1}$$

forms an o.n. basis of $L^2(\mathbb{R})$.

A typical choice for Γ is a standard lattice $\mathbb{Z}^n$. Moreover, we can often reduce to this case by making a linear change of variables. Indeed, suppose that $\Gamma = P\mathbb{Z}^n$ for some $P \in GL_n(\mathbb{R})$. Then, $\psi \in L^2(\mathbb{R}^n)$ is an o.n. wavelet associated with (A, Γ) if and only if $|\det P|^{1/2}\psi(P\cdot)$ is an o.n. wavelet associated with $(P^{-1}AP, \mathbb{Z}^n)$. Hence, the choice of a standard lattice $\Gamma = \mathbb{Z}^n$ is not an essential restriction.

For some of the problems discussed in this section, it is imperative that we allow more than one function generating a wavelet system. Hence, more generally a (A, Γ) wavelet is a finite collection $\{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$, so that the corresponding wavelet system

$$\{\psi^l_{j,k} : l = 1, \dots, L, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$$

is an o.n. basis of $L^2(\mathbb{R}^n)$.

3.1 Known Results

A typical assumption about a dilation A is that it is *expansive* or *expanding*. That is, all of eigenvalues λ of A satisfy $|\lambda| > 1$. This is the class of dilations for which most of the higher dimensional wavelet theory has been developed. In addition, it is often assumed that a dilation A has integer entries, or equivalently

$$A\mathbb{Z}^n \subset \mathbb{Z}^n. \tag{3.2}$$

The latter condition assures that higher dimensional analogue of the classical dyadic wavelet system has nested translation structure across all its scales. Indeed, a wavelet system at scale $j \in \mathbb{Z}$ is invariant under translates by vectors in $A^{-j}\mathbb{Z}^n$. It is often desirable that a wavelet system at $j + 1$ scale, which is invariant under $A^{-j-1}\mathbb{Z}^n$, includes all translations at j scale. This is the main reason for imposing the invariance condition (3.2). For such class of expansive dilations Problems 2.1, 2.3, and 2.5 all remain open.

On the antipodes lie dilations A farthest from preserving the lattice $\mathbb{Z}^n$, satisfying

$$\mathbb{Z}^n \cap (A^T)^j(\mathbb{Z}^n) = \{0\} \qquad \text{for all } j \in \mathbb{Z} \setminus \{0\}, \tag{3.3}$$

where A^T is the transpose of A. Somewhat surprisingly, more is known about wavelets associated with such dilations than those satisfying (3.2).

Theorem 3.12 *Assume that $A \in GL_n(\mathbb{R})$ is an expansive matrix satisfying (3.3). Then, the following hold:*

(i) The collection of all o.n. wavelets associated to $(A, \mathbb{Z}^n)$ is path connected in $L^2(\mathbb{R})$ norm.
(ii) The collection of all Parseval wavelets associated to $(A, \mathbb{Z}^n)$ is path connected in $L^2(\mathbb{R})$ norm.

Proof Bownik [8] and Chui et al. [26] any expansive dilation A satisfying (3.3) admits only minimally supported frequency (MSF) wavelets. That is, any o.n. wavelet associated with A must necessarily be MSF, see also Theorem 3.6. Thus, Problem 2.1 for dilations A satisfying (3.3) is reduced to the connectivity of MSF wavelets in the setting of real expansive dilations. Fortunately, the one dimensional result of Speegle on the connectivity of MSF dyadic wavelets, Theorem 2.3, also works in higher dimensional setting by Speegle [56, Theorem 3.3]. Combining these two results yields part (i).

Part (ii) was shown in [13, Theorem 2.4]. Its proof relies on a fact characterizing L^2 closure of the set of all tight frame wavelets associated with a dilation A satisfying (3.3). A function $f \in L^2(\mathbb{R}^n)$ belong to this closure if and only if its frequency support $W = \operatorname{supp} \hat{f}$ satisfies

$$|W \cap (k + W)| = 0 \qquad \text{for all } k \in \mathbb{Z}^n \setminus \{0\}. \tag{3.4}$$

This enables the reduction of the connectivity problem to the class of MSF Parseval wavelets. This are wavelets of the form $\hat{\psi} = \mathbf{1}_W$, such that:

- the translates $\{W + k\}_{k \in \mathbb{Z}^n}$ pack $\mathbb{R}^n$, i.e., (3.4) holds, and
- $\{(A^T)^j W\}_{j \in \mathbb{Z}}$ is a partition of $\mathbb{R}^n$ modulo null sets.

By the result of Paluszyński et al. [49, Theorem 4.2], the collection of all MSF Parseval wavelets is path connected. Although this result was shown in [49] only for dyadic wavelets in one dimension, it can be generalized to higher dimensions as Speegle's generalizations [56] in the setting of expansive dilations. □

We finish by observing that Problem 2.3 has an immediate affirmative answer for dilations satisfying (3.3). Likewise, Problem 2.5 also has an affirmative answer, for example, using intersection results in [12]. However, it needs to be stressed out that the space of negative dilates $V(\psi)$ does not need to be shift-invariant, see [19].

3.2 Characterization of Dilations

One of the most fundamental problems in wavelet theory asks for a characterization of dilations for which o.n. wavelets exist. Although this problem has been explicitly stated by Speegle [57] and Wang [59], it has been studied earlier in late 1990s.

Problem 3.1 For what dilations $A \in GL_n(\mathbb{R})$ and lattices $\Gamma \subset \mathbb{R}^n$, there exist an orthonormal wavelet associated with (A, Γ)?

A more concrete version of Problem 3.1 asks for a characterization of dilations admitting MSF wavelets.

Definition 3.2 Let (A, Γ) be a dilation-lattice pair. We say that $W \subset \mathbb{R}^n$ is an (A, Γ)-*wavelet set* if

(t) $\{W + \gamma\}_{k \in \Gamma}$ is a partition of $\mathbb{R}^n$ modulo null sets, and
(d) $\{A^j W\}_{j \in \mathbb{Z}}$ is a partition of $\mathbb{R}^n$ modulo null sets.

In analogy to the one dimensional setting, frequency support of an MSF wavelet associated with (A, Γ) is necessarily (A^T, Γ^*)-wavelet set, where Γ^* is the dual lattice of Γ. Hence, we have the following variant of Problem 3.1.

Problem 3.2 Characterize pairs of dilations $A \in GL_n(\mathbb{R})$ and lattices $\Gamma \subset \mathbb{R}^n$ for which wavelet set exists.

Translation tiling (t) exists for any choice of a lattice Γ and is known as a fundamental domain. The existence of dilation tiling (d) has been investigated by Larson et al. [45]. They have shown that there exists a measurable set $W \subset \mathbb{R}^n$ of finite measure satisfying (d) if and only if $|\det A| \neq 1$. Despite these two simple facts, the problem of simultaneous dilation and translation tiling remains open.

The first positive result in this direction was obtained by Dai et al. [29].

Theorem 3.2 *If A is an expansive matrix and Γ is any lattice, then (A, Γ)-wavelet set exists.*

A significant progress toward resolving Problem 3.2 has been obtained by Speegle [57], which was then carried by Ionescu and Wang [39], who have given a complete answer in two dimensions. Here we present a simpler, yet equivalent, formulation of their main result [39, Theorem 1.3].

Theorem 3.3 *Suppose $A \in GL_2(\mathbb{R})$, $|\det A| > 1$, and $\Gamma \subset \mathbb{R}^2$ is a full rank lattice. There exists (A, Γ)-wavelet set $\iff$*

$$V \cap \Gamma = \{0\}, \tag{3.5}$$

where V is the eigenspace corresponding to an eigenvalue λ of A satisfying $|\lambda| < 1$. In particular, if all eigenvalues $|\lambda| \geq 1$, then $V = \{0\}$ and (3.5) holds automatically.

As an illustration of subtleness of Theorem 3.3 we give the following example.

Example 3.1 Let $\Gamma = \mathbb{Z}^2$ and $\alpha \in \mathbb{R}$. Then, the following holds true:

- MSF wavelet does not exist for $A = \begin{bmatrix} 3 & 0 \\ \alpha & 1/2 \end{bmatrix}$ for any $\alpha \in \mathbb{R}$.
- MSF wavelet exists for $A^T = \begin{bmatrix} 3 & \alpha \\ 0 & 1/2 \end{bmatrix} \iff \alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Lemvig and the first author [14] have shown the following result on the ubiquity of MSF wavelets. For any choice of dilation $A \in GL_n(\mathbb{R})$ with $|\det A| \neq 1$, there exists (A, Γ)-wavelet set for almost every full rank lattice Γ. In fact, a slightly stronger result holds.

Theorem 3.4 *Let A be any matrix in $GL_n(\mathbb{R})$ with $|\det A| \neq 1$. Let $\Gamma \subset \mathbb{R}^n$ be any full rank lattice. Then there exists $(A, U\Gamma)$-wavelet set for almost every (in the sense of Haar measure) orthogonal matrix $U \in O(n)$.*

The proof of Theorem 3.4 relies on techniques from geometry of numbers and involves estimates on a number of lattice points in dilates of the unit ball of the form $A^j(B(0, 1))$, where $j \in \mathbb{Z}$. Since $A^j(B(0, 1))$ is a convex symmetric body, this number is at least its volume up to a proportionality constant depending solely on the choice of Γ. If the corresponding upper bound holds

$$\#|\Gamma \cap A^j(B(0,1))| \le C \max(1, |\det A|^j) \qquad \text{for all } j \in \mathbb{Z}, \tag{3.6}$$

then many results in wavelet theory, such as characterizing equations, hold. The main result in [14] shows that (3.6) holds for almost every choice of a lattice Γ, which is then used to prove Theorem 3.4.

The expectation is that the answers to Problems 3.1 and 3.2 are actually the same. In other words, if there exists an o.n. wavelet associated with (A, Γ), then there also exists an MSF wavelet associated with (A, Γ). However, this is unknown since even more basic problem involving Calderón's formula remains open.

3.3 Calderón's Formula

Problem 3.3 was implicitly raised by Speegle [56] and explicitly formulated in [14].

Problem 3.3 Does Calderón's formula

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}((A^T)^j\xi)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n \tag{3.7}$$

hold for any orthonormal (or Parseval) wavelet ψ associated with (A, Γ)?

Bownik and Lemvig [14] Problem 3.3 has affirmative answer for pairs (A, Γ) such that its dual pair (A^T, Γ^*) satisfies the lattice counting estimate (3.6). Indeed, (3.7) is the first of two equations characterizing Parseval wavelets, which has been studied by a large number of authors both for expansive [6, 23, 26, 31, 51] and non-expansive dilations [35, 37]. The second equation states that for all $\alpha \in \Gamma^*$

$$\sum_{j\in\mathbb{Z},(A^T)^{-j}\alpha\in\Gamma^*} \psi((A^T)^{-j}\xi)\overline{\hat{\psi}((A^T)^{-j}(\xi+\alpha))} = \delta_{\alpha,0} \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{3.8}$$

The expectation is that the equations (3.7) and (3.8) characterize Parseval wavelets for all possible pairs (A, Γ). This has been shown for expansive dilations [26], dilations expanding on a subspace [35, 37], and more generally satisfying the lattice counting estimate (3.6). However, Problem 3.3 remains as a formidable obstacle toward this goal. An example in [15, Example 3.1] and more recent work [32] are an evidence of looming difficulties.

3.4 Well-localized Wavelets

A variant of Problem 3.1 asks for a characterization of dilations for which well-localized o.n. wavelets exist. We say that a function $\psi \in L^2(\mathbb{R}^n)$ is *well-localized* if both ψ and $\hat{\psi}$ have polynomial decay. That is, for some large $N > 0$, we have

$$\psi(x) = O(|x|^{-N}) \text{ as } |x| \to \infty \qquad \text{and} \qquad \hat{\psi}(\xi) = O(|\xi|^{-N}) \text{ as } |\xi| \to \infty.$$

Problem 3.4 Let $\Gamma = \mathbb{Z}^n$ be the lattice of translates. For what expansive dilations A do there exist well-localized wavelets (possibly with multiple generators)?

Note that in Problem 3.4 it is imperative that we allow multiple generators of a wavelet system. Indeed, suppose that A is an integer expansive matrix, i.e., (3.2) holds. If $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$ is a well-localized o.n. wavelet associated with an integer dilation, then the number L of generators must be divisible by $|\det A| - 1$. This is a consequence of the fact that the wavelet dimension function defined as

$$\mathcal{D}_\Psi(\xi) := \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^n} |\hat{\psi}^l((A^T)^j(\xi + k))|^2$$

satisfies a higher dimensional analogue of Theorem 2.4. In particular, $\mathcal{D}_\Psi$ is integer-valued and satisfies

$$\int_{[0,1]^n} \mathcal{D}_\Psi(\xi) d\xi = \frac{L}{|\det A| - 1}.$$

If Ψ consists of well-localized functions, then the series defining $\mathcal{D}_\Psi$ converges uniformly and hence it must be constant. Thus, L is divisible by $|\det A| - 1$.

Daubechies [30, Chapter 1] asked whether "there exist orthonormal wavelet bases (necessarily not associated with a multiresolution analysis), with good time-frequency localization, and with irrational a." A partial answer was given by Chui and Shi [27] who showed that all wavelets associated with dilation factors a such that a^j is irrational for all $j \geq 1$ must be minimally supported frequency (MSF). A complete answer was given by the first author [9] who proved the following result.

Theorem 3.5 *Suppose a is an irrational dilation factor, $a > 1$. If $\Psi = \{\psi^1, \dots, \psi^L\}$ is an orthonormal wavelet associated with a, then at least one of ψ^l is poorly localized in time. More precisely, there exists $l = 1, \dots, L$ such that for any $\delta > 0$,*

$$\limsup_{|x| \to \infty} |\psi^l(x)||x|^{1+\delta} = \infty.$$

On the other hand, Auscher [1] proved that there exist Meyer wavelets (smooth and compactly supported in the Fourier domain) for every rational dilation factor. Combining Auscher's result with Theorem 3.5 gives a complete answer to Problem 3.4 in one dimensional case. Well-localized orthonormal wavelets can only exist for rational dilation factors and they are non-existent for irrational dilations.

In higher dimensions Problem 3.4 remains a challenging open problem. A partial answer was given by the first author in [8].

Theorem 3.6 *Suppose A is an expanding matrix such that* (3.3) *holds. If $\Psi = \{\psi^1, \dots, \psi^L\}$ is an o.n. wavelet associated with A, then Ψ is combined MSF, i.e., $\bigcup_{l=1}^{L} \operatorname{supp} \hat{\psi}^l$ has a minimal possible measure (equal to L).*

Since any combined MSF wavelet must satisfy

$$\sum_{l=1}^{L} |\hat{\psi}^l(\xi)|^2 = \chi_W(\xi) \qquad \text{for a.e. } \xi,$$

for some measurable set $W \subset \mathbb{R}^n$, at least one ψ^l is not be well-localized in time. Moreover, Speegle and the first author [19] showed that Theorem 3.6 is sharp, in the sense that it has a converse. The converse result states that if all wavelets associated with an expanding dilation A are MSF, then A must necessarily satisfy (3.3).

To obtain a satisfactory (even partial) answer to Problem 3.4, it is also necessary to construct well-localized wavelets for large classes of expansive dilations. A natural class of well-localized wavelets are r-regular wavelets introduced by Meyer [47]. We recall that a function ψ is r-regular, where $r = 0, 1, 2, \dots$, or ∞, if ψ is C^r with polynomially decaying partial derivatives of orders $\le r$,

$$\partial^\alpha \psi(x) = O(|x|^{-N}) \text{ as } |x| \to \infty \qquad \text{for all } |\alpha| \le r,\ N > 0.$$

For any integer dilation A, which supports a self-similar tiling of $\mathbb{R}^n$, Strichartz [58] constructed r-regular wavelets for all $r \in \mathbb{N}$. However, there are examples in $\mathbb{R}^4$ of dilation matrices without self-similar tiling [40, 41]. In [7] the first author has shown that for every integer dilation and $r \in \mathbb{N}$, there is an r-regular wavelet basis with an associated r-regular multiresolution analysis. However, the question of existence of ∞-regular wavelets in higher dimensions is still open.

3.5 Meyer Wavelets for Integer Dilations

Schwartz class is defined as a collection of all ∞-regular functions on $\mathbb{R}^n$.

Problem 3.5 Do Schwartz class wavelets exist for integer expansive dilations A and lattice $\Gamma = \mathbb{Z}^n$?

One dimensional wavelet in the Schwartz class is a famous example of Meyer [47], which can be adapted to any integer dilation factor $a \geq 2$. In two dimensions an affirmative answer to Problem 3.5 was given by Speegle and the first author [20].

Theorem 3.7 *For every expansive* 2×2 *integer dilation* A*, there exists an o.n. wavelet consisting of* $(|\det A| - 1)$ *band-limited Schwartz class functions.*

Hence Problem 3.5 needs to be answered only in dimensions ≥ 3. It is valid to ask the same question for a larger class of dilations with rational entries. Auscher's result [1] on Meyer wavelets for rational dilations indicates that this might be a valid expectation.

3.6 Schwartz Class Wavelets

We end by stating not that serious, yet curious problem. The only known construction of wavelets in the Schwartz class is a Meyer wavelet which is a band-limited function. Hence, it is natural ask the following question.

Problem 3.6 Suppose ψ is an orthonormal wavelet such that ψ belongs to the Schwartz class. Is $\hat{\psi}$ necessarily compactly supported?

4 Proof of Theorem 2.8

Let ψ be an MRA wavelet as in Definition 2.2. A function φ given in the condition (M5) of Definition 2.2 is called a *scaling function*. For this function there exists a 1-periodic low-pass filter $m \in L^2([0, 1])$ and a 1-periodic high-pass filter $h \in L^2([0, 1])$ such that $\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi)$, $\hat{\psi}(2\xi) = h(\xi)\hat{\varphi}(\xi)$ and the matrix

$$\begin{bmatrix} h(\xi) & h(\xi + \frac{1}{2}) \\ m(\xi) & m(\xi + \frac{1}{2}) \end{bmatrix} \tag{4.1}$$

is unitary for a.e. $\xi \in \mathbb{R}$. In particular,

$$|\hat{\psi}(2\xi)|^2 = |\hat{\varphi}(\xi)|^2 - |\hat{\varphi}(2\xi)|^2 \qquad \text{for a.e. } \xi \in \mathbb{R}. \tag{4.2}$$

Moreover, $\hat{\varphi}$ satisfies the following conditions for a.e. $\xi \in \mathbb{R}$:

(F1) $\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi)$ for some measurable 1-periodic function m,
(F2) $\lim_{j\to\infty} |\hat{\varphi}(2^{-j}\xi)| = 1$,
(F3) $\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + k)|^2 = 1$.

In fact, properties (F1)–(F3) characterize scaling function for an MRA by Hernández and Weiss [38, Theorem 7.5.2].

A *scaling set* is a measurable subset $S \subset \mathbb{R}$, such that the characteristic function $\mathbf{1}_S$ satisfies the above three conditions. This translates into the following conditions (see [50])

(S1) $S \subset 2S$,
(S2) $\lim_{j\to\infty} \mathbf{1}_S(2^{-j}\xi) = 1$,
(S3) $\sum_{k\in\mathbb{Z}} \mathbf{1}_S(\xi + k) = 1$.

If (S1)–(S3) are satisfied, then $W = 2S \setminus S$ is a wavelet set, as defined above Theorem 2.3.

In order to find a wavelet set W in the support of $\hat{\psi}$ we will find a scaling set S in the support of $\hat{\varphi}$ and we shall prove that the wavelet set $W = 2S \setminus S$ is contained in the support of $\hat{\psi}$.

Towards this goal we start with the following basic lemma.

Lemma 4.1 *Let $A \subset \mathbb{R}$ be a set of a finite measure. If K is a measurable subset of $\mathbb{R}$ satisfying* (S3)*, then*

$$\lim_{n\to\infty} \left| \bigcup_{k\in\mathbb{Z}\setminus\{0\}} (A + 2^n k) \cap K \right| = 0.$$

Proof Condition (S3) implies that the measure of K satisfies $|K| = 1$. Let $\epsilon > 0$. Since A and K have a finite measure there is an $M \in \mathbb{N}$ such that $|A \cap [-M, M]^c| \le \frac{\epsilon}{2}$ and $|K \cap [-M, M]^c| \le \frac{\epsilon}{2}$. Let $A_0 = A \cap [-M, M]^c$, $A_1 = A \cap [-M, M]$ and $K_0 = K \cap [-M, M]^c$, $K_1 = K \cap [-M, M]$. We have that

$$\left| \bigcup_{k\in\mathbb{Z}\setminus\{0\}} (A + 2^n k) \cap K \right| \le \left| \bigcup_{k\in\mathbb{Z}\setminus\{0\}} (A + 2^n k) \cap K_0 \right| + \left| \bigcup_{k\in\mathbb{Z}\setminus\{0\}} (A_0 + 2^n k) \cap K_1 \right| + \left| \bigcup_{k\in\mathbb{Z}\setminus\{0\}} (A_1 + 2^n k) \cap K_1 \right|.$$

Clearly, $|\bigcup_{k\in\mathbb{Z}\setminus\{0\}} (A + 2^n k) \cap K_0| \le |K_0| \le \frac{\epsilon}{2}$. Also there is an $N \in \mathbb{N}$ such that $|\bigcup_{k\in\mathbb{Z}\setminus\{0\}} (A_1 + 2^n k) \cap K_1| = 0$ for $n \ge N$. Moreover, since K satisfies (S3), we have

$$\left| \bigcup_{k\in\mathbb{Z}\setminus\{0\}} (A_0 + 2^n k) \cap K_1 \right| \le \left| \bigcup_{k\in\mathbb{Z}} (A_0 + k) \cap K \right| \le \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \mathbf{1}_{A_0}(\xi - k)\mathbf{1}_K(\xi)\, d\xi = \int_{A_0} \sum_{k\in\mathbb{Z}} \mathbf{1}_K(\xi + k)\, d\xi = |A_0| \le \frac{\epsilon}{2}.$$

Therefore, $|\bigcup_{k\in\mathbb{Z}\setminus\{0\}} (A + 2^n k) \cap K| \le \epsilon$, which implies that the limit of the measures is zero. □

For the next lemma we define a 1-periodization of a set $E \subset \mathbb{R}$ by

$$E^P = \bigcup_{k\in\mathbb{Z}} (E + k).$$

Lemma 4.2 *If a measurable set $S' \subset \mathbb{R}$ satisfies* (S1)*,* (S2)*, and*

$$\sum_{k\in\mathbb{Z}} \mathbf{1}_{S'}(\xi + k) \geq 1 \qquad \textit{for a.e. } \xi \in \mathbb{R}, \tag{4.3}$$

then there exists a scaling set contained in S'.

Proof By (4.3) there is a measurable set $K' \subset S'$ satisfying (S3). Let $K_0 = S' \cap [-\frac{1}{2}, \frac{1}{2}]$ and $K = K_0 \cup (K' \setminus K_0^P)$. Clearly, K is a subset of S' satisfying (S3). Moreover, since (S2) holds for S' we conclude that (S2) holds for K_0 and, therefore, for K, which contains K_0.

For an integer $n \geq 0$ define

$$E_n = 2^{-n}K \setminus \bigcup_{j=n+1}^{\infty} ((2^{-j}K)^P \setminus 2^{-j}K).$$

We claim that $S = \bigcup_{n=0}^{\infty} E_n$ is a scaling set contained in S'. For $n \geq 0$ we have that $E_n \subset 2^{-n}K \subset 2^{-n}S' \subset S'$, where the last inclusion follows from (S1) for S'. This proves that $S \subset S'$ and it remains to show that S satisfies (S1)–(S3).

To prove the inclusion $S \subset 2S$, it is enough to check that $E_n \subset 2E_{n+1}$ for $n \geq 0$. Since $2E^P \subset (2E)^P$ holds for every $E \subset \mathbb{R}$ we have that

$$\begin{aligned} E_n &= 2^{-n}K \setminus \bigcup_{j=n+2}^{\infty} ((2^{-j+1}K)^P \setminus 2^{-j+1}K) \\ &\subset 2^{-n}K \setminus \bigcup_{j=n+2}^{\infty} (2(2^{-j}K)^P \setminus 2^{-j+1}K) = 2E_{n+1}. \end{aligned}$$

Thus, (S1) holds for S.

We have already observed that K satisfies (S2). Hence, (S2) for S will follow from the equality $\lim_{j\to\infty} \mathbf{1}_S(2^{-j}\xi) = 1$ for a.e. $\xi \in K$. Let us fix $\xi \in K$. Since we have already proven that $S \subset 2S$, the above equality is satisfied if there is an $n \geq 0$ such that $2^{-n}\xi \in S$. Consider for $n \geq 0$

$$B_n = 2^n \bigcup_{j=n+1}^{\infty} ((2^{-j}K)^P \setminus 2^{-j}K)$$

and $B = \bigcap_{n=0}^{\infty} B_n \cap K$. If $\xi \notin B$, then $\xi \notin B_n \cap K$ for some $n \geq 0$. Since $\xi \in K$, this implies that $2^{-n}\xi \notin 2^{-n}B_n$; that is,

$$2^{-n}\xi \in 2^{-n}K \setminus \bigcup_{j=n+1}^{\infty} ((2^{-j}K)^P \setminus 2^{-j}K) = E_n \subset S.$$

That S satisfies (S2) will follow, if we show that B has measure zero. Clearly, $|B| \leq \liminf_{n\to\infty} |B_n \cap K|$. Since K satisfies (S3) we have

$$(2^{-j}K)^P \setminus 2^{-j}K = \bigcup_{k\in\mathbb{Z}\setminus\{0\}} (2^{-j}K + k)$$

for $j \geq 0$. Therefore, for $n \geq 0$

$$\begin{aligned} B_n = 2^n \bigcup_{j=n+1}^{\infty} \bigcup_{k\in\mathbb{Z}\setminus\{0\}} (2^{-j}K + k) &= \bigcup_{j=n+1}^{\infty} \bigcup_{k\in\mathbb{Z}\setminus\{0\}} (2^{n-j}K + 2^n k) \\ &= \bigcup_{j=1}^{\infty} \bigcup_{k\in\mathbb{Z}\setminus\{0\}} (2^{-j}K + 2^n k) = \bigcup_{k\in\mathbb{Z}\setminus\{0\}} (A + 2^n k), \end{aligned}$$

where $A = \bigcup_{j=1}^{\infty} 2^{-j}K$. Since (S3) for K gives that $|K| = 1$, A is of finite measure $|A| \leq \sum_{j=1}^{\infty} |2^{-j}K| = 1$. From Lemma 4.1 it follows that $\lim_{n\to\infty} |B_n \cap K| = 0$, which shows that B has measure zero. Hence S satisfies (S2).

Proving that S satisfies (S3) is equivalent to showing that modulo null sets $S^P = \mathbb{R}$ and

$$(S + k) \cap S = \emptyset \qquad \text{for } k \in \mathbb{Z} \setminus \{0\}. \tag{4.4}$$

To see that the intersection is empty it is enough to check that

$$(E_n + k) \cap E_m = \emptyset \qquad \text{for } k \in \mathbb{Z} \setminus \{0\},\ m, n \geq 0.$$

Without loss of generality we can assume that $m \leq n$. If $m = n$, then

$$(E_n + k) \cap E_n \subset (2^{-n}K + k) \cap 2^{-n}K = 2^{-n}((K + 2^n k) \cap K) = \emptyset$$

for $k \in \mathbb{Z} \setminus \{0\}$, since K satisfies (S3). Likewise,

$$(E_n + k) \cap 2^{-n}K \subset (2^{-n}K + k) \cap 2^{-n}K = \emptyset$$

for $k \in \mathbb{Z} \setminus \{0\}$. Thus, $E_n + k \subset (2^{-n}K)^P \setminus 2^{-n}K$. If $m < n$, then $E_n + k \subset \bigcup_{j=m+1}^{\infty} (2^{-j}K)^P \setminus 2^{-j}K$. This implies that $(E_n + k) \cap E_m = \emptyset$ for $k \in \mathbb{Z} \setminus \{0\}$ and hence (4.4) holds.

The last step of the proof is to show that $S^P = \mathbb{R}$ (modulo null sets). Since K satisfies (S3), it is enough to prove that $K \subset S^P$. For $n \geq 0$ let

$$C_n = \bigcup_{j=n+1}^{\infty} (2^{-j}K)^P \qquad \text{and} \qquad C = \bigcap_{n=0}^{\infty} C_n \cap K.$$

Let $\xi \in K$. If $\xi \notin C_0$, then $\xi \in K \setminus C_0 \subset E_0 \subset S$. Therefore, we can concentrate on the case when $\xi \in C_0$. Since $C_{n+1} \subset C_n$, it is clear that either $\xi \in C$ or there is an $n' \geq 1$ such that $\xi \in C_{n'-1} \setminus C_{n'}$. If the latter is satisfied, then $\xi \in (2^{-n'}K)^P$. Therefore, there is an $l \in \mathbb{Z}$ such that $\xi + l \in 2^{-n'}K$. Moreover, since ξ does not belong to $C_{n'}$, neither does $\xi + l$. This gives us $\xi + l \in 2^{-n'}K \setminus C_{n'} \subset E_{n'} \subset S$. This proves that $\xi \in S^P$.

We close the proof by showing that C has measure zero. Indeed, $|C| \leq \liminf_{n\to\infty} |C_n \cap K|$. Since K satisfies (S3), for any measurable $E \subset \mathbb{R}$ we have that $|E^P \cap K| = |E|$, therefore

$$|C_n \cap K| \leq \sum_{j=n+1}^{\infty} |(2^{-j}K)^P \cap K| = \sum_{j=n+1}^{\infty} |2^{-j}K| = 2^{-n}|K| = 2^{-n},$$

proving that $|C| = 0$. Thus, we have shown that (S3) holds for S and this ends the proof of the lemma. □

The above lemma immediately yields the following

Proposition 4.3 *For every scaling function φ there exists a scaling set contained in the support of $\hat{\varphi}$.*

Proof Let S' be the support of $\hat{\varphi}$. Since $\hat{\varphi}$ satisfies conditions (F1)–(F3), it is clear that S' satisfies (S1), (S2) and (4.3). Therefore, by Lemma 4.2, there exists a scaling set contained in S'. □

Finally we can conduct the proof of Theorem 2.8.

Proof of Theorem 2.8 Let ψ be an MRA wavelet and φ its associated scaling function. By Proposition 4.3 there is a scaling set S contained in $\operatorname{supp}\hat{\varphi}$. We want to show that the wavelet set $W = 2S \setminus S$ is contained in $\operatorname{supp}\hat{\psi}$.

From (4.2) it follows that $|\hat{\psi}(\xi)|^2 = |\hat{\varphi}(\frac{\xi}{2})|^2 - |\hat{\varphi}(\xi)|^2$. Therefore

$$\operatorname{supp}\hat{\psi} = (2\operatorname{supp}\hat{\varphi}) \setminus D, \qquad \text{where } D = \{\xi \in \mathbb{R} : |\hat{\varphi}(\frac{\xi}{2})| = |\hat{\varphi}(\xi)|\}.$$

Clearly, $W = 2S \setminus S \subset 2\operatorname{supp}\hat{\varphi}$. Thus, in order to establish that $W \subset \operatorname{supp}\hat{\psi}$, it is enough to prove that $W \cap D = \emptyset$ or, equivalently, that $2S \cap D \subset S$. Assume that $\xi \in 2S \cap D$, then $|\hat{\varphi}(\frac{\xi}{2})| = |\hat{\varphi}(\xi)| > 0$, because $\frac{\xi}{2} \in S$. This implies that $|m(\frac{\xi}{2})| = 1$, where m is the low-pass filter of φ. Thus, the unitarity of (4.1) and

1-periodicity of m gives that $m(\frac{\xi}{2}+\frac{k}{2})=0$ for all odd integers k. Hence, for such k we obtain that $\hat{\varphi}(\xi+k)=m(\frac{\xi}{2}+\frac{k}{2})\hat{\varphi}(\frac{\xi}{2}+\frac{k}{2})=0$. This already implies that $\xi\in S$. Indeed, since S satisfies (S3), there is an $l\in\mathbb{Z}$ such that $\xi+l\in S$. Thus, $\hat{\varphi}(\xi+l)\neq 0$, so l must be even. Obviously, $\frac{\xi}{2}+\frac{l}{2}\in\frac{S}{2}\subset S$ and $\frac{\xi}{2}\in S$. Therefore, (S3) for S implies that $l=0$ and $\xi\in S$ follows. □

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When Is Every Quasi-Multiplier a Multiplier?

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Dedicated to the memory of Ronald G. Douglas

Abstract We answer the title question for σ-unital C^*-algebras. The answer is that the algebra must be the direct sum of a dual C^*-algebra and a C^*-algebra satisfying a certain local unitality condition. We also discuss similar problems in the context of Hilbert C^*-bimodules and imprimitivity bimodules and in the context of centralizers of Pedersen's ideal.

Keywords Multiplier · Quasimultiplier · Hilbert C^*-bimodule · Imprimitivity bimodule · Calkin algebra

Mathematics Subject Classification (2010) AMS subject classification: 46L05

1 Introduction

Let A be a C^*-algebra and A^{**} its Banach space double dual, also known as its enveloping von Neumann algebra. An element T of A^{**} is called a multiplier of A if $Ta \in A$ and $aT \in A$, $\forall a \in A$. Also T is a left multiplier if $Ta \in A$, $\forall a \in A$, T is a right multiplier if $aT \in A$, $\forall a \in A$ and T is a quasi-multiplier if $aTb \in A$, $\forall a, b \in A$. The sets of multipliers, left multipliers, right multipliers and quasi-multipliers are denoted respectively by $M(A)$, $LM(A)$, $RM(A)$, and $QM(A)$. More information about multipliers, etc. can be found in [12, §3.12].

We believe that quasi-multipliers were first introduced to operator algebraists in [2]. It was shown there that a self-adjoint element h of A^{**} is a multiplier if and only if $\pm h$ satisfy a certain semicontinuity property, and self-adjoint quasi-

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_5

multipliers are characterized similarly with a weaker semicontinuity property. The fact that these semicontinuity properties are in general different was one of the key "complications" discovered in [2]. We will not use semicontinuity theory in any proofs in this paper.

Multipliers of C^*-algebras have many important applications. In particular they play a crucial role in the theory of extensions of C^*-algebras, as shown in [9], and they are used in KK-theory. Quasi-multipliers, though less important, also have applications as shown, for example, in [3, 13], and [8]. (Note that [8] contains the results of [13]).

It is obvious that $QM(A) = M(A)$ if A is commutative, and more generally if A is n-homogeneous. Also $QM(A) = M(A)$ if A is elementary, and therefore also if A is dual; i.e., if A is the direct sum of elementary C^*-algebras. It follows from [3, Theorem 4.9] that if $LM(A) = M(A)$ for a σ-unital C^*-algebra A, then also $QM(A) = M(A)$. (This is shown also without σ-unitality in Proposition 3.7 below.) Therefore it is sufficient to consider the title question.

My association with Ron Douglas was very beneficial to and influential in my career. In particular my interests in multipliers and Calkin algebras arose from this association.

2 Preliminaries

A C^*-algebra A is called *locally unital* if there is a family $\{I_j\}$ of (closed, two-sided) ideals such that $(\sum I_j)^- = A$ and for each j there is u_j in A such that $(\mathbf{1} - u_j)I_j = I_j(\mathbf{1} - u_j) = \{0\}$. Here $\mathbf{1}$ is the identity of A^{**}. Since this concept may not be completely intuitive, we will explore what it means.

Proposition 2.1 *If I is an ideal of A, then there is u in A such that $(\mathbf{1} - u)I = I(\mathbf{1} - u) = \{0\}$ if and only if there is an ideal J such that $IJ = \{0\}$ and A/J is unital.*

Proof If u is as above, let J be the closed span of $A(\mathbf{1} - u)A$. Then clearly $IJ = \{0\}$. Also the image of u is an identity for A/J, since, for example, the fact that $((\mathbf{1}-u)a)^*(\mathbf{1}-u)a \in J$ implies that $(\mathbf{1}-u)a \in J$. Conversely if J is as above, let u be an element of A whose image is the identity of A/J, then $(\mathbf{1}-u)I \subset I \cap J = \{0\}$, and similarly $I(\mathbf{1} - u) = \{0\}$. □

Note that it follows from the above that u may be taken to be a positive contraction. The following lemma is undoubtedly known but we don't know a reference.

Lemma 2.2 *If I and J are ideals of a C^*-algebra A such that $IJ = \{0\}$, A/I is unital, and A/J is unital, then A is unital.*

Proof Let u and v be elements of A such that the image of u is the identity of A/I and the image of v is the identity of A/J. Then both u and v map to the identity of

$A/(I+J)$. Therefore $u-v=x+y$ with $x \in I$ and $y \in J$. If $w=u-x=v+y$, then w gives the identity both modulo I and modulo J. Therefore w is an identity for A. □

We denote by prim A the primitive ideal space of A. The basic facts about prim A can be found in [12, §4.1].

Proposition 2.3 *The C^*-algebra A is locally unital if and only if:*

(i) *Every compact subset of prim A has compact closure, and*
(ii) *For every closed compact subset of prim A the corresponding quotient algebra is unital.*

Proof If A is locally unital let $\{I_j\}$ be as in the definition. Then $\{\text{prim} I_j\}$ is an open cover of prim$A$. If K is a compact subset of prim A, then there are $I_{j_1}, \ldots, I_{j_n}$ such that $K \subset \bigcup_1^n \text{prim} I_{j_l}$. By Proposition 2.1 and Lemma 2.2 there is an ideal J such that $(I_{j_1}+\cdots+I_{j_n})J=\{0\}$ and A/J is unital. It follows that if $L=\text{hull}(J)$, then L is compact and closed, and $L \supset \bigcup_1^n \text{prim} I_{j_l} \supset K$. This implies both (i) and (ii).

Now assume (i) and (ii). There is an open cover $\{U_j\}$ of prim$A$, such that each U_j is contained in a compact set K_j. If I_j is the ideal corresponding to U_j, then $A=(\sum I_j)^-$. If $L_j=\overline{K_j}$ and $J_j=\ker(L_j)$, then A/J_j is unital and $J_j I_j=\{0\}$. □

Note that a dual C^*-algebra is locally unital if and only if all of the elementary C^*-algebras in its direct sum decomposition are finite dimensional. This follows from the above proposition, or it can be deduced directly from the definition.

If $\mathcal{A}=\{A_x : x \in X\}$ is a continuous field of C^*-algebras over a locally compact Hausdorff space X, then the corresponding C^*-algebra is the set of continuous sections of $\mathcal{A}$ vanishing at ∞. Of course n-homogeneous C^*-algebras arise in this way, where each A_x is isomorphic to the algebra of $n \times n$ matrices. The local unitality of such algebras is discussed in the next proposition.

Proposition 2.4 *Let A be the C^*-algebra arising from a continuous field of C^*-algebras $\{A_x\}$ over a locally compact Hausdorff space X. If each A_x is unital and if the identity section is continuous, then A is locally unital. Conversely, if each A_x is simple and A is locally unital, then each A_x is unital and the identity section is continuous.*

Proof For the first statement let $\{U_j\}$ be an open cover of X such that each $\overline{U_j}$ is compact. If I_j is the set of continuous sections vanishing outside of U_j, then I_j is an ideal of A and $A=(\sum I_j)^-$. If f_j is a continuous scalar–valued function on X vanishing at ∞ such that $f_j(x)=1$, $\forall x \in U_j$, let $u_j=f_j\mathbf{1}$. Then $u_j \in A$ and $(\mathbf{1}-u_j)I_j=I_j(\mathbf{1}-u_j)=\{0\}$.

Now assume each A_x is simple and A is locally unital. Note that prim$A$ can now be identified with X. If K is a compact subset of X, then the corresponding quotient algebra is obtained from the restriction of the continuous field to K. It follows from Proposition 2.3 that this algebra is unital. Therefore each A_x, $x \in K$, is unital and the identity section is continuous on K. Since X is locally compact, the result follows. □

Note that A arises from a continuous field of simple C^*-algebras over a locally compact Hausdorff space if and only if prim$A$ is Hausdorff. We provide an example to show that the simplicity is necessary in the second statement of Proposition 2.4. Let $X = [0, 1]$ and let $A_x = \mathbb{C} \oplus \mathbb{C}$ for $x \neq 0$. Let $A_0 = \mathbb{C}$, identified with $\mathbb{C} \oplus \{0\} \subset \mathbb{C} \oplus \mathbb{C}$. So A is the set of continuous functions f from $[0, 1]$ to $\mathbb{C} \oplus \mathbb{C}$ such that $f(0) \in \mathbb{C} \oplus \{0\}$. Clearly the identity section of this continuous field is not continuous. But A is commutative and therefore locally unital.

We now establish some notations and record some facts that will be used throughout the next section. Let e be a strictly positive element of a σ-unital C^*-algebra A. Strictly positive elements are discussed in [12, §3.10]. One property is that the kernel projection of e in the von Neumann algebra A^{**} is 0. Another is that the sets eA, Ae, and eAe are dense in A.

We will also use the concept of open projection, see [1] or the end of [12, §3.11]. Certain projections in A^{**} are called open, and there is an order–preserving bijection between open projectious and hereditary C^*-subalgebras of A. If p is the open projection for the hereditary C^*-subalgebra B then $B = (pA^{**}p) \cap A$ and any approximate identity for B converges to p in the strong topology of A^{**}. Also p is central in A^{**} if and only if B is an ideal, and for general p the central cover of p in A^{**} is the open projection for the ideal of A generated by B.

If U is an open subset of $(0, \infty)$, then the spectral projection $\chi_U(e)$ is an open projection. The corresponding subalgebra B is the hereditary C^*-algebra B generated by $f(e)$, where f is any continuous function on $[0, \infty)$ such that $U = \{x : f(x) \neq 0\}$. If $U = (\epsilon, \infty)$, we will denote the corresponding subalgebra by B_ϵ, and if $U = (0, \epsilon)$ we will denote the subalgebra by C_ϵ. Also we denote by I_ϵ the ideal generated by C_ϵ. (The reason we are looking at subsets of $(0, \infty)$ instead of $[0, \infty)$ is that the kernel projection of e is 0.)

If p and q are open projections with corresponding subalgebras C and B, let $X(p, q)$ denote the closed linear span of CAB. Two facts that we don't need are that $X(p, q) = (pA^{**}q) \cap A$ and that a (in A) is in $X(p, q)$ if and only if $a^*a \in B$ and $aa^* \in C$. A fact that we do need is that $X(p, q) = \{0\}$ if and only if p and q are centrally disjoint in A^{**}. This follows from the fact that the strong closure of $X(p, q)$ in A^{**} is $pA^{**}q$.

The following lemma is probably known, but we don't know a reference.

Lemma 2.5 *If A is an infinite dimensional C^*-algebra, then A contains an infinite sequence $\{B_n\}$ of mutually orthogonal non-zero hereditary C^*-subalgebras.*

Proof By [10, 4.6.14] A contains a self-adjoint element h whose spectrum, $\sigma(h)$, is infinite. Therefore $\sigma(h)$ contains a cluster point x_0. Then there is a sequence $\{x_n\} \subset \sigma(h)$ such that $x_n \neq x_0$, $x_n \neq x_m$ for $n \neq m$, $x_n \neq 0$, and $\{x_n\}$ converges to x_0. It is a routine exercise to find mutually disjoint open sets U_n such that $x_n \in U_n$. For each n find a continuous function f_n such that $f_n(x_n) \neq 0$, $f_n(0) = 0$, and

$f_n(x) = 0$ for x not in U_n. Then let B_n be the hereditary C^*-algebra generated by $f_n(h)$. □

3 Results and Concluding Remarks

Throughout this section, up to and including the proof of theorem 3.4, A is a $\sigma-$unital C^*-algebra, e is a strictly positive element of A, and the notations of the previous section apply. The following is the main lemma.

Lemma 3.1 *If $\epsilon > 0$ and $\{B_n\}$ is an infinite sequence of mutually orthogonal hereditary C^*-algebras of B_ϵ such that $B_n \cap I_{\frac{1}{n}} \neq \{0\}$, $\forall n$, then $QM(A) \neq M(A)$.*

Proof Let $p_n = \chi_{(0,\frac{1}{n})}(e)$ and let q_n be the open projection for B_n. Since $B_n \cap I_{\frac{1}{n}} \neq \{0\}$ and the central cover of p_n is the open projection for $I_{\frac{1}{n}}$, p_n and q_n are not centrally disjoint in A^{**}. Therefore $X(p_n, q_n) \neq \{0\}$. Since $\chi_{(\theta,\frac{1}{n})}(e)$ converses to p_n as $\theta \searrow 0$, it follows that also $X(\chi_{(\theta,\frac{1}{n})}(e), q_n) \neq 0$ for θ sufficiently small. Then we can recursively choose n_k and θ_k such that $n_k \to \infty$, $\sum_1^\infty \frac{1}{n_k} < \infty$, $0 < \theta_k < \frac{1}{n_k}$, $\frac{1}{n_{k+1}} < \theta_k$, and $X(\chi_{(\theta_k,\frac{1}{n_k})}(e), q_{n_k}) \neq \{0\}$. Choose $a_k \in X(\chi_{(\theta_k,\frac{1}{n_k})}(e), q_{n_k})$ such that $\|a_k\| = 1$. Then the a_k's are mutually orthogonal in the sense that $a_k^* a_l = a_k a_l^* = 0$ for $k \neq l$. Thus $T = \sum_1^\infty a_k$ exists in A^{**}. Since $eT = \sum_1^\infty e a_k$ and $\|e a_k\| \leq \frac{1}{n_k}$, then $eT \in A$. Since Ae is dense in A, this implies that $T \in RM(A) \subset QM(A)$. We claim that $T \notin M(A)$. If f is a continuous function such that $f(0) = 0$ and $f(x) = 1$ for $x \geq \epsilon$, then $Tf(e) = T$. Thus $T \in M(A)$ implies $T \in A$. To see that this is not so, let $r_k = \chi_{(\theta_k,\frac{1}{n_k})}(e)$. Since $\|r_k e\| \to 0$ and eA is dense in A, $T \in A$ would imply $\|r_k T\| \to 0$. But $r_k T = a_k$ and $\|a_k\| = 1$. □

Lemma 3.2 *If $QM(A) = M(A)$ and $I = \bigcap_1^\infty I_{\frac{1}{n}}$, then I is a dual C^*-algebra and a direct summand of A.*

Proof If $\epsilon > 0$, then it is impossible to find infinitely many non-zero mutually orthogonal hereditary C^*-subalgebras of $B_\epsilon \cap I$. By Lemma 2.5 $B_\epsilon \cap I$ is finite dimensional. This implies that id $(B_\epsilon \cap I)$, the ideal generated by $B_\epsilon \cap I$, is the direct sum of finitely many elementary C^*-algebras. Since I is the limit of id $(B_\epsilon \cap I)$ as $\epsilon \searrow 0$, it follows that I is dual. If z is the open central projection for I, the fact that $B_\epsilon \cap I$ is finite dimensional implies that $z\chi_{(\epsilon,\infty)}(e)$ is a finite rank projection in the dual C^*-algebra I. It then follows from spectral theory that $ze \in I$. This implies that $z \in M(A)$, whence I is a direct summand of A. □

Lemma 3.3 *If $QM(A) = M(A)$ and $\bigcap_1^\infty I_{\frac{1}{n}} = \{0\}$, then $\forall \epsilon > 0$, $\exists \delta > 0$ such that $I_\delta B_\epsilon = \{0\}$.*

Proof If $I_\delta \cap B_\epsilon$ is finite dimensional for some $\delta > 0$, then dim $(I_{\frac{1}{n}} \cap B_\epsilon)$ must stabilize at some finite value as n increases. It follows that the set $I_{\frac{1}{n}} \cap B_\epsilon$ also stabilizes, and hence $I_{\frac{1}{n}} \cap B_\epsilon = \{0\}$ for n sufficiently large. Therefore we may assume $I_\delta \cap B_\epsilon$ is infinite dimensional, $\forall \delta > 0$.

Case 1 There is $\delta > 0$ such that $B_\epsilon \cap I_\delta \cap I_{\frac{1}{n}}$, is an essential ideal of $B_\epsilon \cap I_\delta$, $\forall n$. Then, using Lemma 2.5, choose a sequence $\{B_n\}$ of non-zero mutually orthogonal hereditary C^*–subalgebras of $B_\epsilon \cap I_\delta$. By the essential property $B_n \cap I_{\frac{1}{n}} \neq 0$, $\forall n$. So Lemma 3.1 gives a contradiction.

Case 2 For each $\delta > 0$, there is n such that $B_\epsilon \cap I_\delta \cap I_{\frac{1}{n}}$ is not essential in $B_\epsilon \cap I_\delta$. Then we can construct n_k recursively so that $n_{k+1} > n_k$ and $B_\epsilon \cap I_{\frac{1}{n_{k+1}}}$ is not essential in $B_\epsilon \cap I_{\frac{1}{n_k}}$. Then for each k there is a non-zero hereditary C^*–subalgebra B_k of $B_\epsilon \cap I_{\frac{1}{n_k}}$ such that B_k is orthogonal to $I_{\frac{1}{n_{k+1}}}$. Again Lemma 3.1 produces a contradiction. □

Theorem 3.4 *If A is a σ-unital C^*-algebra, then $QM(A) = M(A)$ if and only if A is the direct sum of a dual C^*-algebra and a locally unital C^*-algebra.*

Proof Assume $QM(A) = M(A)$. By Lemma 3.2 $A = I \oplus A_1$ where I is a dual C^*-algebra and A_1 satisfies the hypothesis of Lemma 3.3. For each $\delta > 0$, A_1/I_δ is unital, since $f(e)$ maps to a unit for A_1/I_δ for any continuous function f such that $f(x) = 1$ for $x \geq \delta$. So if J_ϵ is the ideal generated by B_ϵ and $I_\delta B_\epsilon = \{0\}$, then by Lemma 1.1, there is u in A such that $(\mathbf{1} - u)J_\epsilon = J_\epsilon(\mathbf{1} - u) = \{0\}$. Since $A_1 = (\sum J_{\frac{1}{n}})^-$, we have shown that A_1 is locally unital.

Now assume $A = A_0 \oplus A_1$ where A_0 is dual and A_1 is locally unital. (For this part we don't need σ–unitality.) Then $QM(A_0) = M(A_0)$, since A_0 is an ideal in A_0^{**}. Let $A_1 = (\sum I_\alpha)^-$, as in the definition of locally unital. For each α, the weak closure of I_α in A_1^{**} will be denoted by I_α^{**} (to which it is isomorphic). Then I_α^{**} is an ideal in A_1^{**}. If $(u_\alpha - \mathbf{1})I_\alpha = \{0\}$, then also $(u_\alpha - \mathbf{1})I_\alpha^{**} = \{0\}$. If $T \in QM(A_1)$ and $x \in I_\alpha$, then since $Tx \in I_\alpha^{**}$, $Tx = u_\alpha Tx \in A_1$. Therefore $T \in LM(A)$. A symmetrical proof shows also that $T \in RM(A)$. □

Corollary 3.5 *If A is a σ-unital primitive C^*-algebra, then $QM(A) = M(A)$ if and only if A is either elementary or unital.*

Proof If A is primitive, prim A has a dense point. Since singleton sets are compact, Proposition 2.3 implies A is unital if it is locally unital. □

Combining the theorem with [4, Theorem 3.27], we obtain:

Corollary 3.6 *If A is a σ-unital C^*-algebra, then the middle and weak forms of semicontinuity coincide in A^{**} if and only if A is the direct sum of a dual C^*-algebra and a locally unital C^*-algebra.*

It occurred belatedly to us that since many of the applications of quasi-multipliers concern quasi-multipliers of imprimitivity bimodules or Hilbert C^*-bimodules, it might make sense to consider the title question in a broader context. In proposition 3.7 below, whose proof is purely formal, the imprimitivity bimodule case is reduced to what has already been done. Example 3.8 below deals with what are probably the simplest examples of Hilbert C^*-bimodules that are not impimitivity bimodules. Although we have a solution for these examples, it has not immediately inspired a conjecture for the general case. But Example 3.8 does suggest that it may be easier to find when $QM(X) = M(X)$ than to answer the constituent questions whether $LM(X) \subset RM(X)$ or $RM(X) \subset LM(X)$.

Hilbert C^*-bimodules were introduced in [13] as a generalization of imprimitivity bimodules. If X is an $A - B$ Hilbert C^*-bimodule, then X has a linking algebra L. Then L is a C^*-algebra endowed with two multiplier projections p and q, such that $p + q = \mathbf{1}$, pLp is identified with A, qLq is identified with B, and pLq is identified with X. The existence of L may be taken as a working definition of Hilbert C^*-bimodule. Then X is an $A - B$ imprimitivity bimodule if and only if each of A and B generates L as an ideal. Linking algebras of imprimitivity bimodules were introduced in [7].

If X and L are as above, then we define $M(X) = pM(L)q$, $LM(X) = pLM(L)q$, $RM(X) = pRM(L)q$, and $QM(X) = pQM(L)q$. Note that X^{**} can be identified with $pL^{**}q$. It is not hard to see that for $T \in X^{**}$, $T \in M(X)$ if and only if $aT \in X$ and $Tb \in X$, $\forall a \in A, b \in B, T \in LM(X)$ if and only if $Tb \in X$, $\forall b \in B$, $T \in RM(X)$ if and only if $aT \in X$, $\forall a \in A$, and $T \in QM(X)$ if and only if $aTb \in X$, $\forall a \in A$, $b \in B$. Because it is no longer true that $RM(X) = LM(X)^*$, there are more than one question to consider. Since $M(X) = LM(X) \cap RM(X)$, there are actually only two questions. Namely, we ask whether $QM(X) = LM(X)$, which turns out to be equivalent to $RM(X) \subset LM(X)$, and whether $QM(X) = RM(X)$, which is equivalent to $LM(X) \subset RM(X)$.

Proposition 3.7 *Let A be a C^*-algebra.*

(i) *Then $LM(A) = M(A)$ if and only if $QM(A) = M(A)$.*
Let X be an $A - B$ Hilbert C^*–bimodule.
(ii) *Then $QM(X) = LM(X)$ if and only if $RM(X) \subset LM(X)$.*
(iii) *Then $QM(X) = RM(X)$ if and only if $LM(X) \subset RM(X)$*
Let X be an $A - B$ imprimitivity bimodule and L its linking algebra.
(iv) *Then $QM(X) = LM(X)$ if and only if $QM(A) = M(A)$.*
(v) *Then $QM(X) = RM(X)$ if and only if $QM(B) = M(B)$.*
(vi) *Then $QM(X) = M(X)$ if and only if $QM(L) = M(L)$.*

Proof

(i) Assume $LM(A) = M(A)$ and $T \in QM(A)$. If $a \in A$, then $aT \in LM(A)$. Hence $aT \in M(A)$ and $AaT \subset A$. Since $A^2 = A$, this implies $QM(A) \subset RM(A)$. Since $QM(A) = QM(A)^*$ and $LM(A) = RM(A)^*$, we also have $QM(A) \subset LM(A)$, whence $QM(A) = M(A)$.

(ii) Since $RM(X) \subset QM(X)$, one direction is obvious. So assume $RM(X) \subset LM(X)$ and let $T \in QM(X)$. For $b \in B$, $Tb \in RM(X)$. Therefore $TbB \subset X$. Since $B^2 = B$, it follows that $T \in LM(X)$.

(iii) is similar to (ii).

(iv) First assume $QM(X) = LM(X)$ and let $T \in QM(A)$. If $x \in X$, then $Tx \in RM(X) \subset QM(X)$. Therefore $Tx \in LM(X) \subset LM(L)$. So if $y \in X$, then $Txy^* = T\langle x, y\rangle_A \in A$. Since X is an imprimitivity bimodule, $\langle X, X\rangle_A$ spans a dense subset of A. Thus we have shown $QM(A) = M(A)$.

Now assume $QM(A) = M(A)$ and $T \in QM(X)$. If $x \in X$, then $Tx^* \in RM(A) \subset QM(A)$. Therefore $Tx^* \in M(A) \subset M(L)$. So for $y \in X$, $Tx^*y = T\langle x, y\rangle_B \in X$. Since the span of $\langle X, X\rangle_B$ is dense in B, this shows that $T \in LM(X)$.

(v) is similar to (iv)

(vi) Since one direction is obvious, assume $QM(X) = M(X)$. By (iv) and (v), then $QM(A) = M(A)$ and $QM(B) = M(B)$. Note that X^* is a $B - A$ Hilbert C^*–bimodule, $QM(X^*) = QM(X)^*$, and $M(X^*) = M(X)^*$. So for each of the four components of L, we have that every quasi-multiplier is a multiplier. Therefore $QM(L) = M(L)$. □

In connection with (iv) and (v) note that A is strongly morita equivalent to B if and only if an $A - B$ imprimitivity bimodule exists. Several important properties of C^*-algebras are preserved by strong morita equivalence. The property of being a dual C^*-algebra is so preserved but local unitality is not preserved. It is easy to construct examples of strongly morita equivalent separable C^*-algebras A and B such that $QM(A) = M(A)$ but $QM(B) \neq M(B)$

Example 3.8 Let H_1 and H_2 be Hilbert spaces and let $X = \mathbb{K}(H_2, H_1)$, the space of compact operators from H_2 to H_1. Then X is a $\mathbb{K}(H_1) - \mathbb{K}(H_2)$ imprimitivity bimodule, but X can also be made into an $A - B$ Hilbert C^*-bimodule in many ways. We just take A and B to be C^*-subalgebras of $B(H_1)$ and $B(H_2)$ such that $A \supset \mathbb{K}(H_1)$ and $B \supset \mathbb{K}(H_2)$. Now X^{**} can be identified with $B(H_2, H_1)$, the space of all bounded linear operators from H_2 to H_1. (Note that $L \subset B(H_1 \oplus H_2)$.) For $T \in X^{**}$, $T \in LM(X)$ if and only if Tb is compact, $\forall b \in B$. This is equivalent to $T^*TB \subset \mathbb{K}(H_2)$. In other words, $T \in LM(X)$ if and only if the image of T^*T in the Calkin algebra of H_2 is orthogonal to the image of B in this Calkin algebra. Similarly $T \in RM(X)$ if and only if the image of TT^* in the Calkin algebra of H_1 is orthogonal to the image of A in this Calkin algebra. To analyze this, we add some reasonable hypotheses. Assume that each of H_1 and H_2 is separable and infinite dimensional and that each of A and B is σ-unital. Note that any non-zero projection in the Calkin algebra can be lifted to a projection which is necessarily of infinite rank and that if P and Q are infinite rank projections in $B(H_1)$ and $B(H_2)$ there is a partial isometry U such that $UU^* = P$ and $U^*U = Q$. Looking at H_1 for example, we see that there is a Calkin projection, namely $\mathbf{1}$, which fails to annihilate the Calkin image of A if and only if $A \neq \mathbb{K}(H_1)$. Also it was essentially shown in [6] that there is a non-zero Calkin projection which does annihilate the Calkin image of A if and only if $\mathbf{1} \notin A$. Then we see that $LM(X) \subset RM(X)$ if and only if either

$\mathbf{1} \in B$ (which causes $LM(X)$ to be small) or $A = \mathbb{K}(H_1)$ (which causes $RM(X)$ to be big), and $RM(X) \subset LM(X)$ if and only if either $\mathbf{1} \in A$ or $B = \mathbb{K}(H_2)$. Also $QM(X) = M(X)$ if and only if either both A and B contain $\mathbf{1}$ or both $A = \mathbb{K}(H_1)$ and $B = \mathbb{K}(H_2)$. Of course the last case is the case when X is an imprimitivity bimodule.

Although Example 3.8 may deal with the simplest examples, it may actually be exceptional. In support of this, we point out that in [5] the case where A has an infinite dimensional elementary direct summand was the "bad" case.

As an after–afterthought, there is another way to generalize the problem of this paper. Namely, consider questions like the title question in the context of centralizers of Pedersen's ideal. We will discuss this informally. The interested reader can fill in the details with the help of the discussion of centralizers in [12, §3.12] and the discussion of Pedersen's ideal in [12, §5.6]. We think the appropriate question is whether every locally bounded quasi-centralizer of Pedersen's ideal comes from a double centralizer, and the answer, if A is σ-unital, is the same as before. Namely, A must be the direct sum of a dual C^*-algebra and a locally unital C^*-algebra. If we drop the local boundedness hypothesis, then A must be locally unital. The reason for this last assertion is that it was shown in [11] that every double centralizer of Pedersen's ideal is locally bounded, and it was pointed out in [5] that if A has an infinite dimensional elementary direct summand, then there definitely are non-locally bounded quasi-centralizers of Pedersen's ideal. Our reason for preferring the first version of the question is that we consider non-locally bounded centralizers to be pathological.

To prove the forward direction of the assertion above, note that the hypothesis implies that every T in $QM(A)$ comes from a double centralizer C. Since C is locally bounded and agrees with the bounded T on Pedersen's ideal, then C is bounded. Thus C comes from a multiplier and $QM(A) = M(A)$. For the converse, the case of dual C^*-algebras is trivial. If A is locally unital, let I_j and u_j be as in the definition of local unitality. It is not hard to see that each I_j is contained in Pedersen's ideal and that u_j may be taken in Pedersen's ideal. (Actually Pedersen's ideal is $\sum I_j$ in this case.) Then the argument for the converse direction of Theorem 3.4 applies.

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Isomorphism in Wavelets II

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Abstract A scaling function φ_A associated with a $d \times d$ expansive dyadic integral matrix A can be isomorphically embedded into the family of scaling functions associated with a $s \times s, d \leq s$, expansive dyadic integral matrix B. On the other hand, a scaling function φ_A associated with a $d \times d$ expansive dyadic integral matrix A and a finite two scaling relation can be isomorphically embedded into the family of scaling functions associated with expansive dyadic integral $s \times s$ matrix B, for any s. In particular, for $s = 1$ and $B = [2]$. We provide examples for such isomorphisms.

Keywords Parseval frame wavelets · Isomorphism · High dimension · Scaling function

Mathematics Subject Classification (2010) Primary 46N99, 47N99, 46E99; Secondary 42C40, 65T60

1 Introduction

For a vector $\vec{\ell} \in \mathbb{R}^d$, the *translation operator* $T_{\vec{\ell}}$ is defined as

$$(T_{\vec{\ell}} f)(\vec{t}) \equiv f(\vec{t} - \vec{\ell}), \ \forall f \in L^2(\mathbb{R}^d), \ \forall \vec{t} \in \mathbb{R}^d \ .$$

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_6

Let A be a $d \times d$ integral matrix with eigenvalues $\beta_1, \cdots, \beta_d$. A is called *expansive* if $\min\{|\beta_1|, \cdots, |\beta_d|\} > 1$. A is called *dyadic* if $|\det(A)| = 2$. We define the operator D_A as

$$(D_A f)(\vec{t}) \equiv (\sqrt{2}) f(A\vec{t}),\ \forall f \in L^2(\mathbb{R}^d),\ \forall \vec{t} \in \mathbb{R}^d\ .$$

The operators $T_{\vec{\ell}}$ and D_A are unitary operators on $L^2(\mathbb{R}^d)$.

Let $\{s_{\vec{n}} \mid \vec{n} \in \mathbb{Z}^d\}$ be a solution to the following system of equations (1.1) associated with a $d \times d$ expansive dyadic integral matrix A:

$$\begin{cases} \sum_{\vec{n} \in \mathbb{Z}^d} h_{\vec{n}} \overline{h_{\vec{n}+\vec{k}}} = \delta_{\vec{0}\vec{k}},\ \vec{k} \in A\mathbb{Z}^d, \\ \sum_{\vec{n} \in \mathbb{Z}^d} h_{\vec{n}} = \sqrt{2}. \end{cases} \tag{1.1}$$

The set $\Lambda = \{\vec{n} \in \mathbb{Z}^d \mid s_{\vec{n}} \neq 0\}$ is the support of $\{s_{\vec{n}} \mid \vec{n} \in \mathbb{Z}^d\}$. If Λ is a finite set, then $\{s_{\vec{n}}\}$ is called a *finite* solution. Define the operator Ψ on $L^2(\mathbb{R}^d)$ as

$$\Psi \equiv \sum_{\vec{n} \in \Lambda} s_{\vec{n}} D_A T_{\vec{n}}.$$

When Λ is finite the operator Ψ has a non-zero fixed point φ_A (Lawton [4] and Bownik [2]),

$$\varphi_A = \Psi \varphi_A. \tag{1.2}$$

This φ_A is the scaling function associated with matrix A and it induces a Parseval frame wavelet ψ_A associated with matrix A. It satisfies the two-scale relation:

$$\varphi_A = \sum_{\vec{n} \in \Lambda} s_{\vec{n}} D_A T_{\vec{n}} \varphi_A. \tag{1.3}$$

We will say that φ_A is *derived* from the solution $\mathcal{S}$. This scaling function φ_A associated with matrix A is generated by a solution $\mathcal{S} = \{s_{\vec{n}}\}$ to the system of Eqs. (1.1). The scaling function φ_A induces a Parseval frame wavelet ψ_A associated with matrix A as defined in Definition 1.1.

Definition 1.1 Let A be an expansive dyadic integral matrix. A function $\psi_A \in L^2(\mathbb{R}^d)$ is called a Parseval frame wavelet associated with A, if the set

$$\{D_A^n T_{\vec{\ell}}\, \psi_A \mid n \in \mathbb{Z}, \vec{\ell} \in \mathbb{Z}^d\}$$

forms a normalized tight frame for $L^2(\mathbb{R}^d)$. That is

$$\|f\|^2 = \sum_{n \in \mathbb{Z}, \vec{\ell} \in \mathbb{Z}^d} |\langle f, D_A^n T_{\vec{\ell}}\, \psi_A \rangle|^2,\ \forall f \in L^2(\mathbb{R}^d).$$

If the set is also orthogonal, then ψ_A is an orthonormal wavelet for $L^2(\mathbb{R}^d)$ associated with A.

2 Definition of Isomorphisms

Let A be an expansive dyadic integral matrix. Let $\mathcal{W}(A,d)$ be the collection of all scaling functions in $L^2(\mathbb{R}^d)$ associated with A and solutions to the system of Eq. (1.1). Define $\mathcal{W}(d) \equiv \bigcup_A \mathcal{W}(A,d)$. The union is for all $d \times d$ expansive dyadic integral matrices. Define $\mathcal{W} \equiv \bigcup_{d\geq 1} \mathcal{W}(d)$. This is the set of scaling functions in all dimensions.

In particular, let $\mathcal{W}_0(A,d)$ be the collection of all scaling functions in $L^2(\mathbb{R}^d)$ associated with A and *finite* solutions to the system of Eq. (1.1). Define $\mathcal{W}_0(d) \equiv \bigcup_A \mathcal{W}_0(A,d)$. The union is for all $d \times d$ expansive dyadic integral matrices. Define $\mathcal{W}_0 \equiv \bigcup_{d\geq 1} \mathcal{W}_0(d)$.

Let A be a $d \times d$ expansive dyadic integral matrix and $\varphi_A \in \mathcal{W}(A,d)$ which is derived from the solution $\{a_{\vec{n}} \mid \vec{n} \in \mathbb{Z}^d\}$ to (1.1). Denote the support of this solution as Λ_A, and $\mathcal{S}_A = \{a_{\vec{n}} \mid \vec{n} \in \Lambda_A\}$.

A *reduced system of equations* $\mathcal{E}_{(\Lambda_A,A,d)}$ from system of Eqs. (1.1) can be obtained by the following steps:

Step 1 For $\vec{n} \in \mathbb{Z}^d \backslash \Lambda_A$, replace all variables $h_{\vec{n}}$ in (1.1) by 0.
Step 2 Then remove all trivial equations "$0 = 0$".
Step 3 If there are redundant equations, choose and keep one and remove the other identical equations.

Note that, the discussion of the reduced system of equations $\mathcal{E}_{(\Lambda_A,A,d)}$ from the support Λ_A does not depend on the existence of a solution $\mathcal{S}_A$. This gives flexibility in the discussion.

Denote the family of all such reduced systems of equations by $\mathfrak{E}$. A reduced system of equations $\mathcal{E}_{(\Lambda_A,A,d)}$ has the following form:

$$\begin{cases} \sum_{\vec{n}\in\Lambda_A} h_{\vec{n}}\overline{h_{\vec{n}+\vec{k}}} = \delta_{\vec{0}\vec{k}}, \ \vec{k} \in \Lambda_A^E, \\ \sum_{\vec{n}\in\Lambda_A} h_{\vec{n}} = \sqrt{2}. \end{cases} \tag{2.1}$$

The *index set* Λ_A^E in (2.1) is a subset of $A\mathbb{Z}^d$. The equation in $\mathcal{E}_{(\Lambda_A,A,d)}$ that corresponding to $\vec{k} \in \Lambda_A^E$ is

$$\sum_{\vec{n}\in\Lambda_A} h_{\vec{n}}\overline{h_{\vec{n}+\vec{k}}} = \delta_{\vec{0}\vec{k}}.$$

This set Λ_A^E might not be unique due to the Step 3 above. However, it is fixed in discussion. It is clear that $\mathcal{S}_A$ is a solution to (2.1).

Similarly, for an $s \times s$ expansive dyadic integral matrix B and $\Lambda_B \subset \mathbb{Z}^s$, the reduced system of equation is

$$\mathcal{E}_{(\Lambda_B,B,s)}: \quad \begin{cases} \sum_{\vec{m}\in\Lambda_B} h'_{\vec{m}}\overline{h'_{\vec{m}+\vec{\ell}}} = \delta_{\vec{0}\vec{\ell}}, \ \vec{\ell} \in \Lambda_B^E, \\ \sum_{\vec{m}\in\Lambda_B} h'_{\vec{m}} = \sqrt{2}. \end{cases} \tag{2.2}$$

Definition 2.1 $\mathcal{E}_{(\Lambda_A,A,d)}, \mathcal{E}_{(\Lambda_B,B,s)} \in \mathfrak{E}$ are isomorphic, or $\mathcal{E}_{(\Lambda_A,A,d)} \sim \mathcal{E}_{(\Lambda_B,B,s)}$ if there exist

(a) a bijection $\theta : \Lambda_A \to \Lambda_B$ and
(b) a bijection η from an index set Λ_A^E of $\mathcal{E}_{(\Lambda_A,A,d)}$ onto an index set Λ_B^E of $\mathcal{E}_{(\Lambda_B,B,s)}$

with the following properties: for each $\vec{k} \in \Lambda_A^E$, the equation in $\mathcal{E}_{(\Lambda_B,B,s)}$ generated by $\vec{\ell} \equiv \eta(\vec{k})$ is obtained by replacing $h_{\vec{n}}$ by $h'_{\theta(\vec{n})}$ and $\delta_{\vec{0}\vec{k}}$ by $\delta_{\vec{0}\vec{\ell}}$ in the equation in $\mathcal{E}_{(\Lambda_A,A,d)}$ generated by $\vec{k}$.

In each of the examples in Sects. 4 and 5 we will list the corresponding matrices A and B, sets Λ_A, Λ_A, mappings θ, η. The related systems of equations $(\mathcal{S}_A, \mathcal{E}_{(\Lambda_A,A,d)})$ and $(\mathcal{S}_B, \mathcal{E}_{(\Lambda_B,B,s)})$ are reduced. We check each of the cases with computer programs. For simplicity we omit the details.

Let $\mathcal{S}_A = \{a_{\vec{n}} \mid \vec{n} \in \Lambda_A\}$ be a solution to (2.1) and $\mathcal{S}_B = \{b_{\vec{m}} \mid \vec{m} \in \Lambda_B\}$ be a solution to (2.2). Let $\varphi_A, \varphi_B \in \mathcal{W}$ be the scaling functions derived from $(\mathcal{S}_A, \mathcal{E}_{(\Lambda_A,A,d)})$ and $(\mathcal{S}_B, \mathcal{E}_{(\Lambda_B,B,s)})$ respectively. Notice that d and s can be different.

Definition 2.2 The scaling functions φ_A, φ_B are *algebraically isomorphic*, or $\varphi_A \simeq \varphi_B$, if $\mathcal{E}_{(\Lambda_A,A,d)} \sim \mathcal{E}_{(\Lambda_B,B,s)}$ with bijection θ and η. And

$$b_{\theta(\vec{n})} = a_{\vec{n}}, \forall \vec{n} \in \Lambda_A.$$

It is clear that the isomorphism of the reduced system of equations guarantees the isomorphism of the scaling functions derived from the solutions of the reduced system of equations. We have

Lemma 2.3 *For isomorphic systems $\mathcal{E}_{(\Lambda_A,A,d)}$ and $\mathcal{E}_{(\Lambda_B,B,s)}$ with bijection θ from Λ_A to Λ_B, if $\mathcal{S}_A = \{a_{\vec{n}} \mid \vec{n} \in \Lambda_A\}$ is a solution to $\mathcal{E}_{(\Lambda_A,A,d)}$, then the set $\mathcal{S}_B \equiv \{b_{\vec{m}} = a_{\theta^{-1}(\vec{m})} \mid \vec{m} \in \Lambda_B\}$ is a solution to $\mathcal{E}_{(\Lambda_B,B,s)}$. Moreover, the scaling functions derived from $(\mathcal{S}_A, \mathcal{E}_{(\Lambda_A,A,d)})$ and $(\mathcal{S}_B, \mathcal{E}_{(\Lambda_B,B,s)})$ are algebraically isomorphic.*

Definition 2.4 Let $\mathcal{U}$ and $\mathcal{V}$ be subsets of $\mathcal{W}$.

1. The set $\mathcal{U}$ is *isomorphically embedded into* $\mathcal{V}$,

$$\mathcal{U} \sqsubseteq \mathcal{V},$$

if for each φ_U in $\mathcal{U}$ there is an element $\varphi_V \in \mathcal{V}$ such that $\varphi_U \simeq \varphi_V$.

2. If $\mathcal{U} \sqsubseteq \mathcal{V}$ and $\mathcal{V} \sqsubseteq \mathcal{U}$, $\mathcal{U}$ and $\mathcal{V}$ are isomorphically identical, or

$$\mathcal{U} \cong \mathcal{V}.$$

We have

Theorem 2.5

$$\mathcal{W}(1) \sqsubseteq \mathcal{W}(2) \sqsubseteq \mathcal{W}(3) \sqsubseteq \cdots,$$

that is, the sequence $\{\mathcal{W}(d) \mid d \in \mathbb{N}\}$ *is an ascending sequence.*

In [3], we proved that

Theorem 2.6

$$\mathcal{W}_0(1) \cong \mathcal{W}_0(2) \cong \mathcal{W}_0(3) \cong \cdots,$$

that is, each $\varphi \in \mathcal{W}_0$ *is isomorphic to a one dimensional scaling function in* $\mathcal{W}_0(1)$.

The purpose of this paper is to prove Theorem 2.5 and present examples for both Theorems 2.5 and 2.6.

3 Proof of Theorem 2.5

Let s be a natural number and $d \leq s$ and B be a $s \times s$ expansive dyadic integral matrix. To prove Theorem 2.5, we need to find a function $\varphi_B \in \mathcal{W}_0(s)$ for any given $\varphi_A \in \mathcal{W}_0(d)$ such that $\varphi_A \simeq \varphi_B$.

By the Smith Normal Form for integral matrices [1] $A = UDV$, where U, V are integral matrices of determinant ± 1, and D a diagonal matrix with the last diagonal entry 2 and all other diagonal entries 1. Let $\vec{e}_1, \ldots, \vec{e}_d$ be the standard basis for $\mathbb{Z}^d$. Note that $V\mathbb{Z}^d = \mathbb{Z}^d$ and $U\mathbb{Z}^d = \mathbb{Z}^d$. We have

$$\begin{aligned}\mathbb{Z}^d &= span\{\vec{e}_1, \ldots, \vec{e}_{d-1}, 2\vec{e}_d\} \,\dot\cup\, \big(span\{\vec{e}_1, \ldots, \vec{e}_{d-1}, 2\vec{e}_d\} + \vec{e}_d\big) \\ &= D\mathbb{Z}^d \,\dot\cup\, (D\mathbb{Z}^d + \vec{e}_d) = DV\mathbb{Z}^d \,\dot\cup\, (DV\mathbb{Z}^d + \vec{e}_d) \\ &= UDV\mathbb{Z}^d \,\dot\cup\, U(DV\mathbb{Z}^d + \vec{e}_d) = A\mathbb{Z}^d \dot\cup \big(A\mathbb{Z}^d + U\vec{e}_d\big).\end{aligned}$$

Let $\vec{\ell}_A \equiv U\vec{e}_d$. It follows that, for any $d \times d$ expansive dyadic integral matrix A, there exists a vector $\vec{\ell}_A \in \mathbb{Z}^d \backslash A\mathbb{Z}^d$ such that

$$\mathbb{Z}^d = A\mathbb{Z}^d \dot\cup (\vec{\ell}_A + A\mathbb{Z}^d).$$

The same proof shows that there exists a vector $\vec{\ell}_B \in \mathbb{Z}^s \backslash B\mathbb{Z}^s$ such that

$$\mathbb{Z}^s = B\mathbb{Z}^s \cup (\vec{\ell}_B + B\mathbb{Z}^s).$$

Since $d \leq s$, we can consider $\mathbb{R}^d$ as subspace of $\mathbb{R}^s$. Let $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_s\}$ be the standard basis for $\mathbb{R}^s$. We will further assume that the first d vectors of the basis, $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_d\}$ be the standard basis for $\mathbb{R}^d$.

Define the mapping Θ from $\mathbb{Z}^d$ to $\mathbb{Z}^s$.

$$\Theta(\vec{n}) = \begin{cases} BA^{-1}(\vec{n}), & \text{if } \vec{n} \in A\mathbb{Z}^d, \\ \vec{\ell}_B + BA^{-1}(\vec{n} - \vec{\ell}_A), & \text{if } \vec{n} \in \vec{\ell}_A + A\mathbb{Z}^d. \end{cases} \tag{3.1}$$

This is a well-defined mapping on $\mathbb{Z}^d$ since $\mathbb{Z}^d = A\mathbb{Z}^d \cup (\vec{\ell}_A + A\mathbb{Z}^d)$ with range $\Theta(\mathbb{Z}^d) \subset \mathbb{Z}^s$. Since $\det B \neq 0$ and A has an inverse on $A\mathbb{Z}^d$ with range contained in $\Theta(\mathbb{Z}^d) \subseteq \mathbb{Z}^s$, the mapping Θ is an injection. We have $\Theta(\vec{0}) = \vec{0}$. Also, if $\Theta(\vec{x}) = \vec{0}$ for some $\vec{x} \in \mathbb{Z}^d$ then $\vec{x} = \vec{0}$.

Lemma 3.1 *For $\vec{n} \in \mathbb{Z}^d$ and $\vec{k} \in A\mathbb{Z}^d$, we have*

$$\Theta(\vec{n} + \vec{k}) = \Theta(\vec{n}) + \Theta(\vec{k}).$$

Proof Since $\vec{k} \in A\mathbb{Z}^d$, $\Theta(\vec{k}) = BA^{-1}(\vec{k})$. We have

Since $\vec{k} \in A\mathbb{Z}^d$, $\vec{n} + \vec{k} \in A\mathbb{Z}^d$ iff $\vec{n} \in A\mathbb{Z}^d$. Also $\vec{n} + \vec{k} \in \vec{\ell}_A + A\mathbb{Z}^d$ iff $\vec{n} \in \vec{\ell}_A + A\mathbb{Z}^d$. So we have

$$\begin{aligned} \Theta(\vec{n}) + \Theta(\vec{k}) &= BA^{-1}(\vec{k}) + \begin{cases} BA^{-1}(\vec{n}), \ \vec{n} \in A\mathbb{Z}^d \\ \vec{\ell}_B + BA^{-1}(\vec{n} - \vec{\ell}_A), \ \vec{n} \in \vec{\ell}_A + A\mathbb{Z}^d \end{cases} \\ &= \begin{cases} BA^{-1}(\vec{n} + \vec{k}), \ \vec{n} \in A\mathbb{Z}^d \\ \vec{\ell}_B + BA^{-1}((\vec{n} + \vec{k}) - \vec{\ell}_A), \ \vec{n} \in \vec{\ell}_A + A\mathbb{Z}^d \end{cases} \\ &= \Theta(\vec{n} + \vec{k}). \end{aligned}$$

□

Lemma 3.2 *If $\vec{n}_1, \vec{n}_2 \in \mathbb{Z}^d$ and $\vec{\ell} \equiv \Theta(\vec{n}_2) - \Theta(\vec{n}_1) \in B\mathbb{Z}^s$, then there exists a vector $\vec{k} \in A\mathbb{Z}^d$ such that $\vec{\ell} = \Theta(\vec{k})$, and $\vec{n}_2 = \vec{n}_1 + \vec{k}$.*

Proof Since $\Theta(\vec{n}_2) - \Theta(\vec{n}_1) = \vec{\ell} \in B\mathbb{Z}^d$, by Eq. (3.1) we have only two cases.

Case 1 Both $\Theta(\vec{n}_1), \Theta(\vec{n}_2)$ are in $B\mathbb{Z}^s$. By Eq. (3.1), $n_1 = A\lambda_1, n_2 = A\lambda_2$ for some vectors $\lambda_1, \lambda_2 \in \mathbb{Z}^d$. Denote $\vec{k} = A\lambda_2 - A\lambda_1 \in A\mathbb{Z}^d$. So $\vec{\ell} = \Theta(\vec{n}_2) - \Theta(\vec{n}_1) = BA^{-1}(A\lambda_2) - BA^{-1}(A\lambda_1) = BA^{-1}(A\lambda_2 - A\lambda_1) = \Theta(\vec{k})$. We have $\vec{\ell} = \Theta(\vec{k})$ and $\vec{n}_2 - \vec{n}_1 = \vec{k}$.

Case 2 Both $\Theta(\vec{n}_1), \Theta(\vec{n}_2)$ are in $\vec{\ell}_B + B\mathbb{Z}^s$. By Eq. (3.1), $n_1 = \vec{\ell}_A + A\lambda_1, n_2 = \vec{\ell}_A + A\lambda_2$ for some vectors $\lambda_1, \lambda_2 \in \mathbb{Z}^d$. Denote $\vec{k} = A\lambda_2 - A\lambda_1 \in A\mathbb{Z}^d$. So

$\vec{\ell} = \Theta(\vec{n}_2) - \Theta(\vec{n}_1) = (\vec{\ell}_A + BA^{-1}(A\lambda_2)) - (\vec{\ell}_A + BA^{-1}(A\lambda_1)) = BA^{-1}(A\lambda_2 - A\lambda_1) = \Theta(\vec{k})$. We have $\vec{\ell} = \Theta(\vec{k})$ and $\vec{n}_2 - \vec{n}_1 = \vec{k}$. □

For matrix B, the system of equation in (1.1) becomes

$$\begin{cases} \sum_{\vec{m}\in\mathbb{Z}^s} h'_{\vec{m}}\overline{h'_{\vec{m}+\vec{\ell}}} = \delta_{\vec{0}\vec{\ell}}, \ \vec{\ell} \in B\mathbb{Z}^s, \\ \sum_{\vec{m}\in\mathbb{Z}^s} h'_{\vec{m}} = \sqrt{2}. \end{cases} \tag{3.2}$$

Consider the reduced system $\mathcal{E}_{(\Lambda_A,A,d)}$ with index set Λ_A^E. Define $\theta \equiv \Theta|_{\Lambda_A}$ and $\eta \equiv \Theta|_{\Lambda_A^E}$. Denote $\Lambda_B \equiv \theta(\Lambda_A)$. Since Θ is an injection from $\mathbb{Z}^d$ to $\mathbb{Z}^s$, θ is a bijection from Λ_A to Λ_B.

Let $\mathcal{S}_A = \{a_{\vec{n}} \mid \vec{n} \in \Lambda_A\}$ be a solution to $\mathcal{E}_{(\Lambda_A,A,d)}$.

Define

$$b_{\vec{m}} \equiv \begin{cases} a_{\theta^{-1}(\vec{m})}, & \text{if } \vec{m} \in \Lambda_B, \\ 0, & \text{if } \vec{m} \in \mathbb{Z}^s\backslash\Lambda_B. \end{cases}$$

$$\mathcal{S}_B \equiv \{b_{\vec{m}} \mid \vec{m} \in \Lambda_B\}.$$

To prove that $\varphi_B \simeq \varphi_A$, by Definition 2.2, we only need to show that

1. The system of equation

$$\begin{cases} \sum_{\vec{m}\in\Lambda_B} h'_{\vec{m}}\overline{h'_{\vec{m}+\vec{\ell}}} = \delta_{\vec{0}\vec{\ell}}, \ \vec{\ell} \in \eta(\Lambda_A^E), \\ \sum_{\vec{m}\in\Lambda_B} h'_{\vec{m}} = \sqrt{2}. \end{cases} \tag{3.3}$$

   is the reduced system of equations $\mathcal{E}_{(\Lambda_B,B,s)}$. Or equivalently, the set $\eta(\Lambda_A^E)$ is an index set for $\mathcal{E}_{(\Lambda_B,B,s)}$, denoted as Λ_B^E. This is Lemma 3.3 below.
2. The set $\mathcal{S}_B \equiv \{b_{\vec{m}} \mid \vec{m} \in \Lambda_B\}$ is a solution to (3.3) by Lemma 2.3.

Lemma 3.3 *The set $\eta(\Lambda_A^E)$ is an index set for $\mathcal{E}_{(\Lambda_B,B,s)}$.*

Proof Let $\vec{k} \in \Lambda_A^E$. A reduced equation in $\mathcal{E}_{(\Lambda_A,A,d)}$ generated by $\vec{k}$ has the following form:

$$\sum_{\vec{n}\in\Lambda_A} h_{\vec{n}}\overline{h_{\vec{n}+\vec{k}}} = \delta_{\vec{0}\vec{k}}. \tag{3.4}$$

We will show that $\vec{\ell} \equiv \eta(\vec{k}) \in B\mathbb{Z}^s$ generates an reduced equation in $\mathcal{E}_{(\Lambda_B,B,s)}$. We write

$$\sum_{\vec{m}\in\mathbb{Z}^s} h'_{\vec{m}}\overline{h'_{\vec{m}+\vec{\ell}}} = \delta_{\vec{0}\vec{\ell}}$$

Note that $h'_{\vec{m}} = 0$ for $\vec{m} \notin \Lambda_B$, so the above equation is the same as

$$\sum_{\vec{m}\in\Lambda_B} h'_{\vec{m}}\overline{h'_{\vec{m}+\vec{\ell}}} = \delta_{\vec{0}\vec{\ell}}. \tag{3.5}$$

By definition of θ, η and the fact that $\Lambda_B \equiv \theta(\Lambda_A)$, we have

$$\sum_{\vec{n}\in\Lambda_A} h'_{\theta(\vec{n})}\overline{h'_{\theta(\vec{n})+\eta(\vec{k})}} = \delta_{\vec{0}\eta(\vec{k})}$$

By Lemma 3.1, $\theta(\vec{n}+\vec{k}) = \theta(\vec{n}) + \Theta(\vec{k}) = \theta(\vec{n}) + \eta(\vec{k})$, thus

$$\sum_{\vec{n}\in\Lambda_A} h'_{\theta(\vec{n})}\overline{h'_{\theta(\vec{n}+\vec{k})}} = \delta_{\vec{0}\eta(\vec{k})}$$

Replace $h'_{\theta(\vec{n})}$ with $h_{\vec{n}}$, $h'_{\theta(\vec{n}+\vec{k})}$ with $h_{\vec{n}+\vec{k}}$ and $\delta_{\vec{0}\eta(\vec{k})}$ with $\delta_{\vec{0}\vec{k}}$, We obtained the same equation as (3.4). Since (3.4) is non-trivial, (3.5) is non-trivial as well. Furthermore, (3.5) is a reduced equation in $\mathcal{E}_{(\Lambda_B,B,s)}$. It is clear that different elements in $\eta(\Lambda_A^E)$ generate different equations in $\mathcal{E}_{(\Lambda_B,B,s)}$.

Next, we will show that every (non-trivial) equation in $\mathcal{E}_{(\Lambda_B,B,s)}$ can be generated by an element in $\eta(\Lambda_A^E)$. Let the following be a non-trivial equation in $\mathcal{E}_{(\Lambda_B,B,s)}$ generated by $\vec{\ell}_0 \in B\mathbb{Z}^s$:

$$\sum_{\vec{m}\in\Lambda_B} h'_{\vec{m}}\overline{h'_{\vec{m}+\vec{\ell}_0}} = \delta_{\vec{0}\vec{\ell}_0}. \tag{3.6}$$

Denote $\vec{m} = \theta(\vec{n})$, where $\vec{n} \in \Lambda_A \subset \mathbb{Z}^d$:

$$\sum_{\theta(\vec{n})\in\Lambda_B} h'_{\theta(\vec{n})}\overline{h'_{\theta(\vec{n})+\vec{\ell}_0}} = \delta_{\vec{0}\vec{\ell}_0}.$$

By Lemma 3.2, there exists $\vec{k}_0 \in A\mathbb{Z}^d$ such that $\vec{\ell}_0 = \Theta(\vec{k}_0)$:

$$\sum_{\Theta(\vec{n})\in\Lambda_B} h'_{\Theta(\vec{n})}\overline{h'_{\Theta(\vec{n})+\Theta(\vec{k}_0)}} = \delta_{\vec{0}\Theta(\vec{k}_0)}.$$

By Lemma 3.1,

$$\sum_{\Theta(\vec{n})\in\Lambda_B} h'_{\Theta(\vec{n})}\overline{h'_{\Theta(\vec{n}+\vec{k}_0)}} = \delta_{\vec{0}\Theta(\vec{k}_0)}.$$

Replace $h'_{\Theta(\vec{n})}$ with $h_{\vec{n}}$, $h'_{\Theta(\vec{n}+\vec{k}_0)}$ with $h_{\vec{n}+\vec{k}_0}$ and $\delta_{\vec{0}\Theta(\vec{k}_0)}$ with $\delta_{\vec{0}\vec{k}_0}$, we have

$$\sum_{\vec{n}\in\Lambda_A} h_{\vec{n}}\overline{h_{\vec{n}+\vec{k}_0}} = \delta_{\vec{0}\vec{k}_0}.$$

It is clear that this is a reduced non-trivial equation in $\mathcal{E}_{(\Lambda_A,A,d)}$ generated by $\vec{k}_0$. On the other hand, since this is a reduced non-trivial equation in $\mathcal{E}_{(\Lambda_A,A,d)}$, it is generated by an element $\vec{k}$ in its index set Λ_A^E. It follows that $\vec{\ell} \equiv \eta(\vec{k}) \in \Lambda_B^E$ generates the same equation as (3.6). Hence $\Lambda_B^E = \eta(\Lambda_A^E)$ is an index set for $\mathcal{E}_{(\Lambda_B,B,s)}$. □

The proof of Theorem 2.5 is completed.

4 Examples from Higher Dimensions to One Dimension

Examples for Theorem 2.6 are presented in this section.

The sublattice $A\mathbb{Z}^d$ generated by the $d \times d$ expansive dyadic integral matrix A can be further simplified by changing of basis:

Proposition 4.1 ([3]) *Let $d \geq 1$ be a natural number and A a $d \times d$ expansive dyadic integral matrix. Then $\mathbb{R}^d$ has a basis $\{\vec{f}_j \mid j = 1, \ldots, d\}$ with properties that, under this new basis, a vector $\vec{k}$ is in $A\mathbb{Z}^d$ if and only if the last coordinate of $\vec{k}$ is an even number. That is, under this new basis, we have*

$$A\mathbb{Z}^d = \{(\vec{x}, 2n) \mid \vec{x} \in \mathbb{Z}^{d-1}, n \in \mathbb{Z}\}. \tag{4.1}$$

Hence, for simplicity, all matrices discussed in the examples in this section will have this property (4.1). Let A be a $d \times d$ expansive dyadic integral matrix with properties (4.1).

For a natural number $N \geq 1$, define

$$\Lambda_{d,N} \equiv [0, 2^N)^d \cap \mathbb{Z}^d = \big\{(n_1, \cdots, n_d) \mid 0 \leq n_1, \cdots, n_d \leq 2^N - 1\big\}. \tag{4.2}$$

The set $\Lambda_{d,N}$ contains 2^{dN} elements in $\mathbb{Z}^d$.

For vector $\vec{n} = (n_1, n_2, \cdots, n_{d-1}, n_d) \in \mathbb{Z}^d$, define the function $\sigma_{d,N} : \mathbb{Z}^d \to \mathbb{Z}$ as:

$$\sigma_{d,N}(\vec{n}) = \sum_{j=1}^{d} n_j \cdot 4^{(j-1)N}. \tag{4.3}$$

Define $f_{d,N} : \mathbb{Z}^d \to \mathbb{Z}$:

$$f_{d,N}(\vec{x}, y) \equiv \lfloor \frac{y}{2} \rfloor 2^{(2d-3)N+2} + \begin{cases} 2\sigma_{d-1,N}(\vec{x}) & y \text{ even} \\ 2\sigma_{d-1,N}(\vec{x}) + 1 & y \text{ odd} \end{cases} \quad \forall \vec{x} \in \mathbb{Z}^{d-1}, y \in \mathbb{Z} \tag{4.4}$$

where $\lfloor \frac{y}{2} \rfloor$ gives the greatest integer that is less than or equal to $\frac{y}{2}$.

Define mappings $\theta_{d,N}$ and $\eta_{d,N}$ as follows:

$$\theta_{d,N}\big((\vec{x}, y)\big) \equiv f_{d,N}(\vec{x}, y), \ (\vec{x}, y) \in \Lambda_{d,N} \tag{4.5}$$

$$\eta_{d,N}\big((\vec{x}, y)\big) \equiv f_{d,N}(\vec{x}, y), \ (\vec{x}, y) \in \Lambda^E_{d,N}. \tag{4.6}$$

$\theta_{d,N}$, $\eta_{d,N}$ are injections on $\Lambda_{d,N}$ and $\Lambda^E_{d,N}$ respectively.

Denote

$$\begin{aligned} \Lambda_A &= \Lambda_{d,N}. \\ \Lambda^E_A &= \{\vec{n} = (\vec{x}, 2j) \in \mathbb{Z}^d \mid \sigma_{d,N}(\vec{n}) \geq 0; \vec{n} \in (-2^N, 2^N)^d \cap \mathbb{Z}^d\}. \\ \theta &= \theta_{d,N}. \\ \eta &= \eta_{d,N}. \\ \Lambda_1 &= \theta(\Lambda_A). \\ \Lambda^E_1 &= \eta(\Lambda^E_A). \end{aligned}$$

With the above settings, the following Theorem collects some results from Section 4 of [3]. This is a special version of Theorem 2.6.

Theorem 4.2

1. *The systems of equations $\mathcal{E}_{(\Lambda_A, A, d)}$ is a reducing system and Λ^E_A is an index set.*
2. *The systems of equations $\mathcal{E}_{(\Lambda_1, [2], 1)}$ is a reducing system and Λ^E_1 is an index set.*
3. *The systems of equations $\mathcal{E}_{(\Lambda_A, A, d)}$ and $\mathcal{E}_{(\Lambda_1, [2], 1)}$ are isomorphic with bijections θ and η:*

$$\mathcal{E}_{(\Lambda_A, A, d)} \sim \mathcal{E}_{(\Lambda_1, [2], 1)}.$$

Example Let $A = \begin{bmatrix} -1 & 2 \\ -2 & 2 \end{bmatrix}$ and $B = [2]$.

Choose $\Lambda_A = \Lambda_{2,1} = \{(0,0), (0,1), (1,0), (1,1)\}$. It is clear that $\mathcal{E}_{(\Lambda_A, A, 2)}$ below is a reduced system of equation:

$$\mathcal{E}_{(\Lambda_A,A,2)} : \begin{cases} h_{00} + h_{10} + h_{01} + h_{11} = \sqrt{2} \\ h_{00}^2 + h_{10}^2 + h_{01}^2 + h_{11}^2 = 1 \\ h_{00} \cdot h_{10} + h_{01} \cdot h_{11} = 0. \end{cases}$$

The bijections defined in (4.5) and (4.6) become

$$\theta(x, y) = \lfloor \frac{y}{2} \rfloor 4 + \begin{cases} 2x & y \text{ even} \\ 2x + 1 & y \text{ odd} \end{cases} \quad (x, y) \in \Lambda_A;$$

$$\eta(x, y) = \lfloor \frac{y}{2} \rfloor 4 + \begin{cases} 2x & y \text{ even} \\ 2x + 1 & y \text{ odd} \end{cases} \quad (x, y) \in \Lambda_A^E = \{(0,0), (1,0)\}.$$

The mappings are:

Λ_A	$\Lambda_B = \theta(\Lambda_A)$	Λ_A^E	$\Lambda_B^E = \eta(\Lambda_A^E)$
(0, 0)	0	(0, 0)	0
(0, 1)	1		
(1, 0)	2	(1, 0)	2
(1, 1)	3		

Under the above mapping, the corresponding isomorphic systems of equations are

$$\mathcal{E}_{(\Lambda_A,A,2)} : \begin{cases} h_{00} + h_{10} + h_{01} + h_{11} = \sqrt{2} \\ h_{00}^2 + h_{10}^2 + h_{01}^2 + h_{11}^2 = 1 \\ h_{00} \cdot h_{10} + h_{01} \cdot h_{11} = 0. \end{cases} \qquad \mathcal{E}_{(\Lambda_B,B,1)} : \begin{cases} h_0 + h_1 + h_2 + h_3 = \sqrt{2} \\ h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1 \\ h_0 \cdot h_2 + h_1 \cdot h_3 = 0. \end{cases}$$

Example Let $A = \begin{bmatrix} -1 & 2 \\ -2 & 2 \end{bmatrix}$ and $B = [2]$. Choose $\Lambda_A = \Lambda_{2,3} = \{(x, y) \mid 0 \leq x, y \leq 2^3 - 1\}$. The index set Λ_A^E for $\mathcal{E}_{(\Lambda_A,A,2)}$ is $\{(x, y) \mid -7 \leq x \leq 7, y \in \{2, 4, 6\}$ or $0 \leq x \leq 7, y = 0\}\}$.

The bijections defined in (4.5) and (4.6) become

$$\theta(x, y) = \lfloor \frac{y}{2} \rfloor 2^{3+2} + \begin{cases} 2x & y \text{ even} \\ 2x + 1 & y \text{ odd} \end{cases} \quad (x, y) \in \Lambda_A;$$

$$\eta(x, y) = \lfloor \frac{y}{2} \rfloor 2^{3+2} + \begin{cases} 2x & y \text{ even} \\ 2x + 1 & y \text{ odd} \end{cases} \quad (x, y) \in \Lambda_A^E.$$

The mappings are:

$\theta(x,y)$	0	1	2	3	4	5	6	7
0	0	1	32	33	64	65	96	97
1	2	3	34	35	66	67	98	99
2	4	5	36	37	68	69	100	101
3	6	7	38	39	70	71	102	103
4	8	9	40	41	72	73	104	105
5	10	11	42	43	74	75	106	107
6	12	13	44	45	76	77	108	109
7	14	15	46	47	78	79	110	111

$\eta(x,y)$	0	2	4	6
−7		18	50	82
−6		20	52	84
−5		22	54	86
−4		24	56	88
−3		26	58	90
−2		28	60	92
−1		30	62	94
0	0	32	64	96
1	2	34	66	98
2	4	36	68	100
3	6	38	70	102
4	8	40	72	104
5	10	42	74	106
6	12	44	76	108
7	14	46	78	110

For example, $\theta(4,3) = 41$ according to the above mapping table. $\Lambda_B = \theta(\Lambda_A)$ is the content listed in the table for θ and $\Lambda_B^E = \eta(\Lambda_A^E)$ is the content listed in the table for η. The corresponding isomorphic systems of equations can be obtained:

$$\mathcal{E}_{(\Lambda_A,A,2)}: \begin{cases} \sum_{\vec{n}\in\Lambda_A} h_{\vec{n}} &= \sqrt{2} \\ \sum_{\vec{n}\in\Lambda_A} h_{\vec{n}}^2 &= 1 \\ \sum_{\vec{n}\in\Lambda_A} h_{\vec{n}} \cdot h_{\vec{n}+\vec{k}} &= 0,\ \vec{k} \in \Lambda_A^E \end{cases} \qquad \mathcal{E}_{(\Lambda_B,B,1)}: \begin{cases} \sum_{m\in\Lambda_B} h_m &= \sqrt{2} \\ \sum_{m\in\Lambda_B} h_m^2 &= 1 \\ \sum_{m\in\Lambda_B} h_m \cdot h_{m+\ell} &= 0, \ell \in \Lambda_B^E. \end{cases}$$

Example Let $A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ and $B = [2]$. Choose $\Lambda_A = \Lambda_{3,1} = \{(\vec{x}, y) \mid \vec{x} = (n_1, n_2), y = n_3, 0 \leq n_1, n_2, n_3 \leq 2^1 - 1\}$. The index set Λ_A^E for $\mathcal{E}_{(\Lambda_A, A, 3)}$ is $\{(0, 0, 0), (1, 0, 0), (-1, 1, 0), (0, 1, 0), (1, 1, 0)\}$.

The bijections defined in (4.5) and (4.6) become

$$\theta(\vec{x}, y) = \lfloor \frac{y}{2} \rfloor 2^{3+2} + \begin{cases} 2\sigma_{2,1}(\vec{x}) & y \text{ even} \\ 2\sigma_{2,1}(\vec{x}) + 1 & y \text{ odd} \end{cases} \quad (\vec{x}, y) \in \Lambda_A;$$

$$\eta(\vec{x}, y) = \lfloor \frac{y}{2} \rfloor 2^{3+2} + \begin{cases} 2\sigma_{2,1}(\vec{x}) & y \text{ even} \\ 2\sigma_{2,1}(\vec{x}) + 1 & y \text{ odd} \end{cases} \quad (\vec{x}, y) \in \Lambda_A^E.$$

Where $\sigma_{2,1}(n_1, n_2) = \sum_{j=1}^{2} n_j \cdot 4^{(j-1)}$ by Eq. (4.3).

The mappings are:

Λ_A	$\Lambda_B = \theta(\Lambda_A)$
(0, 0, 0)	0
(0, 0, 1)	1
(1, 0, 0)	2
(1, 0, 1)	3
(0, 1, 0)	8
(0, 1, 1)	9
(1, 1, 0)	10
(1, 1, 1)	11

Λ_A^E	$\Lambda_B^E = \eta(\Lambda_A^E)$
(0, 0, 0)	0
(1, 0, 0)	2
(-1, 1, 0)	6
(0, 1, 0)	8
(1, 1, 0)	10

Example Let $A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ and $B = [2]$. Choose $\Lambda_A = \Lambda_{3,2} = \{(\vec{x}, y) \mid \vec{x} = (n_1, n_2), y = n_3, 0 \leq n_1, n_2, n_3 \leq 2^2 - 1\}$. The index set Λ_A^E for $\mathcal{E}_{(\Lambda_A, A, 3)}$ contains 74 elements as shown later.

The bijections defined in (4.5) and (4.6) become

$$\theta(\vec{x}, y) = \lfloor \frac{y}{2} \rfloor 2^{6+2} + \begin{cases} 2\sigma_{2,2}(\vec{x}) & y \text{ even} \\ 2\sigma_{2,2}(\vec{x}) + 1 & y \text{ odd} \end{cases} \quad (\vec{x}, y) \in \Lambda_A;$$

$$\eta(\vec{x}, y) = \lfloor \frac{y}{2} \rfloor 2^{6+2} + \begin{cases} 2\sigma_{2,2}(\vec{x}) & y \text{ even} \\ 2\sigma_{2,2}(\vec{x}) + 1 & y \text{ odd} \end{cases} \quad (\vec{x}, y) \in \Lambda_A^E.$$

Where $\sigma_{2,2}(n_1, n_2) = \sum_{j=1}^{2} n_j \cdot 4^{(j-1)}$ by Eq. (4.3).

The mappings are:

$\theta(\vec{x}, y)$	0	1	2	3
(0, 0)	0	1	256	257
(1, 0)	2	3	258	259
(2, 0)	4	5	260	261
(3, 0)	6	7	262	263
(0, 1)	32	33	288	289
(1, 1)	34	35	290	291
(2, 1)	36	37	292	293
(3, 1)	38	39	294	295
(0, 2)	64	65	320	321
(1, 2)	66	67	322	323
(2, 2)	68	69	324	325
(3, 2)	70	71	326	327
(0, 3)	96	97	352	353
(1, 3)	98	99	354	355
(2, 3)	100	101	356	357
(3, 3)	102	103	358	359

$\eta(\vec{x}, y), y = 0$ $\vec{x} = (x_1, x_2)$	0	1	2	3
−3		26	58	90
−2		28	60	92
−1		30	62	94
0	0	32	64	96
1	2	34	66	98
2	4	36	68	100
3	6	38	70	102

$\eta(\vec{x}, y), y = 2$ $\vec{x} = (x_1, x_2)$	−3	−2	−1	0	1	2	3
−3	154	186	218	250	282	314	346
−2	156	188	220	252	284	316	348
−1	158	190	222	254	286	318	350
0	160	192	224	256	288	320	352
1	162	194	226	258	290	322	354
2	164	196	228	260	292	324	356
3	166	198	230	262	294	326	358

For example, $\theta(3, 2, 1) = 71$, $\eta(2, 1, 0) = 36$, $\eta(2, 1, 2) = 292$ according to the above mapping tables. $\Lambda_B = \theta(\Lambda_A)$ is the content listed in the table for θ and $\Lambda_B^E = \eta(\Lambda_A^E)$ is the content listed in the 2 tables for η. We omit the corresponding isomorphic systems of equations as it can be easily populated from the table content of η.

So far, all examples are with Λ_A of the form $\Lambda_{d,N}$. Next we will show an example with Λ_A a proper subset of $\Lambda_{d,N}$.

Example Let $A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ and $B = [2]$.
Choose $\Lambda_A = \{(0,0,0), (0,0,1), (1,0,0), (1,0,1), (2,3,2), (2,3,3), (3,3,2), (3,3,3)\}$. Notice that this support set Λ_A is properly contained in $\Lambda_{3,2}$, which is the support of the previous example. The index set Λ_A^E for $\mathcal{E}_{(\Lambda_A,A,3)}$ contains 5 elements as shown later.

The mappings are:

Λ_A	$\Lambda_B = \theta(\Lambda_A)$	Λ_A^E	$\Lambda_B^E = \eta(\Lambda_A^E)$
$(0,0,0)$	0	$(0,0,0)$	0
$(0,0,1)$	1		
$(1,0,0)$	2	$(1,0,0)$	2
$(1,0,1)$	3		
		$(1,3,2)$	354
$(2,3,2)$	356	$(2,3,2)$	356
$(2,3,3)$	357		
$(3,3,2)$	358	$(3,3,2)$	358
$(3,3,3)$	359		

The corresponding isomorphic systems of equations are:

$$\mathcal{E}_{(\Lambda_A,A,3)}:$$
$$\begin{cases} h_{0,0,0} + h_{0,0,1} + h_{1,0,0} + h_{1,0,1} + h_{2,3,2} + h_{2,3,3} + h_{3,3,2} + h_{3,3,3} = \sqrt{2} \\ h_{0,0,0}^2 + h_{0,0,1}^2 + h_{1,0,0}^2 + h_{1,0,1}^2 + h_{2,3,2}^2 + h_{2,3,3}^2 + h_{3,3,2}^2 + h_{3,3,3}^2 = 1 \\ h_{0,0,0}h_{1,0,0} + h_{0,0,1}h_{1,0,1} + h_{2,3,2}h_{3,3,2} + h_{2,3,3}h_{3,3,3} = 0 \\ h_{1,0,0}h_{2,3,2} + h_{1,0,1}h_{2,3,3} = 0 \\ h_{0,0,0}h_{2,3,2} + h_{0,0,1}h_{2,3,3} + h_{1,0,0}h_{3,3,2} + h_{1,0,1}h_{3,3,3} = 0 \\ h_{0,0,0}h_{3,3,2} + h_{0,0,1}h_{3,3,3} = 0; \end{cases}$$

$$\mathcal{E}_{(\Lambda_B,B,1)}:$$
$$\begin{cases} h_0 + h_1 + h_2 + h_3 + h_{356} + h_{357} + h_{358} + h_{359} = \sqrt{2} \\ h_0^2 + h_1^2 + h_2^2 + h_3^2 + h_{356}^2 + h_{357}^2 + h_{358}^2 + h_{359}^2 = 1 \\ h_0h_2 + h_1h_3 + h_{356}h_{358} + h_{357}h_{359} = 0 \\ h_2h_{356} + h_3h_{357} = 0 \\ h_0h_{356} + h_1h_{357} + h_2h_{358} + h_3h_{359} = 0 \\ h_0h_{358} + h_1h_{359} = 0. \end{cases}$$

5 From Lower Dimensions to Higher Dimensions

In this section we provide an example for Theorem 2.5.

Example Let $A = \begin{bmatrix} 1 & \text{-}2 \\ 2 & \text{-}2 \end{bmatrix}$, $\ell_A = \begin{bmatrix} \text{-}1 \\ \text{-}1 \end{bmatrix}$, and $B = \begin{bmatrix} 0 & \text{-}1 & 2 \\ 1 & 0 & \text{-}2 \\ 0 & \text{-}1 & 0 \end{bmatrix}$,

$\ell_B = \begin{bmatrix} 1 \\ \text{-}1 \\ 0 \end{bmatrix}$. Choose $\Lambda_A = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 3\}$.

The mappings are:

Λ_A	$\Lambda_B = \theta(\Lambda_A)$	Λ_A^E	$\Lambda_B^E = \eta(\Lambda_A^E)$
(0, 0)	(0, 0, 0)	(0, 0)	(0, 0, 0)
(0, 1)	(1, -2, 0)		
(0, 2)	(0, -2, -1)	(0, 2)	(0, -2, -1)
(0, 3)	(1, -4, -1)		
(1, 0)	(0, 1, 1)	(1, 0)	(0, 1, 1)
(1, 1)	(1, -1, 1)		
(1, 2)	(0, -1, 0)	(1, 2)	(0, -1, 0)
(1, 3)	(1, -3, 0)		

The corresponding isomorphic systems of equations are:

$$\mathcal{E}_{(\Lambda_A, A, 2)}:$$
$$\begin{cases} h_{0,0} + h_{0,1} + h_{0,2} + h_{0,3} + h_{1,0} + h_{1,1} + h_{1,2} + h_{1,3} = \sqrt{2} \\ h_{0,0}^2 + h_{0,1}^2 + h_{0,2}^2 + h_{0,3}^2 + h_{1,0}^2 + h_{1,1}^2 + h_{1,2}^2 + h_{1,3}^2 = 1 \\ h_{0,0}h_{0,2} + h_{0,1}h_{0,3} + h_{1,0}h_{1,2} + h_{1,1}h_{1,3} = 0 \\ h_{0,0}h_{1,0} + h_{0,1}h_{1,1} + h_{0,2}h_{1,2} + h_{0,3}h_{1,3} = 0 \\ h_{0,0}h_{1,2} + h_{0,1}h_{1,3} = 0; \end{cases}$$

$$\mathcal{E}_{(\Lambda_B, B, 3)}:$$
$$\begin{cases} h_{0,0,0} + h_{1,\text{-}2,0} + h_{0,\text{-}2,\text{-}1} + h_{1,\text{-}4,\text{-}1} + h_{0,1,1} + h_{1,\text{-}1,1} + h_{0,\text{-}1,0} + h_{1,\text{-}3,0} = \sqrt{2} \\ h_{0,0,0}^2 + h_{1,\text{-}2,0}^2 + h_{0,\text{-}2,\text{-}1}^2 + h_{1,\text{-}4,\text{-}1}^2 + h_{0,1,1}^2 + h_{1,\text{-}1,1}^2 + h_{0,\text{-}1,0}^2 + h_{1,\text{-}3,0}^2 = 1 \\ h_{0,0,0}h_{0,\text{-}2,\text{-}1} + h_{1,\text{-}2,0}h_{1,\text{-}4,\text{-}1} + h_{0,1,1}h_{0,\text{-}1,0} + h_{1,\text{-}1,1}h_{1,\text{-}3,0} = 0 \\ h_{0,0,0}h_{0,1,1} + h_{1,\text{-}2,0}h_{1,\text{-}1,1} + h_{0,\text{-}2,\text{-}1}h_{0,\text{-}1,0} + h_{1,\text{-}4,\text{-}1}h_{1,\text{-}3,0} = 0 \\ h_{0,0,0}h_{0,\text{-}1,0} + h_{1,\text{-}2,0}h_{1,\text{-}3,0} = 0. \end{cases}$$

Acknowledgments The work of the author "Zhongyan Li" was supported in part by the National Natural Science Foundation of China (Grant No. 11571107).

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Nevanlinna-Pick Families and Singular Rational Varieties

Kenneth R. Davidson and Eli Shamovich

We dedicate this to the memory of Ronald G. Douglas. His work has inspired many of us.

Abstract The goal of this note is to apply ideas from commutative algebra (a.k.a. affine algebraic geometry) to the question of constrained Nevanlinna-Pick interpolation. More precisely, we consider subalgebras $A \subset \mathbb{C}[z_1, \ldots, z_d]$, such that the map from the affine space to the spectrum of A is an isomorphism except for finitely many points. Letting $\mathfrak{A}$ be the weak-$*$ closure of A in $\mathcal{M}_d$—the multiplier algebra of the Drury-Arveson space. We provide a parametrization for the Nevanlinna-Pick family of $M_k(\mathfrak{A})$ for $k \geq 1$. In particular, when $k = 1$ the parameter space for the Nevanlinna-Pick family is the Picard group of A.

1 Introduction

Let $\mathbb{D}$ denote the unit disc in the complex plane. The Nevanlinna-Pick interpolation problem is to find an analytic function $f : \mathbb{D} \to \mathbb{D}$, such that at the prescribed set of points $F = \{z_1, \ldots, z_n\} \subset \mathbb{D}$, the function f attains some prescribed values, namely $f(z_j) = w_j$, with $w_1, \ldots, w_n$ given. A clean and elegant solution for this problem was obtained by Pick [22] and Nevanlinna [20, 21]. Such a function f exists if and only if the Pick matrix $\left[\frac{1-w_i\overline{w_j}}{1-z_i\overline{z_j}}\right]_{i,j=1}^n$ is positive semi-definite. In fact, the same still holds if one replaces the scalars w_i with matrices.

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_7

In [27] Saravson gave an operator theoretic interpretation to the Nevanlinna-Pick interpolation problem. He observed that the problem can be formulated in terms of a commutant lifting problem on a subspace of the Hardy space $H^2 = H^2(\mathbb{D})$. Note that the expression $k(z,w) = \frac{1}{1-z\bar{w}}$ that appears in the Pick matrix is in fact the Szego kernel, the reproducing kernel of H^2. The multiplier algebra of H^2, namely the algebra of all functions f such that $fH^2 \subset H^2$, turns out to be H^∞, the algebra of all bounded functions on the disc. Let B_F be the Blaschke product that vanishes on F. It is a bounded function and thus a multiplier on H^2. Furthermore, multiplication by B_F is an isometry on H^2. Thus we can consider the subspace $M_F = H^2 \ominus B_F H^2$. Let P_{M_F} be the orthogonal projection on M_F. For every $f \in H^\infty$, let M_f be the multiplication operator defined by f on H^2. Let $\mathcal{I}_F \subset H^\infty$ be the ideal of functions that vanish on F. Sarason showed that the map $f \mapsto P_{M_F} M_f P_{M_F}$ induces an isometry on $H^\infty/\mathcal{I}_F$.

Abrahamse in [1] replaced the disc with a multiply connected domain in $\Omega \subset \mathbb{C}$. He showed that in this case one cannot consider a single reproducing kernel Hilbert space, but one must consider a family of kernels parametrized by the torus $\mathbb{R}^g/\mathbb{Z}^g$, where the connectivity of Ω is $g+1$. Each point of the parameter space corresponds to a character χ of the fundamental group $\pi_1(\Omega)$ and we associate to χ the function space of χ-automorphic functions on Ω. These spaces were considered in the case of the annulus by Sarason in [26]. The work of Abrahamse generated a lot of interest (see for example [7, 28] for the treatment of the question of which subspaces of the parameter space are sufficient for a particular interpolation datum and what subspaces are sufficient for all data). The term Nevanlinna-Pick family was coined to describe the situation where a family of kernels is sufficient to solve the Nevanlinna-Pick problem.

The first author, Paulsen, Raghupathi and Singh in [11] observed that Nevanlinna-Pick families arise naturally in the setting of the disc, provided one considers a constrained interpolation question. They consider H_1^∞, the algebra of all bounded analytic functions on the disc with vanishing derivative at the origin. They prove that there is a Nevanlinna-Pick family parametrized by the projective 2-sphere. Following their work, Raghupathi generalized their results to the case of algebras of the form $\mathbb{C}1 + BH^\infty$, where B is a Blaschke product [23], and proved that constrained interpolation on the disc combined with an action of a Fuchsian group yields Abrahamse's result in [24]. In [6] Ball, Bolotnikov and ter Horst have generalized the result of [11] to the matrix-valued case and showed that one can parametrize the Nevanlinna-Pick families for $M_k(H_1^\infty)$ by a disjoint union of Grassmannians. A dual parametrization using test functions was provided by Dritschel and Pickering [16] in the case of the Neil parabola and much more generally by Dritschel and Undrakh [17]. The former was used by Dritschel, Jury and McCullough [15] to study rational dilations on the Neil parabola.

Finally, the first author jointly with Hamilton in [10] used the predual factorization property $\mathbb{A}_1(1)$ to show that every weak-$*$ closed subalgebra $\mathfrak{A}$ of the multiplier algebra of a complete Nevanlinna-Pick space has a Nevanlinna-Pick family. In particular, they have considered the algebra $\mathcal{M}_d$ of multipliers on the

Drury-Arveson space H^2_d. The Drury-Arveson space is a complete Nevanlinna-Pick space by [2, 13]. Arias and Popescu [4] show that $\mathcal{M}_d$ has the property $\mathbb{A}_1(1)$. However, the parametrization in [10] given for the Nevanlinna-Pick family of a weak-$*$ closed subalgebra $\mathfrak{A} \subset \mathcal{M}_d$ is by all $\mathcal{M}_d$-cyclic vectors in H^2_d.

The goal of this note is to provide a concrete description for the parameters of Nevanlinna-Pick families for a certain class of weak-$*$ closed subalgebras of $\mathcal{M}_d$. Note that the algebra H^∞_1 is generated by z^2 and z^3 and can be viewed as the pullback of bounded analytic functions on the cuspidal cubic (also known as the Neil parabola) in $\mathbb{D}^2$. Cusp algebras such as this one were considered first by Agler and McCarthy in [3]. Let us consider algebras $A \subset \mathbb{C}[z_1, \ldots, z_d] = \mathcal{P}_d$, such that the induced map from the affine space to the spectrum of A is a biholomorphism outside finitely many points. The key idea is to consider the largest common ideal in $\mathfrak{c} \subset A \subset \mathcal{P}_d$, called the conductor ideal (see Sect. 2 for the definition). By analogy to [1], we want to consider line (and more generally vector) bundles on the spectrum of A. Fortunately, we can describe them rather easily by considering the free $A/\mathfrak{c}$-submodules $M \subset (\mathcal{P}_d/\mathfrak{c})^{\oplus k}$, such that $M \otimes (\mathcal{P}_d/\mathfrak{c}) \cong (\mathcal{P}_d/\mathfrak{c})^{\oplus k}$ (compare to [9]). We set $\mathfrak{A} \subset \mathcal{M}_d$ to be the weak-$*$ closure of A.

In Sect. 2 we will collect some necessary information about the conductor ideal in the topological setting. In Sect. 3 we show that the parameter space for the Nevanlinna-Pick family is the Picard group of A, i.e, the group of isomorphism classes of line bundles on the spectrum of A. In Sect. 4 we will prove that $M_k(\mathcal{M}_d)$ has the property $\mathbb{A}_1(1)$ acting on $M_k(H^2_d)$ by transposed right multiplication and provide a description of the parameter space for the Nevanlinna-Pick families of $M_k(\mathfrak{A})$ for $k \geq 2$. Lastly, in Sect. 5 we prove the $M_6(\mathcal{M}_d)$ does not have $\mathbb{A}_1(1)$ acting on $\mathcal{H}^2_d \otimes \mathbb{C}^6$. This implies that one cannot avoid considering vector bundles on the singular variety if one wants to do constrained matrix-valued Nevanlinna-Pick interpolation.

2 The Conductor Ideal

The idea of conductor ideal is quite old in commutative algebra and algebraic number theory. Let $A \subset B$ be two rings. Then the conductor of A in B is the ideal of elements $a \in A$, such that $aB \subset A$. Alternatively, the conductor is the annihilator of the A-module B/A. It turns out that the conductor is an ideal of B and it is the largest ideal common to A and B.

We are going to discuss topological algebras, hence we make these definitions in our setting. All algebras considered in this section are unital and commutative. Let $\mathfrak{A} \subset \mathfrak{B} \subset B(\mathcal{H})$ be two weak-$*$ closed operator algebras. We assume that $\mathfrak{A}$ is a closed subalgebra of $\mathfrak{B}$.

Definition 2.1 The conductor ideal is the ideal $\mathfrak{c} \subset \mathfrak{A}$ defined by

$$\mathfrak{c} = \{f \in \mathfrak{A} \mid f\mathfrak{B} \subset \mathfrak{A}\}.$$

For every $g \in \mathfrak{B}$ and every $f \in \mathfrak{c}$ we have $fg\mathfrak{B} \subset f\mathfrak{B} \subset \mathfrak{A}$. Thus $\mathfrak{c}$ is an ideal of $\mathfrak{B}$. Furthermore, it is clearly the largest ideal with this property.

Lemma 2.2 *Let $\mathfrak{A} \subset \mathfrak{B}$ be as above and let $\mathfrak{c}$ be the conductor. Then $\mathfrak{c}$ is weak-$*$ closed and we have the commutative diagram*

$$\begin{array}{ccc} \mathfrak{A} & \longrightarrow & \mathfrak{B} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathfrak{A}/\mathfrak{c} & \longrightarrow & \mathfrak{B}/\mathfrak{c}. \end{array} \tag{2.1}$$

The vertical arrows are quotient maps and the horizontal ones are embeddings. Furthermore, $\mathfrak{A} = \pi^{-1}(\mathfrak{A}/\mathfrak{c})$ as a subspace of $\mathfrak{B}$.

Proof Let $a_\alpha \to a$ be a weak-$*$ convergent net, with $a_\alpha \in \mathfrak{c}$. Since multiplication is separately weak-$*$ continuous, for every $b \in \mathfrak{B}$, $a_\alpha b \to ab$. Since A is closed $ab \in \mathfrak{A}$. Hence $a \in \mathfrak{c}$.

Now it is clear that $\mathfrak{A} \subset \pi^{-1}(\mathfrak{A}/\mathfrak{c})$, so let $f \in \pi^{-1}(A/\mathfrak{c})$. Hence there exists $a \in \mathfrak{A}$, such that $f - a \in \mathfrak{c}$, but $\mathfrak{c} \subset A$ and we are done. □

Recall that Lat($\mathfrak{A}$) denotes the lattice of $\mathfrak{A}$-invariant subspaces of $\mathcal{H}$. Let us write $\mathcal{C} = \overline{\mathfrak{c}\mathcal{H}}$. It is clear that $\mathcal{C}$ is both $\mathfrak{A}$ and $\mathfrak{B}$-invariant subspace. Hence, the compression map $\mathfrak{B} \to B(\mathcal{C}^\perp)$ is a completely contractive and weak-$*$ continuous homomorphism. The kernel of this map contains $\mathfrak{c}$ and thus makes $\mathcal{C}^\perp$ into a $\mathfrak{B}/\mathfrak{c}$-module (and also an $\mathfrak{A}/\mathfrak{c}$-module).

Definition 2.3 Let $\mathcal{N} \subset \mathcal{C}^\perp$ be an $\mathfrak{A}/\mathfrak{c}$-invariant subspace. We define $\mathcal{H}_\mathcal{N} = \mathcal{N} \oplus \mathcal{C}$. It is clear that $\mathcal{H}_\mathcal{N} \in$ Lat($\mathfrak{A}$).

Let us assume that the kernel of the compression to $\mathcal{C}^\perp$ is precisely $\mathfrak{c}$. This is valid for $\mathcal{M}_d$, the multiplier algebra of Drury-Arveson space as the following lemma shows.

Lemma 2.4 *If $\mathcal{J}$ is an ideal of $\mathcal{M}_d$, the multiplier algebra of Drury-Arveson space, then compression to $\overline{\mathcal{I}H^2_d}^\perp$ is completely isometrically isomorphic to $\mathcal{M}_d/\mathcal{J}$.*

Proof We use the fact that $\mathcal{M}_d$ is equal to the compression of the non-commutative analytic Toeplitz algebra $\mathcal{L}_d$ (acting on full Fock space $\mathcal{F}^2_d$) to symmetric Fock space, which is the orthogonal complement of the range of the commutator ideal $\mathcal{C}$. This identification is a special case of [12, Theorem 2.1]. This result shows that if $\mathcal{I} \subset \mathcal{L}_d$ is a weak-$*$ closed ideal, then the kernel of the compression onto the complement of its range is precisely $\mathcal{I}$ and the quotient is completely isometric isomorphic to $\mathcal{L}_d/\mathcal{I}$. The corresponding result for $\mathcal{M}_d$ holds for weak-$*$ closed ideals $\mathcal{J}$ of $\mathcal{M}_d$ by applying the result to the preimage of $\mathcal{J}$ in $\mathcal{L}_d$. □

Note that if $\xi \in \mathcal{H}$ is $\mathfrak{B}$-cyclic, then $\mathcal{C} = \overline{\mathfrak{c}\xi}$ and the cyclic module $\overline{\mathfrak{A}\xi} = \overline{\mathfrak{A}/\mathfrak{c}P_{\mathcal{C}^\perp}\xi} \oplus \mathcal{C}$, where $P_{\mathcal{C}^\perp}$ is the orthogonal projection onto $\mathcal{C}^\perp$. We record this in the following proposition.

Proposition 2.5 *Assume that* $\mathfrak{c}$ *is precisely the kernel of the compression of* $\mathfrak{B}$ *to* $\mathcal{C}^{\perp}$. *Then the cyclic modules* $\mathfrak{A}\xi$, *where* ξ *is* $\mathfrak{B}$*-cyclic are in one-to-one correspondence with cyclic* $\mathfrak{A}/\mathfrak{c}$*-submodules of* $\mathcal{C}^{\perp}$.

3 Nevanlinna-Pick Families: The Scalar Case

Let H^2_d, for $d \in \mathbb{N}$, denote the Drury-Arveson space. Recall that this space is a reproducing kernel space of analytic functions on $\mathbb{B}_d$, the unit ball of $\mathbb{C}^d$, with reproducing kernel $k_d(z,w) = \frac{1}{1-\langle z,w\rangle}$. We will denote by $\mathcal{M}_d$ the algebra of multipliers of H^2_d. In particular, for $d = 1$, $H^2_1 = H^2(\mathbb{D})$ and $\mathcal{M}_1 = H^{\infty}(\mathbb{D})$. We will write simply H^2 and H^{∞} for H^2_1 and $\mathcal{M}_1$, respectively. Note that since $\mathcal{M}_d$ is a multiplier algebra of a reproducing kernel Hilbert space, it is weak-$*$ closed. Let $\mathfrak{A} \subset \mathcal{M}_d$ be a weak-$*$ closed subalgebra. We fix a set of points $F = \{z_1, \dots, z_n\} \subset \mathbb{B}_d$ and assume for simplicity that $\mathfrak{A}$ separates F. Let $\mathcal{I}_F \subset \mathfrak{A}$ be the weak-$*$ closed ideal of functions that vanish on F. We say that $\xi \in H^2_d$ is outer if it is $\mathcal{M}_d$-cyclic. For the convenience of the reader we recall some material from [10].

Definition 3.1 We say that a weak-$*$ closed subalgebra $\mathfrak{A} \subset B(\mathcal{H})$ has property $\mathbb{A}_1(1)$, or alternatively that $\mathfrak{A}$ has property $\mathbb{A}_1(1)$ acting on $\mathcal{H}$, if for every weak-$*$ continuous functional φ on $\mathfrak{A}$ with $\|\varphi\| < 1$, there exist $\xi, \eta \in \mathcal{H}$, such that $\|\xi\|, \|\eta\| < 1$ and $\varphi(f) = \langle f\xi, \eta\rangle$ for every $f \in \mathfrak{A}$.

The following result was established in [4, Proposition 6.2] and an alternative argument is found in [10, Theorem 5.2].

Theorem 3.2 (Arias-Popescu) *Let* $\mathfrak{A} \subset \mathcal{M}_d$ *be a weak-$*$ closed operator algebra. Then* $\mathfrak{A}$ *has* $\mathbb{A}_1(1)$ *and furthermore,* ξ *may be chosen to be outer.*

As was shown in [1], [11] and [23], to understand constrained interpolation we need to consider families of kernels. By [10, Lemma 2.1], every cyclic $\mathfrak{A}$-submodule $L \subset H^2_d$ is a reproducing kernel Hilbert space with respect to a kernel k^L defined on all of $\mathbb{B}_d$.

Definition 3.3 (Davidson-Hamilton) Let $\mathfrak{A} \subset \mathcal{M}_d$ be a weak-$*$ closed subalgebra. We say that a collection of kernels $\{k^{L_j}\}_{j\in J}$ associated with cyclic $\mathfrak{A}$-submodules is a Nevanlinna-Pick family, if for every set of points $F = \{z_1, \dots, z_n\} \subset \mathbb{B}_d$ separated by $\mathfrak{A}$ and complex scalars $w_1, \dots, w_n$, there exists $f \in \mathfrak{A}$, such that $f(z_\ell) = w_\ell$, for all $\ell = 1, \dots, d$ and $\|f\| \le 1$ if and only if the Pick matrices

$$\left[(1 - w_k\overline{w_{ell}})k^{L_j}(z_k, z_\ell)\right]_{k,\ell=1}^{n}$$

are positive for every $j \in J$.

By [10, Theorem 5.5] every weak-$*$ closed subalgebra of $\mathcal{M}_d$ admits a Nevanlinna-Pick family parametrized by the outer functions. This parameter family is not described in detail and our goal is to show that for finitely many constraints, this family is parametrized by a finite-dimensional manifold.

Set $\mathcal{P}_d = \mathbb{C}[z_1, \dots, z_d]$. Let $\psi\colon \mathbb{C}^d \to \mathbb{C}^e$ be a polynomial map that is an isomorphism outside a finite set of points. Let $A \subset \mathcal{P}_d$ be the corresponding algebra, i.e., the image of $\mathcal{P}_e$ under ψ^* and $\mathfrak{A} = \overline{A}^{w*} \subset \mathcal{M}_d$. The spectrum of A is the ring of polynomial functions on the image of $\mathbb{C}^d$. This is a rational variety with finitely many singular points. Let $\alpha_1, \dots, \alpha_\ell$ be the points in the fibers over the singular points in the image. Let $\mathfrak{m}_j$ be the maximal ideal corresponding to α_j. Recall that if $\mathfrak{p} \subset \mathcal{P}_d$ is a prime ideal, an ideal $\mathfrak{q}$ is called $\mathfrak{p}$-primary, if the only associated prime of $\mathcal{P}_d/\mathfrak{q}$ is $\mathfrak{p}$. By [18, Proposition 3.9] we have that there exists $k \in \mathbb{N}$, such that $\mathfrak{p}^k \subset \mathfrak{q} \subset \mathfrak{p}$. By primary decomposition there exist $\mathfrak{m}_j$-primary ideals $\mathfrak{q}_j$, such that $\mathfrak{c} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_\ell$. Thus by the Chinese remainder theorem $\mathcal{P}_d/\mathfrak{c} \cong \mathcal{P}_d/\mathfrak{q}_1 \times \cdots \mathcal{P}_d/\mathfrak{q}_\ell$. Now $\mathcal{P}_d/\mathfrak{q}_j$ is naturally embedded into $\mathcal{P}_d/\mathfrak{m}_j^{k_j}$ and thus the dual space of $\mathcal{P}_d/\mathfrak{q}_j$ can be identified with the quotient of the space spanned by the functionals taking the value of the polynomial and its derivatives up-to total order k_j at the point α_j.

From now on we will assume that all the points of the support of $\mathfrak{c}$ lie in $\mathfrak{B}_d$, the unit ball of $\mathbb{C}^d$. Let $\mathfrak{c} \subset A$ be the conductor ideal of A. Then the conductor of $\mathfrak{A}$ in $\mathcal{M}_d$ is the weak-$*$ closure of $\mathfrak{c}$.

Lemma 3.4 $$A = \mathcal{P}_d \cap \mathfrak{A}.$$

Proof One inclusion is obvious. On the other hand the dual space of $\mathcal{P}_d/\mathfrak{c}$ is spanned by the functionals of evaluation of the polynomial and its derivatives at the points α_j. The fiber square (2.1) tells us that A is precisely the preimage of $A/\mathfrak{c}$ in $\mathcal{P}_d/\mathfrak{c}$. Since $A/\mathfrak{c}$ is a subspace of $\mathcal{P}_d/\mathfrak{c}$ it is given by the vanishing of finitely many functionals. If $f \in \mathcal{P}_d \setminus A$, then there exists a functional in $(\mathcal{P}_d/\mathfrak{c})^*$ that vanishes on $A/\mathfrak{c}$ and does not vanish on the image of f. Now note that the evaluation functionals and the valuation of derivatives at points inside the disk are weak-$*$ continuous functionals on $\mathcal{M}_d$. Therefore we can conclude that if $f \in \mathcal{P}_d \setminus A$, then $f \notin \mathfrak{A}$. □

The following is a generalization of the Helson-Lowdenslager Theorem in the case when $d = 1$ (compare with [11, Theorem 1.3] in the case of H_1^∞).

Theorem 3.5 *Let $d = 1$ and write $\mathcal{P}_1 = \mathbb{C}[z]$ and $\mathcal{M}_1 = H^\infty$. Let $\mathcal{N} \subset L^2(\mathbb{T})$ be an $\mathfrak{A}$-invariant closed subspace, that is not invariant for H^∞. Then there exists an $A/\mathfrak{c}$-submodule $M \subset \mathbb{C}[z]/\mathfrak{c}$, such that $\mathcal{N} = JH^2_M$, where J is an unimodular function.*

Proof The proof is essentially the proof in [11]. Set $\widetilde{\mathcal{N}} = \overline{H^\infty \mathcal{N}}$. Recall that every ideal in $\mathbb{C}[z]$ is principal and thus $\mathfrak{c}$ is generated by $f_c = \prod_{j=1}^k (z - \alpha_j)^{m_j}$. Additionally, we have a Blaschke product B_c, such that $\mathcal{C} = B_c H^2$. By the classical Helson-Lowdenslager theorem, either $\widetilde{\mathcal{N}}$ is $L^2(E)$ for some measurable $E \subset \mathbb{T}$ or $\widetilde{\mathcal{N}} = JH^2$ for some unimodular function J. In the former case, note that z is a unitary on $\widetilde{\mathcal{N}}$ and thus f_c acts as an invertible operator. We conclude that $\widetilde{\mathcal{N}} = f_c\widetilde{\mathcal{N}} \subset \mathcal{N}$ and this contradicts our assumption that $\mathcal{N}$ is not H^∞-invariant.

In the latter case, we have that $\widetilde{\mathcal{N}} = JH^2 \supset \mathcal{N} \supset JB_cH^2$, where all of the containments are strict by assumption (since $\mathfrak{c}$ is an ideal of $\mathbb{C}[z]$ as well). Hence there exists a subspace $0 \neq M \subsetneq \mathbb{C}[z]/\mathfrak{c}$, such that $\mathcal{N} = JM \oplus JB_cH^2$. Since $\mathcal{N}$ is invariant under $\mathfrak{A}$, it then follows immediately that M is in fact an $A/\mathfrak{c}$-submodule of $\mathbb{C}[z]/\mathfrak{c}$. □

By Lemma 2.4, we may apply Proposition 2.5 to the conductor ideal of a subalgebra $\mathfrak{A}$ of the multiplier algebra $\mathcal{M}_d$. Therefore the cyclic $\mathfrak{A}$-submodules of H_d^2 of the form $\overline{\mathfrak{A}\xi}$, where ξ is outer, are parametrized by cyclic $A/\mathfrak{c}$ submodules of $\mathcal{P}_d/\mathfrak{c}$. Since outer functions do not vanish, the image of an outer function in $\mathcal{P}_d/\mathfrak{c}$ is a unit. Let us denote the unit group of the Artinian ring $\mathcal{P}_d/\mathfrak{c}$ by $(\mathcal{P}_d/\mathfrak{c})^\times$ and the unit group of $A/\mathfrak{c}$ by $(A/\mathfrak{c})^\times$.

Now let ξ_1, ξ_2 be outer and assume that $\overline{\mathfrak{A}\xi_1} = \overline{\mathfrak{A}\xi_2}$. Let us denote $f_j = P_{\mathcal{C}^\perp}\xi_j$, for $j = 1, 2$. Since $\overline{\mathfrak{A}\xi_j} = f_j A/\mathfrak{c} \oplus \mathcal{C}$, implies that $f_1 A/\mathfrak{c} = f_2 A/\mathfrak{c}$. Thus there exists a unit $u \in A/fc$, such that $f_1 = f_2 u$. Clearly, if such a unit exists, then $f_1 A/\mathfrak{c} = f_2 A/\mathfrak{c}$ and thus $\overline{\mathfrak{A}\xi_1} = \overline{\mathfrak{A}\xi_2}$. Hence we obtain the following lemma.

Lemma 3.6 *Let $\mathfrak{A} \subset \mathcal{M}_d$ be as above, then $\mathfrak{A}$ has a Nevanlinna-Pick family parametrized by* $(\mathcal{P}_d/\mathfrak{c})^\times \,/\, (A/\mathfrak{c})^\times$.

We can actually give a better geometric description of the parameter space. Consider again the conductor square (2.1) and by [25, Equation 1.4] we have an exact sequence:

$$\begin{array}{ccccccc} 1 & \longrightarrow & A^\times & \xrightarrow{\iota} & \mathcal{P}_d^\times \times (A/\mathfrak{c})^\times & \xrightarrow{\alpha} & (\mathcal{P}_d/\mathfrak{c})^\times \\ & & & & & \swarrow & \\ & & & & \mathrm{Pic}(A) & \longrightarrow & \mathrm{Pic}(A/\mathfrak{c}) \times \mathrm{Pic}(\mathbb{C}[z]). \end{array}$$

Here the first map is $\iota \colon \mathbb{C}^\times \cong A^\times \to \mathcal{P}_d^\times \times (A/\mathfrak{c})^\times \cong \mathbb{C}^\times \times (A/\mathfrak{c})^\times$ is $\iota(t) = (t, t)$. The second map is $\alpha(t, f) = tf^{-1}$. Now it is clear that the image of α is precisely $(A/\mathfrak{c})^\times$ and additionally since $\mathrm{Pic}(\mathcal{P}_d) = 1$ (see for example [19, Theorem 1.6]). Finally, we note that $A/\mathfrak{c}$ is an Artinian ring and thus is a finite product of Artinian local rings. We conclude that $\mathrm{Pic}(A/\mathfrak{c}) = 1$. This allows us to simplify the exact sequence into

$$1 \longrightarrow (A/\mathfrak{c})^\times \longrightarrow (\mathbb{C}[z]/\mathfrak{c})^\times \longrightarrow \mathrm{Pic}(A) \longrightarrow 1\ .$$

Thus $\mathrm{Pic}(A) = (\mathbb{C}[z]/\mathfrak{c})^\times \,/\, (A/\mathfrak{c})^\times$. We summarize this discussion in the following theorem.

Theorem 3.7 *Let $\mathfrak{A} \subset \mathcal{M}_d$ be a weak-$*$ closure of a subring $A \subset \mathcal{P}_d$ that arises from a parametrization of a singular rational variety with isolated singular points. Then $\mathfrak{A}$ admits a Nevanlinna-Pick family parametrized by the Picard group of A.*

This is analogous to the situation in [1] as we have a collection of line bundles parametrizing the Nevanlinna-Pick families for a multiply connected domain.

Example 3.8 In the case of the algebras $\mathfrak{A} = \mathbb{C}1 + BH^\infty$ considered in [23], with $B = \prod_{j=1}^r \left(\frac{z-\lambda_j}{1-z\overline{\lambda_j}}\right)^{k_j}$ a finite Blaschke product, the algebra $A = \mathfrak{A} \cap \mathbb{C}[z]$ is the algebra $A = \mathbb{C}1 + f\mathbb{C}[z]$, where $f = \prod_{j=1}^r \left(z - \lambda_j\right)^{k_j}$ is the (monic) polynomial with the zeroes prescribed by B including the order. Then $\mathfrak{c} = (f)$ and $A/\mathfrak{c} \cong \mathbb{C}$, whereas $\mathbb{C}[z]/\mathfrak{c} \cong \mathbb{C}[z]/(z-\lambda_1)^{k_1} \times \cdots \times \mathbb{C}[z]/(z-\lambda_r)^{k_r}$. The ring $A/\mathfrak{c}$ embeds into $\mathbb{C}[z]/\mathfrak{c}$ diagonally. The units of $\mathbb{C}[z]/\mathfrak{c}$ are precisely the elements with non-zero value at every λ_j. Hence as a space $(\mathbb{C}[z]/\mathfrak{c})^\times \cong \underbrace{\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times}_{r\text{-times}} \times \mathbb{C}^{k_1-1} \times \cdots \times \mathbb{C}^{k_r-1}$. We conclude that the Picard group parametrizing the Nevanlinna-Pick family has dimension $\sum_{j=1}^r k_j - 1$.

In particular, for the algebra H_1^∞ considered in [11] we have that $A = \mathbb{C}[z^2, z^3] = \mathbb{C}1 + z^2\mathbb{C}[z]$ and $\mathrm{Pic}(A) \cong \mathbb{C}$. The disparity with the compact parameter space obtained in [11] follows from the fact that the authors of [11] consider all cyclic non-trivial $A/(z^2)$-submodules of $\mathbb{C}[z]/(z^2)$. This results in a one point compactification of $\mathrm{Pic}(A)$ yielding the complex projective space or a sphere as stated in [11]. If one allows in the above setting all non-trivial cyclic modules, in other words if we replace $(\mathbb{C}[z]/\mathfrak{c})^\times$ with $\mathbb{C}[z]/\mathfrak{c} \setminus \{0\}$, then one will obtain the projective space $\mathbb{P}^{\sum_{j=1}^r k_j}(\mathbb{C})$ that is compact.

Example 3.9 Consider the map $\varphi\colon \mathbb{D} \to \mathbb{D}^2$ given by $\varphi(z) = (z^2, z^5)$. In this case $A = \mathbb{C}[z^2, z^5]$ and $\mathfrak{c} = (z^4)$. Note that $\mathfrak{A}$ is not of the form $\mathbb{C}1 + BH^\infty$. However, the method introduced above allows us to handle this case. We have that $A/(z^4)$ is spanned by $\{1, z^2\}$. Again the ring $\mathbb{C}[z]/(z^4)$ is a local Artinian ring and so is $A/(z^4)$. The invertible elements in $\mathbb{C}[z]/(z^4)$ are those of the form $\alpha + \beta z + \gamma z^2 + \delta z^3$, where $\alpha \neq 0$. The units of $A/(z^4)$ act on these points via

$$(a, 0, b, 0) \cdot (\alpha, \beta, \gamma, \delta) = (a\alpha, a\beta, a\gamma + b\alpha, a\delta + b\beta).$$

Since both α and a are non-zero, the orbit of each point $(\alpha, \beta, \gamma, \delta)$ is the intersection of the plane spanned by $(\alpha, \beta, \gamma, \delta)$ and $(0, 0, \alpha, \beta)$ and the open affine subset of $\mathbb{C}^4$ defined by $\alpha \neq 0$. Note also that the stabilizer of each point is trivial. The coordinate ring of this affine subset is $B = \mathbb{C}[\alpha, \beta, \gamma, \delta, 1/\alpha]$. The quotient by the action of our group corresponds to the subring of fixed elements. It is straightforward to check that $B^{(A/(z^4))^\times} = \mathbb{C}[\beta/\alpha, (\delta\alpha - \gamma\beta)/\alpha^2]$. The map $\mathbb{C}^4 \setminus \{\alpha = 0\} \to \mathbb{C}^2$, given by $(\alpha, \beta, \gamma, \delta) \mapsto (\beta/\alpha, (\delta\alpha - \gamma\beta)/\alpha^2)$ is surjective and thus $\mathbb{C}^2$ is the quotient. For each point $(x, y) \in \mathbb{C}^2$, we can associate an invertible element $f_{x,y} = 1 + xz + yz^3$ in $\mathbb{C}[z]/(z^4)$. The $A/(z^4)$-module generated by $f_{x,y}$ is spanned by $f_{x,y}$ and $z^2 f_{x,y} = z^2 + xz^3$. So we have that $H^2_{f_{x,y}} = \mathrm{Span}\{f_{x,y}, z^2 f_{x,y}\} \oplus z^4 H^2$. Let $g_{x,y}$ and $h_{x,y}$ be an orthonormal basis for the space spanned by $f_{x,y}$ and $z^2 f_{x,y}$. Then the reproducing kernel of this space is

$$k_{x,y}(z,w) = g_{x,y}(z)\overline{g_{x,y}(w)} + h_{x,y}(z)\overline{h_{x,y}(w)} + \frac{z^4\bar{w}^4}{1-z\bar{w}}.$$

If one would like a compact parameter space, one can compactify $\mathbb{C}^2$ by considering $\mathbb{P}^2(\mathbb{C})$ instead, where one identifies $\mathbb{C}^2$ as an open affine subset in $\mathbb{P}^2(\mathbb{C})$ via the map $(x, y) \mapsto (1 : x : y)$. Then our rational map can be extend by $(\alpha, \beta, \gamma, \delta) \mapsto (\alpha^2 : \alpha\beta : \delta\alpha - \gamma\beta)$ to the complement of the plane $\{\alpha = \beta = 0\}$. This subset corresponds to all elements $f \in \mathbb{C}[z]/(z^4)$, such that the $A/(z^4)$-submodule $A/(z^4)f$ is a two dimensional vector space.

Example 3.10 Consider the algebra $A = \mathbb{C}[w, zw, z^2, z^3] \subset \mathcal{P}_2 = \mathbb{C}[z, w]$. The map $\varphi(z, w) = \big(z^3, z^2, zw, w\big)$ is injective from $\mathbb{C}^2$ to $\mathbb{C}^4$. It is also easy to check that the only singularity of this map is at the origin. Recall that the Veronese map of degree 3 from $\mathbb{P}^2(\mathbb{C})$ to $\mathbb{P}^9(\mathbb{C})$ is defined by

$$(z : w : u) \mapsto (z^3 : z^2w : z^2u : zw^2 : zwu : zu^2 : w^3 : w^2u : wu^2 : u^3).$$

Restricting to the open affine subset $u \neq 0$, we get the map

$$(z, w) \mapsto (z^3, z^2w, z^2, zw^2, zw, z, w^3, w^2, w, 1).$$

Now it is easy to see that φ is the map obtained by composition with the projection onto the first, third, fifth and ninth coordinates. Hence the spectrum of A is a projection of the above affine open subset of the degree 3 Veronese variety.

A monomial $z^n w^m \in A$ if $m \neq 0$ and if $m = 0$, then we must have $n > 1$ or $n = 0$. Thus the only monomial not in A is z. Consequently the conductor ideal is generated by w and z^2. The quotient $\mathbb{C}[z, w]/\mathfrak{c} \cong \mathbb{C}[z]/(z^2)$ and $A/\mathfrak{c} \cong \mathbb{C}$. Therefore, the parameter space in this example is isomorphic to the parameter space of H_1^∞. For each point $\alpha + \beta z$, with $|\alpha|^2 + |\beta|^2 = 1$, we get the following kernel

$$k_{\alpha,\beta}(z, w, u, v) = (\alpha + \beta z)\overline{(\alpha + \beta u)} + \frac{w\bar{v} + z^2\bar{u}^2 + zw\bar{u}\bar{v}}{1 - z\bar{u} - w\bar{v}}.$$

The numerator in the fraction is obtained by observing that

$$M_w M_w^* + M_{z^2} M_{z^2}^* + M_{zw} M_{zw}^* = M_w M_w^* + M_z M_z^* - P_z = I - P_1 - P_z.$$

Here P_1 and P_z stand for the orthogonal projections onto the spaces spanned by 1 and z, respectively. Hence this is the orthogonal projection onto $\overline{wH_2^2 + z^2H_2^2}$.

Recall from [5, Definition 2.6] that for a Hilbert submodule $\mathcal{K} \subset H_d^2$, a sequence $f_1, f_2, \ldots \in \mathcal{M}_d$ is called an inner sequence if $\sum_{j=1}^\infty M_{f_j} M_{f_j}^* = P_{\mathcal{K}}$ and for almost every $\zeta \in \partial\mathbb{B}_d$, we have $\sum_{j=1}^\infty |f_j(\zeta)|^2 = 1$. Consider our sequence $f_1(z, w) = w$,

$f_2(z, w) = z^2$ and $f_3(z, w) = zw$ (complemented by zeroes). We have already seen that the first condition of Arveson's definition is satisfied. To see the second note that for every $\zeta = (z, w)$, such that $|z|^2 + |w|^2 = 1$, we have

$$|w|^2 + |z|^4 + |z|^2|w|^2 = |w|^2 + |z|^2(|z|^2 + |w|^2) = |w|^2 + |z|^2 = 1.$$

Hence this is an inner sequence for the submodule $\overline{wH_2^2 + z^2H_2^2}$.

4 Nevanlinna-Pick Families: The Matrix Case

First, we need an analog of [10, Theorem 3.1] in the setting of the Drury-Arveson space. In [10, Lemma 7.4] the first author and Hamilton prove this result for $d = 1$. The proof of the following proposition, just like the proof of [10, Theorem 3.1], is based on [14].

Proposition 4.1 *Let $k \in \mathbb{N}$, then $M_k(\mathcal{M}_d)$ acting on $M_k(H_d^2)$ by transposed right multiplication, i.e., the map $M_k(\mathcal{M}_d) \to M_{k^2}(\mathcal{M}_d)$ is given by $F \mapsto F \otimes I_k$, has $\mathbb{A}_1(1)$. More precisely, for every weak-$*$ continuous functional φ on $M_k(\mathcal{M}_d)$ with $\|\varphi\| < 1$, there exist X, Y in $\left(H_d^2\right)^{\oplus ks}$ such that $s \le k$, $\|X\|, \|Y\| < 1$ and $\varphi(F) = \langle (F \otimes I_s)X, Y\rangle$. Moreover, X can be chosen to be $M_k(\mathcal{M}_d)$-cyclic.*

Proof Let $\mathcal{F}_d^2 = \oplus_{n=0}^{\infty} \left(\mathbb{C}^d\right)^{\otimes n}$ be the full Fock space. The symmetrization map is a projection $P\colon \mathcal{F}_d^2 \to H_d^2$. The cutdown by P induces a complete contractive and weak-$*$ continuous map $q\colon \mathcal{L}_d \to \mathcal{M}_d$, where $\mathcal{L}_d$ stands for the weak-$*$ closed algebra generated by the left creation operators on $\mathcal{F}_d^2$. Let $q_k\colon M_k(\mathcal{L}_d) \to M_k(\mathcal{M}_d)$ denote $q \otimes I_{M_k}$. Any weak-$*$ continuous functional φ on $M_k(\mathcal{M}_d)$ can be pulled-back to $M_k(\mathcal{L}_d)$ without increasing the norm ($\varphi \circ q_k$). By Davidson and Pitts [14] the algebra $M_k(\mathcal{L}_d)$ has $\mathbb{A}_1(1)$. Hence we can find two vectors $\xi, \eta \in \left(\mathcal{F}_d^2\right)^{\oplus k}$, such that $\varphi \circ q_k(T) = \langle T\xi, \eta\rangle$, for every $T \in M_k(\mathcal{L}_d)$. Let $\xi = (\xi_1, \dots, \xi_k)^T$ and let $\mathcal{K} = \sum_{j=1}^k \mathcal{L}_d \xi_j$ be the $\mathcal{L}_d$-invariant subspace generated by the coordinates of ξ. By Davidson and Pitts [14, Theorem 2.1], there exists a row isometry $R\colon \left(\mathcal{F}_d^2\right)^{\oplus s} \to \mathcal{F}_d^2$ with coordinates in $\mathcal{R}_d = \mathcal{L}_d'$, such that $\mathcal{K} = R\left(\mathcal{F}_d^2\right)^{\oplus s}$ and $s \le k$. Set $\xi_j = Ru_j$, for $1 \le j \le k$ and $u_1, \dots, u_k \in \left(\mathcal{F}_d^2\right)^{\oplus s}$. Additionally, let $\eta = (\eta_1, \dots, \eta_k)^T$ and set $v_j = R^*\eta_j$, for $1 \le j \le k$. Note that $u_1, \dots, u_k$ generate $\left(\mathcal{F}_d^2\right)^{\oplus s}$ as a $\mathcal{L}_d$-module. Hence if we let U be the column vector obtained by stacking the u_j and similarly V is the column vector of the v_j, then

$$\varphi \circ q_k(F) = \langle F\xi, \eta\rangle = \langle F(I_k \otimes R)U, \eta\rangle = \langle (F \otimes I_s)U, V\rangle.$$

Now set $X = PU$ and $Y = PV$. Since $u_1, \dots, u_k$ generate the $\mathcal{L}_d$-module $\left(\mathcal{F}_d^2\right)^{\oplus s}$, then U is $M_k(\mathcal{M}_d) \otimes I_s$-cyclic and hence so is X. □

Corollary 4.2 *In the proof above one may consider U as a matrix with columns u_j and similarly V is a matrix with columns v_j. Let $X = PU$ and $Y = PV$. Then for every $F \in M_k(\mathcal{M}_d)$ we have $\varphi(F) = \operatorname{tr}\left(XF^T Y^*\right)$.*

Lemma 4.3 *The rank of the matrix $X(z)$, for $z \in \mathbb{B}_d$ is constant and is equal to s, where s is the number obtained in Proposition 4.1.*

Proof We need to show that for $z \in \mathbb{B}_d$, the matrix $X(z)$ is surjective. Then there exists $w \in \mathbb{C}^s$ non-zero, such that $\langle x_j(z), w\rangle = 0$ for all $1 \le j \le k$, where x_j are the columns of X. That, however, contradicts the fact that the x_j generate $\left(H^2_d\right)^{\oplus s}$ as an $M_s(\mathcal{M}_d)$-module. □

Lemma 4.4 *Let $X \in M_{k,s}(H^2_d)$ be a $M_k(\mathcal{M}_d) \otimes I_s$-cyclic vector and $H^2_{d,X} = \overline{(M_k(\mathfrak{A}) \otimes I_s)\, X}$. If $\widetilde{X} = P_{\mathcal{C}^\perp} X$, then*

$$H^2_{d,X} = (M_k(A/\mathfrak{c}) \otimes I_s)\, \widetilde{X} \oplus M_{k,s}(\mathfrak{C}).$$

Furthermore, the map $\widetilde{X}\colon\ (\mathcal{P}_d/\mathfrak{c})^{\oplus k} \to (\mathcal{P}_d/\mathfrak{c})^{\oplus s}$ is surjective.

Proof To see the first part of the lemma one simply notes that

$$\overline{\left(M_k(\overline{\mathfrak{c}}^{w*}) \otimes I_s\right) X} = \overline{\left(M_k(\overline{\mathfrak{c}}^{w*}) \otimes I_s\right) (M_k(\mathcal{M}_d) \otimes I_s)\, X} = M_{k,s}(\mathfrak{C}).$$

The second follows from Lemma 4.3 and the Nakayama lemma [18, Corollary 4.8]. □

Now given $X_1, X_2 \in M_{k,s}(H^2_d)$ that are $M_k(\mathcal{M}_d) \otimes I_s$-cyclic, we ask when is $H^2_{d,X_1} = H^2_{d,X_2}$? This is if and only if there exists $F, G \in M_k(A/\mathfrak{c})$, such that $\widetilde{X_1} = \widetilde{X_2} F^T$ and $\widetilde{X_2} = \widetilde{X_1} G^T$. In particular, this is true if we can find $F \in \mathrm{GL}_k(A/\mathfrak{c})$, such that $\widetilde{X_1} = \widetilde{X_2} F^T$. However, if $s < k$, then it need not be the case.

Set $\mathcal{Q}_s = \{Z \in \mathrm{Hom}((\mathcal{P}_d/\mathfrak{c})^{\oplus k}, (\mathcal{P}_d/\mathfrak{c})^{\oplus s} \mid \mathrm{rank}(Z) = s\}$. Then from [10, Theorem 7.2] we have that

Theorem 4.5 *Let $\mathfrak{A} \subset \mathcal{M}_d$ be a weak-$*$ closure of a subring $A \subset \mathcal{P}_d$ that arises from a parametrization of a singular rational variety with isolated singular points. Then for every $k \in \mathbb{N}$, the algebra $M_k(\mathfrak{A})$ admits a Nevanlinna-Pick family parametrized by the space $\sqcup_{s<k} \mathcal{Q}_s / (\mathrm{GL}_k(A/\mathfrak{c}) \otimes I_s)$.*

Remark 4.6 Let us fix $k \in \mathbb{N}$ and $s \le k$. Let $Z \in \mathcal{Q}_s$. Consider the columns of Z^T as elements of $(\mathcal{P}_d/\mathfrak{c})^{\oplus k}$ and let M_Z be the $A/\mathfrak{c}$-module generated by these elements. By the assumption Z is surjective and hence $\mathcal{P}/\mathfrak{c} \otimes_{A/\mathfrak{c}} M_Z$ is a free $\mathcal{P}_d/\mathfrak{c}$-module of rank s. Hence this data corresponds to a rank s vector bundle on the spectrum of A as in [9]. However, we might get isomorphic vector bundles by considering different submodules of $(\mathcal{P}_d/\mathfrak{c})^{\oplus s}$. Therefore, in a sense, the Nevanlinna-Pick family is parametrized by all vector bundles on the spectrum of A of rank less than or equal to k.

5 Property $\mathbb{A}_1(1)$ for Matrices

In this section we study weak-$*$ closed operator algebras $\mathfrak{A} \subset B(\mathcal{H})$, such that $M_k(\mathfrak{A})$ has the property $\mathbb{A}_1(1)$ acting on $\mathcal{H}^{\oplus k}$. First we provide a generalization of [8, Corollary 3.5] with bounds on the condition number of the similarity.

Theorem 5.1 *Let $\mathfrak{A} \subset B(\mathcal{H})$ be a weak-$*$ closed unital operator algebra. Assume that $M_k(\mathfrak{A})$ has $\mathbb{A}_1(1)$ acting on $\mathcal{H}^{\oplus k}$. Let $\varphi\colon \mathfrak{A} \to M_k$ be a weak-$*$ continuous completely contractive unital homomorphism. Then for $\epsilon > 0$, there exists a k-dimensional subspace $\mathcal{K}_\epsilon \subset \mathcal{H}$ that is semi-invariant for $\mathfrak{A}$, and an invertible linear map $S_\epsilon\colon \mathbb{C}^k \to \mathcal{K}_\epsilon$, such that $\varphi(a) = S_\epsilon^{-1} P_{\mathcal{K}_\epsilon} a|_{\mathcal{K}_\epsilon} S_\epsilon$ for every $a \in \mathfrak{A}$, and* $\lim_{\epsilon \to 0+} \|S_\epsilon\| \|S_\epsilon^{-1}\| = 1$.

Proof Let $e_1, \dots, e_k$ be an orthonormal basis for $\mathbb{C}^k$ and let $\varphi_{ij}(a) = \langle \varphi(a) e_j, e_i \rangle$ be the matrix coefficients of φ. Since φ is completely contractive and unital it extends to a unital completely positive (ucp) map on $\mathfrak{A} + \mathfrak{A}^*$. We can now construct a state on $M_k(\mathfrak{A})$ from φ. Let $T = \sum_{i,j=1}^k E_{ij} \otimes a_{ij} \in M_k(\mathfrak{A})$, then

$$
\begin{aligned}
s(T) &= \frac{1}{k} \sum_{i,j=1}^{k} \varphi_{ij}(a_{ij}) \\
&= \frac{1}{k} \sum_{i,j=1}^{k} \operatorname{tr}\left(\varphi(a_{ij}) E_{ij}^*\right) \\
&= \langle \left(\mathrm{id}_{M_k} \otimes \varphi\right)(T) I_k, I_k \rangle.
\end{aligned}
$$

In the latter equality, we view M_k as a Hilbert space with the normalized Hilbert-Schmidt product.

Fix $\epsilon > 0$. Since s is a state and $M_k(\mathfrak{A})$ has $\mathbb{A}_1(1)$, there exist $\xi, \eta \in \mathcal{H}^{\oplus k}$, such that $s(T) = \langle T\xi, \eta \rangle$ for every $T \in M_k(\mathfrak{A})$ and $\|\xi\|, \|\eta\| < \sqrt{1+\epsilon}$. Write $\xi = (\xi_1, \dots, \xi_k)^T$ and $\eta = (\eta_1, \dots, \eta_k)^T$. Note that for every $1 \le i, j \le k$,

$$
\frac{1}{k} \varphi_{ij}(a) = s(E_{ij} \otimes a) = \langle \left(E_{ij} \otimes a\right) \xi, \eta \rangle = \langle a \xi_j, \eta_i \rangle.
$$

Following [8] we define the following subspaces of $\mathcal{H}$

$$
\begin{aligned}
\mathcal{N}_\epsilon &= \overline{\textstyle\sum_{j=1}^k \mathfrak{A} \xi_j}, & \mathcal{N}_{*,\epsilon} &= \overline{\textstyle\sum_{j=1}^k \mathfrak{A}^* \eta_j}, \\
\mathcal{G}_\epsilon &= \mathcal{N}_\epsilon \cap \mathcal{N}_{*,\epsilon}, & \mathcal{K}_\epsilon &= \mathcal{N}_\epsilon \ominus \mathcal{G}_\epsilon.
\end{aligned}
$$

Clearly $\mathcal{K}_\epsilon$ is a semi-invariant subspace of $\mathfrak{A}$. Now write $\xi_j = \xi_{j1} + \xi_{j2}$, with $\xi_{j1} \in \mathcal{K}_\epsilon$ and $\xi_{j2} \in \mathcal{G}_\epsilon$. Fix j, then for every $a \in \mathfrak{A}$ and every $1 \le i \le k$ we have

$$
\frac{1}{k} \varphi_{ij}(a) = \langle a \xi_j, \eta_i \rangle = \langle \xi_j, a^* \eta_i \rangle = \langle a \xi_{j1}, \eta_i \rangle.
$$

Hence we can assume that $\xi_1, \dots, \xi_k \in \mathcal{K}_\epsilon$. Since ξ_{1j} is a projection of ξ_j, we will not increase the norm by replacing ξ_j with ξ_{1j}. From the fact that φ is a homomorphism we have that for every $a, b \in \mathfrak{A}$ and every $1 \le i, j \le k$

$$\begin{aligned}\langle a\xi_j, b^*\eta_i\rangle &= \langle ba\xi_j, \eta_i\rangle = \frac{1}{k}\varphi_{ij}(ba) = \frac{1}{k}\sum_{r=1}^{k}\varphi_{ir}(b)\varphi_{rj}(a)\\ &= k\sum_{r=1}^{k}\langle \xi_r, b^*\eta_i\rangle\langle a\xi_j, \eta_r\rangle = \langle k\sum_{r=1}^{k}\langle a\xi_j, \eta_r\rangle\xi_r, b^*\eta_i\rangle\end{aligned}$$

This immediately implies that

$$P_{\mathcal{K}_\epsilon} a\xi_j = k\sum_{r=1}^{k}\langle a\xi_j, \eta_r\rangle\xi_r.$$

Additionally, since the elements of the form $\sum_{r=1}^{k} P_{\mathcal{K}_\epsilon} a_r\xi_r$ are dense in $\mathcal{K}_\epsilon$, we obtain that $\mathcal{K}_\epsilon$ is spanned by $\xi_1, \dots, \xi_k$. Since φ is unital we have that $\langle \xi_j, \eta_i\rangle = \frac{1}{k}\delta_{ij}$. Conclude that $\xi_1, \dots, \xi_k$ is a basis for yy_ϵ.

The last consideration can be slightly refined, namely

$$1 = s(I) = \langle \xi, \eta\rangle = \sum_{r=1}^{k}\langle \xi_r, \eta_r\rangle \le \|\xi\|\|\eta\| \le 1 + \epsilon. \tag{5.1}$$

Now consider the map $S_\epsilon : \mathbb{C}^k \to \mathcal{K}$ given by

$$S_\epsilon e_i = \xi_i \quad \text{for } 1 \le i \le n.$$

It is immediate that S_ϵ is bijective. Furthermore, S_ϵ intertwines $\varphi(a)$ and $P_{\mathcal{K}_\epsilon} a|_{\mathcal{K}_\epsilon}$. To see this we simply observe that

$$\begin{aligned}S_\epsilon\varphi(a)e_i &= \sum_{r=1}^{k}\varphi_{ri}(a)S_\epsilon e_r\\ &= k\sum_{r=1}^{k}\langle a\xi_i, \eta_r\rangle\xi_r = P_{\mathcal{K}_\epsilon} a\xi_i\\ &= P_{\mathcal{K}_\epsilon} aS_\epsilon e_i.\end{aligned}$$

Hence every homomorphism is similar to the compression of $\mathfrak{A}$ to a semi-invariant subspace. It remains to estimate the condition number. Note that as ϵ tends to 0 in (5.1) we have that η tends to ξ. In particular, the norm of each ξ_j tends to $\frac{1}{\sqrt{k}}$ and the inner products $\langle \xi_i, \xi_j \rangle$ for $i \neq j$ tend to 0. Therefore $\sqrt{k} S_\epsilon$ is close to a unitary, and thus the condition number is close to 1. □

Corollary 5.2 *Let $\mathfrak{A} \subset B(\mathcal{H})$ be a unital weak-$*$ closed operator algebra. Assume that $M_k(\mathfrak{A})$ has $\mathbb{A}(1)$ acting on $\mathcal{H}^{\oplus k}$. Let $\varphi \colon \mathfrak{A} \to M_k$ be a weak-$*$ continuous unital homomorphism. Then for every $\epsilon > 0$, there exists a semi-invariant subspace $\mathcal{K}_\epsilon \subset \mathcal{H}$ and an invertible linear map $T_\epsilon \colon \mathbb{C}^k \to \mathcal{K}_\epsilon$, such that for every $a \in \mathfrak{A}$, $\varphi(a) = T_\epsilon^{-1} P_{\mathcal{K}_\epsilon} a|_{\mathcal{K}_\epsilon}$ and $\lim_{\epsilon \to 0+} \|T_\epsilon\| \|T_\epsilon^{-1}\| = \|\varphi\|_{cb}$.*

Proof By Smith's lemma every bounded homomorphism $\varphi \colon \mathfrak{A} \to M_k$ is completely bounded and by a theorem of Paulsen it is similar to a completely contractive homomorphism. Furthermore, there is a similarity S, such that $\psi = S^{-1} \varphi S$ is completely contractive and $\|\varphi\|_{cb} = \|S\| \|S^{-1}\|$. Given $\epsilon > 0$, we apply Theorem 5.1 to ψ to find a semi-invariant subspace $\mathcal{K}_\epsilon \subset \mathcal{H}$ and an invertible linear map $S_\epsilon \colon \mathbb{C}^k \to \mathcal{K}_\epsilon$, such that for every $a \in \mathfrak{A}$, $\psi(a) = S_\epsilon^{-1} P_{\mathcal{K}_\epsilon} a|_{\mathcal{K}_\epsilon} S_\epsilon$. Set $T_\epsilon = S_\epsilon S^{-1}$. It is clear now that T_ϵ is the similarity that we are after. To prove the last statement we note that

$$\|\varphi\|_{cb} \leq \|T_\epsilon\| \|T_\epsilon^{-1}\| \leq \|S\| \|S^{-1}\| \|S_\epsilon\| \|S_\epsilon^{-1}\| = \|\varphi\|_{cb} \|S_\epsilon\| \|S_\epsilon^{-1}\|.$$

Now it remains to apply the fact that $\lim_{\epsilon \to 0+} \|S_\epsilon\| \|S_\epsilon^{-1}\| = 1$. □

The following lemma and corollary demonstrate that the methods employed in the previous section to obtain the property $\mathbb{A}_1(1)$ for $M_k(\mathcal{M}_d)$ are necessary.

Lemma 5.3 *Suppose that $f \in H^2_2$ satisfies*

$$\|f\| = 1 \quad \textit{and} \quad \|z_1 f\|^2 > 1 - \epsilon \quad \textit{and} \quad \|z_2 f\|^2 > 1 - \epsilon$$

Then $|\langle f, 1 \rangle|^2 \geq 1 - 4\epsilon$ and $\operatorname{dist}(f, \mathbb{C}1) \leq 2\sqrt{\epsilon}$.

Proof Write $f = \sum a_{mn} z_1^m z_2^n$. Note that the monomials are orthogonal and $\|z_1^m z_2^n\|^2 = \dfrac{m!\,n!}{(n+m)!}$. So we have

$$1 = \|f\|^2 = \sum |a_{mn}|^2 \frac{m!\,n!}{(n+m)!}.$$

Thus

$$\begin{aligned} \|z_1 f\|^2 + \|z_2 f\|^2 &= \sum |a_{mn}|^2 \frac{m!\,n!}{(n+m)!} \Big(\frac{m+1}{m+n+1} + \frac{n+1}{m+n+1} \Big) \\ &= 1 + \sum |a_{mn}|^2 \frac{m!\,n!}{(n+m)!} \frac{1}{m+n+1} \\ &\leq \frac{3}{2} + \frac{1}{2} |a_{00}|^2. \end{aligned}$$

Hence we have that

$$1 - 4\epsilon \leq |a_{00}|^2 = |\langle f, 1\rangle|^2 \leq 1.$$

It follows that

$$\text{dist}(f, \mathbb{C}1)^2 = \|f\|^2 - |\langle f, 1\rangle|^2 \leq 4\epsilon.$$

□

Corollary 5.4 *The algebra $M_6(\mathcal{M}_d)$ does not have $\mathbb{A}_1(1)$.*

Proof By Theorem 5.1 we need to produce a weak-$*$ continuous completely contractive homomorphism $\varphi\colon \mathcal{M}_d \to M_6$ which cannot be approximated by compressions to semi-invariant subspaces. We will prove this for $\mathcal{M}_2$, and for every other $d > 2$, it follows by embedding $\mathcal{M}_2$ into $\mathcal{M}_d$.

Let us consider first the compression of $\mathcal{M}_2$ to Span$\{1, z_1, z_2\}$. This is a semi-invariant subspace and hence this is a representation that we will denote by π. It is clear that π unital, completely contractive, and weak-$*$ continuous. Let $\varphi = \pi \oplus \pi$. In order to approximate φ by compressions to semi-invariant subspaces, we need to be able to find for every $\epsilon > 0$, two unit vectors $f, g \in \mathcal{H}^2_2$, such that $|\langle f, g\rangle| < \epsilon$ and the compression of $\mathcal{M}_2$ onto Span$\{f, z_1 f, z_2 f\}$ and onto Span$\{g, z_1 g, z_2 g\}$ is ϵ-similar to π. In particular, this implies that both f and g satisfy the assumptions of Lemma 5.3. Multiplying f and g by unimodular scalars will not change the properties of these vectors. Hence we may assume that $f(0), g(0) > 0$. However, this implies that

$$\sqrt{1 - 4\epsilon} \leq \langle f, 1\rangle,\ \langle g, 1\rangle \leq 1.$$

Therefore

$$\begin{aligned}|\langle f, g\rangle| &\geq \langle f, 1\rangle\langle g, 1\rangle - \|P^{\perp}_{\mathbb{C}1} f\| \, \|P^{\perp}_{\mathbb{C}1} g\| \\ &\geq (1 - 4\epsilon) - (2\sqrt{\epsilon})^2 = 1 - 8\epsilon.\end{aligned}$$

This contradicts the fact that $\langle f, g\rangle < \epsilon$ when $\epsilon < 0.1$. □

We suspect that $M_2(\mathcal{M}_d)$ does not have $\mathbb{A}_1(1)$. We state this as an explicit problem:

Question 5.5 Does $M_2(\mathcal{M}_d)$ have $\mathbb{A}_1(1)$ for $d \geq 2$?

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On Certain Commuting Isometries, Joint Invariant Subspaces and C^*-Algebras

B. Krishna Das, Ramlal Debnath, and Jaydeb Sarkar

Dedicated to the memory of Ronald G. Douglas, our teacher, mentor and friend

Abstract In this paper, motivated by the Berger, Coburn and Lebow and Bercovici, Douglas and Foias theory for tuples of commuting isometries, we study analytic representations and joint invariant subspaces of a class of n tuples of commuting isometries and prove that the C^*-algebra generated by the n-tuple of multiplication operators by the coordinate functions restricted to an invariant subspace of finite codimension in $H^2(\mathbb{D}^n)$ is unitarily equivalent to the C^*-algebra generated by the n-tuple of multiplication operators by the coordinate functions on $H^2(\mathbb{D}^n)$.

Keywords Unilateral shift · Commuting isometries · Joint invariant subspaces · Hardy space over unit polydisc · C^*-algebras · Finite codimensional subspaces

Mathematics Subject Classification (2010) Primary 47A13, 47C15, 47L80; Secondary 47A20, 47A45, 47B35, 47A65, 46E22, 46E40, 47A05

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_8

Notations

$\mathcal{H}, \mathcal{E}, \mathcal{E}_*$	Hilbert spaces
$\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$	The space of all bounded linear operators from $\mathcal{E}$ to $\mathcal{E}_*$
$\mathcal{B}(\mathcal{E})$	The space of all bounded linear operators on $\mathcal{E}$
$\mathbb{D}^n$	Open unit polydisc in $\mathbb{C}^n$
$H^2(\mathbb{D}^n)$	Hardy space on $\mathbb{D}^n$
$H^2_{\mathcal{E}}(\mathbb{D}^n)$	$\mathcal{E}$-valued Hardy space on $\mathbb{D}^n$
$H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D}^n)$	Set of all $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$-valued bounded analytic functions on $\mathbb{D}^n$.
$(M_{z_1}, \dots, M_{z_n})$	n-tuple of multiplication operator by the coordinate functions on $H^2(\mathbb{D}^n)$

(1) All Hilbert spaces are assumed to be over the complex numbers.
(2) For a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, we denote by $P_{\mathcal{S}}$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$.
(3) For nested closed subspaces $\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{H}$, the orthogonal projection of $\mathcal{M}_2$ onto $\mathcal{M}_1$ is denoted by $P^{\mathcal{M}_2}_{\mathcal{M}_1}$.

1 Introduction

Tuples of commuting isometries on Hilbert spaces are cental objects of study in (multivariable) operator theory. This paper is concerned with the study of analytic representations, joint invariant subspaces and C^*-algebras of a certain class of tuples of commuting isometries.

To be precise, let $\mathcal{H}$ be a Hilbert space, and let $(V_1, \dots, V_n)$ be an n-tuple of commuting isometries on $\mathcal{H}$. In what follows, we always assume that $n \geq 2$. Set

$$V = \prod_{i=1}^{n} V_i.$$

We say that $(V_1, \dots, V_n)$ is a *pure n-isometry* if V is a unilateral shift. A closed subspace $\mathcal{S} \subseteq H^2(\mathbb{D}^n)$ is said to be an *invariant subspace* of $H^2(\mathbb{D}^n)$ if $M_{z_i}\mathcal{S} \subseteq \mathcal{S}$ for all $i = 1, \dots, n$ where M_{z_i} is the multiplication operator by the coordinate function z_i on $H^2(\mathbb{D}^n)$. Simpler (but complex enough) examples of pure n-isometry can be obtained by taking restrictions of the n-tuple of multiplication operators by coordinate functions $(M_{z_1}, \dots, M_{z_n})$ on $H^2(\mathbb{D}^n)$ to invariant subspaces of $H^2(\mathbb{D}^n)$ as follows. Given an invariant subspace $\mathcal{S}$ of $H^2(\mathbb{D}^n)$, we let

$$R_{z_i} = M_{z_i}|_{\mathcal{S}} \in \mathcal{B}(\mathcal{S}) \qquad (i = 1, \dots, n).$$

Then it is easy to see that $(R_{z_1}, \ldots, R_{z_n})$ is a pure n-isometry. We denote by $\mathcal{T}(\mathcal{S})$ the C^*-algebra generated by the commuting isometries $\{R_{z_1}, \ldots, R_{z_n}\}$. We simply say that $\mathcal{T}(\mathcal{S})$ is the *C^*-algebra corresponding to the invariant subspace $\mathcal{S}$*.

In this paper we aim to address three basic issues of pure n-isometries: (i) analytic and canonical models for pure n-isometries, (ii) an abstract classification of joint invariant subspaces for pure n-isometries, and (iii) the nature of C^*-algebra $\mathcal{T}(\mathcal{S})$ where $\mathcal{S}$ is a finite codimensional invariant subspace in $H^2(\mathbb{D}^n)$. To that aim, for (i) and (ii), we consider the initial approach by Berger et al. [6] from a more modern point of view (due to Bercovici et al. [5]) along with the technique of [20]. For (iii), we will examine Seto's approach [26] more closely from "subspace" approximation point of view.

We now briefly outline the setting and the main contributions of this paper. Let $\mathcal{E}$ be a Hilbert space, and let $\varphi \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$. We say that φ is an inner function if $\varphi(e^{it})^*\varphi(e^{it}) = I_{\mathcal{E}}$ for almost every t (cf. page 196, [21]). Recall that two n-tuples of commuting operators $(A_1, \ldots, A_n)$ on $\mathcal{H}$ and $(B_1, \ldots, B_n)$ on $\mathcal{K}$ are said to be *unitarily equivalent* if there exists a unitary operator $U : \mathcal{H} \to \mathcal{K}$ such that $UA_i = B_iU$ for all $i = 1, \ldots, n$. In [5], motivated by Berger et al. [6], Bercovici, Douglas and Foias proved the following result: A pure n-isometry is unitarily equivalent to a model pure n-isometry. The model pure n-isometries are defined as follows [5]: Consider a Hilbert space $\mathcal{E}$, unitary operators $\{U_1, \ldots, U_n\}$ on $\mathcal{E}$ and orthogonal projections $\{P_1, \ldots, P_n\}$ on $\mathcal{E}$. Let $\{\Phi_1, \ldots, \Phi_n\} \subseteq H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ be bounded $\mathcal{B}(\mathcal{E})$-valued holomorphic functions (polynomials) on $\mathbb{D}$, where

$$\Phi_i(z) = U_i(P_i^\perp + zP_i) \qquad (z \in \mathbb{D}),$$

and $i = 1, \ldots, n$. Then the n-tuple of multiplication operators $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{E}}(\mathbb{D})$ is called a *model pure n-isometry* if the following conditions are satisfied:

(a) $U_iU_j = U_jU_i$ for all $i, j = 1, \ldots n$;
(b) $U_1 \cdots U_n = I_{\mathcal{E}}$;
(c) $P_i + U_i^*P_jU_i = P_j + U_j^*P_iU_j \leq I_{\mathcal{E}}$ for all $i \neq j$; and
(d) $P_1 + U_1^*P_2U_1 + U_1^*U_2^*P_3U_2U_1 + \cdots + U_1^*U_2^* \cdots U_{n-1}^*P_nU_{n-1} \cdots U_2U_1 = I_{\mathcal{E}}$.

It is easy to see that a model pure n-isometry is also a pure n-isometry (see page 643 in [5]).

We refer to Bercovici et al. [3–5] and also [8–10, 12, 14, 15, 17, 19, 22, 26] and [27, 28] for more on pure n-isometries, $n \geq 2$, and related topics.

Our first main result, Theorem 2.1, states that a pure n-isometry is unitarily equivalent to an explicit (and canonical) model pure n-isometry. In other words, given a pure n-isometry $(V_1, \ldots, V_n)$ on $\mathcal{H}$, we explicitly solve the above conditions (a)–(d) for some Hilbert space $\mathcal{E}$, unitary operators $\{U_1, \ldots, U_n\}$ on $\mathcal{E}$ and orthogonal projections $\{P_1, \ldots, P_n\}$ on $\mathcal{E}$ so that the corresponding model pure n-isometry $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ is unitarily equivalent to $(V_1, \ldots, V_n)$. This also gives a new proof of Bercovici, Douglas and Foias theorem. On the one hand, our model pure n-isometry is explicit and canonical. On the other hand, our proof is perhaps more computational than the one in [5]. Another advantage of our approach is the proof of a list of useful equalities related to commuting isometries, which can be useful in other contexts.

Our second main result concerns a characterization of joint invariant subspaces of model pure n-isometries. To be precise, let $\mathcal{W}$ be a Hilbert space, and let $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ be a model pure n-isometry on $H^2_{\mathcal{W}}(\mathbb{D})$. Let $\mathcal{S}$ be a closed subspace of $H^2_{\mathcal{W}}(\mathbb{D})$. In Theorem 3.1, we prove that $\mathcal{S}$ is invariant for $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ if and only if there exist a Hilbert space $\mathcal{W}_*$, an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}_*, \mathcal{W})}(\mathbb{D})$ and a model pure n-isometry $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ on $H^2_{\mathcal{W}_*}(\mathbb{D})$ such that

$$\mathcal{S} = \Theta H^2_{\mathcal{W}_*}(\mathbb{D}),$$

and

$$\Phi_i \Theta = \Theta \Psi_i,$$

for all $i = 1, \ldots, n$. Moreover, the above representation is unique in an appropriate sense (see the remark following Theorem 3.1).

The third and final result concerns C^*-algebras corresponding to finite codimensional invariant subspaces in $H^2(\mathbb{D}^n)$. To be more specific, recall that if $n = 1$ and $\mathcal{S}$ and $\mathcal{S}'$ are invariant subspaces of $H^2(\mathbb{D})$, then $U\mathcal{T}(\mathcal{S})U^* = \mathcal{T}(\mathcal{S}')$ for some unitary $U : \mathcal{S} \to \mathcal{S}'$. Indeed, since $\mathcal{S} = \theta H^2(\mathbb{D})$ for some inner function $\theta \in H^\infty(\mathbb{D})$, it follows, by Beurling theorem, that $U := M_\theta : H^2(\mathbb{D}) \to \mathcal{S}$ is a unitary and hence $U^*\mathcal{T}(\mathcal{S})U = \mathcal{T}(H^2(\mathbb{D}))$. Clearly, the general case follows from this special case. For invariant subspaces $\mathcal{S}$ and $\mathcal{S}'$ of $H^2(\mathbb{D}^n)$, we say that $\mathcal{T}(\mathcal{S})$ and $\mathcal{T}(\mathcal{S}')$ are *isomorphic* as C^*-algebras if $U\mathcal{T}(\mathcal{S})U^* = \mathcal{T}(\mathcal{S}')$ holds for some unitary $U : \mathcal{S} \to \mathcal{S}'$. It is then natural to ask: If $n > 1$ and $\mathcal{S}$ and $\mathcal{S}'$ are invariant subspaces of $H^2(\mathbb{D}^n)$, are $\mathcal{T}(\mathcal{S})$ and $\mathcal{T}(\mathcal{S}')$ isomorphic as C^*-algebras?

In the same paper [6], Berger, Coburn and Lebow asked whether $\mathcal{T}(\mathcal{S})$ is isomorphic to $\mathcal{T}(H^2(\mathbb{D}^2))$ for every finite codimensional invariant subspaces $\mathcal{S}$ in $H^2(\mathbb{D}^2)$. This question was recently answered positively by Seto in [26]. Here we extend Seto's answer from $H^2(\mathbb{D}^2)$ to the general case $H^2(\mathbb{D}^n)$, $n \geq 2$.

The rest of this paper is organized as follows. In Sect. 2 we study and review the analytic construction of pure n-isometries. We also examine a (canonical) model pure n-isometry. A characterization of invariant subspaces is given in Sect. 3. Finally, in Sect. 4, we prove that $\mathcal{T}(\mathcal{S})$ is isomorphic to $\mathcal{T}(H^2(\mathbb{D}^n))$ where $\mathcal{S}$ is a finite codimensional invariant subspaces in $H^2(\mathbb{D}^n)$.

2 Pure *n*-Isometries and Model Pure *n*-Isometries

In this section, we first derive an explicit analytic representation of a pure n-isometry. Then we propose a canonical model for pure n-isometries.

For motivation, let us recall that if X on $\mathcal{H}$ is a bounded linear operator, then X is a unilateral shift operator if and only if X and M_z on $H^2_{\mathcal{W}(X)}(\mathbb{D})$ are unitarily equivalent. Here

$$\mathcal{W}(X) = \ker X^* = \mathcal{H} \ominus X\mathcal{H},$$

is the *wandering subspace* for X (see Halmos [16]) and M_z denotes the multiplication operator by the coordinate function z on $H^2_{\mathcal{W}(X)}(\mathbb{D})$, that is, $(M_z f)(w) = wf(w)$ for all $f \in H^2_{\mathcal{W}(X)}(\mathbb{D})$ and $w \in \mathbb{D}$. Explicitly, if X is a unilateral shift on $\mathcal{H}$, then

$$\mathcal{H} = \bigoplus_{m=0}^{\infty} X^m \mathcal{W}(X).$$

Hence the natural map $\Pi_X : \mathcal{H} \to H^2_{\mathcal{W}(X)}(\mathbb{D})$ defined by

$$\Pi_X(X^m \eta) = z^m \eta,$$

for all $m \geq 0$ and $\eta \in \mathcal{W}(X)$, is a unitary operator and

$$\Pi_X X = M_z \Pi_X.$$

We call Π_X the *Wold-von Neumann decomposition* of the shift X.

Now let $\mathcal{H}$ be a Hilbert space, and let $(V_1, \ldots, V_n)$ be a pure n-isometry on $\mathcal{H}$. Throughout this paper, we shall use the following notation:

$$\tilde{V}_i = \prod_{j \neq i} V_j,$$

for all $i = 1, \ldots, n$. For simplicity, we also use the notation

$$\mathcal{W} = \mathcal{W}(V),$$

and

$$\mathcal{W}_i = \mathcal{W}(V_i) \ \text{ and } \tilde{\mathcal{W}}_i = \mathcal{W}(\tilde{V}_i),$$

for all $i = 1, \ldots, n$. Since $V = \Pi_{i=1}^n V_i$ and $\tilde{V}_i = V_i^* V$ for all $i = 1, \ldots, n$, it is easy to see that

$$\mathcal{W}_i, \tilde{\mathcal{W}}_i \subseteq \mathcal{W},$$

for all $i = 1, \ldots, n$. We denote by $P_{\mathcal{W}_i}$ and $P_{\tilde{\mathcal{W}}_i}$ the orthogonal projections of $\mathcal{W}$ onto the subspaces $\mathcal{W}_i$ and $\tilde{\mathcal{W}}_i$, respectively.

Theorem 2.1 *Let $(V_1, \ldots, V_n)$ be a pure n-isometry on a Hilbert space $\mathcal{H}$, $V = \Pi_{i=1}^n V_i$, and let $\mathcal{W} = \mathcal{W}(V)$. Let $\Pi_V : \mathcal{H} \to H^2_{\mathcal{W}}(\mathbb{D})$ be the Wold-von Neumann decomposition of V. If $\tilde{V}_i = V_i^* V$ and $\tilde{\mathcal{W}}_i = \mathcal{W}(\tilde{V}_i)$, then*

$$\Pi_V V_i = M_{\Phi_i} \Pi_V,$$

where

$$\Phi_i(z) = U_i(P_{\tilde{\mathcal{W}}_i} + z P^{\perp}_{\tilde{\mathcal{W}}_i}),$$

for all $z \in \mathbb{D}$, and

$$U_i = (P_{\mathcal{W}} V_i + \tilde{V}_i^{\,*})|_{\mathcal{W}},$$

is a unitary operator on $\mathcal{W}$ and $i = 1, \ldots, n$. In particular, $(V_1, \ldots, V_n)$ on $\mathcal{H}$ and $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent.

Proof Let $\Pi_V : \mathcal{H} \to H^2_{\mathcal{W}}(\mathbb{D})$ be the Wold-von Neumann decomposition of V. Then

$$\Pi_V V_i \Pi_V^* \in \{M_z\}',$$

and hence there exists $\Phi_i \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ [16, 21] such that $\Pi_V V_i \Pi_V^* = M_{\Phi_i}$ or, equivalently,

$$\Pi_V V_i = M_{\Phi_i} \Pi_V,$$

for all $i = 1, \ldots, n$. Note that M_{Φ_i} on $H^2_{\mathcal{W}}(\mathbb{D})$ is defined by

$$(M_{\Phi_i} f)(z) = \Phi_i(z) f(z), \tag{2.1}$$

for all $f \in H^2_{\mathcal{W}}(\mathbb{D})$, $z \in \mathbb{D}$ and $i = 1, \ldots, n$. We now proceed to compute the bounded analytic functions $\{\Phi_i\}_{i=1}^n$. Our method follows the construction in [20]. In fact, a close variant of Theorem 2.1 below follows from Theorems 3.4 and 3.5 of [20]. We will only sketch the construction, highlighting the essential ingredients for our present purpose. Let $i \in \{1, \ldots, n\}$, $z \in \mathbb{D}$ and $\eta \in \mathcal{W}$. By an abuse of notation, we will also denote the constant function η in $H^2_{\mathcal{W}}(\mathbb{D})$ corresponding to the vector $\eta \in \mathcal{W}$ by η itself. Then from (2.1), we have that

$$\Phi_i(z)\eta = (M_{\Phi_i}\eta)(z) = (\Pi_V V_i \Pi_V^* \eta)(z).$$

Now it follows from the definition of Π_V that $\Pi_V^* \eta = \eta$, and hence $\Phi_i(z)\eta = (\Pi_V V_i \eta)(z)$. But $I_{\mathcal{W}} = P_{\tilde{\mathcal{W}}_i} + \tilde{V}_i \tilde{V}_i^*|_{\mathcal{W}}$ yields that $V_i \eta = V_i P_{\tilde{\mathcal{W}}_i} \eta + V \tilde{V}_i^* \eta$ and thus

$$\begin{aligned}\Pi_V V_i \eta &= \Pi_V (V_i P_{\tilde{\mathcal{W}}_i} \eta + V \tilde{V}_i^* \eta)\\ &= \Pi_V (V_i P_{\tilde{\mathcal{W}}_i} \eta) + \Pi_V (V \tilde{V}_i^* \eta)\\ &= \Pi_V (V_i P_{\tilde{\mathcal{W}}_i} \eta) + M_z \Pi_V (\tilde{V}_i^* \eta),\end{aligned}$$

as $\Pi_V V = M_z \Pi_V$. Now, since $V^*(V_i(I - \tilde{V}_i \tilde{V}_i^*) V_i^*) = 0$ and $V^*(\tilde{V}_i^* \eta) = 0$, it follows that $V_i P_{\tilde{\mathcal{W}}_i} \eta \in \mathcal{W}$ and $\tilde{V}_i^* \eta \in \mathcal{W}$. This implies that

$$\Pi_V V_i \eta = V_i P_{\tilde{\mathcal{W}}_i} \eta + M_z \tilde{V}_i^* \eta,$$

and so $\Phi_i(z)\eta = V_i P_{\tilde{\mathcal{W}}_i} \eta + z \tilde{V}_i^* \eta$. It follows that $\Phi_i(z) = V_i|_{\tilde{\mathcal{W}}_i} + z \tilde{V}_i^*|_{\tilde{V}_i \mathcal{W}_i}$ as $\mathcal{W} = \tilde{V}_i \mathcal{W}_i \oplus \tilde{\mathcal{W}}_i$. Finally, $\mathcal{W} = \mathcal{W}_i \oplus V_i \tilde{\mathcal{W}}_i$ implies that

$$U_i = \begin{bmatrix} \tilde{V}_i^*|_{\tilde{V}_i \mathcal{W}_i} & 0 \\ 0 & V_i|_{\tilde{\mathcal{W}}_i} \end{bmatrix} : \begin{matrix} \tilde{V}_i \mathcal{W}_i \\ \oplus \\ \tilde{\mathcal{W}}_i \end{matrix} \to \begin{matrix} \mathcal{W}_i \\ \oplus \\ V_i \tilde{\mathcal{W}}_i \end{matrix},$$

is a unitary operator on $\mathcal{W}$. Therefore

$$\Phi_i(z) = U_i (P_{\tilde{\mathcal{W}}_i} + z P_{\tilde{\mathcal{W}}_i}^{\perp}),$$

for all $z \in \mathbb{D}$. By definition of U_i, it follows that $U_i = (V_i P_{\tilde{\mathcal{W}}_i} + \tilde{V_i}^*)|_{\mathcal{W}}$. This and

$$V_i P_{\tilde{\mathcal{W}}_i} = P_{\mathcal{W}} V_i, \tag{2.2}$$

yields $U_i = (P_{\mathcal{W}} V_i + \tilde{V_i}^*)|_{\mathcal{W}}$. ■

We now study the coefficients of the one-variable polynomials in Theorem 2.1 more closely and prove that the corresponding pure n-isometry $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ is a model pure n-isometry (see Sect. 1 for the definition of model pure n-isometries).

Let $(V_1, \ldots, V_n)$ be a pure n-isometry on a Hilbert space $\mathcal{H}$. Consider the analytic representation $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ of $(V_1, \ldots, V_n)$ as in Theorem 2.1. First we prove that $\{U_j\}_{j=1}^n$ is a commutative family. Let $p, q \in \{1, \ldots, n\}$ and $p \neq q$. As $\mathcal{W} = \ker V^*$, it follows that

$$\tilde{V}_p^* \tilde{V}_q^*|_{\mathcal{W}} = 0.$$

Then using (2.2) we obtain

$$\begin{aligned} U_pU_q &= (P_{\mathcal{W}}V_p + \tilde{V}_p^*)(P_{\mathcal{W}}V_q + \tilde{V}_q^*)|_{\mathcal{W}} \\ &= (P_{\mathcal{W}}V_pP_{\mathcal{W}}V_q + \tilde{V}_p^*P_{\mathcal{W}}V_q + P_{\mathcal{W}}V_p\tilde{V}_q^*)|_{\mathcal{W}} \\ &= (P_{\mathcal{W}}V_pV_q + \prod_{i\neq p,q} V_i^*P_{\tilde{\mathcal{W}}_q} + V_pP_{\tilde{\mathcal{W}}_p}\tilde{V}_q^*)|_{\mathcal{W}} \\ &= (P_{\mathcal{W}}V_pV_q + (\prod_{i\neq p,q} V_i^*)(P_{\tilde{\mathcal{W}}_q} + \tilde{V}_qP_{\tilde{\mathcal{W}}_p}\tilde{V}_q^*))|_{\mathcal{W}} \\ &= (P_{\mathcal{W}}V_pV_q + (\prod_{i\neq p,q} V_i^*))|_{\mathcal{W}}, \end{aligned}$$

as $(P_{\tilde{\mathcal{W}}_q} + \tilde{V}_qP_{\tilde{\mathcal{W}}_p}\tilde{V}_q^*)|_{\mathcal{W}} = I_{\mathcal{W}}$, and hence

$$U_pU_q = U_qU_p,$$

follows by symmetry. Now if $I \subseteq \{1,\ldots,n\}$, then the same line of arguments as above yields

$$\prod_{i\in I} U_i = (P_{\mathcal{W}}(\prod_{i\in I} V_i) + (\prod_{i\in I^c} V_i^*))|_{\mathcal{W}}. \tag{2.3}$$

In particular, since $P_{\mathcal{W}}V|_{\mathcal{W}} = 0$, we have that

$$\prod_{i=1}^{n} U_i = I_{\mathcal{W}}.$$

The following lemma will be crucial in what follow.

Lemma 2.2 *Fix* $1 \leq j \leq n$. *Let* $I \subseteq \{1,\ldots,n\}$, *and let* $j \notin I$. *Then*

$$(\prod_{i\in I} U_i^*)P_{\tilde{\mathcal{W}}_j}^{\perp}(\prod_{i\in I} U_i) = (\prod_{i\in I^c\setminus\{j\}} V_i)(\prod_{i\in I^c\setminus\{j\}} V_i^*)|_{\mathcal{W}} - (\prod_{i\in I^c} V_i)(\prod_{i\in I^c} V_i^*)|_{\mathcal{W}}.$$

Proof Since $P_{\tilde{\mathcal{W}}_j} = I_{\mathcal{W}} - P_{\mathcal{W}}\tilde{V}_j\tilde{V}_j^*|_{\mathcal{W}}$, we have $P_{\tilde{\mathcal{W}}_j}^{\perp} = P_{\mathcal{W}}\tilde{V}_j\tilde{V}_j^*|_{\mathcal{W}} = \tilde{V}_j\tilde{V}_j^*|_{\mathcal{W}}$. By once again using the fact that $V^*|_{\mathcal{W}} = P_{\mathcal{W}}V|_{\mathcal{W}} = 0$, and by (2.3), one sees that

$$\begin{aligned} (\prod_{i\in I} U_i^*)P_{\tilde{\mathcal{W}}_j}^{\perp}(\prod_{i\in I} U_i) &= [(\prod_{i\in I} V_i^*) + P_{\mathcal{W}}(\prod_{i\in I^c} V_i)]\tilde{V}_j\tilde{V}_j^*[P_{\mathcal{W}}(\prod_{i\in I} V_i) + (\prod_{i\in I^c} V_i^*)]|_{\mathcal{W}} \\ &= (\prod_{i\in I^c\setminus\{j\}} V_i)\tilde{V}_j^*P_{\mathcal{W}}(\prod_{i\in I} V_i)|_{\mathcal{W}} \\ &= (\prod_{i\in I^c\setminus\{j\}} V_i)\tilde{V}_j^*(I - VV^*)(\prod_{i\in I} V_i)|_{\mathcal{W}} \\ &= (\prod_{i\in I^c\setminus\{j\}} V_i)(\prod_{i\in I^c\setminus\{j\}} V_i^*)|_{\mathcal{W}} - (\prod_{i\in I^c} V_i)(\prod_{i\in I^c} V_i^*)|_{\mathcal{W}} \end{aligned}$$

This completes the proof of the lemma. ■

Theorem 2.3 *If $(V_1, \ldots, V_n)$ be an n-isometry on a Hilbert space $\mathcal{H}$, and let $U_1, \ldots, U_n$ be unitary operators as in Theorem 2.1. Then*

(a) $U_p U_q = U_q U_p$ *for* $p, q = 1, \ldots n$,
(b) $\prod_{p=1}^n U_p = I_{\mathcal{W}}$,
(c) $(P^{\perp}_{\tilde{\mathcal{W}}_i} + U_i^* P^{\perp}_{\tilde{\mathcal{W}}_j} U_i) = (P^{\perp}_{\tilde{\mathcal{W}}_j} + U_j^* P^{\perp}_{\tilde{\mathcal{W}}_i} U_j) \leq I_{\mathcal{W}}$ $(1 \leq i < j \leq n)$,
(d) $P^{\perp}_{\tilde{\mathcal{W}}_1} + U_1^* P^{\perp}_{\tilde{\mathcal{W}}_2} U_1 + U_1^* U_2^* P^{\perp}_{\tilde{\mathcal{W}}_2} U_2 U_1 + \cdots + (\Pi_{i=1}^{n-1} U_i^*) P^{\perp}_{\tilde{\mathcal{W}}_n} (\Pi_{i=1}^{n-1} U_i) = I_{\mathcal{W}}$.

Proof By Lemma 2.2 applied to $I = \{p\}$ and $j = q$, where $p, q \in \{1, \ldots, n\}$ and $p \neq q$, we have

$$U_p^* P^{\perp}_{\tilde{\mathcal{W}}_q} U_p = (\prod_{i \neq p,q} V_i)(\prod_{i \neq p,q} V_i^*)|_{\mathcal{W}} - \tilde{V}_p \tilde{V}_p^*|_{\mathcal{W}},$$

hence

$$\begin{aligned}(P^{\perp}_{\tilde{\mathcal{W}}_p} + U_p^* P^{\perp}_{\tilde{\mathcal{W}}_q} U_p) &= P_{\mathcal{W}} \tilde{V}_p \tilde{V}_p^*|_{\mathcal{W}} + (\prod_{i \neq p,q} V_i)(\prod_{i \neq p,q} V_i^*)|_{\mathcal{W}} - P_{\mathcal{W}} \tilde{V}_p \tilde{V}_p^*|_{\mathcal{W}} \\ &= (\prod_{i \neq p,q} V_i)(\prod_{i \neq p,q} V_i^*)|_{\mathcal{W}} \\ &\leq I_{\mathcal{W}}.\end{aligned}$$

Therefore by symmetry, we have

$$(P^{\perp}_{\tilde{\mathcal{W}}_p} + U_p^* P^{\perp}_{\tilde{\mathcal{W}}_q} U_p) = (P^{\perp}_{\tilde{\mathcal{W}}_q} + U_q^* P^{\perp}_{\tilde{\mathcal{W}}_p} U_q) \leq I_{\mathcal{W}}.$$

Finally, we let $I_j = \{1, \ldots, j-1\}$ for all $1 < j \leq n$ and $I_{n+1} = \{1, \ldots, n\}$. Then Lemma 2.2 implies that for $1 < j \leq n$,

$$(\prod_{i \in I_j} U_i) P^{\perp}_{\tilde{\mathcal{W}}_j} (\prod_{i \in I_j} U_i^*) = [(\prod_{i \in I^c_{j+1}} V_i)(\prod_{i \in I^c_{j+1}} V_i^*) - (\prod_{i \in I^c_j} V_i)(\prod_{i \in I^c_j} V_i^*)]|_{\mathcal{W}}.$$

This and $P^{\perp}_{\tilde{\mathcal{W}}_1} = \tilde{V}_1 \tilde{V}_1^*|_{\mathcal{W}}$ imply that

$$P^{\perp}_{\tilde{\mathcal{W}}_1} + U_1^* P^{\perp}_{\tilde{\mathcal{W}}_2} U_1 + U_1^* U_2^* P^{\perp}_{\tilde{\mathcal{W}}_3} U_2 U_1 + \cdots + (\prod_{i=1}^{n-1} U_i^*) P^{\perp}_{\tilde{\mathcal{W}}_n} (\prod_{i=1}^{n-1} U_i) = I_{\mathcal{W}}.$$

This completes the proof of the theorem. ■

As a corollary, we have:

Corollary 2.4 *Let $\mathcal{H}$ be a Hilbert space and $(V_1, \ldots, V_n)$ be a pure n-isometry on $\mathcal{H}$. Let $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ be the pure n-isometry as constructed in Theorem 2.1, and let $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ on $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$, for some Hilbert space $\tilde{\mathcal{W}}$, unitary operators*

$\{\tilde{U}_i\}_{i=1}^n$ and orthogonal projections $\{P_i\}_{i=1}^n$ on $\tilde{\mathcal{W}}$, be a model pure n-isometry. Then:

(a) *$(M_{\Phi_1}, \ldots, M_{\Phi_n})$ is a model pure n-isometry.*
(b) *$(V_1, \ldots, V_n)$ and $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ are unitarily equivalent.*
(c) *$(V_1, \ldots, V_n)$ and $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ are unitarily equivalent if and only if there exists a unitary operator $W : \mathcal{W} \to \tilde{\mathcal{W}}$ such that $WU_i = \tilde{U}_i W$ and $WP_i = \tilde{P}_i W$ for all $i = 1, \ldots, n$.*

Proof Parts (a) and (b) follows directly from the previous theorem. The third part is easy and readily follows from Theorem 4.1 in [20] or Theorem 2.9 in [5]. ■

Combining Corollary 2.4 with Theorem 2.3, we have the following characterization of commutative isometric factors of shift operators.

Corollary 2.5 *Let $\mathcal{E}$ be a Hilbert space, and let $\{\Phi_i\}_{i=1}^n \subseteq H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ be a commutative family of isometric multipliers. Then*

$$M_z = \prod_{i=1}^n M_{\Phi_j},$$

or, equivalently

$$\prod_{i=1}^n \Phi_j(z) = zI_{\mathcal{E}}, \quad (z \in \mathbb{D})$$

if and only if, up to unitary equivalence, $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ is a model pure n-isometry.

In other words, $zI_{\mathcal{E}}$ factors as n commuting isometric multipliers $\{\Phi_i\}_{i=1}^n$ in $H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ if and only if there exist unitary operators $\{U_i\}_{i=1}^n$ on $\mathcal{E}$ and orthogonal projections $\{P_i\}_{i=1}^n$ on $\mathcal{E}$ satisfying the properties (a)–(d) in Theorem 2.3 such that $\Phi_i(z) = U_i(P_i^\perp + zP_i)$ for all $i = 1, \ldots, n$.

3 Joint Invariant Subspaces

Let $\mathcal{W}$ be a Hilbert space. Let $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ be a model pure n-isometry on $H^2_{\mathcal{W}}(\mathbb{D})$, and let $\mathcal{S}$ be a closed invariant subspace for $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$, that is

$$M_{\Phi_i}\mathcal{S} \subseteq \mathcal{S},$$

for all $i = 1, \ldots, n$. Then $(M_{\Phi_1}|_{\mathcal{S}}, \ldots, M_{\Phi_n}|_{\mathcal{S}})$ is an n-tuple of commuting isometries on $\mathcal{S}$. Clearly

$$\prod_{i=1}^n (M_{\Phi_i}|_{\mathcal{S}}) = (\prod_{i=1}^n M_{\Phi_i})|_{\mathcal{S}},$$

and since

$$\prod_{j=1}^{n} M_{\Phi_j} = M_z,$$

it follows that

$$(\prod_{i=1}^{n} M_{\Phi_i})|_{\mathcal{S}} = M_z|_{\mathcal{S}}, \tag{3.1}$$

that is, $\mathcal{S}$ is a invariant subspace for M_z on $H^2_{\mathcal{W}}(\mathbb{D})$. Moreover, since $M_z|_{\mathcal{S}}$ is a unilateral shift on $\mathcal{S}$, the tuple $(M_{\Phi_1}|_{\mathcal{S}}, \ldots, M_{\Phi_n}|_{\mathcal{S}})$ is a pure n-isometry on $\mathcal{S}$. Then by Corollary 2.4 there is a model pure n-isometry $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ on $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$, for some Hilbert space $\tilde{\mathcal{W}}$, such that $(M_{\Phi_1}|_{\mathcal{S}}, \ldots, M_{\Phi_n}|_{\mathcal{S}})$ and $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ are unitarily equivalent. The main purpose of this section is to describe the invariant subspaces $\mathcal{S}$ in terms of the model pure n-isometry $(M_{\Psi_1}, \ldots, M_{\Psi_n})$.

As a motivational example, consider the classical $n = 1$ case. Here the model pure 1-isometry is the multiplication operator M_z on $H^2_{\mathcal{W}}(\mathbb{D})$ for some Hilbert space $\mathcal{W}$. Let $\mathcal{S}$ be a closed subspace of $H^2_{\mathcal{W}}(\mathbb{D})$. Then by the Beurling [7], Lax [18] and Halmos [16] theorem (or see page 239, Theorem 2.1 in [13]), $\mathcal{S}$ is invariant for M_z if and only if there exist a Hilbert space $\mathcal{W}_*$ and an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}_*, \mathcal{W})}(\mathbb{D})$ such that

$$\mathcal{S} = \Theta H^2_{\mathcal{W}_*}(\mathbb{D}).$$

Moreover, in this case, if we set

$$V = M_z|_{\mathcal{S}},$$

then $\mathcal{W}_* = \mathcal{S} \ominus z\mathcal{S}$ and V on $\mathcal{S}$ and M_z on $H^2_{\mathcal{W}_*}(\mathbb{D})$ are unitarily equivalent. This follows directly from the above representation of $\mathcal{S}$. Indeed, it follows that $X = M_\Theta : H^2_{\mathcal{W}_*}(\mathbb{D}) \to \operatorname{ran} M_\Theta = \mathcal{S}$ is a unitary operator and

$$XM_z = VX.$$

Now, we proceed with the general case.

Theorem 3.1 *Let $n > 1$. Let $\mathcal{W}$ be a Hilbert space, $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ be a model pure n-isometry on $H^2_{\mathcal{W}}(\mathbb{D})$, and let $\mathcal{S}$ be a closed subspace of $H^2_{\mathcal{W}}(\mathbb{D})$. Then $\mathcal{S}$ is invariant for $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ if and only if there exist a Hilbert space $\mathcal{W}_*$, an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}_*, \mathcal{W})}(\mathbb{D})$ and a model pure n-isometry $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ on $H^2_{\mathcal{W}_*}(\mathbb{D})$ such that*

$$\mathcal{S} = \Theta H^2_{\mathcal{W}_*}(\mathbb{D}),$$

and

$$\Phi_j \Theta = \Theta \Psi_j,$$

for all $j = 1, \ldots, n$.

Proof Let $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ be a model pure n-isometry on $H^2_{\mathcal{W}}(\mathbb{D})$, and let $\mathcal{S}$ be a closed invariant subspace for $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$. Let

$$\mathcal{W}_* = \mathcal{S} \ominus z\mathcal{S}.$$

Since $\mathcal{S}$ is an invariant subspace for M_z on $H^2_{\mathcal{W}}(\mathbb{D})$ (see Eq. (3.1)), by Beurling, Lax and Halmos theorem, there exists an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}_*, \mathcal{W})}(\mathbb{D})$ such that $\mathcal{S}$ can be represented as

$$\mathcal{S} = \Theta H^2_{\mathcal{W}_*}(\mathbb{D}),$$

If $1 \leq j \leq n$, then

$$\Phi_j \mathcal{S} \subseteq \mathcal{S},$$

implies that ran $(M_{\Phi_j} M_\Theta) \subseteq$ ran M_Θ, and so by Douglas's range and inclusion theorem [11]

$$M_{\Phi_j} M_\Theta = M_\Theta M_{\Psi_j},$$

for some $\Psi_j \in H^\infty_{\mathcal{B}(\mathcal{W}_*)}(\mathbb{D})$. Note that $M_{\Phi_j} M_\Theta$ is an isometry and $\|\Theta \Psi_j f\| = \|\Psi_j f\|$ for each $f \in H^2_{\mathcal{W}_*}(\mathbb{D})$. But then $\|M_{\Psi_j} f\| = \|f\|$ implies that M_{Ψ_j} is an isometry, that is, Ψ_j is an inner function, and hence

$$M_{\Psi_j} = M_\Theta^* M_{\Phi_j} M_\Theta,$$

for all $j = 1, \ldots, n$. So

$$\prod_{i=1}^{n} M_{\Psi_i} = (M_\Theta^* M_{\Phi_1} M_\Theta) \cdots (M_\Theta^* M_{\Phi_n} M_\Theta).$$

Now $P_{\text{ran } M_\Theta} = M_\Theta M_\Theta^*$ and $\Phi_j \Theta H^2_{\mathcal{W}_*}(\mathbb{D}) \subseteq \Theta H^2_{\mathcal{W}_*}(\mathbb{D})$ implies that

$$M_\Theta M_\Theta^* M_{\Phi_j} M_\Theta = M_{\Phi_j} M_\Theta,$$

for all $j = 1, \ldots, n$. Consequently

$$\prod_{j=1}^{n} M_{\Psi_j} = M_\Theta^* (\prod_{j=1}^{n} M_{\Phi_j}) M_\Theta^* = M_\Theta^* M_z M_\Theta = M_\Theta^* M_\Theta M_z = M_z,$$

that is, $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ is a pure n-isometry on $H^2_{\mathcal{W}_*}(\mathbb{D})$. In view of Corollary 2.5, this also implies that the tuple $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ is a model pure n-isometry. This completes the proof of the theorem. ■

The representation of $\mathcal{S}$ is unique in the following sense: if there exist a Hilbert space $\hat{\mathcal{W}}$, an inner multiplier $\hat{\Theta} \in H^\infty_{\mathcal{B}(\hat{\mathcal{W}},\mathcal{W})}(\mathbb{D})$ and a model pure n-isometry $(M_{\hat{\Psi}_1}, \ldots, M_{\hat{\Psi}_n})$ on $H^2_{\hat{\mathcal{W}}}(\mathbb{D})$ such that $\mathcal{S} = \hat{\Theta} H^2_{\hat{\mathcal{W}}}(\mathbb{D})$ and $\Phi_i \hat{\Theta} = \hat{\Theta}\hat{\Psi}_i$ for all $i = 1, \ldots, n$, then there exists a unitary $\tau : \mathcal{W}_* \to \hat{\mathcal{W}}$ such that

$$\Theta = \hat{\Theta}\tau,$$

and

$$\hat{\Psi}_j \tau = \tau \Psi_j \qquad (j = 1, \ldots, n).$$

In other words, the model pure n-isometries $(M_{\hat{\Psi}_1}, \ldots, M_{\hat{\Psi}_n})$ on $H^2_{\hat{\mathcal{W}}}(\mathbb{D})$ and $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ on $H^2_{\mathcal{W}_*}(\mathbb{D})$ are unitary equivalent (under the same unitary τ). Indeed, the existence of the unitary τ along with the first equality follows from the uniqueness of the Beurling, Lax and Halmos theorem (cf. page 239, Theorem 2.1 in [13]). For the second equality, observe that (see the uniqueness part in [19])

$$\hat{\Theta}\tau\Psi_i = \Theta\Psi_i = \Phi_i\Theta = \Phi_i\hat{\Theta}\tau,$$

that is $\hat{\Theta}\tau\Psi_i = \hat{\Theta}\hat{\Psi}_i\tau$, and so

$$\tau\Psi_i = \hat{\Psi}_i\tau,$$

for all $i = 1, \ldots, n$.

It is curious to note that the content of Theorem 3.1 is related to the question [1] and its answer [24] on the classifications of invariant subspaces of Γ-isometries. A similar result also holds for invariant subspaces for the multiplication operator tuple on the Hardy space over the unit polydisc in $\mathbb{C}^n$ (see [19]).

Our approach to pure n-isometries has other applications to n-tuples, $n \geq 2$, of commuting contractions (cf. see [9]) that we will explore in a future paper.

4 C^*-Algebras Generated by Commuting Isometries

In this section, we extend Seto's result [26] on isomorphic C^*-algebras of invariant subspaces of finite codimension in $H^2(\mathbb{D}^2)$ to that in $H^2(\mathbb{D}^n)$, $n \geq 2$. Given a Hilbert space $\mathcal{H}$, the set of all compact operators from $\mathcal{H}$ to itself is denoted by $K(\mathcal{H})$. Recall that, for a closed subspace $\mathcal{S} \subseteq H^2(\mathbb{D}^n)$, we say that $\mathcal{S}$ is an invariant subspace of $H^2(\mathbb{D}^n)$ if $M_{z_i}\mathcal{S} \subseteq \mathcal{S}$ for all $i = 1, \ldots, n$. Also recall that in the case of an invariant subspace $\mathcal{S}$ of $H^2(\mathbb{D}^n)$, $(R_{z_1}, \ldots, R_{z_n})$ is an n-isometry on $\mathcal{S}$ where

$$R_{z_i} = M_{z_i}|_{\mathcal{S}} \in \mathcal{B}(\mathcal{S}) \qquad (i = 1, \ldots, n).$$

Lemma 4.1 *If $\mathcal{S}$ is an invariant subspace of finite codimension in $H^2(\mathbb{D}^n)$, then $K(\mathcal{S}) \subseteq \mathcal{T}(\mathcal{S})$.*

Proof Since $\mathcal{T}(\mathcal{S})$ is an irreducible C^*-algebra (cf. [26, Proposition 2.2]), it is enough to prove that $\mathcal{T}(\mathcal{S})$ contains a non-zero compact operator. As

$$\prod_{i=1}^{n} (I_{H^2(\mathbb{D}^n)} - M_{z_i} M_{z_i}^*) = P_{\mathbb{C}} \in \mathcal{T}(H^2(\mathbb{D}^n)),$$

we are done when $\mathcal{S} = H^2(\mathbb{D}^n)$. Let us now suppose that $\mathcal{S}$ is a proper subspace of $H^2(\mathbb{D}^n)$. For arbitrary $1 \leq i < j \leq n$, we have

$$[R_{z_i}^*, R_{z_j}] = P_{\mathcal{S}} M_{z_i}^* M_{z_j}|_{\mathcal{S}} - P_{\mathcal{S}} M_{z_j} P_{\mathcal{S}} M_{z_i}^*|_{\mathcal{S}} = P_{\mathcal{S}} M_{z_j} P_{\mathcal{S}^\perp} M_{z_i}^*|_{\mathcal{S}} \in K(\mathcal{S}),$$

as $\mathcal{S}^\perp$ is finite dimensional. It remains for us to prove that $[R_{z_i}^*, R_{z_j}] \neq 0$ for some $1 \leq i < j \leq n$. If not, then $\mathcal{S}$ is a proper doubly commuting invariant subspace with finite codimension. As a result, we would have $\mathcal{S} = \varphi H^2(\mathbb{D}^n)$ for some inner function $\varphi \in H^\infty(\mathbb{D}^n)$ ([25]) and hence $\mathcal{S}$ has infinite codimension (see the corollary in page 969, [2]), a contradiction. ■

In what follows, a finite rank operator on a Hilbert space will be denoted by F (without referring to the ambient Hilbert space). Also, if $\mathcal{M}$ is an invariant subspaces of $H^2(\mathbb{D}^n)$, then we set

$$R_{z_i}^{\mathcal{M}} = M_{z_i}|_{\mathcal{M}} \in \mathcal{B}(\mathcal{M}),$$

and simply write R_{z_i}, $i = 1, \ldots, n$, when $\mathcal{M}$ is clear from the context.

Lemma 4.2 *Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are invariant subspaces of $H^2(\mathbb{D}^n)$, $\mathcal{M}_1 \subseteq \mathcal{M}_2$ and $\dim(\mathcal{M}_2 \ominus \mathcal{M}_1) < \infty$. Then $\mathcal{T}(\mathcal{M}_1) = \{P_{\mathcal{M}_1} T|_{\mathcal{M}_1} : T \in \mathcal{T}(\mathcal{M}_2)\}$. Moreover, if $\mathcal{L}$ is a closed subspace of $\mathcal{M}_1$ and $P_{\mathcal{L}}^{\mathcal{M}_2} \in \mathcal{T}(\mathcal{M}_2)$, then $P_{\mathcal{L}}^{\mathcal{M}_1} \in \mathcal{T}(\mathcal{M}_1)$.*

Proof Note that $R_{z_i}^{\mathcal{M}_2}|_{\mathcal{M}_1} = R_{z_i}^{\mathcal{M}_1}$ and so, by taking adjoint, we have

$$P_{\mathcal{M}_1} (R_{z_i}^{\mathcal{M}_2})^*|_{\mathcal{M}_1} = (R_{z_i}^{\mathcal{M}_1})^*,$$

for all $i = 1, \ldots, n$. Then $R_{z_i}^{\mathcal{M}_1} (R_{z_j}^{\mathcal{M}_1})^* = P_{\mathcal{M}_1} R_{z_i}^{\mathcal{M}_2} P_{\mathcal{M}_1}^{\mathcal{M}_2} (R_{z_j}^{\mathcal{M}_2})^*|_{\mathcal{M}_1}$, $i = 1, \ldots, n$. This yields

$$\begin{aligned} R_{z_i}^{\mathcal{M}_1} (R_{z_j}^{\mathcal{M}_1})^* &= P_{\mathcal{M}_1} R_{z_i}^{\mathcal{M}_2} I_{\mathcal{M}_2} (R_{z_j}^{\mathcal{M}_2})^*|_{\mathcal{M}_1} - P_{\mathcal{M}_1} R_{z_i}^{\mathcal{M}_2} P_{\mathcal{M}_2 \ominus \mathcal{M}_1}^{\mathcal{M}_2} (R_{z_i}^{\mathcal{M}_2})^*|_{\mathcal{M}_1} \\ &= P_{\mathcal{M}_1} R_{z_i}^{\mathcal{M}_2} (R_{z_j}^{\mathcal{M}_2})^*|_{\mathcal{M}_1} + F, \end{aligned}$$

for all $i, j = 1, \ldots, n$, as $\dim(\mathcal{M}_2 \ominus \mathcal{M}_1) < \infty$. Similarly $(R_{z_j}^{\mathcal{M}_1})^* R_{z_i}^{\mathcal{M}_1} = P_{\mathcal{M}_1}(R_{z_j}^{\mathcal{M}_2})^* R_{z_i}^{\mathcal{M}_2}|_{\mathcal{M}_1} + F$ for all $i, j = 1, \ldots, n$. Now let $T_1 \in \mathcal{T}(\mathcal{M}_1)$ be a finite word formed from the symbols

$$\{R_{z_i}^{\mathcal{M}_1}, (R_{z_i}^{\mathcal{M}_1})^* : i = 1, \ldots, n\},$$

and let $T_2 \in \mathcal{T}(\mathcal{M}_2)$ be the same word but formed from the corresponding symbols in

$$\{R_{z_i}^{\mathcal{M}_2}, (R_{z_i}^{\mathcal{M}_2})^* : i = 1, \ldots, n\}.$$

Then $T_1 = P_{\mathcal{M}_1} T_2|_{\mathcal{M}_1} + F$. Since both $\mathcal{T}(\mathcal{M}_1)$ and $\{P_{\mathcal{M}_1} T|_{\mathcal{M}_1} : T \in \mathcal{T}(\mathcal{M}_2)\}$ are closed subspaces of $\mathcal{B}(\mathcal{M}_1)$ and both contain all the compact operators in $\mathcal{B}(\mathcal{M}_1)$, it follows that $\mathcal{T}(\mathcal{M}_1) = \{P_{\mathcal{M}_1} T|_{\mathcal{M}_1} : T \in \mathcal{T}(\mathcal{M}_2)\}$. The second assertion now clearly follows from the first one. ■

A thorough understanding of co-doubly commuting invariant subspaces of finite codimension is important to analyze C^*-algebras of invariant subspaces of finite codimension in $H^2(\mathbb{D}^n)$. If $\mathcal{S}$ is a closed invariant subspace of $H^2(\mathbb{D})$, then we know that $\mathcal{S} = \theta H^2(\mathbb{D})$ for some inner function $\theta \in H^\infty(\mathbb{D})$. To simplify notations, for a given inner function $\theta \in H^\infty(\mathbb{D})$, we denote

$$\mathcal{S}_\theta = \theta H^2(\mathbb{D}), \quad \text{and} \quad \mathcal{Q}_\theta = H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D}).$$

Also, given an inner function $\theta_i \in H^\infty(\mathbb{D})$, $1 \leq i \leq n$, denote by M_{θ_i} the multiplication operator

$$(M_{\theta_i} f)(z_1, \ldots, z_n) = \theta_i(z_i) f(z_1, \ldots, z_n)$$

for all $f \in H^2(\mathbb{D}^n)$ and $(z_1, \ldots, z_n) \in \mathbb{D}^n$. Recall now that an invariant subspace $\mathcal{S}$ of $H^2(\mathbb{D}^n)$ is said to be *co-doubly commuting* [23] if $\mathcal{S} = \mathcal{S}_\Phi$ where

$$\mathcal{S}_\Phi = (\mathcal{Q}_{\varphi_1} \otimes \cdots \otimes \mathcal{Q}_{\varphi_n})^\perp, \tag{4.1}$$

and φ_i, $i = 1, \ldots, n$, is either inner or the zero function. We warn the reader that the suffix Φ in $\mathcal{S}_\Phi$ refers to the finite Blaschke products $\{\varphi_i\}_{i=1}^n$. Here, in view of (4.1) (or see [23]), we have

$$(M_{\varphi_p} M_{\varphi_p}^*)(M_{\varphi_q} M_{\varphi_q}^*) = (M_{\varphi_q} M_{\varphi_q}^*)(M_{\varphi_p} M_{\varphi_p}^*),$$

for all $p, q = 1, \ldots, n$, and

$$P_{\mathcal{S}_\Phi} = I_{H^2(\mathbb{D}^n)} - \prod_{i=1}^n (I_{H^2(\mathbb{D}^n)} - M_{\varphi_i} M_{\varphi_i}^*). \tag{4.2}$$

It also follows that

$$\mathcal{S}_\Phi = M_{\varphi_1} H^2(\mathbb{D}^n) + \cdots + M_{\varphi_n} H^2(\mathbb{D}^n).$$

Therefore, $\mathcal{S}_\Phi$ has finite codimension if and only if φ_i is a finite Blashcke product for all $i = 1, \ldots, n$. Moreover, it can be proved following the same line of argument as Lemma 3.1 in [26] that if $\mathcal{S}$ is an invariant subspace of $H^2(\mathbb{D}^n)$ then $\mathcal{S}$ is of finite codimension if and only if there exist finite Blaschke products $\varphi_1, \ldots, \varphi_n$ such that

$$\mathcal{S}_\Phi \subseteq \mathcal{S}.$$

Given $\mathcal{S}_\Phi$ as in (4.1) and $1 \leq i < j \leq n$, we define $\mathcal{Q}_\Phi[i, j]$ by

$$\mathcal{Q}_\Phi[i, j] = \mathcal{Q}_{\varphi_i} \otimes \mathcal{Q}_{\varphi_{i+1}} \otimes \cdots \otimes \mathcal{Q}_{\varphi_j} \subseteq H^2(\mathbb{D}^{j-i+1}).$$

Lemma 4.3 *Let* $\{\varphi_i\}_{i=1}^n$ *be finite Blaschke products. If*

$$\mathcal{L}_1 = \mathcal{Q}_\Phi[1, n-1]^\perp \otimes H^2(\mathbb{D}),\ \mathcal{L}_2 = \mathcal{Q}_\Phi[1, n-1] \otimes \mathcal{S}_{\varphi_n},$$

$$\mathcal{L}_3 = \mathcal{Q}_\Phi[1, n-1] \otimes H^2(\mathbb{D}), \mathcal{L}_2' = \mathcal{Q}_\Phi[1, n-1] \otimes \varphi_n \mathcal{S}_{\varphi_n}$$

and

$$\mathcal{L}_2'' = \mathcal{Q}_\Phi[1, n-1] \otimes \varphi_n \mathcal{Q}_{\varphi_n},$$

then $P_{\mathcal{L}_1}, P_{\mathcal{L}_2}, P_{\mathcal{L}_2'}$ *and* $P_{\mathcal{L}_2''}$ *are in* $\mathcal{T}(H^2(\mathbb{D}^n))$ *and* $P_{\mathcal{L}_1}^{\mathcal{S}_\Phi}, P_{\mathcal{L}_2}^{\mathcal{S}_\Phi}, P_{\mathcal{L}_2'}^{\mathcal{S}_\Phi}$ *and* $P_{\mathcal{L}_2''}^{\mathcal{S}_\Phi}$ *are in* $\mathcal{T}(\mathcal{S}_\Phi)$.

Proof Clearly $\mathcal{S}_\Phi = \mathcal{L}_1 \oplus \mathcal{L}_2$, $H^2(\mathbb{D}^n) = \mathcal{L}_1 \oplus \mathcal{L}_3$ and $\mathcal{L}_2 = \mathcal{L}_2' \oplus \mathcal{L}_2''$. By virtue of Lemma 4.2, we only prove the lemma for $H^2(\mathbb{D}^n)$. Since $\mathcal{L}_2''$ is finite-dimensional, it follows, by Lemma 4.1, that $P_{\mathcal{L}_2''} \in \mathcal{T}(H^2(\mathbb{D}^n))$. Since $\varphi_i \in H^\infty(\mathbb{D})$ is a finite Blaschke product, it follows that φ_i is holomorphic in an open set containing the closure of the disc, and hence $M_{\varphi_i} = \varphi_i(M_{z_i}) \in \mathcal{T}(H^2(\mathbb{D}^n))$ for all $i = 1, \ldots, n$. Then, by (4.2), $P_{\mathcal{S}_\Phi} \in \mathcal{T}(H^2(\mathbb{D}^n))$. In view of $\mathcal{S}_\Phi = \mathcal{L}_1 \oplus \mathcal{L}_2$, it is then enough to prove only that $P_{\mathcal{L}_2} \in \mathcal{T}(H^2(\mathbb{D}^n))$. This readily follows from the equality

$$P_{\mathcal{L}_2} = \Big(\prod_{i=1}^{n-1} (I_{H^2(\mathbb{D}^n)} - M_{\varphi_i} M_{\varphi_i}^*)\Big) M_{\varphi_n} M_{\varphi_n}^*.$$

This completes the proof of the lemma. ∎

In particular, $\mathcal{T}(\mathcal{S}_\Phi)$ contains a wealth of orthogonal projections. This leads to some further observations concerning the C^*-algebra $\mathcal{T}(\mathcal{S}_\Phi)$. First, given $\mathcal{S}_\Phi$ as in (4.1), we consider the unitary operator $U : H^2(\mathbb{D}^n) \to \mathcal{S}_\Phi$ defined by

$$U = \begin{bmatrix} I_{\mathcal{L}_1} & 0 \\ 0 & M_{\varphi_n} \end{bmatrix} : \begin{matrix} \mathcal{L}_1 \\ \oplus \\ \mathcal{L}_3 \end{matrix} \to \begin{matrix} \mathcal{L}_1 \\ \oplus \\ \mathcal{L}_2 \end{matrix}.$$

Then $U = P_{\mathcal{L}_1} + M_{\varphi_n} P_{\mathcal{L}_3}$ and $U^* = P_{\mathcal{L}_1}^{\mathcal{S}_\Phi} + M_{\varphi_n}^* P_{\mathcal{L}_2}^{\mathcal{S}_\Phi}$. We have the following result:

Theorem 4.4 *If $\{\varphi_i\}_{i=1}^n$ are finite Blaschke products, then*

$$U^* \mathcal{T}(\mathcal{S}_\Phi) U = \mathcal{T}(H^2(\mathbb{D}^n)).$$

In particular, $\mathcal{T}(\mathcal{S}_\Phi)$ and $\mathcal{T}(H^2(\mathbb{D}^n))$ are unitarily equivalent.

Proof A simple computation first confirms that

$$U^* R_{z_n} U = M_{z_n} \in \mathcal{T}(H^2(\mathbb{D}^n)),$$

that is

$$M_{z_n} \in U^* \mathcal{T}(\mathcal{S}_\Phi) U \quad \text{and} \quad R_{z_n} \in U \mathcal{T}(H^2(\mathbb{D}^n)) U^*.$$

Next, let $i = 1, \ldots, n-1$. Then

$$R_{z_i} U = M_{z_i} P_{\mathcal{L}_1} + R_{z_i} M_{\varphi_n} P_{\mathcal{L}_3} = M_{z_i} P_{\mathcal{L}_1} + M_{z_i} M_{\varphi_n} P_{\mathcal{L}_3},$$

as $M_{\varphi_n} \mathcal{L}_3 = \mathcal{L}_2 \subseteq \mathcal{S}_\Phi$, and so

$$\begin{aligned} U^* R_{z_i} U &= (P_{\mathcal{L}_1}^{\mathcal{S}_\Phi} + M_{\varphi_n}^* P_{\mathcal{L}_2}^{\mathcal{S}_\Phi})(M_{z_i} P_{\mathcal{L}_1} + M_{z_i} M_{\varphi_n} P_{\mathcal{L}_3}) \\ &= M_{z_i} P_{\mathcal{L}_1} + P_{\mathcal{L}_1} M_{z_i} M_{\varphi_n} P_{\mathcal{L}_3} + M_{\varphi_n}^* P_{\mathcal{L}_2} M_{z_i} M_{\varphi_n} P_{\mathcal{L}_3}, \end{aligned}$$

as $M_{z_i} \mathcal{L}_1 \subseteq \mathcal{L}_1$ and $M_{z_i} M_{\varphi_n} \mathcal{L}_3 = M_{z_i} \mathcal{L}_2 \subseteq \mathcal{S}_\Phi$. Then $U^* R_{z_i} U \in \mathcal{T}(H^2(\mathbb{D}^n))$ for al $i = 1, \ldots, n$, by Lemma 4.3. In particular

$$U^* \mathcal{T}(\mathcal{S}_\Phi) U \subseteq \mathcal{T}(H^2(\mathbb{D}^n)).$$

On the other hand, since $\mathcal{L}_2 = \mathcal{L}_2' \oplus \mathcal{L}_2''$ and $\mathcal{L}_2''$ is finite dimensional, it follows that $P_{\mathcal{L}_2} = P_{\mathcal{L}_2'} + F$, and thus $U^* = U^*|_{\mathcal{L}_1} + U^*|_{\mathcal{L}_2'} + F$. Now $U M_{z_i} U^*|_{\mathcal{L}_1} = U M_{z_i}|_{\mathcal{L}_1} = M_{z_i}|_{\mathcal{L}_1}$ as $z_i \mathcal{L}_1 \subseteq \mathcal{L}_1$ and hence

$$U M_{z_i} U^*|_{\mathcal{L}_1} = R_{z_i}|_{\mathcal{L}_1},$$

and on the other hand

$$UM_{z_i}U^*|_{\mathcal{L}_2'} = U(M_{z_i}M_{\varphi_n}^*|_{\mathcal{L}_2'}) = U(M_{z_i}P_{\mathcal{S}_\Phi}M_{\varphi_n}^*)|_{\mathcal{L}_2'} = U(R_{z_i}R_{\varphi_n}^*)|_{\mathcal{L}_2'},$$

where $R_{\varphi_n} = M_{\varphi_n}|_{\mathcal{S}_\Phi}$. Moreover, since $\mathcal{L}_3 = \mathcal{L}_2 \oplus \mathcal{S}_\Phi^\perp$ and $\mathcal{S}_\Phi^\perp$ is finite dimensional, it follows that $P_{\mathcal{L}_3} = P_{\mathcal{L}_2} + F$, and thus

$$\begin{aligned} UM_{z_i}U^*|_{\mathcal{L}_2'} &= P_{\mathcal{L}_1}R_{z_i}R_{\varphi_n}^*|_{\mathcal{L}_2'} + M_{\varphi_n}P_{\mathcal{L}_3}R_{z_i}R_{\varphi_n}^*|_{\mathcal{L}_2'} \\ &= P_{\mathcal{L}_1}R_{z_i}R_{\varphi_n}^*|_{\mathcal{L}_2'} + M_{\varphi_n}P_{\mathcal{L}_2}R_{z_i}R_{\varphi_n}^*|_{\mathcal{L}_2'} + F \\ &= P_{\mathcal{L}_1}^{\mathcal{S}_\Phi}R_{z_i}R_{\varphi_n}^*|_{\mathcal{L}_2'} + R_{\varphi_n}P_{\mathcal{L}_2}^{\mathcal{S}_\Phi}R_{z_i}R_{\varphi_n}^*|_{\mathcal{L}_2'} + F, \end{aligned}$$

and hence

$$UM_{z_i}U^* = R_{z_i}P_{\mathcal{L}_1}^{\mathcal{S}_\Phi} + P_{\mathcal{L}_1}^{\mathcal{S}_\Phi}R_{z_i}R_{\varphi_n}^*P_{\mathcal{L}_2'}^{\mathcal{S}_\Phi} + R_{\varphi_n}P_{\mathcal{L}_2}^{\mathcal{S}_\Phi}R_{z_i}R_{\varphi_n}^*P_{\mathcal{L}_2'}^{\mathcal{S}_\Phi} + F.$$

By Lemma 4.3, it follows then that $UM_{z_i}U^* \in \mathcal{T}(\mathcal{S}_\Phi)$ and so

$$U\mathcal{T}(H^2(\mathbb{D}^n))U^* \subseteq \mathcal{T}(\mathcal{S}_\Phi).$$

Therefore, the conclusion follows from the fact that $U^*R_{z_n}U = M_{z_n} \in \mathcal{T}(H^2(\mathbb{D}^n))$. ■

Now let $\mathcal{S}$ be an invariant subspace of finite codimension, and let $\mathcal{S}_\Phi \subseteq \mathcal{S}$, as in (4.1), for some finite Blashcke products $\{\varphi_i\}_{i=1}^n$. We proceed to prove that $\mathcal{T}(\mathcal{S})$ is unitarily equivalent to $\mathcal{T}(\mathcal{S}_\Phi)$. Let

$$m := \dim(\mathcal{S} \ominus \mathcal{S}_\Phi).$$

Observe that

$$P_{\mathcal{S}_\Phi} = M_{\varphi_1}M_{\varphi_1}^* + (I_{H^2(\mathbb{D}^n)} - M_{\varphi_1}M_{\varphi_1}^*)\Big(I_{H^2(\mathbb{D}^n)} - \prod_{i=2}^{n}(I_{H^2(\mathbb{D}^n)} - M_{\varphi_i}M_{\varphi_i}^*)\Big),$$

and so

$$\mathcal{S}_\Phi = \Big(\mathcal{S}_{\varphi_1} \otimes H^2(\mathbb{D}^{n-1})\Big) \oplus \Big(\mathcal{Q}_{\varphi_1} \otimes \mathcal{Q}_\Phi[2,n]^\perp\Big).$$

Lemma 4.5 *$P^{\mathcal{S}}_{\mathcal{S}_{\varphi_1}\otimes H^2(\mathbb{D}^{n-1})}, P^{\mathcal{S}}_{\mathcal{Q}_{\varphi_1}\otimes \mathcal{Q}_\Phi[2,n]^\perp} \in \mathcal{T}(\mathcal{S})$ and*

$$P^{\mathcal{S}_\Phi}_{\mathcal{S}_{\varphi_1}\otimes H^2(\mathbb{D}^{n-1})}, P^{\mathcal{S}_\Phi}_{\mathcal{Q}_{\varphi_1}\otimes \mathcal{Q}_\Phi[2,n]^\perp} \in \mathcal{T}(\mathcal{S}_\Phi).$$

Proof First one observes that, by virtue of Lemma 4.2, it is enough to prove the result for $\mathcal{S}$. Note that $M_{\varphi_1}\mathcal{S} \subseteq \mathcal{S}$. Define $R_{\varphi_1} \in \mathcal{B}(\mathcal{S})$ by $R_{\varphi_1} = M_{\varphi_1}|_{\mathcal{S}}$. Then $R_{\varphi_1} = \varphi_1(M_{z_1})|_{\mathcal{S}} \in \mathcal{T}(\mathcal{S})$ and

$$P_{M_{\varphi_1}\mathcal{S}} = R_{\varphi_1} R_{\varphi_1}^* \in \mathcal{T}(\mathcal{S}).$$

Now on the one hand

$$\mathcal{S}_{\varphi_1} \otimes H^2(\mathbb{D}^{n-1}) = M_{\varphi_1} H^2(\mathbb{D}^n) = M_{\varphi_1}\mathcal{S} \oplus \Big(M_{\varphi_1} H^2(\mathbb{D}^n) \ominus M_{\varphi_1}\mathcal{S}\Big),$$

also, $M_{\varphi_1} H^2(\mathbb{D}^n) \ominus M_{\varphi_1}\mathcal{S} = M_{\varphi_1}(H^2(\mathbb{D}^n) \ominus \mathcal{S})$ is finite dimensional, and hence we conclude $P_{\mathcal{S}_{\varphi_1} \otimes H^2(\mathbb{D}^{n-1})} \in \mathcal{T}(\mathcal{S})$. This along with $\dim(\mathcal{S} \ominus \mathcal{S}_\Phi) < \infty$ and the decomposition

$$\mathcal{S} = (\mathcal{S}_{\varphi_1} \otimes H^2(\mathbb{D}^{n-1})) \oplus (\mathcal{Q}_{\varphi_1} \otimes \mathcal{Q}_\Phi[2,n]^\perp) \oplus (\mathcal{S} \ominus \mathcal{S}_\Phi),$$

implies that $P_{\mathcal{Q}_{\varphi_1} \otimes \mathcal{Q}_\Phi[2,n]^\perp} \in \mathcal{T}(\mathcal{S})$. This completes the proof of the lemma. ■

For simplicity, let us introduce some more notation. Given $q \in \mathbb{N}$, let us denote

$$\mathbb{C}^{\otimes q} = \mathbb{C} \otimes \cdots \otimes \mathbb{C} \subseteq H^2(\mathbb{D}^q).$$

Note that $\mathbb{C}^{\otimes q}$ is the one-dimensional subspace consisting of the constant functions in $H^2(\mathbb{D}^q)$. Recalling $\dim(\mathcal{S} \ominus \mathcal{S}_\Phi) = m(< \infty)$, we consider the orthogonal decomposition of $\mathcal{S}_{\varphi_1} \otimes H^2(\mathbb{D}^{n-1})$ as:

$$\mathcal{S}_{\varphi_1} \otimes H^2(\mathbb{D}^{n-1}) = \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{S}_3,$$

where

$$\begin{cases} \mathcal{S}_1 = (\varphi_1 \mathcal{Q}_{z^m}) \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D}) \\ \mathcal{S}_2 = \mathcal{S}_{z^m \varphi_1} \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D}) \\ \mathcal{S}_3 = \mathcal{S}_{\varphi_1} \otimes (\mathbb{C}^{\otimes(n-2)})^\perp \otimes H^2(\mathbb{D}). \end{cases}$$

Finally, we define

$$\mathcal{L} = \mathcal{S}_2 \oplus \mathcal{S}_3 \oplus \Big(\mathcal{Q}_{\varphi_1} \otimes \mathcal{Q}_\Phi[2,n]^\perp\Big).$$

With this notation we have

$$\mathcal{S}_\Phi = \mathcal{S}_1 \oplus \mathcal{L},$$

and

$$\mathcal{S} = (\mathcal{S} \ominus \mathcal{S}_\Phi) \oplus \mathcal{S}_1 \oplus \mathcal{L}.$$

Lemma 4.6 *$P^{\mathcal{S}}_{\mathcal{S}_i} \in \mathcal{T}(\mathcal{S})$ and $P^{\mathcal{S}_\Phi}_{\mathcal{S}_i} \in \mathcal{T}(\mathcal{S}_\Phi)$ for all $i = 1, 2, 3$.*

Proof In view of Lemma 4.2, it is enough to prove that $P^{\mathcal{S}}_{\mathcal{S}_i} \in \mathcal{T}(\mathcal{S})$, $i = 1, 2, 3$. Note that $P_{\mathcal{S}_{\varphi_1} \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D})} \in \mathcal{T}(\mathcal{S})$ as

$$P_{\mathcal{S}_{\varphi_1} \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D})} = P_{\mathcal{S}_{\varphi_1} \otimes H^2(\mathbb{D}^{n-1})} (I_{\mathcal{S}} - X) P_{\mathcal{S}_{\varphi_1} \otimes H^2(\mathbb{D}^{n-1})},$$

where

$$X = \sum_{2 \leq i_1 < \cdots < i_k \leq n-1} (-1)^{k+1} R_{z_{i_1}} \cdots R_{z_{i_k}} R^*_{z_{i_1}} \cdots R^*_{z_{i_k}}.$$

Therefore

$$P_{\mathcal{S}_3} = P_{\mathcal{S}_{\varphi_1} \otimes H^2(\mathbb{D}^{n-1})} - P_{\mathcal{S}_{\varphi_1} \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D})} \in \mathcal{T}(\mathcal{S}).$$

Finally, since $P_{\mathcal{S}_2} = R^m_{z_1} P_{\mathcal{S}_{\varphi_1} \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D})} R^{*m}_{z_1}$ and $\mathcal{S}_1 \oplus \mathcal{S}_2 = \mathcal{S}_{\varphi_1} \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D})$, it follows that $P_{\mathcal{S}_1}$ and $P_{\mathcal{S}_2}$ are in $\mathcal{T}(\mathcal{S})$. ∎

Before we proceed to the unitary equivalence of the C^*-algebras $\mathcal{T}(\mathcal{S})$ and $\mathcal{T}(\mathcal{S}_\Phi)$ we note that

$$\varphi_1 \mathcal{Q}_{z^m} = \text{span}\, \{\varphi_1, \varphi_1 z, \ldots, \varphi_1 z^{m-1}\}.$$

Theorem 4.7 *If $\mathcal{S}$ is a finite co-dimensional invariant subspace of $H^2(\mathbb{D}^n)$ and $\mathcal{S}_\Phi \subseteq \mathcal{S}$ for some finite Blaschke products $\{\varphi_i\}_{i=1}^n$, then $\mathcal{T}(\mathcal{S})$ and $\mathcal{T}(\mathcal{S}_\Phi)$ are unitarily equivalent.*

Proof By noting that $H^2(\mathbb{D}) = \mathbb{C} \oplus \mathcal{S}_z$, we decompose $\mathcal{S}_1$ as $\mathcal{S}_1 = \mathcal{F}_1 \oplus \mathcal{M}_1$ where

$$\mathcal{F}_1 = (\varphi_1 \mathcal{Q}_{z^m}) \otimes \mathbb{C}^{\otimes(n-1)}, \quad \text{and} \quad \mathcal{M}_1 = (\varphi_1 \mathcal{Q}_{z^m}) \otimes \mathbb{C}^{\otimes(n-2)} \otimes \mathcal{S}_z.$$

Taking into consideration $\dim \mathcal{F}_1 = \dim\, (\mathcal{S} \ominus \mathcal{S}_\Phi)$, we have a unitary $V : \mathcal{F}_1 \to \mathcal{S} \ominus \mathcal{S}_\Phi$, and then, using the decompositions

$$\mathcal{S}_\Phi = \mathcal{F}_1 \oplus \mathcal{M}_1 \oplus \mathcal{L}.$$

and

$$\mathcal{S} = (\mathcal{S} \ominus \mathcal{S}_\Phi) \oplus \mathcal{S}_1 \oplus \mathcal{L},$$

we see that

$$U = \begin{bmatrix} V & 0 & 0 \\ 0 & M_{z_n}^* & 0 \\ 0 & 0 & I_{\mathcal{L}} \end{bmatrix} : \mathcal{F}_1 \oplus \mathcal{M}_1 \oplus \mathcal{L} \to (\mathcal{S} \ominus \mathcal{S}_\Phi) \oplus \mathcal{S}_1 \oplus \mathcal{L},$$

defines a unitary from $\mathcal{S}_\Phi$ to $\mathcal{S}$. We claim that $U^*\mathcal{T}(\mathcal{S})U = \mathcal{T}(\mathcal{S}_\Phi)$. First we prove that $U^*\mathcal{T}(\mathcal{S})U \subseteq \mathcal{T}(\mathcal{S}_\Phi)$. Since $\dim\mathcal{F}_1 < \infty$, it suffices to prove that $U^* R_{z_i}^{\mathcal{S}} U|_{\mathcal{M}_1 \oplus \mathcal{L}} \in \mathcal{T}(\mathcal{S}_\Phi)$ for all $i = 1, \cdots, n$. Observe first that $U\mathcal{M}_1 = M_{z_n}^* \mathcal{M}_1 = \mathcal{S}_1 \subseteq \mathcal{S}_\Phi$, $M_{z_n}\mathcal{S}_1 \subseteq \mathcal{S}_1$ and $M_{z_n}\mathcal{L} \subseteq \mathcal{L}$. Since

$$U^* R_{z_n}^{\mathcal{S}} U|_{\mathcal{M}_1 \oplus \mathcal{L}} = U^* M_{z_n} M_{z_n}^*|_{\mathcal{M}_1} + M_{z_n}|_{\mathcal{L}},$$

and $U^* M_{z_n} M_{z_n}^*|_{\mathcal{M}_1} = M_{z_n}^2 M_{z_n}^*|_{\mathcal{M}_1} = M_{z_n}^2 P_{\mathcal{S}_\Phi} M_{z_n}^*|_{\mathcal{M}_1}$, it follows that

$$U^* R_{z_n}^{\mathcal{S}} U|_{\mathcal{M}_1 \oplus \mathcal{L}} = (R_{z_n}^{\mathcal{S}_\Phi})^2 (R_{z_n}^{\mathcal{S}_\Phi})^* P_{\mathcal{M}_1}^{\mathcal{S}_\Phi} + R_{z_n}^{\mathcal{S}_\Phi} P_{\mathcal{L}}^{\mathcal{S}_\Phi} \in \mathcal{T}(\mathcal{S}_\Phi).$$

Now for $1 < i < n$, we have

$$U^* R_{z_i}^{\mathcal{S}} U|_{\mathcal{M}_1 \oplus \mathcal{L}} = U^* M_{z_i} M_{z_n}^*|_{\mathcal{M}_1} + U^* M_{z_i}|_{\mathcal{L}},$$

where $U^* M_{z_i} M_{z_n}^*|_{\mathcal{M}_1} = M_{z_i} M_{z_n}^*|_{\mathcal{M}_1}$ as $z_i \mathcal{S}_1 \subseteq \mathcal{S}_3 \subseteq \mathcal{L}$. On the other hand, since $z_i \mathcal{S}_2 \subseteq \mathcal{S}_3$ we have $z_i \mathcal{L} \subseteq \mathcal{L}$ and hence $U^* M_{z_i}|_{\mathcal{L}} = M_{z_i}|_{\mathcal{L}}$, whence

$$U^* R_{z_i}^{\mathcal{S}} U|_{\mathcal{M}_1 \oplus \mathcal{L}} = R_{z_i}^{\mathcal{S}_\Phi} (R_{z_n}^{\mathcal{S}_\Phi})^* P_{\mathcal{M}_1}^{\mathcal{S}_\Phi} + R_{z_i}^{\mathcal{S}_\Phi} P_{\mathcal{L}}^{\mathcal{S}_\Phi} \in \mathcal{T}(\mathcal{S}_\Phi).$$

Now we decompose $\mathcal{M}_1$ as $\mathcal{M}_1 = \mathcal{K}_1 \oplus \tilde{\mathcal{K}}_1$ where

$$\mathcal{K}_1 = (\varphi_1 \mathcal{Q}_{z^{m-1}}) \otimes \mathbb{C}^{\otimes(n-2)} \otimes \mathcal{S}_z \quad \text{and} \quad \tilde{\mathcal{K}}_1 = (\varphi_1 z^{m-1}\mathbb{C}) \otimes \mathbb{C}^{\otimes(n-2)} \otimes \mathcal{S}_z.$$

Then

$$U^* R_{z_1}^{\mathcal{S}} U|_{\mathcal{M}_1} = U^* M_{z_1} M_{z_n}^*|_{\mathcal{K}_1} + U^* M_{z_1} M_{z_n}^*|_{\tilde{\mathcal{K}}_1} = M_{z_n} M_{z_1} M_{z_n}^*|_{\mathcal{K}_1} + M_{z_1} M_{z_n}^*|_{\tilde{\mathcal{K}}_1},$$

as $M_{z_1} M_{z_n}^* \mathcal{K}_1 \subseteq \mathcal{S}_1$ and $M_{z_1} M_{z_n}^* \tilde{\mathcal{K}}_1 \subseteq \mathcal{S}_2$. On the other hand, $U^* R_{z_1}^{\mathcal{S}} U|_{\mathcal{S}_2 \oplus \mathcal{S}_3} = M_{z_1}|_{\mathcal{S}_2 \oplus \mathcal{S}_3}$ as $M_{z_1}(\mathcal{S}_2 \oplus \mathcal{S}_3) \subseteq \mathcal{S}_2 \oplus \mathcal{S}_3 \subseteq \mathcal{L}$, and finally, by denoting $\mathcal{N} = \mathcal{Q}_{\varphi_1} \otimes \mathcal{Q}_\Phi[2, n]^\perp$, we have

$$U^* R_{z_1}^{\mathcal{S}} U|_{\mathcal{N}} = U^* M_{z_1}|_{\mathcal{N}} = U^*(I_{\mathcal{S}} - P_{\mathcal{S}_1}^{\mathcal{S}}) M_{z_1}|_{\mathcal{N}} + U^* P_{\mathcal{S}_1}^{\mathcal{S}} M_{z_1}|_{\mathcal{N}}.$$

Then $\mathcal{S} \ominus \mathcal{S}_1 = (\mathcal{S} \ominus \mathcal{S}_\Phi) \oplus \mathcal{L}$ and $M_{z_1}\mathcal{N} \subseteq \mathcal{S}_\Phi$ implies that

$$U^* R_{z_1}^{\mathcal{S}} U|_{\mathcal{N}} = P_{\mathcal{L}}^{\mathcal{S}_\Phi} M_{z_1}|_{\mathcal{N}} + M_{z_n} P_{\mathcal{S}_1}^{\mathcal{S}_\Phi} M_{z_1}|_{\mathcal{N}},$$

and so

$$U^* R_{z_1}^{\mathcal{S}} U|_{\mathcal{M}_1 \oplus \mathcal{L}} = R_{z_n}^{\mathcal{S}_\Phi} R_{z_1}^{\mathcal{S}_\Phi} (R_{z_n}^{\mathcal{S}_\Phi})^* P_{\mathcal{K}_1}^{\mathcal{S}_\Phi} + R_{z_1}^{\mathcal{S}_\Phi} (R_{z_n}^{\mathcal{S}_\Phi})^* P_{\tilde{\mathcal{K}}_1}^{\mathcal{S}_\Phi} + R_{z_1}^{\mathcal{S}_\Phi} P_{\mathcal{S}_2 \oplus \mathcal{S}_3}^{\mathcal{S}_\Phi}$$
$$+ P_{\mathcal{L}}^{\mathcal{S}_\Phi} R_{z_1}^{\mathcal{S}_\Phi} P_{\mathcal{N}}^{\mathcal{S}_\Phi} + R_{z_n}^{\mathcal{S}_\Phi} P_{\mathcal{S}_1}^{\mathcal{S}_\Phi} R_{z_1}^{\mathcal{S}_\Phi} P_{\mathcal{N}}^{\mathcal{S}_\Phi} + F.$$

This implies that $U^* R_{z_1}^{\mathcal{S}} U \in \mathcal{T}(\mathcal{S}_\Phi)$, and therefore $U^* \mathcal{T}(\mathcal{S}) U \subseteq \mathcal{T}(\mathcal{S}_\Phi)$. We now proceed to prove the reverse inclusion $U \mathcal{T}(\mathcal{S}_\Phi) U^* \in \mathcal{T}(\mathcal{S})$. Since $\dim(\mathcal{S} \ominus \mathcal{S}_\Phi) < \infty$, it is enough to prove that $U R_{z_i}^{\mathcal{S}_\Phi} U^*|_{\mathcal{S}_1 \oplus \mathcal{L}} \in \mathcal{T}(\mathcal{S})$ for all $i = 1, \ldots, n$. Once again, note that $U^* \mathcal{S}_1 = \mathcal{M}_1 \subseteq \mathcal{S}_\Phi$, $z_n \mathcal{M}_1 \subseteq \mathcal{M}_1$, $z_n \mathcal{S}_1 \subseteq \mathcal{S}_1$ and $z_n \mathcal{L} \subseteq \mathcal{L}$. Hence

$$U R_{z_n}^{\mathcal{S}_\Phi} U^*|_{\mathcal{S}_1 \oplus \mathcal{L}} = U M_{z_n}^2|_{\mathcal{S}_1} + U M_{z_n}|_{\mathcal{L}} = M_{z_n}|_{\mathcal{S}_1} + M_{z_n}|_{\mathcal{L}},$$

that is

$$U R_{z_n}^{\mathcal{S}_\Phi} U^*|_{\mathcal{S}_1 \oplus \mathcal{L}} = R_{z_n}^{\mathcal{S}} P_{\mathcal{S}_1 \oplus \mathcal{L}}^{\mathcal{S}} \in \mathcal{T}(\mathcal{S}).$$

Now, for fixed $1 < i < n$, we have $z_i \mathcal{M}_1 \subseteq \mathcal{S}_3$ and $z_i \mathcal{L} \subseteq \mathcal{L}$. Then

$$\begin{aligned} U R_{z_i}^{\mathcal{S}_\Phi} U^*|_{\mathcal{S}_1 \oplus \mathcal{L}} &= U M_{z_i} M_{z_n}|_{\mathcal{S}_1} + U M_{z_i}|_{\mathcal{L}} \\ &= M_{z_i} M_{z_n}|_{\mathcal{S}_1} + M_{z_i}|_{\mathcal{L}} \\ &= R_{z_i}^{\mathcal{S}} R_{z_n}^{\mathcal{S}} P_{\mathcal{S}_1}^{\mathcal{S}} + R_{z_i}^{\mathcal{S}} P_{\mathcal{L}} \in \mathcal{T}(\mathcal{S}). \end{aligned}$$

Finally, we consider the decomposition $\mathcal{S}_1 = \mathcal{S}_1' \oplus \mathcal{S}_1''$ where

$$\mathcal{S}_1' = (\varphi_1 \mathcal{Q}_{z^{m-1}}) \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D}) \text{ and } \mathcal{S}_1'' = (\varphi_1 z^{m-1} \mathbb{C}) \otimes \mathbb{C}^{\otimes(n-2)} \otimes H^2(\mathbb{D}).$$

Then

$$\begin{aligned} U R_{z_1}^{\mathcal{S}_\Phi} U^*|_{\mathcal{S}_1} &= U M_{z_1} M_{z_n}|_{\mathcal{S}_1'} + U M_{z_1} M_{z_n}|_{\mathcal{S}_1''} \\ &= M_{z_n}^* M_{z_1} M_{z_n}|_{\mathcal{S}_1'} + M_{z_1} M_{z_n}|_{\mathcal{S}_1''} \\ &= M_{z_1}|_{\mathcal{S}_1'} + M_{z_1} M_{z_n}|_{\mathcal{S}_1''}, \end{aligned}$$

as $z_1 z_n \mathcal{S}_1' \subseteq \mathcal{M}_1$ and $z_1 z_n \mathcal{S}_1'' \subseteq \mathcal{S}_2$. Moreover

$$U R_{z_1}^{\mathcal{S}_\Phi} U^*|_{\mathcal{S}_2 \oplus \mathcal{S}_3} = U M_{z_1}|_{\mathcal{S}_2 \oplus \mathcal{S}_3} = M_{z_1}|_{\mathcal{S}_2 \oplus \mathcal{S}_3},$$

as $z_1(\mathcal{S}_2 \oplus \mathcal{S}_3) \subseteq \mathcal{S}_2 \oplus \mathcal{S}_3$. From the definition of $\mathcal{N}$, it follows that

$$U R_{z_1}^{\mathcal{S}_\Phi} U^*|_{\mathcal{N}} = U P_{\mathcal{M}_1}^{\mathcal{S}_\Phi} M_{z_1}|_{\mathcal{N}} + U(I_{\mathcal{S}_\Phi} - P_{\mathcal{M}_1}^{\mathcal{S}_\Phi}) M_{z_1}|_{\mathcal{N}},$$

this in turn implies that

$$U R_{z_1}^{\mathcal{S}_\Phi} U^*|_{\mathcal{N}} = M_{z_n}^* P_{\mathcal{M}_1}^{\mathcal{S}} M_{z_1}|_{\mathcal{N}} + P_{\mathcal{L}}^{\mathcal{S}} M_{z_1}|_{\mathcal{N}} + F,$$

as $\mathcal{S}_\Phi \ominus \mathcal{M}_1 = \mathcal{F}_1 \oplus \mathcal{L}$ and $\mathcal{F}_1$ is finite dimensional. Therefore

$$\begin{aligned} U R_{z_1}^{\mathcal{S}_\Phi} U^*|_{\mathcal{S}_1 \oplus \mathcal{L}} &= R_{z_1}^{\mathcal{S}} P_{\mathcal{S}_1'}^{\mathcal{S}} + R_{z_1}^{\mathcal{S}} R_{z_n}^{\mathcal{S}} P_{\mathcal{S}_1''}^{\mathcal{S}} + R_{z_1}^{\mathcal{S}} P_{\mathcal{S}_2 \oplus \mathcal{S}_3}^{\mathcal{S}} \\ &\quad + (R_{z_n}^{\mathcal{S}})^* P_{\mathcal{M}_1}^{\mathcal{S}} M_{z_1} P_{\mathcal{N}}^{\mathcal{S}} + P_{\mathcal{L}}^{\mathcal{S}} R_{z_1}^{\mathcal{S}} P_{\mathcal{N}}^{\mathcal{S}} + F \in \mathcal{T}(\mathcal{S}). \end{aligned}$$

This completes the proof of the theorem. ■

On combining Theorems 4.4 and 4.7, we have the following:

Theorem 4.8 *If $\mathcal{S}$ is a finite co-dimensional invariant subspace of $H^2(\mathbb{D}^n)$, then $\mathcal{T}(\mathcal{S})$ and $\mathcal{T}(H^2(\mathbb{D}^n))$ are unitarily equivalent.*

In the case $n = 2$, the proof of the above result is considerably simpler and direct than the one by Seto [26] (for instance, if $n = 2$, then $1 < i < n$ case does not appear in the proof of Theorem 4.7).

Acknowledgments The research of the first named author is supported by DST-INSPIRE Faculty Fellowship No. DST/INSPIRE/04/2015/001094. The research of the third named author is supported in part by NBHM (National Board of Higher Mathematics, India) grant NBHM/R.P.64/2014, and the Mathematical Research Impact Centric Support (MATRICS) grant, File No: MTR/2017/000522 and Core Research Grant, File No: CRG/2019/000908, by the Science and Engineering Research Board (SERB), Department of Science & Technology (DST), Government of India.

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Spectral Analysis, Model Theory and Applications of Finite-Rank Perturbations

Dale Frymark and Constanze Liaw

Dedicated to the memory of R.G. Douglas, a magnificent person, administrator and mathematician.

Abstract This survey focuses on two main types of finite-rank perturbations: self-adjoint and unitary. We describe both classical and more recent spectral results, paying special attention to singular self-adjoint perturbations and model representations of unitary perturbations.

Keywords Finite-rank perturbations · Representations · Spectral theory · Model theory

Mathematics Subject Classification (2010) Primary 47A55; Secondary 44A15, 30E20, 47A56, 47A10

1 Introduction

Let $\mathbf{A}$ be a self-adjoint (possibly unbounded) operator on a separable Hilbert space $\mathcal{H}$. Fix a d-dimensional subspace $\mathcal{K} \leq \mathcal{H}$. Consider all self-adjoint perturbations $\mathbf{A} + K$ with $\operatorname{Ran} K \subset \mathcal{K}$. All self-adjoint perturbations $\mathbf{A} + K$ are formally given

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_9

by the *family of self-adjoint finite-rank perturbations*:

$$\mathbf{A}_\Gamma = \mathbf{A} + \mathbf{B}\Gamma\mathbf{B}^* \tag{1.1}$$

for some Hermitian $d \times d$ matrix Γ, where $\mathbf{B} : \mathbb{C}^d \to \mathcal{K}$ is an invertible coordinate operator that takes the standard basis $\{\mathbf{e}_k\}_{k=1}^d$ of $\mathbb{C}^d$ into a basis $\mathbf{B}\mathbf{e}_k$ of $\mathcal{K}$. Reducing our attention to the essence of the problem, we always assume without loss of generality that $\mathcal{K}$ is cyclic for $\mathbf{A}$ on $\mathcal{H}$, that is, $\mathcal{H} = \operatorname{clos\,span}\{(\mathbf{A} - zI)^{-1}\mathcal{K} : z \in \mathbb{C} \setminus \mathbb{R}\}$. See Sect. 5 for a more general definition of $\mathbf{A}_\Gamma$ which applies when the functions $\mathbf{B}\mathbf{e}_k$ do not belong to the Hilbert space $\mathcal{H}$, but are instead taken from a larger space.

The *family of self-adjoint rank-one perturbations* represents a special case of the family of finite-rank perturbations given in Eq. (1.1), and can be formally given by

$$A_\gamma = A + \gamma(\,\cdot\,, \varphi)\varphi, \qquad \varphi \in \mathcal{K} \tag{1.2}$$

with parameter $\gamma \in \mathbb{R}$. See Sect. 3.1 for the precise definition, as well as Sect. 1.1 regarding notation on $\mathbf{A}$ versus A.

Interest in this type of perturbation problem originally arose from the theory of self-adjoint extensions [83]. Natural applications to the variation of boundary conditions of differential operators, in particular Sturm–Liouville operators, were investigated by Aronszajn and Donoghue in the 1950s. Other famous perturbation theoretic results, such as those by von Neumann and Kato–Rosenblum, apply because rank-one perturbations are trace class. The great achievements in this field furnish a rather concrete description of the spectral properties of the perturbed operators A_γ. See Sect. 3 for a sampling of these results.

The spectral theory for quantum mechanical systems (see e.g. [6]), large random matrices (see e.g. [16]) and free probability (see e.g. [15]), and the decoupling of CMV matrices (see e.g. [77, Section 4.5]) present other standard applications. Additional applications to quantum graph theory arise from transforming the graph to a tree by adding *partition vertices* to existing edges and imposing boundary conditions on the partition vertices [19, Ch. 3]. The number of partition vertices that needs to be added in order to transform a graph into a tree is equal to the first Betti or cyclomatic number of the original graph, which equals the number of edges minus the number of vertices plus the number of connected components.

In the late 1980s and early 1990s, a surge of interest took place in perturbation theory following the discovery of the celebrated Simon–Wolff criterion, which was used in a proof of Anderson localization for the discrete random Schrödinger operator in dimension one. A brief discussion of the Simon–Wolff criterion is included in Sect. 8.2.

Given two arbitrary operators on the same Hilbert space, it is generally not easy to find out whether they are related via a rank-one or a finite-rank perturbation. The situation is different if we consider two (so-called) Anderson-type Hamiltonians. We refer the reader to Sect. 8.2 for a definition. For now it suffices to know that they are perturbation problems with a random perturbation that is almost surely non-compact. Under mild assumptions, the essential part of two realizations of an Anderson-type Hamiltonian are related by a rank-one perturbation (almost surely with respect to the product of the probability measures), see [53].

Unitary perturbation theory is the other main topic of this survey. Let $\mathbf{U}$ be a unitary operator on a Hilbert space $\mathcal{H}$. Fix a d-dimensional subspace $\mathcal{R} \leq \mathcal{H}$. Then the set of operators K with $\operatorname{Ran} K \subset \mathcal{R}$ that make $\mathbf{U} + K$ a unitary operator can be parametrized by unitary $d \times d$ matrices. Specifically, there is a bijective coordinate operator $\mathbf{J} : \mathbb{C}^d \to \mathcal{R}$ so that $K = \mathbf{J}(\alpha - I)\mathbf{J}^*\mathbf{U}$ for a unitary $d \times d$ matrix α. The created *family of unitary finite-rank perturbations* of $\mathbf{U}$ is given by

$$\mathbf{U}_\alpha = \mathbf{U} + \mathbf{J}(\alpha - I)\mathbf{J}^*\mathbf{U}, \tag{1.3}$$

with α taken from the unitary $d \times d$ matrices. Without loss of generality, we focus on the domain altered by assuming that $\mathcal{R}$ is a $*$-cyclic subspace for $\mathbf{U}$, i.e. we assume that $\mathcal{H} = \operatorname{clos}\operatorname{span}\{\mathbf{U}^k\mathcal{R} : k \in \mathbb{Z}\}$.

The special case when $d = 1$ is closely related to Aleksandrov–Clark theory, and is described in Sect. 4.1. In this setting, the family of perturbations in Eq. (1.3) reduce to the well-known *family of unitary rank-one perturbations*

$$U_\alpha = U + (\alpha - 1)(\,\cdot\,, U^*\varphi)_{\mathcal{H}}\varphi, \tag{1.4}$$

with $\alpha \in \mathbb{T}$ and $\mathcal{R} = \operatorname{span}\{\varphi\}$. We say that φ is a $*$-cyclic vector for U, i.e. $\mathcal{H} = \operatorname{clos}\operatorname{span}\{U^k\varphi : k \in \mathbb{Z}\}$. Again, see Sect. 1.1 for notation.

While self-adjoint and unitary operators are intimately connected via the Cayley transform, it is well-known (see e.g. [21, Theorem 4.3.1]) that this correspondence is not a bijection between the two operator classes. In fact, even when the mappings are well-defined, the Cayley transform does not explicitly take (1.1) to its analog (1.3). This can be seen for the rank-one setting in Liaw–Treil [58, pp. 124–128]. Also notice that we encounter some inconveniences arising from unbounded operators in the self-adjoint setting. Of course, the unbounded case is exactly what occurs when dealing with boundary conditions of differential operators and several other applications. The unitary setting, on the other hand, is always restricted to bounded operators (see Remark 3.2).

It is therefore somewhat surprising that, in spite of these differences, many results on self-adjoint finite-rank perturbations have analogs in the unitary setting. It is also common to find that the problems raise similar questions, e.g. about the boundary behavior of analytic functions.

Families of rank-one and finite-rank perturbations seem rather elementary, yet their study has revealed a quite subtle nature. Their complexity is verified by connections to several deep fields of analysis: Nehari interpolation problem, holomorphic composition operators, rigid functions, existence of the limit of the Julia–Carathéodory quotient, Carleson embedding, and functional models. Some of these connections are the topic of existing books and surveys, including [23, 58, 71, 73].

While writing this survey, it became evident that a complete account of the subject of finite-rank perturbations is worthy of a whole book due to the connections to many other fields of mathematics. We decided to focus on a few aspects, while only briefly mentioning others. For example, some deserving topics such as related function theoretic nuances are not surveyed in detail. We also often refer to existing surveys and books on the topic such as, e.g. [6, 23, 58, 71, 73, 78], in order to not overlap excessively.

It should be noted that some central objects of perturbation theory, such as Aleksandrov Spectral Averaging and Poltoratski's Theorem, appear in the Appendix for convenience.

Section 2 contains highlights of classical perturbation theory that provide additional context for the more specific results to come. In particular, we focus on aspects of the spectrum that are invariant under different types of perturbations.

Sections 3 and 4 present well-known features of rank-one perturbation theory in the self-adjoint and unitary settings respectively. Section 3 includes a discussion of singular perturbations, some spectral results (including Aronszajn–Donoghue theory) and Nevanlinna–Herglotz functions, which form the backbone of the theory. The unitary setting of Sect. 4 is built upon Aleksandrov–Clark theory and features the Sz.-Nagy–Foiaş and de Branges–Rovnyak approach, as well as the overarching Nikolski–Vasyunin transcription free model theory. The latter reduces to the ones by Sz.-Nagy–Foiaş and de Branges–Rovnyak by choosing a specific weight. These model representations form rather concrete applications of model theory.

Sections 5 through 8 focus on finite-rank perturbations. Where possible, the presentation runs in analogy to Sects. 3 and 4.

For finite-rank self-adjoint perturbations the setup (Sect. 5) is a bit more involved, and we include information on extension theory, as well as a summary of some mathematical physics applications. In Sect. 6 we present known results regarding the spectral analysis of finite-rank perturbations and compare them to Aronszajn–Donoghue theory.

Section 7 contains information about model spaces culminating in the Nikolski–Vasyunin model theoretic representation of unitary finite-rank perturbations. A short exposition on related Krein spaces and reproducing kernel Hilbert spaces is provided. In Sect. 8 relationships between the family of spectra of the perturbation problem and the characteristic function are presented.

In the Appendix we take a moment to convey just the ideas behind several other well-deserving topics in the field. We refer to other literature for more information.

1.1 Notation

We use different notation to help the reader distinguish between the unitary the self-adjoint setting.

In the self-adjoint setting, a rank-one perturbation of an operator A will be denoted as A_γ, where $\gamma \in \mathbb{R}$. We will use "boldface" $\mathbf{A}_\Gamma$ for a finite-rank perturbation that is given by a self-adjoint matrix $(d \times d)$-matrix Γ. The real spectral measures for these cases will be referred to as μ_γ and $\boldsymbol{\mu}_\Gamma$ respectively. An additional superscript will be added when the trace of the matrix-valued spectral measures is required: $\boldsymbol{\mu}_\Gamma^{\mathrm{tr}}$. Also, the subscript will be entirely dropped when referring to objects corresponding to the unperturbed operator $\mathbf{A}$, e.g. $\boldsymbol{\mu} = \boldsymbol{\mu}_{\mathbf{0}}$, $F = F_0$, $\mathbf{F} = \mathbf{F}_{\mathbf{0}}$, etc.

In the unitary setting, a rank-one perturbation of an operator U will be denoted as U_α, where $\alpha \in \mathbb{T}$. A finite-rank perturbation will be given by $\mathbf{U}_\alpha$ with unitary $(d \times d)$-matrix α. Notation similar to the self-adjoint setting will be used for the spectral measures, e.g. μ_α and $\boldsymbol{\mu}_\alpha$. Here, the subscript α indicates that we work with unitary perturbations. Characteristic functions and model spaces will be denoted in the rank-one case by θ and $\mathcal{K}_\theta$, and in the finite-rank case by $\boldsymbol{\theta}$ and $\mathcal{K}_{\boldsymbol{\theta}}$. Dropping the subscript again refers to objects that correspond to the unperturbed operator $\mathbf{U}$, except this operator arises from using $\alpha = I$, e.g. $\boldsymbol{\mu} = \boldsymbol{\mu}_I$, etc. We will simply write I for the identity matrix, with the dimension inferred from context.

Spaces will be written in "mathcal" notation, e.g. $\mathcal{H}$ and $\mathcal{H}_s(\mathbf{A})$. In particular, $\mathcal{D}$ and $\mathcal{D}_*$ refer to the deficiency spaces on the unitary side.

2 Perturbation-Theoretic Background

We begin by presenting some central ideas from classical perturbation theory of self-adjoint operators, in order to better frame later discussions.

A linear operator A from a Banach space $\mathcal{X}$ to a Banach space $\mathcal{Y}$ is said to be *compact* if the image $A(\mathcal{X}_1)$ of any bounded subset $\mathcal{X}_1 \subset \mathcal{X}$ is relatively compact in $\mathcal{Y}$. Consider linear operators acting on a Hilbert space $\mathcal{H}$. The class of compact operators $\mathscr{S}$ is then obtained by taking the closure of the set of finite-rank operators with respect to the operator norm topology. A characterization of the spectrum of self-adjoint operators that differ by a compact perturbation is available. Recall that the *spectrum* of an operator A, denoted by $\sigma(A)$, is the closure of the set of all $\lambda \in \mathbb{C}$ for which operator $A - \lambda I$ is not invertible. The essential spectrum is the spectrum minus the isolated eigenvalues of finite (algebraic) multiplicity.

Theorem 2.1 (von Neumann, See E.g. [21, Theorems 3 and 6 of Ch. 9]) *Let A and B be bounded self-adjoint operators. Then B is compact if and only if the essential spectra of A and $A + B$ are the same.*

In the self-adjoint setting, we can view compact operators as compact perturbations of the zero operator to see that compact operators are characterized as those whose only (possible) accumulation point of eigenvalues is the origin. A more refined standard definition restricts the speed at which the eigenvalues tend to 0. Namely, the *von Neumann–Schatten classes*, $\mathscr{S}_p$, consist of compact operators whose sequence of singular values $\{s_k\}$ belongs to ℓ^p. Here, the singular values of an operator T are defined as the eigenvalues of $|T| = (T^*T)^{1/2}$. Self-adjoint operators thus have the property that $s_k = |\lambda_k|$, where λ is the sequence of eigenvalues.

Theorem 2.2 (Kato [46, Theorem 1] and Rosenblum [72, Theorem 1.6]) *Let A and B be self-adjoint operators and assume $B \in \mathscr{S}_1$. Then the absolutely continuous parts of A and $A + B$ are unitarily equivalent.*

Carey and Pincus [22] characterized trace class, $\mathscr{S}_1$, perturbations $A + B$ of A. Apart from leaving the absolutely continuous spectrum invariant, it must be possible to split the isolated eigenvalues of A and $A + B$ as follows into three categories. The first and second categories are comprised of the eigenvalues of A and of $A + B$, respectively, that have summable distance from the essential spectrum of A. The third category contains all remaining eigenvalues of A and $A+B$. And there must exist a bijection φ mapping those eigenvalues of A in this category to those remaining eigenvalues of $A + B$ so that the sum of $|\lambda - \varphi(\lambda)|$ over all eigenvalues λ of A in this category is finite. In other words the remaining eigenvalues of A have trace class distance to the remaining ones of $A + B$.

To emphasize a dichotomy, we mention that absolutely continuous spectrum can be destroyed by a Hilbert–Schmidt operator of arbitrarily small Hilbert–Schmidt norm:

Theorem 2.3 (Weyl–von Neumann, See E.g. [45, p. 525]) *Let A be a self-adjoint operator. For every $\eta > 0$, there exists a self-adjoint operator B with Hilbert-Schmidt norm less than η so that $A + B$ has pure point spectrum.*

Since the Hilbert–Schmidt norm dominates the standard operator norm, this means that the absolutely continuous spectrum may be unstable under arbitrarily small perturbations.

Theorem 2.3 was first proved by Weyl [83] for compact perturbations and then for the smaller class of Hilbert–Schmidt perturbations by von Neumann [82]. Extensions to normal operators and perturbations were proved by Berg [17] for compact operators and by Voiculescu [80, 81] for Hilbert–Schmidt perturbations. These results form the basis of K-homology theory, which studies the homology of the category consisting of locally compact Hausdorff spaces.

On the side, we mention Baranov [12] where a model representation and a spectral synthesis for rank-one perturbations of normal operators is achieved.

In order to avoid possible confusion, we spell out that we are not (at least not explicitly) reaching for a spectral synthesis, or other questions usually related to K-homology. Instead, we are primarily interested in spectral invariants and describing the spectral measure under perturbations.

3 Aspects of Self-Adjoint Rank-One Perturbations

3.1 *Scales of Hilbert Spaces*

When considering perturbations like Eq. (1.2), it is sometimes convenient to loosen our restrictions on the perturbation vector φ to expand our possible applications, e.g. to changing boundary conditions of differential operators. We say that the perturbation is *bounded* when the vector φ is from the Hilbert space $\mathcal{H}$. The previous sections have dealt exclusively with bounded perturbations. If $\varphi \notin \mathcal{H}$, we say the perturbation is *singular*. These perturbations are significantly more complicated; it is imperative to ensure that the perturbation is well-defined in order to extend the tools that are presented in Sect. 3.2. The description here roughly follows that of [6].

Let A be a self-adjoint (possibly unbounded) operator on a separable Hilbert space $\mathcal{H}$. Consider the non-negative operator $|A| = (A^*A)^{1/2}$, whose domain coincides with the domain of A. Alternatively, if A is bounded from below, the shifted operator $A + kI$, $k \in \mathbb{R}$ sufficiently large, will provide a non-negative operator. We introduce a scale of Hilbert spaces.

Definition 3.1 ([6, Section 1.2.2]) For $s \geq 0$, define the space $\mathcal{H}_s(A)$ to consist of φ from $\mathcal{H}$ for which the s-norm

$$\|\varphi\|_s := \|(|A| + I)^{s/2}\varphi\|_{\mathcal{H}}, \tag{3.1}$$

is bounded. The space $\mathcal{H}_s(A)$ equipped with the norm $\|\cdot\|_s$ is complete. The adjoint spaces, formed by taking the linear bounded functionals on $\mathcal{H}_s(A)$, are used to define these spaces for negative indices, i.e. $\mathcal{H}_{-s}(A) := \mathcal{H}_s^*(A)$. The corresponding norm in the space $\mathcal{H}_{-s}(A)$ is thus defined by (3.1) as well. The collection of these $\mathcal{H}_s(A)$ spaces will be called the *scale of Hilbert spaces associated with the self-adjoint operator A*.

It is not difficult to see that the spaces satisfy the nesting properties

$$\ldots \subset \mathcal{H}_2(A) \subset \mathcal{H}_1(A) \subset \mathcal{H} = \mathcal{H}_0(A) \subset \mathcal{H}_{-1}(A) \subset \mathcal{H}_{-2}(A) \subset \ldots,$$

and that for every two s, t with $s < t$, the space $\mathcal{H}_t(A)$ is dense in $\mathcal{H}_s(A)$ in the norm $\|\cdot\|_s$. Indeed, the operator $(A + 1)^{t/2}$ defines an isometry from $\mathcal{H}_s(A)$ to $\mathcal{H}_{s-t}(A)$. In the rest of the subsection, we will use the brackets $\langle\cdot, \cdot\rangle$ to denote both the inner product in the Hilbert space $\mathcal{H}$ and the action of the functionals. For instance, if $\varphi \in \mathcal{H}_{-s}(A)$, $\psi \in \mathcal{H}_s(A)$, then

$$\langle\varphi, \psi\rangle := \left\langle(|A| + I)^{-s/2}\varphi, (|A| + I)^{s/2}\psi\right\rangle,$$

where the brackets on the right hand side denote the inner product.

Throughout the literature of other fields similar constructions occur under different names. For instance, the pairing of $\mathcal{H}_1(A)$, $\mathcal{H}$, and $\mathcal{H}_{-1}(A)$ is sometimes referred to as a *Gelfand triple* or *rigged Hilbert space*. Also, when A is the derivative operator, these scales are simply Sobolev spaces (with $p = 2$). More details about Hilbert scales can be found in [49].

It is worth noting that these Hilbert scales are related to those generated by so-called left-definite theory [60]. This theory employs powers of a semi-bounded self-adjoint differential operator to create a continuum of operators whereupon spectral properties can be studied. The theory can be applied to self-adjoint extensions of self-adjoint operators, which can be viewed as finite-rank perturbations, see e.g. [29, 30] and the references therein.

Rank-one perturbations of a given operator A arise most commonly when the vectors φ are bounded linear functionals on the domain of the operator A, so many applications are focused on $\mathcal{H}_{-2}(A)$. Here, we only discuss the case $\varphi \in \mathcal{H}_{-1}(A)$ for the sake of simplicity. However, references usually contain information on extensions to $\varphi \in \mathcal{H}_{-2}(A)$, and information on the case when $\varphi \notin \mathcal{H}_{-2}(A)$ can be found in [25, 50].

Remark 3.2 The case $\mathcal{H}_{-1}(A)$ for the self-adjoint setting most closely aligns with unitary perturbations, see [58, pp. 124–128]. It is not immediately clear how the more singular perturbations, $\mathcal{H}_{-n}(A)$ for $n > 1$, translate to the unitary side.

3.2 Spectral Theory of Rank-One Perturbations

A nice overview of what is now known as Aronszajn–Donoghue theory was given in [78]. Extensions of Aronszajn–Donoghue theory to the case when the spectral measure is associated with a perturbation vector $\varphi \in \mathcal{H}_{-2}(A)$ can found in [5] and [47], but here we take $\varphi \in \mathcal{H}_{-1}(A)$ unless otherwise mentioned. The results compare the spectral measures μ and μ_γ of the unperturbed and the perturbed operators and are expressed through the scalar-valued *Borel* transform

$$F_\gamma(z) := \int_{\mathbb{R}} \frac{d\mu_\gamma(t)}{t - z} \qquad \text{for } z \in \mathbb{C}\backslash\mathbb{R}, \tag{3.2}$$

which is abbreviated F for $\gamma = 0$.

One of the standard identities at the heart of the theory is often referred to as the Aronszajn–Krein formula $F_\gamma(z) = F(z)/(1 + \gamma F(z))$. The distinction of whether or not a point has mass is encrypted in the functions F and $G(x) := \displaystyle\int \frac{d\mu(t)}{(x - t)^2}$.

Theorem 3.3 (Aronszajn–Donoghue Theory, E.g. [78, Theorem 12.2]) *When $\gamma \neq 0$, the sets*

$$S_\gamma = \left\{x \in \mathbb{R} \,\middle|\, \lim_{y\to 0} F(x+iy) = -1/\gamma;\, G(x) = \infty\right\},$$

$$P_\gamma = \left\{x \in \mathbb{R} \,\middle|\, \lim_{y\to 0} F(x+iy) = -1/\gamma;\, G(x) < \infty\right\}, \text{ and}$$

$$C = \left\{x \in \mathbb{R} \,\middle|\, \lim_{y\to 0} \operatorname{Im} F(x+iy) \neq 0\right\},$$

contain spectral information of the perturbed operator A_γ as follows:

(i) *For fixed $\gamma \neq 0$, the sets S_γ, P_γ and C are mutually disjoint.*
(ii) *Set P_γ is the set of eigenvalues, and set C (S_γ) is a carrier for the absolutely (singular) continuous measure, respectively.*
(iii) *For $\gamma \neq \beta$ the singular parts of A_γ and A_β are mutually singular.*

Remark 3.4 Set X being a carrier for a measure τ means that $\tau(\mathbb{R}\backslash X) = 0$. Any (measurable) set that contains the support of a measure is also a carrier. Since we do not require a carrier to be closed, there may be carrier sets that are strictly contained in the support of a measure.

The density function of the absolutely continuous measure and the pure point masses of A_γ are completely described by the following result.

Proposition 3.5 *Assume that $\gamma \neq 0$.*

(i) *For $\lambda \in P_\gamma$ we have $\mu_\gamma(\{\lambda\}) = \frac{1}{\gamma^2 G(\lambda)}$.*
(ii) *The density function of the absolutely continuous part of A_γ is given by*

$$\frac{d\mu_\gamma(x)}{dx} = \frac{1}{\pi} \lim_{y\to 0^+} \frac{\operatorname{Im} F(x+iy)}{|1+\gamma F(x+iy)|^2},$$

with respect to Lebesgue a.e. $x \in \mathbb{R}$.

We mention that the limit in part (ii) of the proposition exists with respect to Lebesgue a.e. x. Indeed, by the Aronszajn–Krein formula $\frac{\operatorname{Im} F}{|1+\gamma F|^2} = \operatorname{Im} F_\gamma$, and F_γ is analytic on the upper half-plane.

A characterization of the singular continuous part of A_γ has been sought after but is still outstanding. Only partial results have been established. Instead of elaborating on the details here, we refer the reader to [23, 57, 78] and the references therein. We also point the reader to [58] for a discussion of, and references for, the question: "How unstable can the singular spectrum become?"

The measures μ and μ_γ, which are the spectral measures associated with rank-one perturbations of self-adjoint operators, are associated with scalar Nevanlinna–Herglotz functions. These functions are analytic self-maps of the upper half plane

$\mathbb{C}_+$ and possess the Nevanlinna–Riesz–Herglotz representation

$$\widetilde{F}(z) = c + dz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t),$$

and μ is a measures which satisfies the decay condition $\int_{\mathbb{R}} (1+t^2)^{-1} d\mu(t) < \infty$. The examples that use $\widetilde{F}$ are more singular $\mathcal{H}_{-2}(A)$ perturbations. In order to give the reader additional intuition about these measures, we include some examples from [35, App. A].

[35, App. A]	Borel transform	Spectral Measure $d\mu(t)$
Eq. (5)	$F(z) = -1/z$	$\delta_{\{0\}}(t)dt$
Eq. (6)	$\widetilde{F}(z) = \ln(z)$	$\chi_{(-\infty,0)}(t)dt$
Eq. (7)	$\widetilde{F}(z) = \ln(-1/z)$	$\chi_{(0,\infty)}(t)dt$
Eq. (8)	$\widetilde{F}(z) = z^r - \cos(\frac{r\pi}{2})$	$\|t\|^r \pi^{-1} \sin(r\pi)\chi_{(-\infty,0)} dt, r \in (0, 1)$
Eq. (10)	$\widetilde{F}(z) = \tan(z)$	$\sum_{n\in\mathbb{Z}} \delta_{\{n\pi\}}(t)dt$
Eq. (17)	$F(z) = \ln\left(\frac{z-t_1}{z-t_2}\right)$	$\chi_{[t_1,t_2]}(t)dt$ with $t_1 < t_2$

Examples of Nevanlinna–Herglotz functions and corresponding spectral measures. The examples Eq. (6)–(8) use the principal value of the logarithm. The integration variable is λ. The first column contains references to equations in [35, App. A]. Other examples and their sources can also be found there

4 Aspects of Unitary Rank-One Perturbations and Model Theory

Consider the unitary rank-one perturbation problem given by Eq. (1.4). Let μ_α be the spectral measure of U_α with respect to the $*$-cyclic vector φ, which is simultaneously also $*$-cyclic for U_α for all $\alpha \in \mathbb{T}$. Then, the Spectral Theorem says U_α can be represented by the operator that acts via multiplication by the independent variable on the space $L^2(\mu_\alpha)$.

The operator U_0 is well-known to be a completely non-unitary (i.e. it is not unitary on any of its non-trivial invariant subspaces) contraction. Therefore, it (and hence the family of measures $\{\mu_\alpha\}$) corresponds to the compression of the shift operator in a model representation associated with a characteristic function θ. Studying the intricacies of these model representations emerges as one of the main strategies in this field.

Model spaces are subspaces of a weighted L^2-space, of which we discuss several: the one by Clark, which resembles a simplified Sz.-Nagy–Foiaş model; the one by de Branges–Rovnyak which was e.g. studied by the Sarason school; and an overarching description of model theory developed by Nikolski–Vasyunin. This final formulation essentially incorporates the former ones by choosing an appropriate weight function.

4.1 Aleksandrov–Clark Theory and Sz.-Nagy–Foiaş Model for Perturbations with Purely Singular Spectrum

A seminal paper by Clark [24] laid the foundation that connects rank-one perturbations with reproducing kernel Hilbert spaces. The field has since grown into what is now known as Aleksandrov–Clark theory, honoring the deep insights gained by Aleksandrov about Clark measures—especially in the presence of an absolutely continuous component. A nice exposition of Aleksandrov–Clark theory can be found in [23], which we mostly follow along with in this section. We refer readers interested in a more general exposition of the Sz.-Nagy–Foiaş model spaces to [79]. For roughly the second half of this subsection, we work with characteristic functions that are inner, or equivalently, within the Clark setting of purely singular spectral measures.

For an analytic function $\theta : \mathbb{D} \to \mathbb{D}$ and a point $\alpha \in \mathbb{T}$, the function

$$u_\alpha(z) := \Re\left(\frac{\alpha + \theta(z)}{\alpha - \theta(z)}\right) = \frac{1 - |\theta(z)|^2}{|\alpha - \theta(z)|^2}, \tag{4.1}$$

is positive and harmonic on $\mathbb{D}$. For each α, a theorem by Herglotz [36] says this function corresponds uniquely to a positive measure μ_α with $u_\alpha = P\mu_\alpha$. Here, $P\mu_\alpha = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu_\alpha(\zeta)$ is the Poisson integral of μ_α.

We let $\mathcal{A}_\theta := \{\mu_\alpha : \alpha \in \mathbb{T}\}$ denote the family of measures associated with the function θ. We will call $\mathcal{A}_\theta$ the family of *Clark measures* of θ when θ is an inner function, i.e. a bounded analytic function with unit modulus a.e. on $\mathbb{T}$. Note that when θ is a general analytic self-map of the disk, the family $\mathcal{A}_\theta$ is usually referred to as the *Aleksandrov–Clark family/operators* of θ.

With the Herglotz transformation $(H\mu)(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta)$ of a *measure* $\mu = \mu_1$, it can easily be verified that the function

$$\theta(z) := \frac{(H\mu)(z) - 1}{(H\mu)(z) + 1}, \tag{4.2}$$

is an analytic self map of the disk. The condition $\theta(0) = 0$ is equivalent to each $\mu_\alpha \in \mathcal{A}_\theta$ being a probability measure [23, Proposition 9.1.8].

These Clark measures can be used to describe the unitary perturbations of an important operator. To do so, we define the shift operator $S : H^2 \to H^2$ by $(Sf)(z) = zf(z)$, where $H^2 = H^2(\mathbb{D})$ denotes the Hardy space. Likewise, for later, we define the backward shift operator to be $(S^* f)(z) = \frac{f(z) - f(0)}{z}$. Beurling's Theorem [20] then says that the S-invariant subspaces of H^2 are exactly those that can be written as θH^2 for some inner function θ.

In order to take advantage of this relationship, we now assume θ is an inner function with $\theta(0) = 0$. Assuming that $\theta(0) = 0$ is not essential, but rather a convenience. Sometimes we will refer to such functions as *characteristic* functions.

Given such a θ, the *Sz.-Nagy-Foiaş model space* [79] can then be defined as

$$\mathcal{K}_\theta := H^2 \ominus \theta H^2. \tag{4.3}$$

Beurling's Theorem further implies that S^*-invariant subspaces of H^2 are simply model spaces $\mathcal{K}_\theta$ corresponding to some inner θ.

On the side, we mention two major advances in complex analysis:

(i) Douglas–Shapiro–Shields [27] have shown that for $f \in H^2$, $f \in \mathcal{K}_\theta$ if and only if the meromorphic function f/θ on $\mathbb{D}$ has a pseudo-continuation to a function $\widetilde{f_\theta} \in H^2(\mathbb{C} \setminus \overline{\mathbb{D}})$ with $\widetilde{f_\theta}(\infty) = 0$ (also see [23, Theorem 8.2.5]). The analogous result was also shown there to hold for conjugate pairs of H^p spaces.

(ii) A milestone has been achieved with the Ahern–Clark Theorems [2, 3] with respect to understanding when the Julia–Carathéodory angular derivative exists. Results for higher derivatives were also included. This result was generalized by Fricain–Mashreghi [31], and they proved that if a member of a deBranges–Rovnyak space is continuous on an open arc of the boundary, then it is analytic there. Also see the survey by Garcia–Ross [34, Theorem 6.11] for a summary.

Moving on with our program, let P_θ be the orthogonal projection of H^2 onto $\mathcal{K}_\theta$. The *compression of the shift operator* is thus defined as

$$S_\theta = P_\theta S|_{\mathcal{K}_\theta}.$$

This allows us to write the family of rank-one perturbations on $\mathcal{K}_\theta$:

$$V_\alpha f = S_\theta f + \alpha \left\langle f, \frac{\theta}{z} \right\rangle \mathbf{1}, \qquad \text{with} \qquad \alpha \in \mathbb{T}. \tag{4.4}$$

In particular, the following theorem of Clark says that these are the only unitary rank-one perturbations of S_θ.

Theorem 4.1 (Clark [24, Remark 2.3]) *Any operator X that is both unitary and a rank-one perturbation of S_θ can be written as $X = V_\alpha$ for some $\alpha \in \mathbb{T}$.*

Let μ_α be the Clark measure associated with the inner function θ and the point $\alpha \in \mathbb{T}$. Since V_α is a cyclic unitary operator, the spectral theorem says that V_α can be represented as multiplication by the independent variable on some $L^2(\nu)$ space. It turns out that the space $L^2(\nu)$ can be canonically identified with $L^2(\mu_\alpha)$. Let M be the operator on the space $L^2(\mu_\alpha)$ acting via multiplication by the independent variable. Then, the unitary operator that intertwines, $C_\alpha M = V_\alpha C_\alpha$, and maps the constant function $\mathbf{1} \in L^2(\mu_\alpha)$ to some vector in the defect space $\operatorname{Ran}(I - S_\theta^* S_\theta)^{1/2}$ is called the *adjoint Clark operator*. It is given by the *normalized Cauchy transform*

$$C_\alpha : L^2(\mu_\alpha) \to \operatorname{Hol}(\mathbb{D}) \qquad \text{with} \qquad (C_\alpha g)(z) := \frac{K(g d\mu_\alpha)}{K\mu_\alpha},$$

where K is the Cauchy transform $(K\nu)(z) = \int_{\mathbb{T}} \frac{d\nu(\zeta)}{1-z\bar{\zeta}}$. The Clark operator is often denoted by Φ in literature, so that $C_\alpha = \Phi^*$.

These representations gives us access to spectral information regarding the Clark family $\{\mu_\alpha\}$, $\alpha \in \mathbb{T}$.

Theorem 4.2 (See E.g. [23, Proposition 9.1.14] and [34, Proposition 8.3]) *In the above setting we have:*

1. $(d\mu_\alpha)_{\mathrm{ac}} = u_\alpha dm$ *(with* $u_\alpha(z) = (1 - |\theta(z)|^2)|\alpha - \theta(z)|^{-2}$ *as in* (4.1)*).*
2. $\mu_\alpha \perp \mu_\beta$ *for all* $\alpha \neq \beta$*,* $\beta \in \mathbb{T}$*.*
3. μ_α *has a point mass at* $\zeta \in \mathbb{T}$ *if and only if* $\theta(\zeta) = \alpha$ *and* $|\theta'(\zeta)| < \infty$*. In that case this point mass is given by* $\mu_\alpha(\{\zeta\}) = |\theta'(\zeta)|^{-1}$*.*
4. *The set* $\{\zeta \in \mathbb{T} : \lim_{r\to 1^-} \theta(r\zeta) = \alpha\}$ *is a carrier for* μ_α*. (Recall that* μ_α *is purely singular in the Clark setting.)*

This result is in direct correspondence with the Aronszajn–Donoghue Theorem 3.3 above. Also, observe that a point mass equals the reciprocal magnitude of the derivative of the Borel transform in the self-adjoint setting, and of the Cauchy transform in the unitary setting. In fact, [23, Item (1) of Corollary 9.1.24] offers a finer carrier of the singular spectrum in terms of the lower Dini derivative of μ_α.

The Sz.-Nagy–Foiaş representation simplifies to the setting described in this subsection precisely when operator V_1 has no absolutely continuous part (or, equivalently, when the characteristic function θ is inner). This poses a significant restriction. The de Branges–Rovnyak model is an alternative representation of the situation under weaker conditions. In the most general Aleksandrov–Clark situation, one is required to deal with the full two-storied Sz.-Nagy–Foiaş model space

$$\mathcal{K}_\theta = \begin{pmatrix} H^2 \\ \operatorname{clos} \Delta L^2 \end{pmatrix} \ominus \begin{pmatrix} \theta \\ \Delta \end{pmatrix} H^2,$$

instead of just the first component as in (4.3). The defect function Δ is $\Delta(z) = (1 - \theta(z)^*\theta(z))^{1/2}$ for $z \in \mathbb{T}$. For further reference, see [65, Section 1.3.5].

4.2 de Branges–Rovnyak Model and Perturbations in the Extreme Case

In this subsection, we assume that the characteristic function θ is an *extreme* point, i.e. that $\int_{\mathbb{T}} \ln(1 - |\theta(z)|)dm(z) = -\infty$. It is well-known that θ extreme if and only $L^2(\mu) = H^2(\mu)$ for the corresponding Aleksandrov–Clark measure $\mu = \mu_1$. This situation is ideal for the de Branges–Rovnyak model space, as it now reduces from two components

$$\mathcal{K}_\theta = \left\{ \begin{pmatrix} g_+ \\ g_- \end{pmatrix} : g_+ \in H^2,\ g_- \in H^2_-,\ g_- - \theta^* g_+ \in \Delta L^2 \right\}$$

to a one component space. Here we used the notation $H^2_- := L^2 \ominus H^2$.

We describe the reduced one-component de Branges–Rovnyak model space: So, assume $\theta : H^2 \to H^2$ is extreme. Then the de Branges–Rovnyak model space $\mathcal{H}(\theta) \subset H^2$ consists of functions in the range space of the defect operator, i.e. $\mathcal{H}(\theta) = (I - |\theta|^2)^{1/2} H^2$. The canonical norm on this space is the *range norm* which arises by taking the minimal norm of the pre-image of an element from $\mathcal{H}(\theta)$. Much of the success of this approach is based upon the fact that $\mathcal{H}(\theta)$ is a reproducing kernel Hilbert space with reproducing kernel $k_w^\theta(z) = \frac{1-\overline{\theta(w)}\theta(z)}{1-\bar{w}z}$. The deep structure of this space is the focus of [74]. We also point the reader to the books by Fricain–Mashreghi [32, 33] for a modern treatment of the de Branges–Rovnyak spaces, which is both comprehensive and accessible. Here we only mention a few items relevant to perturbation theory. We will omit other interesting topics such as multipliers of $\mathcal{H}(\theta)$, the theory regarding the Julia–Carathéodory angular derivatives and Denjoy–Wolff points—all of which are detailed in [74].

The connection with the corresponding Aleksandrov–Clark measure μ is made through Eq. (4.2), see e.g. [74, Chapter III]. Much of the development in this area is attributed to the dissertation of Ball [11]. For instance, it was shown there that the measure μ has an atom at a point $z_0 \in \mathbb{T}$ if and only if the function $\frac{\theta(z)-1}{z-z_0}$ belongs to $\mathcal{H}(\theta)$, see e.g. [74, Section (III-12)].

4.3 *General Perturbations and Nikolski–Vasyunin Model Theory*

Not all rank-one perturbations satisfy any of the conditions under which we can use the representations detailed in Sects. 4.1 and 4.2. Model theory for unitary perturbations in the general setting is much more complicated. Instead of a one-story model space, the general setting requires a two-story model space. While this description is superior in abstraction and admits more general settings, the models discussed in Sects. 4.1 and 4.2 have provided many deep insights over the years.

An overarching treatise of the Sz.-Nagy–Foiaş, the de Branges–Rovnyak model space and other model spaces (e.g. the one studied by Pavlov) was achieved by Nikolski–Vasyunin [65–68]. There, a general so-called *transcription free* model space was introduced as a subspace of a (possibly) two-storied weighted space $L^2(\mathcal{D}_* \oplus \mathcal{D}, W)$ on the unit circle. Here, the defect spaces of contraction V_0 are given by $\mathcal{D} = \operatorname{clos}\operatorname{Ran}(I - S_\theta^* S_\theta)^{1/2}$ and $\mathcal{D}_* = \operatorname{clos}\operatorname{Ran}(I - S_\theta S_\theta^*)^{1/2}$. We also note that the defect spaces $\mathcal{D}$ and $\mathcal{D}_*$ were identified with $\mathbb{T}$ in Sects. 4.1 and 4.2. This L^2 space then reduces to the Sz.-Nagy–Foiaş, the de Branges–Rovnyak, the

Pavlov model spaces, and other transcriptions by making specific choices of the weight W. The connection to rank-one perturbations comes from the dependence of this W on the characteristic function θ.

General rank-one perturbations were studied in Liaw–Treil [55]. This subject is included in the lecture notes by Liaw–Treil [58] on the relationship between rank-one perturbations and singular integral operators. Instead of repeating large chunks of information here, we refer the reader to those lecture notes.

5 Self-Adjoint Finite-Rank Perturbations

We adapt the self-adjoint finite-rank setup given in (1.1) to account for singular perturbation vectors, which are useful in many applications. We mostly follow along with [6, Ch. 3]. We begin by defining the coordinate operator $\mathbf{B} : \mathbb{C}^d \to \mathcal{H}_{-2}(\mathbf{A})$ that takes the standard basis $\{\mathbf{e}_k\}_{k=1}^d \subset \mathbb{C}^d$ to $\{\varphi_k\}_{k=1}^d \subset \mathcal{H}_{-2}(\mathbf{A})$. Note that we are changing notation slightly from Sect. 1, as we used to think of $\mathbf{B}$ as an operator $\mathbf{B} : \mathbb{C}^d \to \operatorname{Ran}\mathbf{B}$ that was invertible. As before, we assume without loss of generality the invertibility of $\mathbf{B}$ on its range.

Consider finite-rank perturbations of a self-adjoint operator $\mathbf{A}$ on the separable Hilbert space $\mathcal{H}$ given by

$$\mathbf{A}_\Gamma = \mathbf{A} + \mathbf{B}\Gamma\mathbf{B}^*, \tag{5.1}$$

where Γ is a Hermitian $d \times d$ matrix and the operator $\mathbf{B}\Gamma\mathbf{B}^*$ is an operator of rank d from the Hilbert space $\mathcal{H}_2(\mathbf{A})$ to the Hilbert space $\mathcal{H}_{-2}(\mathbf{A})$. Note that we can assume without loss of generality that the matrix Γ is invertible. If Γ is not invertible, then the orthogonal complement to the kernel of the operator Γ yields a finite-rank operator of rank strictly less than d determined by a non-degenerate Hermitian matrix.

The vectors φ_k can be thought of as modifying the domain of $\mathbf{A}$ by d dimensions that are in $\mathcal{H}_{-2}(\mathbf{A})$. However, to ensure that each of these vectors are non-degenerate and adding new dimensions, we will call the set of vectors $\varphi_k \in \mathcal{H}_{-2}(\mathbf{A})\backslash\mathcal{H}$, $k = 1, \ldots, d$, *$\mathcal{H}$-independent* if and only if the equality

$$\sum_{k=1}^{d} c_k\varphi_k \in \mathcal{H},\ c_k \in \mathbb{C},$$

implies $c_1 = c_2 = \cdots = c_d = 0$. If a desired set is not $\mathcal{H}$-independent, then the matrix $\mathbf{B}\Gamma\mathbf{B}^*$ will not be invertible and define a degenerate perturbation of rank strictly less than d. For this reason, we consider only $\mathcal{H}$-independent perturbations.

5.1 Singular Finite-Rank Perturbations

The operator $\mathbf{A}_\Gamma$ on the domain $\mathrm{Dom}(\mathbf{A})$ is symmetric as an operator acting from $\mathcal{H}_2(\mathbf{A}) = \mathrm{Dom}(\mathbf{A})$ to $\mathcal{H}_{-2}(\mathbf{A})$. The self-adjoint operator given by Eq. (5.1) coincides with one of the self-adjoint extensions of the operator $\mathbf{A}^0$ equal to the operator $\mathbf{A}$ restricted to the domain

$$\mathrm{Dom}(\mathbf{A}^0) = \mathrm{Dom}(\mathbf{A}) \cap \mathrm{Ker}(\mathbf{B}\Gamma\mathbf{B}^*).$$

On the side we mention that $\mathrm{Ker}(\mathbf{B}\Gamma\mathbf{B}^*) = \mathrm{Ker}(\mathbf{B}^*)$, because we are assuming Γ to be invertible and $\mathcal{H}$-independence of φ_k.

Lemma 5.1 ([6, Lemma 3.1.1]) *Suppose that the vectors $\varphi_k \in \mathcal{H}_{-2}(\mathbf{A})\backslash\mathcal{H}$, $k = 1, \dots, d$, are $\mathcal{H}$-independent and form an orthonormal system in $\mathcal{H}_{-2}(\mathbf{A})$. Then the restriction $\mathbf{A}^0$ of the operator $\mathbf{A}$ to the domain $\mathrm{Dom}(\mathbf{A}^0)$ is a densely defined symmetric operator with the deficiency indices (d, d).*

Note that the vectors φ_k having unit norm in $\mathcal{H}_{-2}(\mathbf{A})$ is not a restriction, as every $\mathcal{H}$-independent system $\{\varphi_k\}$ can be orthonormalized. We assume unit norm in the following discussions and results.

If we let the vectors φ_k, $k = 1, \dots, d$ be $\mathcal{H}$-independent, all vectors $\psi \in \mathrm{Dom}(\mathbf{A}^{0*})$ can be represented as:

$$\psi = \widehat{\psi} + \sum_{k=1}^{d} \Big(a_{+k}(\psi)(\mathbf{A} - iI)^{-1}\varphi_k + a_{-k}(\psi)(\mathbf{A} + iI)^{-1}\varphi_k \Big), \tag{5.2}$$

where $\widehat{\psi} \in \mathrm{Dom}(\mathbf{A}^0)$, $a_\pm(\psi) \in \mathbb{C}$.

The theory of self-adjoint extensions of symmetric differential operators, commonly referred to as Glazman–Krein–Naimark theory [4, 64], should be compared to this setup. The $\mathrm{Dom}(\mathbf{A}^0)$ should be thought of as a "minimal" domain for the operator $\mathbf{A}$, as the domain is unaffected by the perturbation $\mathbf{B}\Gamma\mathbf{B}^*$ and will be contained in the domains of all extensions. Likewise, the "maximal" domain is represented by $\mathrm{Dom}(\mathbf{A}^{0*})$ and Eq. (5.2) is a modified version of the classical von Neumann's formula (the maximal domain is the direct sum of the minimal domain and the defect spaces). The key space $\mathrm{Dom}(\mathbf{A}^{0*})$ should thus be considered as a finite dimensional extension of the space $\mathcal{H}_2(\mathbf{A})$ in the sense that $\mathrm{Dom}(\mathbf{A}^{0*})$ is isomorphic to the direct sum of $\mathcal{H}_2(\mathbf{A})$ and $\mathbb{C}^d$.

We also emphasize that the spaces $\mathbf{A}^0$, defined via Lemma 5.1, and $\mathbf{A}^{0*}$, are dependent on the choice of the vectors $\{\varphi_k\}_{k=1}^d$. We can thus formulate a second scale of Hilbert spaces

$$\mathrm{Dom}(\mathbf{A}) = \mathcal{H}_2(\mathbf{A}) \subset \mathrm{Dom}(\mathbf{A}^{0*}) \subset \mathcal{H} \subset \mathrm{Dom}(\mathbf{A}^{0*})^* \subset \mathcal{H}_{-2}(\mathbf{A}) = \mathrm{Dom}(\mathbf{A})^*,$$

which is constructed using both the operators $\mathbf{A}$ and $\mathbf{B}\Gamma\mathbf{B}^*$. The norms in $\mathcal{H}_{-2}(\mathbf{A})$ and $\mathcal{H}_2(\mathbf{A})$ are the standard norms from Definition 3.1. We avoid most of the specific properties of these spaces and operators, but point out that the norm in the space $\text{Dom}(\mathbf{A}^{0*})^*$ is listed in [6, Equation (3.11)], near other pertinent facts.

5.2 Self-Adjoint Extensions

The self-adjoint finite-rank perturbation given by (5.1) can be adapted as an application to self-adjoint extension theory. Namely, self-adjoint extensions of the operator $\mathbf{A}^0$ are parametrized by $d \times d$ unitary matrices by the classical Glazman–Krein–Naimark theory [4, 64]. Let V be such a matrix and the vector notation $\vec{a}_\pm \equiv \{a_\pm\}_{k=1}^d$ denote the coefficients from Eq. (5.2). The corresponding self-adjoint operator $\mathbf{A}(V)$ coincides with the restriction of the operator $\mathbf{A}^{0*}$ to the domain

$$\text{Dom}(\mathbf{A}(V)) = \{\psi \in \text{Dom}(\mathbf{A}^{0*}) \ : \ -V\vec{a}_-(\psi) = \vec{a}_+(\psi)\}. \tag{5.3}$$

We present an explicit connection between V and Γ in Lemma 5.3 below.

The extension given by the matrix $V = I$ coincides with the original operator $\mathbf{A}$. This case is handled by classical self-adjoint extension theory. However, when the perturbing vectors $\{\varphi_k\}_{k=1}^d$ belong to $\mathcal{H}_{-1}(\mathbf{A})$, descriptions of the corresponding domains become more difficult.

Theorem 5.2 ([6, Theorem 3.1.1]) *Let $\varphi_k \in \mathcal{H}_{-1}(\mathbf{A})\backslash\mathcal{H}$ be an $\mathcal{H}$-independent basis such that $\langle(\mathbf{A} - iI)^{-1}\varphi_j, (\mathbf{A} + iI)^{-1}\varphi_k\rangle = \delta_{jk}$, and let Γ be a Hermitian invertible matrix. Then the self-adjoint operator $\mathbf{A}_\Gamma = \mathbf{A} + \mathbf{B}\Gamma\mathbf{B}^*$ is the self-adjoint restriction of the operator $\mathbf{A}^{0*}$ to the following domain*

$$\begin{aligned}&\text{Dom}(\mathbf{A}_\Gamma)\\ =&\{\psi \in \text{Dom}(\mathbf{A}^{0*}) \ : \ \vec{a}_+(\psi) = -(\Gamma^{-1} + \mathbf{F}(i))^{-1}(\Gamma^{-1} - \mathbf{F}^*(i))\vec{a}_-(\psi)\},\end{aligned}$$

where $\mathbf{F}(i) = \mathbf{B}(\mathbf{A}_\Gamma - iI)^{-1}\mathbf{B}^*$.

The notation $\mathbf{F}(i)$ comes from the Borel transform, which we focus on in Sect. 6.

We have $\mathbf{A}_0 = \mathbf{A}$ when $\Gamma = 0$. Further note that the matrix $V = (\Gamma^{-1} + \mathbf{F}(i))^{-1}(\Gamma^{-1} - \mathbf{F}^*(i))$ is unitary. Hence, the theorem says that if the vectors φ_j φ_j and the desired perturbation Γ are known, then the domain of the self-adjoint extension can be written via the explicit unitary matrix V, as in the classical theory.

However, this leads to the natural question: Given the domain of a self-adjoint extension in terms of V, can we recover the perturbation Γ responsible for this domain? The answer is given by the following result.

Lemma 5.3 ([6, Lemma 3.1.2]) *Let $\varphi_k \in \mathcal{H}_{-1}(\mathbf{A})\backslash\mathcal{H}$, $k = 1, \ldots, d$, be an $\mathcal{H}$-independent orthogonal system. If*

$$\det\Big(V + [iI + \mathrm{Re}(\mathbf{F}(i))]^{-1}[iI - \mathrm{Re}(\mathbf{F}(i))]\Big) \neq 0$$

then the operator $\mathbf{A}^{0*}$ restricted to the domain of functions

$$\{\psi \in \mathrm{Dom}(\mathbf{A}^{0*}) \ : \ -V\vec{a}_-(\psi) = \vec{a}_+(\psi)\}$$

is a finite dimensional additive perturbation of the operator $\mathbf{A}$. In particular, the Hermitian invertible matrix Γ is given by

$$\Gamma = \Big(-\mathrm{Re}(\mathbf{F}(i)) + i(I - V)^{-1}(I + V)\Big)^{-1}.$$

The last formula necessitates the analysis of whether $I - V$ is invertible. This distinction is handled in the proof, where it is determined that if $I - V$ is not invertible, then there is a degeneracy in the choice of the vectors φ_k. This means that the set of vectors $\{\varphi_k\}_{k=1}^d$ contains extra elements because we can find a new set of elements $\{\varphi_k^*\}_{k=1}^{d^*}$, $d^* < d$, such that the corresponding matrix V^* has a trivial eigensubspace.

The description of domains of self-adjoint extensions resulting from finite-rank perturbations with vectors from $\mathcal{H}_{-2}(\mathbf{A})$ are much more involved (see e.g. [7]), and while very interesting in their own right, fall outside the scope of our discussion.

5.3 Some Applications of Singular Finite Rank Perturbations

The singular finite-rank perturbation setup employed in this section has a wide array of applications. Perhaps, their most common uses include point interactions for differential operators via connections to distribution theory and singular potentials of Schrödinger operators. This is immediately evident from the rank-one case when considering changing boundary conditions of regular Sturm–Liouville operators, see [76, Section 11.6].

Several contributions to the finite-rank case can be found in [6]. These include the analysis of operators with generalized delta interactions to achieve both spectral and scattering results. It is also possible to consider infinite-rank perturbations, under some simplifying assumptions, to help approach problems given by two-body, three-body and few-body models.

Finally, we should mention that singular perturbations can be transcribed into the theory of rigged Hilbert spaces, i.e. [48]. This theory places a larger emphasis on properties of singular quadratic forms, which can also describe self-adjoint extensions. Specific extensions, such as the Friedrichs or von Neumann–Krein cases, are sometimes easier to formulate in this context. Various aspects of spectral theory for singular finite-rank perturbations of self-adjoint operators are detailed in [48, Section 9].

6 Spectral Theory of Self-Adjoint Finite-Rank Perturbations

Consider the family of finite-rank perturbations $\mathbf{A}_\Gamma = \mathbf{A} + \mathbf{B}\Gamma\mathbf{B}^*$, see (1.1), with cyclic subspace Ran $\mathbf{B}$. It is well-known that Ran $\mathbf{B}$ is then also cyclic for A_Γ for all symmetric Γ. For simplicity let us focus on bounded perturbations in this section. By the Spectral Theorem, this perturbation family corresponds to a family of matrix-valued spectral measures $\boldsymbol{\mu}_\Gamma$ through

$$\mathbf{B}^*(\mathbf{A}_\Gamma - zI)^{-1}\mathbf{B} = \int_{\mathbb{R}} \frac{d\boldsymbol{\mu}_\Gamma(t)}{t-z} \qquad \text{for } z \in \mathbb{C} \setminus \mathbb{R}.$$

The right hand side is the matrix-valued Borel transform, $\mathbf{F}_\Gamma(z) := \int_{\mathbb{R}}(t - z)^{-1} d\boldsymbol{\mu}_\Gamma(t)$. We obtain the *scalar spectral measures* μ_Γ by taking the trace of $\boldsymbol{\mu}_\Gamma$. This trace is a scalar-valued measure which recovers the spectrum of $\mathbf{A}_\Gamma$ via $\sigma(\mathbf{A}_\Gamma) = \operatorname{supp} \mu_\Gamma$. However, to access more subtle information, we formulate some of the results of the field we define the family of matrix-valued functions W_Γ by $d\boldsymbol{\mu}_\Gamma(t) = W_\Gamma(t) d\mu_\Gamma(t)$. Finally, we arrive at $(W_\Gamma)_{\text{ac}} := d\boldsymbol{\mu}_\Gamma/dx$ by taking a component-wise Randon–Nikodym derivative.

6.1 Absolutely Continuous Spectrum and Scattering Theory

The unitary equivalence of the absolutely continuous spectrum of operators that differ by a finite-rank perturbations is available through simply applying the Kato–Rosenblum Theorem 2.2. In the more general setting of compact perturbations, the standard proof relies on the existence of the wave operators. Namely, let P_{ac} denote the orthogonal projection from the Hilbert space onto the absolutely continuous part of A. For self-adjoint A and compact self-adjoint K it was shown that the strong operator topology limit of $e^{it(A+K)}e^{-itA}P_{\text{ac}}$ exists (see [72, Theorem 1.6] and [46, Theorem 1]), which in turn yields the Kato–Rosenblum theorem.

For finite-rank perturbations, a quicker proof of unitary equivalence of the absolutely continuous spectrum is available. This proof uses the Aronszajn–Krein relation

$$\mathbf{F}_\Gamma = (I + \mathbf{F}\Gamma)^{-1}\mathbf{F} = \mathbf{F}(I + \mathbf{F}\Gamma)^{-1}, \tag{6.1}$$

of the matrix-valued Borel transforms and was first discovered by Kuroda [51]. Via efficient notation, and in a slightly different language, Liaw and Treil [56, Appendix A.1] present this proof in a format appropriate for a graduate course.

Of course, scattering theory is able to give us more information by relating how wave operators, e.g. $s - \lim_{t\to\infty} e^{it(A+K)}e^{-itA}P_{\mathrm{ac}}$, and their packets are affected by the perturbation. For an interesting exposition of scattering theory for finite-rank perturbations confer, e.g. [50, Ch. 4]. Applications in Mathematical Physics can also be found in [50, Ch. 5–7]. Alternatively, the scattering theory of finite-rank perturbations can be analyzed using boundary triples, see e.g. [14].

Validating the observation that the behavior of the absolutely continuous spectrum is one of the easier objects to capture, we conclude this subsection with its full perturbation theoretic characterization. The density of the matrix-valued spectral measure of the perturbed operator $(W_\Gamma)_{\mathrm{ac}}$ is determined (see [56, Lemma A.3]) in terms of that of the unperturbed operator W_{ac} by

$$\left(W_\Gamma\right)_{\mathrm{ac}}(x) = \lim_{y\to 0^+}(I + \mathbf{F}(x+iy)^*\Gamma)^{-1}W_{\mathrm{ac}}(x)\lim_{y\to 0^+}(I + \Gamma\mathbf{F}(x+iy))^{-1},$$

with respect to Lebesgue a.e. $x \in \mathbb{R}$.

In Eq. (8.1) below, we also include a full description of the perturbed operator's matrix-density in terms of the matrix characteristic function of a corresponding model representation.

6.2 *Vector Mutually Singular Parts*

As evidenced by much research in the field, working with the singular spectrum will require a more subtle analysis than is necessary for the absolutely continuous part. From a naive perspective, the task at hand is to attempt to obtain some information about non-tangential boundary values $z \to \lambda$ of matrix-valued analytic functions on $\mathbb{D}$ for $(\mu_\Gamma)_{\mathrm{s}}$-a.e. $\lambda \in \mathbb{T}$. As we discuss in Sect. 8.2, Poltoratski's Theorem does not hold in the matrix-valued setting. Yet some positive results prevail.

Recall the Aronszajn–Donoghue Theorem, which states the mutual singularity of the singular parts under rank-one perturbations, see item (iii) of Theorem 3.3. For finite-rank perturbations it is easy to construct examples for which two different perturbed operators have the same eigenvalue by taking direct sums of rank-one perturbations. The eigenvalues of the different components are completely independent from one another. Hence, a literal extension of this Aronszajn–Donoghue result cannot be true for the scalar-valued spectral measure. Through defining a vector-valued analog of the mutual singularity of matrix measures, Liaw–Treil [56, Theorem 6.2] achieved such a generalization of the Aronszajn–Donoghue Theorem. The scalar-valued spectral measures are also restricted:

Theorem 6.1 ([56, Theorem 6.3]) *Fix a singular scalar Radon measure ν, and $d \times d$-matrices $\Gamma > 0$ and self-adjoint Γ_0. Then the scalar spectral measures of*

$\mathbf{A}_{\Gamma_0+t\Gamma}$ *are mutually singular with respect to* ν *for all except maybe countably many* $t \in \mathbb{R}$.

6.3 Equivalence Classes and Spectral Multiplicity

In [35], Gesztesy–Tsekanovskii obtained structural results for Nevanlinna–Herglotz functions that are applicable to finite-rank perturbations. Under the assumption that $\ker(I + \Gamma\mathbf{F}(z)) = \{0\}$ for all $z \in \mathbb{C}_+$, some of these results resemble the Kato–Rosenblum Theorem 2.2 and Aronszajn–Donoghue Theorem 3.3.

We begin by introducing the following sets, where $1 \le r \le d$:

$$S_{\mathrm{r}}\,(\boldsymbol{\mu})_{\mathrm{ac}} = \Bigg\{ x \in \mathbb{R} \,\Bigg|\, \lim_{y\to 0^+} \mathbf{F}(x+iy) \text{ exists finitely, and}$$

$$\lim_{y\to 0^+} \operatorname{rank}(\operatorname{Im}(\mathbf{F}(x+i0))) = r \Bigg\},$$

$$S\,(\boldsymbol{\mu})_{\mathrm{ac}} = \bigcup_{r=1}^{d} S_{\mathrm{r}}\,(\boldsymbol{\mu})_{\mathrm{ac}}\,.$$

Here, the existence of matrix limits are understood entrywise. Consider the equivalence classes of $S_{\mathrm{r}}(\boldsymbol{\mu}_\Gamma)_{\mathrm{ac}}$ and $S(\boldsymbol{\mu}_\Gamma)_{\mathrm{ac}}$ associated with $\mathbf{F}_\Gamma(z)$; and denote them by $E_r(\boldsymbol{\mu}_\Gamma)_{\mathrm{ac}}$ and $E(\boldsymbol{\mu}_\Gamma)_\Gamma)_{\mathrm{ac}}$, respectively.

In this setting, Gesztesy–Tsekanovskii [35, Theorem 6.6][1] have shown that:

1. For $1 \le r \le d$, the classes $E_r(\boldsymbol{\mu}_\Gamma)_{\mathrm{ac}}$, and $E(\boldsymbol{\mu}_\Gamma)_{\mathrm{ac}}$ are independent of Γ.
2. Suppose $\boldsymbol{\mu}_{\Gamma_1}$ is a discrete point measure for some Γ_1. Then $\boldsymbol{\mu}_\Gamma$ is a discrete point measure for all Γ.
3. The set of those $x \in \mathbb{R}$ for which, simultaneously, there is no Γ such that $\lim_{y\to 0^+} \operatorname{Im}\big(\mathbf{F}_\Gamma(x+iy)\big)$ exists and $\lim_{y\to 0^+} \det\big(\operatorname{Im}\big(\mathbf{F}_\Gamma(x+iy)\big)\big) = 0$, is a subset of $E_d(\boldsymbol{\mu}_\Gamma)_{\mathrm{ac}}$.

7 Model Theory of Finite-Rank Unitary Perturbations

Taking a different route than Clark theory, we follow [55] to set up the problem. This perspective is more natural here, since we are interested in perturbation theory. It allows us to bypass some minor technical road blocks that arise for finite-rank

[1] Gesztesy–Tsekanovskii present these results for a slightly more general setting, when $\mathbf{F}$ and $\mathbf{F}_\Gamma$ are related by a certain linear fractional transformation. Their presentation reduces to ours upon making the choices $\Gamma_{1,1} = \Gamma_{2,2} = I$ and $\Gamma_{2,1} = \mathbf{0}$ and $\Gamma_{1,2} = \Gamma$.

perturbations (when connecting the family measures with the family of operators). Some of the model theory of rank-one perturbations carries over to model theory of finite-rank setting with the added complication that one has to keep track of the order of matrix products. For example, the description of the absolutely continuous part in terms of the characteristic function has an analog for finite-rank perturbations. Other results such as identifying when the extreme situation occurs (when the de Branges–Rovnyak transcription simplifies) need to be slightly adjusted. For this particular question, taking the trace will be appropriate.

In Sect. 7.2 we briefly mention some other representations using Krein spaces and reproducing kernel Hilbert spaces.

7.1 Setup and Model Spaces

Recall the setting for unitary finite-rank perturbations $\mathbf{U}_\alpha = \mathbf{U} + \mathbf{J}(\alpha - I)\mathbf{J}^*\mathbf{U}$ with unitary α, as detailed in and around (1.3). It is well-known that $\operatorname{Ran}\mathbf{J}$ also forms a $*$-cyclic subspace for the perturbed operators $\mathbf{U}_\alpha$. Let $\boldsymbol{\mu}_\alpha$ be the family of matrix-valued spectral measures on $\mathbb{T}$ given by the Spectral Theorem through

$$\mathbf{J}^*(I - z\mathbf{U}_\alpha^*)^{-1}\mathbf{J} = \int_{\mathbb{T}} \frac{d\boldsymbol{\mu}_\alpha(\zeta)}{1 - z\bar{\zeta}} \qquad \text{for } z \in \mathbb{C} \setminus \mathbb{T}. \tag{7.1}$$

It is not hard to see that the operator $\mathbf{U}_\alpha$ is a completely non-unitary contraction for matrices α with $\|\alpha\| < 1$. This provides us access to the associated model theory. Referring the reader to [55, Sections 3 and 4], we omit the details of showing that operator $\mathbf{U}_\mathbf{0}$ corresponds to the matrix-valued characteristic function

$$\boldsymbol{\theta}(z) = (K\boldsymbol{\mu}(z) - I)(K\boldsymbol{\mu}(z))^{-1}. \tag{7.2}$$

Here, the identity I maps $\mathcal{D} \to \mathcal{D}$ and K is the Cauchy transform of a matrix-valued measure $(K\boldsymbol{\nu})(z) = \int_{\mathbb{T}} \frac{d\boldsymbol{\nu}(\zeta)}{1-z\bar{\zeta}}$.

It is not hard to see that the relation in (7.2) is equivalent to the Herglotz formula

$$(H\boldsymbol{\mu})(z) = (I + \boldsymbol{\theta}(z))(I - \boldsymbol{\theta}(z))^{-1}, \tag{7.3}$$

with the Herglotz transformation of a matrix-valued measure $(H\boldsymbol{\nu})(z) = \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} d\boldsymbol{\nu}(\zeta)$. Now, one can reason that replacing $\boldsymbol{\mu}$ by $\boldsymbol{\mu}_\alpha$ in (7.3) will result in replacing $\boldsymbol{\theta}$ by $\boldsymbol{\theta}\alpha^*$. And we arrive at the starting point of Aleksandrov–Clark Theory, see e.g. [62, Eq. (2.5)] when $\boldsymbol{\theta}(0) = \mathbf{0}$.

It is worth mentioning that in starting with (1.3) we do not really make a hidden assumption. We would recover the general starting point of [62] by taking $\mathbf{U}_\alpha$ with strict contraction α instead of $\mathbf{U}_\mathbf{0}$ with $\boldsymbol{\alpha} = \mathbf{0}$. As when dealing with rank-one perturbations, operator $\mathbf{U_0}$ is unitarily equivalent to the compressed shift operator on a transcription free model space.

Similar to the rank-one setting, here, the Sz.-Nagy–Foiaş model space reduces to $H^2(\mathbb{C}^d) \ominus \boldsymbol{\theta} H^2(\mathbb{C}^d)$, if and only if $\boldsymbol{\theta}$ is inner (i.e. has non-tangential boundary values that are unitary with respect to Lebesgue measure a.e. on $\mathbb{T}$), if and only if $\mathbf{U}$ has purely singular spectrum. See e.g. [55, Corollary 5.8] for a reference of the second equivalence. Also see [26].

The de Branges–Rovnyak model space reduces to one-story if and only if $\boldsymbol{\theta}$ is an extreme point, and if and only if $\int_{\mathbb{T}} \operatorname{tr}(\ln(I - |\boldsymbol{\theta}(z)|)dm(z) = -\infty$ (see [62, Theorem 4.3.1]). There seems to be no immediate description of the extreme property in terms of the operator $\mathbf{U}$ or the perturbation family $\mathbf{U}_\alpha$.

In any case, the de Branges–Rovnyak model space reduces at times when Sz.-Nagy–Foiaş model does not. When dealing with the general case of finite-rank unitary perturbations, no such reduction can be assumed a priori. This general case is the subject of Liaw–Treil [55] and some of Martin [62] holds in this generality.

In [55] Liaw–Treil study the general Nikolski–Vasyunin model of finite-rank Aleksandrov–Clark perturbations. Determining the unitary operator realizing this representation yields a generalization of the Clark-type operator and its adjoint. For the adjoint, the transcription choice leading to the full Sz.-Nagy–Foiaş model features a generalization of the normalized Cauchy transform.

7.2 *Krein Spaces and Reproducing Kernel Hilbert Spaces in Applications*

Krein spaces are indefinite inner product spaces; spaces which possess a Hermitian sesquilinear form that allows elements to have positive or negative values for their "norm." A Hilbert inner product can be canonically defined on Krein spaces, so they can be viewed as a direct sum of Hilbert spaces [18]. In particular, Krein spaces are naturally defined as extension spaces for symmetric operators with equal deficiency indices and have their own tools to determine spectral properties. Applications to the spectral analysis of direct sums of indefinite Sturm–Liouville operators is possible because so-called definitizable operators in Krein spaces are stable under finite-rank perturbations [13]. Furthermore, compact perturbations of self-adjoint operators in Krein spaces also preserve certain spectral points [10], and the spectral subspaces corresponding to sufficiently small surrounding neighborhoods of these points are actually Pontryagin spaces (simpler versions of Krein spaces).

Representations of symmetric operators with equal deficiency indices are also possible in reproducing kernel Hilbert spaces; Hilbert spaces of functions where point evaluation is a continuous linear functional. Among other results, Aleman–Martin–Ross [9] carried out representations for Sturm–Liouville and Schrödinger (differential) operators, Toeplitz operators and infinite Jacobi matrices. The idea becomes that for each such example, the structure of the model space hosts the full information (including spectral properties) of the symmetric operator. In [9, Section 5], the characteristic functions corresponding to these examples are computed explicitly; so that the de Branges–Rovnyak model space (which is a reproducing kernel Hilbert spaces) is completely determined.

Representations in the Herglotz space constitute another interesting topic in [9].

8 Spectral Theory of Finite-Rank Unitary Perturbations

Consider the setting of Sect. 7.1. Recall that $\mathbf{U}_\alpha$ for unitary α is a unitary rank d perturbation of a unitary operator $\mathbf{U}$, and recall that (7.1) defines the family of associated matrix-valued spectral measures $\boldsymbol{\mu}_\alpha$. In analogy to the self-adoint setting, we define the family of matrix-valued functions W_α by $d\boldsymbol{\mu}_\alpha(t) = W_\alpha(t)d\mu_\alpha(t)$. Taking a component-wise Randon–Nikodym derivative, we arrive at $(W_\alpha)_{\rm ac} := d\boldsymbol{\mu}_\alpha/dx$. Further, recall that $\boldsymbol{\theta}$ is the matrix-valued characteristic function of the completely non-unitary contraction $\mathbf{U_0}$, and that $\Delta(z) = (I - \boldsymbol{\theta}^*(z)\boldsymbol{\theta}(z))^{1/2}$.

A complete explicit description of the matrix-valued spectral measures of $\mathbf{U}_\alpha$ in terms of the characteristic function is currently not available. In fact, the theory for finite-rank perturbations is lagging behind what is known for rank-one perturbations, see Theorem 4.2. This problem has been in recent years and continues to be a field of active study. Here we explain some results in this direction.

8.1 Spectral Properties in Terms of the Characteristic Function

The location of the spectrum of the perturbed operator is captured by:

Theorem 8.1 (See Mitkovski [63, Corollary 4.4]) *The spectrum of* $\mathbf{U}_\alpha$ *consists of those points* $\lambda \in \mathbb{T}$ *at which either* $\boldsymbol{\theta}$ *cannot be analytically continued across* λ*, or* $\boldsymbol{\theta}(\lambda)$ *is analytically continuable with* $\boldsymbol{\theta}(\lambda) - \alpha$ *not invertible.*

In combination with von Neumann's theorem, Theorem 2.1, a characterization by Lifshitz [59, Theorem 4] of the essential spectrum of $U_\mathbf{0}$ says that it consists of those points $\lambda \in \mathbb{T}$ for which (at least) one of the following conditions fails:

- $\boldsymbol{\theta}$ is analytic on some open neighborhood of λ,
- there is a neighborhood N_λ of λ so that $\boldsymbol{\theta}$ is unitary for all $\lambda \in N_\lambda \cap \mathbb{T}$.

For the absolutely continuous part of the perturbed operator's spectral measure, a full matrix-version becomes available upon combination of Liaw–Treil [55, Theorem 5.6] with the Herglotz formula (7.3) for $\mathbf{U}_\alpha$, which is obtained from that for $\mathbf{U}$ by simultaneously replacing $\boldsymbol{\mu}$ by $\boldsymbol{\mu}_\alpha$ and $\boldsymbol{\theta}$ by $\boldsymbol{\theta}\alpha^*$. Namely, we have

$$(I_n - \alpha b(\lambda)^*)W_\alpha(\lambda)(I_n - b(\lambda)\alpha^*) = (\Delta(\lambda))^2 \qquad \text{for Lebesgue a.e. } \lambda \in \mathbb{T}, \tag{8.1}$$

in the sense of non-tangential boundary limits. In particular, for the absolutely continuous part, the multiplicity function is given by a non-tangential limit rank $(W_\alpha(\lambda))_{\rm ac} = \operatorname{rank} \lim_{z\to\lambda}$. Slightly weaker results are contained in Douglas–Liaw [26].

8.2 Singular Part in Terms of the Characteristic Function

As for rank-one perturbations, capturing the singular part is a more difficult venture. The main problem here is that Poltoratski's Theorem requires a major adjustment (see Sect. 8.2 for a discussion). As a result, a description of the singular part in terms of the characteristic function is still outstanding.

For regular points (i.e. those that lie in the complement of the essential spectrum), both eigenvalues and eigenvectors of $\mathbf{U}_\alpha$ are described in Martin [62, Proposition 5.2.2]. Namely, a regular point $\lambda \in \mathbb{T}$ is an eigenvalue of $\mathbf{U}_\alpha$ if and only if $\lim_{z\to\lambda}(\alpha\boldsymbol{\theta}^*(z) - \mathbf{U}^*)$ exists and is not invertible. Eigenvectors are those functions $\chi_{\{\lambda\}}\mathbf{x}$ with $\mathbf{x} \in \mathbb{C}^d \cap \ker(\alpha\boldsymbol{\theta}^*(\lambda) - \mathbf{U}^*)$ and where χ denotes the characteristic function. In that same proposition, a necessary and sufficient condition is provided for a point to not be an eigenvalue of $\mathbf{U}_\alpha$ for any unitary α.

There are many open questions remaining in this area. Some of them were subject of investigation in [54].

Acknowledgments The authors would like to thank Hari Bercovici and Tracy Weyand for pointing out references and suggesting phrases for some of the wording referring to their field of research. Special thanks to Alan Sola for reading large parts of this survey and for making useful suggestions.

The work of Constanze Liaw was supported by the US National Science Foundation under the grant DMS-1802682.

Appendix: Brief Summaries of Other Closely Related Topics

We discuss Aleksandrov Spectral Averaging and Poltoratski's Theorem. These are both central tools in the field. Thereafter, we briefly illuminate the Simon–Wolff Theorem to which we attribute some of the popularity of the topic among mathematical physicists. We wrap up with a promising direction connecting the field to modern function theoretic operator theory.

Aleksandrov Spectral Averaging

Undoubtedly one of the most celebrated results of the field is the following averaging formula. On the side we mention that we can retrieve restrictions on the Aleksandrov–Clark family of spectral measures, e.g., by choosing the function g to be the characteristic function of a set of Lebesgue measure zero. We being by considering the rank-one setting and will then turn to finite-rank. For part of this subsection we follow [23].

Theorem A.1 (Aleksandrov Disintegration Theorem, See [8] and [23, Theorem 9.4.11]) *For* $g \in L^1(\mathbb{T})$ *we have*

$$\int \left(\int g(\zeta) d\mu_\alpha(\zeta) \right) dm(\alpha) = \int g(\zeta) dm(\zeta). \tag{A.1}$$

For a bounded Borel function f on $\mathbb{T}$, let

$$(Gf)(\alpha) := \int f(\zeta) d\mu_\alpha(\zeta). \tag{A.2}$$

It is one of the main aspects of Theorem A.1 that for $f \in L^1(\mathbb{T})$, the function Gf makes sense for Lebesgue a.e. $\alpha \in \mathbb{T}$ and that it is integrable.

It turns out that G satisfies even more subtle mapping properties. We briefly summarize those before we explain what is known for finite-rank perturbations.

Due to the assumption that $\theta(0) = 0$, we see from Sect. 4.1 that $\|\mu_\alpha\| = 1$ and so

$$\|Gf\|_\infty \le \|f\|_\infty.$$

Note also that the function Gf is continuous whenever f is continuous. The Monotone Class Theorem (see i.e. [23, Theorem 9.4.3]) can be used to show that if f is a bounded Borel function, then Gf is also a bounded Borel function. Hence, the integral

$$\int_{\mathbb{T}} (Gf)(\alpha) dm(\alpha),$$

makes sense. In fact, the transformation G in (A.2) can be extended to many classes of functions. Not only do we have $GC \subset C$, $CL^\infty \subset L^\infty$, and $GL^1 \subset L^1$, but also $GL^p \subset L^p$ $(1 \le p \le \infty)$, $G(BMO) \subset BMO$, $G(VMO) \subset VMO$, and $GB^s_{pq} \subset B^s_{pq}$, where B^s_{pq} are the Besov classes, see [8].

Now, let us turn to what is known about Aleksandrov Spectral Averaging for finite-rank perturbations.

In the unitary setting, a generalization of the Aleksandrov Spectral Averaging formula for continuous functions was obtained in Elliot [28] under extra conditions and in Martin [62, Theorem 3.2.3].

For self-adjoint operators a Aleksandrov-type Spectral Averaging formula was proved, Liaw–Treil [56, Theorems 4.1, 4.6]. These formulas imply restrictions on the singular parts of families of Aleksandrov–Clark measures.

Aleksandrov Spectral Averaging for the Drury–Arveson space in the setting for inner characteristic functions was achieved by Jury [39, Theorem 2.9]. For more on Aleksandrov–Clark theory for the Drury–Arveson space see Sect. 8.2 and the references therein.

Poltoratski's Theorem

Deep at the heart of many results in Aleksandrov–Clark theory lies the celebrated result (proved by Poltoratski in [69], also see [23, Section 10.3]) stating that for a Radon measure τ on $\mathbb{T}$ and $f \in L^2(\tau)$ the normalized Cauchy transform $\frac{\mathcal{C}f\tau(z)}{\mathcal{C}\tau(z)}$ possesses non-tangential boundary values $z \to \lambda$ for τ_s-a.e. $\lambda \in \mathbb{T}$. This result is so important, because it empowers us to study the behavior of the spectral measure on sets that are of Lebesgue measure zero. In particular, one can sometimes use Poltoratski's Theorem to retrieve information about the singular parts of the spectral measures.

Direct sum examples of scalar characteristic functions immediately show that a literal extension of the statement of Poltoratski's Theorem is not possible to the finite-rank setting. Nonetheless, Kapustin–Poltoratski [44, Theorem 3] have proved a finite-rank analog which features a matrix-valued numerator alongside a scalar-valued denominator as well as an multiplication by a left inverse of the coordinate map $\mathbf{J}$ in (1.3). This left inverse 'automatically' annihilates directions in which the limit of the ratio does not exist.

Simon–Wolff Criterion

In [75, Theorem 3 of Section 2] Simon and Wolff provided a characterization—formulated in terms of the spectral measure μ—of when rank-one perturbation problems A_γ are pure point for Lebesgue a.e. parameters $\gamma \in \mathbb{R}$. They applied their result to showing that the one-dimensional discrete random Schrödinger operator exhibits so-called Anderson localization, see [37, 78]. The idea of the Simon–Wolff localization proof was to sweep through the parameter domain for the perturbed operators' random coupling constants.

In Poltoratski [70] the Simon–Wolff Theorem was extended from the rank-one to the finite-rank setting.

Simon–Wolff's celebrated work initiated further applications of rank-one perturbations to a generalization of random Schrödinger operators called Anderson-type Hamiltonians, see e.g. [37]. Anderson-type Hamiltonians are obtained from perturbing a self-adjoint operator by countably infinitely many rank-one perturbations, each coupled by a random variable. More concretely, they are of the form $A_\omega = A + \sum \omega_i \langle \cdot, \varphi_i \rangle \varphi_i$, where $\{\varphi_i\}$ forms an orthonormal basis of $\mathcal{H}$, and ω_i are independent random variables that are chosen in accordance with an identical probability distribution. In view of Sect. 2, the fact that the discrete random Schrödinger operator features an almost surely non-compact perturbation operator underlines the level of difficulty in dealing with such objects.

In Jaksiç–Last [37, 38], these methods are utilized to prove the almost sure cyclicity of the singular spectrum of the Anderson-type Hamiltonian. And in Abakumov–Liaw–Poltoratski [1], it is shown that under some condition any non-trivial vector is cyclic. In Liaw [52] these results are applied to numerically support a delocalization conjecture for the two-dimensional discrete random Schrödinger operator.

Functions of Several Variables

Recall that the Drury–Arveson space $H^2(\mathbb{B}^n)$ is the reproducing kernel Hilbert space of functions on the open unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$, $n \in \mathbb{N}$, that arises from the reproducing kernel $k(z, w) = (1 - \langle z, w \rangle_{\mathbb{C}^n})^{-1}$ with $z, w \in \mathbb{B}^n$.

Jury [39] extended much of the de Branges–Rovnyak construction of Clark theory to $H^2(\mathbb{B}^n)$. As before, the de Branges–Rovnyak model spaces $\mathcal{H}(\boldsymbol{\theta})$ are contractively contained in $H^2(\mathbb{B}^n)$. The family of Clark measure is replaced by a family of states on some noncommutative operator system. The backward shift is replaced by a canonical solution to the Gleason problem in $\mathcal{H}(\boldsymbol{\theta})$. An extension of some of Jury's work to non-inner but so-called quasi-extreme characteristic functions was carried out in Jury–Martin [40]. There, the Aleksandrov–Clark measures are necessarily generalized to certain positive linear functionals. For related work on function analytic noncommutative operator theory, we refer the reader to a series of papers by Jury and Martin [41–43].

On the side we mention that it is not immediately clear whether a perturbation problem corresponds to this Aleksandrov–Clark theory for functions of several variables.

The state of affairs for self-adjoint finite-rank perturbation problems is similar. The conditions and explicit formulas necessary to pose a well-defined problem in this area have not been investigated, to the best knowledge of the authors. However, there exists a generalization of Nevanlinna–Herglotz functions to several variables (see e.g. [61]) whose integral representation should form a framework for the analog of the Borel transform in (3.2).

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Invariance of the Essential Spectra of Operator Pencils

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Abstract The essential spectrum of operator pencils with bounded coefficients in a Hilbert space is studied. Sufficient conditions in terms of the operator coefficients of two pencils are derived which guarantee the same essential spectrum. This is done by exploiting a strong relation between an operator pencil and a specific linear subspace (linear relation).

Keywords Operator pencil · Essential spectrum · Linear relations · Singular sequence · Fredholm operator · Pseudo-inverse

Mathematics Subject Classification (2010) 47A06, 47A53, 47A56

1 Introduction

Its is a well-known fact that the essential spectrum of a linear operator is invariant under compact perturbations. Here we understand the essential spectrum as the complement of the (semi-) Fredholm domain. More precisely, we investigate four kinds of essential spectra: the Fredholm essential spectrum, the upper and the lower semi Fredholm essential spectrum and the semi Fredholm essential spectrum. For simplicity, we refer to those four kinds just as the "essential spectra".

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_10

In many applications, e.g. in mathematical physics or in transport theory, one is interested in the (essential) spectrum of operator pencils, see, e.g., [8, 9]. A linear operator pencil is a first order polynomial with bounded operators as coefficients, that is, it is of the form

$$\mathcal{A}_1(\lambda) = \lambda S_1 - T_1,$$

where $\lambda \in \mathbb{C}$ and S_1 and T_1 are bounded operators acting between two normed spaces. By definition (see, e.g., [13, 14]) a complex number λ is in the spectrum of the pencil $\mathcal{A}_1$ if zero is in the spectrum of the operator $\lambda S_1 - T_1$. In the same way the essential spectrum of $\mathcal{A}_1$ is defined as the set of all $\lambda \in \mathbb{C}$ such that the operator $\lambda S_1 - T_1$ is no (semi-) Fredholm operator.

We investigate the question which perturbations of the coefficients do not change the essential spectrum. For this, consider a second operator pencil

$$\mathcal{A}_2(\lambda) = \lambda S_2 - T_2,$$

where S_2 and T_2 are bounded operators acting between the same spaces as S_1 and T_1. If $S_1 - S_2$ and $T_1 - T_2$ are two compact operators, then obviously also the difference

$$\mathcal{A}_1(\lambda) - \mathcal{A}_2(\lambda) = \lambda(S_1 - S_2) - (T_1 - T_2)$$

is compact and, hence, the essential spectra of $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide. But the essential spectrum of two operator pencils may coincide even if the difference of the coefficients is substantial. For example, let M be a bounded and boundedly invertible operator. Then obviously

$$\mathcal{A}_1(\lambda) = \lambda I - T \quad \text{and} \quad \mathcal{A}_2(\lambda) = \lambda M - TM = \mathcal{A}_1(\lambda)M$$

have the same essential spectrum.

Here we make use of the following simple observation: Let $S, T : X \to Y$ be bounded linear operators between two Hilbert spaces X and Y such that the upper semi Fredholm essential spectrum of the pencil $\mathcal{A}(\lambda) := \lambda S - T$ is not $\mathbb{C}$. Then the essential spectra of $\mathcal{A}$ and TS^{-1} coincide (see Corollary 3.5 below). Note, that in general S is not invertible and here S^{-1} and TS^{-1} are understood in the sense of linear relations (or, what is the same, multivalued mappings, see [1, 5, 15]). That is, S^{-1} and TS^{-1} are subspaces of $Y \times X$ and $Y \times Y$, respectively, given by

$$S^{-1} := \{\{Sx, x\} : x \in X\}, \text{ and}$$

$$TS^{-1} := \Big\{\{x, z\} : \{x, y\} \in S^{-1},\ \{y, z\} \in T,\ \text{for some } y \in X\Big\} = \operatorname{ran} \begin{bmatrix} S \\ T \end{bmatrix}.$$

Addition and multiplication of two subspaces are defined in analogy to the addition and multiplication of two linear mappings. In particular, we have for $\lambda \in \mathbb{C}$

$$TS^{-1} - \lambda = \{\{Sx, Tx - \lambda Sx\} : x \in X\}$$

and the notion of (essential) spectrum and resolvent set for linear relations are defined similarly as for linear operators, for details we refer to Sect. 2 below.

Therefore, the relationship of the essential spectra of two linear operator pencils $\mathcal{A}_1$ and $\mathcal{A}_2$ is the same as the relationship of the essential spectra of the linear relations $T_1S_1^{-1}$ and $T_2S_2^{-1}$. Now one can utilize known results for linear relations (see, e.g., [2]): If the difference of the two orthogonal projections onto the subspaces $T_1S_1^{-1}$ and $T_2S_2^{-1}$ is compact, then the essential spectra of the two pencils coincide. This difference can be expressed with the (pseudo-) inverse Z_j of the operator $S_j^*S_j + T_j^*T_j$, $j = 1, 2$, and it has the form

$$\begin{bmatrix} S_1Z_1S_1^* - S_2Z_2S_2^* & S_1Z_1T_1^* - S_2Z_2T_2^* \\ T_1Z_1S_1^* - T_2Z_2S_2^* & T_1Z_1T_1^* - T_2Z_2T_2^* \end{bmatrix}. \tag{1.1}$$

The first main result (cf. Sect. 5 below) shows that if (1.1) is compact then the essential spectra of $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide.

The second main result of this paper (cf. Sect. 5 below) makes use of the so-called singular sequences (cf. Sect. 2 below). If S_1 and S_2 are Fredholm, then the pseudo-inverses $S_1^\dagger$ and $S_2^\dagger$ exist. If, in addition,

$$(T_2 - T_1)S_2^\dagger S_1, \quad (T_2 - T_1)S_1^\dagger S_2, \quad T_1S_2^\dagger(S_1 - S_2) \quad \text{and} \quad T_2S_1^\dagger(S_1 - S_2)$$

are compact, then the upper semi Fredholm essential spectra of $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide. We prove similar results also for the lower semi Fredholm essential spectrum.

2 Preliminaries on Linear Relations

Let X, Y and Z be Banach spaces. The set of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. As usual, we set $\mathcal{L}(X) := \mathcal{L}(X, X)$. A *linear relation* L from X into Y is a subspace of $X \times Y$ and the set of all linear relations from X into Y is denoted by $LR(X, Y)$. Moreover, $CR(X, Y)$ is the set of all closed linear relations from X into Y. Also here, we set $LR(X) := LR(X, X)$ and $CR(X) := CR(X, X)$. Each $T \in \mathcal{L}(X, Y)$ is identified with an element in $CR(X, Y)$ via its graph.

Given a linear relation $L \in LR(X, Y)$, we introduce the following sets:

$$\begin{aligned} \operatorname{dom} L &= \{x \in X \,:\, \{x, y\} \in L \text{ for some } y \in Y\}, \\ \ker L &= \{x \in X : \{x, 0\} \in L\}, \\ \operatorname{ran} L &= \{y \in Y \,:\, \{x, y\} \in L \text{ for some } x \in X\}, \\ \operatorname{mul} L &= \{y \in Y : \{0, y\} \in L\}, \end{aligned}$$

which are called the *domain*, the *kernel*, the *range* and the *multivalued part* of L, respectively. The *inverse* of the linear relation L is given by

$$L^{-1} := \{\{y, x\} \in Y \times X : \{x, y\} \in L\}. \tag{2.1}$$

The linear relation αL with $\alpha \in \mathbb{C}$ is defined by

$$\alpha L := \{\{x, \alpha y\} \in X \times Y : \{x, y\} \in L\}. \tag{2.2}$$

The (operator-like) sum of two linear relations $L, M \in LR(X, Y)$ is defined as

$$L + M := \{\{x, y + y'\} \in X \times Y : \{x, y\} \in L, \{x, y'\} \in M\}. \tag{2.3}$$

If we assume that $X = Y$ then in view of (2.2) and (2.3) we have

$$L - \lambda = L - \lambda I = \{\{x, y - \lambda x\} : \{x, y\} \in L\}. \tag{2.4}$$

The product of two linear relations $L \in LR(Y, Z)$ and $M \in LR(X, Y)$ is defined by

$$LM := \{\{x, z\} \in X \times Z : \{x, y\} \in M,\ \{y, z\} \in L \text{ for some } y \in Y\}.$$

We recall some basic notions from Fredholm theory for linear relations, see [5].

Definition 2.1 Let $L \in LR(X, Y)$. The *nullity* and the *deficiency* of L are defined as follows

$$\operatorname{nul} L := \dim \ker L, \quad \text{and}$$

$$\operatorname{def} L := \operatorname{codim} \operatorname{ran} L := \dim Y/\operatorname{ran} L.$$

If either $\operatorname{nul} L < \infty$ or $\operatorname{def} L < \infty$, we define the *index* of a linear relation as follows

$$\operatorname{ind} L := \operatorname{nul} L - \operatorname{def} L,$$

where the value of the difference is taken to be $\operatorname{ind} L := \infty$ if $\operatorname{nul} L$ is infinite and $\operatorname{ind} L := -\infty$ if $\operatorname{def} L$ is infinite.

Furthermore we define the set of *upper* (*lower*) *semi Fredholm* relations, see e.g. [5],

$$\Phi_+(X, Y) := \{L \in CR(X, Y) : \operatorname{nul} L < \infty \text{ and } \operatorname{ran} L \text{ is closed in } Y\},$$
$$\Phi_-(X, Y) := \{L \in CR(X, Y) : \operatorname{def} L < \infty \text{ and } \operatorname{ran} L \text{ is closed in } Y\},$$

and the set of *Fredholm relations* as

$$\Phi(X, Y) := \Phi_+(X, Y) \cap \Phi_-(X, Y).$$

If $X = Y$, we write briefly $\Phi_+(X)$, $\Phi_-(X)$, and $\Phi(X)$, respectively. The following characterization of $\Phi_+(X, Y)$ is based on [5, Theorem V.1.11].

Proposition 2.2 *Let $L \in CR(X, Y)$ where X and Y are Hilbert spaces, then the following are equivalent:*

(i) $L \notin \Phi_+(X, Y)$.
(ii) *There exists a sequence $(\{x_n, y_n\})$ in L such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$, $x_n \rightharpoonup 0$ and $y_n \to 0$.*
(iii) *There exists a sequence $(\{x_n, y_n\})$ in L such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$, $x_n \rightharpoonup 0$ and $\operatorname{dist}(y_n, \operatorname{mul} L) \to 0$.*

Proof For the proof of (i)⇒(ii), assume first that $\dim\ker L = \infty$ and choose an infinite orthonormal system (x_n) in $\ker L$. Then $\{x_n, 0\} \in L$ is a sequence as required in (ii). Second, assume that $\operatorname{ran} L$ is not closed. Then there exist a sequence (z_n) in $\operatorname{ran} L$ and some $z \in Y \backslash \operatorname{ran} L$ such that $z_n \to z$. Choose $u_n \in (\ker L)^\perp$ such that $\{u_n, z_n\} \in L$ for each $n \in \mathbb{N}$. If (u_n) is bounded, then (u_n) has a subsequence (u_{n_k}) such that $u_{n_k} \rightharpoonup u$ for some $u \in X$. Then the closedness of L and $\{u_{n_k}, z_{n_k}\} \rightharpoonup \{u, z\}$ imply that $\{u, z\} \in L$ and thus $z \in \operatorname{ran} L$, which is a contradiction. Hence, (u_n) is unbounded. It is no restriction to assume that $\|u_n\| \to \infty$ as $n \to \infty$. We set $x_n := u_n/\|u_n\| \in (\ker L)^\perp$ and $y_n := z_n/\|u_n\|$. Then $\{x_n, y_n\} \in L$, $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $y_n \to 0$ as $n \to \infty$. Then a subsequence of (x_n) converges weakly, hence we may assume that $x_n \rightharpoonup x$ for some $x \in (\ker L)^\perp$. As $\{x_n, y_n\} \rightharpoonup \{x, 0\}$ and L is closed, it follows that $x = 0$.

The implication (ii)⇒(iii) is trivial. Thus, let us prove (iii)⇒(i). For this, let $(\{x_n, y_n\}) \subset L$ be a sequence as in (iii). Suppose that $\dim\ker L < \infty$ and that $\operatorname{ran} L$ is closed. Consider the linear relation

$$M := L \cap \left[(\ker L)^\perp \times (\operatorname{mul} L)^\perp\right].$$

Then M is obviously closed and (the graph of) an operator. Moreover, $\ker M = \{0\}$ and $\operatorname{ran} M = \operatorname{ran} L \cap (\operatorname{mul} L)^\perp$ is closed. Hence M, considered as an operator from $\operatorname{dom} M$, equipped with the graph norm, is a bounded upper semi Fredholm operator. Let $x_n = u_n + v_n$ and $y_n = w_n + z_n$, where $u_n \in \ker L$, $v_n \in (\ker L)^\perp$, $w_n \in \operatorname{mul} L$, and $z_n \in (\operatorname{mul} L)^\perp$, $n \in \mathbb{N}$. Then $x_n \rightharpoonup 0$ and $\dim\ker L < \infty$ imply $u_n \to 0$

and $\|v_n\| \to 1$. Also, $\|z_n\| = \operatorname{dist}(y_n, \operatorname{mul} L) \to 0$. We have $\{v_n, z_n\} \in M$, that is, $v_n \in \operatorname{dom} M$ and $Mv_n = z_n \to 0$, which is a contradiction to the fact that M is an upper semi Fredholm operator (cf. [4, XI Theorem 2.5]). □

In what follows, we introduce the adjoint of a linear relation. For this we assume in addition that the spaces X and Y are Hilbert spaces equipped with inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$, respectively. If no confusion arises, we use for simplicity just the notion $(\cdot,\cdot)$. The adjoint L^* of $L \in LR(X,Y)$ is a linear relation from Y to X, defined by

$$L^* = \{\{y, x\} \in Y \times X : (y, v)_Y = (x, u)_X \text{ for all } \{u, v\} \in L\}.$$

Note that always $L^* \in CR(Y, X)$. The following identities for $L \in LR(X, Y)$ are straightforward (see also [15, Section 14.1], [3, Proposition 2.4], and [12])

$$\begin{aligned}
(L^*)^{-1} &= (L^{-1})^*, \\
(\lambda L)^* &= \overline{\lambda} L^*, \quad \lambda \neq 0, \\
\ker L^* &= (\operatorname{ran} L)^\perp, && (2.5) \\
(\operatorname{ran} L^*)^\perp &= \ker \overline{L}, && (2.6) \\
L^* &= -(L^\perp)^{-1}. && (2.7)
\end{aligned}$$

The range of L is closed if and only if the range of L^* is closed, see, e.g. [3, Proposition 2.5]. This together with (2.5) and (2.6) implies that for all $L \in CR(X, Y)$

$$L \in \Phi_\pm(X, Y) \quad \text{if and only if} \quad L^* \in \Phi_\mp(Y, X). \tag{2.8}$$

Next, we define the spectrum of a linear relation and introduce different types of essential spectra as in [16], see also [6] for the operator case.

Definition 2.3 Let $L \in LR(X)$. The *spectrum* and the *resolvent set* of L are defined by

$$\sigma(L) := \{\lambda \in \mathbb{C} : (L - \lambda)^{-1} \in \mathcal{L}(X)\} \qquad \text{and} \qquad \rho(L) := \mathbb{C} \setminus \sigma(L),$$

respectively. The essential spectra of L are defined as

$$\begin{aligned}
\sigma_{e1}(L) &:= \{\lambda \in \mathbb{C} \ : \ L - \lambda \notin \Phi_+(X) \cup \Phi_-(X)\}, \\
\sigma_{e2}^{\pm}(L) &:= \{\lambda \in \mathbb{C} \ : \ L - \lambda \notin \Phi_\pm(X)\}, \\
\sigma_{e3}(L) &:= \{\lambda \in \mathbb{C} \ : \ L - \lambda \notin \Phi(X)\}.
\end{aligned}$$

Note that $L-\lambda \in \Phi_{\pm}(X)$ requires $L-\lambda$ (and thus L) to be closed. Hence, if L is not closed, we have $\sigma(L) = \sigma_{e1}(L) = \sigma_{e2}^{\pm}(L) = \sigma_{e3}(L) = \mathbb{C}$. Also, we obviously have

$$\sigma_{e1}(L) = \sigma_{e2}^{+}(L) \cap \sigma_{e2}^{-}(L) \qquad \text{and} \qquad \sigma_{e3}(L) = \sigma_{e2}^{+}(L) \cup \sigma_{e2}^{-}(L).$$

In particular,

$$\sigma_{e1}(L) \subset \sigma_{e2}^{\pm}(L) \subset \sigma_{e3}(L).$$

3 Essential Spectra of the Operator Pencil $\lambda S - T$ and the Linear Relation TS^{-1}

Throughout this section let X and Y be Banach spaces. Given $S, T \in \mathcal{L}(X, Y)$, we will establish a relationship between the (essential) spectra of the operator pencil $\mathcal{A}(\lambda) = \lambda S - T$ and the associated linear relation

$$TS^{-1} \in LR(Y).$$

Note that S^{-1} is the inverse of the graph of S viewed as a linear relation. Then it follows from (2.1) and (2.4) that

$$\begin{aligned} TS^{-1} &= \Big\{\{y, z\} : \{y, x\} \in S^{-1},\ \{x, z\} \in T \text{ for some } x \in X\Big\} \\ &= \{\{Sx, Tx\} : x \in X\} \qquad (3.1) \\ &= \operatorname{ran} \begin{bmatrix} S \\ T \end{bmatrix}. \qquad (3.2) \end{aligned}$$

From this it is immediate that

$$\operatorname{dom}(TS^{-1}) = \operatorname{ran} S, \quad \ker(TS^{-1}) = S \ker T,$$
$$\operatorname{ran}(TS^{-1}) = \operatorname{ran} T, \quad \operatorname{mul}(TS^{-1}) = T \ker S.$$

The spectrum and the essential spectra for a linear operator pencil are defined similarly as for linear relations.

Definition 3.1 For an operator pencil $\mathcal{A}(\lambda) = \lambda S - T$ with $S, T \in \mathcal{L}(X, Y)$ the *spectrum* $\sigma(\mathcal{A})$ and the *resolvent set* $\rho(\mathcal{A})$ are defined as

$$\sigma(\mathcal{A}) := \{\lambda \in \mathbb{C} \ :\ \lambda S - T \text{ is not boundedly invertible}\},$$
$$\rho(\mathcal{A}) := \mathbb{C} \setminus \sigma(\mathcal{A}).$$

The essential spectra of $\mathcal{A}$ are given by

$$\sigma_{e1}(\mathcal{A}) := \{\lambda \in \mathbb{C} \,:\, \lambda S - T \notin \Phi_+(X,Y) \cup \Phi_-(X,Y)\},$$
$$\sigma_{e2}^{\pm}(\mathcal{A}) := \{\lambda \in \mathbb{C} \,:\, \lambda S - T \notin \Phi_\pm(X,Y)\},$$
$$\sigma_{e3}(\mathcal{A}) := \{\lambda \in \mathbb{C} \,:\, \lambda S - T \notin \Phi(X,Y)\}.$$

The next proposition shows how the spectra of $\mathcal{A}$ and TS^{-1} are related to each other.

Proposition 3.2 *Let $\mathcal{A}(\lambda) = \lambda S - T$ be an operator pencil with $S, T \in \mathcal{L}(X,Y)$ and $\lambda \in \mathbb{C}$ then the following holds.*

(a) $\ker(TS^{-1} - \lambda) = S \ker \mathcal{A}(\lambda)$.
(b) $\operatorname{ran}(TS^{-1} - \lambda) = \operatorname{ran} \mathcal{A}(\lambda)$.
(c) We have

$$\dim \ker(TS^{-1} - \lambda) = \dim \frac{\ker \mathcal{A}(\lambda)}{\ker S \cap \ker T}.$$

(d) If $\sigma_{e2}^{+}(\mathcal{A}) \neq \mathbb{C}$, then TS^{-1} is closed, i.e., $TS^{-1} \in CR(Y)$. This is in particular the case if $\rho(\mathcal{A}) \neq \emptyset$.
(e) We have $\sigma(TS^{-1}) \subset \sigma(\mathcal{A})$.
(f) If $\ker S \cap \ker T = \{0\}$*, then*

$$\sigma(TS^{-1}) = \sigma(\mathcal{A}).$$

Proof From (2.3) and (3.1) it is easy to see

$$TS^{-1} - \lambda = \{\{Sx, Tx - \lambda Sx\} : x \in X\}$$

which implies (a) and (b). Observe that the map $[x] \mapsto Sx$ from $\frac{\ker(\lambda S - T)}{\ker S \cap \ker T}$ to $S \ker(\lambda S - T)$ is bijective which proves (c).

In order to prove (d) set $N_0 := \ker S \cap \ker T$ and let $\lambda \in \mathbb{C}$ such that $\mathcal{A}(\lambda) \in \Phi_+(X,Y)$. Then $\ker \mathcal{A}(\lambda)$ is finite dimensional and, hence, closed. It has a complementary subspace and we have

$$\ker \mathcal{A}(\lambda) = N_0 \dotplus N_1 \qquad \text{and} \qquad X = \ker \mathcal{A}(\lambda) \dotplus M$$

with closed subspaces $N_1 \subset \ker \mathcal{A}(\lambda)$ and $M \subset X$. Let $(\{y_n, z_n\})$ be a sequence in TS^{-1} which converges to $\{y, z\} \in Y \times Y$. Then, by (3.1), we find a sequence (x_n) in X with

$$y_n = Sx_n \quad \text{and} \quad z_n = Tx_n.$$

We have to prove that there exists some $x \in X$ such that $Sx_n \to Sx$ and $Tx_n \to Tx$. To this end, we write $x_n = u_n + v_n + w_n$ with $u_n \in N_0$, $v_n \in N_1$ and $w_n \in M$. Since $\mathcal{A}(\lambda)$ maps M bijectively onto its (closed) range and $A(\lambda)w_n = A(\lambda)x_n = \lambda Sx_n - Tx_n \to \lambda y - z$, it follows that (w_n) converges to some $w \in M$. Hence, (Sw_n) and (Tw_n) converge and therefore (Sv_n) converges. Since $\ker(S|_{N_1}) = \{0\}$, (v_n) converges to some $v \in N_1$ and we obtain $Sx_n = S(v_n + w_n) \to S(v + w)$ and $Tx_n = T(v_n + w_n) \to T(v + w)$.

For the proof of (e) let $\lambda \in \rho(\mathcal{A})$. Then TS^{-1} is closed by (d) and $\ker(TS^{-1} - \lambda) = \{0\}$, $\operatorname{ran}(TS^{-1} - \lambda) = Y$ by (a) and (b). Hence,

$$\operatorname{mul}(TS^{-1} - \lambda)^{-1} = \ker(TS^{-1} - \lambda) = \{0\}$$

and $(TS^{-1} - \lambda)^{-1}$ is a closed operator in Y with domain Y. By the closed graph theorem, it is an element of $\mathcal{L}(Y)$. This proves (e). For (f), assume that $\lambda \in \rho(TS^{-1})$ and, in addition, that $\ker S \cap \ker T = \{0\}$. Then $\operatorname{ran}\mathcal{A}(\lambda) = Y$ by (b) and $\ker\mathcal{A}(\lambda) = \ker S \cap \ker T = \{0\}$ by (c). □

Remark 3.3 Note that the condition $\ker S \cap \ker T = \{0\}$ in (f) is necessary for $\rho(\mathcal{A})$ to be non-empty. In fact, if $x \in \ker S \cap \ker T$, $x \neq 0$, then $x \in \ker\mathcal{A}(\lambda)$ for all $\lambda \in \mathbb{C}$ and thus $\rho(\mathcal{A}) = \emptyset$.

The following proposition shows that also the essential spectra of the pencil $\lambda S - T$ and the linear relation TS^{-1} are intimately connected to each other.

Proposition 3.4 *Let $\mathcal{A}(\lambda) = \lambda S - T$ be an operator pencil with $S, T \in \mathcal{L}(X, Y)$ and $\lambda \in \mathbb{C}$. Then we have*

$$\sigma_{e2}^{+}(TS^{-1}) \subset \sigma_{e2}^{+}(\mathcal{A}) \qquad \text{and} \qquad \sigma_{e2}^{-}(TS^{-1}) \supset \sigma_{e2}^{-}(\mathcal{A}). \tag{3.3}$$

If TS^{-1} is closed, then

$$\sigma_{e2}^{-}(TS^{-1}) = \sigma_{e2}^{-}(\mathcal{A}). \tag{3.4}$$

If $\dim(\ker S \cap \ker T) < \infty$, *then*

$$\sigma_{e2}^{+}(TS^{-1}) = \sigma_{e2}^{+}(\mathcal{A}). \tag{3.5}$$

Hence, if TS^{-1} is closed and $\dim(\ker S \cap \ker T) < \infty$, *then*

$$\sigma_{e1}(TS^{-1}) = \sigma_{e1}(\mathcal{A}) \qquad \text{and} \qquad \sigma_{e3}(TS^{-1}) = \sigma_{e3}(\mathcal{A}).$$

Proof From Proposition 3.2 (b) it follows that $\operatorname{ran}(TS^{-1} - \lambda)$ is closed if and only if $\operatorname{ran}\mathcal{A}(\lambda)$ is closed and $\operatorname{def}(TS^{-1} - \lambda) = \operatorname{def}\mathcal{A}(\lambda)$. This proves the second relation in (3.3). If $\mathcal{A}(\lambda) \in \Phi_+(X, Y)$ for some $\lambda \in \mathbb{C}$, then TS^{-1} is closed by Proposition 3.2 (d) and from Proposition 3.2 (a) we conclude $\operatorname{nul}(TS^{-1} - \lambda) \leq \operatorname{nul}(\mathcal{A}(\lambda))$. Hence, $TS^{-1} - \lambda \in \Phi_+(Y)$ and (3.3) is proved.

If TS^{-1} is closed, then obviously $\mathcal{A}(\lambda) \in \Phi_-(X, Y)$ implies $TS^{-1}-\lambda \in \Phi_-(Y)$, which shows (3.4). If $\dim(\ker S \cap \ker T) < \infty$, then $TS^{-1} - \lambda \in \Phi_+(Y)$ implies $\dim\ker \mathcal{A}(\lambda) < \infty$ (see Proposition 3.2 (c)) and therefore $\mathcal{A}(\lambda) \in \Phi_+(X, Y)$. □

The following corollary follows from Proposition 3.2 (d) $\mathcal{A}(\lambda) \in \Phi_+(X, Y)$ implies $\dim(\ker S \cap \ker T) < \infty$.

Corollary 3.5 *If $\sigma_{e2}^+(\mathcal{A}) \neq \mathbb{C}$ (in particular, if $\rho(\mathcal{A}) \neq \emptyset$), then*

$$\sigma_{e2}^+(TS^{-1}) = \sigma_{e2}^+(\mathcal{A}) \qquad \text{and} \qquad \sigma_{e2}^-(TS^{-1}) = \sigma_{e2}^-(\mathcal{A}),$$

and therefore also

$$\sigma_{e1}(TS^{-1}) = \sigma_{e1}(\mathcal{A}) \qquad \text{and} \qquad \sigma_{e3}(TS^{-1}) = \sigma_{e3}(\mathcal{A}).$$

4 Essential Spectrum of Linear Relations Under Perturbations

In this section we let X and Y be Hilbert spaces. We say that $L, M \in CR(X, Y)$ are *compact perturbations of each other* if $P_L - P_M$ is compact. Here, P_L denotes the orthogonal projection onto the closed subspace L. If $\rho(L) \cap \rho(M) \neq \emptyset$, this is equivalent to $(L - \mu)^{-1} - (M - \mu)^{-1}$ being compact for some (and hence for all) $\mu \in \rho(L) \cap \rho(M)$ (see [2]).

Lemma 4.1 *Two linear relations $L, M \in CR(X, Y)$ in the Hilbert spaces X, Y are compact perturbations of each other if and only if L^* and M^* are compact perturbations of each other.*

Proof Relation (2.7) and the unitary mapping $U : X \times Y \to Y \times X$ which is given by

$$U(x, y) := (y, -x)$$

yield $L^* = UL^{\perp}$. Therefore

$$P_{L^*} - P_{M^*} = P_{UL^{\perp}} - P_{UM^{\perp}} = U(P_{L^{\perp}} - P_{M^{\perp}})U^* = U(P_L - P_M)U^*.$$

Hence, $P_{L^*} - P_{M^*}$ is compact if and only if $P_L - P_M$ is compact. □

Proposition 4.2 *Let X, Y be Hilbert spaces and let $L, M \in CR(X, Y)$ be compact perturbations of each other. Then $L \in \Phi_\pm(X, Y)$ if and only if $M \in \Phi_\pm(X, Y)$. In particular,*

$$\sigma_{e2}^+(L) = \sigma_{e2}^+(M) \qquad \text{and} \qquad \sigma_{e2}^-(L) = \sigma_{e2}^-(M),$$

and therefore also

$$\sigma_{e1}(L) = \sigma_{e1}(M) \qquad \textit{and} \qquad \sigma_{e3}(L) = \sigma_{e3}(M).$$

Proof Let $L \notin \Phi_+(X, Y)$. Due to Proposition 2.2 there exists a sequence $(\{x_n, y_n\})$ in L with $\|x_n\| = 1$ for all $n \in \mathbb{N}$, $x_n \rightharpoonup 0$, and $y_n \to 0$. Set $\{x_n', y_n'\} := P_M\{x_n, y_n\} \in M$, $n \in \mathbb{N}$. Since $\{x_n, y_n\} \rightharpoonup 0$, we conclude from

$$\{x_n', y_n'\} = (P_M - P_L)\{x_n, y_n\} + \{x_n, y_n\}$$

and the compactness of $P_M - P_L$ that $\|x_n'\| \to 1$, $y_n' \to 0$ as $n \to \infty$, and $x_n' \rightharpoonup 0$. Setting $x_n'' := x_n'/\|x_n'\|$ and $y_n'' := y_n'/\|x_n'\|$, we obtain $\{x_n'', y_n''\} \in L$ with $\|x_n''\| = 1$ for all $n \in \mathbb{N}$, $x_n'' \rightharpoonup 0$, and $y_n'' \to 0$. Hence, Proposition 2.2 implies that $M \notin \Phi_+(X, Y)$. This shows that $L \in \Phi_+(X, Y)$ if and only if $M \in \Phi_+(X, Y)$. Using this, Lemma 4.1, and (2.8), we obtain the same statement with $\Phi_+(X, Y)$ replaced by $\Phi_-(X, Y)$.

The remaining statements on the essential spectra follow from Proposition 4.3 in [2] which implies that L and M are compact perturbations of each other if and only if $L - \lambda$ and $M - \lambda$ are compact perturbations of each other. □

5 Essential Spectrum of Operator Pencils Under Perturbations

In this section we give sufficient conditions for the equality of the essential spectra of two operator pencils $\mathcal{A}_1$ and $\mathcal{A}_2$

$$\mathcal{A}_1(\lambda) = \lambda S_1 - T_1 \qquad \text{and} \qquad \mathcal{A}_2(\lambda) = \lambda S_2 - T_2$$

in terms of their coefficients $S_1, S_2, T_1, T_2 \in \mathcal{L}(X, Y)$. In the proofs of our main theorems we use the above-established concept of the relationship between operator pencils and linear relations.

The first statement is obvious and follows from the well-known fact that $\mathcal{L}(X, Y) \cap \Phi_\pm(X, Y)$ is invariant under compact perturbations.

Proposition 5.1 *Assume that $T_2 - T_1$ and $S_2 - S_1$ are compact. Then*

$$\sigma_{e1}(\mathcal{A}_1) = \sigma_{e1}(\mathcal{A}_2), \quad \sigma_{e2}^{\pm}(\mathcal{A}_1) = \sigma_{e2}^{\pm}(\mathcal{A}_2), \quad \textit{and} \quad \sigma_{e3}(\mathcal{A}_1) = \sigma_{e3}(\mathcal{A}_2).$$

Let $A \in \mathcal{L}(X, Y)$. It follows from $\ker A = \ker A^*A$ and the closed range theorem that A has closed range if and only if the same is true for A^*A. In this case, $X = \ker A \oplus \operatorname{ran} A^*$, $Y = \ker A^* \oplus \operatorname{ran} A$ and the restriction $A_0 = A|_{\operatorname{ran} A^*}$: $\operatorname{ran} A^* \to \operatorname{ran} A$ is boundedly invertible. Recall that the *pseudo-inverse* $A^\dagger$ of A is then defined by

$$A^{\dagger} := A_0^{-1} P_{\operatorname{ran} A}.$$

For an overview of equivalent definitions of the pseudo-inverse of linear operators we refer to [7, Chapter II]. It is immediate that

$$P_{\operatorname{ran} A} = AA^{\dagger} \tag{5.1}$$

and one can show, see e.g. [11, Theorem 4], that

$$(A^{\dagger})^* = (A^*)^{\dagger}. \tag{5.2}$$

Moreover we have from [7, Theorem 2.1.5] that

$$A^{\dagger} = (A^*A)^{\dagger}A^* = A^*(AA^*)^{\dagger}. \tag{5.3}$$

Our first main theorem is the following.

Theorem 5.2 *Let X, Y be Hilbert spaces and $S_1, S_2, T_1, T_2 \in \mathcal{L}(X, Y)$ with corresponding pencils*

$$\mathcal{A}_1(\lambda) = \lambda S_1 - T_1 \quad \text{and} \quad \mathcal{A}_2(\lambda) = \lambda S_2 - T_2.$$

Assume that for both $j = 1, 2$ the operator $S_j^* S_j + T_j^* T_j \in \mathcal{L}(X)$ has closed range and that the operator

$$\begin{bmatrix} S_1 Z_1 S_1^* - S_2 Z_2 S_2^* & S_1 Z_1 T_1^* - S_2 Z_2 T_2^* \\ T_1 Z_1 S_1^* - T_2 Z_2 S_2^* & T_1 Z_1 T_1^* - T_2 Z_2 T_2^* \end{bmatrix} \in \mathcal{L}(Y \times Y) \tag{5.4}$$

is compact, where

$$Z_j := (S_j^* S_j + T_j^* T_j)^{\dagger}, \quad j = 1, 2.$$

Then

$$\sigma_{e2}^{-}(\mathcal{A}_1) = \sigma_{e2}^{-}(\mathcal{A}_2).$$

If, in addition, $S_j^* S_j + T_j^* T_j \in \Phi_+(X)$ for $j = 1, 2$, then

$$\sigma_{e2}^{+}(\mathcal{A}_1) = \sigma_{e2}^{+}(\mathcal{A}_2).$$

Proof Let $j = 1, 2$ and set $A_j := \left[\begin{smallmatrix} S_j \\ T_j \end{smallmatrix}\right]$. Then $A_j^* A_j = S_j^* S_j + T_j^* T_j$ implies that A_j has closed range which means that the relation $T_j S_j^{-1}$ is closed. As discussed before, we find with (5.3) that

$$A_jA_j^{\dagger} = A_j(A_j^*A_j)^{\dagger}A_j^* = A_jZ_jA_j^* = \begin{bmatrix} S_j \\ T_j \end{bmatrix} Z_j \begin{bmatrix} S_j^* & T_j^* \end{bmatrix} = \begin{bmatrix} S_jZ_jS_j^* & S_jZ_jT_j^* \\ T_jZ_jS_j^* & T_jZ_jT_j^* \end{bmatrix}$$

is the orthogonal projection onto $\operatorname{ran} A_j = T_jS_j^{-1}$. Hence, the operator in (5.4) is the difference of the orthogonal projections onto the closed subspaces $T_1S_1^{-1}$ and $T_2S_2^{-1}$ of $Y \times Y$. Also note that $\ker S_j \cap \ker T_j = \ker A_j = \ker A_j^*A_j$. Now, the statements of Theorem 5.2 follow from Propositions 4.2 and 3.4. □

Example

(a) Let us consider the example from the introduction, where $X = Y$ and $\mathcal{A}_1(\lambda) = \lambda I - T$ and $\mathcal{A}_2(\lambda) = (\lambda I - T)M$ with $T, M \in \mathcal{L}(X)$ and M boundedly invertible. Clearly, all the essential spectra of $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide, respectively. We have $S_1 = I$, $T_1 = T$, $S_2 = M$ and $T_2 = TM$. Then both $S_1^*S_1 + T_1^*T_1 = I + T^*T$ and $S_2^*S_2 + T_2^*T_2 = M^*(I + T^*T)M$ are boundedly invertible and the operator matrix in (5.4) is the zero matrix. Indeed, we have

$$T_2S_2^{-1} = \operatorname{ran} \begin{bmatrix} M \\ TM \end{bmatrix} = \operatorname{ran} \begin{bmatrix} I \\ T \end{bmatrix} = T_1S_1^{-1}.$$

(b) Let X, Y be Hilbert spaces and let $M_1, M_2 \in \mathcal{L}(X, Y)$ be boundedly invertible. Let $K_S, K_T \in \mathcal{L}(Y)$ be compact such that $-1 \notin \sigma(K_S) \cap \sigma(K_T)$. Then the operator $R := (I + K_S)^*(I + K_S) + (I + K_T)^*(I + K_T)$ is boundedly invertible. Indeed, R is a compact perturbation of $2I$ and therefore Fredholm with index zero and the condition $-1 \notin \sigma(K_S) \cap \sigma(K_T)$ guarantees that $\ker R = \{0\}$. Consider

$$S_1 = T_1 = M_1, \quad \text{and} \quad S_2 = (I + K_S)M_2, \quad T_2 = (I + K_T)M_2.$$

Using the invertibility of M_1, M_2, we note

$$T_1S_1^{-1} = \operatorname{ran} \begin{bmatrix} S_1 \\ T_1 \end{bmatrix} = \operatorname{ran} \begin{bmatrix} M_1 \\ M_1 \end{bmatrix} = \operatorname{ran} \begin{bmatrix} I \\ I \end{bmatrix}$$

and

$$T_2S_2^{-1} = \operatorname{ran} \begin{bmatrix} S_2 \\ T_2 \end{bmatrix} = \operatorname{ran} \begin{bmatrix} (I + K_S)M_2 \\ (I + K_T)M_2 \end{bmatrix} = \operatorname{ran} \begin{bmatrix} I + K_S \\ I + K_T \end{bmatrix}.$$

Set $Z_2 := ((I + K_S)^*(I + K_S) + (I + K_T)^*(I + K_T))^{-1}$. In this case, the operator in (5.4) reads as

$$\begin{bmatrix} \frac{1}{2}I - (I + K_S)Z_2(I + K_S)^* & \frac{1}{2}I - (I + K_S)Z_2(I + K_T)^* \\ \frac{1}{2}I - (I + K_T)Z_2(I + K_S)^* & \frac{1}{2}I - (I + K_T)Z_2(I + K_T)^* \end{bmatrix}.$$

Obviously, this operator is compact as

$$\tfrac{1}{2}I - Z_2$$

is compact. Hence, the conditions in Theorem 5.2 are satisfied and all essential spectra of the two pencils

$$\mathcal{A}_1(\lambda) = \lambda S_1 - T_1 \qquad \text{and} \qquad \mathcal{A}_2(\lambda) = \lambda S_2 - T_2$$

coincide.

Lemma 5.3 *Let X, Y be Hilbert spaces, $S, T \in \mathcal{L}(X, Y)$, $S \in \Phi_+(X, Y)$, and $\lambda \in \mathbb{C}$. Assume furthermore that TS^{-1} is closed. Then we have $\lambda \in \sigma_{e2}^{+}(TS^{-1})$ if and only if there exists a sequence (y_n) in $(\ker S)^{\perp}$ such that $\|Sy_n\| \to 1$, $y_n \rightharpoonup 0$, and $(\lambda S - T)y_n \to 0$ as $n \to \infty$.*

Proof Assume that $TS^{-1} - \lambda \notin \Phi_+(X, Y)$. By Proposition 2.2 there exists a sequence $\{x_n, z_n\} \in TS^{-1} - \lambda$ with $\|x_n\| = 1$ for all $n \in \mathbb{N}$, $x_n \rightharpoonup 0$, and $z_n \to 0$ as $n \to \infty$. As $TS^{-1} - \lambda = \{\{Sx, Tx - \lambda Sx\} : x \in X\}$ (see (2.3) and (3.1)), there exists a sequence (v_n) in X such that $\|Sv_n\| = 1$ for all $n \in \mathbb{N}$, $Sv_n \rightharpoonup 0$, and $Tv_n - \lambda Sv_n \to 0$ as $n \to \infty$. For $n \in \mathbb{N}$ let $v_n = u_n + y_n$ with $u_n \in \ker S$ and $y_n \in (\ker S)^{\perp}$. Then $\|Sy_n\| = 1$ and $Sy_n \rightharpoonup 0$. Since S maps $(\ker S)^{\perp}$ bijectively onto the closed subspace $\operatorname{ran} S$, it follows that $y_n \rightharpoonup 0$. Hence, $Ty_n - \lambda Sy_n \rightharpoonup 0$ so that $Tv_n - \lambda Sv_n \to 0$ implies that $Tu_n \rightharpoonup 0$. But (Tu_n) is contained in the finite-dimensional subspace $T \ker S$ and thus $Tu_n \to 0$ as $n \to \infty$, which implies $(\lambda S - T)y_n \to 0$.

Conversely, let (y_n) in $(\ker S)^{\perp}$ be a sequence as in the lemma. Set $y_n' := \|Sy_n\|^{-1} y_n$ and $x_n := Sy_n'$ as well as $z_n := \lambda Sy_n' - Ty_n'$. Then $\{x_n, z_n\} \in TS^{-1} - \lambda$, $\|x_n\| = 1$ for all $n \in \mathbb{N}$, $x_n \rightharpoonup 0$, and $z_n \to 0$ as $n \to \infty$. Hence, $TS^{-1} - \lambda \notin \Phi_+(X, Y)$ by Proposition 2.2. □

The following proposition is the second main result of this paper.

Proposition 5.4 *Let X, Y be Hilbert spaces and $S_1, S_2, T_1, T_2 \in \mathcal{L}(X, Y)$. Assume that the following assumptions are satisfied.*

1. $S_1 \in \Phi_+(X, Y)$.
2. $S_2 \in \Phi(X, Y)$.
3. $(T_2 - T_1)S_2^{\dagger}S_1$ *is a compact operator.*
4. $T_1 S_2^{\dagger}(S_1 - S_2)$ *is a compact operator.*

Then $T_1S_1^{-1}$ and $T_2S_2^{-1}$ both are closed and

$$\sigma_{e2}^{+}(T_1S_1^{-1}) \subset \sigma_{e2}^{+}(T_2S_2^{-1}).$$

Proof Let $j \in \{1,2\}$. For $\lambda \in \mathbb{C}\setminus\{0\}$ we have $\mathcal{A}_j(\lambda) = \lambda S_j - T_j = \lambda(S_j - \frac{T_j}{\lambda})$. Since $S_j \in \Phi_+(X,Y)$, for $|\lambda|$ sufficiently large we have that $\mathcal{A}_j(\lambda) \in \Phi_+(X,Y)$ (see [10, Theorem IV-5.31]). Therefore, $T_jS_j^{-1}$ is closed by Proposition 3.2 (d).

Assume that $\lambda \in \sigma_{e2}^{+}(T_1S_1^{-1})$. Then by Lemma 5.3 there exists $y_n \in (\ker S_1)^{\perp}$ such that $\|S_1y_n\| \to 1$, $y_n \rightharpoonup 0$, and $(\lambda S_1 - T_1)y_n \to 0$ as $n \to \infty$. We set $y_n' := S_2^{\dagger}S_1y_n \in \operatorname{ran} S_2^* = (\ker S_2)^{\perp}$, $n \in \mathbb{N}$. Obviously, $y_n' \rightharpoonup 0$. Since $\dim\ker S_2^* < \infty$ and $y_n \rightharpoonup 0$, it follows from (5.1)

$$\|S_2y_n'\| = \|P_{\operatorname{ran} S_2}S_1y_n\| = \|S_1y_n - P_{\ker S_2^*}S_1y_n\| \to 1$$

as $n \to \infty$. Also, setting $K := T_2 - T_1$,

$$\begin{aligned} T_2y_n' - \lambda S_2y_n' &= T_2S_2^{\dagger}S_1y_n - \lambda(S_1y_n - P_{\ker S_2^*}S_1y_n) \\ &= KS_2^{\dagger}S_1y_n + T_1S_2^{\dagger}S_1y_n - \lambda S_1y_n + \lambda P_{\ker S_2^*}S_1y_n \\ &= KS_2^{\dagger}S_1y_n + T_1(S_2^{\dagger}S_1 - I)y_n + \lambda P_{\ker S_2^*}S_1y_n - (\lambda S_1 - T_1)y_n. \end{aligned}$$

Now, the claim follows from Lemma 5.3, the compactness of $KS_2^{\dagger}S_1$ and $P_{\ker S_2^*}$ and the fact that $S_2^{\dagger}S_1 - I = S_2^{\dagger}(S_1 - S_2) - P_{\ker S_2}S_2^*$. ◻

Theorem 5.5 *Let X, Y be Hilbert spaces and $S_1, S_2, T_1, T_2 \in \mathcal{L}(X,Y)$ and let $S_1, S_2 \in \Phi(X,Y)$.*

(a) If $(T_2-T_1)S_2^{\dagger}S_1$, $(T_2-T_1)S_1^{\dagger}S_2$, $T_1S_2^{\dagger}(S_1-S_2)$, and $T_2S_1^{\dagger}(S_1-S_2)$ are compact, then

$$\sigma_{e2}^{+}(\mathcal{A}_1) = \sigma_{e2}^{+}(\mathcal{A}_2). \tag{5.5}$$

(b) If $S_1S_2^{\dagger}(T_2-T_1)$, $S_2S_1^{\dagger}(T_2-T_1)$, $(S_1-S_2)S_2^{\dagger}T_1$ and $(S_1-S_2)S_1^{\dagger}T_2$ are compact, then

$$\sigma_{e2}^{-}(\mathcal{A}_1) = \sigma_{e2}^{-}(\mathcal{A}_2). \tag{5.6}$$

Proof From Proposition 5.4 we obtain $\sigma_{e2}^{+}(T_1S_1^{-1}) = \sigma_{e2}^{+}(T_2S_2^{-1})$ and (5.5) is a consequence of Proposition 3.4.

By assumption (cf. (2.8)) we have $S_1^*, S_2^* \in \Phi(Y, X)$. The assumptions in (ii) and (5.2) imply the compactness of $T_1^*(S_2^*)^\dagger(S_1^* - S_2^*)$ and of $T_2^*(S_1^*)^\dagger(S_1^* - S_2^*)$. Proposition 5.4 yields

$$\sigma_{e2}^+(T_1^*(S_1^*)^{-1}) = \sigma_{e2}^+(T_2^*(S_2^*)^{-1}).$$

Hence we have together with Corollary 3.5 that

$$\sigma_{e2}^+(\mathcal{A}_1^*) = \sigma_{e2}^+(T_1^*(S_1^*)^{-1}) = \sigma_{e2}^+(T_2^*(S_2^*)^{-1}) = \sigma_{e2}^+(\mathcal{A}_2^*)$$

with $\mathcal{A}_i^*(\lambda) := \lambda S_i^* - T_i^*$ for $i = 1, 2$. Therefore, $\overline{\lambda} \in \sigma_{e2}^+(\mathcal{A}_1^*)$ if and only if $\overline{\lambda} \in \sigma_{e2}^+(\mathcal{A}_2^*)$. Now, (5.6) follows from (2.8) applied to the operators $\mathcal{A}_1^*(\overline{\lambda})$ and $\mathcal{A}_2^*(\overline{\lambda})$. □

Remark 5.6 Let $S \in \mathcal{L}(X, Y)$ and let T be a densely defined closed linear operator in X. Set $\mathcal{A}(\lambda) := \lambda S - T$. Assume that $\mu \in \rho(\mathcal{A})$. Then we have by definition

$$(TS^{-1} - \mu)^{-1} = \{\{Tx - \mu Sx, Sx\} : x \in \operatorname{dom} T\} = \{\{y, S(T - \mu S)^{-1}y\} : y \in X\}.$$

Using compactness of the perturbation of the corresponding linear relations we obtain the following result: For $i = 1, 2$ let $\mathcal{A}_i(\lambda) = \lambda S_i - T_i$ with $S_i \in \mathcal{L}(X, Y)$ bounded and T_i closed and densely defined from X to Y and let $\mu \in \rho(\mathcal{A}_1) \cap \rho(\mathcal{A}_2)$ with

$$S_1(T_1 - \mu S_1)^{-1} - S_2(T_2 - \mu S_2)^{-1} \quad \text{compact}$$

then $\sigma_{e2}^\pm(\mathcal{A}_1) = \sigma_{e2}^\pm(\mathcal{A}_2)$ (cf. Propositions 3.4 and 4.2). Note that the compactness of the resolvent difference does not depend on the choice of μ. Furthermore, we have no inclusion assumption on the multivalued parts as in [16].

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Decomposition of the Tensor Product of Two Hilbert Modules

Soumitra Ghara and Gadadhar Misra

This paper is dedicated to the memory of Ronald G. Douglas

Abstract Given a pair of positive real numbers α, β and a sesqui-analytic function K on a bounded domain $\Omega \subset \mathbb{C}^m$, in this paper, we investigate the properties of the sesqui-analytic function

$$\mathbb{K}^{(\alpha,\beta)} := K^{\alpha+\beta}\big(\partial_i\bar{\partial}_j \log K\big)_{i,j=1}^{m}$$

taking values in $m \times m$ matrices. One of the key findings is that $\mathbb{K}^{(\alpha,\beta)}$ is non-negative definite whenever K^α and K^β are non-negative definite. In this case, a realization of the Hilbert module determined by the kernel $\mathbb{K}^{(\alpha,\beta)}$ is obtained. Let $\mathcal{M}_i, i = 1, 2$, be two Hilbert modules over the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$. Then $\mathbb{C}[z_1, \ldots, z_{2m}]$ acts naturally on the tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$. The restriction of this action to the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$ obtained using the restriction map $p \mapsto p_{|\Delta}$ leads to a natural decomposition of the tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$, which is investigated. Two of the initial pieces in this decomposition are identified.

Keywords Cowen-Douglas class · Non negative definite kernels · Jet construction · Tensor product · Hilbert modules

The work of the first named author was supported by CSIR SPM Fellowship (Ref. No. SPM-07/079(0242)/2016-EMR-I). The work of the second named author was supported by the JC Bose National Fellowship and the MATRICS grant of the SERB as well as the CAS II of the UGC. Many of the results in this paper are from the PhD thesis of the first named author submitted to the Indian Institute of Science in the year 2018.

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_11

2010 Mathematics Subject Classification 47B32, 47B38

1 Introduction

1.1 Hilbert Module

We will find it useful to state many of our results in the language of Hilbert modules. The notion of a Hilbert module was introduced by R.G. Douglas (cf. [11]), which we recall below. We point out that in the original definition, the module multiplication was assumed to be continuous in both the variables. However, for our purposes, it would be convenient to assume that it is continuous only in the second variable.

Definition 1.1 (Hilbert Module) A Hilbert module $\mathcal{M}$ over a unital, complex algebra $\mathbb{A}$ consists of a complex Hilbert space $\mathcal{M}$ and a map $(a, h) \mapsto a \cdot h$, $a \in \mathbb{A}, h \in \mathcal{M}$, such that

(i) $1 \cdot h = h$
(ii) $(ab) \cdot h = a \cdot (b \cdot h)$
(iii) $(a + b) \cdot h = a \cdot h + b \cdot h$
(iv) for each a in $\mathbb{A}$, the map $\mathbf{m}_a : \mathcal{M} \to \mathcal{M}$, defined by $\mathbf{m}_a(h) = a \cdot h,\ h \in \mathcal{M}$, is a bounded linear operator on $\mathcal{M}$.

A closed subspace $\mathcal{S}$ of $\mathcal{M}$ is said to be a submodule of $\mathcal{M}$ if $\mathbf{m}_a h \in \mathcal{S}$ for all $h \in \mathcal{S}$ and $a \in \mathbb{A}$. The quotient module $\mathcal{Q} := \mathcal{H}/\mathcal{S}$ is the Hilbert space $\mathcal{S}^\perp$, where the module multiplication is defined to be the compression of the module multiplication on $\mathcal{H}$ to the subspace $\mathcal{S}^\perp$, that is, the module action on $\mathcal{Q}$ is given by $\mathbf{m}_a(h) = P_{\mathcal{S}^\perp}(\mathbf{m}_a h)$, $h \in \mathcal{S}^\perp$. Two Hilbert modules $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathbb{A}$ are said to be isomorphic if there exists a unitary operator $U : \mathcal{M}_1 \to \mathcal{M}_2$ such that $U(a \cdot h) = a \cdot Uh$, $a \in \mathbb{A}, h \in \mathcal{M}_1$.

Let $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ be a sesqui-analytic (that is holomorphic in first m-variables and anti-holomorphic in the second set of m-variables) non-negative definite kernel on a bounded domain $\Omega \subset \mathbb{C}^m$. It uniquely determines a Hilbert space $(\mathcal{H}, K)$ consisting of holomorphic functions on Ω taking values in $\mathbb{C}^k$ possessing the following properties. For $w \in \Omega$,

(i) the vector valued function $K(\cdot, w)\zeta$, $\zeta \in \mathbb{C}^k$, belongs to the Hilbert space $\mathcal{H}$;
(ii) $\langle f, K(\cdot, w)\zeta\rangle_{\mathcal{H}} = \langle f(w), \zeta\rangle_{\mathbb{C}^k}$, $f \in (\mathcal{H}, K)$.

Assume that the operator of multiplication M_{z_i} by the ith coordinate function z_i is bounded on the Hilbert space $(\mathcal{H}, K)$ for $i = 1, \ldots, m$. Then $(\mathcal{H}, K)$ may be realized as a Hilbert module over the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$ with the module action given by the point-wise multiplication:

$$\mathbf{m}_p(h) = ph,\ \ h \in (\mathcal{H}, K),\ \ p \in \mathbb{C}[z_1, \ldots, z_m].$$

Let K_1 and K_2 be two scalar valued non-negative definite kernels defined on $\Omega \times \Omega$. It turns out that $(\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2)$ is the reproducing kernel Hilbert space with the reproducing kernel $K_1 \otimes K_2 : (\Omega \times \Omega) \times (\Omega \times \Omega) \to \mathbb{C}$ given by the formula

$$(K_1 \otimes K_2)(z, \zeta; w, \rho) = K_1(z, w)K_2(\zeta, \rho), \quad z, \zeta, w, \rho \in \Omega.$$

Assume that the multiplication operators M_{z_i}, $i = 1, \ldots, m$, are bounded on $(\mathcal{H}, K_1)$ as well as on $(\mathcal{H}, K_2)$. Then $(\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2)$ may be realized as a Hilbert module over $\mathbb{C}[z_1, \ldots, z_{2m}]$ with the module action defined by

$$\mathbf{m}_p(h) = ph, \ h \in (\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2), \ p \in \mathbb{C}[z_1, \ldots, z_{2m}].$$

The module $(\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2)$ admits a natural direct sum decomposition as follows.

For a non-negative integer k, let $\mathcal{A}_k$ be the subspace of $(\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2)$ defined by

$$\mathcal{A}_k := \Big\{f \in (\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2) : \Big(\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z, \zeta)\Big)_{|\Delta} = 0, \ |\boldsymbol{i}| \le k\Big\}, \tag{1.1}$$

where $\boldsymbol{i} \in \mathbb{Z}_+^m$, $|\boldsymbol{i}| = i_1 + \cdots + i_m$, $\big(\frac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} = \frac{\partial^{|\boldsymbol{i}|}}{\partial \zeta_1^{i_1} \cdots \partial \zeta_m^{i_m}}$, and $\Big(\big(\frac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z, \zeta)\Big)_{|\Delta}$ is the restriction of $\big(\frac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z, \zeta)$ to the diagonal set $\Delta := \{(z, z) : z \in \Omega\}$. It is easily verified that each of the subspaces $\mathcal{A}_k$ is closed and invariant under multiplication by any polynomial in $\mathbb{C}[z_1, \ldots, z_{2m}]$ and therefore they are sub-modules of $(\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2)$. Setting $\mathcal{S}_0 = \mathcal{A}_0^{\perp}$, $\mathcal{S}_k := \mathcal{A}_{k-1} \ominus \mathcal{A}_k$, $k = 1, 2, \ldots$, we obtain a direct sum decomposition of the Hilbert space

$$(\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2) = \bigoplus_{k=0}^{\infty} \mathcal{S}_k.$$

In this decomposition, the subspaces $\mathcal{S}_k \subseteq (\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2)$ are not necessarily sub-modules. Indeed, one may say they are semi-invariant modules following the terminology commonly used in Sz.-Nagy–Foias model theory for contractions. We study the compression of the module action to these subspaces analogous to the ones studied in operator theory. Also, such a decomposition is similar to the Clebsch-Gordan formula, which describes the decomposition of the tensor product of two irreducible representations, say ϱ_1 and ϱ_2 of a group G when restricted to the diagonal subgroup in $G \times G$:

$$\varrho_1(g) \otimes \varrho_2(g) = \bigoplus_k d_k \pi_k(g),$$

where π_k, $k \in \mathbb{Z}_+$, are irreducible representation of the group G and d_k, $k \in \mathbb{Z}_+$, are natural numbers. However, the decomposition of the tensor product of two Hilbert modules cannot be expressed as the direct sum of submodules. Noting that $\mathcal{S}_0$ is a quotient module, describing all the semi-invariant modules $\mathcal{S}_k$, $k \geq 1$, would appear to be a natural question. To describe the equivalence classes of $\mathcal{S}_0, \mathcal{S}_1, \ldots$ etc., it would be useful to recall the notion of the push-forward of a module.

Let $\iota : \Omega \to \Omega \times \Omega$ be the map $\iota(z) = (z, z)$, $z \in \Omega$. Any Hilbert module $\mathcal{M}$ over the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$ may be thought of as a module $\iota_\star\mathcal{M}$ over the ring $\mathbb{C}[z_1, \ldots, z_{2m}]$ by re-defining the multiplication: $\mathbf{m}_p(h) = (p \circ \iota)h$, $h \in \mathcal{M}$ and $p \in \mathbb{C}[z_1, \ldots, z_{2m}]$. The module $\iota_\star\mathcal{M}$ over $\mathbb{C}[z_1, \ldots, z_{2m}]$ is defined to be the push-forward of the module $\mathcal{M}$ over $\mathbb{C}[z_1, \ldots, z_m]$ under the inclusion map ι.

In [1], Aronszajn proved that the Hilbert space $(\mathcal{H}, K_1K_2)$ corresponding to the point-wise product K_1K_2 of two non-negative definite kernels K_1 and K_2 is obtained by the restriction of the functions in the tensor product $(\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2)$ to the diagonal set Δ. Building on his work, it was shown in [10] that the restriction map is isometric on the subspace $\mathcal{S}_0$ onto $(\mathcal{H}, K_1K_2)$ intertwining the module actions on $\iota_\star(\mathcal{H}, K_1K_2)$ and $\mathcal{S}_0$. However, using the jet construction given below, it is possible to describe the quotient modules $\mathcal{A}_k^\perp$, $k \geq 0$. We reiterate that one of the main questions we address is that of describing the semi-invariant modules, namely, $\mathcal{S}_1, \mathcal{S}_2, \ldots$. We have succeed in describing only $\mathcal{S}_1$ only after assuming that the pair of kernels is of the form K^α, K^β, $\alpha, \beta > 0$, where the real power of a non-negative definite kernel is defined below.

Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a non-zero sesqui-analytic function. Let t be a real number. The function K^t is defined in the usual manner, namely $K^t(z, w) = \exp(t \log K(z, w))$, $z, w \in \Omega$, assuming that a continuous branch of the logarithm of K exists on $\Omega \times \Omega$. Clearly, K^t is also sesqui-analytic. However, if K is non-negative definite, then K^t need not be non-negative definite unless t is a natural number. A direct computation, assuming the existence of a continuous branch of logarithm of K on $\Omega \times \Omega$, shows that for $1 \leq i, j \leq m$,

$$\partial_i \bar{\partial}_j \log K(z, w) = \frac{K(z, w)\partial_i \bar{\partial}_j K(z, w) - \partial_i K(z, w)\bar{\partial}_j K(z, w)}{K(z, w)^2}, \quad z, w \in \Omega,$$

where ∂_i and $\bar{\partial}_j$ denote $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial \bar{w}_j}$, respectively.

For a sesqui-analytic function $K : \Omega \times \Omega \to \mathbb{C}$ satisfying $K(z, z) > 0$, an alternative interpretation of $K(z, w)^t$ (resp. $\log K(z, w)$) is possible using the notion of polarization. The real analytic function $K(z, z)^t$ (resp. $\log K(z, z)$) defined on Ω extends to a unique sesqui-analytic function in some neighbourhood U of the diagonal set $\{(z, z) : z \in \Omega\}$ in $\Omega \times \Omega$. If the principal branch of logarithm of K exists on $\Omega \times \Omega$, then it is easy to verify that these two definitions of $K(z, w)^t$ (resp. $\log K(z, w)$) agree on the open set U. In the particular case, when $K_1 = (1 - z\bar{w})^{-\alpha}$ and $K_2 = (1 - z\bar{w})^{-\beta}$, $\alpha, \beta > 0$, the description of the semi-invariant modules $\mathcal{S}_k$, $k \geq 0$, is obtained from somewhat more general results of Ferguson and Rochberg.

Theorem 1.2 (Ferguson-Rochberg, [13]) *Suppose that* $K_1(z,w) = \frac{1}{(1-z\bar{w})^\alpha}$ *and* $K_2(z,w) = \frac{1}{(1-z\bar{w})^\beta}$ *on* $\mathbb{D} \times \mathbb{D}$ *for some* $\alpha, \beta > 0$. *Then the Hilbert modules* $\mathcal{S}_n$ *and* $\iota_\star(\mathcal{H}, (1-z\bar{w})^{-(\alpha+\beta+2n)})$ *are isomorphic.*

In this paper, first we show that if K^α and K^β, $\alpha, \beta > 0$, are two non-negative definite kernels on Ω, then function $\mathbb{K}^{(\alpha,\beta)} : \Omega \times \Omega \to \mathcal{M}_m(\mathbb{C})$ defined by

$$\mathbb{K}^{(\alpha,\beta)}(z,w) = K^{\alpha+\beta}(z,w)\Big(\big(\partial_i \bar{\partial}_j \log K\big)(z,w) \Big)_{i,j=1}^{m}, \;\; z, w \in \Omega,$$

is also a non-negative definite kernel. In this case, a description of the Hilbert module $\mathcal{S}_1$ is obtained. Indeed, it is shown that the Hilbert modules $\mathcal{S}_1$ and $\iota_\star\big(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\big)$ are isomorphic.

1.2 The Jet Construction

For a bounded domain $\Omega \subset \mathbb{C}^m$, let K_1 and K_2 be two scalar valued non-negative definite kernels defined on $\Omega \times \Omega$. Assume that the multiplication operators M_{z_i}, $i = 1, \ldots, m$, are bounded on $(\mathcal{H}, K_1)$ as well as on $(\mathcal{H}, K_2)$. For a non-negative integer k, let $\mathcal{A}_k$ be the subspace defined in (1.1).

Let d be the cardinality of the set $\{\boldsymbol{i} \in \mathbb{Z}_+^m, |\boldsymbol{i}| \le k\}$, which is $\binom{m+k}{m}$. Define the linear map $J_k : (\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2) \to \text{Hol}(\Omega \times \Omega, \mathbb{C}^{\text{d}})$ by

$$(J_k f)(z,\zeta) = \sum_{|\boldsymbol{i}| \le k} \big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z,\zeta) \otimes e_{\boldsymbol{i}}, \;\; f \in (\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2), \tag{1.2}$$

where $\{e_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathbb{Z}_+^m, |\boldsymbol{i}| \le k}$ is the standard orthonormal basis of $\mathbb{C}^d$. Define the map $R : \text{ran} J_k \to \text{Hol}(\Omega, \mathbb{C}^{\text{d}})$ to be the restriction map, that is, $R(\mathbf{h}) = \mathbf{h}_{|\Delta}$, $\mathbf{h} \in \text{ran} J_k$. Clearly, $\ker RJ_k = \mathcal{A}_k$. Hence the map $RJ_k : \mathcal{A}_k^\perp \to \text{Hol}(\Omega, \mathbb{C}^{\text{d}})$ is one to one. Therefore we can give a natural inner product on $\text{ran} RJ_k$, namely,

$$\langle RJ_k(f), RJ_k(g) \rangle = \langle P_{\mathcal{A}_k^\perp} f, P_{\mathcal{A}_k^\perp} g \rangle, \;\; f, g \in (\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2).$$

In what follows, we think of $\text{ran} RJ_k$ as a Hilbert space equipped with this inner product. The theorem stated below is a straightforward generalization of one of the main results from [10].

Theorem 1.3 ([10, Proposition 2.3]) *Let* $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ *be two non-negative definite kernels. Then* $\text{ran} RJ_k$ *is a reproducing kernel Hilbert space and its reproducing kernel* $J_k(K_1, K_2)_{|\text{res}\,\Delta}$ *is given by the formula*

$$J_k(K_1, K_2)_{|\text{res}\,\Delta}(z,w) := \big(K_1(z,w) \partial^{\boldsymbol{i}} \bar{\partial}^{\boldsymbol{j}} K_2(z,w)\big)_{|\boldsymbol{i}|,|\boldsymbol{j}|=0}^{k}, \;\; z, w \in \Omega.$$

Now for any polynomial p in z, ζ, define the operator $\mathcal{T}_p$ on $\operatorname{ran} RJ_k$ as

$$(\mathcal{T}p)(RJ_k f) = \sum_{|\boldsymbol{l}|\leq k} \Big(\sum_{\boldsymbol{q}\leq \boldsymbol{l}} \binom{\boldsymbol{l}}{\boldsymbol{q}} \Big(\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{q}} p(z,\zeta) \Big)_{|\Delta} \Big(\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{l}-\boldsymbol{q}} f(z,\zeta) \Big)_{|\Delta} \Big) \otimes e_{\boldsymbol{l}},$$

where $f \in (\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2)$, $\boldsymbol{l} = (l_1, \ldots, l_m)$ and $\boldsymbol{q} = (q_1, \ldots, q_m) \in \mathbb{Z}_+^m$. Here, $\boldsymbol{q} \leq \boldsymbol{l}$ means $q_i \leq l_i,\ i = 1, \ldots, m$ and $\binom{\boldsymbol{l}}{\boldsymbol{q}} = \binom{l_1}{q_1} \cdots \binom{l_m}{q_m}$. The proof of the proposition below follows from a straightforward computation using the Leibniz rule, the details are on page 378–379 of [10].

Proposition 1.4 *For any polynomial p in $\mathbb{C}[z_1, \ldots, z_{2m}]$, $P_{\mathcal{A}_k^\perp} M_{p|\mathcal{A}_k^\perp}$ is unitarily equivalent to $\mathcal{T}_p$ on* $(\operatorname{ran} RJ_k)$.

In Sect. 4, we prove a generalization of the theorem of Salinas for all kernels of the form $J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}$. In particular, we show that if $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ are two sharp kernels (resp. generalized Bergman kernels), then so is the kernel $J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}$.

In Sect. 5, we introduce the notion of a generalized Wallach set for an arbitrary non-negative definite kernel K defined on a bounded domain $\Omega \subset \mathbb{C}^m$. Recall that the ordinary Wallach set associated with the Bergman kernel B_Ω of a bounded symmetric domain Ω is the set $\{t > 0 : B_\Omega^t$ is non-negative definite$\}$. Replacing the Bergman kernel in the definition of the Wallach set by an arbitrary non-negative definite kernel K, we define the ordinary Wallach set $\mathcal{W}(K)$. More importantly, we introduce the generalized Wallach set $G\mathcal{W}(K)$ associated to the kernel K to be the set $\{t \in \mathbb{R} :\ K^t\big(\partial_i \bar{\partial}_j \log K\big)_{i,j=1}^m$ is non-negative definite$\}$, where we have assumed that K^t is well defined for all $t \in \mathbb{R}$. In the particular case of the Euclidean unit ball $\mathbb{B}_m$ in $\mathbb{C}^m$ and the Bergman kernel, the generalized Wallach set $G\mathcal{W}(B_{\mathbb{B}_m})$, $m > 1$, is shown to be the set $\{t \in \mathbb{R} : t \geq 0\}$. If $m = 1$, then it is the set $\{t \in \mathbb{R} : t \geq -1\}$.

In Sect. 6, we study quasi-invariant kernels. Let $J : \mathrm{Aut}(\Omega) \times \Omega \to GL_k(\mathbb{C})$ be a function such that $J(\varphi, \cdot)$ is holomorphic for each φ in $\mathrm{Aut}(\Omega)$, where $\mathrm{Aut}(\Omega)$ is the group of all biholomorphic automorphisms of Ω. A non-negative definite kernel $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ is said to be quasi-invariant with respect to J if K satisfies the following transformation rule:

$$J(\varphi, z) K(\varphi(z), \varphi(w)) J(\varphi, w)^* = K(z, w),\ z, w \in \Omega,\ \varphi \in \mathrm{Aut}(\Omega).$$

It is shown that if $K : \Omega \times \Omega \to \mathbb{C}$ is a quasi-invariant kernel with respect to $J : \mathrm{Aut}(\Omega) \times \Omega \to \mathbb{C} \setminus \{0\}$, then the kernel $K^t\big(\partial_i \bar{\partial}_j \log K\big)_{i,j=1}^m$ is also quasi-invariant with respect to $\mathbb{J}$ whenever $t \in G\mathcal{W}(K)$, where $\mathbb{J}(\varphi, z) = J(\varphi, z)^t D\varphi(z)^{\mathrm{tr}}$, $\varphi \in \mathrm{Aut}(\Omega)$, $z \in \Omega$. In particular, taking $\Omega \subset \mathbb{C}^m$ to be a bounded symmetric domain and setting K to be the Bergman kernel B_Ω, in the language of [20] (see also [2, 3, 5]), we conclude that the multiplication tuple $\boldsymbol{M}_z$ on $(\mathcal{H}, \boldsymbol{B}_\Omega^{(t)})$, where $\boldsymbol{B}_\Omega^{(t)}(z, w) := (B_\Omega^t \partial_i \bar{\partial}_j \log B_\Omega)_{i,j=1}^m$, is homogeneous with respect to the group $\mathrm{Aut}(\Omega)$ for t in $G\mathcal{W}(B_\Omega)$.

2 A New Non-negative Definite Kernel

The scalar version of the following lemma is well-known. However, the easy modifications necessary to prove it in the case of $k \times k$ matrices are omitted.

Lemma 2.1 (Kolmogorov) *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain, and let $\mathcal{H}$ be a Hilbert space. If $\phi_1, \phi_2, \ldots, \phi_k$ are anti-holomorphic functions from Ω into $\mathcal{H}$, then $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ defined by $K(z, w) = \big(\langle \phi_j(w), \phi_i(z)\rangle_{\mathcal{H}}\big)_{i,j=1}^k$, $z, w \in \Omega$, is a sesqui-analytic non-negative definite kernel.*

For any reproducing kernel Hilbert space $(\mathcal{H}, K)$, the following proposition, which is Lemma 4.1 of [8] is a basic tool in what follows.

Proposition 2.2 *Let $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ be a non-negative definite kernel. For every $\boldsymbol{i} \in \mathbb{Z}_+^m$, $\eta \in \mathbb{C}^k$ and $w \in \Omega$, we have*

(i) $\bar\partial^{\boldsymbol{i}} K(\cdot, w)\eta$ *is in* $(\mathcal{H}, K)$,
(ii) $\big\langle f, \bar\partial^{\boldsymbol{i}} K(\cdot, w)\eta\big\rangle_{(\mathcal{H},K)} = \big\langle (\partial^{\boldsymbol{i}} f)(w), \eta\big\rangle_{\mathbb{C}^k}$, $f \in (\mathcal{H}, K)$.

Here and throughout this paper, for any non-negative definite kernel $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ and $\eta \in \mathbb{C}^k$, let $\bar\partial^{\boldsymbol{i}} K(\cdot, w)\eta$ denote the function $\big(\frac{\partial}{\partial \bar w_1}\big)^{i_1} \cdots \big(\frac{\partial}{\partial \bar w_m}\big)^{i_m} K(\cdot, w)\eta$ and $(\partial^{\boldsymbol{i}} f)(z)$ be the function

$$\big(\tfrac{\partial}{\partial z_1}\big)^{i_1} \cdots \big(\tfrac{\partial}{\partial z_m}\big)^{i_m} f(z), \boldsymbol{i} = (i_1, \ldots, i_m) \in \mathbb{Z}_+^m.$$

Proposition 2.3 *Let Ω be a bounded domain in $\mathbb{C}^m$ and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function. Suppose that K^α and K^β, defined on $\Omega \times \Omega$, are non-negative definite for some $\alpha, \beta > 0$. Then the function*

$$K^{\alpha+\beta}(z, w)\Big(\big(\partial_i \bar\partial_j \log K\big)(z, w)\Big)_{i,j=1}^m, \quad z, w \in \Omega,$$

is a non-negative definite kernel on $\Omega \times \Omega$ taking values in $\mathcal{M}_m(\mathbb{C})$.

Proof First, set $\phi_i(z) = \beta \bar\partial_i K^\alpha(\cdot, z) \otimes K^\beta(\cdot, z) - \alpha K^\alpha(\cdot, z) \otimes \bar\partial_i K^\beta(\cdot, z)$, $i = 1, \ldots, m$. From Proposition 2.2, it follows that each ϕ_i is a function from Ω into the Hilbert space $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$. Then we have

$$\begin{aligned}\big\langle \phi_j(w), \phi_i(z)\big\rangle &= \beta^2 \partial_i \bar\partial_j K^\alpha(z, w) K^\beta(z, w) + \alpha^2 K^\alpha(z, w) \partial_i \bar\partial_j K^\beta(z, w)\\ &\quad - \alpha\beta\big(\partial_i K^\alpha(z, w) \bar\partial_j K^\beta(z, w) + \bar\partial_j K^\alpha(z, w) \partial_i K^\beta(z, w)\big)\\ &= \beta^2\big(\alpha(\alpha - 1)K^{\alpha+\beta-2}(z, w)\partial_i K(z, w)\bar\partial_j K(z, w)\\ &\quad + \alpha K^{\alpha+\beta-1}(z, w)\partial_i \bar\partial_j K(z, w)\big)\\ &\quad + \alpha^2\big(\beta(\beta - 1)K^{\alpha+\beta-2}(z, w)\partial_i K(z, w)\bar\partial_j K(z, w)\end{aligned}$$

$$+ \beta K^{\alpha+\beta-1}(z,w)\partial_i\bar{\partial}_j K(z,w)\big)$$
$$- 2\alpha^2\beta^2 K^{\alpha+\beta-2}(z,w)\partial_i K(z,w)\bar{\partial}_j K(z,w)$$
$$= (\alpha^2\beta+\alpha\beta^2)K^{\alpha+\beta-2}(z,w)\big(K(z,w)\partial_i\bar{\partial}_j K(z,w)$$
$$- \partial_i K(z,w)\bar{\partial}_j K(z,w)\big)$$
$$= \alpha\beta(\alpha+\beta)K^{\alpha+\beta}(z,w)\partial_i\bar{\partial}_j \log K(z,w).$$

An application of Lemma 2.1 now completes the proof. □

The particular case, when $\alpha = 1 = \beta$ occurs repeatedly in the following. We therefore record it separately as a corollary.

Corollary 2.4 *Let Ω be a bounded domain in $\mathbb{C}^m$. If $K : \Omega \times \Omega \to \mathbb{C}$ is a non-negative definite kernel, then*

$$K^2(z,w)\Big(\big(\partial_i\bar{\partial}_j \log K\big)(z,w)\Big)_{i,j=1}^m$$

is also a non-negative definite kernel, defined on $\Omega \times \Omega$, taking values in $\mathcal{M}_m(\mathbb{C})$.

A more substantial corollary is the following, which is taken from [4]. Here we provide a slightly different proof. Recall that a non-negative definite kernel $K : \Omega \times \Omega \to \mathbb{C}$ is said to be *infinitely divisible* if for all $t > 0$, K^t is also non-negative definite.

Corollary 2.5 *Let Ω be a bounded domain in $\mathbb{C}^m$. Suppose that $K : \Omega \times \Omega \to \mathbb{C}$ is an infinitely divisible kernel. Then the function $\Big(\big(\partial_i\bar{\partial}_j \log K\big)(z,w)\Big)_{i,j=1}^m$ is a non-negative definite kernel taking values in $\mathcal{M}_m(\mathbb{C})$.*

Proof For $t > 0$, $K^t(z,w)$ is non-negative definite by hypothesis. Then it follows, from Corollary 2.4, that $\big(K^{2t}\partial_i\bar{\partial}_j \log K^t(z,w)\big)_{i,j=1}^m$ is non-negative definite. Hence $\big(K^{2t}\partial_i\bar{\partial}_j \log K(z,w)\big)_{i,j=1}^m$ is non-negative definite for all $t > 0$. Taking the limit as $t \to 0$, we conclude that $\big(\partial_i\bar{\partial}_j \log K(z,w)\big)_{i,j=1}^m$ is non-negative definite. □

Remark 2.6 It is known that even if K is a positive definite kernel,

$$\Big(\big(\partial_i\bar{\partial}_j \log K\big)(z,w)\Big)_{i,j=1}^m$$

need not be a non-negative definite kernel. In fact, $\Big(\big(\partial_i\bar{\partial}_j \log K\big)(z,w)\Big)_{i,j=1}^m$ is non-negative definite if and only if K is infinitely divisible (see [4, Theorem 3.3]).

Let $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ be the positive definite kernel given by $K(z, w) = 1 + \sum_{i=1}^{\infty} a_i z^i \bar{w}^i$, $z, w \in \mathbb{D}$, $a_i > 0$. For any $t > 0$, a direct computation gives

$$\begin{aligned}
&\big(K^t \partial\bar{\partial} \log K\big)(z, w)\\
&\quad = \big(1 + \sum_{i=1}^{\infty} a_i z^i \bar{w}^i\big)^t \; \partial\bar{\partial}\big(\sum_{i=1}^{\infty} a_i z^i \bar{w}^i - \tfrac{(\sum_{i=1}^{\infty} a_i z^i \bar{w}^i)^2}{2} + \cdots\big)\\
&\quad = (1 + t a_1 z\bar{w} + \cdots)(a_1 + 2(2a_2 - a_1^2) z\bar{w} + \cdots)\\
&\quad = a_1 + (4a_2 + (t-2)a_1^2) z\bar{w} + \cdots .
\end{aligned}$$

Thus, if $t < 2$, one may choose $a_1, a_2 > 0$ such that $4a_2 + (t-2)a_1^2 < 0$. Hence $\big(K^t \partial\bar{\partial} \log K\big)(z, w)$ cannot be a non-negative definite kernel. Therefore, in general, for $\big(\big(K^t \partial_i \bar{\partial}_j \log K\big)(z, w) \big)_{i,j=1}^m$ to be non-negative definite, it is necessary that $t \geq 2$.

2.1 Boundedness of the Multiplication Operator on $(\mathcal{H}, \mathbb{K})$

For $\alpha, \beta > 0$, let $\mathbb{K}^{(\alpha,\beta)}$ denote the kernel $\big(K^{\alpha+\beta}\big(\partial_i \bar{\partial}_j \log K\big)(z, w)\big)_{i,j=1}^m$. If $\alpha = 1 = \beta$, then we write $\mathbb{K}$ instead of $\mathbb{K}^{(1,1)}$. For a holomorphic function $f : \Omega \to \mathbb{C}$, the operator M_f of multiplication by f on the linear space $\mathrm{Hol}(\Omega, \mathbb{C}^k)$ is defined by the rule $M_f h = f h$, $h \in \mathrm{Hol}(\Omega, \mathbb{C}^k)$, where $(f h)(z) = f(z)h(z)$, $z \in \Omega$. The boundedness criterion for the multiplication operator M_f restricted to the Hilbert space $(\mathcal{H}, K)$ is well-known for the case of positive definite kernels. In what follows, often we have to work with a kernel which is merely non-negative definite. A precise statement is given below. The first part is from [22] and the second part follows from the observation that the boundedness of the operator $\sum_{i=1}^m M_i M_i^*$ is equivalent to the non-negative definiteness of the kernel $(c^2 - \langle z, w\rangle)K(z, w)$ for some positive constant c.

Lemma 2.7 *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ be a non-negative definite kernel.*

(i) *For any holomorphic function $f : \Omega \to \mathbb{C}$, the operator M_f of multiplication by f is bounded on $(\mathcal{H}, K)$ if and only if there exists a constant $c > 0$ such that $\big(c^2 - f(z)\overline{f(w)}\big)K(z, w)$ is non-negative definite on $\Omega \times \Omega$. In case M_f is bounded, $\|M_f\|$ is the infimum of all $c > 0$ such that $\big(c^2 - f(z)\overline{f(w)}\big)K(z, w)$ is non-negative definite.*

(ii) *The operator M_{z_i} of multiplication by the ith coordinate function z_i is bounded on $(\mathcal{H}, K)$ for $i = 1, \ldots, m$, if and only if there exists a constant $c > 0$ such that $\big(c^2 - \langle z, w\rangle\big)K(z, w)$ is non-negative definite.*

As we have pointed out, the distinction between the non-negative definite kernels and the positive definite ones is very significant. Indeed, as shown in [8, Lemma 3.6], it is interesting that if the operator $\boldsymbol{M}_z := (M_{z_1}, \ldots, M_{z_m})$ is bounded on $(\mathcal{H}, K)$ for some non-negative definite kernel K such that $K(z, z)$, $z \in \Omega$, is invertible, then K is positive definite. A direct proof of this statement, different from the inductive proof of Curto and Salinas is in the PhD thesis of the first named author [14].

It is natural to ask if the operator M_f is bounded on $(\mathcal{H}, K)$, then if it remains bounded on the Hilbert space $(\mathcal{H}, \mathbb{K})$. From the Theorem stated below, in particular, it follows that the operator M_f is bounded on $(\mathcal{H}, \mathbb{K})$ whenever it is bounded on $(\mathcal{H}, K)$.

Theorem 2.8 *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a non-negative definite kernel. Let $f : \Omega \to \mathbb{C}$ be an arbitrary holomorphic function. Suppose that there exists a constant $c > 0$ such that $\big(c^2 - f(z)\overline{f(w)}\big)K(z, w)$ is non-negative definite on $\Omega \times \Omega$. Then the function $\big(c^2 - f(z)\overline{f(w)}\big)^2\mathbb{K}(z, w)$ is non-negative definite on $\Omega \times \Omega$.*

Proof Without loss of generality, we assume that f is non-constant and K is non-zero. The function $G(z, w) := \big(c^2 - f(z)\overline{f(w)}\big)K(z, w)$ is non-negative definite on $\Omega \times \Omega$ by hypothesis. We claim that $|f(z)| < c$ for all z in Ω. If not, then by the open mapping theorem, there exists an open set $\Omega_0 \subset \Omega$ such that $|f(z)| > c$, $z \in \Omega_0$. Since $\big(c^2 - |f(z)|^2\big)K(z, z) \geq 0$, it follows that $K(z, z) = 0$ for all $z \in \Omega_0$. Now, let h be an arbitrary vector in $(\mathcal{H}, K)$. Clearly, $|h(z)| = |\langle h, K(\cdot, z)\rangle| \leq \|h\|\|K(\cdot, z)\| = \|h\|K(z, z)^{\frac{1}{2}} = 0$ for all $z \in \Omega_0$. Consequently, $h(z) = 0$ on Ω_0. Since Ω is connected and h is holomorphic, it follows that $h = 0$. This contradicts the assumption that K is non-zero verifying the validity of our claim.

From the claim, we have that the function $c^2 - f(z)\overline{f(w)}$ is non-vanishing on $\Omega \times \Omega$. Therefore, the kernel K can be written as the product

$$K(z, w) = \frac{1}{\big(c^2 - f(z)\overline{f(w)}\big)}G(z, w),\ \ z, w \in \Omega.$$

Since $|f(z)| < c$ on Ω, the function $\frac{1}{\big(c^2 - f(z)\overline{f(w)}\big)}$ has a convergent power series expansion, namely,

$$\frac{1}{\big(c^2 - f(z)\overline{f(w)}\big)} = \sum_{n=0}^{\infty} \frac{1}{c^{2(n+1)}} f(z)^n\overline{f(w)}^n,\quad z, w \in \Omega.$$

Therefore it defines a non-negative definite kernel on $\Omega \times \Omega$. Note that

$$\begin{aligned}&\big(K(z, w)^2\partial_i\bar{\partial}_j \log K(z, w)\big)_{i,j=1}^m\\ =&\big(K(z, w)^2\partial_i\bar{\partial}_j \log \frac{1}{\big(c^2 - f(z)\overline{f(w)}\big)}\big)_{i,j=1}^m + \big(K(z, w)^2\partial_i\bar{\partial}_j \log G(z, w)\big)_{i,j=1}^m\end{aligned}$$

$$= \frac{1}{\big(c^2 - f(z)\overline{f(w)}\big)^2} \Big(K(z,w)^2 \big(\partial_i f(z) \overline{\partial_j f(w)}\big)_{i,j=1}^m + G(z,w)^2 \big(\partial_i \bar{\partial}_j \log G(z,w)\big)_{i,j=1}^m \Big),$$

where for the second equality, we have used that

$$\partial_i \bar{\partial}_j \log \frac{1}{\big(c^2 - f(z)\overline{f(w)}\big)} = \frac{\partial_i f(z)\overline{\partial_j f(w)}}{\big(c^2 - f(z)\overline{f(w)}\big)^2},\ \ z, w \in \Omega,\ \ 1 \le i, j \le m.$$

Thus

$$\big(c^2 - f(z)\overline{f(w)}\big)^2 \mathbb{K}(z,w) = K(z,w)^2 \Big(\partial_i f(z)\overline{\partial_j f(w)} \Big)_{i,j=1}^m + \Big(G(z,w)^2 \partial_i \bar{\partial}_j \log G(z,w) \Big)_{i,j=1}^m.$$

By Lemma 2.1, the function $\big(\partial_i f(z)\overline{\partial_j f(w)}\big)_{i,j=1}^m$ is non-negative definite on $\Omega \times \Omega$. Thus the product $K(z,w)^2\big(\partial_i f(z)\overline{\partial_j f(w)}\big)_{i,j=1}^m$ is also non-negative definite on $\Omega \times \Omega$. Since G is non-negative definite on $\Omega \times \Omega$, by Corollary 2.4, the function $\big(G(z,w)^2 \partial_i \bar{\partial}_j \log G(z,w)\big)_{i,j=1}^m$ is also non-negative definite on $\Omega \times \Omega$. The proof is now complete since the sum of two non-negative definite kernels remains non-negative definite. □

A sufficient condition for the boundedness of the multiplication operator on the Hilbert space $\big(\mathcal{H}, \mathbb{K}\big)$ is an immediate Corollary.

Corollary 2.9 *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a non-negative definite kernel. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Suppose that the multiplication operator M_f on $(\mathcal{H}, K)$ is bounded. Then the multiplication operator M_f is also bounded on $(\mathcal{H}, \mathbb{K})$.*

Proof Since the operator M_f is bounded on $(\mathcal{H}, K)$, by Lemma 2.7, we find a constant $c > 0$ such that $\big(c^2 - f(z)\overline{f(w)}\big)K(z,w)$ is non-negative definite on $\Omega \times \Omega$. Then, by Theorem 2.8, it follows that $\big(c^2 - f(z)\overline{f(w)}\big)^2 \mathbb{K}(z,w)$ is non-negative definite on $\Omega \times \Omega$. Also, from the proof of Theorem 2.8, we have that $\big(c^2 - f(z)\overline{f(w)}\big)^{-1}$ is non-negative definite on $\Omega \times \Omega$ (assuming that f is non-constant). Hence $\big(c - f(z)\overline{f(w)}\big)\mathbb{K}(z,w)$, being the product of two non-negative definite kernels, is non-negative definite on $\Omega \times \Omega$. An application of Lemma 2.7, a second time, completes the proof. □

A second Corollary provides a sufficient condition for the positive definiteness of the kernel $\mathbb{K}$.

Corollary 2.10 *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a non-negative definite kernel satisfying $K(w,w) > 0$, $w \in \Omega$. Suppose that the* *multiplication operator M_{z_i} on $(\mathcal{H}, K)$ is bounded for $i = 1, \ldots, m$. Then the kernel $\mathbb{K}$ is positive definite on $\Omega \times \Omega$.*

Proof *By Corollary 2.4, we already have that $\mathbb{K}$ is non-negative definite. Moreover, since M_{z_i} on $(\mathcal{H}, K)$ is bounded for $i = 1, \ldots, m$, it follows from Theorem 2.9 that M_{z_i} is bounded on $(\mathcal{H}, \mathbb{K})$ also. Therefore, using [8, Lemma 3.6], we see that $\mathbb{K}$ is positive definite if $\mathbb{K}(w, w)$ is invertible for all $w \in \Omega$. To verify this, set*

$$\phi_i(w) = \bar{\partial}_i K(\cdot, w) \otimes K(\cdot, w) - K(\cdot, w) \otimes \bar{\partial}_i K(\cdot, w), \ 1 \leq i \leq m.$$

We have $\mathbb{K}(w, w) = \frac{1}{2}\big(\langle \phi_j(w), \phi_i(w)\rangle\big)_{i,j=1}^m$ from the proof of Proposition 2.3. Therefore $\mathbb{K}(w, w)$ is invertible if the vectors $\phi_1(w), \ldots, \phi_m(w)$ are linearly independent. Note that for $w = (w_1, \ldots, w_m)$ in Ω and $j = 1, \ldots, m$, we have $(M_{z_j} - w_j)^* K(\cdot, w) = 0$. Differentiating this equation with respect to $\bar{w}_i$, we obtain

$$(M_{z_j} - w_j)^* \bar{\partial}_i K(\cdot, w) = \delta_{ij} K(\cdot, w), \ 1 \leq i, j \leq m.$$

Thus

$$\big((M_{z_j} - w_j)^* \otimes I\big)\big(\phi_i(w)\big) = \delta_{ij} K(\cdot, w) \otimes K(\cdot, w), \ 1 \leq i, j \leq m. \tag{2.1}$$

Now assume that $\sum_{i=1}^m c_i \phi_i(w) = 0$ for some scalars $c_1, \ldots, c_m$. Then, for $1 \leq j \leq m$, we have that $\sum_{i=1}^m \big((M_{z_j} - w_j)^* \otimes I\big)\big(\phi_i(w)\big) = 0$. Thus, using (2.1), we see that $c_j K(\cdot, w) \otimes K(\cdot, w) = 0$. Since $K(w, w) > 0$, we conclude that $c_j = 0$. Hence the vectors $\phi_1(w), \ldots, \phi_m(w)$ are linearly independent. This completes the proof. □

Remark 2.11 Recall that an operator T is said to be a 2-hyper contraction if $I - T^*T \geq 0$ and $I - 2T^*T + T^{*2}T^2 \geq 0$. If $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ is a non-negative definite kernel, then it is not hard to verify that the adjoint M_z^* of the multiplication by the coordinate function z is a 2-hyper contraction on $(\mathcal{H}, K)$ if and only if $(1 - z\bar{w})^2 K$ is non-negative definite. It follows from Theorem 2.8 that if M_z^* on $(\mathcal{H}, K)$ is a contraction, then M_z^* on $(\mathcal{H}, \mathbb{K})$ is a 2-hyper contraction.

3 Realization of $\big(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\big)$

Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function. Suppose that the functions K^α and K^β are non-negative definite for some $\alpha, \beta > 0$. In this section, we give a description of the Hilbert space $\big(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\big)$. As before, we set

$$\phi_i(w) = \beta \bar{\partial}_i K^\alpha(\cdot, w) \otimes K^\beta(\cdot, w) - \alpha K^\alpha(\cdot, w) \otimes \bar{\partial}_i K^\beta(\cdot, w), \ 1 \leq i \leq m, \ w \in \Omega.$$

Let $\mathcal{N}$ be the subspace of $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$ which is the closed linear span of the vectors

$$\big\{ \phi_i(w) : w \in \Omega,\ 1 \le i \le m \big\}.$$

From the definition of $\mathcal{N}$, it is not easy to determine which vectors are in it. A useful alternative description of the space $\mathcal{N}$ is given below.

Recall that $K^\alpha \otimes K^\beta$ is the reproducing kernel for the Hilbert space $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$, where the kernel $K^\alpha \otimes K^\beta$ on $(\Omega \times \Omega) \times (\Omega \times \Omega)$ is given by

$$K^\alpha \otimes K^\beta(z, \zeta; z', \zeta') = K^\alpha(z, z') K^\beta(\zeta, \zeta'),$$

$z = (z_1, \dots, z_m)$, $\zeta = (\zeta_1, \dots, \zeta_m)$, $z' = (z_{m+1}, \dots, z_{2m})$, $\zeta' = (\zeta_{m+1}, \dots, \zeta_{2m})$ are in Ω. We realize the Hilbert space $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$ as a space consisting of holomorphic functions on $\Omega \times \Omega$. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be the subspaces defined by

$$\mathcal{A}_0 = \big\{ f \in (\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta) : f_{|\Delta} = 0 \big\}$$

and

$$\mathcal{A}_1 = \big\{ f \in (\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta) : f_{|\Delta} = (\partial_{m+1} f)_{|\Delta} = \dots = (\partial_{2m} f)_{|\Delta} = 0 \big\},$$

where Δ is the diagonal set $\{(z, z) \in \Omega \times \Omega : z \in \Omega\}$, $\partial_i f$ is the derivative of f with respect to the ith variable, and $f_{|\Delta}$, $(\partial_i f)_{|\Delta}$ denote the restrictions to the set Δ of the functions f, $\partial_i f$, respectively. It is easy to see that both $\mathcal{A}_0$ and $\mathcal{A}_1$ are closed subspaces of the Hilbert space $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$ and $\mathcal{A}_1$ is a closed subspace of $\mathcal{A}_0$. Now observe that, for $1 \le i \le m$, we have

$$\begin{aligned} \bar\partial_i (K^\alpha \otimes K^\beta)(\cdot, (z', \zeta')) &= \bar\partial_i K^\alpha(\cdot, z') \otimes K^\beta(\cdot, \zeta'),\ z', \zeta' \in \Omega \\ \bar\partial_{m+i} (K^\alpha \otimes K^\beta)(\cdot, (z', \zeta')) &= K^\alpha(\cdot, z') \otimes \bar\partial_i K^\beta(\cdot, \zeta'),\ z', \zeta' \in \Omega. \end{aligned} \tag{3.1}$$

Hence, taking $z' = \zeta' = w$, we see that

$$\phi_i(w) = \beta \bar\partial_i (K^\alpha \otimes K^\beta)(\cdot, (w, w)) - \alpha \bar\partial_{m+i} (K^\alpha \otimes K^\beta)(\cdot, (w, w)). \tag{3.2}$$

We now state a useful lemma on the Taylor coefficients of an analytic functions. The straightforward proof follows from the chain rule [23, page 8], which is omitted.

Lemma 3.1 *Suppose that $f : \Omega \times \Omega \to \mathbb{C}$ is a holomorphic function satisfying $f_{|\Delta} = 0$. Then*

$$(\partial_i f)_{|\Delta} + (\partial_{m+i} f)_{|\Delta} = 0, \quad 1 \le i \le m.$$

An alternative description of the subspace $\mathcal{N}$ of $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$ is provided below.

Proposition 3.2 $\mathcal{N} = \mathcal{A}_0 \ominus \mathcal{A}_1$.

Proof For all $z \in \Omega$, we see that

$$\phi_i(w)(z,z) = \alpha\beta K^{\alpha+\beta-1}(z,w)\bar{\partial}_i K(z,w) - \alpha\beta K^{\alpha+\beta-1}(z,w)\bar{\partial}_i K(z,w) = 0.$$

Hence each $\phi_i(w)$, $w \in \Omega, 1 \leq i \leq m$, belongs to $\mathcal{A}_0$ and consequently, $\mathcal{N} \subset \mathcal{A}_0$. Therefore, to complete the proof of the proposition, it is enough to show that $\mathcal{A}_0 \ominus \mathcal{N} = \mathcal{A}_1$.

To verify this, note that $f \in \mathcal{N}^\perp$ if and only if $\langle f, \phi_i(w)\rangle = 0$, $1 \leq i \leq m$, $w \in \Omega$. Now, in view of (3.2) and Proposition 2.2, we have that

$$\begin{aligned}\langle f, \phi_i(w)\rangle =& \big\langle f, \beta\bar{\partial}_i(K^\alpha \otimes K^\beta)(\cdot,(w,w)) - \alpha\bar{\partial}_{m+i}(K^\alpha \otimes K^\beta)(\cdot,(w,w))\big\rangle \\ =& \beta(\partial_i f)(w,w) - \alpha(\partial_{m+i} f)(w,w),\ \ 1 \leq i \leq m,\ \ w \in \Omega.\end{aligned}$$

Thus f is in $\mathcal{N}^\perp$ if and only if the function $\beta\,(\partial_i f)_{|\Delta} - \alpha\,(\partial_{m+i} f)_{|\Delta} = 0$, $1 \leq i \leq m$. Combining this with Lemma 3.1, we see that any $f \in \mathcal{A}_0 \ominus \mathcal{N}$, satisfies

$$\begin{aligned}&\beta(\partial_i f)_{|\Delta} - \alpha(\partial_{m+i} f)_{|\Delta} = 0,\\ &(\partial_i f)_{|\Delta} + (\partial_{m+i} f)_{|\Delta} = 0,\end{aligned}$$

for $1 \leq i \leq m$. Therefore, we have $(\partial_i f)_{|\Delta} = (\partial_{m+i} f)_{|\Delta} = 0$, $1 \leq i \leq m$. Hence f belongs to $\mathcal{A}_1$.

Conversely, let $f \in \mathcal{A}_1$. In particular, $f \in \mathcal{A}_0$. Hence invoking Lemma 3.1 once again, we see that

$$(\partial_i f)_{|\Delta} + (\partial_{m+i} f)_{|\Delta} = 0,\ \ 1 \leq i \leq m.$$

Since f is in $\mathcal{A}_1$, $(\partial_{m+i} f)_{|\Delta} = 0$, $1 \leq i \leq m$, by definition. Therefore, $(\partial_i f)_{|\Delta} = (\partial_{m+i} f)_{|\Delta} = 0,\ 1 \leq i \leq m$, which implies

$$\beta(\partial_i f)_{|\Delta} - \alpha(\partial_{m+i} f)_{|\Delta} = 0,\ \ 1 \leq i \leq m.$$

Hence $f \in \mathcal{A}_0 \ominus \mathcal{N}$, completing the proof. □

We now give a description of the Hilbert space $\big(\mathcal{H},\ \mathbb{K}^{(\alpha,\beta)}\big)$. Define a linear map $\mathcal{R}_1 : (\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta) \to \mathrm{Hol}(\Omega, \mathbb{C}^m)$ by setting

$$\mathcal{R}_1(f) = \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}}\begin{pmatrix}(\beta\partial_1 f - \alpha\partial_{m+1} f)_{|\Delta}\\ \vdots \\ (\beta\partial_m f - \alpha\partial_{2m} f)_{|\Delta}\end{pmatrix} \tag{3.3}$$

for $f \in (\mathcal{H}, K^{\alpha}) \otimes (\mathcal{H}, K^{\beta})$ and note that

$$\mathcal{R}_1(f)(w) = \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \begin{pmatrix} \langle f, \phi_1(w)\rangle \\ \vdots \\ \langle f, \phi_m(w)\rangle \end{pmatrix}, \ w \in \Omega. \tag{3.4}$$

From Eq. (3.4), it is easy to see that $\ker \mathcal{R}_1 = \mathcal{N}^{\perp}$. We have $\mathcal{N} = \mathcal{A}_0 \ominus \mathcal{A}_1$, see Proposition 3.2. Therefore, $\ker \mathcal{R}_1^{\perp} = \mathcal{A}_0 \ominus \mathcal{A}_1$ and the map $\mathcal{R}_{1|\mathcal{A}_0 \ominus \mathcal{A}_1} \to \mathrm{ran}\mathcal{R}_1$ is bijective. Require this map to be a unitary by defining an appropriate inner product on $\mathrm{ran}\mathcal{R}_1$, that is, Set

$$\langle \mathcal{R}_1(f), \mathcal{R}_1(g)\rangle := \left\langle P_{\mathcal{A}_0 \ominus \mathcal{A}_1} f, P_{\mathcal{A}_0 \ominus \mathcal{A}_1} g\right\rangle, \ f, g \in (\mathcal{H}, K^{\alpha}) \otimes (\mathcal{H}, K^{\beta}), \tag{3.5}$$

where $P_{\mathcal{A}_0 \ominus \mathcal{A}_1}$ is the orthogonal projection of $(\mathcal{H}, K^{\alpha}) \otimes (\mathcal{H}, K^{\beta})$ onto the subspace $\mathcal{A}_0 \ominus \mathcal{A}_1$. This choice of the inner product on the range of $\mathcal{R}_1$ makes the map $\mathcal{R}_1$ unitary.

Theorem 3.3 *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function. Suppose that the functions K^{α} and K^{β} are non-negative definite for some $\alpha, \beta > 0$. Let $\mathcal{R}_1$ be the map defined by* (3.3). *Then the Hilbert space determined by the non-negative definite kernel $\mathbb{K}^{(\alpha,\beta)}$ coincides with the space* ran $\mathcal{R}_1$ *and the inner product given by* (3.5) *on* ran $\mathcal{R}_1$ *agrees with the one induced by the kernel $\mathbb{K}^{(\alpha,\beta)}$.*

Proof Let $\{e_1, \ldots, e_m\}$ be the standard orthonormal basis of $\mathbb{C}^m$. From the proof of Proposition 2.3, for $1 \le i, j \le m$, we have

$$\begin{aligned}\left\langle \phi_j(w), \phi_i(z)\right\rangle &= \alpha\beta(\alpha+\beta) K^{\alpha+\beta}(z,w) \partial_i \bar{\partial}_j \log K(z,w) \\ &= \alpha\beta(\alpha+\beta) \left\langle \mathbb{K}^{(\alpha,\beta)}(z,w) e_j, e_i \right\rangle_{\mathbb{C}^m}, \ z, w \in \Omega.\end{aligned}$$

Therefore, from (3.4), it follows that for all $w \in \Omega$ and $1 \le j \le m$,

$$\mathcal{R}_1(\phi_j(w)) = \sqrt{\alpha\beta(\alpha+\beta)} \mathbb{K}^{(\alpha,\beta)}(\cdot, w) e_j.$$

Hence, for all $w \in \Omega$ and $\eta \in \mathbb{C}^m$, $\mathbb{K}^{(\alpha,\beta)}(\cdot, w)\eta$ belongs to $\mathrm{ran}\mathcal{R}_1$. Let $\mathcal{R}_1(f)$ be an arbitrary element in $\mathrm{ran}\mathcal{R}_1$ where $f \in \mathcal{A}_0 \ominus \mathcal{A}_1$. Then

$$\begin{aligned}\left\langle \mathcal{R}_1(f), \mathbb{K}^{(\alpha,\beta)}(\cdot, w) e_j \right\rangle &= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \left\langle \mathcal{R}_1(f), \mathcal{R}_1(\phi_j(w))\right\rangle \\ &= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \left\langle f, \phi_j(w)\right\rangle\end{aligned}$$

$$= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}}(\beta\partial_j f(w,w) - \alpha\partial_{m+j} f(w,w))$$
$$= \left\langle \mathcal{R}_1(f)(w), e_j \right\rangle_{\mathbb{C}^m},$$

where the second equality follows since both f and $\phi_j(w)$ belong to $\mathcal{A}_0 \ominus \mathcal{A}_1$. This completes the proof. □

We obtain the density of polynomials in $\left(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\right)$ as a consequence of this theorem. Let $z = (z_1, \ldots, z_m)$ and let $\mathbb{C}[z] := \mathbb{C}[z_1, \ldots, z_m]$ denote the ring of polynomials in m-variables. The following proposition gives a sufficient condition for density of $\mathbb{C}[z] \otimes \mathbb{C}^m$ in the Hilbert space $\left(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\right)$.

Proposition 3.4 *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are non-negative definite on $\Omega \times \Omega$ for some $\alpha, \beta > 0$. Suppose that both the Hilbert spaces $(\mathcal{H}, K^\alpha)$ and $(\mathcal{H}, K^\beta)$ contain the polynomial ring $\mathbb{C}[z]$ as a dense subset. Then the Hilbert space $\left(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\right)$ contains the ring $\mathbb{C}[z] \otimes \mathbb{C}^m$ as a dense subset.*

Proof Since $C[z]$ is dense in both the Hilbert spaces $(\mathcal{H}, K^\alpha)$ and $(\mathcal{H}, K^\beta)$, it follows that $C[z] \otimes C[z]$, which is $\mathbb{C}[z_1, \ldots, z_{2m}]$, is contained in the Hilbert space $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$ and is dense in it. Since $\mathcal{R}_1$ maps $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$ onto $\left(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\right)$, to complete the proof, it suffices to show that $\mathcal{R}_1(\mathbb{C}[z_1, \ldots, z_{2m}]) = \mathbb{C}[z] \otimes \mathbb{C}^m$. It is easy to see that $\mathcal{R}_1(\mathbb{C}[z_1, \ldots, z_{2m}]) \subseteq \mathbb{C}[z] \otimes \mathbb{C}^m$. Conversely, if $\sum_{i=1}^m p_i(z_1, \ldots, z_m) \otimes e_i$ is an arbitrary element of $\mathbb{C}[z] \otimes \mathbb{C}^m$, then it is easily verified that the function $p(z_1, \ldots, z_{2m}) := \sqrt{\frac{\alpha\beta}{\alpha+\beta}} \sum_{i=1}^m (z_i - z_{m+i}) p_i(z_1, \ldots, z_m)$ belongs to $\mathbb{C}[z_1, \ldots, z_{2m}]$ and $\mathcal{R}_1(p) = \sum_{i=1}^m p_i(z_1, \ldots, z_m) \otimes e_i$. Therefore $\mathcal{R}_1(\mathbb{C}[z_1, \ldots, z_{2m}]) = \mathbb{C}[z] \otimes \mathbb{C}^m$ completing the proof. □

3.1 Description of the Hilbert Module $\mathcal{S}_1$

In this subsection, we give a description of the Hilbert module $\mathcal{S}_1$ in the particular case when $K_1 = K^\alpha$ and $K_2 = K^\beta$ for some sesqui-analytic function K defined on $\Omega \times \Omega$ and a pair of positive real numbers α, β.

Theorem 3.5 *Let $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β, defined on $\Omega \times \Omega$, are non-negative definite for some $\alpha, \beta > 0$. Suppose that the multiplication operators $M_{z_i}, i = 1, 2, \ldots, m$, are bounded on both $(\mathcal{H}, K^\alpha)$ and $(\mathcal{H}, K^\beta)$. Then the Hilbert module $\mathcal{S}_1$ is isomorphic to the push-forward module $\iota_\star\left(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\right)$ via the module map $\mathcal{R}_{1|\mathcal{S}_1}$.*

Proof From Theorem 3.3, it follows that the map $\mathcal{R}_1$ defined in (3.3) is a unitary map from $\mathcal{S}_1$ onto $(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$. Now we will show that $\mathcal{R}_1 P_{\mathcal{S}_1}(ph) = (p \circ \iota)\mathcal{R}_1 h,\ h \in \mathcal{S}_1, p \in \mathbb{C}[z_1, \ldots, z_{2m}]$. Let h be an arbitrary element of $\mathcal{S}_1$. Since $\ker \mathcal{R}_1 = \mathcal{S}_1^\perp$ (see the discussion before Theorem 3.3), it follows that $\mathcal{R}_1 P_{\mathcal{S}_1}(ph) =$

$\mathcal{R}_1(ph),\ p \in \mathbb{C}[z_1, \dots, z_{2m}]$. Hence

$$\begin{aligned}\mathcal{R}_1 P_{\mathcal{S}_1}(ph) &= \mathcal{R}_1(ph)\\ &= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}}\sum_{j=1}^{m}(\beta\partial_j(ph) - \alpha\partial_{m+j}(ph))_{|\Delta} \otimes e_j\\ &= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}}\Big(\sum_{j=1}^{m} p_{|\Delta}(\beta\partial_j h - \alpha\partial_{m+j}h)_{|\Delta} \otimes e_j\\ &\qquad + \sum_{j=1}^{m} h_{|\Delta}(\beta\partial_j p - \alpha\partial_{m+j}p)_{|\Delta} \otimes e_j\Big)\\ &= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}}\sum_{j=1}^{m} p_{|\Delta}(\beta\partial_j h - \alpha\partial_{m+j}h)_{|\Delta} \otimes e_j \quad (\text{since } h \in \mathcal{S}_1)\\ &= (p \circ \iota)\mathcal{R}_1 h,\end{aligned}$$

completing the proof. □

Notation 3.6 *For $1 \le i \le m$, let $M_i^{(1)}$ and $M_i^{(2)}$ denote the operators of multiplication by the coordinate function z_i on the Hilbert spaces $(\mathcal{H}, K_1)$ and $(\mathcal{H}, K_2)$, respectively. If $m = 1$, we let $M^{(1)}$ and $M^{(2)}$ denote the operators $M_1^{(1)}$ and $M_1^{(2)}$, respectively.*

In case $K_1 = K^\alpha$ and $K_2 = K^\beta$, let $M_i^{(\alpha)}$, $M_i^{(\beta)}$ and $M_i^{(\alpha+\beta)}$ denote the operators of multiplication by the coordinate function z_i on the Hilbert spaces $(\mathcal{H}, K^\alpha)$, $(\mathcal{H}, K^\beta)$ and $(\mathcal{H}, K^{\alpha+\beta})$, respectively. If $m = 1$, we write $M^{(\alpha)}$, $M^{(\beta)}$ and $M^{(\alpha+\beta)}$ instead of $M_1^{(\alpha)}$, $M_1^{(\beta)}$ and $M_1^{(\alpha+\beta)}$, respectively.

Finally, let $\mathbb{M}_i^{(\alpha,\beta)}$ denote the operator of multiplication by the coordinate function z_i on $(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$. Also let $\mathbb{M}^{(\alpha,\beta)}$ denote the operator $\mathbb{M}_1^{(\alpha,\beta)}$ whenever $m = 1$.

Remark 3.7 It is verified that $(M_i^{(\alpha)} \otimes I)^*(\phi_j(w)) = \bar{w}_i\phi_j(w) + \beta\delta_{ij}K^\alpha(\cdot, w) \otimes K^\beta(\cdot, w)$ and $(I \otimes M_i^{(\beta)})^*(\phi_j(w)) = \bar{w}_i\phi_j(w) - \alpha\delta_{ij}K^\alpha(\cdot, w) \otimes K^\beta(\cdot, w)$, $1 \le i, j \le m, w \in \Omega$. Therefore,

$$P_{\mathcal{S}_1}(M_i^{(\alpha)} \otimes I)_{|\mathcal{S}_1} = P_{\mathcal{S}_1}(I \otimes M_i^{(\beta)})_{|\mathcal{S}_1},\ i = 1, 2, \dots, m.$$

Corollary 3.8 *The m-tuple $\big(P_{\mathcal{S}_1}(M_1^{(\alpha)} \otimes I)_{|\mathcal{S}_1}, \dots, P_{\mathcal{S}_1}(M_m^{(\alpha)} \otimes I)_{|\mathcal{S}_1}\big)$ is unitarily equivalent to the m-tuple $(\mathbb{M}_1^{(\alpha,\beta)}, \dots, \mathbb{M}_m^{(\alpha,\beta)})$ on $\big(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\big)$. In particular, if either the m-tuple of operators $(M_1^{(\alpha)}, \dots, M_m^{(\alpha)})$ on $(\mathcal{H}, K^\alpha)$ or the*

m-tuple of operators $(M_{(1)}^{(\beta)}, \ldots, M_m^{(\beta)})$ on $(\mathcal{H}, K^\beta)$ is bounded, then the m-tuple $(\mathbb{M}_1^{(\alpha,\beta)}, \ldots, \mathbb{M}_m^{(\alpha,\beta)})$ is also bounded on $\big(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\big)$.

Proof The proof of the first statement follows from Theorem 3.5 and the proof of the second statement follows from the first together with Remark 3.7. □

3.2 Description of the Quotient Module $\mathcal{A}_1^\perp$

In this subsection, we give a description of the quotient module $\mathcal{A}_1^\perp$. Let $(\mathcal{H}, K^{\alpha+\beta})\widehat{\oplus}(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$ be the Hilbert module, which is the Hilbert space $(\mathcal{H}, K^{\alpha+\beta})\oplus(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$ equipped with the multiplication over the polynomial ring $\mathbb{C}[z_1, \ldots, z_{2m}]$ induced by the $2m$-tuple $(T_1, \ldots, T_m, T_{m+1}, \ldots, T_{2m})$ described below. First, for any polynomial $p \in \mathbb{C}[z_1, \ldots, z_{2m}]$, let $p^*(z) := (p \circ \iota)(z) = p(z, z)$, $z \in \Omega$ and let $S_p : (\mathcal{H}, K^{\alpha+\beta}) \to (\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$ be the operator given by

$$S_p(f_0) = \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \sum_{j=1}^{m} (\beta(\partial_j p)^* - \alpha(\partial_{m+j} p)^*) f_0 \otimes e_j,\ f_0 \in (\mathcal{H}, K^{\alpha+\beta}).$$

On the Hilbert space $(\mathcal{H}, K^{\alpha+\beta}) \oplus (\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$, let $T_i = \begin{pmatrix} M_{z_i} & 0 \\ S_{z_i} & M_{z_i} \end{pmatrix}$, and $T_{m+i} = \begin{pmatrix} M_{z_i} & 0 \\ S_{z_{m+i}} & M_{z_i} \end{pmatrix}$, $1 \le i \le m$. Now, a straightforward verification shows that the module multiplication induced by these $2m$-tuple of operators is given by the formula:

$$\mathbf{m}_p(f_0 \oplus f_1) = \begin{pmatrix} M_{p^*} f_0 & 0 \\ S_p f_0 & M_{p^*} f_1 \end{pmatrix},\ f_0 \oplus f_1 \in (\mathcal{H}, K^{\alpha+\beta}) \oplus (\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}). \tag{3.6}$$

Clearly, this module multiplication is distinct from the one induced by the $M_p \oplus M_p$, $p \in \mathbb{C}[z_1, \ldots, z_m]$ on the direct sum $(\mathcal{H}, K^{\alpha+\beta})\oplus(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$.

Theorem 3.9 *Let $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β, defined on $\Omega \times \Omega$, are non-negative definite for some $\alpha, \beta > 0$. Suppose that the multiplication operators M_{z_i}, $i = 1, 2, \ldots, m$, are bounded on both $(\mathcal{H}, K^\alpha)$ and $(\mathcal{H}, K^\beta)$. Then the quotient module $\mathcal{A}_1^\perp$ and the Hilbert module $(\mathcal{H}, K^{\alpha+\beta})\widehat{\oplus}(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$ are isomorphic.*

Proof The proof is accomplished by showing that, for an arbitrary polynomial p in $\mathbb{C}[z_1, \ldots, z_{2m}]$, the compression operator $P_{\mathcal{A}_1^\perp} M_{p|\mathcal{A}_1^\perp}$ is unitarily equivalent to the operator $\begin{pmatrix} M_{p^*} & 0 \\ S_p & M_{p^*} \end{pmatrix}$ on $(\mathcal{H}, K^{\alpha+\beta}) \bigoplus(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$.

We recall that the map $\mathcal{R}_0 : (\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta) \to (\mathcal{H}, K^{\alpha+\beta})$ given by $\mathcal{R}_0(f) = f_{|\Delta}$, f in $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$, defines a unitary map from $\mathcal{S}_0$ onto $(\mathcal{H}, K^{\alpha+\beta})$, and it intertwines the operators $P_{\mathcal{S}_0} M_{p|\mathcal{S}_0}$ on $\mathcal{S}_0$ and M_{p^*} on

$(\mathcal{H}, K^{\alpha+\beta})$, that is, $M_{p^*}\mathcal{R}_{0|\mathcal{S}_0} = \mathcal{R}_{0|\mathcal{S}_0} P_{\mathcal{S}_0} M_{p|\mathcal{S}_0}$. Combining this with Theorem 3.3, we conclude that the map $\mathcal{R} = \begin{pmatrix} \mathcal{R}_{0|\mathcal{S}_0} & 0 \\ 0 & \mathcal{R}_{1|\mathcal{S}_1} \end{pmatrix}$ is unitary from $\mathcal{S}_0 \bigoplus \mathcal{S}_1$ (which is $\mathcal{A}_1^\perp$) to $(\mathcal{H}, K^{\alpha+\beta}) \bigoplus (\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$. Since $\mathcal{S}_0$ is invariant under M_p^*, it follows that $P_{\mathcal{S}_1} M^*_{p|\mathcal{S}_0} = 0$. Hence

$$\mathcal{R} P_{\mathcal{A}_1^\perp} M^*_{p|\mathcal{A}_1^\perp} \mathcal{R}^* = \begin{pmatrix} \mathcal{R}_0 P_{\mathcal{S}_0} M^*_{p|\mathcal{S}_0} \mathcal{R}_0^* & \mathcal{R}_0 P_{\mathcal{S}_0} M^*_{p|\mathcal{S}_1} \mathcal{R}_1^* \\ 0 & \mathcal{R}_1 P_{\mathcal{S}_1} M^*_{p|\mathcal{S}_1} \mathcal{R}_1^* \end{pmatrix}$$

on $\mathcal{S}_0 \bigoplus \mathcal{S}_1$. We have $\mathcal{R}_0 P_{\mathcal{S}_0} M^*_{p|\mathcal{S}_0} \mathcal{R}_0^* = (M_{p^*})^*$, already, on $(\mathcal{H}, K^{\alpha+\beta})$. From Theorem 3.5, we see that $\mathcal{R}_1 P_{\mathcal{S}_1} M^*_{p|\mathcal{S}_1} \mathcal{R}_1^* = (M_{p^*})^*$ on $(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$. To prove this, note that $\mathcal{R}_0 P_{\mathcal{S}_0} M^*_{p|\mathcal{S}_1} \mathcal{R}_1^* = S_p^*$. Recall that $\mathcal{R}_1^*(\mathbb{K}^{(\alpha,\beta)}(\cdot, w)e_j) = \phi_j(w)$. Consequently, an easy computation gives

$$\begin{aligned} &\mathcal{R}_0 P_{\mathcal{S}_0} M^*_{p|\mathcal{S}_1} \mathcal{R}_1^* (\mathbb{K}^{(\alpha,\beta)}(\cdot, w)e_j) \\ &\quad = \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} (\overline{\beta(\partial_j p)(w, w) - \alpha(\partial_{m+j} p)(w, w)}) K^{\alpha+\beta}(\cdot, w). \end{aligned}$$

Set $S_p^\sharp = \mathcal{R}_1 P_{\mathcal{S}_1} M_{p|\mathcal{S}_0} \mathcal{R}_0^*$. Then for $1 \le j \le m$, and $w \in \Omega$, we get

$$\begin{aligned} &(S_p^\sharp)^*(\mathbb{K}^{(\alpha,\beta)}(\cdot, w)e_j) \\ &\quad = \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} (\overline{\beta(\partial_j p)(w, w) - \alpha(\partial_{m+j} p)(w, w)}) K^{\alpha+\beta}(\cdot, w). \end{aligned}$$

For f in $(\mathcal{H}, K^{\alpha+\beta})$, we have

$$\begin{aligned} \langle S_p^\sharp f(z), e_j \rangle &= \langle S_p^\sharp f, \mathbb{K}^{(\alpha,\beta)}(\cdot, z)e_j \rangle \\ &= \langle f, (S_p^\sharp)^*(\mathbb{K}^{(\alpha,\beta)}(\cdot, z)e_j) \rangle \\ &= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} (\beta(\partial_j p)(z, z) - \alpha(\partial_{m+j} p)(z, z)) \langle f, K^{\alpha+\beta}(\cdot, z) \rangle \\ &= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \big(\beta(\partial_j p)(z, z) - \alpha(\partial_{m+j} p)(z, z)\big) f(z). \end{aligned}$$

Hence $S_p^\sharp = S_p$ completing the proof of the theorem. □

Corollary 3.10 *Let $\Omega \subset \mathbb{C}$ be a bounded domain. The compression operator $P_{\mathcal{A}_1^\perp}(M^{(\alpha)} \otimes I)_{|\mathcal{A}_1^\perp}$ is unitarily equivalent to the operator $\begin{pmatrix} M^{(\alpha+\beta)} & 0 \\ \delta\,\mathsf{inc} & \mathbb{M}^{(\alpha,\beta)} \end{pmatrix}$ on*

$(\mathcal{H}, K^{\alpha+\beta}) \bigoplus (\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$, where $\delta = \frac{\beta}{\sqrt{\alpha\beta(\alpha+\beta)}}$ and inc *is the inclusion operator from $(\mathcal{H}, K^{\alpha+\beta})$ into $(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$.*

4 Generalized Bergman Kernels

We now discuss an important class of operators introduced by Cowen and Douglas in the very influential paper [6]. The case of 2 variables was discussed in [7], while a detailed study in the general case appeared later in [8]. The definition below is taken from [8]. Let $\boldsymbol{T} := (T_1, \ldots, T_m)$ be a m-tuple of commuting bounded linear operators on a separable Hilbert space $\mathcal{H}$. Let $D_{\boldsymbol{T}} : \mathcal{H} \to \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ be the operator defined by $D_{\boldsymbol{T}}(x) = (T_1 x, \ldots, T_m x)$, $x \in \mathcal{H}$.

Definition 4.1 (Cowen-Douglas Class Operator) Let $\Omega \subset \mathbb{C}^m$ be a bounded domain. The operator $\boldsymbol{T}$ is said to be in the Cowen-Douglas class $B_n(\Omega)$ if $\boldsymbol{T}$ satisfies the following requirements:

(i) $\dim \ker D_{\boldsymbol{T}-w} = n, \quad w \in \Omega$
(ii) $\mathrm{ran} D_{\boldsymbol{T}-w}$ is closed for all $w \in \Omega$
(iii) $\bigvee \big\{ \ker D_{\boldsymbol{T}-w} : w \in \Omega \big\} = \mathcal{H}$.

If $\boldsymbol{T} \in B_n(\Omega)$, then for each $w \in \Omega$, there exist functions $\gamma_1, \ldots, \gamma_n$ holomorphic in a neighbourhood $\Omega_0 \subseteq \Omega$ containing w such that $\ker D_{\boldsymbol{T}-w'} = \bigvee \{\gamma_1(w'), \ldots, \gamma_n(w')\}$ for all $w' \in \Omega_0$ (cf. [7]). Consequently, every $\boldsymbol{T}$ in $B_n(\Omega)$ corresponds to a rank n holomorphic Hermitian vector bundle $E_{\boldsymbol{T}}$ defined by

$$E_{\boldsymbol{T}} = \{(w, x) \in \Omega \times \mathcal{H} : x \in \ker D_{\boldsymbol{T}-w}\}$$

and $\pi(w, x) = w$, $(w, x) \in E_{\boldsymbol{T}}$.

For a bounded domain Ω in $\mathbb{C}^m$, let $\Omega^* = \{z : \bar{z} \in \Omega\}$. It is known that if T is an operator in $B_n(\Omega^*)$, then for each $w \in \Omega$, T is unitarily equivalent to the adjoint of the multiplication tuple $(M_{z_1}, \ldots, M_{z_m})$ on some reproducing kernel Hilbert space $(\mathcal{H}, K) \subseteq \mathrm{Hol}(\Omega_0, \mathbb{C}^n)$ for some open subset $\Omega_0 \subseteq \Omega$ containing w. Here the kernel K can be described explicitly as follows. Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a holomorphic frame of the vector bundle $E_{\boldsymbol{T}}$ on a neighbourhood $\Omega_0^* \subseteq \Omega^*$ containing $\bar{w}$. Define $K_\Gamma : \Omega_0 \times \Omega_0 \to \mathcal{M}_n(\mathbb{C})$ by $K_\Gamma(z, w) = \big(\langle \gamma_j(\bar{w}), \gamma_i(\bar{z}) \rangle\big)_{i,j=1}^n$, $z, w \in \Omega_0$. Setting $K = K_\Gamma$, one may verify that the operator $\boldsymbol{T}$ is unitarily equivalent to the adjoint of the m-tuple of multiplication operators $(M_{z_1}, \ldots, M_{z_m})$ on the Hilbert space $(\mathcal{H}, K)$.

If $T \in B_1(\Omega^*)$, the curvature matrix $\mathcal{K}_T(\bar{w})$ at a fixed but arbitrary point $\bar{w} \in \Omega^*$ is defined by

$$\mathcal{K}_T(\bar{w}) = \big(\partial_i \bar{\partial}_j \log \|\gamma(\bar{w})\|^2\big)_{i,j=1}^m,$$

where γ is a holomorphic frame of E_T defined on some open subset $\Omega_0^* \subseteq \Omega^*$ containing $\bar{w}$. If T is realized as the adjoint of the multiplication tuple $(M_{z_1}, \ldots, M_{z_m})$ on some reproducing kernel Hilbert space $(\mathcal{H}, K) \subseteq \mathrm{Hol}(\Omega_0)$, where $w \in \Omega_0$, the curvature $\mathcal{K}_T(\bar{w})$ is then equal to

$$\big(\partial_i \bar{\partial}_j \log K(w, w)\big)_{i,j=1}^m.$$

The study of operators in the Cowen-Douglass class using the properties of the kernel functions was initiated by Curto and Salinas in [8]. The following definition is taken from [24].

Definition 4.2 (Sharp Kernel and Generalized Bergman Kernel) A positive definite kernel $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ is said to be sharp if

(i) the multiplication operator M_{z_i} is bounded on $(\mathcal{H}, K)$ for $i = 1, \ldots, m$,
(ii) $\ker D_{(\boldsymbol{M}_z - w)^*} = \mathrm{ran} K(\cdot, w)$, $w \in \Omega$,

where $\boldsymbol{M}_z$ denotes the m-tuple $(M_{z_1}, M_{z_2}, \ldots, M_{z_m})$ on $(\mathcal{H}, K)$. Moreover, if $\mathrm{ran} D_{(\boldsymbol{M}_z - w)^*}$ is closed for all $w \in \Omega$, then K is said to be a generalized Bergman kernel.

We start with the following lemma (cf. [9, page 285]) which provides a sufficient condition for the sharpness of a non-negative definite kernel K.

Lemma 4.3 *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ be a non-negative definite kernel. Assume that the multiplication operator M_{z_i} on $(\mathcal{H}, K)$ is bounded for $1 \le i \le m$. If the vector valued polynomial ring $\mathbb{C}[z_1, \ldots, z_m] \otimes \mathbb{C}^k$ is contained in $(\mathcal{H}, K)$ as a dense subset, then K is a sharp kernel.*

Corollary 4.4 *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are non-negative definite on $\Omega \times \Omega$ for some $\alpha, \beta > 0$. Suppose that either the m-tuple of operators $(M_1^{(\alpha)}, \ldots, M_m^{(\alpha)})$ on $(\mathcal{H}, K^\alpha)$ or the m-tuple of operators $(M_1^{(\beta)}, \ldots, M_m^{(\beta)})$ on $(\mathcal{H}, K^\beta)$ is bounded. If both the Hilbert spaces $(\mathcal{H}, K^\alpha)$ and $(\mathcal{H}, K^\beta)$ contain the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$ as a dense subset, then the kernel $\mathbb{K}^{(\alpha,\beta)}$ is sharp.*

Proof By Corollary 3.8, the m-tuple of operators $(\mathbb{M}_1^{(\alpha,\beta)}, \ldots, \mathbb{M}_m^{(\alpha,\beta)})$ is bounded on $\big(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\big)$. If both the Hilbert spaces $(\mathcal{H}, K^\alpha)$ and $(\mathcal{H}, K^\beta)$ contain the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$ as a dense subset, then by Proposition 3.4, we see that the ring $\mathbb{C}[z_1, \ldots, z_m] \otimes \mathbb{C}^m$ is contained in $(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$ and is dense in it. An application of Lemma 4.3 now completes the proof. □

Some of the results in this paper generalize, among other things, one of the main results of [24], which is reproduced below.

Theorem 4.5 (Salinas, [24, Theorem 2.6]) *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain. If $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ are two sharp kernels (resp. generalized Bergman kernels), then $K_1 \otimes K_2$ and $K_1 K_2$ are also sharp kernels (resp. generalized Bergman kernels).*

For two scalar valued non-negative definite kernels K_1 and K_2, defined on $\Omega\times\Omega$, the jet construction (Theorem 1.3) gives rise to a family of non-negative kernels $J_k(K_1, K_2)_{|\text{res}\,\Delta}$, $k \geq 0$, where

$$J_k(K_1, K_2)_{|\text{res}\,\Delta}(z, w) := \big(K_1(z, w)\partial^{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{j}} K_2(z, w)\big)_{|\boldsymbol{i}|,|\boldsymbol{j}|=0}^{k}, \quad z, w \in \Omega.$$

In the particular case when $k = 0$, it coincides with the point-wise product K_1K_2. In this section, we generalize Theorem 4.5 for all kernels of the form $J_k(K_1, K_2)_{|\text{res}\,\Delta}$. First, we discuss two important corollaries of the jet construction which will be used later in this paper.

For $1 \leq i \leq m$, let J_kM_i denote the operator of multiplication by the ith coordinate function z_i on the Hilbert space $\big(\mathcal{H}, J_k(K_1, K_2)_{|\text{res}\,\Delta}\big)$. In case $m = 1$, we write J_kM instead of J_kM_1.

Taking $p(z, \zeta)$ to be the ith coordinate function z_i in Proposition 1.4, we obtain the following corollary.

Corollary 4.6 *Let $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ be two non-negative definite kernels. Then the m-tuple of operators $\big(P_{\mathcal{A}_k^\perp}(M_1^{(1)} \otimes I)_{|\mathcal{A}_k^\perp}, \ldots, P_{\mathcal{A}_k^\perp}(M_m^{(1)} \otimes I)_{|\mathcal{A}_k^\perp}\big)$ is unitarily equivalent to the m-tuple $(J_kM_1, \ldots, J_kM_m)$ on the Hilbert space $\big(\mathcal{H}, J_k(K_1, K_2)_{|\text{res}\,\Delta}\big)$.*

Combining this with Corollary 3.10 we obtain the following result.

Corollary 4.7 *Let $\Omega \subset \mathbb{C}$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are non-negative definite on $\Omega \times \Omega$ for some $\alpha, \beta > 0$. The following operators are unitarily equivalent:*

(i) *the operator $P_{\mathcal{A}_1^\perp}(M^{(\alpha)} \otimes I)_{|\mathcal{A}_1^\perp}$*
(ii) *the multiplication operator J_1M on $\big(\mathcal{H}, J_1(K^\alpha, K^\beta)_{|\text{res}\,\Delta}\big)$*
(iii) *the operator $\begin{pmatrix} M^{(\alpha+\beta)} & 0 \\ \delta\,\mathsf{inc} & \mathbb{M}^{(\alpha,\beta)} \end{pmatrix}$ on $(\mathcal{H}, K^{\alpha+\beta}) \bigoplus (\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$ where $\delta = \frac{\beta}{\sqrt{\alpha\beta(\alpha+\beta)}}$ and $\mathsf{inc} : (\mathcal{H}, K^{\alpha+\beta}) \to (\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$ is the inclusion operator.*

We need the following lemmas for the generalization of Theorem 4.5.

Lemma 4.8 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and T be a bounded linear operator on $\mathcal{H}_1$. Then*

$$\ker(T \otimes I_{\mathcal{H}_2}) = \ker T \otimes \mathcal{H}_2.$$

Proof It is easily seen that $\ker T \otimes \mathcal{H}_2 \subset \ker(T \otimes I_{\mathcal{H}_2})$. To establish the opposite inclusion, let x be an arbitrary element in $\ker(T \otimes I_{\mathcal{H}_2})$. Fix an orthonormal basis $\{f_i\}$ of $\mathcal{H}_2$. Note that x is of the form $\sum v_i \otimes f_i$ for some v_i's in $\mathcal{H}_1$. Since $x \in \ker(T \otimes I_{\mathcal{H}_2})$, we have $\sum Tv_i \otimes f_i = 0$. Moreover, since $\{f_i\}$ is an orthonormal basis of $\mathcal{H}_2$, it follows that $Tv_i = 0$ for all i. Hence x belongs to $\ker(T) \otimes \mathcal{H}_2$, completing the proof of the lemma. □

Lemma 4.9 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. If $B_1, \dots, B_m$ are closed subspaces of $\mathcal{H}_1$, then*

$$\bigcap_{l=1}^{m}(B_l \otimes \mathcal{H}_2) = \Big(\bigcap_{l=1}^{m} B_l\Big) \otimes \mathcal{H}_2.$$

Proof We only prove the non-trivial inclusion, namely, $\cap_{l=1}^{m}(B_l \otimes \mathcal{H}_2) \subseteq \big(\cap_{l=1}^{m} B_l\big) \otimes \mathcal{H}_2$. Let $\{f_j\}_j$ be an orthonormal basis of $\mathcal{H}_2$ and x be an arbitrary element in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Recall that x can be written uniquely as $\sum x_j \otimes f_j$, $x_j \in \mathcal{H}_1$.

Claim If x belongs to $B_l \otimes \mathcal{H}_2$, then x_j belongs to B_l for all j.

To prove the claim, assume that $\{e_i\}_i$ is an orthonormal basis of B_l. Since $\{e_i \otimes f_j\}_{i,j}$ is an orthonormal basis of $B_l \otimes \mathcal{H}_2$ and x can be written as $\sum x_{ij} e_i \otimes f_j = \sum_j (\sum_i x_{ij} e_i) \otimes f_j$. Then, the uniqueness of the representation $x = \sum x_j \otimes f_j$, ensures that $x_j = \sum_i x_{ij} e_i$. In particular, x_j belongs to B_l for all j. Thus the claim is verified.

Now let y be any element in $\cap_{l=1}^{m}(B_l \otimes \mathcal{H}_2)$. Let $\sum y_j \otimes f_j$ be the unique representation of y in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then from the claim, it follows that $y_j \in \cap_{l=1}^{m} B_l$. Consequently, $y \in (\cap_{l=1}^{m} B_l) \otimes \mathcal{H}_2$. This completes the proof. □

The proof of the following lemma is straightforward and therefore it is omitted.

Lemma 4.10 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Let $A : \mathcal{H}_1 \to \mathcal{H}_1$ be a bounded linear operator and $B : \mathcal{H}_1 \to \mathcal{H}_2$ be a unitary operator. Then*

$$\ker BAB^* = B(\ker A).$$

The lemma given below is a generalization of [6, Lemma 1.22 (i)] to commuting tuples. Recall that for a commuting m-tuple $\boldsymbol{T} = (T_1, \dots, T_m)$, the operator $\boldsymbol{T}^{\boldsymbol{i}}$ is defined by $T_1^{i_1} \cdots T_m^{i_m}$, where $\boldsymbol{i} = (i_1, \dots, i_m) \in \mathbb{Z}_+^m$.

Lemma 4.11 *If $K : \Omega \times \Omega \to \mathbb{C}$ is a positive definite kernel such that the m-tuple of multiplication operators $\boldsymbol{M}_z = (M_{z_1}, \dots, M_{z_m})$ on $(\mathcal{H}, K)$ is bounded, then for $w \in \Omega$ and $\boldsymbol{i} = (i_1, \dots, i_m)$, $\boldsymbol{j} = (j_1, \dots, j_m)$ in $\mathbb{Z}_+^m$,*

(i) $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}} \bar{\partial}^{\boldsymbol{j}} K(\cdot, w) = 0$ *if* $|\boldsymbol{i}| > |\boldsymbol{j}|$,
(ii) $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}} \bar{\partial}^{\boldsymbol{j}} K(\cdot, w) = \boldsymbol{j}! \delta_{\boldsymbol{i}\boldsymbol{j}} K(\cdot, w)$ *if* $|\boldsymbol{i}| = |\boldsymbol{j}|$.

Proof First, we claim that if $i_l > j_l$ for some $1 \le l \le m$, then

$$(M_{z_l}^* - \bar{w}_l)^{i_l} \bar{\partial}_l^{j_l} K(\cdot, w) = 0.$$

The claim is verified by induction on j_l. The case $j_l = 0$ holds trivially since $(M_{z_l}^* - \bar{w}_l) K(\cdot, w) = 0$. Now assume that the claim is valid for $j_l = p$. We have to show that it is true for $j_l = p + 1$ also. Suppose $i_l > p + 1$. Then $i_l - 1 > p$. Hence, by the induction hypothesis, $(M_{z_l}^* - \bar{w}_l)^{i_l - 1} \bar{\partial}_l^{p} K(\cdot, w) = 0$. Differentiating this with respect to $\bar{w}_l$, we see that

$$(i_l-1)(M_{z_l}^*-\bar{w}_l)^{i_l-2}(-1)\bar{\partial}_l^p K(\cdot,w)+(M_{z_l}^*-\bar{w}_l)^{i_l-1}\bar{\partial}_l^{p+1}K(\cdot,w)=0.$$

Applying $(M_{z_l}^*-\bar{w}_l)$ to both sides of the equation above, we obtain

$$(i_l-1)(M_{z_l}^*-\bar{w}_l)^{i_l-1}(-1)\bar{\partial}_l^p K(\cdot,w)+(M_{z_l}^*-\bar{w}_l)^{i_l}\bar{\partial}_l^{p+1}K(\cdot,w)=0.$$

Therefore, using the induction hypothesis once again, we conclude that $(M_{z_l}^*-\bar{w}_l)^{i_l}\bar{\partial}_l^{p+1}K(\cdot,w)=0$. Hence the claim is verified.

Now, to prove the first part of the lemma, assume that $|\boldsymbol{i}|>|\boldsymbol{j}|$. Then there exists a l such that $i_l > j_l$. Hence from the claim, we have $(M_{z_l}^*-\bar{w}_l)^{i_l}\bar{\partial}_l^{j_l}K(\cdot,w)=0$. Differentiating with respect to all other variables except $\bar{w}_l$, we get $(M_{z_l}^*-\bar{w}_l)^{i_l}\bar{\partial}^{\boldsymbol{j}}K(\cdot,w)=0$. Applying the operator $(\boldsymbol{M}_z^*-\bar{w})^{\boldsymbol{i}-i_le_l}$, where e_l is the lth standard unit vector of $\mathbb{C}^m$, we see that $(\boldsymbol{M}_z^*-\bar{w})^{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{j}}K(\cdot,w)=0$, completing the proof of the first part.

For the second part, assume that $|\boldsymbol{i}|=|\boldsymbol{j}|$ and $\boldsymbol{i}\neq\boldsymbol{j}$. Then there is at least one l such that $i_l>j_l$. Hence by the argument used in the last paragraph, we conclude that $(\boldsymbol{M}_z^*-\bar{w})^{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{j}}K(\cdot,w)=0$. Finally, if $\boldsymbol{i}=\boldsymbol{j}$, we use induction on $\boldsymbol{i}$ to proof the lemma. There is nothing to prove if $\boldsymbol{i}=0$. For the proof by induction, now, assume that $(\boldsymbol{M}_z^*-\bar{w})^{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{i}}K(\cdot,w)=\boldsymbol{i}!K(\cdot,w)$ for some $\boldsymbol{i}\in\mathbb{Z}_+^m$. To complete the induction step, we have to prove that $(\boldsymbol{M}_z^*-\bar{w})^{\boldsymbol{i}+e_l}\bar{\partial}^{\boldsymbol{i}+e_l}K(\cdot,w)=(\boldsymbol{i}+e_l)!K(\cdot,w)$. By the first part of the lemma, we have $(\boldsymbol{M}_z^*-\bar{w})^{\boldsymbol{i}+e_l}\bar{\partial}^{\boldsymbol{i}}K(\cdot,w)=0$. Differentiating with respect to $\bar{w}_l$, we get that

$$(\boldsymbol{M}_z^*-\bar{w})^{\boldsymbol{i}+e_l}\bar{\partial}^{\boldsymbol{i}+e_l}K(\cdot,w)-(i_l+1)(\boldsymbol{M}_z^*-\bar{w})^{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{i}}K(\cdot,w)=0.$$

Hence, by the induction hypothesis, we conclude that

$$(\boldsymbol{M}_z^*-\bar{w})^{\boldsymbol{i}+e_l}\bar{\partial}^{\boldsymbol{i}+e_l}K(\cdot,w)=(\boldsymbol{i}+e_l)!K(\cdot,w).$$

This completes the proof. □

Corollary 4.12 *Let $K:\Omega\times\Omega\to\mathbb{C}$ be a positive definite kernel. Suppose that the m-tuple of multiplication operators $\boldsymbol{M}_z$ on $(\mathcal{H},K)$ is bounded. Then, for all $w\in\Omega$, the set $\left\{\bar{\partial}^{\boldsymbol{i}}K(\cdot,w):\boldsymbol{i}\in\mathbb{Z}_+^m\right\}$ is linearly independent. Consequently, the matrix $\left(\partial^{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{j}}K(w,w)\right)_{\boldsymbol{i},\boldsymbol{j}\in\Lambda}$ is positive definite for any finite subset Λ of $\mathbb{Z}_+^m$.*

Proof Let w be an arbitrary point in Ω. It is enough to show that the set $\left\{\bar{\partial}^{\boldsymbol{i}}K(\cdot,w):\boldsymbol{i}\in\mathbb{Z}_+^m,|\boldsymbol{i}|\leq k\right\}$ is linearly independent for each non-negative integer k. Since K is positive definite, there is nothing to prove if $k=0$. To complete the proof by induction on k, assume that the set $\left\{\bar{\partial}^{\boldsymbol{i}}K(\cdot,w):\boldsymbol{i}\in\mathbb{Z}_+^m,|\boldsymbol{i}|\leq k\right\}$ is linearly independent for some non-negative integer k. Suppose that $\sum_{|\boldsymbol{i}|\leq k+1} a_{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{i}} K(\cdot, w) = 0$ for some $a_{\boldsymbol{i}}$'s in $\mathbb{C}$. Then

$$(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{q}}\big(\sum_{|\boldsymbol{i}|\leq k+1} a_{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{i}} K(\cdot, w)\big) = 0,$$

for all $\boldsymbol{q} \in \mathbb{Z}_+^m$ with $|\boldsymbol{q}| \leq k+1$. If $|\boldsymbol{q}| = k+1$, by Lemma 4.11, we have that $a_{\boldsymbol{q}}\ \boldsymbol{q}!K(\cdot, w) = 0$. Consequently, $a_{\boldsymbol{q}} = 0$ for all $\boldsymbol{q} \in \mathbb{Z}_+^m$ with $|\boldsymbol{q}| = k+1$. Hence, by the induction hypothesis, we conclude that $a_{\boldsymbol{i}} = 0$ for all $\boldsymbol{i} \in \mathbb{Z}_+^m$, $|\boldsymbol{i}| \leq k+1$ and the set $\left\{ \bar{\partial}^{\boldsymbol{i}} K(\cdot, w) : \boldsymbol{i} \in \mathbb{Z}_+^m, |\boldsymbol{i}| \leq k+1 \right\}$ is linearly independent, completing the proof of the first part of the corollary.

If Λ is a finite subset of $\mathbb{Z}_+^m$, then it follows from the linear independence of the vectors $\left\{\bar{\partial}^{\boldsymbol{i}} K(\cdot, w) : \boldsymbol{i} \in \Lambda\right\}$ that $\left(\left\langle\bar{\partial}^{\boldsymbol{j}} K(\cdot, w), \bar{\partial}^{\boldsymbol{i}} K(\cdot, w)\right\rangle\right)_{\boldsymbol{i},\boldsymbol{j}\in\Lambda}$ is a positive definite matrix. Now the proof is complete since

$$\left\langle\bar{\partial}^{\boldsymbol{j}} K(\cdot, w), \bar{\partial}^{\boldsymbol{i}} K(\cdot, w)\right\rangle = \partial^{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{j}} K(w, w),$$

see Proposition 2.2. □

The following proposition is also a generalization to the multi-variate setting of [6, Lemma 1.22 (ii)](see also [7]).

Proposition 4.13 *If $K : \Omega \times \Omega \to \mathbb{C}$ is a sharp kernel, then for every $w \in \Omega$*

$$\bigcap_{|\boldsymbol{j}|=k+1} \ker\,(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{j}} = \bigvee \left\{\bar{\partial}^{\boldsymbol{j}} K(\cdot, w) : |\boldsymbol{j}| \leq k\right\}.$$

Proof The inclusion $\bigvee\{\bar{\partial}^{\boldsymbol{j}} K(\cdot, w) : |\boldsymbol{j}| \leq k\} \subseteq \bigcap_{|\boldsymbol{j}|=k+1} \ker\,(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{j}}$ follows from part (i) of Lemma 4.11. We use induction on k for the opposite inclusion. From the definition of sharp kernel, this inclusion is evident if $k = 0$. Assume that

$$\bigcap_{|\boldsymbol{j}|=k+1} \ker\,(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{j}} \subseteq \bigvee \left\{\bar{\partial}^{\boldsymbol{j}} K(\cdot, w) : |\boldsymbol{j}| \leq k\right\}$$

for some non-negative integer k. To complete the proof by induction, we show that the inclusion remains valid for $k+1$ as well. Let f be an arbitrary element of $\bigcap_{|\boldsymbol{i}|=k+2} \ker(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}}$. Fix a $\boldsymbol{j} \in \mathbb{Z}_+^m$ with $|\boldsymbol{j}| = k+1$. Then it follows that $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{j}} f$ belongs to $\cap_{l=1}^m \ker(M_{z_l}^* - \bar{w}_l)$. Since K is sharp, we see that $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{j}} f = c_{\boldsymbol{j}} K(\cdot, w)$ for some constant $c_{\boldsymbol{j}}$ depending on w. Therefore

$$\begin{aligned}(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{j}}\Big(f - \sum_{|\boldsymbol{q}|=k+1} \frac{c_{\boldsymbol{q}}}{\boldsymbol{q}!}\bar{\partial}^{\boldsymbol{q}} K(\cdot, w)\Big)& \\ = c_{\boldsymbol{j}} K(\cdot, w) - \sum_{|\boldsymbol{q}|=k+1} \frac{c_{\boldsymbol{q}}}{\boldsymbol{q}!}(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{j}}\bar{\partial}^{\boldsymbol{q}} K(\cdot, w)&\end{aligned}$$

$$= c_{\boldsymbol{j}} K(\cdot, w) - \sum_{|\boldsymbol{q}|=k+1} c_{\boldsymbol{q}} \delta_{\boldsymbol{j}\boldsymbol{q}} \frac{\boldsymbol{j}!}{\boldsymbol{q}!} K(\cdot, w)$$
$$= 0,$$

where the last equality follows from Lemma 4.11. Or, in other words, the vector $f - \sum_{|\boldsymbol{q}|=k+1} \frac{c_{\boldsymbol{q}}}{\boldsymbol{q}!} \bar{\partial}^{\boldsymbol{q}} K(\cdot, w)$ belongs to $\bigcap_{|\boldsymbol{j}|=k+1} \ker(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{j}}$. Thus by the induction hypothesis, $f - \sum_{|\boldsymbol{q}|=k+1} \frac{c_{\boldsymbol{q}}}{\boldsymbol{q}!} \bar{\partial}^{\boldsymbol{q}} K(\cdot, w) = \sum_{|\boldsymbol{j}|\le k} d_{\boldsymbol{j}} \bar{\partial}^{\boldsymbol{j}} K(\cdot, w)$. Hence f belongs to $\bigvee\{\bar{\partial}^{\boldsymbol{j}} K(\cdot, w) : |\boldsymbol{j}| \le k+1\}$. This completes the proof. □

For a m-tuple of bounded operators $\mathbf{T} = (T_1, \dots, T_m)$ on a Hilbert space $\mathcal{H}$, we define an operator $D^{\mathbf{T}} : \mathcal{H} \bigoplus \cdots \bigoplus \mathcal{H} \to \mathcal{H}$ by

$$D^{\boldsymbol{T}}(x_1, \dots, x_m) = \sum_{i=1}^m T_i x_i, \quad x_1, \dots, x_m \in \mathcal{H}.$$

A routine verification shows that $(D_{\boldsymbol{T}})^* = D^{\boldsymbol{T}^*}$. The following lemma is undoubtedly well known, however, we provide a proof for the sake of completeness.

Lemma 4.14 *Let $K : \Omega \times \Omega \to \mathbb{C}$ be a positive definite kernel such that the m-tuple of multiplication operators $\boldsymbol{M}_z$ on $(\mathcal{H}, K)$ is bounded. Let $w = (w_1, \dots, w_m)$ be a fixed but arbitrary point in Ω and let $\mathcal{V}_w$ be the subspace given by $\{f \in (\mathcal{H}, K) : f(w) = 0\}$. Then K is a generalized Bergman kernel if and only if for every $w \in \Omega$,*

$$\mathcal{V}_w = \Big\{ \textstyle\sum_{i=1}^m (z_i - w_i) g_i : g_i \in (\mathcal{H}, K) \Big\}. \tag{4.1}$$

Proof First, observe that the right-hand side of (4.1) is equal to $\mathrm{ran} D^{\boldsymbol{M}_z - w}$. Hence it suffices to show that K is a generalized Bergman kernel if and only if $\mathcal{V}_w = \mathrm{ran} D^{\boldsymbol{M}_z - w}$. In any case, we have the following inclusions

$$\begin{aligned} \mathrm{ran} D^{\boldsymbol{M}_z - w} = \mathrm{ran}(D_{(\boldsymbol{M}_z - w)^*})^* &\subseteq \overline{\mathrm{ran}(D_{(\boldsymbol{M}_z - w)^*})^*} \\ &= \ker D_{(\boldsymbol{M}_z - w)^*}{}^{\perp} \subseteq \{cK(\cdot, w) : c \in \mathbb{C}\}^{\perp} \\ &= \mathcal{V}_w. \end{aligned}$$

Hence it follows that $\mathcal{V}_w = \mathrm{ran} D^{\boldsymbol{M}_z - w}$ if and only if equality is forced everywhere in these inclusions, that is, $\mathrm{ran}(D_{(\boldsymbol{M}_z - w)^*})^* = \overline{\mathrm{ran}(D_{(\boldsymbol{M}_z - w)^*})^*}$ and $\ker D_{(\boldsymbol{M}_z - w)^*}{}^{\perp} = \{cK(\cdot, w) : c \in \mathbb{C}\}^{\perp}$. Now note that $\mathrm{ran}(D_{(\boldsymbol{M}_z - w)^*})^* = \overline{\mathrm{ran}(D_{(\boldsymbol{M}_z - w)^*})^*}$ if and only if $\mathrm{ran}(D_{(\boldsymbol{M}_z - w)^*})^*$ is closed. Recall that, if $\mathcal{H}_1, \mathcal{H}_2$ are two Hilbert spaces, and an operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ has closed range, then T^* also has closed range. Therefore, $\mathrm{ran}(D_{(\boldsymbol{M}_z - w)^*})^*$ is closed if and only if $\mathrm{ran} D_{(\boldsymbol{M}_z - w)^*}$ is closed. Finally, note that $\ker D_{(\boldsymbol{M}_z - w)^*}{}^{\perp} = \{cK(\cdot, w) : c \in \mathbb{C}\}^{\perp}$ holds if and only if $\ker D_{(\boldsymbol{M}_z - w)^*} = \{cK(\cdot, w) : c \in \mathbb{C}\}$. This completes the proof. □

Recall that $M_i^{(1)}, M_i^{(2)}, J_k M_i$, $1 \leq i \leq m$, are the operators of multiplication by the coordinate function z_i on the Hilbert spaces $(\mathcal{H}, K_1), (\mathcal{H}, K_2)$ and $(\mathcal{H}, J_k(K_1, K_2)_{|\text{res}\,\Delta})$, respectively. Set $\boldsymbol{M}^{(1)} = (M_1^{(1)}, \ldots, M_m^{(1)})$, $\boldsymbol{M}^{(2)} = (M_1^{(2)}, \ldots, M_m^{(2)})$ and $\boldsymbol{J}_k\boldsymbol{M} = (J_k M_1, \ldots, J_k M_m)$. For the sake of brevity, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the Hilbert spaces $(\mathcal{H}, K_1)$ and $(\mathcal{H}, K_2)$, respectively for the rest of this section.

The following is the main tool to prove that the kernel $J_k(K_1, K_2)_{|\text{res}\,\Delta}$ is sharp whenever K_1 and K_2 are sharp.

Lemma 4.15 *If $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ are two sharp kernels, then for all $w = (w_1, \ldots, w_m) \in \Omega$,*

$$\bigcap_{p=1}^{m} \ker\Big(\big((M_p^{(1)} - w_p)^* \otimes I\big)_{|\mathcal{A}_k^{\perp}}\Big)$$
$$= \bigcap_{|\boldsymbol{i}|=1} \ker\big(\boldsymbol{M}^{(1)} - w\big)^{*\boldsymbol{i}} \otimes \bigcap_{|\boldsymbol{i}|=k+1} \ker\big(\boldsymbol{M}^{(2)} - w\big)^{*\boldsymbol{i}}$$
$$= \bigvee \big\{K_1(\cdot, w) \otimes \bar{\partial}^{\boldsymbol{i}} K_2(\cdot, w) : |\boldsymbol{i}| \leq k\big\}.$$

Proof Since K_1 and K_2 are sharp kernels, by Proposition 4.13, it follows that

$$\bigcap_{|\boldsymbol{i}|=1} \ker(\boldsymbol{M}^{(1)} - w)^{*\boldsymbol{i}} \otimes \bigcap_{|\boldsymbol{i}|=k+1} \ker(\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}}$$
$$= \bigvee\{K_1(\cdot, w) \otimes \bar{\partial}^{\boldsymbol{j}} K_2(\cdot, w) : |\boldsymbol{j}| \leq k\}. \tag{4.2}$$

Therefore, we will be done if we can show that

$$\bigcap_{p=1}^{m} \ker\Big(\big((M_p^{(1)} - w_p)^* \otimes I\big)_{|\mathcal{A}_k{}^{\perp}}\Big)$$
$$= \bigcap_{|\boldsymbol{i}|=1} \ker(\boldsymbol{M}^{(1)} - w)^{*\boldsymbol{i}} \otimes \bigcap_{|\boldsymbol{i}|=k+1} \ker(\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}}. \tag{4.3}$$

To prove this, first note that

$$\bigcap_{p=1}^{m} \ker\Big(\big((M_p^{(1)} - w_p)^* \otimes I\big)_{|\mathcal{A}_k^{\perp}}\Big) = \Big(\bigcap_{p=1}^{m} \ker\big((M_p^{(1)} - w_p)^* \otimes I\big)\Big) \bigcap \mathcal{A}_k^{\perp}$$
$$= \Big(\bigcap_{p=1}^{m} \big(\ker(M_p^{(1)} - w_p)^* \otimes \mathcal{H}_2\big)\Big) \bigcap \mathcal{A}_k^{\perp}$$

$$= \Big(\Big(\bigcap_{p=1}^{m} \ker(M_p^{(1)} - w_p)^*\Big) \otimes \mathcal{H}_2\Big) \bigcap \mathcal{A}_k^{\perp}$$

$$= \Big(\ker D_{(\boldsymbol{M}^{(1)}-w)^*} \otimes \mathcal{H}_2\Big) \bigcap \mathcal{A}_k^{\perp}.$$

Here the second equality follows from Lemma 4.8 and the third equality follows from Lemma 4.9. In view of the above computation, to verify (4.3), it is enough to show that

$$\Big(\ker D_{(\boldsymbol{M}^{(1)}-w)^*} \otimes \mathcal{H}_2\Big) \bigcap \mathcal{A}_k^{\perp} = \bigcap_{|\boldsymbol{i}|=1} \ker\,(\boldsymbol{M}^{(1)} - w)^{*\boldsymbol{i}} \otimes \bigcap_{|\boldsymbol{i}|=k+1} \ker\,(\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}}. \tag{4.4}$$

Since K_1 is a sharp kernel, $\ker D_{(\boldsymbol{M}^{(1)}-w)^*}$ is spanned by the vector $K_1(\cdot, w)$. It is also easy to see that the vector $K_1(\cdot, w) \otimes \bar{\partial}^{\boldsymbol{j}} K_2(\cdot, w)$ belongs to $\mathcal{A}_k^{\perp}$ and hence, it is in $\Big(\ker D_{(\boldsymbol{M}^{(1)}-w)^*} \otimes \mathcal{H}_2\Big) \bigcap \mathcal{A}_k^{\perp}$ for all $\boldsymbol{j}$ in $\mathbb{Z}_+^m$ with $|\boldsymbol{j}| \leq k$. Therefore, by (4.2), we have the inclusion

$$\bigcap_{|\boldsymbol{i}|=1} \ker\,(\boldsymbol{M}^{(1)} - w)^{*\boldsymbol{i}} \otimes \bigcap_{|\boldsymbol{i}|=k+1} \ker\,(\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}} \subseteq \Big(\ker D_{(\boldsymbol{M}^{(1)}-w)^*} \otimes \mathcal{H}_2\Big) \bigcap \mathcal{A}_k^{\perp}. \tag{4.5}$$

Now to prove the opposite inclusion, first note that an arbitrary vector of $\big(\ker D_{(\boldsymbol{M}^{(1)}-w)^*} \otimes \mathcal{H}_2\big) \bigcap \mathcal{A}_k^{\perp}$ can be taken to be of the form $K_1(\cdot, w) \otimes g$, where $g \in \mathcal{H}_2$ is such that $K_1(\cdot, w) \otimes g \in \mathcal{A}_k^{\perp}$. We claim that such a vector g must be in $\bigcap_{|\boldsymbol{i}|=k+1} \ker\,(\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}}$.

As before, we realize the vectors of $\mathcal{H}_1 \otimes \mathcal{H}_2$ as holomorphic functions in $z = (z_1, \ldots, z_m)$, $\zeta = (\zeta_1, \ldots, \zeta_m)$ in Ω. Fix any $\boldsymbol{i} \in \mathbb{Z}_+^m$ with $|\boldsymbol{i}| = k+1$. Then $(\zeta - z)^{\boldsymbol{i}} = (\zeta_{q_1} - z_{q_1})(\zeta_{q_2} - z_{q_2}) \cdots (\zeta_{q_{k+1}} - z_{q_{k+1}})$ for some $1 \leq q_1, q_2, \ldots, q_{k+1} \leq m$. Since $M_i^{(1)}$ and $M_i^{(2)}$ are bounded for $1 \leq i \leq m$, for any $h \in \mathcal{H}_1 \otimes \mathcal{H}_2$, we see that the function $(\zeta - z)^{\boldsymbol{i}} h$ belongs to $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then

$$\begin{aligned}
&\big\langle K_1(\cdot, w) \otimes g, (\zeta_{q_1} - z_{q_1})(\zeta_{q_2} - z_{q_2}) \cdots (\zeta_{q_{k+1}} - z_{q_{k+1}})h\big\rangle \\
&= \Big\langle M^*_{(\zeta_{q_1} - z_{q_1})}(K_1(\cdot, w) \otimes g), (\zeta_{q_2} - z_{q_2}) \cdots (\zeta_{q_{k+1}} - z_{q_{k+1}})h\Big\rangle \\
&= \Big\langle (I \otimes M_{q_1}^{(2)*} - M_{q_1}^{(1)*} \otimes I)K_1(\cdot, w) \otimes g, (\zeta_{q_2} - z_{q_2}) \cdots (\zeta_{q_{k+1}} - z_{q_{k+1}})h\Big\rangle \\
&= \Big\langle K_1(\cdot, w) \otimes M_{q_1}^{(2)*} g - \bar{w}_{q_1} K_1(\cdot, w) \otimes g, (\zeta_{q_2} - z_{q_2}) \cdots (\zeta_{q_{k+1}} - z_{q_{k+1}})h\Big\rangle \\
&= \Big\langle K_1(\cdot, w) \otimes (M_{q_1}^{(2)} - w_{q_1})^* g, (\zeta_{q_2} - z_{q_2}) \cdots (\zeta_{q_{k+1}} - z_{q_{k+1}})h\Big\rangle.
\end{aligned}$$

Repeating this process, we get

$$\left\langle K_1(\cdot, w) \otimes g, (\zeta - z)^{\boldsymbol{i}} h \right\rangle = \left\langle K_1(\cdot, w) \otimes (\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}} g, h \right\rangle.$$

Since $|\boldsymbol{i}| = k+1$, it follows that the element $(\zeta - z)^{\boldsymbol{i}} h$ belongs to $\mathcal{A}_k$. Furthermore, since $K_1(\cdot, w) \otimes g \in \mathcal{A}_k^{\perp}$, from the above equality, we have

$$\left\langle K_1(\cdot, w) \otimes (\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}} g, h \right\rangle = 0$$

for any $h \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Taking $h = K_1(\cdot, w) \otimes K_2(\cdot, u)$, $u \in \Omega$, we get $K_1(w, w)\big((\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}} g\big)(u) = 0$ for all $u \in \Omega$. Since $K_1(w, w) > 0$, it follows that $(\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}} g = 0$. Since this is true for all $\boldsymbol{i} \in \mathbb{Z}_+^m$ with $|\boldsymbol{i}| = k+1$, it follows that $g \in \bigcap_{|\boldsymbol{i}|=k+1} \ker (\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}}$. Hence $K_1(\cdot, w) \otimes g$ belongs to

$$\bigcap_{|\boldsymbol{i}|=1} \ker (\boldsymbol{M}^{(1)} - w)^{*\boldsymbol{i}} \otimes \bigcap_{|\boldsymbol{i}|=k+1} \ker (\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}},$$

proving the opposite inclusion of (4.5). This completes the proof of equality in (4.3). □

Theorem 4.16 *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain. If $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ are two sharp kernels, then so is the kernel $J_k(K_1, K_2)_{|\text{res}\,\Delta}$, $k \geq 0$.*

Proof Since the tuple $\boldsymbol{M}^{(1)}$ is bounded, by Corollary 4.6, it follows that the tuple $\boldsymbol{J}_k\boldsymbol{M}$ is also bounded. Now we will show that the kernel $J_k(K_1, K_2)_{|\text{res}\,\Delta}$ is positive definite on $\Omega \times \Omega$. Since K_2 is positive definite, by Corollary 4.12, we obtain that the matrix $\big(\partial^{\boldsymbol{i}} \bar{\partial}^{\boldsymbol{j}} K_2(w, w)\big)_{|\boldsymbol{i}|,|\boldsymbol{j}|=0}^{k}$ is positive definite for $w \in \Omega$. Moreover, since K_1 is also positive definite, we conclude that $J_k(K_1, K_2)_{|\text{res}\,\Delta}(w, w)$ is positive definite for $w \in \Omega$. Hence, by [8, Lemma 3.6], we conclude that the kernel $J_k(K_1, K_2)_{|\text{res}\,\Delta}$ is positive definite.

To complete the proof, we need to show that

$$\ker D_{(\boldsymbol{J}_k\boldsymbol{M} - w)^*} = \text{ran} J_k(K_1, K_2)_{|\text{res}\,\Delta}(\cdot, w), \;\; w \in \Omega.$$

Note that, by the definition of R and J_k (see the discussion before Theorem 1.3), we have

$$R J_k(K_1(\cdot, w) \otimes \bar{\partial}^{\boldsymbol{i}} K_2(\cdot, w)) = J_k(K_1, K_2)_{|\text{res}\,\Delta}(\cdot, w) e_{\boldsymbol{i}}, \;\; \boldsymbol{i} \in \mathbb{Z}_+^m, \;\; |\boldsymbol{i}| \leq k.$$

In the computation below, the third equality follows from Lemma 4.10, the injectivity of the map $R J_{k|\mathcal{A}_k^{\perp}}$ implies the fourth equality, the fifth equality follows from Lemma 4.15 and finally the last equality follows from (4):

$$\begin{aligned}\ker D_{(\boldsymbol{J}_k\boldsymbol{M}-w)^*} &= \bigcap_{p=1}^{m} \ker(J_k M_p - w_p)^* \\ &= \bigcap_{p=1}^{m} \ker\Big((RJ_k)P_{\mathcal{A}_k^\perp}\big((M_p^{(1)} - w_p)^* \otimes I\big)_{|\mathcal{A}_k^\perp}(RJ_k)^*\Big) \\ &= \bigcap_{p=1}^{m} (RJ_k)\Big(\ker\big(P_{\mathcal{A}_k^\perp}\big((M_p^{(1)} - w_p)^* \otimes I\big)_{|\mathcal{A}_k^\perp}\big)\Big) \\ &=(RJ_k)\Big(\bigcap_{p=1}^{m} \ker\big(P_{\mathcal{A}_k^\perp}\big((M_p^{(1)} - w_p)^* \otimes I\big)_{|\mathcal{A}_k^\perp}\big)\Big) \\ &=(RJ_k)\Big(\bigvee\big\{\, K_1(\cdot, w) \otimes \bar\partial^{\boldsymbol{i}} K_2(\cdot, w) : |\boldsymbol{j}| \le k\big\}\Big) \\ &=\operatorname{ran} J_k(K_1, K_2)_{|\operatorname{res}\Delta}(\cdot, w).\end{aligned}$$

This completes the proof. □

The lemma given below is the main tool to prove Theorem 4.18.

Lemma 4.17 *Let $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ be two generalized Bergman kernels, and let $w = (w_1, \ldots, w_m)$ be an arbitrary point in Ω. Suppose that f is a function in $\mathcal{H}_1 \otimes \mathcal{H}_2$ satisfying $\big(\big(\frac{\partial}{\partial\zeta}\big)^{\boldsymbol{i}} f(z, \zeta)\big)_{|z=\zeta=w} = 0$ for all $\boldsymbol{i} \in \mathbb{Z}_+^m$, $|\boldsymbol{i}| \le k$. Then*

$$f(z, \zeta) = \sum_{j=1}^{m}(z_j - w_j) f_j(z, \zeta) + \sum_{|\boldsymbol{q}|=k+1} (z - \zeta)^{\boldsymbol{q}} f_{\boldsymbol{q}}^{\sharp}(z, \zeta)$$

for some functions $f_j, f_{\boldsymbol{q}}^{\sharp}$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$, $j = 1, \ldots, m$, $\boldsymbol{q} \in \mathbb{Z}_+^m$, $|\boldsymbol{q}| = k + 1$.

Proof Since K_1 and K_2 are generalized Bergman kernels, by Theorem 4.5, we have that $K_1 \otimes K_2$ is also a generalized Bergman kernel. Therefore, if f is a function in $\mathcal{H}_1 \otimes \mathcal{H}_2$ vanishing at (w, w), then using Lemma 4.14, we find functions $f_1, \ldots, f_m$, and $g_1, \ldots, g_m$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that

$$f(z, \zeta) = \sum_{j=1}^{m}(z_j - w_j) f_j + \sum_{j=1}^{m}(\zeta_j - w_j) g_j.$$

Equivalently, we have

$$f(z, \zeta) = \sum_{j=1}^{m}(z_j - w_j)(f_j + g_j) + \sum_{j=1}^{m}(z_j - \zeta_j)(-g_j).$$

Thus the statement of the lemma is verified for $k = 0$. To complete the proof by induction on k, assume that the statement is valid for some non-negative integer k. Let f be a function in $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that

$$\Big(\big(\frac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z, \zeta)\Big)_{|z=\zeta=w} = 0$$

for all $\boldsymbol{i} \in \mathbb{Z}_+^m,\ |\boldsymbol{i}| \leq k + 1$. By induction hypothesis, we can write

$$f(z, \zeta) = \sum_{j=1}^{m}(z_j - w_j) f_j(z, \zeta) + \sum_{|\boldsymbol{q}|=k+1} (z - \zeta)^{\boldsymbol{q}} f_{\boldsymbol{q}}^{\sharp}(z, \zeta) \tag{4.6}$$

for some $f_j, f_{\boldsymbol{q}}^{\sharp} \in \mathcal{H}_1 \otimes \mathcal{H}_2,\ j = 1, \ldots, m,\ \boldsymbol{q} \in \mathbb{Z}_+^m,\ |\boldsymbol{q}| = k + 1$. Fix a $\boldsymbol{i} \in \mathbb{Z}_+^m$ with $|\boldsymbol{i}| = k + 1$. Applying $\big(\frac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}}$ to both sides of (4.6), we see that

$$\begin{aligned}
&\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z, \zeta)\\
&= \sum_{j=1}^{m}(z_j - w_j)\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f_j(z, \zeta) + \sum_{|\boldsymbol{q}|=k+1} \big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}}\big((z - \zeta)^{\boldsymbol{q}} f_{\boldsymbol{q}}^{\sharp}(z, \zeta)\big)\\
&= \sum_{j=1}^{m}(z_j - w_j)\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f_j(z, \zeta) + \sum_{|\boldsymbol{q}|=k+1} \sum_{\boldsymbol{p} \leq \boldsymbol{i}} \tbinom{\boldsymbol{i}}{\boldsymbol{p}}\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{p}}(z - \zeta)^{\boldsymbol{q}}\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}-\boldsymbol{p}} f_{\boldsymbol{q}}^{\sharp}(z, \zeta).
\end{aligned}$$

Putting $z = \zeta = w$, we obtain

$$\Big(\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z, \zeta)\Big)_{|z=\zeta=w} = (-1)^{|\boldsymbol{i}|}\boldsymbol{i}!\ f_{\boldsymbol{i}}^{\sharp}(w, w),$$

where we have used the identity: $\Big(\big(\frac{\partial}{\partial \zeta}\big)^{\boldsymbol{p}}(z - \zeta)^{\boldsymbol{q}}\Big)_{|z=\zeta=w} = \delta_{\boldsymbol{pq}}(-1)^{|\boldsymbol{p}|}p!$.

Since $\big(\big(\frac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z, \zeta)\big)_{|z=\zeta=w} = 0$, we conclude that $f_{\boldsymbol{i}}^{\sharp}(w, w) = 0$. Since the statement of the lemma has been shown to be valid for $k = 0$, it follows that

$$f_{\boldsymbol{i}}^{\sharp}(z, \zeta) = \sum_{j=1}^{m}(z_j - w_j)\big(f_{\boldsymbol{i}}^{\sharp}\big)_j(z, \zeta) + \sum_{j=1}^{m}(z_j - \zeta_j)\big(f_{\boldsymbol{i}}^{\sharp}\big)_j^{\sharp}(z, \zeta) \tag{4.7}$$

for some $\big(f_{\boldsymbol{i}}^{\sharp}\big)_j, \big(f_{\boldsymbol{i}}^{\sharp}\big)_j^{\sharp} \in \mathcal{H}_1 \otimes \mathcal{H}_2,\ j = 1, \ldots, m$. Since (4.7) is valid for any $\boldsymbol{i} \in \mathbb{Z}_+^m,\ |\boldsymbol{i}| = k + 1$, replacing the $f_{\boldsymbol{q}}^{\sharp}$'s in (4.6) by $\sum_{j=1}^m (z_j - w_j)\big(f_{\boldsymbol{q}}^{\sharp}\big)_j(z, \zeta) + \sum_{j=1}^m (z_j - \zeta_j)\big(f_{\boldsymbol{q}}^{\sharp}\big)_j^{\sharp}(z, \zeta)$, we obtain the desired conclusion after some straightforward algebraic manipulation. □

Theorem 4.18 *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain. If $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ are generalized Bergman kernels, then so is the kernel $J_k(K_1, K_2)_{|\text{res}\,\Delta}$, $k \geq 0$.*

Proof By Theorem 4.16, we will be done if we can show that $\text{ran} D_{(\boldsymbol{J}_k \boldsymbol{M} - w)^*}$ is closed for every $w \in \Omega$. Fix a point $w = (w_1, \dots, w_m)$ in Ω. Let $\boldsymbol{X} := \big(P_{\mathcal{A}_k^\perp}(M_1^{(1)} \otimes I)_{|\mathcal{A}_k^\perp}, \dots, P_{\mathcal{A}_k^\perp}(M_m^{(1)} \otimes I)_{|\mathcal{A}_k^\perp}\big)$. By Corollary 4.6, we see that $\text{ran} D_{(\boldsymbol{J}_k \boldsymbol{M} - w)^*}$ is closed if and only if $\text{ran} D_{(\boldsymbol{X} - w)^*}$ is closed. Moreover, since $(D_{(\boldsymbol{X}-w)^*})^* = D^{(\boldsymbol{X}-w)}$, we conclude that $\text{ran} D_{(\boldsymbol{X}-w)^*}$ is closed if and only if $\text{ran} D^{(\boldsymbol{X}-w)}$ is closed. Note that X satisfies the following equality:

$$\ker D_{(\boldsymbol{X}-w)^*}{}^\perp = \overline{\text{ran}(D_{(\boldsymbol{X}-w)^*})^*} = \overline{\text{ran} D^{(\boldsymbol{X}-w)}}.$$

Therefore, to prove $\text{ran} D^{(\boldsymbol{X}-w)}$ is closed, it is enough to show that $\ker D_{(\boldsymbol{X}-w)^*}{}^\perp \subseteq \text{ran} D^{\boldsymbol{X}-w}$. To prove this, note that

$$D^{(\boldsymbol{X}-w)}(g_1 \oplus \cdots \oplus g_m) = P_{\mathcal{A}_k^\perp}\Big(\sum_{i=1}^m (z_i - w_i) g_i\Big),\ \ g_i \in \mathcal{A}_k^\perp, i = 1, \dots, m.$$

Thus

$$\text{ran} D^{(\boldsymbol{X}-w)} = \Big\{P_{\mathcal{A}_k^\perp}\Big(\sum_{i=1}^m (z_i - w_i) g_i : g_1, \dots, g_m \in \mathcal{A}_k^\perp\Big\}. \tag{4.8}$$

Now, let f be an arbitrary element of $\ker D_{(\boldsymbol{X}-w)^*}{}^\perp$. Then, by Lemma 4.15 and Proposition 2.2, we have $\big(\big(\frac{\partial}{\partial \bar\zeta}\big)^{\boldsymbol{i}} f(z,\zeta)\big)_{|z=\zeta=w} = 0$ for all $\boldsymbol{i} \in \mathbb{Z}_+^m$, $|\boldsymbol{i}| \leq k$. By Lemma 4.17,

$$f(z,\zeta) = \sum_{j=1}^m (z_j - w_j) f_j(z,\zeta) + \sum_{|\boldsymbol{q}|=k+1} (z-\zeta)^{\boldsymbol{q}} f_{\boldsymbol{q}}^\sharp(z,\zeta)$$

for some functions $f_j, f_{\boldsymbol{q}}^\sharp$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$, $j = 1, \dots, m$; $\boldsymbol{q}$ in $\mathbb{Z}_+^m$, $|\boldsymbol{q}| = k+1$. Note that the element $\sum_{|\boldsymbol{q}|=k+1} (z-\zeta)^{\boldsymbol{q}} f_{\boldsymbol{q}}^\sharp$ belongs to $\mathcal{A}_k$. Hence $f = P_{\mathcal{A}_k^\perp}(f) = P_{\mathcal{A}_k^\perp}\big(\sum_{j=1}^m (z_j - w_j) f_j\big)$. Furthermore, since the subspace $\mathcal{A}_k$ is invariant under $(M_j^{(1)} - w_j)$, $j = 1, \dots, m$, we see that

$$\begin{aligned} f = P_{\mathcal{A}_k^\perp}\big(\textstyle\sum_{j=1}^m (z_j - w_j) f_j\big) &= P_{\mathcal{A}_k^\perp}\Big(\textstyle\sum_{j=1}^m (z_j - w_j)\big(P_{\mathcal{A}_k^\perp} f_j + P_{\mathcal{A}_k} f_j\big)\Big) \\ &= P_{\mathcal{A}_k^\perp}\big(\textstyle\sum_{j=1}^m (z_j - w_j)(P_{\mathcal{A}_k^\perp} f_j)\big). \end{aligned}$$

Therefore, from (4.8), we conclude that $f \in \text{ran} D^{(\boldsymbol{X}-w)}$. This completes the proof. □

4.1 The Class $\mathcal{F}B_2(\Omega)$

In this subsection, first we will use Theorem 4.18 to prove that, if $\Omega \subset \mathbb{C}$, and K^α, K^β, defined on $\Omega \times \Omega$, are generalized Bergman kernels, then so is the kernel $\mathbb{K}^{(\alpha,\beta)}$. The following proposition, which is interesting on its own right, is an essential tool in proving this theorem. The notation below is chosen to be close to that of [16].

Proposition 4.19 *Let $\Omega \subset \mathbb{C}$ be a bounded domain. Let T be a bounded linear operator of the form $\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ on $H_0 \bigoplus H_1$. Suppose that T belongs to $B_2(\Omega)$ and T_0 belongs to $B_1(\Omega)$. Then T_1 belongs to $B_1(\Omega)$.*

Proof First, note that, for $w \in \Omega$,

$$(T - w)(x \oplus y) = ((T_0 - w)x + Sy) \oplus (T_1 - w)y. \tag{4.9}$$

Since $T \in B_2(\mathbb{D})$, $T - w$ is onto. Hence, from the above equality, it follows that $(T_1 - w)$ is onto.

Now we claim that $\dim \ker(T_1 - w) = 1$ for all $w \in \Omega$. From (4.9), we see that $(x \oplus y)$ belongs to $\ker(T - w)$ if and only if $(T_0 - w)x + Sy = 0$ and $y \in \ker(T_1 - w)$. Therefore, if $\dim \ker(T_1 - w)$ is 0, it must follow that $\ker(T - w) = \ker(T_0 - w)$, which is a contradiction. Hence the dimension of $\ker(T_1 - w)$ is at least 1. Now assume that $\dim \ker(T_1 - w) > 1$. Let $v_1(w)$ and $v_2(w)$ be two linearly independent vectors in $\ker(T_1 - w)$. Since $(T_0 - w)$ is onto, there exist $u_1(w), u_2(w) \in H_0$ such that $(T_0 - w)u_i(w) + Sv_i(w) = 0$, $i = 1, 2$. Hence the vectors $(u_1(w) \oplus v_1(w))$, $(u_2(w) \oplus v_2(w))$ belong to $\ker(T - w)$. Also, since $\dim \ker(T_0 - w) = 1$, there exists $\gamma(w) \in H_0$, such that $(\gamma(w) \oplus 0)$ belongs to $\ker(T - w)$. It is easy to verify that the vectors $\{(u_1(w) \oplus v_1(w)), (u_2(w) \oplus v_2(w)), (\gamma(w) \oplus 0)\}$ are linearly independent. This is a contradiction since $\dim \ker(T - w) = 2$. Therefore $\dim \ker(T_1 - w) \leq 1$. In consequence, $\dim \ker(T_1 - w) = 1$.

Finally, to show that $\overline{\bigvee}_{w \in \Omega} \ker(T_1 - w) = H_1$, let y be an arbitrary vector in H_1 which is orthogonal to $\overline{\bigvee}_{w \in \Omega} \ker(T_1 - w)$. Then it follows that $(0 \oplus y)$ is orthogonal to $\ker(T - w)$, $w \in \Omega$. Consequently, $y = 0$. This completes the proof. □

Theorem 4.20 *Let $\Omega \subset \mathbb{C}$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are positive definite on $\Omega \times \Omega$ for some $\alpha, \beta > 0$. Suppose that the operators $M^{(\alpha)*}$ on $(\mathcal{H}, K^\alpha)$ and $M^{(\beta)*}$ on $(\mathcal{H}, K^\beta)$ belong to $B_1(\Omega^*)$. Then the operator $\mathbb{M}^{(\alpha,\beta)*}$ on $(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$ belongs to $B_1(\Omega^*)$. Equivalently, if K^α and K^β are generalized Bergman kernels, then so is the kernel $\mathbb{K}^{(\alpha,\beta)}$.*

Proof Since the operators $M^{(\alpha)*}$ and $M^{(\beta)*}$ belong to $B_1(\Omega^*)$, it follows from Theorem 4.18 that the kernel $J_1(K^\alpha, K^\beta)_{|\text{res}\,\Delta}$ is a generalized Bergman kernel. Therefore, from Corollary 4.7, we see that the operator $\begin{pmatrix} M^{(\alpha+\beta)*} & \eta\, \mathsf{inc}^* \\ 0 & \mathbb{M}^{(\alpha,\beta)*} \end{pmatrix}$ belongs to $B_2(\Omega^*)$, where $\eta = \frac{\beta}{\sqrt{\alpha\beta(\alpha+\beta)}}$ and inc is the inclusion operator from $(\mathcal{H}, K^{\alpha+\beta})$ into

$(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$. Also, by Theorem 4.5, the operator $M^{(\alpha+\beta)^*}$ on $(\mathcal{H}, K^{\alpha+\beta})$ belongs to $B_1(\Omega^*)$. Proposition 4.19, therefore shows that the operator $\mathbb{M}^{(\alpha,\beta)^*}$ on $(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$ belongs to $B_1(\Omega^*)$. □

A smaller class of operators $\mathcal{F}B_n(\Omega)$ from $B_n(\Omega)$, $n \geq 2$, was introduced in [16]. A set of tractable complete unitary invariants and concrete models were given for operators in this class. We give below examples of a large class of operators in $\mathcal{F}B_2(\Omega)$. In case Ω is the unit disc $\mathbb{D}$, these examples include the homogeneous operators of rank 2 in $B_2(\mathbb{D})$ which are known to be in $\mathcal{F}B_2(\mathbb{D})$.

Definition 4.21 An operator T on $H_0 \bigoplus H_1$ is said to be in $\mathcal{F}B_2(\Omega)$ if it is of the form $\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$, where $T_0, T_1 \in B_1(\Omega)$ and S is a non-zero operator satisfying $T_0 S = S T_1$.

Theorem 4.22 *Let $\Omega \subset \mathbb{C}$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are positive definite on $\Omega \times \Omega$ for some $\alpha, \beta > 0$. Suppose that the operators $M^{(\alpha)^*}$ on $(\mathcal{H}, K^\alpha)$ and $M^{(\beta)^*}$ on $(\mathcal{H}, K^\beta)$ belong to $B_1(\Omega^*)$. Then the operator $(J_1 M)^*$ on $(\mathcal{H}, J_1(K^\alpha, K^\beta)_{|\text{res}\,\Delta})$ belongs to $\mathcal{F}B_2(\Omega^*)$.*

Proof By Theorem 4.18, the operator $(J_1 M)^*$ on $(\mathcal{H}, J_1(K^\alpha, K^\beta)_{|\text{res}\,\Delta})$ belongs to $B_2(\Omega^*)$, and by Corollary 4.7, it is unitarily equivalent to the operator $\begin{pmatrix} M^{(\alpha+\beta)^*} & \eta\, \mathsf{inc}^* \\ 0 & \mathbb{M}^{(\alpha,\beta)^*} \end{pmatrix}$ on $(\mathcal{H}, K^{\alpha+\beta}) \bigoplus (\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$. By Theorem 4.5, the operator $M^{(\alpha+\beta)^*}$ on $(\mathcal{H}, K^{\alpha+\beta})$ belongs to $B_1(\Omega^*)$ and by Theorem 4.20, the operator $\mathbb{M}^{(\alpha,\beta)^*}$ on $(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$ belongs to $B_1(\Omega^*)$. The adjoint of the inclusion operator inc clearly intertwines $M^{(\alpha+\beta)^*}$ and $\mathbb{M}^{(\alpha,\beta)^*}$. Therefore the operator $(J_1 M)^*$ on $(\mathcal{H}, J_1(K^\alpha, K^\beta)_{|\text{res}\,\Delta})$ belongs to $\mathcal{F}B_2(\Omega^*)$. □

Let $\Omega \subset \mathbb{C}$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^{α_1}, K^{α_2}, K^{β_1} and K^{β_2} are positive definite on $\Omega \times \Omega$ for some $\alpha_i, \beta_i > 0, i = 1, 2$. Suppose that the operators $M^{(\alpha_i)^*}$ on $(\mathcal{H}, K^{\alpha_i})$ and $M^{(\beta_i)^*}$ on $(\mathcal{H}, K^{\beta_i})$, $i = 1, 2$, belong to $B_1(\Omega^*)$. Let $\mathcal{A}_1(\alpha_i, \beta_i)$ be the subspace $\mathcal{A}_1$ of the Hilbert space $(\mathcal{H}, K^{\alpha_i}) \otimes (\mathcal{H}, K^{\beta_i})$ for $i = 1, 2$. Then we have the following corollary.

Corollary 4.23 *The operator $\left(M^{(\alpha_1)} \otimes I\right)^*_{|\mathcal{A}_1(\alpha_1,\beta_1)^\perp}$ is unitarily equivalent to the operator $\left(M^{(\alpha_2)} \otimes I\right)^*_{|\mathcal{A}_1(\alpha_2,\beta_2)^\perp}$ if and only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.*

Proof If $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, then there is nothing to prove. For the converse, assume that the operators $\left(M^{(\alpha_1)} \otimes I\right)^*_{|\mathcal{A}_1(\alpha_1,\beta_1)^\perp}$ and $\left(M^{(\alpha_2)} \otimes I\right)^*_{|\mathcal{A}_1(\alpha_2,\beta_2)^\perp}$ are unitarily equivalent. Then, by Corollary 3.10, we see that the operator $\begin{pmatrix} M^{(\alpha_1+\beta_1)^*} & \eta_1\, (\mathsf{inc})_1^* \\ 0 & \mathbb{M}^{(\alpha_1,\beta_1)^*} \end{pmatrix}$ on $(\mathcal{H}, K^{\alpha_1+\beta_1}) \bigoplus (\mathcal{H}, \mathbb{K}^{(\alpha_1,\beta_1)})$ is unitarily equivalent to $\begin{pmatrix} M^{(\alpha_2+\beta_2)^*} & \eta_2\, (\mathsf{inc})_2^* \\ 0 & \mathbb{M}^{(\alpha_2,\beta_2)^*} \end{pmatrix}$ on $(\mathcal{H}, K^{\alpha_2+\beta_2}) \bigoplus (\mathcal{H}, \mathbb{K}^{(\alpha_2,\beta_2)})$, where $\eta_i = \frac{\beta_i}{\sqrt{\alpha_i \beta_i (\alpha_i + \beta_i)}}$ and $(\mathsf{inc})_i$ is the inclusion operator from $(\mathcal{H}, K^{\alpha_i+\beta_i})$ into $(\mathcal{H}, \mathbb{K}^{(\alpha_i,\beta_i)})$, $i = 1, 2$.

Since $M^{(\alpha_i)*}$ on $(\mathcal{H}, K^{\alpha_i})$ and $M^{(\beta_i)*}$ on $(\mathcal{H}, K^{\beta_i})$, $i = 1, 2$, belong to $B_1(\Omega^*)$, by Theorem 4.22, we conclude that the operator $\begin{pmatrix} M^{(\alpha_i+\beta_i)*} & \eta_i\ (\mathsf{inc})_i^* \\ 0 & \mathbb{M}^{(\alpha_i,\beta_i)*} \end{pmatrix}$ belongs to $\mathcal{F}B_2(\Omega^*)$ for $i = 1, 2$. Therefore, by [16, Theorem 2.10], we obtain that

$$\mathcal{K}_{M^{(\alpha_1+\beta_1)*}} = \mathcal{K}_{M^{(\alpha_2+\beta_2)*}} \quad \text{and} \quad \frac{\eta_1\ \|(\mathsf{inc})_1^*(t_1)\|^2}{\|t_1\|^2} = \frac{\eta_2\ \|(\mathsf{inc})_2^*(t_2)\|^2}{\|t_2\|^2},$$

where $\mathcal{K}_{M^{(\alpha_i+\beta_i)*}}$, $i = 1, 2$, is the curvature of the operator $M^{(\alpha_i+\beta_i)*}$, and t_1 and t_2 are two non-vanishing holomorphic sections of the vector bundles $E_{\mathbb{M}^{(\alpha_1,\beta_1)*}}$ and $E_{\mathbb{M}^{(\alpha_2,\beta_2)*}}$, respectively. Note that, for $i = 1, 2$, $t_i(w) = \mathbb{K}^{(\alpha_i,\beta_i)}(\cdot, w)$ is a holomorphic non-vanishing section of the vector bundle $E_{\mathbb{M}^{(\alpha_i,\beta_i)*}}$, and also $(\mathsf{inc})_i^*(\mathbb{K}^{(\alpha_i,\beta_i)}(\cdot, w)) = K^{\alpha_i+\beta_i}(\cdot, w)$, $w \in \Omega$. Therefore the second equality in (4.1) implies that

$$\frac{\eta_1 K^{\alpha_1+\beta_1}(w, w)}{K^{\alpha_1+\beta_1}(w, w)\partial\bar{\partial}\log K(w, w)} = \frac{\eta_2 K^{\alpha_2+\beta_2}(w, w)}{K^{\alpha_2+\beta_2}(w, w)\partial\bar{\partial}\log K(w, w)}, \quad w \in \Omega,$$

or equivalently $\eta_1 = \eta_2$. Furthermore, it is easy to see that $\mathcal{K}_{M^{(\alpha_1+\beta_1)*}} = \mathcal{K}_{M^{(\alpha_2+\beta_2)*}}$ if and only if $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$. Hence, from (4.1), we see that

$$\alpha_1 + \beta_1 = \alpha_2 + \beta_2 \quad \text{and} \quad \eta_1 = \eta_2. \tag{4.10}$$

Then a simple calculation shows that (4.10) is equivalent to $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, completing the proof. □

5 The Generalized Wallach Set

Let Ω be a bounded domain in $\mathbb{C}^m$. Recall that the Bergman space $A^2(\Omega)$ is the Hilbert space of all square integrable analytic functions defined on Ω. The inner product of $A^2(\Omega)$ is given by the formula

$$\langle f, g\rangle := \int_\Omega f(z)\overline{g(z)}\, \mathrm{dV(z)},\ \mathrm{f, g} \in \mathrm{A}^2(\Omega),$$

where $\mathrm{dV(z)}$ is the normalized area measure on $\mathbb{C}^m$. The evaluation linear functional $f \mapsto f(w)$ is bounded on $A^2(\Omega)$ for all $w \in \Omega$. Consequently, the Bergman space is a reproducing kernel Hilbert space. The reproducing kernel of the Bergman space $A^2(\Omega)$ is called the Bergman kernel of Ω and is denoted by B_Ω.

If $\Omega \subset \mathbb{C}^m$ is a bounded symmetric domain, then the ordinary Wallach set $\mathcal{W}_\Omega$ is defined as $\{t > 0 : B_\Omega^t$ is non-negative definite$\}$. Here B_Ω^t, $t > 0$, makes sense since every bounded symmetric domain Ω is simply connected and the Bergman kernel on it is non-vanishing. If Ω is the Euclidean unit ball $\mathbb{B}_m$, then the Bergman kernel is given by

$$B_{\mathbb{B}_m}(z, w) = (1 - \langle z, w\rangle)^{-(m+1)}, \quad z, w \in \mathbb{B}_m, \tag{5.1}$$

and the Wallach set $\mathcal{W}_{\mathbb{B}_m} = \{t \in \mathbb{R} : t > 0\}$. But, in general, there are examples of bounded symmetric domains, like the open unit ball in the space of all $m \times n$ matrices, $m, n > 1$, with respect to the operator norm, where the Wallach set is a proper subset of $\{t \in \mathbb{R} : t > 0\}$. An explicit description of the Wallach set $\mathcal{W}_\Omega$ for a bounded symmetric domain Ω is given in [12].

Replacing the Bergman kernel in the definition of the Wallach set by an arbitrary scalar valued non-negative definite kernel K, we define the ordinary Wallach set $\mathcal{W}(K)$ to be the set

$$\{t > 0 : K^t \text{ is non-negative definite}\}.$$

Here we have assumed that there exists a continuous branch of logarithm of K on $\Omega \times \Omega$ and therefore K^t, $t > 0$, makes sense. Clearly, every natural number belongs to the Wallach set $\mathcal{W}(K)$. In [4], it is shown that K^t is non-negative definite for all $t > 0$ if and only if $\big(\partial_i \bar\partial_j \log K(z, w)\big)_{i,j=1}^m$ is non-negative definite. Therefore it follows from the discussion in the previous paragraph that there are non-negative definite kernels K on $\Omega \times \Omega$ for which $\big(\partial_i \bar\partial_j \log K(z, w)\big)_{i,j=1}^m$ need not define a non-negative definite kernel on $\Omega \times \Omega$. However, it follows from Proposition 2.3 that $K^{t_1+t_2}\big(\partial_i \bar\partial_j \log K(z, w)\big)_{i,j=1}^m$ is a non-negative kernel on $\Omega \times \Omega$ as soon as t_1 and t_2 are in the Wallach set $\mathcal{W}(K)$. Therefore it is natural to introduce the generalized Wallach set for any scalar valued kernel K defined on $\Omega \times \Omega$ as follows:

$$G\mathcal{W}(K) := \big\{t \in \mathbb{R} : \ K^{t-2}\mathbb{K} \text{ is non-negative definite}\big\}, \tag{5.2}$$

where, as before, we have assumed that K^t is well defined for all $t \in \mathbb{R}$. Clearly, we have the following inclusion

$$\big\{t_1 + t_2 \ : \ t_1, t_2 \in \mathcal{W}(K)\big\} \subseteq G\mathcal{W}(K).$$

5.1 *Generalized Wallach Set for the Bergman Kernel of the Euclidean Unit Ball in $\mathbb{C}^m$*

In this section, we compute the generalized Wallach set for the Bergman kernel of the Euclidean unit ball in $\mathbb{C}^m$. In the case of the unit disc $\mathbb{D}$, the Bergman kernel $B_{\mathbb{D}}(z, w) = (1 - z\bar w)^{-2}$ and $\partial\bar\partial \log B_{\mathbb{D}}(z, w) = 2(1 - z\bar w)^{-2}$, $z, w \in \mathbb{D}$. Therefore t is in $G\mathcal{W}(B_{\mathbb{D}})$ if and only if $(1 - z\bar w)^{-(2t+2)}$ is non-negative definite on $\mathbb{D} \times \mathbb{D}$. Consequently, $G\mathcal{W}(B_{\mathbb{D}}) = \{t \in \mathbb{R} : t \geq -1\}$. For the case of the Bergman kernel

$B_{\mathbb{B}_m}$ of the Euclidean unit ball $\mathbb{B}_m$, $m \geq 2$, we have shown that $G\mathcal{W}(B_{\mathbb{B}_m}) = \{t \in \mathbb{R} : t \geq 0\}$. The proof is obtained by putting together a number of lemmas which are of independent interest.

Before computing the generalized Wallach set $G\mathcal{W}(B_{\mathbb{B}_m})$ for Bergman kernel of the Euclidean ball $\mathbb{B}_m$, we point out that the result is already included in [21, Theorem 3.7], see also [15, 19]. The justification for our detailed proofs in this particular case is that it is direct and elementary in nature.

As before, we write $K \succeq 0$ to denote that K is a non-negative definite kernel. For two non-negative definite kernels $K_1, K_2 : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$, we write $K_1 \preceq K_2$ if $K_2 - K_1$ is a non-negative definite kernel on $\Omega \times \Omega$. Analogously, we write $K_1 \succeq K_2$ if $K_1 - K_2$ is non-negative definite.

Lemma 5.1 *Let Ω be a bounded domain in $\mathbb{C}^m$, and $\lambda_0 > 0$ be an arbitrary constant. Let $\{K_\lambda\}_{\lambda \geq \lambda_0}$ be a family of non-negative definite kernels, defined on $\Omega \times \Omega$, taking values in $\mathcal{M}_k(\mathbb{C})$ such that*

(i) *if $\lambda \geq \lambda' \geq \lambda_0$, then $K_{\lambda'} \preceq K_\lambda$,*
(ii) *for $z, w \in \Omega$, $K_\lambda(z, w)$ converges to $K_{\lambda_0}(z, w)$ entrywise as $\lambda \to \lambda_0$.*

Any $f : \Omega \to \mathbb{C}^k$ which is holomorphic and is in $(\mathcal{H}, K_\lambda)$ for all $\lambda > \lambda_0$ belongs to $(\mathcal{H}, K_{\lambda_0})$ if and only if $\sup_{\lambda > \lambda_0} \|f\|_{(\mathcal{H}, K_\lambda)} < \infty$.

Proof Recall that if K and K' are two non-negative definite kernels satisfying $K \preceq K'$, then $(\mathcal{H}, K) \subseteq (\mathcal{H}, K')$ and $\|h\|_{(\mathcal{H}, K')} \leq \|h\|_{(\mathcal{H}, K)}$ for $h \in (\mathcal{H}, K)$ (see [22, Theorem 6.25]). Therefore, by the hypothesis, we have that

$$(\mathcal{H}, K_{\lambda'}) \subseteq (\mathcal{H}, K_\lambda) \quad \text{and} \quad \|h\|_{(\mathcal{H}, K_\lambda)} \leq \|h\|_{(\mathcal{H}, K_{\lambda'})}, \tag{5.3}$$

whenever $\lambda \geq \lambda' \geq \lambda_0$ and $h \in (\mathcal{H}, K_{\lambda'})$.

Now assume that $f \in (\mathcal{H}, K_{\lambda_0})$. Then, clearly $\|f\|_{(\mathcal{H}, K_\lambda)} \leq \|f\|_{(\mathcal{H}, K_{\lambda_0})}$ for all $\lambda > \lambda_0$. Consequently, $\sup_{\lambda > \lambda_0} \|f\|_{(\mathcal{H}, K_\lambda)} \leq \|f\|_{(\mathcal{H}, K_{\lambda_0})} < \infty$. For the converse, assume that $\sup_{\lambda > \lambda_0} \|f\|_{(\mathcal{H}, K_\lambda)} < \infty$. Then, from (5.3), it follows that $\lim_{\lambda \to \lambda_0} \|f\|_{(\mathcal{H}, K_\lambda)}$ exists and is equal to $\sup_{\lambda > \lambda_0} \|f\|_{(\mathcal{H}, K_\lambda)}$. Since f is in $(\mathcal{H}, K_\lambda)$ for all $\lambda > \lambda_0$, by [22, Theorem 6.23], we have that

$$f(z) f(w)^* \preceq \|f\|^2_{(\mathcal{H}, K_\lambda)} K_\lambda(z, w).$$

Taking limit as $\lambda \to \lambda_0$ and using part (ii) of the hypothesis, we obtain

$$f(z) f(w)^* \preceq \sup_{\lambda > \lambda_0} \|f\|^2_{(\mathcal{H}, K_\lambda)} K_{\lambda_0}(z, w).$$

Hence, using [22, Theorem 6.23] once again, we conclude that $f \in (\mathcal{H}, K_{\lambda_0})$ completing the proof. □

If $m \geq 2$, then from (5.1), we have

$$\Big(\big(B^t_{\mathbb{B}_m}\partial_i\bar{\partial}_j \log B_{\mathbb{B}_m}\big)(z,w)\Big)_{i,j=1}^m$$
$$= \frac{m+1}{(1-\langle z,w\rangle)^{t(m+1)+2}}\begin{pmatrix} 1-\sum_{j\neq 1} z_j\bar{w}_j & z_2\bar{w}_1 & \cdots & z_m\bar{w}_1 \\ z_1\bar{w}_2 & 1-\sum_{j\neq 2} z_j\bar{w}_j & \cdots & z_m\bar{w}_2 \\ \vdots & \vdots & \vdots & \vdots \\ z_1\bar{w}_m & z_2\bar{w}_m & \cdots & 1-\sum_{j\neq m} z_j\bar{w}_j \end{pmatrix}. \tag{5.4}$$

For $m \geq 2$, $\lambda \in \mathbb{R}$ and $z, w \in \mathbb{B}_m$, set

$$\mathbb{K}_\lambda(z,w) := \frac{1}{(1-\langle z,w\rangle)^{\lambda}}\begin{pmatrix} 1-\sum_{j\neq 1} z_j\bar{w}_j & z_2\bar{w}_1 & \cdots & z_m\bar{w}_1 \\ z_1\bar{w}_2 & 1-\sum_{j\neq 2} z_j\bar{w}_j & \cdots & z_m\bar{w}_2 \\ \vdots & \vdots & \vdots & \vdots \\ z_1\bar{w}_m & z_2\bar{w}_m & \cdots & 1-\sum_{j\neq m} z_j\bar{w}_j \end{pmatrix}. \tag{5.5}$$

In view (5.4) and (5.5), for $\lambda > 2$, we have

$$\mathbb{K}_\lambda = \frac{2}{t(m+1)}\Big((B^{\frac{t}{2}}_{\mathbb{B}_m})^2 \partial_i\bar{\partial}_j \log B^{\frac{t}{2}}_{\mathbb{B}_m} \Big)_{i,j=1}^m,$$

where $t = \frac{\lambda-2}{m+1} > 0$. Since $B^{t/2}_{\mathbb{B}_m}$ is positive definite on $\mathbb{B}_m \times \mathbb{B}_m$ for $t > 0$, it follows from Corollary 2.4 that $\mathbb{K}_\lambda$ is non-negative definite on $\mathbb{B}_m \times \mathbb{B}_m$ for $\lambda > 2$. Since $\mathbb{K}_\lambda(z,w) \to \mathbb{K}_2(z,w)$, $z, w \in \mathbb{B}_m$, entrywise as $\lambda \to 2$, we conclude that $\mathbb{K}_2$ is also non-negative definite on $\mathbb{B}_m \times \mathbb{B}_m$.

Let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbb{C}^m$. The lemma given below finds the norm of the vector $z_2 \otimes e_1$ in $(\mathcal{H}, \mathbb{K}_\lambda)$ when $\lambda > 2$.

Lemma 5.2 *For each $\lambda > 2$, the vector $z_2 \otimes e_1$ belongs to $(\mathcal{H}, \mathbb{K}_\lambda)$ and*

$$\|z_2 \otimes e_1\|_{(\mathcal{H},\mathbb{K}_\lambda)} = \sqrt{\frac{\lambda-1}{\lambda(\lambda-2)}}.$$

Proof By a straight forward computation, we obtain

$$\bar{\partial}_1\mathbb{K}_\lambda(\cdot,0)e_2 = z_2 \otimes e_1 + (\lambda-1)z_1 \otimes e_2$$

and

$$\bar{\partial}_2\mathbb{K}_\lambda(\cdot,0)e_1 = (\lambda-1)z_2 \otimes e_1 + z_1 \otimes e_2.$$

Thus we have

$$(\lambda-1)\bar{\partial}_2\mathbb{K}_\lambda(\cdot,0)e_1 - \bar{\partial}_1\mathbb{K}_\lambda(\cdot,0)e_2 = (\lambda^2-2\lambda)z_2 \otimes e_1. \tag{5.6}$$

By Proposition 2.2, the vectors $\bar\partial_2\mathbb{K}_\lambda(\cdot,0)e_1$ and $\bar\partial_1\mathbb{K}_\lambda(\cdot,0)e_2$ belong to $(\mathcal{H},\mathbb{K}_\lambda)$. Since $\lambda > 2$, from (5.6), it follows that the vector $z_2\otimes e_1$ belongs to $(\mathcal{H},\mathbb{K}_\lambda)$. Now, taking norm in both sides of (5.6) and using Proposition 2.2 a second time, we obtain

$$\begin{aligned}(\lambda^2-2\lambda)^2\|z_2\otimes e_1\|^2 \\ &= (\lambda-1)^2\langle\partial_2\bar\partial_2\mathbb{K}_\lambda(0,0)e_1,e_1\rangle-(\lambda-1)\langle\partial_1\bar\partial_2\mathbb{K}_\lambda(0,0)e_1,e_2\rangle \\ &\quad -(\lambda-1)\langle\bar\partial_1\partial_2\mathbb{K}_\lambda(0,0)e_2,e_1\rangle+\langle\partial_1\bar\partial_1\mathbb{K}_\lambda(0,0)e_2,e_2\rangle \qquad (5.7)\end{aligned}$$

By a routine computation, we obtain

$$\partial_i\bar\partial_j\mathbb{K}_\lambda(0,0)=(\lambda-1)\delta_{ij}I_m+E_{ji},$$

where δ_{ij} is the Kronecker delta function, I_m is the identity matrix of order m, and E_{ji} is the matrix whose (j,i)th entry is 1 and all other entries are 0. Hence, from (5.7), we see that

$$\begin{aligned}(\lambda^2-2\lambda)^2||z_2\otimes e_1||^2 \\ &= (\lambda-1)^2(\lambda-1)-2(\lambda-1)+(\lambda-1) \\ &= (\lambda-1)(\lambda^2-2\lambda).\end{aligned}$$

Hence $||z_2\otimes e_1||=\sqrt{\frac{\lambda-1}{\lambda(\lambda-2)}}$, completing the proof of the lemma. □

Lemma 5.3 *The multiplication operator by the coordinate function z_2 on $(\mathcal{H},\mathbb{K}_2)$ is not bounded.*

Proof Since $\mathbb{K}_2(\cdot,0)e_1=e_1$, we have that the constant function e_1 is in $(\mathcal{H},\mathbb{K}_2)$. Hence, to prove that M_{z_2} is not bounded on $(\mathcal{H},\mathbb{K}_2)$, it suffices to show that the vector $z_2\otimes e_1$ does not belong to $(\mathcal{H},\mathbb{K}_2)$.

Consider the family of non-negative definite kernels $\{\mathbb{K}_\lambda\}_{\lambda\geq 2}$. Observe that for $\lambda\geq\lambda'\geq 2$,

$$\mathbb{K}_\lambda(z,w)-\mathbb{K}_{\lambda'}(z,w)=\Big((1-\langle z,w\rangle)^{-(\lambda-\lambda')}-1\Big)\mathbb{K}_{\lambda'}(z,w). \qquad (5.8)$$

It is easy to see that if $\lambda\geq\lambda'$, then $(1-\langle z,w\rangle)^{-(\lambda-\lambda')}-1\succeq 0$. Thus the right hand side of (5.8), being a product of a scalar valued non-negative definite kernel with a matrix valued non-negative definite kernel, is non-negative definite. Consequently, $K_{\lambda'}\preceq K_\lambda$. Also since $\mathbb{K}_\lambda(z,w)\to\mathbb{K}_2(z,w)$ entry-wise as $\lambda\to 2$, by Lemma 5.1, it follows that $z_2\otimes e_1\in(\mathcal{H},\mathbb{K}_2)$ if and only if $\sup_{\lambda>2}\|z_2\otimes e_1\|_{(\mathcal{H},\mathbb{K}_\lambda)}<\infty$. By Lemma 5.2, we have $\|z_2\otimes e_1\|_{(\mathcal{H},\mathbb{K}_\lambda)}=\sqrt{\frac{\lambda-1}{\lambda(\lambda-2)}}$. Thus $\sup_{\lambda>2}||z_2\otimes e_1||_{(\mathcal{H},\mathbb{K}_\lambda)}=\infty$. Hence the vector $z_2\otimes e_1$ does not belong to $(\mathcal{H},\mathbb{K}_2)$ and the operator M_{z_2} on $(\mathcal{H},\mathbb{K}_\lambda)$ is not bounded. □

The following theorem describes the generalized Wallach set for the Bergman kernel of the Euclidean unit ball in $\mathbb{C}^m$, $m \geq 2$.

Theorem 5.4 *If $m \geq 2$, then $G\mathcal{W}(B_{\mathbb{B}_m}) = \{t \in \mathbb{R} : t \geq 0\}$.*

Proof In view of (5.4) and (5.5), we see that $t \in G\mathcal{W}(B_{\mathbb{B}_m})$ if and only if $\mathbb{K}_{t(m+1)+2}$ is non-negative definite on $\mathbb{B}_m \times \mathbb{B}_m$. Hence we will be done if we can show that $\mathbb{K}_\lambda$ is non-negative if and only if $\lambda \geq 2$.

From the discussion preceding Lemma 5.2, we have that $\mathbb{K}_\lambda$ is non-negative definite on $\mathbb{B}_m \times \mathbb{B}_m$ for $\lambda \geq 2$.

To prove the converse, assume that $\mathbb{K}_\lambda$ is non-negative definite for some $\lambda < 2$. Note that $\mathbb{K}_2$ can be written as the product

$$\mathbb{K}_2(z, w) = (1 - \langle z, w\rangle)^{-(2-\lambda)}\mathbb{K}_\lambda(z, w),\ z, w \in \mathbb{B}_m. \tag{5.9}$$

Also, the multiplication operator M_{z_2} on $(\mathcal{H}, (1 - \langle z, w\rangle)^{-(2-\lambda)})$ is bounded. Hence, by Lemma 2.7, there exists a constant $c > 0$ such that $(c^2 - z_2\bar{w}_2)(1 - \langle z, w\rangle)^{-(2-\lambda)}$ is non-negative definite. Consequently, we see that the product $(c^2 - z_2\bar{w}_2)(1 - \langle z, w\rangle)^{-(2-\lambda)}\mathbb{K}_\lambda$, which is $(c^2 - z_2\bar{w}_2)\mathbb{K}_2$, is non-negative. Hence, again by Lemma 2.7, it follows that the operator M_{z_2} is bounded on $(\mathcal{H}, \mathbb{K}_2)$. This is a contradiction to the Lemma 5.3. Hence our assumption that $\mathbb{K}_\lambda$ is non-negative for some $\lambda < 2$, is not valid. This completes the proof. □

6 Quasi-Invariant Kernels

In this section, we show that if K a is quasi-invariant kernel with respect to some J, then $K^{t-2}\mathbb{K}$ is also a quasi-invariant kernel with respect to

$$\mathbb{J} := J(\varphi, z)^t D\varphi(z)^{\mathrm{tr}},\ \varphi \in \mathrm{Aut}(\Omega),\ \mathrm{z} \in \Omega,$$

whenever t is in the generalized Wallach set $G\mathcal{W}(K)$. The lemma given below, which will be used in the proof of the Proposition 6.2, follows from applying the chain rule [23, page 8] twice.

Lemma 6.1 *Let $\phi = (\phi_1, \ldots, \phi_m) : \Omega \to \mathbb{C}^m$ be a holomorphic map and g : $\mathrm{ran}\phi \to \mathbb{C}$ be a real analytic function. If $h = g \circ \phi$, then*

$$\Big(\big(\partial_i\bar{\partial}_j h\big)(z)\Big)_{i,j=1}^m = (D\phi(z))^{\mathrm{tr}}\Big(\big(\partial_i\bar{\partial}_j g\big)(\varphi(z))\Big)_{i,j=1}^m\overline{(D\phi(z))},$$

where $(D\phi)(z)^{\mathrm{tr}}$ is the transpose of the derivative of ϕ at z.

Proposition 6.2 *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain. Let $K : \Omega \times \Omega \to \mathbb{C}$ be a non-negative definite kernel and $J : \mathrm{Aut}(\Omega) \times \Omega \to \mathbb{C} \setminus \{0\}$ be a function such that $J(\varphi, \cdot)$ is holomorphic for each φ in $\mathrm{Aut}(\Omega)$. Suppose that K is quasi-invariant* *with respect to J. Then the kernel $K^{t-2}\mathbb{K}$ is also quasi-invariant with respect to $\mathbb{J}$ whenever $t \in G\mathcal{W}_\Omega(K)$, where $\mathbb{J}(\varphi, z) = J(\varphi, z)^t D\varphi(z)^{\mathrm{tr}}$, $\varphi \in \mathrm{Aut}(\Omega),\ \mathrm{z} \in \Omega$.*

Proof Since K is quasi-invariant with respect to J, we have

$$\log K(z, z) = \log |J(\varphi, z)|^2 + \log K(\varphi(z), \varphi(z)),\ \ \varphi \in \mathrm{Aut}(\Omega),\ z \in \Omega.$$

Also, $J(\varphi, \cdot)$ is a non-vanishing holomorphic function on Ω, therefore, for $1 \leq i, j \leq m$, $\partial_i \bar{\partial}_j \log |J(\varphi, z)|^2 = 0$. Hence

$$\partial_i \bar{\partial}_j \log K(z, z) = \partial_i \bar{\partial}_j \log K(\varphi(z), \varphi(z)),\ \ \varphi \in \mathrm{Aut}(\Omega),\ z \in \Omega. \tag{6.1}$$

Any biholomorphic automorphism φ of Ω is of the form $(\varphi_1, \ldots, \varphi_m)$, where $\varphi_i : \Omega \to \mathbb{C}$ is holomorphic, $i = 1, \ldots, m$. By setting $g(z) = \log K(z, z)$, $z \in \Omega$, and using Lemma 6.1, we obtain

$$\begin{aligned}&\big(\partial_i \bar{\partial}_j \log K(\varphi(z), \varphi(z))\big)_{i,j=1}^m \\ &\qquad = D\varphi(z)^{\mathrm{tr}}\big(\big(\partial_l \bar{\partial}_p \log K\big)(\varphi(z), \varphi(z))\big)_{l,p=1}^m \overline{D\varphi(z)}.\end{aligned}$$

Combining this with (6.1), we obtain

$$\big(\partial_i \bar{\partial}_j \log K(z, z)\big)_{i,j=1}^m = D\varphi(z)^{\mathrm{tr}}\big(\big(\partial_l \bar{\partial}_p \log K\big)(\varphi(z), \varphi(z))\big)_{l,p=1}^m \overline{D\varphi(z)}. \tag{6.2}$$

Multiplying $K(z, z)^t$ both sides and using the quasi-invariance of K, a second time, we obtain

$$\begin{aligned}&\big(K(z, z)^t \partial_i \bar{\partial}_j \log K(z, z)\big)_{i,j=1}^m \\ &= J(\varphi, z)^t D\varphi(z)^{\mathrm{tr}} K(\varphi(z), \varphi(z))^t \big(\big(\partial_l \bar{\partial}_p \log K\big)(\varphi(z), \varphi(z))_{l,p=1}^m \overline{J(\varphi, z)^t D\varphi(z)}.\end{aligned}$$

Equivalently, we have

$$K^{t-2}(z, z)\mathbb{K}(z, z) = \mathbb{J}(\varphi, z) K^{t-2}(\varphi(z), \varphi(z))\mathbb{K}(\varphi(z), \varphi(z))\mathbb{J}(\varphi, z)^*, \tag{6.3}$$

where $\mathbb{J}(\varphi, z) = J(\varphi, z)^t D\varphi(z)^{\mathrm{tr}}$, $\varphi \in \mathrm{Aut}(\Omega),\ z \in \Omega$. Therefore, polarizing both sides of the above equation, we have the desired conclusion. □

Remark 6.3 The function J in the definition of quasi-invariant kernel is said to be a projective cocycle if it is a Borel map satisfying

$$J(\varphi\psi, z) = m(\varphi, \psi) J(\psi, z) J(\varphi, \psi z),\ \ \varphi, \psi \in \mathrm{Aut}(\Omega), z \in \Omega, \tag{6.4}$$

where $m : \mathrm{Aut}(\Omega) \times \mathrm{Aut}(\Omega) \to \mathbb{T}$ is a multiplier, that is, m is Borel and satisfies the following properties:

(i) $m(e, \varphi) = m(\varphi, e) = 1$, where $\varphi \in \mathrm{Aut}(\Omega)$ and e is the identity in $\mathrm{Aut}(\Omega)$
(ii) $m(\varphi_1, \varphi_2)m(\varphi_1\varphi_2, \varphi_3) = m(\varphi_1, \varphi_2\varphi_3)m(\varphi_2, \varphi_3)$, $\varphi_1, \varphi_2, \varphi_3 \in \mathrm{Aut}(\Omega)$.

J is said to be a cocycle if it is a projective cocycle with $m(\varphi, \psi) = 1$ for all φ, ψ in $\mathrm{Aut}(\Omega)$.

If $J : \mathrm{Aut}(\Omega) \times \Omega \to \mathbb{C} \setminus \{0\}$ in Proposition 6.2 is a cocycle, then it is verified that the function $\mathbb{J}$ is a projective co-cycle. Moreover, if t is a positive integer, then $\mathbb{J}$ is also a cocycle.

For the preceding to be useful, one must exhibit non-negative definite kernels which are quasi-invariant. It is known that the Bergman kernel B_Ω of any bounded domain Ω is quasi-invariant with respect to J, where $J(\varphi, z) = \det D\varphi(z)$, $\varphi \in \mathrm{Aut}(\Omega)$, $z \in \Omega$.

Lemma 6.4 ([18, Proposition 1.4.12]) *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $\varphi : \Omega \to \Omega$ be a biholomorphic map. Then*

$$B_\Omega(z, w) = \det D\varphi(z) B_\Omega(\varphi(z), \varphi(w)) \overline{\det D\varphi(w)},\ z, w \in \Omega.$$

The following proposition follows from combining Proposition 6.2 and Lemma 6.4, and therefore the proof is omitted.

Proposition 6.5 *Let Ω be a bounded domain $\mathbb{C}^m$. If t is in $G\mathcal{W}(B_\Omega)$, then the kernel*

$$\boldsymbol{B}^{(t)}_\Omega(z, w) := \big(\, B^t_\Omega(z, w) \partial_i \bar{\partial}_j \log B_\Omega(z, w)\, \big)_{i,j=1}^m$$

is quasi-invariant with respect to $(\det D\varphi(z))^t D\varphi(z)^{\mathrm{tr}},\ \varphi \in \mathrm{Aut}(\Omega),\ z \in \Omega.$

For a fixed but arbitrary $\varphi \in \mathrm{Aut}(\Omega)$, let U_φ be the linear map on $\mathrm{Hol}(\Omega, \mathbb{C}^k)$ defined by

$$U_\varphi(f) = J\big(\varphi^{-1}, \cdot\big) f \circ \varphi^{-1}, \quad f \in \mathrm{Hol}(\Omega, \mathbb{C}^k). \tag{6.5}$$

The following proposition is a basic tool in defining unitary representations of the automorphism group $\mathrm{Aut}(\Omega)$. The straightforward proof for the case of unit disc $\mathbb{D}$ appears in [17]. The proof for the general domain Ω follows in exactly the same way.

Proposition 6.6 *The linear map U_φ is unitary on $(\mathcal{H}, K)$ for all φ in* $\mathrm{Aut}(\Omega)$ *if and only if the kernel K is quasi-invariant with respect to J.*

Let $Q : \Omega \to \mathcal{M}_k(\mathbb{C})$ be a real analytic function such that $Q(w)$ is positive definite for $w \in \Omega$. Let $\mathcal{H}$ be the Hilbert space of $\mathbb{C}^k$ valued holomorphic functions on Ω which are square integrable with respect to $Q(w)dV(w)$, that is,

$$\mathcal{H} = \big\{f \in \mathrm{Hol}(\Omega, \mathbb{C}^k) : \|f\|^2 := \int_\Omega \langle Q(w)f(w), f(w)\rangle_{\mathbb{C}^k} dV(w) < \infty\big\},$$

where dV is the normalized volume measure on $\mathbb{C}^m$. Assume that the constant functions are in $\mathcal{H}$. The operator U_φ, defined in (6.5) is unitary if and only if

$$\begin{aligned}\|U_\varphi f\|^2 &= \int_\Omega \langle Q(w)(U_\varphi f)(w), (U_\varphi f)(w)\rangle dV(w)\\ &= \int_\Omega \langle \overline{J(\varphi^{-1}, w)}^{\mathrm{tr}} Q(w)J(\varphi^{-1}, w) f(\varphi^{-1}(w)), f(\varphi^{-1}(w))\rangle dV(w)\\ &= \int_\Omega \langle Q(w)f(w), f(w)\rangle dV(w),\end{aligned}$$

that is, if and only if Q transforms according to the rule

$$\overline{J(\varphi^{-1}, w)}^{\mathrm{tr}} Q(w)J(\varphi^{-1}, w) = Q(\varphi^{-1}(w))|\det(D\varphi^{-1})(w)|^2. \tag{6.6}$$

Set

$$J(\varphi^{-1}, w) = \det(D\varphi^{-1}(w))^t D\varphi^{-1}(w)^{\mathrm{tr}},\ Q^{(t)}(w) := B_\Omega(w, w)^{1-t}\mathcal{K}(w, w)^{-1},$$

where $\mathcal{K}(z, w) := \big(\partial_i \bar\partial_j \log B_\Omega(z, w)\big)_{i,j=1}^m$, $t > 0$. Then $Q^{(t)}$ transforms according to the rule (6.6) since $\mathcal{K}$ transforms according to (6.2) and B_Ω transforms as in Lemma 6.4. If for some $t > 0$, the Hilbert space $L^2_{\mathrm{hol}}(\Omega, Q^{(t)}\,dV)$ determined by the measure is nontrivial, then the corresponding reproducing kernel is of the form $B^t_\Omega(z, w)\mathcal{K}(z, w)$.

Let Ω be a bounded symmetric domain in $\mathbb{C}^m$. Note that if $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ is a quasi-invariant kernel with respect to some J and the commuting tuple $\boldsymbol{M}_z = (M_{z_1}, \dots, M_{z_m})$ on $(\mathcal{H}, K)$ is bounded, then the commuting tuple $\boldsymbol{M}_\varphi := (M_{\varphi_1}, \dots, M_{\varphi_m})$ is unitarily equivalent to $\boldsymbol{M}_z$ via the unitary map U_φ, where $\varphi = (\varphi_1, \dots, \varphi_m)$ is in $\mathrm{Aut}(\Omega)$. If t is in $G\mathcal{W}(B_\Omega)$ and the operator of multiplication M_{z_i} by the coordinate function z_i is bounded on the Hilbert space $(\mathcal{H}, B^{t/2}_\Omega)$, then it follows from Corollary 2.9 that the operator M_{z_i} on the Hilbert space $\big(\mathcal{H}, \boldsymbol{B}^{(t)}_\Omega\big)$ is bounded as well. Therefore, in the language of [20], we conclude that the multiplication tuple $\boldsymbol{M}_z$ on $(\mathcal{H}, \boldsymbol{B}^{(t)}_\Omega)$ is homogeneous with respect to the group $\mathrm{Aut}(\Omega)$. In particular, if Ω is the Euclidean unit ball in $\mathbb{C}^m$, and t is any positive real number, then the multiplication tuple $\boldsymbol{M}_z$ on $(\mathcal{H}, B^{t/2}_{\mathbb{B}_m})$ is bounded. Also, from Theorem 5.4, it follows that $\boldsymbol{B}^{(t)}_{\mathbb{B}_m}$ is non-negative definite. Consequently, the commuting m-tuple of operators $\boldsymbol{M}_z$ must be homogeneous with respect to the group $\mathrm{Aut}(\mathbb{B}_m)$.

Acknowledgments The authors gratefully acknowledge the many hours of very fruitful discussion with Surjit Kumar, Dinesh Kumar Keshari and Kui Ji on the topics of this paper.

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A Survey on Classification of C*-Algebras with the Ideal Property

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Dedicated to the memory of Professor Ronald G. Douglas

Abstract In this paper, we give a survey on classification of C*-algebras with the ideal property. All simple, unital C*-algebras and real rank zero C*-algebras have the ideal property. We will review some results on classification of C*-algebras including real rank zero C*-algebras and unital, simple C*-algebras. Then we will present some invariants and discuss the relations between them. Our goal is to try to give a picture of current research about classifying AH algebras with the ideal property. We will review characterization theorem of the ideal property for AH algebra of Pasnicu (Pacific J. Math. 192:159–182, 2000), the reduction theorems for AH algebra [see G. Gong et al. (J. Funct. Anal. 258(6):2119–2143, 2010; Int. Math. Res. Not. IMRN 24:7606–7641, 2018), and C. Jiang (Canad. Math. Bull. 60(4):791–806, 2017)], and the most recent classification results for AH algebras and related C*-algebras [see K. Ji and C. Jiang (Canad. J. Math. 63:381–412, 2011), C. Jiang and K. Wang (J. Ramanujan Math. Soc. 27(3):305–354, 2012), G. Gong et al. (*A classification of inductive limit* C*-*algebras with the ideal property*, https://arxiv.org/pdf/1607.07581.pdf))].

Mathematics Subject Classification (2000) 46L35

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_12

1 Introduction

Recently some sweeping progresses have been made in the Elliott program [11], the program of classifying separable amenable C*-algebras by using the Elliott invariant (a K-theoretical set of invariant) (see [33, 64] and [20]). These are the results of decades of work by many mathematicians (see also [33, 64] and [20] for the historical discussion there). These progresses could be summarized briefly as the following: Two unital, finite, separable, simple C*-algebras A and B with finite nuclear dimension which satisfy the UCT are isomorphic if and only if their Elliott invariants, Ell(A) and Ell(B), are isomorphic. Moreover, every weakly unperforated Elliott invariant can be achieved by some finite separable simple C*-algebra in the UCT class with finite nuclear dimension. (In fact the algebra can be constructed as the so-called ASH-algebra, see [33]). Combining with the previous classification results by Kirchberg and Phillips [41, 59] of purely infinite, simple C*-algebras, now all unital, separable, simple C*-algebras in the UCT class with finite nuclear dimension are classified by the Elliott invariant.

After the above mentioned successful classification results of simple C*-algebras, it is natural to consider how to extend various classification theorems of simple C*-algebras to certain classes of non simple C*-algebras. Let us recall that at the beginning of Elliott classification program, there are two parallel and related classes to be considered: one is the class of C*-algebras of real rank zero (and stable rank one) and the other is the class of simple C*-algebras. G. Elliott even made the classification conjecture for the class of C*-algebras of real rank zero and stable rank one before his famous conjecture for simple C*-algebras. In fact, AH algebras, with certain restriction on dimension growth (which is related to the regularity conditions such as $\mathcal{Z}$-stability in the case of simple C*-algebras) have been classified much earlier. Namely on one hand is the classification of the real rank zero AH algebras (of no dimension growth, this condition can be slightly relaxed) (see [4–6, 10, 15, 16, 21, 22, 24, 26, 27, 42, 43], the last paper contains the general classification result). On the other hand is the classification obtained in [28] and [18] for the simple, unital AH algebras (of no dimension growth, this condition can be slightly relaxed) (also see [13, 14, 17, 44–47]).

In Elliott program for non simple C*-algebras, we think now there are several directions to explore. On one hand, we can continue to dig out theories for real rank zero C*-algebras (see the most recent work in [1, 2]). On the other hand, It would be important to unify and generalize the current known classification results for the above mentioned two classes: unital, simple C*-algebras and real rank zero C*-algebras. A natural way is to consider C*-algebras with the ideal property. In this paper, we will try to give a rough picture of this direction.

Definition 1.1 A C*-algebra is said to have the ideal property if each of its closed two-sided ideals is generated (as a closed two-sided ideal) by projections inside the ideal.

Definition 1.2 A C*-algebra A is said to have real rank zero (or to be of real rank zero) if the self-adjoint elements of A with finite spectrum are dense in the self-adjoint elements of A.

It is obvious that both simple, unital C*-algebras and real rank zero C*-algebras have the ideal property. There are many other examples of C*-algebras arising from dynamical systems which have the ideal property but are neither of real rank zero nor simple (see [31, 52, 61]). Pasnicu had intensively studied the class of C*-algebras with the ideal properties (see also [49, 51, 53–58] etc.).

This paper is organized as the following. In Sect. 2, we will review some background of classification of simple C*-algebras and real rank zero C*-algebras. Related invariants will be introduced. In Sect. 3, we will recall three characterization theorems in terms of the spectrum distribution property for simple AH algebra, real rank zero AH algebra and AH algebra with the ideal property. From those theorems, we can find that the ideal property is a natural generalization of simple and real rank zero properties. In Sect. 4, we will bring reduction theorems into our readers' sight. The reduction theorems play very important roles in classification theorems. Based on those theorems, it is enough to classify AH algebras of lower dimensional local spectra. In Sect. 5, we will introduce the extended Elliott invariant and Stevens-Jiang invariant and review the relation between them. Both of the invariants have played important role in the classification theory. The Stevens-Jiang invariant is more convenient to use while the extended Elliott invariant broadens the range of traditional Elliott invariant. In Sect. 6, we will see from a counter example that none of the invariants mentioned previously is enough for classifying all AH algebras with the ideal property. We will define the invariant $inv(\cdot)$ and present some classification results of AH algebras with the ideal property.

2 Background

We will discuss various invariants for classification of C*-algebras including the Elliott invariant in this section. The Elliott classification program began with the classification of all A$\mathbb{T}$ algebras of real rank zero in 1989 by Elliott using the scaled ordered K-theory $(K_*(A), K_*(A)^+, \sum A)$. In 1993, Elliott also classified all simple AI algebras by using now so called Elliott invariant. About the classification program, there are many significant classification results worth to be reviewed, we only choose the most related ones. First, we introduce some notations and the original Elliott invariant.

Definition 2.1 Let A be a C*-algebra. Let $\mathcal{P}(A)$ be the set of all projections in A. Let $\mathrm{K}_0(A)$ be the K_0-group of A and $\mathrm{K}_1(A)$ be the K_1-group of A. Denote $K_0(A)^+$ the semigroup of $K_0(A)$ generated by $[p] \in \mathrm{K}_0(A)$, where $p \in \mathcal{P}_\infty(A)$. Define

$$\Sigma A = \{[p] \in K_0(A)^+ : \ p \text{ is a projection in } A\}.$$

Then $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+, \Sigma A)$ is a scaled ordered group. Denote

$$K_*(A) := K_0(A) \oplus K_1(A) \cong K_0(A \otimes C(S^1))$$

and

$$K_*(A)^+ = K_0(A \otimes C(S^1))^+.$$

(The above version of positive cone $K_*(A)^+$ was introduced by Dadarlat and Nemethi in [9]. Elliott used a different but equivalent description.)

Definition 2.2 Let A be a C*-algebra. A weight on A is a function $\phi : A^+ \to [0, +\infty]$ such that

(i) $\phi(\alpha x) = \alpha\phi(x)$, if $x \in A^+$ and $\alpha \in \mathbb{R}^+$;
(ii) $\phi(x + y) = \phi(x) + \phi(y)$, if x and y belong to A^+.

Moreover, ϕ is lower semi-continuous if for each $\alpha \in \mathbb{R}^+$ the set

$$\{x \in A^+ |\ \phi(x) \leq \alpha\}$$

is closed.

Notation 2.3 *Let A be a C^*-algebra and $A^\sim$ is the unitization of A. Following notations will be used in this paper.*

$T(A) = \{\tau : A \to \mathbb{C} \mid \tau$ *is a positive linear trace satisfying* $\tau(1) = 1\}$

$T_F(A) = \{$*All finite traces on* $A\}$

$T_E(A) = \{\phi : A^+ \to [0, +\infty] \mid \phi$ *is a lower semi-continuous weight satisfying* $\phi(u^*xu) = \phi(x)$ *for all* $x \in A^+$ *and unitary* $u \in A^\sim\}$.

AffT$(A) = \{f : T(A) \to \mathbb{C} \mid f$ *is a continuous affine map*$\}$

Any affine map $\xi : X \to Y$ *induces a linear map* $\xi^* : \mathit{Aff}(Y) \to \mathit{Aff}(X)$ *by*

$$\xi^*(f)(\tau) = f(\xi(\tau)),$$

for all $f \in \mathit{Aff}(Y)$ *and* $\tau \in X$. *We still denote* ξ^* *by* ξ *if there is no confusion.*

Define $\rho_A : K_0(A) \to \mathit{AffT}(A)$ *by*

$$\big(\rho_A([p] - [q])\big)(\tau) = \sum_{i=1}^{n} \tau(p_{ii}) - \tau(q_{ii})$$

for $[p] - [q] \in K_0(A)$ *represented by the difference of two equivalence classes of* $n \times n$ *projection matrices* $p = (p_{ij})_{n\times n}$ *and* $q = (q_{ij})_{n\times n}$.

We can also define the map (still denoted by ρ_A) $\rho_A : K_0(A)^+ \to AffT_E(A)$ by

$$\big(\rho_A([p])\big)(\tau) = \sum_{i=1}^{n} \tau(p_{ii})$$

for $[p] \in K_0(A)^+$ represented by an $n \times n$ projection matrix $p = (p_{ij})_{n\times n}$.

Definition 2.4 Let A, B be two C^*-algebras. Let $\alpha : \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ be a homomorphism, and $\xi : \mathrm{T}(B) \to \mathrm{T}(A)$ (or $\xi : \mathrm{T}_E(B) \to \mathrm{T}_E(A)$) be an affine map. We say that α and ξ are compatible if

$$\tau(\alpha(x)) = (\xi(\tau))(x)$$

for all $x \in \mathrm{K}_0(A)$ (or $x \in \mathrm{K}_0(A)^+$) and $\tau \in \mathrm{T}(B)$ (or $\tau \in \mathrm{T}_E(B)$). That is the following diagram is commutative:

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\rho_A} & \mathrm{AffT}(A) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\xi} \\ K_0(B) & \xrightarrow{\rho_A} & \mathrm{AffT}(B), \end{array} \tag{2.1}$$

or

$$\begin{array}{ccc} K_0(A)^+ & \xrightarrow{\rho_A} & \mathrm{AffT}_E(A) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\xi} \\ K_0(B)^+ & \xrightarrow{\rho_A} & \mathrm{AffT}_E(B). \end{array} \tag{2.2}$$

2.5 (Elliott Invariant) *is defined to be*

$$Ell(A) = (K_0(A), K_0(A)^+, \Sigma A, K_1(A), AffT(A), \rho_A).$$

Definition 2.6 A C^*-algebra A is called an AH (approximate homogeneous) algebra if it is an inductive limit of

$$A_1 \to A_2 \to A_3 \to \cdots$$

where $A_n = \bigoplus_{i=1}^{k_n} P_{n_i} M_{[n,i]}(C(X_{n_i}))P_{n_i}$, X_{n_i} are compact metrizable spaces, P_{n_i} are projections of $M_{[n,i]}(C(X_{n_i}))$.

It is called an AF algebra if $X_{n_i} = \{pt\}$; it is called an AI algebra if $X_{n_i} = [0, 1]$; it is called an A$\mathbb{T}$ algebra if $X_{n_i} = S^1$.

We say that the AH inductive limit C*-algebra A defined above is of no dimension growth if

$$\sup_{n,i}\{\dim(X_{n_i})\} < \infty.$$

2.7 (Elliott Conjecture)

1. *($K_*(\cdot), K_*(\cdot)^+, \Sigma(\cdot)$) is a complete invariant for separable nuclear C*-algebras of real rank zero and stable rank one.*
2. *Ell($\cdot$) is a complete invariant for simple separable nuclear C*-algebras.*

For simple case of the conjecture, Elliott, Gong and Li have completely classified all simple AH algebras with no dimension growth in 2007 (see Theorem 2.8 below), which is based on the reduction theorem of Gong. The simple, real rank zero case was done by Elliott and Gong earlier in [15] with the reduction theorem of Dadarlat and Gong (see [4] and [26]).

Theorem 2.8 ([18]) *If A and B are both simple AH algebras with no dimension growth, then*

$$A \cong B \Longleftrightarrow Ell(A) \cong Ell(B).$$

(With the reduction theorem of Gong, see [28]*.)*

Recently, the Elliott program has culminated in the simple, unital, UCT case, with a definitive classification which merely assumes the abstract regularity hypothesis of finite nuclear dimension (this result combines theorems of Gong-Lin-Niu [33], Elliott-Gong-Lin-Niu [20], and White-Winter-Tikuisis [64], while also incorporating the earlier Kirchberg-Phillips classification on the purely infinite side [41, 59]). Their results have exhausted all possible simple, unital, separable C*-algebras that could be classified by the Elliott invariant. Thus, it is desirable to work on the non-simple C*-algebras for the future classification projects.

Real rank zero C*-algebras may be non-simple. Elliott conjecture for real rank zero C*-algebras has a different story because Gong's counter example to the Elliott conjecture (see Theorem 2.9 below) tells us that the proposed invariant is not enough for classifying all real rank zero AH algebras (even with no dimension growth). Subsequently a new invariant of a C*-algebra A—called the invariant of total K-theory of A, was introduced (see [6, 7, 10] and [5]) and all real rank zero AH algebras with no dimension growth were classified by Dadarlat and Gong (see Theorem 2.12 below).

Theorem 2.9 ([27]) *There are two AH algebras A and B with dimension of local spectra of at most two such that*

$$(K_*(A), K_*(A)^+, \Sigma(A)) \cong (K_*(B), K_*(B)^+, \Sigma(B))$$

but A is not isomorphic to B.

Definition 2.10 For a C*-algebra A, let

$$\underline{K}(A) = K_*(A) \bigoplus \bigoplus_{k=2}^{+\infty} K_*(A, \mathbb{Z}/k\mathbb{Z})$$

be as in [5] (see [6, 10], and [7] also), where $K_*(A, \cdot) = K_0(A, \cdot) \oplus K_1(A, \cdot)$. Let $\wedge$ be the Bockstein operation on $\underline{K}(A)$ (see [5, 4.1]). It is well known that

$$K_*(A, Z \oplus \mathbb{Z}/k\mathbb{Z}) = K_0(A \otimes C(W_k \times S^1)),$$

where $W_k = T_{II,k}$ is a connected finite simplicial complex with

$$H^1(T_{II,k}) = 0 \text{ and } H^2(T_{II,k}) = \mathbb{Z}/k\mathbb{Z}.$$

As in [5], let

$$K_*(A, Z \oplus \mathbb{Z}/k\mathbb{Z})^+ = K_0(A \otimes C(W_k \times S^1))^+$$

and let $\underline{K}(A)^+$ be the semigroup generated by $\{K_*(A, \mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z})^+, k = 2, 3, \cdots\}$. We denote the invariant of **total K-theory** of A by

$$(\underline{K}(A), \underline{K}(A)^+, \Sigma A)_\wedge.$$

Definition 2.11 For C*-algebras A and B, let $Hom_\wedge(\underline{K}(A), \underline{K}(B))$ be the set of homomorphisms between $\underline{K}(A)$ and $\underline{K}(B)$ compatible with Bockstein operation $\wedge$. There is a surjective map (see [7])

$$\Gamma : KK(A, B) \to Hom_\wedge(\underline{K}(A), \underline{K}(B)).$$

Following Rørdam (see [60]), we denote by $KL(A, B) := KK(A, B)/ker\ \Gamma$. For two C^*-algebras A and B, by a "homomorphism"

$$\alpha : (\underline{K}(A), \underline{K}(A)^+, \Sigma A)_\wedge \to (\underline{K}(B), \underline{K}(B)^+, \Sigma B)_\wedge,$$

we mean a system of maps:

$$\alpha_k^i : K_i(A, \mathbb{Z}/k\mathbb{Z}) \longrightarrow K_i(B, \mathbb{Z}/k\mathbb{Z}), \quad i = 0, 1, \quad k = 0, 2, 3, \cdots$$

which are compatible with Bockstein operations and $\alpha = \bigoplus_{k,i} \alpha_k^i$ satisfying

$$\alpha(\underline{K}(A)^+) \subset \underline{K}(B)^+ \text{ and } \alpha_0^0(\Sigma A) \subset \Sigma B.$$

Theorem 2.12 ([5, Theorem 9.1]) *If A and B are AH algebras of real rank zero with no dimension growth, then*

$$A \cong B \Longleftrightarrow (\underline{K}(A), \underline{K}(A)^+, \Sigma A)_\wedge \cong (\underline{K}(B), \underline{K}(B)^+, \Sigma B)_\wedge.$$

3 Characterization of Simple, Real Rank Zero and Ideal Properties

We know that both simple AH algebras and real rank zero AH algebras have the ideal property. In this section, we will compare the spectrum distribution properties for simple AH algebra, real rank zero AH algebra and AH algebra with the ideal property. From those characterization theorems (see Theorems 3.4, 3.5 and 3.6 below), we can see clearly that the ideal property generalizes simple and real rank zero properties naturally. These results are also used frequently in the proof of classification theorems. First, let's review definitions of spectrum.

Definition 3.1 Let X and Y be compact metrizable spaces. Let $P \in M_{k_1}(C(Y))$ be a projection with $\mathrm{rank}(P) = k \leq k_1$ and let $\psi : C(X) \to PM_{k_1}(C(Y))P$ be a unital homomorphism. For any given point $y \in Y$, there are $x_1(y), x_2(y), \cdots, x_k(y) \in X$ (may be repeat) and a unitary U_y such that

$$\psi(f)(y) = P(y)U_y \begin{pmatrix} f(x_1(y)) & & & \\ & \ddots & & \\ & & f(x_k(y)) & \\ & & & \mathbf{0}_{n-r} \end{pmatrix} U_y^* P(y).$$

In other words, there are k mutually orthogonal rank 1 projections $p_1, p_2, \cdots, p_k$ with $\sum_{i=1}^k p_i = P(y)$ and $x_1(y), x_2(y), \cdots, x_k(y) \in X$ such that

$$\psi(f)(y) = \sum_{i=1}^{k} f(x_i(y))p_i, \ \text{for} \ \forall f \in C(X).$$

We write $\mathrm{Sp}\psi_y := \{x_1(y), x_2(y), \cdots, x_k(y)\}$, where we also count multiplicity. We shall call $Sp\psi_y$ **the spectrum of** ψ **at the point** y.

3.2 ([15, 1.4.3]) *Let us consider a unital homomorphism $\psi : M_l(C(X)) \to PM_{k_1}(C(Y))P$ with rank$(P) = k \leq k_1$. (It is necessary that k is a multiple of l.) We know that ψ is completely determined by*

$$\psi_1 = \psi|_{e_{11}M_l(C(X))e_{11}} : C(X) \to \psi(e_{11})M_{k_1}(C(Y))\psi(e_{11})$$

up to unitary equivalence. Let us define $\mathrm{Sp}\psi_y := \mathrm{Sp}\psi_{1y}$.

Definition 3.3 If $\psi : QM_{l_1}(C(X))Q \to PM_{k_1}(C(Y))P$ is a unital homomorphism with $\text{rank}(P) = k$ a multiple of $\text{rank}(Q) = l$, then for each y, $\psi(f)(y)$ only depends on the value of $f \in QM_{l_1}(C(X))Q$ at finitely many points $x_1, x_2, \cdots, x_{k/l}$, where $x_i \in X$ may repeat. Namely, if we identify $Q(x_i)M_{l_1}(\mathbb{C})Q(x_i)$ with $M_l(\mathbb{C})$, and still denote the image of $f(x_i)$ in $M_l(\mathbb{C})$ by $f(x_i)$, then there is a unitary $U_y \in M_{k_1}(C(Y))$ such that

$$\psi(f)(y) = P(y)U_y \begin{pmatrix} f(x_1)_{l\times l} & & & \\ & \ddots & & \\ & & f(x_k)_{l\times l} & \\ & & & \mathbf{0}_{n-r} \end{pmatrix} U_y^* P(y).$$

Certainly, U_y depends on the identification of $Q(x_i)M_{l_1}(\mathbb{C})Q(x_i)$ and $M_l(\mathbb{C})$, too. Let us define $\text{Sp}\psi_y := \{x_1, x_2, \cdots, x_{k/l}\}$, where we also count multiplicity.

Theorem 3.4 ([8, Proposition 2.1]) *Let $A = \lim\limits_{\to}(A_n, \phi_{n,m})$ be an inductive limit system with no dimension growth, where $A_n = \bigoplus_{i=1}^{k_n} P_{n,i}M_{[n,i]}C(X_{n,i})P_{n,i}$, $X_{n,i}$ are finite CW complexes and $\phi_{n,m}$ are injective. Then the following conditions are equivalent:*

(i) A is simple.
(ii) For any A_n and any $\eta > 0$, there is $m_0 > 0$ such that for all $m \geq m_0$

$$\text{Sp}(\phi_{n,m})_y \textit{ is } \eta\textit{-dense in } \text{Sp}(A_n)$$

for any $y \in \text{Sp}(A_m)$.

Theorem 3.5 *Let $A = \lim\limits_{\to}(A_n, \phi_{n,m})$ be an inductive limit system with no dimension growth, where $A_n = \bigoplus_{i=1}^{k_n} P_{n,i}M_{[n,i]}C(X_{n,i})P_{n,i}$, $X_{n,i}$ are connected, finite CW complexes. Then the following conditions are equivalent:*

(i) A is of real rank zero.
(ii) For any A_n and any $\eta > 0$, there is $m > 0$ such that

$$\text{Sp}(\phi_{n,m}^{i,j})_{y_1} \textit{ and } \text{Sp}(\phi_{n,m}^{i,j})_{y_2} \textit{ can be paired within } \eta$$

for any partial map of $\phi_{n,m}$ and $y_1, y_2 \in \text{Sp}A_m^j = X_{m,j}$.

In the above theorem, (i) implies (ii) is due to Su (see [63, Theorem 2.5]) and (ii) implies (i) is due to Elliott and Gong (see 1.4.5, 1.4.6 and 2.25 in [15]). The following characterization theorem of AH algebras with the ideal property is proved by C. Pasnicu.

Theorem 3.6 ([50, Theorem 3.1]) *Let $A = \lim\limits_{\rightarrow}(A_n, \phi_{n,m})$ be an AH algebra, with $A_n = \bigoplus_{i=1}^{k_n} A_n^i$, $A_n^i = P_{n,i} M_{[n,i]} C(X_{n,i}) P_{n,i}$ where $X_{n,i}$ are connected, finite CW complexes and $P_{n,i} \in C(X_{n,i}, M_{[n,i]})$ are projections. Then the following are equivalent:*

(i) Any ideal of A is generated by its projections.

(ii) For any fixed n and any fixed $F = \bar{F} \subseteq U = \overset{\circ}{U} \subseteq \mathrm{Sp}(A_n) = \sqcup_{i=1}^{k_n} X_{n,i}$ there is $m_0 > n$ such that for any $m \geq m_0$ any partial map $\phi_{n,m}^j : A_n \to A_m^j$ satisfies either:

$$\mathrm{Sp}(\phi_{n,m}^j)_{y_1} \subseteq B_\eta(\mathrm{Sp}(\phi_{n,m}^j)_{y_2})$$

for any $y_1, y_2 \in \mathrm{Sp}(A_m^j) = X_{m,j}$, where $B_\eta(\mathrm{Sp}(\phi_{n,m}^j)_{y_2})$ is the η-ball of $\mathrm{Sp}(\phi_{n,m}^j)_{y_2}$.

4 Reduction Theorem

This section is dedicated to reduction theorems. By applying a reduction theorem, we can rewrite an inductive limit C*-algebra as an inductive limit of certain special type of algebras. We only present the reduction theorem for AH algebra with the ideal property (Theorems 4.1 and 4.8 below), which generalize previous reduction theorems for simple AH algebras and also for real rank zero AH algebras.

Many reduction theorems tell us it is enough to classify inductive limit C*-algebras which is a limit of direct sum of matrix algebras over three or less dimensional spaces. This is because the cancellation of projections holds for those spaces. We will remind you the cancellation property and give a full proof here (see Theorem 4.5 below, originally it was stated in the remark 3.26 of [15] without a detailed proof).

Theorem 4.1 ([32, Theorem 4.2]) *Suppose that* $\lim(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i}))$, $\phi_{n,m})$ *is an AH inductive limit with* $\dim(X_{n,i}) \leq M$ *for a fixed positive integer M such that the limit algebra has the ideal property. Then there is another inductive limit system* $(\lim(B_n = \bigoplus\limits_{i=1}^{s_n} M_{[n,i]}(C(Y_{n,i})), \psi_{n,m}))$ *with the same limit algebra as the above system, where all the $Y_{n,i}$ are spaces of the form* $\{pt\}$, $[0, 1]$, S^1, S^2, $T_{II,k}$, $T_{III,k}$.

Here $T_{II,k}$ ($T_{III,k}$, respectively) is a connected finite simplicial complex with

$$H^1(T_{II,k}) = 0 \text{ and } H^2(T_{II,k}) = \mathbb{Z}/k\mathbb{Z}$$

($H^1(T_{III,k}) = 0 = H^2(T_{III,k})$ and $H^3(T_{III,k}) = \mathbb{Z}/k\mathbb{Z}$, respectively).

Theorem 4.5 in the following is originally stated in [15, Remark 3.26]. It is a well-known result for topologists but not for operator algebraists. Since there is no precise reference in [15] and we can not find a precise reference, we give a complete proof here. We need the following theorems.

Theorem 4.2 ([34, Chapter 9, Theorem 1.5]) *Let ξ_1 and ξ_2 be two k-dimensional complex vector bundles on a n-dimensional CW complex. Let m be the smallest integer with $\frac{n}{2} \leq m$. Suppose $m \leq k$ and*

$$\xi_1 \oplus \theta^l \cong \xi_2 \oplus \theta^l$$

for some l-dimensional trivial bundle, then

$$\xi_1 \cong \xi_2.$$

Lemma 4.3 *Let X be a 3-dimensional finite simplicial complex with 2-skeleton $X^{(2)}$ and E be a 1-dimensional complex bundle over X. Then E is trivial if and only if $E|_{X^{(2)}}$ is trivial.*

Proof Let $E \setminus \{0\} \subset E$ be the collection of all non zero vectors of all fibres. The vector bundle $E \to X$ is trivial if and only if there exists a cross section $s : X \to E \setminus \{0\}$ which means s is a nowhere vanish cross section of the bundle $E \to X$.

Suppose $E|_{X^{(2)}}$ is trivial and $s : X^{(2)} \to E \setminus \{0\}|_{X^{(2)}}$ is a nowhere vanish cross section. We will show that s can be extended to a nowhere vanish cross section $\tilde{s} : X \to E \setminus \{0\}$. This can be done simplex by simplex. We only need to prove for each three dimensional simplex $\triangle$ with $\partial\triangle \subseteq X^{(2)}$,

$$s|_{\partial\triangle} : \partial\triangle \to E \setminus \{0\}|_{\partial\triangle}$$

can be extended to a nowhere vanish section

$$\tilde{s}|\triangle : \triangle \to E \setminus \{0\}|_{\triangle}.$$

Let

$$T' : E|_{\triangle} \to \triangle \times \mathbb{C}$$

be a trivialization and let

$$T : E \setminus \{0\}|_{\triangle} \to \triangle \times (\mathbb{C} \setminus \{0\})$$

be the restriction of T'. Let

$$p : \triangle \times (\mathbb{C} \setminus \{0\}) \to \mathbb{C} \setminus \{0\}$$

be the projection. Then the map $s' := p \circ T \circ s : \partial\triangle \to \mathbb{C} \setminus \{0\}$ satisfies

$$T(x) = (x, s'(x)) \in \triangle \times (\mathbb{C} \setminus \{0\})$$

for all $x \in \partial \triangle$.

Notice that $\partial\triangle$ is homeomorphic to S^2 and $\pi_2(\mathbb{C} \setminus \{0\}) = 0$, there is an extension $\widetilde{s'} : \triangle \to \mathbb{C} \setminus \{0\}$ with $\widetilde{s'}|_{\partial\triangle} = s'$. Define $\widetilde{s}|_{\triangle} : \triangle \to E \setminus \{0\}|_{\triangle}$ by

$$\widetilde{s}(x) = T^{-1}(x, \widetilde{s'}(x)), \text{ for all } x \in \triangle.$$

Evidently, $\widetilde{s}|_{\triangle}$ is a desired extension of $s|_{\partial\triangle}$. □

Remark 4.4 For any vector spaces V_1, V_2, we use $\mathrm{Hom}(V_1, V_2)$ to denote the set of all linear maps from V_1 to V_2, which is a vector space of dimension $\dim V_1 \times \dim V_2$.

Let $E \to X$, $F \to X$ be two complex vector bundles and let $\mathrm{Hom}(E, F) \to X$ be the vector bundle with fibre $\mathrm{Hom}(E_x, F_x)$ for each $x \in X$. If both E and F are 1-dimensional vector bundles, then $\mathrm{Hom}(E, F)$ is a 1-dimensional vector bundle.

Evidently, for 1-dimensional vector bundles E and F, E is isomorphic to F if and only if $\mathrm{Hom}(E, F)$ has a non zero cross section, and if and only if $\mathrm{Hom}(E, F)$ is trivial.

Theorem 4.5 *The collection of complex vector bundles $V(X)$ over a connected 3-dimensional CW complex X has the property of cancellation, i.e.,*

$$E_1 \oplus F \cong E_2 \oplus F \text{ implies } E_1 \cong E_2,$$

where $E_1, E_2, F \in V(X)$.

Proof By Swan theorem and the assumption, we know

$$E_1 \oplus \theta^l \cong E_2 \oplus \theta^l$$

for some l-dimensional trivial bundle.

Let k be the dimension of the bundle E_1. If $k \geq 2$, applying Theorem 4.2 for $n = 3$, (then $m = 2$ which is smaller or equal to k), we have $E_1 \cong E_2$ immediately. Therefore, it is enough to prove the theorem for the case $k = 1$.

Now suppose $k = 1$. Since

$$(E_1 \oplus \theta^l)|_{X^{(2)}} \cong (E_2 \oplus \theta^l)|_{X^{(2)}},$$

where $X^{(2)}$ is a 2-skeleton of X, applying Theorem 4.2 for $n = 2$, $m = 1$, we have

$$E_1|_{X^{(2)}} \cong E_2|_{X^{(2)}}.$$

By Remark 4.4, this is equivalent to the 1-dimensional bundle $\mathrm{Hom}(E_1, E_2)$ is trivial on $X^{(2)}$. By Lemma 4.3, $\mathrm{Hom}(E_1, E_2)$ is trivial on X. Therefore, by Remark 4.4 again, we have $E_1 \cong E_2$. □

Remark 4.6 This result appears as Remark 3.26 of [15]. After that paper was published, several operator algebraists asked Gong for detailed proof, since the standard result in topology (Theorem 4.2 above) only gives the cancellation property for vector bundle over 2-dimensional space. This is the reason we present the complete proof here. This result is an improvement of Theorem 4.2 for the case of dimension $n = 3$. But let us point out that, for dimension n larger than 3, the number m in Theorem 4.2 is a sharp bound and can not be improved.

Definition 4.7 A dimension drop algebra I_k is of the following form

$$I_k = \{f \in C([0,1], M_k(\mathbb{C})),\ f(0) = \lambda 1_k,\ f(1) = \mu 1_k,\ \lambda, \mu \in \mathbb{C}\}.$$

The following reduction theorem tells us for an AH algebra A with the ideal property we can rewrite it as an inductive limit C*-algebra of certain special subhomogeneous algebras.

Theorem 4.8 ([37, Theorem 3.1]) *Suppose* $\varinjlim(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})),\ \phi_{n,m})$ *is an AH inductive limit with* $X_{n,i}$ *being among the spaces* $\{pt\}$, $[0,1]$, S^1, S^2, $\{T_{II,k}\}_{k=2}^{\infty}$, $\{T_{III,k}\}_{k=2}^{\infty}$, *such that the limit algebra A has ideal property. Then there is another inductive system* $(B_n = \oplus B_n^i, \psi_{n,m})$ *with same limit algebra, where* B_n^i *are either* $M_{[n,i]'}(C(Y_{n,i}))$ *with* $Y_{n,i}$ *being one of* $\{pt\}$, $[0,1]$, S^1, $\{T_{II,k}\}_{k=2}^{\infty}$, *(but without* $\{T_{III,k}\}_{k=2}^{\infty}$ *and* S^2*) or dimension drop algebra* $M_{[n,i]'}(I_{k(n,i)})$.

5 Extended Elliott Invariant and Stevens-Jiang Invariant

In this section, we will introduce and compare the extended Elliott invariant and Stevens-Jiang invariant. We know that for simple C*-algebras, traces are assumed to be bounded in the unital cases, and lower semicontinuous and densely defined in the non-unital case. But neither of them will suffice for the classification of non-simple C*-algebras. Thus, the extended valued traces need to be included in the traditional Elliott invariant, which we call extended Elliott invariant (see [65]). The extended Elliott invariant essentially contains more information than Stevens-Jiang invariant in general (see Remark 5.7 below), while the Stevens-Jiang invariant is usually more convenient to use when proving classification theorems (see Theorem 5.9 and Remark 5.10 below).

Definition 5.1 For a C*-algebra A, the **extended Elliott invariant** of A, denoted by $\mathbf{eEll(A)}$, is defined to be

$$eEll(A) = ((\underline{K}(A), \underline{K}(A)^{+}, \Sigma A)_{\wedge}, \mathrm{T}_E(A), \rho_A).$$

Definition 5.2 For a C*-algebra A, the **Stevens-Jiang invariant** of A, denoted by $\mathbf{inv^0(A)}$, is defined to be

$$inv^0(A) = \{(\underline{K}(A), \underline{K}(A)^{+}, \Sigma A)_{\wedge}, \{\mathrm{AffT}(pAp)\}_{p\in\Sigma(A)}\}.$$

The invariant is introduced by Jiang in [35] (also see [36]) based on some earlier results of Stevens in [62].

Definition 5.3 Let A, B be two C^*-algebras. We say that A and B have isomorphic Stevens-Jiang invariants if: (1) There is a scaled ordered isomorphism $\alpha : \mathrm{K}_0(A) \to \mathrm{K}_0(B)$; (2) For each pair $([p], [\overline{p}]) \in \Sigma A \times \Sigma B$ with $\alpha([p]) = [\overline{p}]$, there is an associate unital positive linear map

$$\xi^{p,\overline{p}} : \mathrm{AffT}(pAp) \longrightarrow \mathrm{AffT}(\overline{p}B\overline{p})$$

satisfying the condition if $p < q$, then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{AffT}(pAp) & \xrightarrow{\xi^{p,\overline{p}}} & \mathrm{AffT}(\overline{p}B\overline{p}) \\ \iota\downarrow & & \iota\downarrow \\ \mathrm{AffT}(qAq) & \xrightarrow{\xi^{q,\overline{q}}} & \mathrm{AffT}(\overline{q}B\overline{q}), \end{array} \tag{5.1}$$

where ι stands for the inclusion map.

Remark 5.4 (See [35, 1.11]) If A and B have isomorphic Stevens-Jiang invariant, then α and $\xi^{e,f}$ are compatible for all $e \in \mathcal{P}(A)$, $f \in \mathcal{P}(B)$ with $\alpha([e]) = [f]$.

Theorem 5.5 ([65, Theorem 1]) *Let A be a stably finite C*-algebra with the ideal property. Then the Stevens-Jiang invariant of A is equivalent to the extended Elliott invariant of A.*

Theorem 5.6 ([65, Theorem 2]) *Let A, B be two stably finite C*-algebras with the ideal property. If A and B have isomorphic extended Elliott invariant, then A and B have isomorphic Stevens-Jiang invariant—and vise versa.*

Remark 5.7 In [65], the author actually shows that for any stably finite C*-algebra, its Stevens-Jiang invariant can always be derived from its extended Elliott invariant in a functor manner. But the converse is not true in general. In fact, there are C*-algebras without the ideal property whose Elliott invariant cannot be derived from its Stevens-Jiang invariant (see Example 5.3 in [65]).

Theorem 5.9 and Remark 5.10 below talk about applications of Stevens-Jiang invariant in classification theory for C*-algebras.

Definition 5.8 A splitting interval algebra is any C^*-algebra of the form

$$\mathcal{S}(\overline{n}_0; \overline{n}_1) = \{f \in M_n(C[0,1]) : f(x) \in \bigoplus_{i=1}^{r_x} M_{n_{x_i}}(C), x = 0, 1\}$$

where each $\overline{n}_x = (n_{x_1}, \cdots, n_{x_{r_x}})$, is a partition of n.

Theorem 5.9 ([39]) *Let A, B be two ASI algebras (inductive limits of splitting interval algebras) with the ideal property. Then*

$$A \cong B \Longleftrightarrow inv^0(A) \cong inv^0(B) \Longleftrightarrow$$

$$\{K_0(A), K_0(A)^+, \Sigma A, \{AffT(pAp)\}\} \cong \{K_0(B), K_0(B)^+, \Sigma B, \{AffT(\bar{p}B\bar{p})\}\}$$

Remark 5.10 The special case of above theorem for AI algebras was done by Ji-Jiang (see [35])—the paper removed restrictions of unital condition and approximate divisible condition in Stevens' classification theorem [62] and built up a framework for the further classification proofs (see [35]).

6 Classification of AH Algebras with the Ideal Property

Now we are ready to talk about classification of AH algebras with the ideal property. There are examples to show that none of the invariants list above is sufficient for classifying all AH algebras with the ideal property (see Example 6.4 below). Thus, we need to introduce extra ingredients for the invariant. Moreover, we will talk about the relation between those invariants and propose some open questions.

Definition 6.1 For C*-algebras A, B, let $DU(A)$ be the commutator subgroup of $U(A)$ and $\overline{DU(A)}$ be its closure. That is,

$$\overline{DU(A)} = \text{closure of the subgroup generated by } uvu^*v^* \text{ where } u, v \in U(A).$$

One can introduce the following metric D_A on $U(A)/\overline{DU(A)}$ (see [48, §3]). For $u, v \in U(A)/\overline{DU(A)}$

$$D_A(u, v) = \inf\{\|uv^* - c\|; c \in \overline{DU(A)}\},$$

where, on the right hand side of the equation, we use u, v to denote any elements in $U(A)$, which represent the elements $u, v \in U(A)/\overline{DU(A)}$.

Definition 6.2 Let A be a unital C^*-algebra. Let Aff$\mathrm{T}A$ be defined as in Notation 2.3, and

$$\rho : K_0(A) \longrightarrow \mathrm{AffT}A$$

the canonical map. The metric d_A on $\mathrm{AffT}A/\overline{\rho K_0(A)}$ is defined as follows (see [48, §3]). Let d' denote the quotient metric on $\mathrm{AffT}A/\overline{\rho K_0(A)}$, i.e,

$$d'(f,g) = \inf\{\|f-g-h\| : \; h \in \overline{\rho K_0(A)}\}$$

for $f, g \in \mathrm{AffT}A/\overline{\rho K_0(A)}$. Define d_A by

$$d_A(f,g) = \begin{cases} 2, & if\ d'(f,g) \geq \frac{1}{2} \\ |e^{2\pi i d'(f,g)} - 1|, & if\ d'(f,g) < \frac{1}{2}. \end{cases}$$

Obviously, $d_A(f,g) \leq 2\pi d'(f,g)$.

Definition 6.3 Denote

$$\Big((\underline{K}(A), \underline{K}(A)_+, \Sigma A),\; \{\mathrm{AffT}(pAp)\}_{p\in A}, \{U(pAp)/\overline{DU(pAp)}\}_{p\in\Sigma A}\Big)$$

by $Inv(A)$. By a map from $Inv(A)$ to $Inv(B)$, we mean there is a "homomorphism"

$$\alpha : (\underline{K}(A), \underline{K}(A)^+, \Sigma A) \longrightarrow (\underline{K}(B), \underline{K}(B)^+, \Sigma B)$$

as in Definition 2.11, and for each pair $([p], [\overline{p}]) \in \Sigma A \times \Sigma B$ with $\alpha([p]) = [\overline{p}]$, there is an associate unital positive linear map

$$\xi^{p,\overline{p}} : \mathrm{AffT}(pAp) \longrightarrow \mathrm{AffT}(\overline{p}B\overline{p})$$

and an associate contractive group homomorphism

$$\chi^{p,\overline{p}} : U(pAp)/\overline{DU(pAp)} \longrightarrow U(\overline{p}B\overline{p})/\overline{DU(\overline{p}B\overline{p})}$$

satisfying if $p < q$, then the following diagrams

$$\begin{array}{ccc} \mathrm{AffT}(pAp) & \xrightarrow{\xi^{p,\overline{p}}} & \mathrm{AffT}(\overline{p}B\overline{p}) \\ \iota\downarrow & & \iota\downarrow \\ \mathrm{AffT}(qAq) & \xrightarrow{\xi^{q,\overline{q}}} & \mathrm{AffT}(\overline{q}B\overline{q}). \end{array} \tag{6.1}$$

$$\begin{array}{ccc} U(pAp)/\overline{DU(pAp)} & \xrightarrow{\chi^{p,\bar{p}}} & U(\bar{p}B\bar{p})/\overline{DU(\bar{p}B\bar{p})} \\ \downarrow \iota_* & & \downarrow \iota_* \\ U(qAq)/\overline{DU(qAq)} & \xrightarrow{\chi^{q,\bar{q}}} & U(\bar{q}B\bar{q})/\overline{DU(\bar{q}B\bar{q})} \end{array} \tag{6.2}$$

$$\begin{array}{ccc} \mathrm{AffT}(pAp)/\overline{\rho K_0(pAp)} & \longrightarrow & U(pAp)/\overline{DU(pAp)} \\ \xi^{p,\overline{p}} \downarrow & & \chi^{p,\overline{p}} \downarrow \\ \mathrm{AffT}(\overline{p}B\overline{p})/\overline{\rho K_0(\overline{p}B\overline{p})} & \longrightarrow & U(\overline{p}B\overline{p})/\overline{DU(\overline{p}B\overline{p})} \end{array} \tag{6.3}$$

and

$$\begin{array}{ccc} U(pAp)/\overline{DU(pAp)} & \longrightarrow & K_1(pAp)/tor\ K_1(pAp) \\ \chi^{p,\overline{p}} \downarrow & & \alpha_1 \downarrow \\ U(\overline{p}B\overline{p})/\overline{DU(\overline{p}B\overline{p})} & \longrightarrow & K_1(\overline{p}B\overline{p})/tor\ K_1(\overline{p}B\overline{p}) \end{array} \tag{6.4}$$

commute, where α_1 is induced by α and ι_* is induced by $u \mapsto u \oplus (q-p) \in U(qAq)$.

Example 6.4 ([29]) There are A$\mathbb{T}$ algebras A and B with ideal property such that $Inv^0(A)$ and $Inv^0(B)$ are isomorphic. But $Inv(A) \ncong Inv(B)$, and consequently $A \ncong B$.

Theorem 6.5 ([30]) *Let A, B be two AH algebras with the ideal property with no dimension growth. Then*

$$A \cong B \Longleftrightarrow Inv(A) \cong Inv(B).$$

In [30], the above invariant is called $Inv'(A)$ and the invariant $Inv(A)$ used in [30] is a reduced version of the above invariant (also see [3] and [38] for more details).

Remark 6.6 Note that the above result does not cover Theorem 5.9. It is desirable to have a theorem to combine and generalize these two classification results.

By results in [30] and [65], we know more examples could be classified by the extended Elliott invariant:

Theorem 6.7 *Let A, B be two AH algebras with the ideal property. Suppose that $K_1(A) = tor K_1(A)$, then*

$$A \cong B \Longleftrightarrow Inv^0(A) \cong Inv^0(B) \Longleftrightarrow eEll(A) \cong eEll(B).$$

Remark 6.8 In [66], the third named author showed that if A, B are two AH algebras with the ideal property and one of them has finitely many ideals, then

$$A \cong B \Longleftrightarrow Inv^0(A) \cong Inv^0(B) \Longleftrightarrow eEll(A) \cong eEll(B).$$

(Note that the two C^* algebras A and B in [29] have countably infinitely many ideals.)

6.9 (Open Questions)

- *As in* [65]*, we can include the extended valued traces in the Elliott invariant to get extended Elliott invariant, which is equivalent to the Stevens-Jiang invariant* $inv^0(\cdot)$*. Can we modify the extended Elliott invariant to recover the invariant* $inv(\cdot)$*?*
- *What is the range of classifiable* C^**-algebra by using the invariant* $inv(\cdot)$*?*

Acknowledgments The first named author was a Ph.D. student of Professor Ronald G. Douglas. It was Douglas who introduced him into the field of operator algebras and non-commutative geometry. The second named author had been working on many problems of Douglas for a long period of time and had many mathematical discussions with Douglas. The third named author was a postdoctoral in Texas A&M University. She was in many lectures given by Professor Douglas and also discussed mathematics with him. All three authors benefited greatly from mathematical insight of Douglas. This paper is dedicated to the memory of our beloved teacher and friend Ronald G. Douglas.

We would also like to thank our coauthors on the topic of C*-algebras with the ideal property for their collaboration: B. Cui, K. Ji, L. Li and C. Pasnicu.

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A Survey on the Arveson-Douglas Conjecture

Kunyu Guo and Yi Wang

Dedicated to the memory of Ronald G. Douglas

Abstract The Arveson-Douglas Conjecture is a conjecture about essential normality of submodules, and quotient modules of some analytic Hilbert modules on polynomial rings. It originated from multi-variable operator theory but turns out to have various connections outside this field. In this survey, we will introduce the conjecture, describe its backgrounds, applications, and recent developments. Several related questions are also included in this survey.

Keywords Arveson-Douglas Conjecture · Essential normality · Index theory · Holomorphic extension · Row contractions

2010 AMS Subject Classification 47A13, 32A50, 30H20, 32D15, 47B35

1 The Arveson-Douglas Conjecture

The Arveson-Douglas Conjecture refers to a set of conjectures involving essential normality of submodules of analytic function spaces. In this section, we will list a few representative conjectures that are generally referred to as the Arveson-Douglas Conjecture. Before that, let us give some definitions.

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_13

Definition 1.1

(1) Let $\mathbb{C}[z_1, \cdots, z_n]$ be the ring of analytic polynomials of n variables. By a *($\mathbb{C}[z_1, \cdots, z_n]$-)Hilbert module* we mean a Hilbert space $\mathcal{H}$ equipped with a ring homomorphism

$$\sigma : \mathbb{C}[z_1, \cdots, z_n] \to \mathcal{B}(\mathcal{H}),$$

where $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded linear operators on $\mathcal{H}$.

(2) For $1 < p < \infty$, the Hilbert module $\mathcal{H}$ is said to be *p-essentially normal* if

$$[\sigma(z_i), \sigma^*(z_j)] = \sigma(z_i)\sigma^*(z_j) - \sigma^*(z_j)\sigma(z_i) \in \mathcal{C}_p, \quad i, j = 1, \cdots, n.$$

Here $\mathcal{C}_p$ denotes the ideal of Schatten p class operators.

(3) The Hilbert module $\mathcal{H}$ is said to be *essentially normal* if

$$[\sigma(z_i), \sigma^*(z_j)] \in \mathcal{K}, \quad i, j = 1, \cdots, n.$$

Here $\mathcal{K}$ denotes the ideal of compact operators on $\mathcal{H}$.

(4) Let $\mathcal{H}$ be a Hilbert module. By a *(Hilbert) submodule* of $\mathcal{H}$ we mean a Hilbert subspace $\mathcal{P} \subset \mathcal{H}$ that is invariant under the module actions, i.e.,

$$\sigma(q)\mathcal{P} \subset \mathcal{P}, \quad \forall q \in \mathbb{C}[z_1, \cdots, z_n],$$

or equivalently,

$$\sigma(z_i)\mathcal{P} \subset \mathcal{P}, \quad i = 1, \cdots, n.$$

Naturally, the submodule $\mathcal{P}$ inherits a Hilbert module structure by

$$\sigma_{\mathcal{P}}(q) = \sigma(q)|_{\mathcal{P}}, \quad \forall q \in \mathbb{C}[z_1, \cdots, z_n].$$

(5) Given a submodule $\mathcal{P}$ of $\mathcal{H}$. The corresponding *(Hilbert) quotient module* is the orthogonal complement $\mathcal{Q} := \mathcal{P}^{\perp}$ (which, can be identified with the quotient $\mathcal{H}/\mathcal{P}$), equipped with the compressed module action

$$\sigma_{\mathcal{Q}}(q) = Q\sigma(q)|_{\mathcal{Q}}, \quad \forall q \in \mathbb{C}[z_1, \cdots, z_n].$$

Here Q denotes the projection operator onto $\mathcal{Q}$.

The Arveson-Douglas Conjecture concerns essential normality, and p-essential normality, of submodules, and quotient modules, of the following Hilbert modules: the Drury-Arveson module, the Hardy module, the Bergman module and the weighted Bergman modules. Let us give the definitions. Let $\mathbb{B}_n$ denote the open unit ball of $\mathbb{C}^n$.

Definition 1.2

(1) The *Drury-Arveson space* H_n^2 is the reproducing kernel Hilbert space on $\mathbb{B}_n$ with reproducing kernels

$$K_z(w) = \frac{1}{1 - \langle w, z \rangle}, \quad w \in \mathbb{B}_n, z \in \mathbb{B}_n.$$

See [58] for a detailed exposure of the Drury-Arveson space.

(2) The *Hardy space* $H^2(\mathbb{B}_n)$ is the Hilbert space consisting of holomorphic functions f on $\mathbb{B}_n$ such that

$$\|f\|_{H^2}^2 = \sup_{0<r<1} \int_S |f(r\zeta)|^2 d\sigma(\zeta) < \infty.$$

Here $S = \partial\mathbb{B}_n$ is the unit sphere and σ is the normalized surface measure on S.

(3) For $s > -1$, the *weighted Bergman space* $L_{a,s}^2(\mathbb{B}_n)$ consists of holomorphic functions f on $\mathbb{B}_n$ such that

$$\|f\|_{L_{a,s}^2}^2 = \int_{\mathbb{B}_n} |f(z)|^2 (1 - |z|^2)^s dv(z) < \infty.$$

Here v denotes the normalized Lebesgue measure on $\mathbb{B}_n$. We write $L_a^2(\mathbb{B}_n)$ for $L_{a,0}^2(\mathbb{B}_n)$, which we call the Bergman space.

(4) When $q \in \mathbb{C}[z_1, \cdots, z_n]$, we define multiplication operator

$$M_q f = qf$$

on each above mentioned function space. Therefore,each such function space has a $\mathbb{C}[z_1, \cdots, z_n]$-Hilbert module structure.

More generally, for a positive integer r, the operators

$$M_q \otimes I_{\mathbb{C}^r}, \quad q \in \mathbb{C}[z_1, \cdots, z_n]$$

define Hilbert module structures on the vector-valued function spaces $H_n^2 \otimes \mathbb{C}^r$, $H^2(\mathbb{B}_n) \otimes \mathbb{C}^r$ and $L_{a,s}^2(\mathbb{B}_n) \otimes \mathbb{C}^r$.

(5) Let $\mathcal{H}$ be one of the spaces in (4). A *graded submodule* $\mathcal{P}$ is a submodule of $\mathcal{H}$ generated by a set of vector polynomials in $\mathbb{C}[z_1, \cdots, z_n] \otimes \mathbb{C}^r$ that are homogeneous. Equivalently, $\mathcal{P}$ is the closure in $\mathcal{H}$, of some graded submodule in $\mathbb{C}[z_1, \cdots, z_n] \otimes \mathbb{C}^r$.

It is well-known that all three types of modules in (4) are p-essentially normal for all $p > n$. In [7, 8], Arveson proposed the following conjecture.

Conjecture 1 (The Arveson's Conjecture) Suppose $\mathcal{P}$ is a graded submodule of the vector-valued Drury-Arveson module $H_n^2 \otimes \mathbb{C}^r$. Then $\mathcal{P}$ is p-essentially normal for all $p > n$.

Arveson raised this conjecture in the course of his studies of geometric invariants for commuting operator tuples [4–9]. In fact, by a method of linearization, Shalit [57] reduced the conjecture to the scalar case (up to a small modification of the range of p), and hence one only needs to consider the scalar version of the conjecture. Then in [22], Douglas showed that the conjecture was well-motivated for a different reason—it leads to a new kind of index theorem. Based on this motivation, Douglas described what he believes is a natural setting for analogues of Arveson's Conjecture.

Conjecture 2 (The Arveson-Douglas Conjecture) Let I be a homogeneous ideal in $\mathbb{C}[z_1, \cdots, z_n]$ and $[I]$ denote its closure in $L_a^2(\mathbb{B}_n)$. Then the graded quotient module $\mathcal{Q}_I := [I]^\perp$ is p-essentially normal for all $p > \dim_{\mathbb{C}} Z(I)$, where

$$Z(I) = \{z \in \mathbb{C}^n : p(z) = 0, \forall p \in I\}.$$

Conjecture 2 is the best-known form of the Arveson-Douglas Conjecture. Results on Conjecture 2 include [8, 15, 29, 34–36, 39–41, 52, 57] and many others. For $p > n$, a simple matrix calculation [8, 21] shows that the p-essential normality of the submodule is equivalent to the p-essential normality of the quotient module. We remark that, in general, the submodule $\mathcal{P}$ is not p-essentially normal for any $p \leq n$. The improvement of the lower bound when considering the quotient module, instead of the submodule, indicates that the quotient modules are more closely related to the zero varieties $Z(I)$. We will discuss this in detail in the following section.

In the case when $r = 1$ and I is a homogeneous radical ideal, using the Hilbert's Nullstellensatz, one can show that

$$\mathcal{P}_I = \{f \in L_a^2(\mathbb{B}_n) : f|_{\mathcal{Z}_I} = 0\},$$

where

$$\mathcal{Z}_I = Z(I) \cap \mathbb{B}_n.$$

Therefore

$$\mathcal{Q}_I = \overline{\text{span}}\{K_z : z \in \mathcal{Z}_I\}.$$

Here

$$K_z(w) = \frac{1}{(1 - \langle w, z\rangle)^{n+1}}$$

is the reproducing kernel of $L_a^2(\mathbb{B}_n)$ at the point z. The following geometric version of the Arveson-Douglas Conjecture was studied by various researchers [28, 31, 33, 38, 51, 63, 64].

Conjecture 3 (Geometric Arveson-Douglas Conjecture) Let $M \subset \mathbb{B}_n$ be a homogeneous variety. Define the quotient module

$$\mathcal{Q}_M = \overline{\text{span}}\{K_z : z \in M\} \subset L_a^2(\mathbb{B}_n).$$

Then $\mathcal{Q}_M$ is p-essentially normal for all $p > \dim_{\mathbb{C}} M$.

We remark that although we have stated Conjecture 1 on the Drury-Arveson module and Conjectures 2 and 3 on the Bergman module, one can, of course, consider their analogues on all other modules in Definition 1.2. However, it makes no difference as long as we consider graded submodules and $p > n$.

Lemma 1.3 *Suppose $I \subset \mathbb{C}[z_1, \cdots, z_n]$ is a homogeneous ideal and let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_{3s}$ denote their closures in the Drury-Arveson module H_n^2, the Hardy module $H^2(\mathbb{B}_n)$ and the weighted Bergman module $L_{a,s}^2(\mathbb{B}_n)$ for a non-negative integer s. Then*

(1) $\mathcal{P}_1$ is essentially normal if and only if $\mathcal{P}_2$ is essentially normal, if and only if $\mathcal{P}_{3s}$ is essentially normal.
(2) For $p > n$, $\mathcal{P}_1$ is p-essentially normal if and only if $\mathcal{P}_2$ is p-essentially normal, if and only if $\mathcal{P}_{3s}$ is p-essentially normal.

This equivalence is well-known and the proof is simple. We will give the proof here for future reference.

Proof Take the pair of modules H_n^2 and $L_a^2(\mathbb{B}_n)$ for example. Let I be a homogeneous ideal. It is well-known that

$$\|z^\alpha\|_{H_n^2}^2 = \frac{\alpha!}{|\alpha|!}$$

and

$$\|z^\alpha\|_{L_a^2(\mathbb{B}_n)}^2 = \frac{n!\alpha!}{(n + |\alpha|)!}.$$

Define the unitary operator

$$A : H_n^2 \to L_a^2(\mathbb{B}_n), z^\alpha \mapsto \left(\frac{(n + |\alpha|)!}{n!|\alpha|!}\right)^{1/2} z^\alpha.$$

It is easy to check that $A(I) = I$. Let $M_{z_i}^{H_n^2}$ be the coordinate multiplication operators on H_n^2 and $M_{z_i}^{L_a^2}$ be the coordinate multiplication operators on $L_a^2(\mathbb{B}_n)$.

Let $P_{[I]}^{H_n^2}$ and $P_{[I]}^{L_a^2}$ denote the projection operators to the closure of I in each space. By Proposition 4.1 in [8], for $p > n$, the p-essential normality of the closure of I in H_n^2 is equivalent to the membership $[M_{z_i}^{H_n^2}, P_{[I]}^{H_n^2}] \in \mathcal{C}_{2p}$ for all i, which is, obviously, equivalent to $[AM_{z_i}^{H_n^2}A^{-1}, P_{[I]}^{L_a^2}] \in \mathcal{C}_{2p}$ for all i. One can verify that $AM_{z_i}^{H_n^2}A^{-1} - M_{z_i}^{L_a^2} \in \mathcal{C}_{2p}$ for all $p > n$. Thus the above membership is equivalent to $[M_{z_i}^{L_a^2}, P_{[I]}^{L_a^2}] \in \mathcal{C}_{2p}$ for all i. Applying Proposition 4.1 in [8] again, we get the desired result. □

Non-graded submodules are also studied substantively [23, 28, 29, 31, 35, 36, 46, 62–65]. In general, it is not true that any submodule is essentially normal (cf. [37]). However, various conditions have been shown to imply essential normality. The study of essential normality of general submodules and quotient modules are sometimes also referred to as the Arveson-Douglas Conjecture.

Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly pseudo-convex domain with smooth boundary. One can also define the Bergman space $L_a^2(\Omega)$, the weighted Bergman spaces $L_{a,s}^2(\Omega)$ and the Hardy space $H^2(\Omega)$ on Ω (cf. [59]). The pointwise multiplication operators also define $\mathbb{C}[z_1, \cdots, z_n]$-Hilbert module structures on the spaces. The following conjecture is a natural generalization of Conjecture 2.

Conjecture 4 Suppose I is an ideal of $\mathbb{C}[z_1, \cdots, z_n]$. Let $\mathcal{P}_{\Omega,I}$ be the closure of I in $L_a^2(\Omega)$. Let $\mathcal{Q}_{\Omega,I}$ be its quotient module. Then $\mathcal{Q}_{\Omega,I}$ is p-essentially normal for all $p > \dim_{\mathbb{C}} \mathcal{Z}_{\Omega,I}$. Here

$$\mathcal{Z}_{\Omega,I} = Z(I) \cap \Omega.$$

The conjectures above concern submodules generated by polynomials. One can also consider submodules with other types of generators.

Conjecture 5 Suppose I is an ideal of $\mathrm{Hol}(\overline{\Omega})$, the ring of holomorphic functions in neighborhoods of $\overline{\Omega}$. Let $\mathcal{P}_{\Omega,I}$ be the closure of I in $L_a^2(\Omega)$. Let $\mathcal{Q}_{\Omega,I}$ be its quotient module. Then $\mathcal{Q}_{\Omega,I}$ is p-essentially normal for all $p > \dim_{\mathbb{C}} \mathcal{Z}_{\Omega,I}$. Here

$$\mathcal{Z}_{\Omega,I} = Z(I) \cap \Omega.$$

A geometric version of Conjecture 5 is the following.

Conjecture 6 Suppose $\tilde{M}$ is a complex analytic subset of an open neighborhood of $\overline{\Omega}$ and $M = \tilde{M} \cap \Omega$. Define

$$\mathcal{P}_{\Omega,M} = \{f \in L_a^2(\Omega) : f|_M = 0\}$$

and

$$\mathcal{Q}_{\Omega,M} = \mathcal{P}_{\Omega,M}^{\perp}.$$

Then $\mathcal{Q}_{\Omega,M}$ is p-essentially normal for all $p > \dim_{\mathbb{C}} M$.

The study of essentially normal Hilbert modules can be traced back to Douglas and Paulsen's book "Hilbert modules over function algebras" [26]. In Chapter 6 of [26], there is a detailed explanation of the K-homology group of module spectrum. In the 1980s, Douglas paid more attention on essential normality of quotient modules of Hardy module over the polydisc. Compared to the existing results on Arveson's conjecture over the unit ball, the situations over the polydisc are totally different. Roughly speaking, over the unit ball, most submodules generated by polynomials and their associated quotient modules are believed to be essentially normal. However, over the polydisc, no non-trivial submodule and few quotient modules are essentially normal. In fact, when $n \geq 2$, any nonzero submodule of Hardy modules $H^2(\mathbb{D}^n)$ is not essentially normal. This is essentially because all M_{z_i} are isometries of infinite multiplicity. Therefore one can only consider the essential normality of quotient modules. By direct calculations, Douglas and Misra found that $[(z-w)^2]^{\perp}$ and $[z^k - w^l]^{\perp}$ in $H^2(\mathbb{D}^2)$ are essentially normal, while $[z^2]^{\perp}$ is not [25]. By techniques of restricting $H^2(\mathbb{D}^n)$ functions on the diagonal, Clark [19] proved that the quotient module $[B_1(z_1) - B_2(z_2), \cdots, B_{n-1}(z_{n-1}) - B_n(z_n)]^{\perp}$ is essentially normal for all B_i being finite Blaschke products. Furthermore, P. Wang [61] showed that if η_i are nonconstant inner functions, then the quotient module $[\eta_1(z_1) - \eta_2(z_2), \cdots, \eta_{n-1}(z_{n-1}) - \eta_n(z_n)]^{\perp}$ is essentially normal only if each η_i is a finite Blaschke product. Let I be an ideal in $\mathbb{C}[z_1, \cdots, z_n]$. Intuitively, in the case of polydisc, essential normality of a quotient module $[I]^{\perp}$ strongly relies on the distribution of $Z(I)$ on the distinguished boundary $\mathbb{T}^n$ of $\mathbb{D}^n$. This is closely related to the theory of distinguished varieties introduced by Agler and McCarthy [1]. Therefore, it is a challenging problem to completely characterize essential normality of $[I]^{\perp}$ over the polydisc.

2 Backgrounds and Applications

2.1 Geometric Invariants for Row Contractions

Conjecture 1 was raised by Arveson in his course of studies of row contractions. Let us give the definition.

Definition 2.1 A *n-contraction* (row contraction) is a n-tuple $T = (T_1, \cdots, T_n)$ of mutually commuting operators acting on a common Hilbert space $\mathcal{H}$ satisfying the inequality

$$\|T_1\xi_1 + \cdots + T_n\xi_n\|^2 \leq \|\xi_1\|^2 + \cdots + \|\xi_n\|^2, \quad \xi_1, \cdots, \xi_n \in \mathcal{H}.$$

In other words, the "row operator" $(T_1, \cdots, T_n)$, viewed as an operator from $\oplus_{i=1}^n \mathcal{H}$ to $\mathcal{H}$, is a contraction.

The n-contractions are one of the natural generalizations of contractions in multivariable operator theory and have been widely studied. A n-contraction T defines a complete positive map

$$\Theta_T : B(H) \to B(H), \quad \Theta_T(A) = \sum_{i=1}^{n} T_i A T_i^*.$$

The n-contraction T is said to be *pure* if $\Theta_T^k(I) \to 0$ in the strong operator topology, as k tends to infinity.

Given a n-tuple $(T_1, \cdots, T_n)$ of mutually commuting operators acting on a common Hilbert space $\mathcal{H}$, one can naturally define a $\mathbb{C}[z_1, \cdots, z_n]$-Hilbert module structure on $\mathcal{H}$: a polynomial q acts on $\mathcal{H}$ by the operator $q(T_1, \cdots, T_n)$.

For example, let $\mathcal{H}$ be one of the function spaces in Definition 1.2. Then the n-tuple $M := (M_{z_1}, \cdots, M_{z_n})$ is a row contraction. The Hilbert module structure constructed as above by M coincides with the one in Definition 1.2.

In the landmarking paper [4], Arveson showed that the "n-shift", that is, the n-contraction $(M_{z_1}, \cdots, M_{z_n})$ acting on H_n^2, has the following "universal property" among pure row contractions.

Theorem 2.2 ([58]) *Let $T = (T_1, \cdots, T_n)$ be a pure row contraction on a Hilbert space $\mathcal{H}$. Then there exists a subspace $\mathcal{K} \subset H_n^2 \otimes \mathcal{D}_T$ that is invariant for $M^* = (M_{z_1}^*, \cdots, M_{z_n}^*)$, such that T is unitarily equivalent to the compression of M to K. In other words, there is an isometry $W : \mathcal{H} \to H_n^2 \otimes \mathcal{D}_T$ such that $W(\mathcal{H}) = \mathcal{K}$ and*

$$T_i^* = W^* M_i^* W, \quad i = 1, \cdots, n.$$

Here $\mathcal{D}_T$ is the defect space of T (cf. [58]).

For a commuting n-tuple T, Arveson [5, 6] introduced a curvature invariant $K(T)$ developed through integrating the trace of certain operators associated with T, and an Euler characteristic $\chi(T)$, defined as the alternating sum of ranks of free modules determined by T. Under the condition that T has finite rank, that is, the positive operator $\Delta = I - \sum_{i=1}^{n} T_i T_i^*$ has finite rank, and is pure and graded, Arveson proved that the two invariants coincide. This can be viewed as an operator-theoretic version of the Gauss-Bonnet-Chern formula from Riemannian geometry. Along his track of investigation, Arveson [7] defined a Dirac operator associated with a n-tuple T and showed that, under some additional assumptions, the curvature invariant $K(T)$ and the Fredholm index of the Dirac operator are equal. The essential normality of the n-tuple T (that is, the essential normality of the Hilbert module determined by T) naturally implies the Fredholmness of the Dirac operator T. Thus an affirmative answer to Conjecture 1 will lead to significant progress on generalizing the Gauss-Bonnet-Chern formula in operator theory [7, 8].

2.2 A New Kind of Index Theorem

The two invariants above defined by Arveson introduced a geometric point of view into the study of multi-variable operator theory. Douglas observed that one can also think in the opposite direction. In [22], he pointed out that essential normal quotient modules define a new kind of index theory for complex varieties. Moreover, such applications work not only for analytic function spaces on $\mathbb{B}_n$, but also for analytic function spaces on a more general type of domains—the bounded strongly pseudo-convex domains with smooth boundary. Since the Drury-Arveson space is not naturally defined on such domains, one usually formulates the conjecture on the Bergman space, as stated in Conjectures 4–6.

The index theory given in [22] is a consequence of the BDF theory introduced by Brown, Douglas and Fillmore (cf. [17]). In general, let $\mathcal{H}$ be an essentially normal Hilbert module and $\mathcal{T}(\mathcal{H})$ be the C^*-algebra generated by $\{\sigma(q) : q \in \mathbb{C}[z_1, \cdots, z_n]\}$, where σ is the associated module action. Let $\mathcal{K}(\mathcal{H})$ be the ideal of compact operators on $\mathcal{H}$. Since $\mathcal{H}$ is essentially normal, we have the short exact sequence

$$0 \to \mathcal{K}(\mathcal{H}) \to \mathcal{T}(\mathcal{H}) + \mathcal{K}(\mathcal{H}) \to C(X_{\mathcal{H}}) \to 0,$$

where $X_{\mathcal{H}}$ is a compact Hausdorff space. By the BDF theory, the exact sequence defines an odd K-homology element, denoted $[\mathcal{H}]$, in $K_1(X_{\mathcal{H}})$. Under the assumption of Conjecture 4–6, if $\mathcal{Q}_M$ (or $\mathcal{Q}_I$) is essentially normal, then we have

$$\overline{M} \cap \partial\Omega \subset X_{\mathcal{Q}_M} \subset \tilde{M} \cap \partial\Omega \text{ (or } \overline{Z(I) \cap \Omega} \cap \partial\Omega \subset X_{\mathcal{Q}_M} \subset Z(I) \cap \partial\Omega).$$

See [29] for a proof on $\mathbb{B}_n$. The following question will be of interest.

Question 7 Under the assumptions of Conjectures 4–6, if $\mathcal{Q}_M$ ($\mathcal{Q}_I$) is essentially normal, then what is $X_{\mathcal{Q}_M}$ ($X_{\mathcal{Q}_I}$)?

In the case when $\tilde{M}$ (or $Z(I)$) intersects $\partial\Omega$ transversely, the two sides of the inclusion above coincide. A good characterization of the element $[\mathcal{Q}_M]$ (or $[\mathcal{Q}_I]$) will be certainly of interest. It is known that $[L_a^2(\mathbb{B}_n)]$ is the fundamental class. Evidences show that the element $[\mathcal{Q}_M]$ ($[\mathcal{Q}_I]$) is non-trivial [21, 24, 28, 40, 41, 46]. An exception may be that it induces the trivial index element [32] for Beurling-type quotient modules of the Hardy module over the n-dimensional unit ball. The best result concerning the index element, is perhaps [28], in which they gave an analytic Grothendieck-Riemann-Roch Theorem.

2.3 Connection with Holomorphic Extension Theorems

Recent developments have shown that Conjecture 6 has strong connections with a classic problem in several complex variables—the L^2-extension problem [11, 20, 55, 71].

The L^2-extension problem asks whether there exist bounded extensions from a space of holomorphic functions on a subvariety to a space of holomorphic functions on a domain of higher dimension. We mention the paper [55] by Oshawa and Takegoshi entitled "On the extension of L^2 holomorphic functions" in 1987. In this paper, they proved the following result.

Theorem 2.3 *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $\psi : \Omega \to \mathbb{R} \cup \{-\infty\}$ a plurisubharmonic function and $H \subset \mathbb{C}^n$ a complex hyperplane. Then there exist a constant C depending only on the diameter of Ω such that: for any holomorphic function f on $\Omega \cap H$ satisfying*

$$\int_{\Omega\cap H} e^{-\psi}|f|^2 dV_{n-1} < \infty,$$

there exists a holomorphic function F on Ω satisfying $F|_{\Omega\cap H} = f$ and

$$\int_{\Omega} e^{-\psi}|F|^2 dV_n \leq C \int_{\Omega\cap H} e^{-\psi}|f|^2 dV_{n-1}.$$

The theorem was then generalized in various directions, known as theorems of Ohsawa-Takegoshi type. In particular, general subvarieties were considered and the requirements on the weight functions were improved. Perhaps the result that is closest to our point of view is in a paper by Beatrous [11]. This result was extended and used in [28] to obtain results on the Geometric Arveson-Douglas Conjecture. The following theorem is a version of strongly pseudoconvex domain associated with L^2-extensions (cf. [11, 13]).

Theorem 2.4 *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with smooth boundary and let $\tilde{M}$ be a d-dimensional complex submanifold of a neighborhood of $\overline{\Omega}$ which intersects $\partial\Omega$ transversely. Let $M = \tilde{M} \cap \Omega$ and let R be the restriction mapping from holomorphic functions on Ω to holomorphic functions on M. Let $s = n - d$ and $L^2_{a,s}(M)$ be the weighted Bergman space on M defined by the weight function $|r(z)|^s$, here $r(z)$ is a defining function for Ω. Then*

(1) The restriction map $R : L^2_a(\Omega) \to L^2_{a,s}(M)$ is bounded.
(2) There is a linear operator $E : L^2_{a,s}(M) \to L^2_a(\Omega)$ such that RE is the identity operator on $L^2_{a,s}(M)$.

The L^2-extension problem is of fundamental importance in several complex variables. Applications of the problem include, but are not restricted to, Suita's Conjecture, properties of Bergman kernels on weakly pseudoconvex domains, the strong openness conjecture, etc. (cf. [71]).

In [31], Douglas and the second author proved the following theorem, which makes connections between the Geometric Arveson-Douglas Conjecture and the L^2-extension problem.

Theorem 2.5 (Douglas, Y. Wang) *Let M be a complex analytic subset of $\mathbb{B}_n$ and let $\mathcal{Q}_M$ be the quotient module defined as in Conjecture 3. Suppose there exists a positive Borel measure μ on M such that the $L^2(\mu)$ norm and the quotient norm on $\mathcal{Q}_M$ are equivalent, i.e., there exists constants $C > c > 0$ such that for any $f \in \mathcal{Q}_M$,*

$$c\|f\|^2 \leq \int_M |f(z)|^2 d\mu(z) \leq C\|f\|^2,$$

then the quotient module $\mathcal{Q}_M$ is essentially normal. In this case we call μ an "equivalent measure" for M.

Remark 2.6 Standard argument shows that, under the assumption that the restriction map

$$R : L_a^2(\mathbb{B}_n) \to L^2(\mu),\ f \mapsto f|_M$$

is bounded, the condition above is equivalent to the existence of a bounded linear operator E : Range$(R) \to L_a^2(\mathbb{B}_n)$ such that $RE = Id$. Under the additional assumption that $\ker(R) = \mathcal{P}_M$, the condition is equivalent to that Range(R) is closed. Given that in the case M intersects $\partial\mathbb{B}_n$ transversely, the weighted measure $\mu_M = |r(z)|^s v_M$ is almost the only reasonable candidate of an equivalent measure, one is essentially asking:

Question 8 Is Range(R) closed in $L_{a,s}^2(M)$?

From this point of view, the Geometric Arveson-Douglas Conjecture is connected to Question 8, which can be considered as a weak version of the L^2-extension problem. On the other hand, it is already known that there exist complex analytic varieties such that equivalent measures do not exist. For example, the union of two homogeneous complex varieties that are tangent at a boundary point. Under these circumstances, the next reasonable thing to ask is

Question 9 Find an equivalent norm of the quotient norm on $\mathcal{Q}_M$ that is intrinsic. That is, find a norm that only uses data on M, for example, values on M of the function itself or its partial derivatives.

Question 9 can be viewed as a generalization of the L^2-Extension Problem. Essentially, a positive answer to Question 9 will also lead to a positive answer to the Geometric Arveson-Douglas Conjecture. Therefore the two problems are combined in this approach.

In [8], Arveson supported his conjecture by showing that a submodule of H_n^2 generated by a set of monomials is p-essentially normal for all $p > n$. Over the years, many exciting results have been obtained. In the following sections, we introduce the known results on the Arveson-Douglas Conjecture. We divide them into three categories, contained in Sects. 3–5.

3 The Case of Low Dimensions or Co-dimensions

Early results on the Arveson-Douglas Conjecture involve submodules with zero varieties of low dimensions or co-dimensions.

3.1 Principal Submodules

In the case when the submodule $\mathcal{P}$ is generated by a single function, the zero locus of $\mathcal{P}$ is an analytic subset of co-dimension 1. Various results have been obtained about essential normality of principal submodules. The general setting is the following. Let Ω be a bounded strongly pseudo-convex domain in $\mathbb{C}^n$ with smooth boundary. Let $\mathcal{H}$ be one of the Hardy space $H^2(\Omega)$, the Bergman space $L_a^2(\Omega)$ or the weighted Bergman spaces $L_{a,s}^2(\Omega)$. For $h \in \mathcal{H}$, denote $\mathcal{P}_{h,\Omega}$ the principal submodule generated by h, that is, the smallest submodule of $\mathcal{H}$ that contains h.

Question 10 When is $\mathcal{P}_{h,\Omega}$ essentially normal, or p-essentially normal?

Early results on the Arveson-Douglas Conjecture were mainly about submodules of the Drury-Arveson module H_n^2. In the case when h is a monomial, Arveson [8] showed that the principal submodule generated by h in H_n^2 is p-essentially normal for all $p > n$. The second result in this category is by the first author. In [39], he showed that any graded submodule of H_2^2 is p-essentially normal for all $p > 2$. Note that almost all graded submodules in H_2^2 are principal by Beurling's representation of ideals in two variables [18]. Then in [40], the first author and K. Wang completely solved the case of principal homogeneous submodules. They showed that if h is a homogeneous polynomial, then the principal submodule of H_n^2 generated by h is p-essentially normal for all $p > n$. The proof in [40] involves detailed trace estimates and some operator inequalities. The results in [40] were applied to the principal submodule $[z_1^n+z_2^n-z_3^n]$ of the 3-shift Hilbert module to obtain a geometric invariant for noncommutative Fermat curve $X^n + Y^n = Z^n$ considered by Arveson [7, 9].

The following surprising result was proved by Douglas and K. Wang in [29]. Their proof used ideas from harmonic analysis and $\bar{\partial}$-estimates.

Theorem 3.1 (Douglas, K. Wang) *Suppose h is a polynomial, not necessarily homogeneous. Then the principal submodule $\mathcal{P}_h$ of $L_a^2(\mathbb{B}_n)$ generated by h is p-essentially normal for all $p > n$.*

Douglas and K. Wang's result has motivated some further examinations on principal submodules. In [35] and [36], Fang and Xia extended Douglas and K.

Wang's result to polynomial-generated principal submodules of a large class of weighted analytic function spaces, including the Hardy space $H^2(\mathbb{B}_n)$, and, under additional assumptions, the Drury-Arveson space H_n^2. Then Douglas and the authors [23] showed the following.

Theorem 3.2 (Douglas, Guo, Y. Wang) *Suppose $\Omega \subseteq \mathbb{C}^n$ is a bounded strongly pseudoconvex domain with smooth boundary, h is a holomorphic function defined in an open neighborhood of $\overline{\Omega}$. Then the principal submodule $\mathcal{P}_{h,\Omega}$ of the Bergman module $L_a^2(\Omega)$, generated by h, is p-essentially normal for all $p > n$.*

The proof of Theorem 3.2 is based on an inequality of a new type, proved in [23]. In the case when $\Omega = \mathbb{B}_n$, the inequality is the following.

Theorem 3.3 *Suppose h is a holomorphic function defined in a neighborhood of $\overline{\mathbb{B}_n}$. Then there exist a constant $C > 0$ and a positive integer N, such that for any $z, w \in \mathbb{B}_n$ and any $f \in \mathrm{Hol}(\mathbb{B}_n)$, we have*

$$|h(z)f(w)| \leq C \frac{|1-\langle z, w\rangle|^N}{(1-|w|^2)^{n+1+N}} \int_{D(w,1)} |h(\lambda)f(\lambda)|dv(\lambda). \tag{3.1}$$

where the set $D(w, 1)$ is the ball centered at w, under the Bergman metric, of radius 1.

In [65], Xia and the second author extended this result to the Hardy module. The following question will be of interest.

Question 11 Under the assumption of Theorem 3.2, characterize the element $[\mathcal{P}_{h,\Omega}^{\perp}]$ in the odd K-homology group.

3.2 Case of Low Dimension

In addition to the results on principal submodules, the first author and K. Wang also showed in [40] that Conjecture 2 holds if $n \leq 3$ or $\dim_{\mathbb{C}} Z(I) \leq 1$. Before continuing, we introduce the notion of quasi-homogeneous polynomials. For a given integer $n \geq 2$, let

$$\mathbf{K} = (K_1, K_2, \cdots, K_n) \in \mathbb{N}^n,$$

be an n-tuple, which will be called the weight. For this given weight, we assign the $\mathbf{K}$-degree $\langle \alpha, \mathbf{K}\rangle = \sum_{j=1}^n \alpha_j K_j$ to each monomial $z^\alpha \in \mathbb{C}[z_1, \cdots, z_n]$, where $\alpha \in \mathbb{Z}_+^n \setminus \{0\}$. We say that a polynomial p is $\mathbf{K}$-quasi-homogeneous of $\mathbf{K}$-degree $m > 0$ if it is a linear combination of monomials of $\mathbf{K}$-degree m. Zero varieties of quasi-homogeneous polynomials are connected to the study of topological spheres. Milnor [54] and Brieskorn [16] showed that, when the dimension $n \neq 3$, a zero variety of the form

$$\Sigma_\alpha = \{z \in \mathbb{C}^n : z_1^{\alpha_1} + \cdots + z_n^{\alpha_n} = 0\} \cap \partial\mathbb{B}_n,$$

is homeomorphic to the sphere S^{2n-3} where all integers $\alpha_i \geq 2$ if and only if it has the same homology as a sphere. In case $n = 3$, it remains to be a problem of interest in low dimensional topology. There exists a natural link between the theory of topological spheres and geometric analysis of Hilbert modules [41].

In the quasi-homogeneous case, the first author and Zhao showed in [46] that for a broad class of analytic Hilbert modules on the unit ball $\mathbb{B}^n$, including the Hardy module, the Bergman module and the Drury-Arveson module, quasi-homogeneous principal submodules are p-essentially normal for all n, $(p > n)$. They also showed that a quasi-homogeneous submodule is p-essentially normal if either $n \leq 3$ or the complex dimension of zero variety ≤ 1.

4 The Geometric Arveson-Douglas Conjecture

Another direction of research for the Arveson-Douglas Conjecture that has been fruitful is the study of submodules with smooth varieties, or varieties with nice singular points.

In [33], Engliš and Eschmeier solved Conjecture 3 under the additional assumption that M is smooth away from the origin. In fact, they proved the result for submodules of a large class of weighted modules. In the case of the Drury-Arveson module, their result is the following.

Theorem 4.1 (Engliš, Eschmeier) *Let M be one of the following.*

(1) a homogeneous variety in $\mathbb{C}^n$ such that $M\backslash\{0\}$ is a complex submanifold of $\mathbb{C}^n$ of dimension d;
(2) a smooth variety in $\mathbb{C}^n$ of dimension d that intersects $\partial\mathbb{B}_n$ transversely.

Let $\mathcal{P}_M$ be the submodule of H_n^2 consisting of all the functions that vanish on $M \cap \mathbb{B}_n$. Let $\mathcal{Q}_M$ be the corresponding quotient module. Then $\mathcal{Q}_M$ is p-essentially normal for all $p > d$.

The proof of Engliš and Eschmeier mainly relies on two ingredients: the theory of generalized Toeplitz operators of Boutet de Monvel and Guillemin [13, 14], and the results of Beatrous about restrictions of Bergman-type functions to subvarieties [12].

Almost the same time, Douglas, Tang and Yu proved the following theorem, by extending Beatrous's extension theorem (Theorem 2.4) and Baum, Douglas and Taylor's results in [10].

Theorem 4.2 (Douglas, Tang, Yu) *Suppose I is an ideal of $\mathbb{C}[z_1, \cdots, z_n]$ generated by k polynomials: $p_1, \cdots, p_k$, $k \leq n-2$. Assume the following.*

(1) The Jacobian matrix $(\partial p_i/\partial z_j)_{i,j}$ is of maximal rank on $Z(I) \cap \partial\mathbb{B}_n$.
(2) $Z(I)$ intersects $\partial\mathbb{B}_n$ transversely.

Then the submodule $\mathcal{P}_I$ of the Bergman module $L^2_a(\mathbb{B}_n)$ is essentially normal. Moreover, the quotient module $\mathcal{Q}_I$ and the weighted Bergman module $L^2_{a,k}(\Omega_I)$ correspond to the same class in $K_1(\partial\Omega_I)$, where $\Omega_I = Z(I) \cap \mathbb{B}_n$.

In fact, more is proved in [28]. They showed that the element $[L^2_{a,k}(\Omega_I)]$ in $K_1(\partial\Omega_I)$ is the fundamental class of $\partial\Omega_I$ defined by the CR-structure on $\partial\Omega_I$. They also showed that $[L^2_{a,k}(\Omega_I)]$ is the image under the boundary map ∂ : $K_0(\overline{\Omega}_I, \partial\Omega_I) \to K_1(\partial\Omega_I)$, of the element $[D_N]$, where $[D_N]$ was introduced by Baum, Douglas and Taylor in [10] from the $\bar{\partial}$-operator on the Dolbeault complex of Ω_I with the Neumann boundary condition. This can be viewed as an analytic version of the Grothendieck-Riemann-Roch theorem. In a recent paper [24], Douglas, Jabbari, Tang and Yu proved an index theorem for the quotient module of a monomial ideal by resolving the monomial ideal by means of a sequence of essentially normal Hilbert modules, each of which is a direct sum of (weighted) Bergman spaces on balls.

Then in [31], Douglas and the second author proved the following theorem.

Theorem 4.3 (Douglas, Y. Wang) *Suppose $\tilde{M}$ is a complex analytic subset of an open neighborhood of $\overline{\mathbb{B}_n}$ satisfying the following conditions:*

(1) $\tilde{M}$ intersects $\partial\mathbb{B}_n$ transversely.
(2) $\tilde{M}$ has no singular points on $\partial\mathbb{B}_n$.

Let $M = \tilde{M} \cap \mathbb{B}_n$. Then the quotient module $\mathcal{Q}_M$ of $L^2_a(\mathbb{B}_n)$ associated with M, is p-essentially normal for all $p > 2\dim_{\mathbb{C}} M$.

The proof is based on a combination of harmonic analysis and operator theory.

5 Decomposition of Modules and Varieties

Another approach to the Arveson-Douglas Conjecture is by decomposing submodules or quotient modules, into sums of nice parts. By nice parts we mean submodules or quotient modules discussed about in the previous two sections. The paper [57] by Shalit was perhaps the first paper in this direction. One of the advantages of considering decomposition is that one can incorporate techniques from the previous two types of approaches to obtain nontrivial examples. Another advantage is that, in many cases, a good decomposition leads to a decomposition of the corresponding index elements. See, for example, [24, 30], etc.

5.1 Decomposition of Submodules

To decompose a submodule $\mathcal{P}$ into nice parts, one essentially needs to write $\mathcal{P}$ as a sum of nice submodules (cf. [49]). In [57], Shalit considered the following stable division property for submodules of the Drury-Arveson module H_n^2.

Definition 5.1 Let $\mathcal{P}$ be a submodule of H_n^2. We say that $\mathcal{P}$ has the *stable division property* if there are a set $\{f_1, \cdots, f_k\} \subset \mathcal{P}$ that generates $\mathcal{P}$ as a module, and a constant C such that, for every $h \in \mathcal{P}$, there are functions $g_1, \cdots, g_k \in H_n^2$, such that

(1) $h = \sum_{i=1}^k f_i g_i$, and
(2) $\sum_{i=1}^k \|f_i g_i\|^2 \leq C\|h\|^2$.

Shalit showed that a graded submodule $\mathcal{P}_I$ in the Drury-Arveson module with the stable division property is p-essentially normal for all $p > \dim_{\mathbb{C}} Z(I)$. The result was then extended by Biswas and Shalit to quasi-homogeneous cases in [15]. In the same paper, they also defined a approximate stable division property and showed that it implies essential normality. In a recent preprint [62] by the second author, the following asymptotic stable division property was defined and shown to imply essential normality, under an additional assumption.

Definition 5.2 Suppose $\mathcal{P}$ is a submodule of the Bergman module $L_a^2(\mathbb{B}_n)$. $\mathcal{P}$ is said to have the *asymptotic stable division property* if there exist an invertible operator T on $\mathcal{P}$, a subset $\{h_i\}_{i\in\Lambda} \subset \mathcal{P}$, finite or countably infinite, and constants C_1, C_2, such that for any $f \in \mathcal{P}$, there exists $\{g_i\}_{i\in\Lambda} \subset \mathrm{Hol}(\mathbb{B}_n)$ with the following properties.

(1) $Tf = \sum_{i\in\Lambda} h_i g_i$, where the convergence is pointwise if Λ is countably infinite.
(2)

$$\int_{\mathbb{B}_n} \left(\sum_{i\in\Lambda} |h_i(z)g_i(z)| \right)^2 dv(z) \leq C_1 \|f\|^2_{L_a^2(\mathbb{B}_n)}.$$

(3)

$$\int_{\mathbb{B}_n} \left(\sum_{i\in\Lambda} |h_i(z)g_i(z)| \right)^2 (1-|z|^2) dv(z) \leq C_2 \|f\|^2_{L_{a,1}^2(\mathbb{B}_n)}.$$

The following theorem was shown by Y. Wang in [62].

Theorem 5.3 *Suppose $\mathcal{P}$ is a submodule of $L_a^2(\mathbb{B}_n)$ with the asymptotic stable division property. If the generating functions h_i are all defined in a neighborhood of $\overline{\mathbb{B}_n}$ and the controlling constants C_i, N_i, determined by h_i (as in Theorem 3.3), are uniformly bounded for all $i \in \Lambda$, then the submodule $\mathcal{P}$ is p-essentially normal for all $p > n$. In particular, if the generating functions h_i are polynomials of uniformly bounded degrees, then $\mathcal{P}$ is p-essentially normal for all $p > n$.*

Theorem 5.3 provides a unified proof of most known results of the Arveson-Douglas Conjecture (the $p > n$ case), including those in [23, 28, 29, 31, 33, 57]. Moreover, the following new result was obtained in [62].

Theorem 5.4 *Suppose I is an ideal in $\mathbb{C}[z_1, \cdots, z_n]$ with primary decomposition $I = \cap_{j=1}^{k} I_j^{m_j}$, where I_j are prime ideals. Assume the following.*

(1) For each $j = 1, \cdots, k$, $Z(I_j)$ has no singular points on $\partial\mathbb{B}_n$ and intersects $\partial\mathbb{B}_n$ transversely.
(2) Any pair of the varieties $\{Z(I_j)\}$ does not intersect on $\partial\mathbb{B}_n$.

Then the submodule $\mathcal{P}_I$ has the asymptotic stable division property with generators of uniformly bounded degrees. As a consequence, $\mathcal{P}_I$ is p-essentially normal for all $p > n$.

Let M be a submodule of the Drury-Arveson module H_n^2. Arveson [6] showed that there exist a sequence of multipliers $\{\varphi_k : k = 1, 2, \cdots\} \subset M$ such that

$$P_M = (SOT) \sum_k M_{\varphi_k} M_{\varphi_k}^*. \tag{5.1}$$

Then the Arveson's conjecture is equivalent to

$$\left[(SOT) \sum_k M_{\varphi_k} M_{\varphi_k}^*, M_{z_i} \right] \in \mathcal{C}_{2p} \tag{5.2}$$

for $p > n$ and $i = 1, 2, \cdots, n$. Applying the construction in Section 4.1 of [37], there is a counterexample that, (5.2) fails to be true for some nonhomogeneous submodule. From the representation of the projection P_M, we see that the submodule M is the closed linear span of those principal submodules $[\phi_k]$ for $k = 1, \cdots$, that is $M = \bigvee_k [\phi_k]$. In particular, when M is homogeneous, each ϕ_k can be chosen as homogeneous. For a homogeneous submodule M of H_n^2, although we do not know whether or not (5.2) holds, but on the Hardy space and Bergman space on the unit ball, Zhao and Yu [70] proved that (5.2) holds for those operators of the form $A = (SOT) \sum_k M_{\varphi_k} M_{\varphi_k}^*$ by trace estimation.

Theorem 5.5 *Let $A = (SOT) \sum_k M_{\varphi_k} M_{\varphi_k}^*$ be a bounded linear operator on the Bergman space or Hardy space on the unit ball $\mathbb{B}_n$, where $\varphi_k \in H^\infty(\mathbb{B}_n)$ $k = 1, 2, \cdots$. Then the commutator $[A, M_{z_i}]$ belongs to Schatten class $\mathcal{C}_{2p}$ for $p > n$ and $i = 1, \cdots, n$, and there is a constant C depending only on p and n such that*

$$\|[A, T_{z_i}]\|_{2p} \le C\|A\|.$$

By applying this theorem, Zhao [69] proved that for approximately representable homogeneous submodules M of the Bergman module on the unit ball, M is p-essentially normal. By definition a submodule M of the Bergman module is called approximately representable if there exist $\varphi_k \in H^\infty(\mathbb{B}_n)$ $k = 1, 2, \cdots$, and positive constants C_1, C_2 such that

$$C_1 P_M \leq (SOT) \sum_k M_{\varphi_k} M_{\varphi_k}^* \leq C_2 P_M.$$

5.2 Decomposition of Quotient Modules

Besides submodules, there are also studies on decompositions of quotient modules. From the view of Conjecture 6, decomposition of quotient modules essentially corresponds to decomposition of varieties. In [51], Kennedy and Shalit proved the following result.

Theorem 5.6 (Kennedy, Shalit) *Suppose $V_1, \cdots, V_k$ are homogeneous varieties in $\mathbb{C}^n$ and $\mathcal{P}_{V_i}$, $\mathcal{Q}_{V_i}$ are the corresponding submodules and quotient modules in H_n^2. Suppose $\mathcal{Q}_{V_i}$ is p-essentially normal, $i = 1, \cdots, k$.*

(1) If $p > \max\{\dim_{\mathbb{C}} V_1, \cdots, \dim_{\mathbb{C}} V_k\}$ and the algebraic sum $\mathcal{P}_{V_1} + \cdots + \mathcal{P}_{V_k}$ is closed, then $\mathcal{Q}_{V_1 \cap \cdots \cap V_k}$ is also p-essentially normal.
(2) If $p > \dim V_1 \cup \cdots \cup V_k$ and the algebraic sum $\mathcal{Q}_{V_1} + \cdots + \mathcal{Q}_{V_k}$ is closed, then $\mathcal{Q}_{V_1 \cup \cdots \cup V_k}$ is also p-essentially normal.

The theorem was then applied to prove that the Arveson-Douglas Conjecture holds for submodules corresponding to varieties that decompose into linear subspaces, and varieties that decompose into components with mutually disjoint linear spans.

From Theorem 5.6 (2), one can see that in order to obtain a nice decomposition, one needs to ensure that the algebraic sum of the quotient spaces to be closed. In [63], Douglas and the second author proved the following theorem, using the techniques developed in [31].

Theorem 5.7 (Douglas, Y. Wang) *Suppose $\tilde{M}_1$ and $\tilde{M}_2$ are two analytic subsets of an open neighborhood of $\overline{\mathbb{B}_n}$. Let $\tilde{M}_3 = \tilde{M}_1 \cap \tilde{M}_2$. Assume that*

(i) $\tilde{M}_1$ and $\tilde{M}_2$ intersect transversely with $\partial\mathbb{B}_n$ and have no singular points on $\partial\mathbb{B}_n$.
(ii) $\tilde{M}_3$ also intersects transversely with $\partial\mathbb{B}_n$ and has no singular points on $\partial\mathbb{B}_n$.
(iii) $\tilde{M}_1$ and $\tilde{M}_2$ intersect cleanly on $\partial\mathbb{B}_n$.

Let $M_i = \tilde{M}_i \cap \mathbb{B}_n$ and $Q_i = \overline{\text{span}}\{K_\lambda : \lambda \in M_i\}$, $i = 1, 2, 3$. Then $Q_1 \cap Q_2 / Q_3$ is finite dimensional and $Q_1 + Q_2$ is closed. As a consequence, $Q_1 + Q_2$ is p-essentially normal for $p > 2d$, where $d = \max\{\dim M_1, \dim M_2\}$.

The lower bound $2d$ was then refined to d in [64].

6 Quotient Modules over the Polydisc, Distinguished Varieties and Boundary Representations

A non-empty set V in $\mathbb{C}^2$ is a distinguished variety if there is a polynomial p in $C[z, w]$ such that $V = Z(p)$ and $V \cap \partial\mathbb{D}^2 = V \cap \mathbb{T}^2$, that is, the variety exits the bidisc through the distinguished boundary $\mathbb{T}^2$. Such a polynomial p is said to be distinguished. This notion introduced by Agler and McCarthy is of fundamental importance in the study of Ando's inequality and extremal Pick problems on the bidisk [1]. In the two variable case, let I be a homogenous ideal of $\mathbb{C}[z, w]$, then there is a homogenous polynomial q such that $[q] \ominus [I]$ is of finite dimension. Therefore, the essential normality of $[I]^\perp$ is equivalent to that of $[q]^\perp$. Since q is homogenous, the distinguished part of q can be factored out, that is, q can be decomposed as

$$q = q_1 \cdot q_2,$$

where $Z(q_1) \cap \partial\mathbb{D}^2 \subset \mathbb{T}^2$ and $Z(q_2) \cap \mathbb{T}^2 = \emptyset$. The following theorem was given by the first author and P. Wang [43].

Theorem 6.1 *Let q be a two variable homogenous polynomial with the above mentioned decomposition. Then the quotient module $[q]^\perp$ is essentially normal if and only if q_2 has the one of the following forms.*

1. *q_2 is a nonzero constant;*
2. *$q_2 = \alpha z - \beta w$, with $|\alpha| \neq |\beta|$;*
3. *$q_2 = c(z - \alpha w)(w - \beta z)$ with $|\alpha| < 1$ and $|\beta| < 1$.*

When a homogenous ideal I of $\mathbb{C}[z_1, \cdots, z_n]$ satisfies that $Z(I) \cap \partial\mathbb{D}^n \subseteq \mathbb{T}^n$, P. Wang and Zhao proved that the quotient module $[I]^\perp$ in $H^2(\mathbb{D}^n)$ is essentially normal [66]. For a general homogenous ideal I, P. Wang and Zhao proved that if the quotient module $[I]^\perp$ in $H^2(\mathbb{D}^n)$ is essentially normal, then the dimension of the zero variety $\dim_{\mathbb{C}} Z(I) \leq 1$. Furthermore, by a careful geometric analysis of the distribution of the zero variety $Z(I)$ on the boundary $\partial\mathbb{D}^n$, they gave a complete characterization for essential normality of $[I]^\perp$ [67]. Along this direction, they also completely characterized essential normality of quasi-homogeneous quotient modules on the polydisc [68].

An interesting problem is to characterize essential normality for principal quotient modules in $H^2(\mathbb{D}^2)$. That is, for which $g \in H^2(\mathbb{D}^2)$, is the quotient module $[g]^\perp$ of the submodule $[g]$ generated by g essentially normal? Even in the case of distinguished polynomials, it remains unknown whether or not $[q]^\perp$ is essentially normal for a distinguished polynomial q.

In the case of 2 variables, the essential normality of Beurling-type quotient modules was studied by the first author and K. Wang [42]. They proved that $[\theta]^\perp$ is essentially normal if and only if the inner function θ is a rational inner function of degree at most $(1, 1)$. Precisely, θ has one of the following forms:

$$\theta(z, w) = \beta\phi_a(z);\ \ \beta\phi_a(w),\ \ or\ \ \beta\phi_a(z)\phi_b(w);$$

where $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$ and $|\beta| = 1$, $|a|, |b| < 1$ or

$$\theta(z, w) = \beta \frac{zw + az + bw + c}{1 + \bar{a}w + \bar{b}z + \bar{c}zw}$$

for $|\beta| = 1$ and $c \neq ab$. An interesting consequence of this result is able to give a complete characterization of the boundary representations for Toeplitz C^*-algebras on Beurling type quotient modules $[\theta]^\perp$. The notion of the boundary representations was introduced by Arveson [2, 3] to study the noncommutative Choquet boundary in the representation theory of C^*-algebras. The results on the boundary representation in [42] extended Arveson's results to Toeplitz algebras on Beurling type quotient modules over the bidisc (cf. [2, 3]). For boundary representations of Toeplitz algebras on Bergman quotient modules on the unit disc, W. He gave a complete characterization [47] by using Zhu's theorem. Zhu's theorem [72] says that a submodule M of $L^2_a(\mathbb{D})$ is essentially normal if and only if $\dim M \ominus zM < \infty$.

Also in [45], the first author, K. Wang and Zhang studied p-essentially normal properties of quotient modules on the bidisk. They established some trace formulas for self-commutators of Toeplitz-type operators on quotient modules. In [44, 68], some results on essential normality of quasi-homogeneous quotient modules over the polydisc are proved. It is worth pointing out that Izuchi and Yang [48] also obtained some results about essential normality of quotient modules over the bidisc. For the study of essential normality and Dixmier trace over bounded symmetric domains, see Upmeier and Wang's paper [60]. An earlier and selective survey on essential normality of Hilbert modules is presented in [38].

Acknowledgments We would like to thank Professor Xiang Tang for a beneficial discussion about this topic. The expert comments and suggestions by the referee of an earlier version of this survey are greatly acknowledged. The first author is partially supported by NSF in China, and the second author is partially supported by NSF.

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The Pieri Rule for GL_n Over Finite Fields

Shamgar Gurevich and Roger Howe

Dedicated to the memory of Ronald Douglas

Abstract The Pieri rule gives an explicit formula for the decomposition of the tensor product of irreducible representation of the complex general linear group $GL_n(\mathbb{C})$ with a symmetric power of the standard representation on $\mathbb{C}^n$. It is an important and long understood special case of the Littlewood-Richardson rule for decomposing general tensor products of representations of $GL_n(\mathbb{C})$.

In our recent work Gurevich and Howe (Rank and Duality in Representation Theory. *Takagi lectures Vol. 19, Japanese Journal of Mathematics (2017)*) and Gurevich and Howe (Harmonic Analysis on GL_n over Finite Fields. *Accepted (2019)*) on the organization of representations of the general linear group over a finite field $\mathbb{F}_q$ using small representations, we used a generalization of the Pieri rule to the context of this latter group.

In this note, we demonstrate how to derive the Pieri rule for $GL_n(\mathbb{F}_q)$. This is done in two steps; the first, reduces the task to the case of the symmetric group S_n, using the natural relation between the representations of S_n and the *spherical principal series* representations of $GL_n(\mathbb{F}_q)$; while in the second step, inspired by a remark of Nolan Wallach, the rule is obtained for S_n invoking the S_l-$GL_n(\mathbb{C})$ Schur duality.

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_14

Along the way, we advertise an approach to the representation theory of the symmetric group which emphasizes the central role played by the *dominance order* on Young diagrams. The ideas leading to this approach seem to appear first, without proofs, in Howe and Moy (Harish-Chandra homomorphisms for p-adic groups. *CBMS Regional Conference Series in Mathematics 59 (1986)*).

1 Introduction

Two basic tasks in the representation theory of a finite group G are: the parameterization of its set $\widehat{G}$ (of isomorphism classes) of irreducible representations (irreps); and the decomposition into direct sum of irreps of certain of its naturally arising representations.

The Pieri rule that we formulate and prove in this note addresses a particular instance of the second task mentioned above, for the case of the general linear group $GL_n = GL_n(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$. It can be used to give a recursive solution to the general problem of decomposing the permutation actions of GL_n on functions on flag manifolds.

The Pieri rule can be useful in other ways. Indeed, in [Gurevich-Howe17, Gurevich-Howe19] we developed a precise notion of "size" for irreps of GL_n, called "tensor rank". This is an integer $0 \leq k \leq n$ that is naturally attached to an irreducible representation (irrep) and helps to compute important analytic properties such as its dimension and character values on certain elements of interest. In particular, in *loc. cit.* the Pieri rule for GL_n enabled us to give an effective formula for the irreps of GL_n of a given tensor rank k.

We proceed to consider the subgroups involved in the construction of representations that appear in the formulation of the Pieri rule.

1.1 Young Diagrams and Parabolic Subgroups

The representations we are interested in are naturally realized on spaces constructed using standard parabolic subgroups [Borel69] of general linear groups, that we will now describe.

Fix an integer $0 \leq k \leq n$, and denote by $\mathcal{Y}_k$ the collection of Young diagrams of size k [Fulton97]. In more detail, by a *Young diagram* (or *partition*) $D \in \mathcal{Y}_k$, we mean an ordered list of non-negative integers

$$D = (d_1 \geq \ldots \geq d_r), \text{ with } d_1 + \ldots + d_r = k. \tag{1.1}$$

It is common to visualize—see Fig. 1 for illustration—the diagram D with the help of a drawing of r rows of square boxes, each row one on top of the other, starting at the left upper corner, in such a way that the i-th row contains d_i boxes.

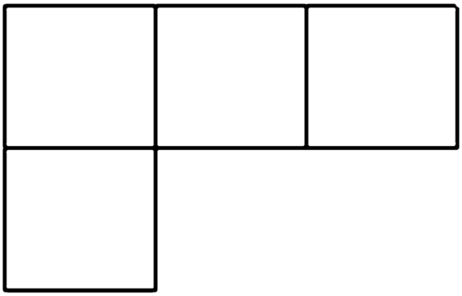


Fig. 1 The Young diagram $D = (3, 1) \in \mathcal{Y}_4$

Fig. 2 The parabolic $P_D \subset GL_4$, $D = (3, 1)$, has $A_3 \in GL_3$, $A_1 \in GL_1$, $U_{3,1} \in M_{3,1}$

To the diagram D (1.1) we can attach the following increasing sequence F_D of subspaces of the k-dimensional vector space $\mathbb{F}_q^k$:

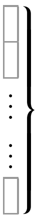
$$F_D \ : \ 0 \subset \mathbb{F}_q^{d_1} \subset \mathbb{F}_q^{d_1+d_2} \subset \ldots \subset \mathbb{F}_q^k, \tag{1.2}$$

and call it the *standard flag* attached to D. In particular, having D we can form—see Fig. 2 for illustration[1] —the stabilizer subgroup

$$P_D = Stab_{GL_k}(F_D) \subset GL_k, \tag{1.3}$$

that we will call the *standard parabolic subgroup attached to* D.

Probably the most important example from this class of parabolic subgroups is the Borel subgroup B of upper triangular matrices in GL_k, which is just P_D with

$D =$  k times.

Next, we describe the specific type of representations that the Pieri rule attempts to decompose.

[1]We denote $M_{k,n} = M_{k,n}(\mathbb{F})$ the space of $k \times n$ matrices over a field $\mathbb{F}$.

1.2 The Pieri Problem

Take $D \in \mathcal{Y}_k$, denote by **1** the trivial representation of P_D, and consider the induced representation

$$I_D = Ind_{P_D}^{GL_k}(\mathbf{1}),$$

which is given by the space of complex valued functions on GL_k/P_D, equipped with the standard left action of GL_k on it.

In the case when $P_D = B$ is the Borel subgroup, the set GL_k/B is the flag variety, and we will call the collection of irreps that appear inside $Ind_B^{GL_k}(\mathbf{1})$ the *spherical principal series (SPS).*

There is a natural recipe (that we will recall in detail below) to parametrize the *SPS* by Young diagrams. Note that for each Young diagram $D \in \mathcal{Y}_k$, we have $I_D < Ind_B^{GL_k}(\mathbf{1})$, where $<$ denotes subrepresentation. Interestingly, each I_D contains (with multiplicity one) a well defined “largest” irreducible subrepresentation ρ_D. We will leave the details of that story for the body of the note, but the collection $\{\rho_D;\ D \in \mathcal{Y}_k\}$ realizes the totality of SPS representations of GL_k.

We proceed to formulate the Pieri problem.

Fix $0 \leq k \leq n$, and denote by $P_{k,n-k} \subset GL_n$ the standard parabolic fixing the first k coordinate subspace of $\mathbb{F}_q^n$. There is a natural surjective map $P_{k,n-k} \twoheadrightarrow GL_k \times GL_{n-k}$. Take an SPS representation ρ_D of GL_k, and denote by $\mathbf{1}_{n-k}$ the trivial representation of GL_{n-k}. Pull back the representation $\rho_D \otimes \mathbf{1}_{n-k}$ from $GL_k \times GL_{n-k}$ to $P_{k,n-k}$ and form the induced representation

$$I_{\rho_D} = Ind_{P_{k,n-k}}^{GL_n}(\rho_D \otimes \mathbf{1}_{n-k}). \tag{1.4}$$

Now we can write down the natural,

Problem 1.2.1 (Pieri Problem) Decompose the representation I_{ρ_D} into irreducibles.

It is easy to see that the components of I_{ρ_D} (1.4) are SPS representations of GL_n, so we are looking for a solution to Problem 1.2.1 in terms of Young diagrams, i.e., members of $\mathcal{Y}_n$.

In this note we present a solution to the Pieri problem for GL_n in two steps. First we explain why it is enough to solve the analogous problem for the representations of the symmetric group S_n. Then, in the second step, we demonstrate that the Pieri rule holds for S_n, invoking the Schur (a.k.a. Schur-Weyl) duality for S_l-$GL_n(\mathbb{C})$, and a use of the classical Pieri rule for $GL_n(\mathbb{C})$ [Howe92, Pieri1893, Weyman89]. We note that in [Ceccherini-Silberstein-Scarabotti-Tolli10], Section 3.5, there is a proof of the Pieri rule for S_n based on a quite different approach.

2 Representations of S_n

The standard parametrization of the irreps of S_n is done using Young diagrams [Sagan91]. We will discuss various aspects of the construction leading to this parametrization, emphasizing the role played by the dominance relation on the set $\mathcal{Y}_n$ of Young diagrams. We will follow closely ideas formulated (without proofs) in Appendix 2 of [Howe-Moy86].

2.1 *The Young Modules*

Recall that partitioning the set $\{1, .., n\}$ into r disjoint subsets of size d_i each, and assigning these numbers, respectively, to the rows of the Young diagram $D = (d_1 \geq ... \geq d_r) \in \mathcal{Y}_n$, gives rise to a *Young tabloid* [Fulton97]. Let us denote by $\mathcal{T}_D$ the collection of all Young tabloids that one can make using D. The natural action of the group S_n on $\mathcal{T}_D$ is transitive. Moreover, we can identify

$$\mathcal{T}_D = S_n / S_D,$$

where $S_D \subset S_n$ is the stabilizer subgroup

$$S_D = Stab_{S_n}(T_D), \tag{2.1}$$

of the tabloid T_D that obtained by assigning to the first row of D the numbers $1, .., d_1$, to the second $d_1 + 1, ..., d_1 + d_2$, etc. The group S_D is naturally isomorphic to the product $S_{d_1} \times ... \times S_{d_r}$ embedded in S_n in the usual way.

Now, we consider the induced representation, called the *Young module associated to D*,

$$Y_D = Ind_{S_D}^{S_n}(\mathbf{1}), \tag{2.2}$$

where $\mathbf{1}$ stands for the trivial representation of S_D. It is naturally realized as the permutation representation of S_n on the space of functions on $\mathcal{T}_D$.

2.2 *Properties of the Young Modules*

We derive basic properties of the family of Young modules (2.2). They give, in particular, as a corollary the standard classification of the irreps of S_n, and, as we mentioned earlier, they can be effectively understood using the important *dominance* relation $\preceq$ on the set $\mathcal{Y}_n$ of Young diagrams, which we recall now.

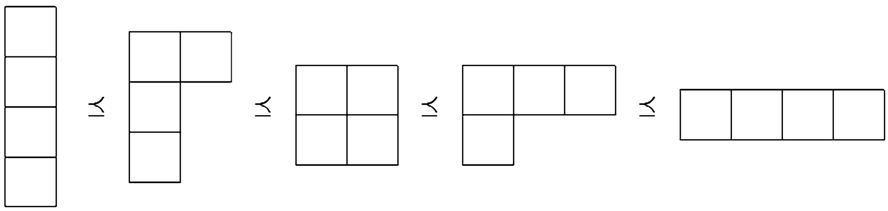


Fig. 3 The set $\mathcal{Y}_4$ is totally ordered by $\preceq$ (this is not true for $\mathcal{Y}_n$, $n \geq 6$)

Suppose—see Fig. 3 for illustration—we have a Young diagram D which is obtained from another diagram by moving one of the boxes of D' to a (perhaps new) lower row, then we write

$$D \preceq D', \tag{2.3}$$

and $\preceq$ on $\mathcal{Y}_n$ is the order generated from all the inequalities of the form (2.3).

Now, using the terminology afforded by the dominance relation, we can formulate the main technical results concerning the Young modules.

For two representations π and τ of a finite group G, let us denote by $\langle \pi, \tau \rangle$ their *intertwining number* [Serre77]

$$\langle \pi, \tau \rangle = \dim Hom(\pi, \tau). \tag{2.4}$$

In addition, we denote the *sign* representation of S_n by sgn, and introduce the *twisted* Young module $Y_E(sgn) = Ind_{S_E}^{S_n}(sgn)$ attached to $E \in \mathcal{Y}_n$. Finally, let us denote by D^t the diagram in $\mathcal{Y}_n$ which is *transpose* to D. That is, D^t is gotten from D by reflecting across the downward diagonal from the top left box; in other words, the columns of D become the rows of D^t. Then,

Theorem 2.2.1 *For any two Young diagrams $D, E \in \mathcal{Y}_n$, we have,*

(1) *Intertwinity:* $\langle Y_E(sgn), Y_D \rangle = \begin{cases} 0, & \text{iff } E \npreceq D^t; \\ 1, & \text{if } E = D^t, \end{cases}$

and,

(2) *Monotonicity:* $D \precneqq E$ *if and only if* $Y_E \lneqq Y_D$.

For a proof of Theorem 2.2.1, see Appendix A.1.

2.3 The Irreducible Representations of S_n

Part (1) of Theorem 2.2.1 produces the standard classification, by Young diagrams, of the unitary dual (i.e., the set of irreps) $\widehat{S_n}$ of S_n, due to Frobenius and others

[Frobenius68]. Indeed, for each $D \in \mathcal{Y}_n$, let us denote by σ_D the unique joint component of $Y_{D^t}(sgn)$ and Y_D. Then,

Corollary 2.3.1 (Classification) *The irreps*

$$\sigma_D, \quad D \in \mathcal{Y}_n, \tag{2.5}$$

are pairwise non-isomorphic and exhaust $\widehat{S_n}$.

For a proof of Corollary 2.3.1 see Appendix A.1.

2.4 The Grothendieck Group of S_n

In Sect. 3 we will draw certain conclusions for the representation theory of the general linear group $GL_n = GL_n(\mathbb{F}_q)$, using the properties obtained in this section for the representations of S_n. An effective way to formulate this passage from S_n to GL_n, is to use the formalism of the Grothendieck group of representations, and in particular to describe consequences of Theorem 2.2.1 to the structure of this group in the case of S_n.

Given a finite group G, we can consider the Abelian group $K(G)$ generated from the set $\widehat{G}$ of isomorphism classes of irreps of G using the direct sum operation $\oplus$. Note that $K(G)$ has a natural partial order $<$ given by the sub-representation relation, and it comes equipped with a bilinear form $\langle \, , \rangle$, giving any two representations π, τ, their intertwining number $\langle \pi, \tau \rangle$ (2.4).

In particular, $K(S_n)$ is a free $\mathbb{Z}$-module with basis $\widehat{S_n} = \{\sigma_D, \ D \in \mathcal{Y}_n\}$, where σ_D are the irreps (2.5). However, $K(S_n)$ has another natural $\mathbb{Z}$-basis, i.e.,

Proposition 2.4.1 *The collection of Young modules Y_D, $D \in \mathcal{Y}_n$, forms a $\mathbb{Z}$-basis for $K(S_n)$.*

Proposition 2.4.1 follows from the following two consequences of Theorem 2.2.1:

Scholium 2.4.2 *The following hold,*

(1) *Spectrum: The irrep σ_E (2.5), appears in the Young module Y_D if and only if $D \preceq E$.*
(2) *Characterization: The irrep σ_D (2.5) is the only irrep that appears in Y_D but not in Y_E for every $D \precneqq E$.*

In particular, from Part (2) of Scholium 2.4.2 we deduce that the collection of Young modules is a minimal generating set of $K(S_n)$, confirming Proposition 2.4.1.

We proceed to describe a class of irreps of GL_n, that in a formal sense behave as if they also form $K(S_n)$.

3 Spherical Principal Series Representations of GL_n

In this section we want first to construct/classify the spherical principal series representations, and second, to recast certain properties of this collection. Both tasks involve, as in the case of S_n, the dominance relation on the set $\mathcal{Y}_n$, of Young diagrams with n boxes.

3.1 The Spherical Principal Series

Inside $GL_n = GL_n(\mathbb{F}_q)$, consider the Borel subgroup B [Borel69] of upper triangular matrices

$$B = \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix}.$$

Recall, see Sect. 1.2, that by definition an irreducible representation ρ of GL_n belongs to the *spherical principal series* (SPS) if it appears inside the induced representation $Ind_B^{GL_n}(\mathbf{1})$, where $\mathbf{1}$ denotes the trivial representation of B.

The construction of the SPS representations, and the verification of some of their properties can be done intrinsically (e.g., see in Section 10.5. of [Gurevich-Howe17]), without the relation to the representation theory of S_n. However, for purposes of this note, we prefer to get all the information from what was obtained already for S_n in Sect. 2. This, in particular, will enable us to derive the Pieri rule for GL_n from that of S_n.

3.2 The Grothendieck Group of the Spherical Principal Series

Let us denote by $K_B(GL_n)$ the Abelian group generated, using the operation of direct sum $\oplus$, from the SPS representations. The notion of subrepresentation induces a partial order $<$ on $K_B(GL_n)$ and the intertwining number pairing $\langle \, , \rangle$ (2.4) gives on it an inner product structure.

We proceed to give an effective description of $K_B(GL_n)$.

Recall, see Sect. 1.2, that the group $K_B(GL_n)$ has a distinguished collection of members in the form of induced representations that are associated to Young diagrams. Indeed, to a Young diagram $D \in \mathcal{Y}_n$ one attaches in a natural a way a flag F_D in $\mathbb{F}_q^n$, see Eq. (1.2), and a corresponding parabolic subgroup $P_D = Stab_{GL_n}(P_D) \subset GL_n$. Then, we can consider the trivial representation $\mathbf{1}$ of P_D, and induce to obtain

$$I_D = Ind_{P_D}^{GL_n}(\mathbf{1}). \tag{3.1}$$

Of course each I_D sits inside $Ind_B^{GL_n}(\mathbf{1})$, but we can say much more on the relation between the various I_D's. Indeed, for a given Young diagram $D = (d_1 \geq ... \geq d_r) \in \mathcal{Y}_n$, we have defined in (2.1) the subgroup $S_D \simeq S_{d_1} \times ... \times S_{d_r} \subset S_n$ and the corresponding Young module $Y_D = Ind_{S_D}^{S_n}(\mathbf{1})$. Then, the Bruhat decomposition [Borel69, Bruhat56] gives a bijection between the double cosets

$$P_D \diagdown GL_n \diagup P_E \text{ and } S_D \diagdown S_n \diagup S_E, \tag{3.2}$$

for every $D, E \in \mathcal{Y}_n$.

But, the cardinalities of the double cosets in (3.2) are exactly the dimensions of, respectively, the intertwining spaces $Hom_{GL_n}(I_D, I_E)$ and $Hom_{S_n}(Y_D, Y_E)$, so we conclude that,

Proposition 3.2.1 (Bruhat Decomposition) *For any two Young diagrams $D, E \in \mathcal{Y}_n$, we have,*

$$\langle I_D, I_E \rangle = \langle Y_D, Y_E \rangle . \tag{3.3}$$

One way to interpret identity (3.3) is as follows:

Corollary 3.2.2 *The correspondence*

$$Y_D \longmapsto I_D, \quad D \in \mathcal{Y}_n, \tag{3.4}$$

induces an order preserving isometry

$$\iota : K(S_n) \xrightarrow{\sim} K_B(GL_n). \tag{3.5}$$

On how to deduce Corollary 3.2.2 from Proposition 3.2.1, see the next section.

3.3 The Grothendieck Groups of S_n and of the Spherical Principal Series

We confirm Corollary 3.2.2, and along the way construct the SPS representations, and deduce various other facts on this collection.

Consider the map ι (3.5), extended by (integral) linearity from the correspondence (3.4). Denote by

$$\rho_D = \iota(\sigma_D), \quad D \in \mathcal{Y}_n, \tag{3.6}$$

the element of $K_B(GL_n)$ corresponding to the irrep (2.5) of S_n. Note that,

- $\rho_D < I_D$; and,
- $\langle \rho_D, I_D \rangle = 1$,

so, in particular, ρ_D is irreducible. In fact, the corresponding properties for S_n imply that

- $\left\langle \rho_D, Ind_B^{GL_n}(\mathbf{1}) \right\rangle = \dim(\sigma_D)$; and,
- we have,

$$\{\rho_D\} = \widehat{I}_D \smallsetminus \bigcup_{D \precneqq E} \widehat{I}_E, \tag{3.7}$$

i.e., ρ_D is the unique irrep that sits in I_D (we denote by $\widehat{I}_D$ the set of irreps inside I_D) but not in I_E, for any Young diagram $E \in \mathcal{Y}_n$ that strictly dominates D.

Remark 3.3.1 In fact, Property (3.7) characterizes the representation ρ_D, and is useful, e.g., you can compute out of it explicitly the dimension of ρ_D and find that (we use bold-face letters to denote the corresponding algebraic groups [Borel69]) it is equal to $\dim(\rho_D) = q^{\dim(\mathbf{GL_n}/\mathbf{P}_D)} + o(...)$, as $q \to \infty$, a fact that in turn characterizes (again, asymptotically) ρ_D uniquely among all irreps in I_D.

How do we know we get all the SPS?

A possible answer is that, as we already said, each I_D has a unique irrep that does not occur in the induced module I_E corresponding to any strictly dominating diagram $E \succneqq D$, namely, $\rho_D = \iota(\sigma_D)$. On the S_n side, the irreps σ_D, $D \in \mathcal{Y}_n$, completely decompose each of the induced representations. By Bruhat, this transfers to GL_n, so we get complete decompositions over there also. In particular, we get a complete decomposition of $Ind_B^{GL_n}(\mathbf{1}) = I_{(1,...,1)}$, the constituents of which are exactly the SPS representations.

Finally, the above discussion also validates Corollary 3.2.2.

Having at our disposal the understanding that the SPS representations and the representations of S_n are in some formal sense the same thing, we can proceed to discuss the Pieri rule.

4 The Pieri Rule

Fix $0 \leq k \leq n$, and denote by $P_{k,n-k} \subset GL_n$ the parabolic subgroup fixing the first k coordinate subspace of $\mathbb{F}_q^n$. There is a natural surjective map $P_{k,n-k} \twoheadrightarrow GL_k \times GL_{n-k}$. Take a Young diagram $D \in \mathcal{Y}_k$, and consider the irreducible SPS representation ρ_D of GL_k defined by (3.6). Denote by $\mathbf{1}_{n-k}$ the trivial representation of GL_{n-k}. Pull back the representation $\rho_D \otimes \mathbf{1}_{n-k}$ from $GL_k \times GL_{n-k}$ to $P_{k,n-k}$

and form the induced representation

$$I_{\rho_D} = Ind_{P_{k,n-k}}^{GL_n}(\rho_D \otimes \mathbf{1}_{n-k}). \tag{4.1}$$

Recall (see Problem 1.2.1 in Sect. 1.2) that, the narrative of the story we are telling in this note is that, we are seeking to compute the decomposition of I_{ρ_D} (4.1) into irreps. Moreover, it is easy to see that all constituents of the representation I_{ρ_D} are SPS, so we are seeking an answer to the decomposition problem in term of Young diagrams.

To arrive at our goal, after introducing some needed terminology, we will

(a) State the Pieri rule for the long established case of the complex general linear group $GL_n(\mathbb{C})$.
(b) Recall Schur duality.
(c) State and prove the Pieri rule for representations of S_n.

Our proof was suggested by a remark of **Nolan Wallach**, and uses the Schur (a.k.a. Schur-Weyl) duality, to deduce the result from the Pieri rule for $GL_n(\mathbb{C})$.

We note that, recently, in [Ceccherini-Silberstein-Scarabotti-Tolli10], the authors gave a different proof of the Pieri rule for S_n. Their treatment uses the Okounkov-Vershik approach [Okounkov-Vershik05] through the branching rule from S_n to S_{n-1} and Gelfand-Tsetlin basis.

We would also like to remark that, nowadays the Pieri rule for S_n can be understood as a particular case of the celebrated Littlewood-Richardson rule [Littlewood-Richardson34, Macdonald79], but was known [Pieri1893] a long time before this general result.

(d) Derive the Pieri rule for the Group $GL_n = GL_n(\mathbb{F}_q)$, using the equivalence, discussed in Sect. 3.2, between the representation theory of the spherical principal series, and that of S_n.

4.1 Skew-Diagrams and Horizontal Strips

The various formulations we present of the Pieri rule use the notions of skew diagram and horizontal strip, that we recall here.

Suppose we have Young diagrams $E \in \mathcal{Y}_n$ and $D \in \mathcal{Y}_k$ such that E contains D, denoted $E \supset D$, i.e., each row of E is at least as long as the corresponding row of D. Then, by removing from E all the boxes belonging to D, we obtain a configuration, denoted $E - D$, called *skew-diagram* [Macdonald79]. If, in addition—see Fig. 4 for illustration, each column of E is at most one box longer than the corresponding column of D, then we call $E - D$ a *horizontal strip* (or *horizontal m-strip* if $E - D$ has m boxes).

Remark 4.1.1 In [Ceccherini-Silberstein-Scarabotti-Tolli10] the term that is being used for "horizontal strip" is "totally disconnected skew diagram".

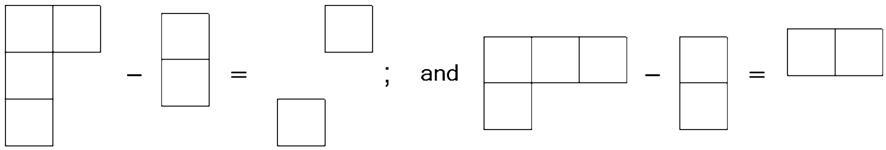


Fig. 4 In $\mathcal{Y}_4$: (2, 1, 1), (3, 1), contain $(1, 1) \in \mathcal{Y}_2$, with difference a horizontal 2-strip

4.2 The Pieri Rule for $GL_n(\mathbb{C})$

The Pieri rule for $GL_n(\mathbb{C})$ is a (very) special case of the general Littlewood-Richardson rule [Howe-Lee12, Littlewood-Richardson34, Macdonald79] for decomposing the tensor product of any pair of irreducible finite dimensional representations of $GL_n(\mathbb{C})$. The Pieri rule has been known since the nineteenth century [Pieri1893], and is relatively easy to establish [Fulton-Harris91, Howe92, Weyman89].

There is a standard way to label the irreducible representations of $GL_n(\mathbb{C})$. It is by their highest weights (see, for example [Fulton-Harris91, Howe92, Weyl46]). A highest weight for $GL_n(\mathbb{C})$ is specified by a decreasing sequence

$$d_1 \geq ... \geq d_n,$$

of integers.

When all the d_j are non-negative, the above sequence can be thought of as specifying a Young diagram D, with j-th row having length d_j. The number of boxes in D can be arbitrarily large, but the number of rows is bounded by n. Irreps of $GL_n(\mathbb{C})$ corresponding to sequences with all d_j non-negative are called *polynomial* representations. These are exactly all the irreps of $GL_n(\mathbb{C})$ that appear in the tensor powers $(\mathbb{C}^n)^{\otimes^l}$ of $\mathbb{C}^n$ for some $l \geq 0$. (Any irrep of $GL_n(\mathbb{C})$ is isomorphic to a twist by a power of determinant of a polynomial representation.). We will denote by

$$\pi_n^D, \quad D = (d_1 \geq ... \geq d_n \geq 0), \tag{4.2}$$

the polynomial representation of $GL_n(\mathbb{C})$ whose highest weight corresponds to the diagram D. The one-rowed diagrams, given by $(d_1, 0, ..., 0)$ correspond to the symmetric powers $S^{d_1}(\mathbb{C}^n)$.

The Pieri rule for $GL_n(\mathbb{C})$ describes the decomposition of a tensor product $\pi_n^D \otimes S^d(\mathbb{C}^n)$ of a general polynomial irrep with a symmetric power $S^d(\mathbb{C}^n)$.

Proposition 4.2.1 (Pieri Rule for $GL_n(\mathbb{C})$) *The representation $\pi_n^D \otimes S^d(\mathbb{C}^n)$ is multiplicity free. Moreover, we have,*

$$\pi_n^D \otimes S^d(\mathbb{C}^n) \simeq \sum_E \pi_n^E, \tag{4.3}$$

where E runs through all diagrams such that

(1) $D \subset E$; *and,*
(2) $E - D$ *is an horizontal d-strip.*

4.3 Schur-Weyl Duality

The group $GL_n(\mathbb{C})$ is defined in terms of its action on $\mathbb{C}^n$. By taking tensor products, this action gives rise naturally to an action on the l-fold tensor product $(\mathbb{C}^n)^{\otimes^l}$ of $\mathbb{C}^n$ with itself (a.k.a., the l-th tensor power of $\mathbb{C}^n$). Clearly, the permutation group S_l also acts on $(\mathbb{C}^n)^{\otimes^l}$ by permuting the factors of the product. This action of S_l clearly commutes with the action of $GL_n(\mathbb{C})$. Schur-Weyl duality [Howe92, Schur27, Weyl46] says that

Proposition 4.3.1 (Schur-Weyl Duality—Non-explicit Form) *The actions of S_l and $GL_n(\mathbb{C})$ on $(\mathbb{C}^n)^{\otimes^l}$ generate mutual commutants of each other.*

From this, Burnside's double commutant theorem [Burnside1905, Weyl46] lets us conclude that, as an $S_l \times GL_n(\mathbb{C})$-module, we have a decomposition

$$(\mathbb{C}^n)^{\otimes^l} \simeq \sum_D \sigma_l^D \otimes \tau_n^D, \tag{4.4}$$

where $D \in \mathcal{Y}_l$ runs through diagrams with l boxes, σ_l^D are the associated irreps (2.5) of S_l, and the τ_n^D are appropriate irreps of $GL_n(\mathbb{C})$. Some computation then shows that, remarkably, τ_n^D is equal to the representation π_n^D (provided of course that D does not have more than n rows; otherwise $\tau_n^D = 0$) given by Eq. (4.2). Thus, we can rewrite (4.4), and obtain

Proposition 4.3.2 (Schur-Weyl Duality—Explicit Form) *As an $S_l \times GL_n(\mathbb{C})$-module, we have the decomposition*

$$(\mathbb{C}^n)^{\otimes^l} \simeq \sum_D \sigma_l^D \otimes \pi_n^D, \tag{4.5}$$

where D runs over all diagrams in $\mathcal{Y}_l$ with at most n rows.

4.4 The Pieri Rule for S_n

With the usual notation, consider $k < n$, and $S_k \subset S_n$, in the standard way, as the group that fixes the last $n - k$ letters on which S_n acts. Then the symmetric group on these letters is S_{n-k}, and we have the product $S_k \times S_{n-k} \subset S_n$.

Take a partition/Young diagram D of size k, and let σ_D be the associated irreducible representation (2.5) of S_k. Let $\mathbf{1}_{n-k}$ be the trivial representation of S_n. Form the induced representation

$$I_{\sigma_D} = Ind_{S_k \times S_{n-k}}^{S_n}(\sigma_D \otimes \mathbf{1}_{n-k}), \tag{4.6}$$

of S_n.

The Pieri Rule for S_n describes the decomposition of this induced representation into irreducible subrepresentations.

Theorem 4.4.1 (Pieri Rule for S_n) *The representation I_{σ_D} (4.6) is multiplicity-free. It consists of one copy of each representation σ_E of S_n, for diagrams $E \in \mathcal{Y}_n$, such that*

(1) $D \subset E$; *and,*
(2) $E - D$ *is an horizontal* $(n-k)$*-strip.*

In Appendix A.2 we give our proof of Theorem 4.4.1, demonstrating how it follows from the Pieri rule for $GL_n(\mathbb{C})$, invoking the Schur-Weyl duality.

Remark 4.4.2 (Description of the Young Module) Theorem 4.4.1 can be used to give a recursive description of the Young module Y_D (2.2).

Given a Young diagram $D \in \mathcal{Y}_n$ with n boxes, let D_s be the diagram consisting of the first s rows of D, and let k_s be the number of boxes in D_s. Suppose that in D there are r rows in all, so that $k_r = n$. Suppose we know how to decompose Y_{D_s}. Then, if we apply the Pieri rule to each component of Y_{D_s} and k_{s+1} is the number of boxes in D_{s+1}, we learn how to decompose $Y_{D_{s+1}}$. Starting with $s = 1$, we can successively decompose the Y_{D_s} for all s up to r, at which point we will have found the decomposition of Y_D.

For example, the above methods provides us with the following combinatorial description of the multiplicity of the irrep σ_E, $E \in \mathcal{Y}_n$, in Y_D: it is the number of ways to fill E with a nested family of sub-diagrams E_s, such that

- $E_s \subset E_{s+1}$; and,
- $E_{s+1} - E_s$ is a horizontal strip with $k_{s+1} - k_s$ boxes.

From this, we can see, again, that the multiplicity of σ_D in Y_D is 1.

4.5 The Pieri Rule for $GL_n(\mathbb{F}_q)$

Now we can finish our story, and deliver the answer to the introduction's motivating problem of decomposing I_{ρ_D} (4.1).

Note that the isomorphism ι (3.5), between the representation groups $K(S_n)$ and $K_B(GL_n)$, sends I_{σ_D} to I_{ρ_D}. So the Pieri rule for S_n, implies the Pieri rule for

$GL_n(\mathbb{F}_q)$, i.e., the same description as in Theorem 4.4.1, just replace there, S_n by $GL_n(\mathbb{F}_q)$, and σ_D, σ_E, by ρ_D, ρ_E, respectively.

Remark 4.5.1 (Decomposing Permutation Representation on Flag Variety) Replacing the Young module Y_D, in Remark 4.4.2, by the induced representation $I_D = Ind_{P_D}^{GL_n}(\mathbf{1})$, which is the space of functions on the flag variety GL_n/P_D. We get a recursive formula for the decomposition into irreps of the permutation representation of GL_n on functions on a fairly general flag variety.

Acknowledgments The material presented in this note is based upon work supported in part by the National Science Foundation under Grants No. DMS-1804992 (S.G.) and DMS-1805004 (R.H.).

This note was written during the winter of 2019–20 while S.G. was visiting the Math Department and the College of Education at Texas A&M University, and he would like to thank these institutions, and to thank personally R. Howe.

Appendix A Proofs

A.1 Proofs for Sect. 2

Proof of Theorem 2.2.1

For a set X let us denote by $L(X)$ the space of complex valued functions on X. We also use this notation to denote the standard permutation representation of a group G, in case it acts on X.

Now we can proceed to give the proof.

Proof

Part (1) Let us analyze the space of intertwiners $Hom(Y_E(sgn), Y_D)$. This has a "geometric" description from which the information we are after can be read.

- First, recall that we can realize Y_D as the permutation representation $L(\mathcal{T}_D)$ associated with the action of S_n on the set $\mathcal{T}_D$ of all tabloids that one can make out of D (see Sect. 2.1). In the same way, $Y_E(sgn)$ can be realized on the space $L(\mathcal{T}_E)$ with the permutation action of S_n on it twisted by sgn.
- Second, using the bases of delta functions of $L(\mathcal{T}_E)$ and $L(\mathcal{T}_D)$, we can associate to every intertwiner $Hom(Y_E(sgn), Y_D)$ a kernel function (i.e., a matrix) K on $\mathcal{T}_D \times \mathcal{T}_E$ that satisfies

$$K(s(T_D), s(T_E)) = sgn(s)K(T_D, T_E), \tag{A.1}$$

for every $s \in S_n$, $T_D \in \mathcal{T}_D$ and $T_E \in \mathcal{T}_E$.

Let us denote by $L(\mathcal{T}_D \times \mathcal{T}_E)^{\mathbf{1}\otimes sgn}$ the collection of all K satisfying Identity (A.1).

In summary, we obtained,

$$Hom(Y_E(sgn), Y_D) = L(\mathcal{T}_D \times \mathcal{T}_E)^{\mathbf{1} \otimes sgn}. \tag{A.2}$$

This is a geometric description of the space of intertwiners.

Now suppose we have K from (A.2), and suppose there are $T_D \in \mathcal{T}_D$ and $T_E \in \mathcal{T}_E$ with rows, one of T_D and one of T_E, that share two numbers $i, j \in \{1, ..., n\}$. Then, the permutation that transposes i and j must preserve $K(T_D, T_E)$, and also must change its sign. Therefore, $K(T_D, T_E) = -K(T_D, T_E)$, so $K(T_D, T_E) = 0$. In other words $K(T_D, T_E) \neq 0$ only if

- each number from the first row of E, should sit in a different row of D, so,
  $E_1 \leq D_1^t$,
  i.e., the length E_1 of the first row of E is not more than that of the first row of D^t.
  and
- each number from the second row of E should sit in a different row of D, so we also have,
  $E_1 + E_2 \leq D_1^t + D_2^t$.
- etc...

Namely, for the space (A.2) to be non-trivial, it is necessary to have
$(*)$ $E \preceq D^t$.

Next, assuming $E = D^t$, we want to show that the intertwining space (A.2) is one dimensional.

Let us first give one orbit in $\mathcal{T}_D \times \mathcal{T}_{D^t}$ that supports a non-trivial intertwiner:

- Take the Young diagram D and fill each box of it with numbers from $\{1, .., n\}$. The object we obtained in this way is called Young *tableau* [Fulton97]. From it we can make in a natural way a Young tabloid T_D by grouping together the numbers in each line of the tableau.
- We could also first "transpose" the filled D to obtain a tableau associated with D^t, and then, in the same way as above, form the corresponding tabloid T_{D^t}.

It is clear that, any two rows, one of T_D and one of T_{D^t}, share no more than one number in common. Hence, the group S_n acts freely on the orbit

$$\mathcal{O}_{T_D, T_{D^t}} \subset \mathcal{T}_D \times \mathcal{T}_{D^t} \tag{A.3}$$

of (T_D, T_{D^t}), and, in particular, there exists an intertwiner K from (A.2) which is supported on it.

Now, let us show that (A.3) is the only orbit that supports such K. Indeed, take $s \in S_n$, such that $s(T_{D^t}) \neq T_{D^t}$. But then, there are rows, one of $s(T_{D^t})$, and one of T_D, that share two numbers in common, then, as was explained earlier, the orbit of $(T_D, s(T_{D^t}))$ does not support an intertwiner.

Finally, let us show that if $E \preceq D^t$, then the space (A.2) is non-zero, i.e., the condition $(*)$ is also sufficient. It is enough to examine the case when $E = (D^t)^\circ$ obtained from D^t, by moving one box down to form a new lower row. Now, look at the tabloids T_D and T_{D^t}, that we used in the paragraph just above, and the natural tabloid $T_{(D^t)^\circ}$ one obtains from the filling of D^t by numbers as we did above in order to create T_{D^t}. Then, as we argued above, the orbit of $(T_D, T_{(D^t)^\circ})$ supports a non-trivial intertwiner.

Part (2) If D is not dominated by E, then from Part (1) we see that $\langle Y_{E^t}(sgn), Y_D\rangle = 0$, and in particular, again by Part (1), Y_E cannot be a subrepresentation of Y_D.

On the other hand, let us assume that D is strictly dominated by E, and show that $Y_E \lneqq Y_D$.

First, we realize the space of intertwiners between Y_E and Y_D geometrically,

$$Hom(Y_E, Y_D) = L(\mathcal{T}_D \times \mathcal{T}_E)^{S_n},$$

where on the right hand side of the equality we have, the space of S_n-invariant kernels K on $\mathcal{T}_D \times \mathcal{T}_E$, or equivalently the space of functions on the set of orbits $S_n \diagdown (\mathcal{T}_D \times \mathcal{T}_E)$.

Second, we can parametrize the above set of orbits as follows. Take $T_D \in \mathcal{T}_D$, and $T_E \in \mathcal{T}_E$, and denote by $R_i(T_D)$ and $R_j(T_E)$, the i-th row of T_D, and j-th row of T_E, respectively. Then, we can define the *intersection matrix*

$$R_{T_D,T_E} = (r_{ij}), \quad r_{ij} = \#\left(R_i(T_D) \cap R_j(T_E)\right), \tag{A.4}$$

i.e., r_{ij} is the number of elements common to both rows. It is clear that R_{T_D,T_E} is an invariant of the orbit. Moreover, it gives a complete invariant. Indeed, it is not difficult to see that if $R_{T_D,T_E} = R_{T'_D,T'_E}$, then there exists $s \in S_n$ such that $s(T_D) = T'_D$, and $s(T_E) = T'_E$.

A direct computation, using the parametrization (A.4), reveals that,

Claim A.1.1 Consider the Young diagrams $D_{n-k,k} = (n-k, k)$ and $D_{n-k',k'} = (n-k', k')$, where $0 \le k, k' \le \frac{n}{2}$. Then,

$$\left\langle Y_{D_{n-k,k}}, Y_{D_{n-k',k'}}\right\rangle = \min\{k+1, k'+1\}.$$

So $Y_{D_{n,0}}$ contains 1 representation—the trivial representation. Then $Y_{D_{n-1,1}}$ contains two representations, one of which is the trivial representation. Since $Y_{D_{n-2,2}}$ has intertwining number 1 with $Y_{D_{n,0}}$ and 2 with $Y_{D_{n-1,1}}$, it must contain the two representations of $Y_{D_{n-1,1}}$ with multiplicity 1 each. Since its self intertwining number is 3, it contains 3 representations, each with multiplicity 1. Then we can continue like that: $Y_{D_{n-3,3}}$ contains each of the representations of $Y_{D_{n-2,2}}$ with multiplicity 1, and then one new representation, and so on. So in particular:

(∗∗) $Y_{D_{n-k-1,k+1}}$ contains $Y_{D_{n-k,k}}$ when $k+1 \leq \frac{n}{2}$.

Now take any diagram D, containing two rows R and R', with R' (which might be of length equal to 0) at least two boxes shorter than R. Then we can form Y_D by first forming the representation $Y_{D_{R,R'}}$ of $S_{R+R'}$, and then extending to be trivial on the stabilizers of the other rows, and then inducing up to S_n. So if we replace R and R' with $R-1$ and $R'+1$, we will get a larger representation, using Fact (∗∗). This completes the verification of Part (2), and of Theorem 2.2.1. □

Proof of Corollary 2.3.1

Proof Note that the dominance order on $\mathcal{Y}_n$ is a partial order, and in particular, is anti-symmetric, i.e., for every $E, D \in \mathcal{Y}_n$, if $E \preceq D$ and $E \succeq D$, then $E = D$. But, if $\sigma_E \simeq \sigma_D$, then, by the "iff" of Part (1) of Theorem 2.2.1, $E \preceq D$ and $E \succeq D$, so the Corollary follows. □

A.2 Proofs for Sect. 4

Proof of Theorem 4.4.1

Proof Schur duality for $S_k \times GL_n(\mathbb{C})$ on the k-fold tensor product $(\mathbb{C}^n)^{\otimes^k}$ says (Proposition 4.3.2, Eq. (4.5)) that we have

$$(\mathbb{C}^n)^{\otimes^k} \simeq \sum_{D \in \mathcal{Y}_k} \sigma_k^D \otimes \pi_n^D,$$

where π_n^D is the irrep of $GL_n(\mathbb{C})$ with highest weight corresponding to the diagram D.

We can also apply Schur duality to the action of $S_{n-k} \times GL_n(\mathbb{C})$ on $(\mathbb{C}^n)^{\otimes^{n-k}}$. Then the space of fixed vectors for S_{n-k} is the $S_{n-k} \times GL_n(\mathbb{C})$-module $\mathbf{1}_{n-k} \otimes \pi_n^{(n-k)}$, corresponding to the diagram with one row of length $n-k$. This is just the $(n-k)$-th symmetric power of the standard action on $\mathbb{C}^n$.

Now consider,

$$(\mathbb{C}^n)^{\otimes^n} \simeq (\mathbb{C}^n)^{\otimes^k} \otimes (\mathbb{C}^n)^{\otimes^{n-k}},$$

as an S_n-module, again with S_n acting by permutation of the factors. If we take the isotypic component of σ_D inside $(\mathbb{C}^n)^{\otimes^k}$, and the space of fixed vectors for S_{n-k} inside $(\mathbb{C}^n)^{\otimes^{n-k}}$, their tensor product will be the isotypic component inside $(\mathbb{C}^n)^{\otimes^n}$ of the representation $\sigma_D \otimes \mathbf{1}_{n-k}$ of $S_k \times S_{n-k}$.

On the other hand, the action of $GL_n(\mathbb{C})$ on the indicated tensor product is described by (a multiple of) the tensor product $\pi_n^D \otimes \pi_n^{(n-k)}$. By the Pieri rule for $GL_n(\mathbb{C})$ (see Proposition 4.2.1) this decomposes into a multiplicity free sum of irreps for $GL_n(\mathbb{C})$ whose highest weights are given by diagrams E having the form indicated in the statement of the proposition: E has n boxes, contains D, and $E - D$ consists of a horizontal $(n-k)$-strip. Thus, the $S_k \times S_{n-k}$ isotypic component for $\sigma_D \otimes \mathbf{1}_{n-k}$ of $(\mathbb{C}^n)^{\otimes^n}$ has the structure

$$\sum_E \sigma_D \otimes \mathbf{1}_{n-k} \otimes \pi_n^E, \tag{A.5}$$

as $S_k \times S_{n-k} \times GL_n(\mathbb{C})$-module, where E runs over the diagrams specified in the statement of the proposition.

Now consider the representation of S_n generated by this space. By Schur duality for S_n, it will be

$$\sum_E \sigma_E \otimes \pi_n^E,$$

as $S_n \times GL_n(\mathbb{C})$-module. Comparing this with Formula (A.5), we conclude that each representation σ_E of S_n contains one copy of the representation $\sigma_D \otimes \mathbf{1}_{n-k}$ when restricted to $S_k \times S_{n-k}$, and that these are the only representations of S_n that do contain $\sigma_D \otimes \mathbf{1}_{n-k}$. By Frobenius reciprocity, this is equivalent to the statement of Theorem 4.4.1. The proof is complete. □

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Cauchy-Riemann Equations for Free Noncommutative Functions

S. ter Horst and E. M. Klem

Dedicated to the memory of Ron Douglas, in recognition to his many deep contributions to mathematics

Abstract In classical complex analysis analyticity of a complex function f is equivalent to differentiability of its real and imaginary parts u and v, respectively, together with the Cauchy-Riemann equations for the partial derivatives of u and v. We extend this result to the context of free noncommutative functions in operator systems. In this context, the real and imaginary parts become so called real noncommutative functions, as appeared recently in the context of Löwner's theorem in several noncommutative variables. Additionally, as part of our investigation of real noncommutative functions, we show that real noncommutative functions are in fact noncommutative functions.

Keywords Cauchy-Riemann equations · Free noncommutative functions · Real noncommutative functions

Mathematics Subject Classification (2010) Primary 32A10; Secondary 46L52, 26B05

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_15

1 Introduction

Over the last decade a theory of free noncommutative (nc) functions that are evaluated in tuples of matrices of arbitrary size was developed. The theory becomes particularly rich when the functions have a domain that is assumed to be right (or left) admissible, in which case the functions admit a Taylor expansion and, under mild boundedness assumptions, are analytic. We refer to [15] for the first book that presents a comprehensive account of the theory, as well the seminal papers [27, 28] by J.L. Taylor that foreshadowed much of the work on this topic of the last decade. Precise definitions will be given a little further in this introduction.

More recently, in connection with Löwner's theorem [18–20], the notion of real nc functions appeared. These functions have domains that consist of tuples on Hermitian matrices, precluding the right (or left) admissibility property, and satisfy slightly different conditions. Another instance where real nc function come up in a natural way is as the real and imaginary part of an nc function. In the present paper we derive the noncommutative Cauchy-Riemann equations for the real and imaginary part of an nc function and consider the question when two real nc functions satisfying the noncommutative Cauchy-Riemann equations appear as the real and imaginary part of an nc function.

We will now provide more precise definitions and state our main result. Throughout $\mathcal{H}$ and $\mathcal{K}$ are complex Hilbert spaces. Denote by $\mathcal{B}(\mathcal{H},\mathcal{K})$ the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, abbreviated to $\mathcal{B}(\mathcal{H})$ in case $\mathcal{H}=\mathcal{K}$. Let $\mathcal{V}\subset\mathcal{B}(\mathcal{H})$ be an operator system, i.e., a norm closed (complex) linear subspace of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, which is closed under taking adjoints and contains the identity operator $I_{\mathcal{H}}$ of $\mathcal{B}(\mathcal{H})$. We denote the real subspace of self-adjoint operators in $\mathcal{V}$ by $\mathcal{V}^{\rm sa}$.

Note that for every operator system $\mathcal{V}\subset\mathcal{B}(\mathcal{H})$ there exists a sequence of norms $\|\cdot\|_n$ on $\mathcal{V}^{n\times n}$, $n=1,2,\ldots$, such that

$$\|X\oplus Y\|_{n+m}=\max\{\|X\|_n,\|Y\|_m\}\quad\text{for all } X\in\mathcal{V}^{n\times n}, Y\in\mathcal{V}^{m\times m}, \tag{1.1}$$

and

$$\|SXT\|_n\leqslant\|S\|\|X\|_n\|T\|\quad\text{for all } X\in\mathcal{V}^{n\times n}\text{ and } S,T\in\mathbb{C}^{n\times n}, \tag{1.2}$$

where $\mathbb{C}^{n\times n}$ denotes the vector space of $n\times n$ complex matrices and $\|\cdot\|$ denotes the operator norm of $\mathbb{C}^{n\times n}$ with respect to the standard Euclidean norm of $\mathbb{C}^n$. Here, for $X\in\mathcal{V}^{m\times n}$ and $S\in\mathbb{C}^{r\times m}$, $T\in\mathbb{C}^{n\times s}$ the product SXT is to be interpreted as $(I_{\mathcal{H}}\otimes S)X(I_{\mathcal{H}}\otimes T)\in\mathcal{V}^{r\times s}$. For more details see [6, p. 21].

We consider functions with domains in

$$\mathcal{V}_{\rm nc}:=\coprod_{n=1}^{\infty}\mathcal{V}^{n\times n}\quad\text{or}\quad\mathcal{V}_{\rm nc}^{\rm sa}:=\coprod_{n=1}^{\infty}(\mathcal{V}^{n\times n})^{\rm sa}.$$

A subset $\mathcal{D}$ of $\mathcal{V}_{\text{nc}}$ or $\mathcal{V}_{\text{nc}}^{\text{sa}}$ is said to be an nc set in case it *respects direct sums*:

$$X, Y \in \mathcal{D} \quad \Longrightarrow \quad X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \mathcal{D}.$$

In some papers the converse implication as well as additional features are also assumed, cf., [19, 20]. See Lemma 2.7 below as well as the paragraph preceding this lemma. For an nc set $\mathcal{D}$ and a positive integer n we define $\mathcal{D}_n := \mathcal{D} \cap \mathcal{V}^{n\times n}$. An nc set $\mathcal{D} \subset \mathcal{V}_{\text{nc}}$ is called *right admissible* in case

$$X \in \mathcal{D}_n, Y \in \mathcal{D}_m, Z \in \mathcal{V}^{n\times m} \Longrightarrow \begin{bmatrix} X & rZ \\ 0 & Y \end{bmatrix} \in \mathcal{D}_{n+m} \text{ for some } 0 \neq r \in \mathbb{C}. \tag{1.3}$$

In case the nc set $\mathcal{D}$ is right admissible and closed under similarity, then the "for some" part in the right-hand side of (1.3) can be replaced by "for all." There is a dual notion of left admissibility, see page 18 and onwards in [15], but we will not need this notion in the present paper.

Let $\mathcal{V} \subset \mathcal{B}(\mathcal{H})$ and $\mathcal{W} \subset \mathcal{B}(\mathcal{K})$ be operator systems. A function $w : \mathcal{D} \to \mathcal{W}_{\text{nc}}$ whose domain $\mathcal{D}$ is an nc set in $\mathcal{V}_{\text{nc}}$ is called an *nc function* in case it has the following properties:

(NC-i) w is *graded*, i.e., $w(\mathcal{D}_n) \subset \mathcal{W}^{n\times n}$ for $n = 1, 2, \ldots$;

(NC-ii) w *respects direct sums*, i.e., for all $X, Y \in \mathcal{D}$ we have

$$w(X \oplus Y) = w(X) \oplus w(Y);$$

(NC-iii) w *respects similarities*, i.e., for all $X \in \mathcal{D}_n$, $S \in \mathbb{C}^{n\times n}$ invertible so that $SXS^{-1} \in \mathcal{D}_n$, we have

$$w(SXS^{-1}) = Sw(X)S^{-1}.$$

For more on operator systems, and more generally operator spaces, as well as matrices over operator systems and operator spaces see [6, 22]. As a specific example let $\mathcal{V}$ be the operator system of diagonal matrices in $\mathbb{C}^{d\times d}$, which can be identified with $\mathbb{C}^d$. In that case $\mathcal{V}^{n\times n}$ can be identified with d-tuples of $n \times n$ matrices, and hence $\mathcal{V}_{\text{nc}}$ corresponds to the set of square complex matrices of arbitrary size. Furthermore, in that case $\mathcal{V}_{\text{nc}}^{\text{sa}}$ corresponds to $\coprod_{n=1}^{\infty}(\mathcal{H}_n)^d$, with $\mathcal{H}_n$ denoting the real space of $n \times n$ Hermitian matrices.

Much of the theory of nc functions developed in [15] is for nc functions whose domains are right (or left) admissible, in which case for each X, Y, Z and $r \neq 0$ as in (1.3) one can define the right difference-differential operator $\Delta w(X, Y)$ at the point Z via

$$w\left(\begin{bmatrix} X & rZ \\ 0 & Y \end{bmatrix}\right) = \begin{bmatrix} w(X) & r\Delta w(X,Y)(Z) \\ 0 & w(Y) \end{bmatrix}, \tag{1.4}$$

with the zero and two block diagonal entries following from (NCi)–(NCiii). This right difference-differential operator is linear in Z and provides a difference formula for w leading to the so-called Taylor-Taylor expansion of w, and, under certain boundedness assumptions on w, provides the Gâteaux-derivative of w; see [15] for an elaborate treatment. Recall that the Gâteaux- or G-derivative of a function $g : \mathcal{D}_g \to \mathcal{Y}$ with domain $\mathcal{D}_g \subset \mathcal{X}$, with $\mathcal{X}$ and $\mathcal{Y}$ Banach spaces over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, at a point $X \in \mathcal{D}_g$ in the direction $Z \in \mathcal{X}$ is given by

$$Dg(X)(Z) := \lim_{\mathbb{K} \ni t \to 0} \frac{g(X + tZ) - g(X)}{t}, \tag{1.5}$$

provided the limit exist. Then g is said to be Gâteaux- or G-differentiable in case $\mathcal{D}_g$ is open and $Dg(X)(Z)$ exists for all $X \in \mathcal{D}_g$ and all $Z \in \mathcal{X}$. In the case of an nc function w on $\mathcal{D}$, G-differentiability means that for each positive integer n, on the restriction of w to $\mathcal{D}_n$ should be G-differentiable; see Sect. 3 for further details and references on G-differentiability as well as Fréchet- or F-differentiability.

A function $w : \mathcal{D}_w \to \mathcal{W}_{\text{nc}}^{\text{sa}}$ is called a *real nc function* in case its domain $\mathcal{D}$ is an nc set contained in $\mathcal{V}_{\text{nc}}^{\text{sa}}$ which is graded and respects direct sums, i.e., (NC-i) and (NC-ii) above hold, and

(RNC-iii) w *respects unitary equivalence*, i.e., for all $X \in \mathcal{D}_n$, $U \in \mathbb{C}^{n \times n}$ unitary so that $UXU^* \in \mathcal{D}_n$, we have

$$w(UXU^*) = Uw(X)U^*.$$

Despite the seeming limitation of unitary equivalence over similarity, one of the contributions of the present paper is the observation that real nc functions are also nc functions, see Theorem 2.1 below. Hence (NC-i), (NC-ii) and (RNC-iii) imply (NC-iii). This result relies heavily on the fact that the domains of real nc functions consist of self-adjoint operators only. The latter also implies that the domains of real nc functions are 'nowhere right admissible,' and hence much of the theory developed in [15] does not apply to real nc functions.

The combination of the conditions (NC-i), (NC-ii) and (RNC-iii) without the restriction to self-adjoint operators has appeared in the literature before, for instance in work on extensions of the continuous functional calculus to the noncommutative setting [9, 10] building on results on noncommutative Gelfand-Naimark theory [3, 7, 26]. More recently, the combination of these three conditions appeared in the work of Davidson and Kennedy on noncommutative Choquet theory [4] and in work of Klep and Špenko on free function theory [16].

Now, given an nc function f on a right admissible domain $\mathcal{D}_f \subset \mathcal{V}_{\rm nc}$, we write

$$f(A+iB) = u(A,B) + iv(A,B), \quad A+iB \in \mathcal{D}_f, \tag{1.6}$$

for $A, B \in \mathcal{V}_{\rm nc}^{\rm sa}$ of the same size and with

$$u(A,B) := \mathrm{Re}\, f(A+iB) \quad \text{and} \quad v(A,B) := \mathrm{Im}\, f(A+iB).$$

This defines real nc functions u and v on domain

$$\mathcal{D} = \{(A,B) \in (\mathcal{V}_{\rm nc}^{\rm sa})^2 \colon A+iB \in \mathcal{D}_f\}.$$

Here $(\mathcal{V}_{\rm nc}^{\rm sa})^2$ is to be interpreted as 2×2 block diagonals with $\mathcal{V}_{\rm nc}^{\rm sa}$ entries, that is, $(\mathcal{V}_{\rm nc}^{\rm sa})^2 = (\mathcal{V}^2)_{\rm nc}^{\rm sa}$. Note that $\mathcal{D}$ is open in $(\mathcal{V}_{\rm nc}^{\rm sa})^2$ precisely when $\mathcal{D}_f$ is open in $\mathcal{V}_{\rm nc}$; in both cases open means that the restriction of the domain to $n \times n$ operator matrices is open in $((\mathcal{V}^{\rm sa})^{n\times n})^2$ and $\mathcal{V}^{n\times n}$, respectively. Furthermore, in case f is G-differentiable, then so are u and v and their G-derivatives satisfy the following noncommutative Cauchy-Riemann equations

$$\begin{gathered} Du(A,B)(Z_1,Z_2) = Dv(A,B)(-Z_2,Z_1), \\ \text{for } (A,B) \in \mathcal{D}_n,\ Z_1, Z_2 \in (\mathcal{V}^{n\times n})^{\rm sa},\ n \in \mathbb{N}. \end{gathered} \tag{1.7}$$

See Theorem 4.1 for these claims as well as additional results.

Conversely, one may wonder whether G-differentiable real nc functions u and v with open domains $\mathcal{D}_u$ and $\mathcal{D}_v$, respectively, in $(\mathcal{V}_{\rm nc}^{\rm sa})^2$ that satisfy (1.7) on $\mathcal{D} = \mathcal{D}_u \cap \mathcal{D}_v$ define an nc function f via (1.6). For this purpose, G-differentiability does not seem to be the appropriate notion of differentiability, and we will rather assume the stronger notion of F-differentiability, in which case the derivative is still obtained via (1.5); see Sect. 3 for further details. Even in classical complex analysis this phenomenon occurs, see [5, 8] as well as Remark 5.6 below. Our main result is the following theorem.

Theorem 1.1 *Let u and v be real nc functions with open domains $\mathcal{D}_u$ and $\mathcal{D}_v$, respectively, in $(\mathcal{V}_{\rm nc}^{\rm sa})^2$ that are F-differentiable and satisfy the nc Cauchy-Riemann equations* (1.7) *on $\mathcal{D} = \mathcal{D}_u \cap \mathcal{D}_v$. Define f on $\mathcal{D}_f = \{A+iB \in \mathcal{V}_{\rm nc} \colon (A,B) \in \mathcal{D}\}$ via* (1.6). *Then f is a F-differentiable nc function.*

Apart from the present introduction, this paper consists of four sections. In Sect. 2 we prove that real nc functions are nc functions, consider some examples and look at domain extensions. Next, in Sect. 3 we review the notions of Gâteaux- and Fréchet differentiability for nc functions. The domains of real nc functions are not right-admissible so that the G-derivative cannot be determined algebraically through the difference-differential operator. In the following section we derive properties of the real and imaginary parts of an nc function, including the nc Cauchy-

Riemann equations. Finally, in Sect. 5 we consider the converse direction and prove Theorem 1.1.

2 Real nc Functions Are nc Functions

In this section we focus on real nc functions only, without assuming any form of differentiability. Our main result is the following theorem.

Theorem 2.1 *Real nc functions are nc functions.*

In order to prove this result we first show that real nc functions also respect intertwining. Throughout let $\mathcal{V} \subset \mathcal{B}(\mathcal{H})$ and $\mathcal{W} \subset \mathcal{B}(\mathcal{K})$ be operator systems.

Proposition 2.2 *A graded function $w : \mathcal{D} \to \mathcal{W}_{\text{nc}}^{\text{sa}}$ on an nc set $\mathcal{D} \subset \mathcal{V}_{\text{nc}}^{\text{sa}}$ respects direct sums and unitary equivalence if and only if it respects intertwining: if $X \in \mathcal{D}_n$, $Y \in \mathcal{D}_m$, and $T \in \mathbb{C}^{n\times m}$ so that $XT = TY$, then $w(X)T = Tw(Y)$.*

Proof The necessity follows from Proposition 2.1 in [15]. Assume w respects direct sums and unitary equivalence, i.e., w is a real nc function. Let $X \in \mathcal{D}_n$, $Y \in \mathcal{D}_m$, and $T_0 \in \mathbb{C}^{n\times m}$ so that $XT_0 = T_0Y$. If $T_0 = 0$, then it is trivial that $w(X)T_0 = T_0w(Y)$, so assume $T_0 \neq 0$. Set $T = \|T_0\|^{-1}T_0$ so that $\|T\| = 1$. Let $D_T := (I - T^*T)^{1/2}$ and $D_{T^*} := (I - TT^*)^{1/2}$ be the defect operators of the contractions T and T^*, respectively. Since X and Y are self-adjoint we have

$$T^*X = YT^*.$$

Therefore

$$XD_{T^*}^2 = X(I - TT^*) = X - TYT^* = X - TT^*X = (I - TT^*)X = D_{T^*}^2X,$$

and similarly $YD_T^2 = D_T^2Y$. By the spectral theorem we have $XD_{T^*} = D_{T^*}X$ and $YD_T = D_TY$. Let U_T be the unitary rotation matrix associated with T:

$$U_T = \begin{bmatrix} T & D_{T^*} \\ D_T & -T^* \end{bmatrix}.$$

Then

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} U_T = \begin{bmatrix} XT & XD_{T^*} \\ YD_T & -YT^* \end{bmatrix} = \begin{bmatrix} TY & D_{T^*}X \\ D_TY & -T^*X \end{bmatrix} = U_T \begin{bmatrix} Y & 0 \\ 0 & X \end{bmatrix}.$$

Hence

$$U_T^* \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} U_T = \begin{bmatrix} Y & 0 \\ 0 & X \end{bmatrix} \in \mathcal{D}_{n+m}.$$

Since w respects direct sums and unitary equivalence, we have that

$$\begin{bmatrix} w(Y) & 0 \\ 0 & w(X) \end{bmatrix} = w\left(\begin{bmatrix} Y & 0 \\ 0 & X \end{bmatrix}\right) = w\left(U_T^* \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} U_T\right)$$
$$= U_T^* w\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) U_T = U_T^* \begin{bmatrix} w(X) & 0 \\ 0 & w(Y) \end{bmatrix} U_T.$$

This shows that

$$\begin{bmatrix} w(X)T & w(X)D_{T^*} \\ w(Y)D_T & -w(Y)T^* \end{bmatrix} = \begin{bmatrix} w(X) & 0 \\ 0 & w(Y) \end{bmatrix} U_T = U_T \begin{bmatrix} w(Y) & 0 \\ 0 & w(X) \end{bmatrix} =$$
$$= \begin{bmatrix} Tw(Y) & D_{T^*}w(X) \\ D_T w(Y) & -T^* w(X) \end{bmatrix}.$$

Comparing the left-upper corners in the above identity yields $w(X)T = Tw(Y)$, and thus

$$w(X)T_0 = \|T_0\| w(X)T = \|T_0\| T w(Y) = T_0 w(Y)$$

as desired. □

Proof of Theorem 2.1 This is now straightforward. By assumption the function w is graded and respects direct sums. Let $X \in \mathcal{D}_n$ and $T \in \mathbb{C}^{n\times n}$ invertible so that $Y := T^{-1}XT \in \mathcal{D}_n$. Then $XT = TY$, and thus $w(X)T = Tw(Y)$ holds by Proposition 2.2. Therefore, we have

$$w(T^{-1}XT) = T^{-1}Tw(Y) = T^{-1}w(X)T. \qquad \square$$

Remark 2.3 Theorem 2.1 shows that assumptions (NC-i), (NC-ii) and (RNC-iii) imply (NC-iii), that is: For $X \in \mathcal{D}_n$, $S \in \mathbb{C}^{n\times n}$ invertible so that $SXS^{-1} \in \mathcal{D}_n$, we have

$$w(SXS^{-1}) = Sw(X)S^{-1}.$$

An important feature here is that $Y := SXS^{-1} \in \mathcal{D}_n$ implies, in particular, that Y is self-adjoint. In this case, X and Y are not only similar, but also unitarily equivalent. In the finite dimensional case this follows from [13, Problem 4.1.P3]; for completeness we included the result in the current setting as Lemma 2.4 below. Consequently, we have $w(SXS^{-1}) = w(UXU^*) = Uw(X)U^*$. However, to arrive at $w(SXS^{-1}) = Sw(X)S^{-1}$ it still seems necessary to have a result like Proposition 2.2, at least for the case of positive definite similarities.

Lemma 2.4 *Let $Y, X \in \mathcal{V}_n^{\mathrm{sa}}$ and $S \in \mathbb{C}^{n\times n}$ invertible so that $Y = SXS^{-1}$. Then $Y = UXU^*$, where $U \in \mathbb{C}^{n\times n}$ is the unitary part of the polar decomposition of S.*

Proof Let $S = U\Lambda$ be the polar decomposition of S with U unitary and Λ positive definite. Since X and Y are self-adjoint, we have

$$U\Lambda X\Lambda^{-1}U^* = SXS^{-1} = Y = Y^* = (S^*)^{-1}XS^* = U\Lambda^{-1}X\Lambda U^*.$$

Thus

$$\Lambda X\Lambda^{-1} = \Lambda^{-1}X\Lambda, \quad \text{so that} \quad \Lambda^2 X = X\Lambda^2.$$

This implies that $\Lambda X = X\Lambda$, which yields

$$Y = SXS^{-1} = U\Lambda X\Lambda^{-1}U^* = UXU^*,$$

as claimed. □

Example 2.5 It also follows from Theorem 2.1 that real nc functions are only distinguishable from other nc functions by the fact that their domains are contained in $\mathcal{V}_{\rm nc}^{\rm sa}$. Simple examples show that the assumption $\mathcal{D} \subset \mathcal{V}_{\rm nc}^{\rm sa}$ cannot be removed without Theorem 2.1 losing its validity. Any one of the functions

$$w_1(X) = X^*, \quad w_2(X) = (X^*X)^{\frac{1}{2}}$$

can be defined on $\mathcal{V}_{\rm nc}$, where they satisfy (NC-i), (NC-ii) and (RNC-iii) but not (NC-iii), hence they are not nc functions on $\mathcal{V}_{\rm nc}$, but their restrictions to $\mathcal{V}_{\rm nc}^{\rm sa}$ are, by Theorem 2.1.

Example 2.6 For $\mathcal{V} = \mathcal{W} = \mathbb{C}$, so that $\mathcal{V}_{\rm nc}$ and $\mathcal{V}_{\rm nc}^{\rm sa}$ are complex and Hermitian matrices of all sizes, respectively, more intricate examples can easily be constructed. Via the continuous functional calculus, any continuous function w with domain in $\mathbb{R}$ can be extended to a real nc function on the nc set of Hermitian matrices whose spectrum is contained in the domain of w, even when it is not differentiable. Clearly the resulting real nc function is also not differentiable in case w is not.

It is not directly clear how a continuous function of several real variables can be extended to a real nc function, except when the domain is restricted to tuples of commuting matrices. In passing, we note that a (unintentional) non-example is given in [14], where an extension of a function in several real variables to a noncommutative domain is considered, which, after some minor modifications, can be restricted to an nc domain in $\coprod_{n=1}^{\infty}(\mathcal{H}_n)^d$, leading to a non-graded function (it maps $(\mathcal{H}_n)^d$ to $\mathcal{H}_{nd}$) which does satisfy conditions (NC-ii) and (RNC-iii).

Domain Extensions Since a real nc function w with domain $\mathcal{D}$ is an nc function, it follows from Proposition A.3 in [15] that w can be uniquely extended to an nc function, also denoted by w, on the similarity invariant envelop of $\mathcal{D}$:

$$\mathcal{D}^{(\rm si)} := \{SXS^{-1} \colon X \in \mathcal{D}_n,\ S \in \mathbb{C}^{n\times n} \text{ invertible}\}$$

via

$$w(Y) = w(SXS^{-1}) := Sw(X)S^{-1} \quad (Y = SXS^{-1} \in \mathcal{D}^{(\mathrm{si})}).$$

However, in general, $\mathcal{D}^{(\mathrm{si})}$ will not be contained in $\mathcal{V}_{\mathrm{nc}}^{\mathrm{sa}}$, although all operators in $\mathcal{D}^{(\mathrm{si})}$ have real spectrum and the only nilpotent operator in $\mathcal{D}^{(\mathrm{si})}$ is the zero operator 0, assuming $0 \in \mathcal{D}$. In the context of real nc functions it may be more natural to consider the extension of w to the unitary equivalence invariant envelop

$$\mathcal{D}^{(\mathrm{ue})} := \{UXU^* \colon X \in \mathcal{D}_n,\ U \in \mathbb{C}^{n\times n} \text{ unitary}\} = \mathcal{D}^{(\mathrm{si})} \cap \mathcal{V}_{\mathrm{nc}}^{\mathrm{sa}},$$

with w extended as before. The fact that $\mathcal{D}^{(\mathrm{ue})} = \mathcal{D}^{(\mathrm{si})} \cap \mathcal{V}_{\mathrm{nc}}^{\mathrm{sa}}$ follows from Lemma 2.4. As this is just the restriction to $\mathcal{D}^{(\mathrm{ue})}$ of the extension of w to $\mathcal{D}^{(\mathrm{si})}$, clearly we end up with a real nc function extension of w to $\mathcal{D}^{(\mathrm{ue})}$ which is uniquely determined by w.

In [19, 20] real free sets (in a slightly different setting) are nc sets $\mathcal{D}$ that are closed under unitary equivalence and have the following property:

(a) For $X, Y \in \mathcal{V}_{\mathrm{nc}}^{\mathrm{sa}}$ we have $X, Y \in \mathcal{D}$ if and only of $X \oplus Y \in \mathcal{D}$.

One implication is true by the assumption that $\mathcal{D}$ is an nc set, but the other direction need not be true for the unitary equivalence envelop of an nc set contained in $\mathcal{V}_{\mathrm{nc}}^{\mathrm{sa}}$.

Lemma 2.7 *Let $\mathcal{D} \subset \mathcal{V}_{\mathrm{nc}}^{\mathrm{sa}}$ be an nc set. Then the unitary equivalence envelop $\mathcal{D}^{(\mathrm{ue})}$ of $\mathcal{D}$ is a real free set if and only if it is closed under left injective intertwining: If $X \in \mathcal{D}_n^{(\mathrm{ue})}$, $Y \in \mathcal{V}_{\mathrm{nc}}^{\mathrm{sa}}$ and $S \in \mathbb{C}^{n\times m}$ injective so that $XS = SY$, then $Y \in \mathcal{D}^{(\mathrm{ue})}$.*

In [2, Definition 2.4] an nc set which is closed under left injective intertwining is called a full nc set, and such sets play an important role in the study of interpolation theory in the noncommutative Schur-Agler class [2].

Proof of Lemma 2.6 Assume $\mathcal{D}^{(\mathrm{ue})}$ is closed under injective intertwining. Since $\mathcal{D}$ is an nc set, so is $\mathcal{D}^{(\mathrm{ue})}$, by [15, Proposition A.1]. Hence it remains to show that for $X, Y \in \mathcal{V}_{\mathrm{nc}}^{\mathrm{sa}}$ with $X \oplus Y \in \mathcal{D}$ also $X, Y \in \mathcal{D}$. This follows by taking $S = S_1 = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$ and $S = S_2 = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$, respectively, with sizes compatible with the decomposition of $X \oplus Y$. Indeed, clearly S_1 and S_2 are injective and we have $(X \oplus Y)S_1 = S_1 X$ and $(X \oplus Y)S_2 = S_2 Y$. Thus $\mathcal{D}^{(\mathrm{ue})}$ is an nc set in $\mathcal{V}_{\mathrm{nc}}^{\mathrm{sa}}$ which is closed under unitary equivalence and satisfies (a), hence it is a real free set.

For the converse direction, assume $\mathcal{D}^{(\mathrm{ue})}$ is a real free set. Take $X \in \mathcal{D}_n^{(\mathrm{ue})}$, $Y \in \mathcal{V}_{\mathrm{nc}}^{\mathrm{sa}}$ and $S \in \mathbb{C}^{n\times m}$ injective so that $XS = SY$. Since $\mathcal{D}^{(\mathrm{ue})}$ is closed under unitary equivalence and S is injective, without loss of generality $S = \left[\begin{smallmatrix} S_1 \\ 0 \end{smallmatrix}\right]$ with S_1 invertible. Then $XS = SY$ implies $\mathrm{Ran}(S)$ is invariant for X. However, X is self-adjoint, so that $\mathrm{Ran}(S)$ is in fact a reducing subspace for X. Hence $X = X_1 \oplus X_2$ with respect to the same decomposition as for S. Then property (a) implies $X_1 \in (\mathcal{V}^{m\times m})^{\mathrm{sa}}$ is in $\mathcal{D}^{(\mathrm{ue})}$, and $XS = SY$ yields $X_1 S_1 = S_1 Y$, i.e., $X_1 = S_1 Y S_1^{-1}$. Hence X_1 and Y are similar. Since X_1 and Y are self-adjoint, X_1 and Y are also unitarily equivalent, by Lemma 2.4. Hence Y is in $\mathcal{D}^{(\mathrm{ue})}$. □

Remark 2.8 In Chapter 9 of [15], for an nc set $\mathcal{D}$ in $\mathcal{V}_{\text{nc}}$ (as well as in more general settings) the *direct summand extension* of $\mathcal{D}$ is defined as

$$\mathcal{D}_{\text{d.s.e.}} = \{X \in \mathcal{V}_{\text{nc}} \colon X \oplus Y \in \mathcal{D} \text{ for some } Y \in \mathcal{V}_{\text{nc}}\}.$$

Clearly, $\mathcal{D}_{\text{d.s.e.}}$ satisfies property (a) above. It is shown, in [15, Proposition 9.1], that in case $\mathcal{D}$ respects similarities, $\mathcal{D}_{\text{d.s.e.}}$ is right admissible and finitely open, whenever $\mathcal{D}$ is right admissible and finitely open, respectively. Furthermore, by Proposition 9.2 in [15] an nc function on the similarity respecting nc set $\mathcal{D}$ extends uniquely to an nc function on $\mathcal{D}_{\text{d.s.e.}}$. However, in the context of the present paper, we are mainly interested in nc set consisting of self-adjoint operators which may or may not respect unitary equivalence, but will not likely respect similarities.

3 Differentiability of nc Functions

For differentiation of vector-valued functions several notions exist, and these may differ for real and complex vector spaces. We refer to Section III.3 in [12], Section 5.3 in [1] and Sections 2.3 and 2.4 in [21] for elaborate treatments. In this paper we will only encounter Gâteaux (G-)differentiability and Fréchet (F-)differentiability. In the context of nc functions over complex Banach spaces these notions are discussed in Chapter 7 of [15], with a few remarks dedicated to the case of real Banach spaces.

For the remainder of this section, let $\mathcal{V} \subset \mathcal{B}(\mathcal{H})$ and $\mathcal{W} \subset \mathcal{B}(\mathcal{K})$ be operator systems. We start with the definitions of G-differentiability and F-differentiability, not distinguishing whether the field $\mathbb{K}$ we work over is $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, where in the case of $\mathbb{K} = \mathbb{R}$ we consider nc functions with domains contained in $\mathcal{V}_{\text{nc}}^{\text{sa}}$ and for $\mathbb{K} = \mathbb{C}$ the nc functions are assumed to have a domain in $\mathcal{V}_{\text{nc}}$. Now let w be an nc function defined on an open domain $\mathcal{D}$ in $\mathcal{V}_{\text{nc}}^{\text{sa}}$ (for $\mathbb{K} = \mathbb{R}$) or in $\mathcal{V}_{\text{nc}}$ (for $\mathbb{K} = \mathbb{C}$) and taking values in $\mathcal{W}_{\text{nc}}$ (contained in $\mathcal{W}_{\text{nc}}^{\text{sa}}$ if $\mathbb{K} = \mathbb{R}$). Then for each $X \in \mathcal{D}_n$ and $Z \in \mathcal{V}^{n\times n}$ (in $(\mathcal{V}^{n\times n})^{\text{sa}}$ for $\mathbb{K} = \mathbb{R}$) we define the *G-derivative* of w at X in direction Z as the limit

$$Dw(X)(Z) := \lim_{\mathbb{K}\ni t\to 0} \frac{w(X+tZ) - w(X)}{t} = \frac{d}{dt} w(X+tZ)\bigg|_{t=0}, \tag{3.1}$$

provided the limit exists, in which case we say that w is *G-differentiable at X in direction Z*. Furthermore, if w is G-differentiable at $X \in \mathcal{D}$ in all directions Z, then we call w *G-differentiable at X*, and in case w is G-differentiable at all $X \in \mathcal{D}$ we say that w is *G-differentiable*. We shall usually refer to Z as the *directional variable*.

Following [12], we say that the nc function w is *F-differentiable in $X \in \mathcal{D}_n$* in case w is G-differentiable in X, the G-derivative $Dw(X)$ at X is linear and continuous in the directional variable and satisfies

$$\lim_{\|Z\|\to 0} \frac{\|w(X+Z) - w(X) - Dw(X)(Z)\|_n}{\|Z\|_n} = 0. \tag{3.2}$$

Note that if for $X \in \mathcal{D}$ there exists a continuous, linear map $Z \mapsto Dw(X)(Z)$ that satisfies (3.2), then it must satisfy (3.1), so that w is G-differentiable at X. Hence, existence of a continuous, linear map $Z \mapsto Dw(X)(Z)$ satisfying (3.2) can be used as another definition of F-differentiability. Even in case w is F-differentiable, we will refer to (3.1) as the G-derivative of w.

The Case $\mathcal{D} \subset \mathcal{V}_{\mathbf{nc}}$ ($\mathbb{K} = \mathbb{C}$) This case is discussed in detail in Chapter 7 of [15]. We just mention a few specific results relevant to the present paper and to illustrate the contrast with the case of real nc functions. Firstly, as observed in [11, 29], in case the G-derivative of w exists at X, then $Dw(X)(Z)$ is automatically linear in Z (homogeneity is always satisfied, but additivity may fail if $\mathbb{K} = \mathbb{R}$). In fact, if, in addition, $Dw(X)(Z)$ is continuous in Z, by Theorem 2.4 in [30], G-differentiability and F-differentiability of w coincide.

Also, since the domain $\mathcal{D}$ of w is assumed to be open in $\mathcal{V}_{\rm nc}$ it must be right-admissible and hence the difference-differential operator $\Delta w(X, Y)(Z)$ defined via (1.4) exists for all $X \in \mathcal{D}_n$, $Y \in \mathcal{D}_m$ and $Z \in \mathcal{V}^{n\times m}$. By Theorem 7.2 in [15], w is G-differentiable in case w is *locally bounded on slices*, that is, if for any n, $X \in \mathcal{D}_n$ and any $Z \in \mathcal{V}^{n\times n}$ there exists a $\varepsilon > 0$ so that $t \mapsto w(X + tZ)$ is bounded for $|t| < \varepsilon$. Moreover, in that case we have $Dw(X)(Z) = \Delta w(X, X)(Z)$, and hence the G-derivative can be determined algebraically by evaluating w in $\left[\begin{smallmatrix} X & rZ \\ 0 & X \end{smallmatrix}\right]$ for small r. Furthermore, by Theorem 7.4 in [15], w is F-differentiable in case w is *locally bounded*, that is, if for any n, $X \in \mathcal{D}_n$ there exists a $\delta > 0$ so that w is bounded on the set of $Y \in \mathcal{D}_n$ with $\|X - Y\|_n < \delta$.

The Case $\mathcal{D} \subset \mathcal{V}_{\mathbf{nc}}^{\mathbf{sa}}$ ($\mathbb{K} = \mathbb{R}$) The domains of real nc functions are 'nowhere right admissible', hence one cannot in general define the difference-differential operator Δw of a real nc function w in the way it is done for nc functions defined on a right admissible nc set. Nonetheless, Proposition 2.5 in [20] provides a difference formula for real nc functions, provided they are F-differentiable.

As pointed out in Example 2.6, any continuous function with domain in $\mathbb{R}$ can be extended to a real nc function. Clearly G- or F-differentiability will not follow under local boundedness properties; consider, for instance, the function w_2 in Example 2.5. The theory of G- and F-differentiability for functions between real Banach spaces is treated in Section 5.3 in [1] and Sections 2.3 and 2.4 in [21]. It is not the case here that G- and F-differentiability coincide. Also, the G-derivative need not be linear in the directional variable. By Proposition 5.3.4 in [1] or Proposition 2.51 in [21], a sufficient condition under which G-differentiability at a point $X \in \mathcal{D}_n$ implies F-differentiability at X is that the map $Y \mapsto Dw(Y)$ from $\mathcal{D}_n$ into the space of linear operators from $(\mathcal{V}^{n\times n})^{\rm sa}$ to $(\mathcal{W}^{n\times n})^{\rm sa}$ is continuous at X. Even if

w is F-differentiable, there does not appear to be a general way to determine Dw algebraically, since there is no difference-differential operator.

The formula presented in the next proposition can be seen as complementary to the difference formula in [20, Proposition 2.5].

Proposition 3.1 *Let $w : \mathcal{D} \to \mathcal{W}_{nc}^{\mathrm{sa}}$ be a G-differentiable real nc function on an open domain $\mathcal{D} \subset \mathcal{V}_{\mathrm{nc}}^{\mathrm{sa}}$. For $X \in \mathcal{D}_n$ and $Z \in \mathcal{V}_n^{\mathrm{sa}}$, with n arbitrary, we have*

$$Dw\left(\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}\right)\left(\begin{bmatrix} 0 & Z \\ Z & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & Dw(X)(Z) \\ Dw(X)(Z) & 0 \end{bmatrix}.$$

Proof Note that

$$V^* \begin{bmatrix} X + tZ & 0 \\ 0 & X - tZ \end{bmatrix} V = \begin{bmatrix} X & tZ \\ tZ & X \end{bmatrix}, \quad \text{where} \quad t \in \mathbb{R},\ V = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ I & -I \end{bmatrix}.$$

Since $\mathcal{D}$ is open and $X \in \mathcal{D}$, for small t both 2×2 block operator matrices are in $\mathcal{D}$. Hence, because V is unitary, we have

$$\begin{aligned} w\left(\begin{bmatrix} X & tZ \\ tZ & X \end{bmatrix}\right) &= w\left(V^* \begin{bmatrix} X + tZ & 0 \\ 0 & X - tZ \end{bmatrix} V\right) \\ &= V^* \begin{bmatrix} w(X + tZ) & 0 \\ 0 & w(X - tZ) \end{bmatrix} V \\ &= \frac{1}{2} \begin{bmatrix} w(X + tZ) + w(X - tZ) & w(X + tZ) - w(X - tZ) \\ w(X + tZ) - w(X - tZ) & w(X + tZ) + w(X - tZ) \end{bmatrix}. \end{aligned}$$

Using this formula we obtain

$$\begin{aligned} Dw\left(\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}\right)\left(\begin{bmatrix} 0 & Z \\ Z & 0 \end{bmatrix}\right) &= \lim_{t \to 0} \frac{w\left(\begin{bmatrix} X & tZ \\ tZ & X \end{bmatrix}\right) - w\left(\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}\right)}{t} \\ &= \frac{1}{2} \lim_{t \to 0} \begin{bmatrix} \frac{w(X+tZ)+w(X-tZ)-2w(X)}{t} & \frac{w(X+tZ)-w(X-tZ)}{t} \\ \frac{w(X+tZ)-w(X-tZ)}{t} & \frac{w(X+tZ)+w(X-tZ)-2w(X)}{t} \end{bmatrix} \\ &= \frac{1}{2} \lim_{t \to 0} \begin{bmatrix} \frac{w(X+tZ)-w(X)}{t} - \frac{w(X-tZ)-w(X)}{-t} & \frac{w(X+tZ)-w(X)}{t} + \frac{w(X-tZ)-w(X)}{-t} \\ \frac{w(X+tZ)-w(X)}{t} + \frac{w(X-tZ)-w(X)}{-t} & \frac{w(X+tZ)-w(X)}{t} - \frac{w(X-tZ)-w(X)}{-t} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} Dw(X)(Z) - Dw(X)(Z) & Dw(X)(Z) + Dw(X)(Z) \\ Dw(X)(Z) + Dw(X)(Z) & Dw(X)(Z) - Dw(X)(Z) \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 0 & Dw(X)(Z) \\ Dw(X)(Z) & 0 \end{bmatrix}. \qquad \square$$

4 Real and Imaginary Part of an nc Function

Throughout this section, let $\mathcal{V} \subset \mathcal{B}(\mathcal{H})$ and $\mathcal{W} \subset \mathcal{B}(\mathcal{K})$ be operator systems and let f be an nc function with domain $\mathcal{D}_f \subset \mathcal{V}_{\rm nc}$. As in the introduction, we define the real and imaginary parts of f as

$$u : \mathcal{D}_u \to \mathcal{W}_{\rm nc}^{\rm sa}, \quad v : \mathcal{D}_v \to \mathcal{W}_{\rm nc}^{\rm sa}, \text{ with} \tag{4.1}$$

$$\mathcal{D}_u = \mathcal{D}_v = \mathcal{D} := \coprod_{n=1}^{\infty} \{(A, B) \colon A, B \in (\mathcal{V}^{n\times n})^{\rm sa},\ A + iB \in \mathcal{D}_f\} \subset (\mathcal{V}_{\rm nc}^{\rm sa})^2,$$

with u and v defined for $(A, B) \in \mathcal{D}$ by

$$\begin{aligned} u(A, B) &:= \operatorname{Re} f(A + iB) = \frac{1}{2}(f(A + iB) + f(A + iB)^*), \\ v(A, B) &:= \operatorname{Im} f(A + iB) = \frac{1}{2i}(f(A + iB) - f(A + iB)^*). \end{aligned} \tag{4.2}$$

In particular, u, v and f satisfy (1.6). The following theorem is the main result of this section.

Theorem 4.1 *Let f be a G-differentiable nc function defined on an open nc set $\mathcal{D}_f \subset \mathcal{V}_{\rm nc}$ and define u and v as in (4.1) and (4.2). Then u and v are G-differentiable real nc functions, whose G-derivatives at $(A, B) \in \mathcal{D}_n$ in direction $Z = (Z_1, Z_2) \in ((\mathcal{V}^{n\times n})^{\rm sa})^2$, for any n, are given by*

$$\begin{aligned} Du(A, B)(Z_1, Z_2) &= \operatorname{Re} Df(A + iB)(Z_1 + iZ_2), \\ Dv(A, B)(Z_1, Z_2) &= \operatorname{Im} Df(A + iB)(Z_1 + iZ_2), \end{aligned} \tag{4.3}$$

and Du, Dv are $\mathbb{R}$-linear in the directional variable and satisfy the nc Cauchy-Riemann equations: For all $n \in \mathbb{N}$, $(A, B) \in \mathcal{D}_n$ and $Z_1, Z_2 \in (\mathcal{V}^{n\times n})^{\rm sa}$

$$Du(A, B)(Z_1, Z_2) = Dv(A, B)(-Z_2, Z_1). \tag{4.4}$$

Finally, if f is F-differentiable, then u and v are F-differentiable as well.

In order to prove this result we first prove a lemma that will also be useful in the sequel. The result may be well-known, but we could not find it in the literature, hence we add a proof for completeness.

Lemma 4.2 *For $Z = Z_1 + iZ_2 \in \mathcal{V}_{\rm nc}$ with $Z_1, Z_2 \in \mathcal{V}_{\rm nc}^{\rm sa}$ we have*

$$\|(Z_1, Z_2)\|_{{\rm sa},n} \leq \|Z_1 + iZ_2\|_n \leq 2\|(Z_1, Z_2)\|_{\rm sa,n}. \tag{4.5}$$

Proof Set

$$\delta = \|Z_1 + iZ_2\|_n = \|Z\|_n, \quad \rho = \|(Z_1, Z_2)\|_{{\rm sa},n} = \max\{\|Z_1\|_{{\rm sa},n}, \|Z_2\|_{{\rm sa},n}\}.$$

Then

$$\delta^2 I_n \geq Z^*Z = Z_1^2 + Z_2^2 + [iZ_1, Z_2] \quad \text{and} \quad \delta^2 I_n \geq ZZ^* = Z_1^2 + Z_2^2 - [iZ_1, Z_2].$$

Here $[T_1, T_2]$ is the commutator of the operators T_1, T_2, i.e., $[T_1, T_2] = T_1T_2 - T_2T_1$. Taking the average of the above two inequalities gives

$$\delta^2 I_n \geq Z_1^2 + Z_2^2.$$

Hence $Z_j^2 \leq \delta^2 I_n$, or equivalently, $\|Z_j\|_{{\rm sa},n} \leq \delta$ for both $j = 1, 2$. Therefore, we have $\|(Z_1, Z_2)\|_{{\rm sa},n} \leq \delta = \|Z_1 + iZ_2\|_n$. For the second inequality, note that $Z_j^2 \leq \rho^2 I_n$ for $j = 1, 2$. Also, we have

$$\|[iZ_1, Z_2]\|_{{\rm sa},n} = \|Z_1Z_2 - Z_2Z_1\|_{{\rm sa},n} \leq 2\|Z_1\|_{{\rm sa},n}\|Z_2\|_{{\rm sa},n} \leq 2\rho^2.$$

This implies $-2\rho^2 I_n \leq [iZ_1, Z_2] \leq 2\rho^2 I_n$, since $[iZ_1, Z_2] \in (\mathcal{V}^{n\times n})^{\rm sa}$. We then obtain

$$0 \leq Z^*Z = Z_1^2 + Z_2^2 + [iZ_1, Z_2] \leq 4\rho^2 I_n,$$

so that $\|Z\|_n \leq 2\rho = 2\|(Z_1, Z_2)\|_{{\rm sa},n}$. □

Since the inequalities in (4.6) provide a comparison between the norms in $\mathcal{V}_{\rm nc}$ and $\mathcal{V}_{\rm nc}^{\rm sa}$, the following corollary is immediate.

Corollary 4.3 *The nc set $\mathcal{D}_f$ is open if and only if $\mathcal{D}$ is open.*

By applying the inequalities of Lemma 4.2 to both the denominator and numerator, we obtain the following corollary.

Corollary 4.4 *Let $Z_1, Z_2 \in (\mathcal{V}^{n\times n})^{\rm sa}$ and $T_1, T_2 \in (\mathcal{W}^{m\times m})^{\rm sa}$. Then*

$$\frac{1}{2}\frac{\|(T_1, T_2)\|_{{\rm sa},m}}{\|(Z_1, Z_2)\|_{{\rm sa},n}} \leq \frac{\|T_1 + iT_2\|_m}{\|Z_1 + iZ_2\|_n} \leq 2\frac{\|(T_1, T_2)\|_{{\rm sa},m}}{\|(Z_1, Z_2)\|_{{\rm sa},n}}. \tag{4.6}$$

Proof of Theorem 4.1 The proof is divided into four parts.

Part 1: u and v Are Real nc Functions It is straightforward to check that u and v are graded and respect direct sums, since f has these properties. Clearly $\mathcal{D}$ is contained in $(\mathcal{V}_{\rm nc}^{\rm sa})^2$. It remains to verify that u and v respect unitary equivalence. Let $(A, B) \in \mathcal{D}_n$ and $U \in \mathbb{C}^{n\times n}$ unitary so that $(UAU^*, UBU^*) \in \mathcal{D}_n$. Set $X = A + iB \in \mathcal{D}_f$. By definition of $\mathcal{D}$ we have $UXU^* \in \mathcal{D}_f$, and since f respects similarities, and hence unitary equivalence, we have

$$f(UXU^*) = Uf(X)U^*.$$

The left hand side specifies to

$$f(UXU^*) = f(UAU^* + iUBU^*) = u(UAU^*, UBU^*) + iv(UAU^*, UBU^*),$$

while on the right hand side we get

$$Uf(X)U^* = Uf(A + iB)U^* = Uu(A, B)U^* + iUv(A, B)U^*.$$

Since the values of u and v are self-adjoint and $(\mathcal{V}^{n\times n})^{\rm sa}$ is closed under unitary equivalence, it follows that

$$u(UAU^*, UBU^*) = Uu(A, B)U^* \quad \text{and} \quad v(UAU^*, UBU^*) = Uv(A, B)U^*.$$

Hence, u and v respect unitary equivalence.

Part 2: Proof of (4.3) Let $X = A + iB \in \mathcal{D}_{f,n}$, $Z = Z_1 + iZ_2 \in \mathcal{V}^{n\times n}$ with $A, B, Z_1, Z_2 \in \mathcal{V}_{\rm nc}^{\rm sa}$. Assume f is G-differentiable at X in direction Z. In this part we show that u and v are G-differentiable at (A, B) in the direction (Z_1, Z_2) and that their G-derivatives satisfy

$$Df(A + iB)(Z_1 + iZ_2) = Du(A, B)(Z_1, Z_2) + iDv(A, B)(Z_1, Z_2). \tag{4.7}$$

This proves (4.3) and shows that u and v are G-differentiable in case f is G-differentiable.

To see that our claim holds, note that for $0 \neq t \in \mathbb{R}$ we have

$$\begin{aligned}\frac{f(X + tZ) - f(X)}{t} &= \frac{f(A + iB + t(Z_1 + iZ_2)) - f(A + iB)}{t} \\ &= \frac{u(A + tZ_1, B + tZ_2) + iv(A + tZ_1, B + tZ_2) - u(A, B) - iv(A, B)}{t} \\ &= \frac{u(A + tZ_1, B + tZ_2) - u(A, B)}{t} + i\frac{v(A + tZ_1, B + tZ_2) - v(A, B)}{t}.\end{aligned}$$

The result follows by letting t go to 0, and noting that in the right most side of the above identities the limits of the real and imaginary parts are independent. From (4.3) and the fact that Df is $\mathbb{C}$-linear in the directional variable, it is straightforward to see that Du and Dv are $\mathbb{R}$-linear in their directional variable.

Part 3: Cauchy-Riemann Equations The proof follows along the same lines as the classical complex analysis proof. For $X = A + iB$, $Z = Z_1 + iZ_2$ and $h \in \mathbb{R}$ we have

$$\begin{aligned} f(X + ihZ) - f(X) &= f(A + iB + ih(Z_1 + iZ_2) - f(A + iB) \\ &= f(A - hZ_2 + i(B + hZ_1)) - f(A + iB) \\ &= u(A - hZ_2, B + hZ_1) + iv(A - hZ_2, B + hZ_1) - u(A, B) - iv(A, B) \\ &= u(A - hZ_2, B + hZ_1) - u(A, B) + i(v(A - hZ_2, B + hZ_1) - v(A, B)). \end{aligned}$$

Dividing by ih and taking $h \to 0$ we obtain

$$\begin{aligned} Df(X)(Z) &= \lim_{h\to 0} \frac{f(X + ihZ) - f(X)}{ih} \\ &= \lim_{h\to 0} \frac{v(A - hZ_2, B + hZ_1) - v(A, B)}{h} + \\ &\qquad - i \lim_{h\to 0} \frac{u(A - hZ_2, B + hZ_1) - u(A, B)}{h} \\ &= Dv(A, B)(-Z_2, Z_1) - iDu(A, B)(-Z_2, Z_1). \end{aligned}$$

Comparing with (4.7) provides the desired equations.

Part 4: F-differentiability Assume f is F-differentiable. This implies that f is G-differentiable and hence u and v are G-differentiable, by Part 2. Since Df is $\mathbb{C}$-linear in the directional variable, it is clear from (4.3) that Du and Dv are $\mathbb{R}$-linear in the directional variable. Now let $X = A + iB$ with $(A, B) \in \mathcal{D}_n$ and $Z = Z_1 + iZ_2$ with $Z_1, Z_2 \in (\mathcal{V}^{n\times n})^{\mathrm{sa}}$. Then

$$\begin{aligned} & f(X + Z) - f(X) - Df(X)(Z) = \\ &\quad = u(A + Z_1, B + Z_2) - u(A, B) - Du(A, B)(Z_1, Z_2) + \\ &\qquad + i(v(A + Z_1, B + Z_2) - v(A, B) - Dv(A, B)(Z_1, Z_2)). \end{aligned}$$

Now apply Corollary 4.4 with Z_1 and Z_2 as above and

$$\begin{aligned} T_1 &= u(A + tZ_1, B + Z_2) - u(A, B) - Du(A, B)(Z_1, Z_2), \\ T_2 &= v(A + tZ_1, B + Z_2) - v(A, B) - Dv(A, B)(Z_1, Z_2), \end{aligned} \tag{4.8}$$

and note that $Z \to 0$ if and only if $(Z_1, Z_2) \to 0$, by Lemma 4.2. It then follows that

$$\lim_{\|Z\|_n \to 0} \frac{\|f(X+Z) - F(X) - Df(X)(Z)\|_n}{\|Z\|_n} = 0 \tag{4.9}$$

holds if and only if

$$\lim_{\|(Z_1,Z_2)\|_{\mathrm{sa},n} \to 0} \frac{\|u(A+tZ_1, B+tZ_2) - u(A,B) - Du(A,B)(Z_1,Z_2)\|_{\mathrm{sa},n}}{\|(Z_1,Z_2)\|_{\mathrm{sa},n}} = 0$$

and

$$\lim_{\|(Z_1,Z_2)\|_{\mathrm{sa},n} \to 0} \frac{\|v(A+tZ_1, B+tZ_2) - v(A,B) - Dv(A,B)(Z_1,Z_2)\|_{\mathrm{sa},n}}{\|(Z_1,Z_2)\|_{\mathrm{sa},n}} = 0.$$

In particular, since (4.9) holds, and $(A, B) \in \mathcal{D}_n$ and $Z_1, Z_2 \in (\mathcal{V}^{n\times n})^{\mathrm{sa}}$ were chosen arbitrarily, it follows that u and v are F-differentiable. □

The fact that the G-derivative of a G-differentiable nc function on a complex-open domain (and hence right-admissible) can be computed algebraically, via block upper triangular matrices, provides additional structure for its real and imaginary parts, which enables us to compute their G-derivatives algebraically as well.

Proposition 4.5 *Let f be an nc function defined on an open nc set $\mathcal{D}_f \subset \mathcal{V}_{\mathrm{nc}}$ and define u and v as in (4.1)–(4.2). Let $X = A + iB \in \mathcal{D}_{f,n}$ and $Y = C + iD \in \mathcal{D}_{f,m}$ and $Z \in \mathcal{V}^{n\times m}$ such that $\left[\begin{smallmatrix} X & Z \\ 0 & Y \end{smallmatrix}\right] \in \mathcal{D}_{f,n+m}$. Then*

$$\left(\begin{bmatrix} A & \frac{1}{2}Z \\ \frac{1}{2}Z^* & C \end{bmatrix}, \begin{bmatrix} B & -\frac{i}{2}Z \\ \frac{i}{2}Z^* & D \end{bmatrix} \right) \in \mathcal{D} \tag{4.10}$$

and there exist operators $T_{X,Y,1}$ and $T_{X,Y,2}$ from $\mathcal{V}^{n\times m}$ to $\mathcal{W}^{n\times m}$ so that

$$\begin{aligned} u\left(\begin{bmatrix} A & \frac{1}{2}Z \\ \frac{1}{2}Z^* & C \end{bmatrix}, \begin{bmatrix} B & -\frac{i}{2}Z \\ \frac{i}{2}Z^* & D \end{bmatrix} \right) &= \begin{bmatrix} u(A,B) & T_{X,Y,1} \\ T^*_{X,Y,1} & u(C,D) \end{bmatrix}, \\ v\left(\begin{bmatrix} A & \frac{1}{2}Z \\ \frac{1}{2}Z^* & C \end{bmatrix}, \begin{bmatrix} B & -\frac{i}{2}Z \\ \frac{i}{2}Z^* & D \end{bmatrix} \right) &= \begin{bmatrix} v(A,B) & T_{X,Y,2} \\ T^*_{X,Y,2} & v(C,D) \end{bmatrix}. \end{aligned} \tag{4.11}$$

Moreover, if $X = Y$, $Z = Z_1 + iZ_2$ with $Z_1, Z_2 \in (\mathcal{V}^{n\times n})^{\mathrm{sa}}$ and f is locally bounded on slices, then

$$Du(A,B)(Z_1,Z_2) = T_{X,Y,1} + T^*_{X,Y,1}, \quad Dv(A,B)(Z_1,Z_2) = T_{X,Y,2} + T^*_{X,Y,2}.$$

Proof The decomposition

$$\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} = \begin{bmatrix} A + iB & Z_1 + iZ_2 \\ 0 & C + iD \end{bmatrix} = \begin{bmatrix} A & \frac{1}{2}Z \\ \frac{1}{2}Z^* & C \end{bmatrix} + i \begin{bmatrix} B & -\frac{i}{2}Z \\ \frac{i}{2}Z^* & D \end{bmatrix}, \tag{4.12}$$

together with $\left[\begin{smallmatrix} X & Z \\ 0 & Y \end{smallmatrix}\right] \in \mathcal{D}_{f,n+m}$ yields (4.10). Since f is an nc function, we have

$$f\left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}\right) = \begin{bmatrix} f(X) & \Delta f(X, Y)(Z) \\ 0 & f(Y) \end{bmatrix},$$

with $\Delta f(X, Y)(Z)$ the right nc difference-differential operator applied to f, at the point (X, Y) and direction Z. Note that

$$\begin{aligned} \begin{bmatrix} f(X) & \Delta f(X, Y)(Z) \\ 0 & f(Y) \end{bmatrix} &= \begin{bmatrix} \frac{1}{2}(f(X) + f(X)^*) & \frac{1}{2}\Delta f(X, Y)(Z) \\ \frac{1}{2}\Delta f(X, Y)(Z)^* & \frac{1}{2}(f(Y) + f(Y)^*) \end{bmatrix} + \\ &\quad + i \begin{bmatrix} -\frac{i}{2}(f(X) - f(X)^*) & -\frac{i}{2}\Delta f(X, Y)(Z) \\ \frac{i}{2}\Delta f(X, Y)(Z)^* & -\frac{i}{2}(f(Y) - f(Y)^*) \end{bmatrix} \\ &= \begin{bmatrix} u(A, B) & \frac{1}{2}\Delta f(X, Y)(Z) \\ \frac{1}{2}\Delta f(X, Y)(Z)^* & u(C, D) \end{bmatrix} + \\ &\quad + i \begin{bmatrix} v(A, B) & -\frac{i}{2}\Delta f(X, Y)(Z) \\ \frac{i}{2}\Delta f(X, Y)(Z)^* & v(C, D) \end{bmatrix}. \end{aligned}$$

This formula for $f(\left[\begin{smallmatrix} X & Z \\ 0 & Y \end{smallmatrix}\right])$ together with (4.12) proves (4.11), where we take $T_{X,Y,1} = \frac{1}{2}\Delta f(X, Y)(Z)$ and $T_{X,Y,2} = -\frac{1}{2}\Delta f(X, Y)(Z)$.

Now assume $X = Y$ and f is locally bounded on slices. Then f is G-differentiable and $\Delta f(X, Y)(Z) = Df(X)(Z)$. It now follows by Theorem 4.1 that

$$T_{X,Y,1} + T_{X,Y,1}^* = \operatorname{Re} Df(X)(Z) = Du(A, B)(Z_1, Z_2),$$

and, similarly, $Dv(A, B)(Z_1, Z_2) = T_{X,Y,2} + T_{X,Y,2}^*$. □

Not all real nc functions "respect diagonals" as in (4.11). Also, one may wonder whether (4.11) in some form extends beyond points of the form (4.10) in case u and v are the real and imaginary parts of an nc function. This is also not the case in general. We illustrate this in the following example.

Example 4.6 Consider the following three real nc functions

$$u(A, B) = A^2 - B^2, \quad v(A, B) = AB + BA, \quad w(A, B) = A^2 \quad ((A, B) \in (\mathcal{V}_{\text{nc}}^{\text{sa}})^2).$$

Then u and v are the real and imaginary part of the nc function $f(X) = X^2$. For an arbitrary 2×2 block point

$$(E, F) := \left(\begin{bmatrix} A & Z_1 \\ Z_1^* & C \end{bmatrix}, \begin{bmatrix} B & Z_2 \\ Z_2^* & D \end{bmatrix} \right) \in (\mathcal{V}_{\text{nc}}^{\text{sa}})^2$$

we obtain:

$$u(E, F) = \begin{bmatrix} A^2 - B^2 + Z_1 Z_1^* - Z_2 Z_2^* & AZ_1 - BZ_2 + Z_1 C - Z_2 D \\ Z_1^* A - Z_2^* B + C Z_1^* - D Z_2^* & C^2 - D^2 + Z_1^* Z_1 - Z_2^* Z_2 \end{bmatrix},$$

$$v(E, F) = \begin{bmatrix} AB + BA + Z_1 Z_2^* + Z_2 Z_1^* & BZ_1 + AZ_2 + Z_2 C + Z_1 D \\ Z_1^* B + Z_2^* A + C Z_2^* + D Z_1^* & CD + DC + Z_1^* Z_2 + Z_2^* Z_1 \end{bmatrix},$$

$$w(E, F) = \begin{bmatrix} A^2 + Z_1 Z_1^* & AZ_1 + Z_1 C \\ Z_1^* A + C Z_1^* & C^2 + Z_1 Z_1^* \end{bmatrix}.$$

It follows that $u(E, F) = \left[\begin{smallmatrix} u(A,B) & * \\ * & u(C,D) \end{smallmatrix}\right]$ holds if and only if

$$Z_1 Z_1^* = Z_2 Z_2^* \quad \text{and} \quad Z_1^* Z_1 = Z_2^* Z_2, \tag{4.13}$$

while $v(E, F) = \left[\begin{smallmatrix} v(A,B) & * \\ * & v(C,D) \end{smallmatrix}\right]$ holds if and only if

$$Z_1 Z_2^* = -Z_2 Z_1^* \quad \text{and} \quad Z_1^* Z_2 = -Z_2^* Z_1. \tag{4.14}$$

Both conditions are true in case $Z_2 = \pm i Z_1$. Conversely, these conditions on Z_1 and Z_2 together imply $Z_2 = \pm i Z_1$, but, in general, neither implies $Z_2 = \pm i Z_1$ by itself. Indeed, the identities in (4.13) imply that the kernels and co-kernels of Z_1 and Z_2 coincide, so that we can reduce to the case where Z_1 and Z_2 are invertible. In that case, by Douglas' Lemma, (4.13) is equivalent to the existence of unitary operators U and V so that $Z_1 = UZ_2 = Z_2 V$. Assume U and V are like this, and Z_1, Z_2 invertible. Then (4.14) implies

$$Z_2 Z_2^* U Z_2 Z_2^* = Z_2 Z_2^* Z_1 Z_2^* = -Z_2 Z_1^* Z_2 Z_2^* = -Z_2 Z_2^* U^* Z_2 Z_2^*.$$

However, Z_2 is invertible, hence $Z_2 Z_2^*$ is invertible. Thus we find that $U = -U^*$, which implies $U = \pm i I$. Hence $Z_1 = \pm i Z_2$.

On the other hand, we have $w(E, F) = \left[\begin{smallmatrix} w(A,B) & * \\ * & w(C,D) \end{smallmatrix}\right]$ precisely when $Z_1 = 0$. Hence (4.11) holds with u or v replaced by w if and only if $Z = 0$, which is true for any real nc function.

5 Cauchy-Riemann Equations: Sufficiency

In this section we prove Theorem 1.1. Throughout, let

$$u : \mathcal{D}_u \to \mathcal{V}_{\mathrm{nc}}^{\mathrm{sa}} \quad \text{and} \quad v : \mathcal{D}_v \to \mathcal{V}_{\mathrm{nc}}^{\mathrm{sa}} \tag{5.1}$$

be real nc functions. For notational convenience we introduce the nc set

$$\mathcal{D} := \mathcal{D}_u \cap \mathcal{D}_v.$$

Now we define f on $\mathcal{D}_f := \{A + iB \colon (A, B) \in \mathcal{D}\}$ by

$$f(A + iB) = u(A, B) + iv(A, B) \quad (A + iB \in \mathcal{D}_f). \tag{5.2}$$

It is easy to see that f is graded, respects direct sums as well as unitary equivalence, since u and v have these properties. However, it is not necessarily the case that f respects similarities, despite the fact that u and v do. The following proposition sums up the properties that f has without further assumptions on u and v (except G-differentiability in the last part). The claims follow directly from (5.2), hence we omit the proof.

Proposition 5.1 *Let u and v be real nc functions as in* (5.1) *and define f as in* (5.2)*. Then f is graded, respects direct sums and respects unitary equivalence. Moreover, in case $X = A + iB$ with $(A, B) \in \mathcal{D}_n$, $Z = Z_1 + iZ_2$ with $Z_1, Z_2 \in (\mathcal{V}^{n\times n})^{\mathrm{sa}}$, for any $n \in \mathbb{N}$, and u and v are G-differentiable at (A, B) in direction (Z_1, Z_2), then*

$$\lim_{\mathbb{R}\ni t\to 0} \frac{f(X + tZ) - f(X)}{t} = Du(A, B)(Z_1, Z_2) + iDv(A, B)(Z_1, Z_2). \tag{5.3}$$

Remark 5.2 Without additional assumptions on u and v it is possible to prove something slightly stronger than the fact that f respects unitary equivalence. If $X = A+iB \in \mathcal{D}_{f,n}$ and $S \in \mathbb{C}^{n\times n}$ is invertible, such that $C := SAS^{-1}$ and $D := SBS^{-1}$ are in $(\mathcal{V}^{n\times n})^{\mathrm{sa}}$, then it still follows easily that $f(SXS^{-1}) = Sf(X)S^{-1}$, using the fact that u and v respect similarity. Note that in this case (A, B) and (C, D) are not only similar via S, but also unitarily equivalent via the unitary matrix in the polar decomposition of S, cf., Remark 2.3. In general, of course, it will not be the case that C and D are self-adjoint.

To prove, under the conditions of Theorem 1.1, that f respects similarity, and hence is an nc function, we will use Lemma 2.3 of [20]. To apply this lemma, we need to prove that f has the following two properties:

(i) f is F-differentiable;
(ii) the following identity holds

$$Df(X)([T, X]) = [T, f(X)], \quad X \in \mathcal{D}_{f,n}, \ T \in \mathbb{C}^{n\times n}, \ n = 1, 2, \ldots. \tag{5.4}$$

As before, $[S, Q]$ denotes the commutator of the operators S and Q. Note that if S and Q are self-adjoint, then $[S, Q]$ is skew-adjoint, and hence $[iS, Q] = i[S, Q]$ is self-adjoint.

That nc functions satisfy (5.4) is a consequence of the first order difference formula obtained in [15, Theorem 2.11], which in fact goes back to the work of J.L. Taylor in [28].

To achieve more than in Proposition 5.1 we require the nc Cauchy-Riemann equations (1.7) which, for convenience, we recall here: For $n = 1, 2, \dots$

$$Du(A, B)(Z_1, Z_2) = Dv(A, B)(-Z_2, Z_1),\ (A, B) \in \mathcal{D}_n,\ Z_1, Z_2 \in (\mathcal{V}^{n\times n})^{\mathrm{sa}}. \tag{5.5}$$

From Proposition 5.1 it is clear what the G-derivative of f should be in case f is F-differentiable. For $X = A + iB \in \mathcal{D}_{f,n}$ and $Z_1 + iZ_2 \in \mathcal{V}^{n\times n}$ we define

$$\widetilde{D} f(A + iB)(Z_1 + iZ_2) := Du(A, B)(Z_1, Z_2) + i Dv(A, B)(Z_1, Z_2), \tag{5.6}$$

provided the G-derivatives of u and v exist in (A, B). As a first step we show that $\widetilde{D} f(X)(Z)$ is $\mathbb{C}$-linear in Z.

Lemma 5.3 *Let u and v be G-differentiable, real nc functions that satisfy the nc Cauchy-Riemann equations (5.5) and assume that Du, Dv are $\mathbb{R}$-linear in the directional variable. Then the map $\widetilde{D} f(X)(Z)$ defined in (5.6) is $\mathbb{C}$-linear in the directional variable Z.*

Proof Since the maps Du and Dv are $\mathbb{R}$-linear in the directional variable, we have that $\widetilde{D} f$ is additive and $\mathbb{R}$-homogeneous in the directional variable. Write $z \in \mathbb{C}$ as $z = re^{i\theta}$ with $r \geq 0$ and $\theta \in [0, 2\pi]$. Note that

$$\begin{aligned} e^{i\theta} Z &= (\cos\theta + i \sin\theta)(Z_1 + iZ_2) \\ &= (Z_1 \cos\theta - Z_2 \sin\theta) + i(Z_1 \sin\theta + Z_2 \cos\theta). \end{aligned}$$

Set $Z_{1,\theta} := Z_1 \cos\theta - Z_2 \sin\theta$ and $Z_{2,\theta} := Z_1 \sin\theta + Z_2 \cos\theta$. It follows that

$$\begin{aligned} \widetilde{D} f(X)(zZ) &= Du(A, B)(rZ_{1,\theta}, rZ_{2,\theta}) + i Dv(A, B)(rZ_{1,\theta}, rZ_{2,\theta}) \\ &= r(Du(A, B)(Z_{1,\theta}, Z_{2,\theta}) + i Dv(A, B)(Z_{1,\theta}, Z_{2,\theta})). \end{aligned} \tag{5.7}$$

Using that G-derivatives Du and Dv are $\mathbb{R}$-linear in the directional variables together with the Cauchy-Riemann equations (5.5) yields

$$\begin{aligned} Du(A, B)(Z_{1,\theta}, Z_{2,\theta}) &= \cos\theta\, Du(A, B)(Z_1, Z_2) + \sin\theta\, Du(A, B)(-Z_2, Z_1) \\ &= \cos\theta\, Du(A, B)(Z_1, Z_2) - \sin\theta\, Dv(A, B)(Z_1, Z_2). \end{aligned}$$

Similarly, we have

$$\begin{aligned}Dv(A,B)(Z_{1,\theta},Z_{2,\theta}) &= \cos\theta\, Dv(A,B)(Z_1,Z_2)+\sin\theta\, Dv(A,B)(-Z_2,Z_1)\\ &= \cos\theta\, Dv(A,B)(Z_1,Z_2)+\sin\theta\, Du(A,B)(Z_1,Z_2).\end{aligned}$$

Combining these formulas shows

$$\begin{aligned}&Du(A,B)(Z_{1,\theta},Z_{2,\theta})+i\,Dv(A,B)(Z_{1,\theta},Z_{2,\theta}) =\\ &\quad= (\cos\theta+i\sin\theta)Du(A,B)(Z_1,Z_2)+((\cos\theta+i\sin\theta))i\,Dv(A,B)(Z_1,Z_2)\\ &\quad= e^{i\theta}(Du(A,B)(Z_1,Z_2)+i\,Dv(A,B)(Z_1,Z_2)).\end{aligned} \tag{5.8}$$

Together with (5.7) this yields

$$\widetilde{D}f(X)(zZ) = z(Du(A,B)(Z_1,Z_2)+i\,Dv(A,B)(Z_1,Z_2)),$$

so that $\widetilde{D}$ is $\mathbb{C}$-homogeneous in the directional variable, and hence $\mathbb{C}$-linear. □

With linearity out of the way, it is straightforward to prove f is F-differentiable in case u and v are F-differentiable.

Lemma 5.4 *Let u and v be F-differentiable real nc functions that satisfy the nc Cauchy-Riemann equations (5.5). Then f given by (5.2) is F-differentiable with G-derivative given by $Df(X)(Z) = \widetilde{D}f(X)(Z)$ as in (5.6).*

Proof The proof is similar to the last part of the proof of Theorem 4.1. Since u and v are F-differentiable, they are G-differentiable, and thus $\widetilde{D}f$ is $\mathbb{C}$-linear in the directional variable. To see that f is F-differentiable, note that for $X = A + iB \in \mathcal{D}_{f,n}$ and $Z = Z_1 + iZ_2$, $Z_1, Z_2 \in (\mathcal{V}^{n\times n})^{\text{sa}}$, we have

$$\begin{aligned}&f(X+Z)-f(X)-\widetilde{D}f(X)(Z) =\\ &\quad= (u(A+Z_1,B+Z_2)-u(A,B)-Du(A,B)(Z_1,Z_2))+\\ &\qquad + i(v(A+Z_1,B+Z_2)-v(A,B)-Dv(A,B)(Z_1,Z_2)).\end{aligned}$$

Using T_1 and T_2 as in (4.8) the same argument applies, in the opposite direction, to conclude that F-differentiability of u and v implies F-differentiability of f. □

Lemma 5.5 *Let u and v be F-differentiable, real nc functions that satisfy the nc Cauchy-Riemann equations (5.5). Define f as in (5.2). Then (5.4) holds.*

Proof Let $X = A + iB$ and $T = T_1 + iT_2$. Then

$$[T,X] = ([iT_1,B]+[iT_2,A])+i([iT_1,-A]+[iT_2,B]).$$

Set $Z_1 = [iT_1, B] + [iT_2, A]$ and $Z_2 = [iT_1, -A] + [iT_2, B]$. By Lemma 5.4 we obtain

$$Df(X)([T, X]) = Du(A, B)(Z_1, Z_2) + iDv(A, B)(Z_1, Z_2).$$

Note that

$$\begin{aligned} Du(A, B)(Z_1, Z_2) &= Du(A, B)([iT_1, B] + [iT_2, A], [iT_1, -A] + [iT_2, B]) \\ &= Du(A, B)([iT_2, A], [iT_2, B]) + Du(A, B)([iT_1, B], [iT_1, -A]) \\ &= Du(A, B)([iT_2, (A, B)]) + Du(A, B)([iT_1, (B, -A)]). \end{aligned}$$

Applying the Cauchy-Riemann equations (5.5) to the second summand gives

$$Du(A, B)(Z_1, Z_2) = Du(A, B)([iT_2, (A, B)]) + Dv(A, B)([iT_1, (A, B)]).$$

Now use that Part (a) Lemma 2.3 of [20] applies to u and v. This yields

$$Du(A, B)(Z_1, Z_2) = [iT_2, u(A, B)] + [iT_1, v(A, B)].$$

Similarly, for $Dv(A, B)(Z_1, Z_2)$ we get

$$\begin{aligned} Dv(A, B)(Z_1, Z_2) &= Dv(A, B)([iT_2, (A, B)]) + Dv(A, B)([iT_1, (B, -A)]) \\ &= Dv(A, B)([iT_2, (A, B)]) + Dv(A, B)([iT_1, (-A, -B)]) \\ &= Dv(A, B)([iT_2, (A, B)]) - Du(A, B)([iT_1, (A, B)]) \\ &= [iT_2, v(A, B)] - [iT_1, u(A, B)]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &Df(X)([T, X]) = \\ &= [iT_2, u(A, B)] + [iT_1, v(A, B)] + i([iT_2, v(A, B)] - [iT_1, u(A, B)]) \\ &= [iT_2, u(A, B)] - i[iT_1, u(A, B)] + [iT_1, v(A, B)] + i[iT_2, v(A, B)] \\ &= [T_1 + iT_2, u(A, B)] + [T_1 + iT_2, iv(A, B)] \\ &= [T, u(A, B) + iv(A, B)] = [T, f(X)], \end{aligned}$$

as claimed. □

Proof of Theorem 1.1 The proof of this theorem is now straightforward. The fact that f is graded and respects direct sums follows from Proposition 5.1. Lemma 5.4 yields the F-differentiability of f. Finally, from Lemma 5.5 we have that (5.4) holds and combining this with the fact that f is F-differentiable we can apply Lemma 2.3

of [20] to conclude that f respects similarities. Therefore, f is a F-differentiable nc function. □

Remark 5.6 As pointed out in [5], in classical complex analysis, G-differentiability of u and v together with the Cauchy-Riemann equations is not strong enough to prove analyticity of f. Continuity of the partial derivatives provides F-differentiability, which is strong enough; this corresponds to the approach taken in the present paper. The Looman-Menchoff theorem, cf., [24, p. 199], states that continuity of f, and hence of u and v, is also sufficient. This in turn implies that u and v were F-differentiable from the start. As the proof of the Looman-Menchoff theorem requires the Baire category theorem and Lebesgue integration, it is not clear whether a similar relaxation of Theorem 1.1 can be achieved in the context considered here. In particular, the theory of integration of nc functions does not appear to be well developed so far. We are just aware of the paper [23] on the nc Hardy space over the unitary matrices and the recent PhD thesis [25] on antiderivatives of higher order nc functions.

Acknowledgments This work is based on research supported in part by the National Research Foundation of South Africa (NRF); Grant Numbers 118583 and 127364 and the DSI-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS). Any opinion, finding and conclusion or recommendation expressed in this material is that of the authors and the NRF and CoE-MaSS do not accept any liability in this regard.

The authors thank the anonymous referees for their useful comments and correction, and, in particular, for their suggestion to extend our results to the level of operator systems.

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Uniform Roe Algebras and Geometric RD Property

Ronghui Ji and Guoliang Yu

Dedicated to the memory of Ronald G. Douglas

Abstract We survey an early work of the authors on the notion of geometric RD property for a uniformly locally finite metric space. We show that metric spaces of polynomial growth satisfy this property. Associated to a metric space X with the geometric RD property we define a Fréchet space $BS_2(X)$ which in fact is a smooth and dense subalgebra of the uniform Roe algebra of the space X. This resulted an alternate proof of a result of the first named author on the nonexistence of positive scalar curvature on certain manifolds.

1 Introduction

This is an early note of the authors on smooth dense subalgebras of Roe algebras. We introduce a notion of geometrically rapid decaying metric spaces and its associated smooth subalgebra of the Roe algebra for the space. We verify that metric spaces of polynomial growth are geometrically rapidly decaying. Using this property we can reprove some results of the second named author on the nonexistence of positive scalar curvature on manifolds, which are uniformly contractible, of polynomial volume growth, and of bounded geometry with polynomial growing contractibility radius [13]. Initially we felt that hyperbolic metric spaces should satisfy this geometric RD property but V. Lafforgue informed us that free groups are not of this geometric RD property and in a later published paper [4], Chen and Wei give

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_16

a complete characterization of this class of spaces and proved that this class is precisely the set of metric spaces of polynomial growth. Since their results relied on some of our results, we decide to publish this note as a survey of the program initiated by the second named author, and for the purpose of the completeness of the subject. It also serves as a memory of Ron G. Douglas, who guided us through our careers. Since the program is still at its infancy we would like to invite readers to investigate positive scalar curvature problem by using smooth dense subalgebras and cyclic cohomology theory of Connes [5].

2 Geometric Rapid Decay Spaces and Spectral Invariance

In this section we define geometric rapid decay property for discrete metric spaces using dense subalgebras in uniform Roe algebras. We will prove that if the geometric RD conditions are satisfied, the dense subalgebra of the uniform Roe algebra is stable under holomorphic functional calculus.

Definition 2.1 Let (X, d) be a discrete metric space. Let $\ell^2(X)$ be the natural ℓ^2-space of X.

(a) Let $BS_2(X)$ be the Fréchet space of functions on $X \times X$.

$$\{k : X \times X \longrightarrow \mathbb{C} | \sup_{x \in X} \sum_{y \in X} |k(x, y)|^2 (1 + d(x, y))^{2k} < \infty, k = 1, 2, \ldots\}.$$

The seminorms $\| \cdot \|_s$ are defined by

$$\|k\|_s = \big(\sup_{x \in X} \sum_{y \in X} |k(x, y)|^2 (1 + d(x, y))^{2s}\big)^{1/2},$$

where $k \in BS_2(X)$ and $s > 0$.

(b) Given a function $k : X \times X \longrightarrow \mathbb{C}$. k is said *finitely propagated* if there is a constant $c_k > 0$, such that, $k(x, y) = 0$ whenever $d(x, y) > c_k$. k is said *bounded* if it defines a bounded operator on $\ell^2(X)$ by convolution. That is, $k : \ell^2(X) \longrightarrow \ell^2(X)$ defined by

$$k \star \xi(x) = \sum_{y \in X} k(x, y) \xi(y)$$

is a bounded operator. The precompleted uniform Roe algebra of X is defined to be

$$C_u(X) = \{k : X \times X \longrightarrow \mathbb{C} |\ k \text{ is bounded and finitely propagated}\}.$$

Let $C_u^*(X)$ be the norm closure of $C_u(X)$ in $B(\ell^2(X))$, and is called the *uniform Roe algebra*.

(c) X is said *geometrically rapidly decaying* if $BS_2(X) \subset C_u^*(X)$.

We remark that $C_u(X) \subset BS_2(X)$.

Lemma 2.2 *Each element in $C_u^*(X)$ is represented by a kernel function $k : X \times X \longrightarrow \mathbb{C}$ such that $k \in \ell^\infty(X, \ell^2(X)) = \ell^{\infty,2}(X \times X)$.*

Proof Let $T \in C_u^*(X)$. Then there exists a sequence $\varphi_n \in C_u(X)$ such that φ_n converges to T in norm. Therefore, for each $y \in X$, $\varphi_n \star \delta_y$ converges to $T \star \delta_y$ in the L^2-norm, where δ_y is the characteristic function on X at y. Now define $\varphi_T(x, y) = T \star \delta_y(x)$. Clearly, $\varphi_T \star \xi = T \star \xi$. □

We remark again, by virtue of this lemma and the closed graph theorem, that (X, d) is rapidly decaying if and only if there exist a $C > 0$ and an $s > 0$, such that $\|\varphi\|_{C_u^*(X)} \leq C\|\varphi\|_s$ for all $\varphi \in BS_2(X)$.

Unlike the case of rapidly decaying groups, it is not clear how to show directly that $BS_2(X)$ is a convolution algebra if (X, d) is rapidly decaying. But this follows from the following theorem as a byproduct.

Theorem 2.3 *Suppose that (X, d) is rapidly decaying, then $BS_2(X)$ is a smooth and dense subalgebra (under convolution) of $C_u^*(X)$. In other words, $BS_2(X) \otimes M_n(\mathbb{C})$ is stable under holomorphic functional calculus in $C_u^*(X) \otimes M_n(\mathbb{C})$ for all $n > 0$. Moreover, the inclusion induces an isomorphism between their K-theories.*

Proof We will use Connes-Moscovici's argument for hyperbolic groups [6] in our context. Let $x_0 \in X$ be a fixed point. Define an (in general) unbounded and self adjoint operator D^{x_0} on $\ell^2(X)$ by

$$D^{x_0}\xi(y) = d(y, x_0)\xi(y), \text{ for } y \in X.$$

Let $\delta^{x_0} : B(\ell^2(X)) \longrightarrow B(\ell^2(X))$ be the unbounded derivation defined by $\delta^{x_0}A = i[D^{x_0}, A]$. Set

$$\text{Domain}(\delta^n) = \{\varphi \in C_u^*(X) | \sup_{x_0} \|(\delta^{x_0})^n(\varphi)\| < \infty\}.$$

It is clear that $\cap\text{Domain}(\delta^n)$ is an algebra by the virtue of the derivation property of δ^{x_0} and by the fact that

$$\sup_x \|a(x)b(x)\| \leq \sup_x \|a(x)\| \cdot \sup_x \|b(x)\|$$

for any families of operators $a(x), b(x)$ depending on the parameter x. We also denote

$$BS(X) = \bigcap_{n=0}^{\infty} \text{Domain}(\delta^n) \cap C_u^*(X),$$

and claim that $BS(X) = BS_2(X)$. The smoothness of $BS(X)$ in $C_u^*(X)$ can be established by the same proof of Theorem (1.2) in [8]. Thus, the claim will end the proof of the theorem.

In fact, the claim is straightforward once we notice the following formula:

$$(\delta^{x_0})^n(\varphi)(\xi)(x) = i^n \sum_{y\in X} \varphi(x,y)\xi(y)(d(x,x_0) - d(y,x_0))^n.$$

If $\varphi \in BS_2(X) \subset C_u^*(X)$, then

$$\sum_{x\in X} |(\delta^{x_0})^n(\varphi)\xi(x)|^2 \le \sum_{x\in X}\Big(\sum_{y\in X} |\varphi(x,y)||\xi(y)|d(x,y)^n\Big)^2 \le \|\varphi_n\|^2_{C_u^*(X)}\|\xi\|^2,$$

Where $\varphi_n(x,y) = |\varphi(x,y)|d(x,y)^n$. Since $\varphi \in BS_2(X) \subset C_u^*(X), \varphi_n \in BS_2(X) \subset C_u^*(X)$. Therefore, $\varphi \in \text{Domain}(\delta^n)$, for all n, consequently $\varphi \in BS(X)$.

If $\varphi \in BS(X)$, then there exists $C_n > 0$, which is independent of x_0, such that for all $\xi \in \ell^2(X)$,

$$C_n\|\xi\|^2 \ge \sum_{x\in X} |(\delta^{x_0})^n(\varphi)\xi(x)|^2.$$

In particular, take $\xi(x) = \delta_0(x) = \{^1_0 \quad {}^{x=x_0}_{\text{otherwise}}$, then

$$\infty > C_n \ge \sum_{x\in X} |\varphi(x,x_0)d(x,x_0)^n|^2$$

for all n. By induction, $\sup_{x_0\in X}\sum_{x\in X}|\varphi(x,x_0)|^2(1+d(x,x_0))^{2n} < \infty$ for all n. Thus, $\varphi \in BS_2(X)$. □

3 Examples of Geometrically Rapid Decay Spaces

In this section we will show that polynomial growth spaces are of geometric RD property.

For a metric space (X,d), we will denote by V the range of the distance function $d(\cdot,\cdot)$ in X. That is,

$$V = \{d(x,y)|x,y \in X\}.$$

If the space is countable, V is clearly countable.

Definition 3.1 A discrete metric space (X, d) is said to be of polynomial growth if there exists a real polynomial $P(x)$ such that for any $r > 0$,

$$\sup_{x_0} |\{x \in X | d(x, x_0) < r\}| \leq P(r).$$

For a space of polynomial growth, it is clear that the space is countable and that the polynomial $P(x)$ in the definition can be chosen to be $C(1 + x)^n$ for some constant $C > 0$ and an integer $n > 1$. Moreover, we will require that the space (X, d) has the following property that

$$\sup_{y\in X} \sum_{x\in X} (1 + d(x, y))^{-2N} < \infty \tag{$\star$}$$

for some integer $N > 1$.

Proposition 3.2 *A metric space* (X, d) *of polynomial growth with the property* $(\star)$ *is geometrically rapidly decaying.*

Proof Let $\varphi \in BS_2(X)$. We have to show that $\|\varphi\|_{C_u^*(X)} < \infty$. In fact, for $\xi \in \ell^2(X)$, we have

$$\begin{aligned}
\|\varphi \star \xi\|_2^2 &= \sum_{x\in X} |\sum_{y\in X} \varphi(x, y)\xi(y)|^2 \\
&\leq \sum_{x\in X} \Bigg(\sum_{y\in X} |\varphi(x, y)|(1 + d(x, y))^N (1 + d(x, y))^{-N} |\xi(y)| \Bigg)^2 \\
&\leq \sum_{x\in X} \Bigg(\sum_{y\in X} |\varphi(x, y)|^2 (1 + d(x, y))^{2N} \Bigg) \Bigg(\sum_{y\in X} (1 + d(x, y))^{-2N} |\xi(y)|^2 \Bigg) \\
&\leq \Bigg(\sup_{x\in X} \sum_{y\in X} |\varphi(x, y)|^2 (1 + d(x, y))^{2N} \Bigg) \sum_{x\in X} \sum_{y\in X} (1 + d(x, y))^{-2N} |\xi(y)|^2 \\
&\leq \Bigg(\sup_{x\in X} \sum_{y\in X} |\varphi(x, y)|^2 (1 + d(x, y))^{2N} \Bigg) \Bigg(\sup_{y\in X} \sum_{x\in X} (1 + d(x, y))^{-2N} \Bigg) \|\xi\|_2^2 \\
&= C \Bigg(\sup_{x\in X} \sum_{y\in X} |\varphi(x, y)|^2 (1 + d(x, y))^{2N} \Bigg) \|\xi\|_2^2,
\end{aligned}$$

where $C = \sup_{y\in X} \sum_{x\in X} (1 + d(x, y))^{-2N} < \infty$.

This proves the proposition. □

Corollary 3.3 *If* Γ *is a group of polynomial growth, then* Γ *is also a geometrically rapidly decaying space as a metric space with respect to any length function on* Γ.

4 Quasi-Isometry and Geometrically Rapid Decay Spaces

In this section we show that geometric RD property for metric spaces is a quasi-isometric invariant.

Theorem 4.1 *Suppose that* (X, d) *and* (Y, d') *are quasi-isometric metric spaces, which are uniformly locally finite. Then* X *is rapidly decaying if and only if* Y *is.*

Proof We note first that a quasi-isometric bijection between two metric spaces preserves the rapid-decaying property. Without loss of generality, we may assume that (X, d) is rapidly decaying.

First of all, if X is rapidly decaying so are its subspaces. This is because that if $X_0 \subset X$, then $C_u^*(X_0)$ is contained in $C_u^*(X)$ naturally. Using the restricted distance function d_0 on X_0, we see that $BS_2(X_0) \subset BS_2(X)$. If $BS_2(X) \subset C_u^*(X)$, then $BS_2(X_0) \subset BS_2(X) \subset C_u^*(X)$. But the support of $\varphi \in BS_2(X_0)$ is contained in $X_0 \times X_0$, thus, $BS_2(X_0) \subset C_u^*(X_0)$.

Next, let f be a quasi-isometry from X to Y. For any $x_0 \in X$ the subset $S_{x_0} = \{x \in X | f(x) = f(x_0)\}$ is finite, and $\sup\{|S_x| | x \in X\} = M < \infty$ by the uniformly local finiteness. Let X_0 be a subset of X such that $X = \amalg_{x \in X_0} S_x$. Thus, $X_0 \hookrightarrow X$ is an injective quasi-isometry. This implies that $f : X_0 \longrightarrow Y$ is also an injective quasi-isometry.

Let $d_0 > 0$ be such that $d'(f(X_0), y) < d_0$ for any $y \in Y$, and $k > 0$ be a positive integer which is an upper bound of $|B(y, d_0)|$ for all $y \in Y$, where $B(y, d_0) = \{z \in Y | d'(z, y) < d_0\}$. The reason that k is finite is because that Y is uniformly locally finite. Let $Y_0 = f(X_0)$. We set $F = \{0, 1, 2, \dots, k-1\}$ with the standard metric, and $Z = X_0 \times F$ with the metric defined by $d''((x, i), (x', j)) = d(x, x') + |i - j|$ for all (x, i) and (x', j) in Z.

Let us define an embedding $\rho : Y \hookrightarrow Z$ as follows. First, $\rho(y) = (f^{-1}(y), 0)$ for all $y \in Y_0$. Let $Y_0 = \{y_1, \dots, y_n, \dots\}$. Let $y_{11}, \dots, y_{1i_1} \in Y$ such that $0 < d'(y_{1j}, y_1) < k$, where $1 \leq j \leq i_1 \leq k$. For y_2, let $y_{21}, \dots, y_{2i_2} \in Y$ so that $0 < d'(y_{2j}, y_2) < k$, and $y_{2j} \neq y_{1i}$ for any i, j. Proceeds inductively that for each y_n, let $y_{n1}, \dots, y_{ni_n} \in Y$ so that $0 < d'(y_{nj}, y_n) < k$, and $y_{nj} \neq y_{mi}$ for any $1 \leq m < n$ and any i, j. Now we define $\rho(y_{ij}) = (f^{-1}(y_i), j)$. This is a well-defined map from Y into Z. Since f is a quasi-isometry, it is not difficult to check that ρ is a quasi-isometric embedding.

Now since (X, d) is rapidly decaying, so is (X_0, d). Since F is finite, $BS_2(X_0 \times F) \cong BS_2(X_0) \otimes M_k(\mathbb{C})$ and $C_u^*(Z) \cong C_u^*(X_0) \otimes M_k(\mathbb{C})$ canonically. This implies that (Z, d'') is also rapidly decaying. Therefore, (Y, d') is too. □

Remark Theorem 4.1 will remain valid if we assume weak quasi-isometry instead of quasi-isometry provided that the positive valued functions ρ_1, ρ_2 in the definition of weak quasi-isometry are of polynomial growth. A positive function ρ on $\mathbb{R}_+$ is said to be of polynomial growth if $\rho(t) \leq p(t)$ for some fixed polynomial $p(t)$, where $t \geq 0$. On the other hand, if one assumes that all metric spaces under consideration are quasi-geodesic, then by Gromov [7] that weak quasi-isometry is equivalent to quasi-isometry.

5 The Product Space of Geometric RD Spaces

In this section all metric spaces will be assumed *countable*. We wish to establish that the product of two such rapidly decaying spaces is again rapidly decaying. This property is a reminiscence of the same property for rapid-decaying groups [10]. To this end we need to extend the construction of a uniform Roe algebra to a more general context.

Definition 5.1 ([9]) Let A be any normed algebra, and let (X, d) be a metric space.

(a) The uniform Roe algebra on X with coefficients in A is defined to be the convolution algebra of the following set of kernel functions with values in A:

$$C_u(X; A) = \{k : X \times X \longrightarrow A |\, k \text{ is bounded and finitely propagated}\}.$$

(b) Suppose that A is a Fréchet algebra with increasing seminorms $\{\|\cdot\|_m | m = 1, 2, \ldots\}$. The Fréchet space of rapidly decaying functions on X with values in A is defined to be:

$$BS_2(X, A) = \{k : X \times X \longrightarrow A |\, \sup_{x\in X} \sum_{y\in X} \|k(x, y)\|_m^2 (1 + d(x, y))^{2k} < \infty, k, m = 1, 2, \ldots, \}.$$

(c) The seminorms $\|\cdot\|_{s,m}$ are defined by

$$\|k\|_{s,m} = \Big(\sup_{x\in X} \sum_{y\in X} \|k(x, y)\|_m^2 (1 + d(x, y))^{2k}\Big)^{1/2},$$

where $k \in BS_2(X)$ and $s = 0, 1, 2, \ldots$.

Remark Suppose that A is a C^*-algebra. Let $\ell^2(X, A)$ be the Hilbert module with the A-valued inner product

$$\langle \xi, \psi \rangle_A = \sum_{x\in X} \xi^\star(x)\psi(x).$$

Then $C_u(X; A)$ acts on $\ell^2(X, A)$ as a bounded convolution algebra. Its completion in $B(\ell^2(X, A))$ will be denoted by $C_u^*(X; A)$.

Lemma 5.2 *If (X, d) is rapidly decaying, then $BS_2(X, A)$ is naturally contained in $C_u^*(X; A)$ for any C^*-algebra A.*

Proof We note that $BS_2(X) \otimes_{alg} A \subset C_u^*(X) \otimes_{alg} A \subset C_u^*(X; A)$, and each containment is norm dense. □

Lemma 5.3 *Let (X, d) and (Y, d') be metric spaces which are rapidly decaying. Then there are natural isomorphisms:*

(a) $BS_2(X \times Y) \cong BS_2(X, BS_2(Y))$;
(b) $C_u^*(X \times Y) \cong C_u^*(X; C_u^*(Y))$.

Proof We observe that $BS_2(X) \otimes BS_2(Y) \subset BS_2(X \times Y)$ as a dense subset as topological vector spaces and similarly for part (b). □

Theorem 5.4 *If (X, d) and (Y, d') are rapid-decaying metric spaces, so is $(X \times Y, d + d')$.*

Proof This follows immediately from Lemmas 5.2 and 5.3. □

6 The Fundamental Cyclic Cocycle

This section is devoted to a survey of the program initiated by the second named author on positive scalar curvature problem. Let M be a noncompact complete Riemannian manifold. Recall from [13] that the uniform Roe algebra $C_u^*(M)$ consists of smoothing operators acting on $L^2(M)$ with kernel k such that $k(x, y)$ vanishes when the distance between x and y is greater than some constant depending on the operator k and $\triangle^n k$ is uniformly bounded for all positive integers n, where $\triangle^n k$ is the Laplace operator on $M \times M$. Similarly, one defines the uniform Roe algebra $C_u^*(S)$ of the Clifford bundle S on the manifold M. As was shown in [13] that if the Clifford bundle S has bounded geometry [11] then the index Ind D of the Dirac operator D on S lies in $C_u^*(S)$ [13, Lemma 3.1]. Therefore, it defines an element in $K_0(C_u^*(S))$. With the same condition we have a natural embedding $i : C_u^*(S) \hookrightarrow C_u^*(M) \otimes M_n(\mathbb{C})$ for some n [13, Lemma 3.2]. This implies that the index Ind D defines a K-theory element in $K_0(C_u^*(M))$. The K-theoretic indices of Dirac operators have important applications to the spectral theory of Laplacian, homotopy invariance of higher signature for certain group cohomology classes and the scalar curvature problem [11–14]. The Chern character of Ind$D$ lives in the cyclic homology of $C_u^*(M)$. But for analytical reason it is more useful to consider the pairing of cyclic cocycle with Ind$D$. In [13] the second author computed the cyclic cohomoloy of $C_u^*(M)$. Let us recall the result of calculation.

Recall that Γ is called a net in M if Γ is a discrete subspace of M and $\exists \epsilon > 0$ such that $\sup_{\gamma \in \Gamma} m(x, \Gamma) \le \epsilon$ for all $x \in M$. This concept is introduced by Connes, Gromov and Moscovici.

Definition 6.1 For any $d \ge 0$, the Rips' polyhedron $P_d(\Gamma)$ associated to Γ is the simplicial polyhedron whose set of vertices (i.e. the 0-skeleton) equals to Γ and where a finite subset $\{\gamma_0, \cdots, \gamma_n\} \subset \Gamma$ spans a n-simplex $[\gamma_0, \cdots, \gamma_n]$ if and only if every two points in $\{\gamma_0, \cdots, \gamma_n\}$ have distance less than or equal to d.

The Rips' polyhedron was first introduced in the studying of hyperbolic groups. Set $P(\Gamma) = \cup_{d \geq 0} P_d(\Gamma)$. Define $H^*_{inf}(P(\Gamma))$ to be the simplicial cohomology (with infinite support and uniformly bounded coefficients) of $P(\Gamma)$.

Theorem 6.2 ([13]) *The cyclic cohomology $HC^n(C^*_u(M))$ of $C^*_u(M)$ is naturally isomorphic to $\oplus_{k \leq 0} H^{n+2k,u}_{inf}(P(\Gamma))$.*

The second author also obtained a formula for the pairing between cyclic cocycles and Ind$D$. We remark that Roe has previously constructed many interesting cyclic cocycles and has computed their pairing with Ind D [12].

For the purpose of applications we need to know when the K-theoretic indices are nonzero in the K-theory of $C^*_u(M)$, the C^* closure of $C_u(M)$. This can be achieved by extending the cyclic cocycles to some dense holomorphically closed subalgebras of $C^*_u(M)$.

The most interesting cyclic cocycle in this connection is the fundamental cyclic cocycle for a uniformly contractible manifold constructed in [13]. The extendability of the fundamental cyclic cocycle would imply the zero-in-the-spectrum conjecture in the even dimensional case, which claims that the spectrum of the Laplacian operator acting on the L^2 forms contains 0 for a uniformly contractible manifold; and the conjecture that there is no metric with strictly positive scalar curvature in the quasi-isometry class of the given metric on M.

Recall that a metric space M is called uniformly contractible if for any $r > 0$, $\exists r' > r$ such that $B(x, r)$ can be contracted to a point in $B(x, r')$ for any $x \in M$, where $B(x, r)$ is the ball of radius r in M. The universal cover of a compact Riemmannian $K(\Gamma, 1)$ manifold is such an example.

The fundamental cyclic cocycle on a uniformly contractible manifold is constructed as follows.

Let M be a uniformly contractible manifold and let Γ be a net in M. We shall define a continuous map ϕ from $P_d(\Gamma)$ to M such that $\phi(\gamma) = \gamma$ for all $\gamma \in \Gamma$ and for any $k > 0$, $\exists d_k > d$, $\phi([\gamma_0, \cdots, \gamma_k])$ is contained in $B(\{\gamma_0, \cdots, \gamma_k\}, d_k)$ for any k-simplex $[\gamma_0, \cdots, \gamma_k]$ in $P_d(\Gamma)$. We shall define ϕ from $P^n_d(\Gamma)$ to M by induction on n. First define ϕ from $P^0_d(\Gamma)$ to M by the obvious embedding. Assume that by induction hypothesis we have defined ϕ for $n < k$. By the uniform contractibility there exists d' such that $d' > d_{k-1} > d$ and $\phi([x_0, \cdots, x_{k-1}])$ can be contracted to a point in $B(\{x_0, \cdots, x_{k-1}\}, d')$ for any $(k-1)$-simplex $[x_0, \cdots, x_{k-1}]$ in $P_d(\Gamma)$. For any given k-simplex $[\gamma_0, \cdots, \gamma_k]$ in $P_d(\Gamma)$, by using the above fact we can extend ϕ continuously from $\cup_i [\gamma_0, \cdots, \hat{\gamma}_i, \cdots, \gamma_k]$ to $[\gamma_0, \cdots, \gamma_k]$ such that $\phi([\gamma_0, \cdots, \gamma_k]) \subseteq B(\{\gamma_0, \cdots, \gamma_k\}, d')$. Thus we have completed our induction process. If we give $P_d(\Gamma)$ the obvious piece-wise smooth structure as usual, then clearly ϕ can be made piece-wise smooth. In particular, ϕ restricts to a smooth mapping on each simplex $[\gamma_0, \cdots, \gamma_k]$.

Let $m = \dim M$ and let θ be a compactly supported closed differential form on M representing the generator in $H^m_c(M)$, such that $\int_M \theta = 1$. We define *the* *fundamental cyclic cocycle* τ_M for a uniformly contractible complete Riemannian manifold M as follows:

$$\tau_M(a_0, \cdots, a_m) =$$
$$\sum_{[\gamma_0,\cdots,\gamma_m]} \sum_{\sigma\in S_m} (-1)^\sigma \int_{x_0,\cdots,x_m} a_0(x_0, x_1)a_1(x_1, x_2)\cdots a_m(x_m, x_0)$$
$$\times \phi_{\gamma_{\sigma(0)}}(x_0)\cdots\phi_{\gamma_{\sigma(m)}}(x_m)dx_0\cdots dx_m \int_{\phi([\gamma_{\sigma(0)},\cdots,\gamma_{\sigma(m)}])} \theta$$

for $a_0, \cdots, a_m \in B_M$, where $\{\phi_\gamma\}_{\gamma\in\Gamma}$ is a partition of unity subordinate to the open cover $\{B(\gamma, r)\}_{\gamma\in\Gamma}$ for some large $r > 0$.

The following result indicates the significance of the fundamental cyclic cocycle whose proof is based on a local index theorem of Connes and Moscovici [6].

Theorem 6.3 ([13]) *Let M be a uniformly contractible complete Riemannian manifold and τ_M be the fundamental cyclic cocycle. Assume that D is the Dirac operator on M (or signature operator if M is even dimensional). Then*

$$\tau_M(\mathrm{Ind}D) \neq 0.$$

In order to utilize this result for geometric applications, one must show that the fundamental cyclic cocycle τ_M extends to a subalgebra of $C_u^*(M)$ which contains $C_u(M)$ and is closed under holomorphic functional calculus in $C_u^*(M)$. In this way, one would establish that the K-theoretic index Ind$D$ be nonzero in $K_0(C_u^*(M))$ [12].

Definition 6.4 Let M be a second countable and locally compact metric space. We say that M is a rapidly decaying space if there is a δ-separated net X in M such that X as a metric space is rapidly decaying, where the metric on X is induced from that on M.

Remark It is clear that the definition is independent of the choice of the δ-separated net in M by Theorem 4.1.

Lemma 6.5 *Let M be a uniformly contractible complete Riemannian manifold with bounded geometry, and let X be a δ-separated net in M. Suppose that M is rapidly decaying. Then $C_u^*(M) \cong C_u^*(X) \otimes K$, where K is the C^*-algebra of all compact operators on some infinite dimensional Hilbert space H. Moreover, under this isomorphism, there is natural inclusion $BS_2(X) \otimes \mathcal{C}^1(H) \hookrightarrow C_u^*(M)$, such that the inclusion is closed under holomorphic functional calculus where $\mathcal{C}^1(H)$ is the set of trace class operators on H.*

Proof The first statement of this lemma is given in the proof of Theorem 4.4 of [13], and the second is a consequence of the first and Theorem 2.4. □

To extend the fundamental cyclic cocycle, one has to estimate the growth of the cyclic cocycle. To do that we need the following concept introduced by Gromov.

Definition 6.6 The contractibility radius $R(r)$ for a uniformly contractible Riemannian manifold M is the infimal radius $R \geq r$ such that for any $x_0 \in M$, $B(x_0, r)$ can be contracted to a point in $B(x_0, R)$.

Theorem 6.7 ([13]) *The fundamental cyclic cocycle is dominated by the contractibility radius in the sense that*

$$|\tau_M(a_0, \cdots, a_m)| \leq$$
$$c \int_{dist(x_i, y_0) \leq R(\sum_i r(a_i)) + c'} |a_0(x_0, x_1) \cdots a_m(x_m, x_0)| dx_0 \cdots dx_m$$

for some constants c, c', where y_0 is a fixed reference point in X, and $r(a_i)$ is the infimal r such that any (x, y) in the support of the kernel of a_i satisfies $dist(x, y) \leq r$.

Theorem 6.8 *Suppose that the manifold M is rapidly decaying. If M has bounded geometry and is uniformly contractible, then the fundamental cyclic cocycle can be extended to a holomorphically closed dense subalgebra of $C_u^*(M)$. Therefore, the K-theoretical index* Ind D *of the Dirac operator D on the uniformly contractible Riemannian manifold M is nonzero in $K_0(C_u^*(M))$.*

Proof This follows the same as the proof in Theorem 4.4 in [13]. For the reader's convenience we will include a modified proof in the context of geometrically rapidly decay conditions. Let Γ be a δ-separated net in M for some $\delta > 0$. There exists a uniformly bounded Borel cover $\{C_\gamma\}_{\gamma \in \Gamma}$ of M such that and $C_\gamma \cap C_{\gamma'} = \emptyset$ if $\gamma \neq \gamma'$. We can decompose $L^2(M) = \bigoplus_{\gamma \in \Gamma} L^2(C_\gamma)$. We identify $L^2(C_\gamma)$ with an abstract Hilbert space H so that $L^2(M) = \ell^2(\Gamma) \otimes H$. Let $BS^2(\Gamma)$ be the space of functions k on $\Gamma \times \Gamma$ such that

$$\|k\|_s = \left(\sup_{x \in X} \sum_{y \in X} |k(x, y)|^2 (1 + d(x, y))^{2s}\right)^{1/2} < \infty,$$

for all $s > 0$. Let $C_u^*(\gamma)$ be the closure of the algebra of all kernel operators acting on $\ell^2(\Gamma)$ with bounded propagation. Since Γ is geometrically rapidly decaying, $BS^2(\Gamma) \subset C_u^*(\Gamma)$. This shows that $BS^2(\Gamma)\hat{\otimes}\mathcal{C}^1(\Gamma) \subset C_u^*(\Gamma)$ and is closed under holomorphic functional calculus. Now we consider the fundamental cyclic cocycle τ_M, when evaluated on $BS^2(\Gamma)\hat{\otimes}\mathcal{C}^1(\Gamma)$ we have

$$\tau_M(b_0 \otimes c_0, b_1 \otimes c_1, \ldots, b_m \otimes c_m) = \sum_{[\gamma_0, \cdots, \gamma_m]} b_0(\gamma_0, \gamma_1) b_1(\gamma_1, \gamma_2) \cdots b_m(\gamma_m, \gamma_0)$$
$$\times tr(\phi_{\gamma_0}(c_0) \cdots \phi_{\gamma_m}(c_m)) \int_{\phi([\gamma_0, \cdots, \gamma_m])} \theta$$

for $b_0, \cdots, b_m \in BS^2(\Gamma)$, $c_i \in \mathcal{C}^1(H)$, and where $\{\phi_\gamma\}_{\gamma\in\Gamma}$ is a partition of unity subordinate to some open cover $\{B(\gamma, r)\}_{\gamma\in\Gamma}$ for some large $r > 0$. Finally, the theorem follows from Theorem 6.3 since $\mathrm{Ind}(D) \in K_0(C_u^*(M)) = K_0(BS^2(\Gamma))$. □

7 The Recent Developments

Since our early investigation of dense subalgebras of the uniform Roe algebra of a metric space described in this survey, there are several new ideas emerged. in [4] the authors showed that geometrically rapid-decay metric spaces must be of polynomial growth. This in turn reconfirms Lafforgue's observation that free group of two generators are not geometrically rapid-decaying. In [3] the authors used the method of "band truncation" of Fourier series of elements in the group C^*-algebra of groups with polynomial H-growth [1] to construct a family of spectral invariant Banach subalgebras in the uniform Roe algebra of the group as a metric space. They also proved that when the group is of subexponential growth the Wiener algebra of the group is inside the uniform Roe algebra and is spectral invariant. In the more recent paper [2] the authors constructed a spectral invariant subalgebra (consisting of elements with kernels) of the uniform Roe algebra on a discrete group that has property RD. In particular, there is a spectral invariant dense subalgebra (consisting of elements with kernels) of the uniform Roe algebra of a hyperbolic group, considered as a metric space! This is to date the most promising construction of dense subalgebras of uniform Roe algebra on general metric spaces.

Acknowledgments The second author "Guoliang Yu" is partially supported by NSF 1700021 and NSF 1564398.

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Integral Curvature and Similarity of Cowen-Douglas Operators

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In Memory of R. G. Douglas

Abstract In 1978, M. J. Cowen and R. G. Douglas introduced a class of geometric operators. In their influential paper (M. J. Cowen and R. G. Douglas, Acta Math. 141:187–261, 1978), they give complete unitary invariants involving curvature and its covariant derivatives for this kind of operators. In this paper, we introduce a new concept named by integral curvature. By using this new invariant, we give a similarity classification for Cowen-Douglas operators with index one. Two operators T and S are called $U + K$ similarity equivalent if there exists a unitary operator U and a compact operator K such that $X := U + K$ is an invertible operator which satisfies $XT = SX$. By considering the difference of the corresponding curvatures, we also study the $U + K$ similarity problems for Cowen-Douglas operators with index one.

Keywords Curvature · Similarity · $U + K$ similarity · Subharmonic function

Mathematics Subject Classification (2010) Primary 47C15, 47B37; Secondary 47B48, 47L40

1 Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the set of bounded linear operators on $\mathcal{H}$. The Grassmann manifold denoted by $\mathrm{Gr}(n, \mathcal{H})$ is the set of all n-dimensional subspaces of the Hilbert space $\mathcal{H}$. For an open bounded connected subset Ω of the complex plane $\mathbb{C}$, and $n \in \mathbb{N}$, a map $t : \Omega \to \mathrm{Gr}(n, \mathcal{H})$ is called

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_17

a holomorphic curve, if there exist n holomorphic functions $\gamma_1, \gamma_2, \cdots, \gamma_n$ on Ω taking values in $\mathcal{H}$ such that $t(w) = \bigvee\{\gamma_1(w), \cdots, \gamma_n(w)\}$, $w \in \Omega$. Given a holomorphic curve $t : \Omega \to \mathrm{Gr}(n, \mathcal{H})$, one can find an n-dimensional Hermitian holomorphic vector bundle E_t over Ω, namely,

$$E_t = \{(x, w) \in \mathcal{H} \times \Omega \mid x \in t(w)\} \text{ and } \pi : E_t \to \Omega, \text{ where } \pi(x, w) = w.$$

M. J. Cowen and R. G. Douglas introduced the class $B_n(\Omega)$ of Cowen-Douglas operators in their very influential paper [2]. An operator T acting on $\mathcal{H}$ is said to be a Cowen-Douglas operator of index n associated with the open bounded subset Ω (denoted $T \in B_n(\Omega)$), if $T - w$ is surjective, $\dim \ker(T - w) = n$ and $\bigvee\limits_{w\in\Omega} \ker(T - w) = \mathcal{H}$, for any $w \in \Omega$.

The class of Cowen-Douglas operators is very rich. In fact, the norm closure of Cowen-Douglas operators contains the collection of all quasi-triangular operators with connected spectrum by using the famous similarity orbit theorem given by C. Apostol, L. A. Fialkow, D.A. Herrero and D. Voiculescu [1].

The following examples of Cowen-Douglas operators are well known:

Example 1.1 Let $\mathbb{D}$ denote the open unit disk and let S_1^* be the backward shift operator, $S_1^* e_n = e_{n-1}, n = 1, 2, \cdots$, where $\{e_n\}_{n=0}^{\infty}$ is the ONB of a Hilbert space $\mathcal{H}$. Then $S^* \in B_1(\mathbb{D})$.

Example 1.2 Let $\mathcal{A}_\alpha$ be a weighted Bergman space and S_z denote the multiplication operator on $\mathcal{A}_\alpha$, that is, $S_z(f)(z) = zf(z)$, $f \in \mathcal{A}_\alpha$. Then $S_z^* \in B_1(\mathbb{D})$ and $\sigma(S_z^*) = \bar{\mathbb{D}}$. Furthermore, if f is an analytic function on $\bar{\mathbb{D}}$ such that

$$f(z) = \sum_{i=0}^{\infty} a_{n_i} z^{n_i}.$$

Then T_f^* is a Cowen-Douglas operator with index k, for some $k \in \mathbb{N}$. If $f(z) = z^m g(z)$, where $m = g.c.d_{i\geq 0} n_i$, then T_g^* is a Cowen-Douglas operator with index $k - m$.

It is well known that each operator T in $B_n(\Omega)$ also give rise to an n-dimensional Hermitian holomorphic vector bundle E_T over Ω (see in [2]),

$$E_T = \{\ker(T - w) : w \in \Omega, \pi(\ker(T - w)) = w\}.$$

Two holomorphic curves $t, \tilde{t} : \Omega \to \mathrm{Gr}(n, \mathcal{H})$ are said to be congruent if there exists a local isometric holomorphic bundle map $V(w)$ such that $V(w)t(w) = \tilde{t}(w)$, $w \in \Omega$. Furthermore, t and $\tilde{t}$ are unitarily equivalent (denoted by $t \sim_u \tilde{t}$), if there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $U(w)t(w) = \tilde{t}(w)$, where $U(w) := U|_{E_t(w)}$ is the restriction of the unitary operator U to the fiber $E_t(w)$. It is an easily fact, by using the Rigidity Theorem in [2], t and $\tilde{t}$ are congruent if and only if t and $\tilde{t}$ are unitarily equivalent.

Two holomorphic curves t and $\tilde{t}$ are said to be similarity equivalent (denoted by $t \sim_s \tilde{t}$), if there exists an invertible operator $X \in \mathcal{B}(\mathcal{H})$ such that $X(w)t(w) = \tilde{t}(w)$, where $X(w) := X_{|E_t(w)}$ is the restriction of X to the fiber $E_t(w)$. Then $X(w)$ is an isomorphism but it is no longer an isometry. In this case we say that the vector bundles E_t and $E_{\tilde{t}}$ are similarity equivalent.

For an open bounded connected subset Ω of $\mathbb{C}$, a Cowen-Douglas operator T with index n determines an non-constant holomorphic curve $t : \Omega \to \mathrm{Gr}(n, \mathcal{H})$, namely, $t(w) = \ker(T - w)$, $w \in \Omega$. Then the unitary and similarity invariants for the operator T are obtained from those of vector bundle E_T.

To describe these invariants, M. J. Cowen and R. G. Douglas introduced the curvature along with its covariant derivatives for the Cowen-Douglas operator T. Let us recall some of these notions as given in [2].

The Hermitian structure of the holomorphic bundle E_T, with respect to a holomorphic frame γ is given by the matrix of inner products

$$h_\gamma(w) = \big(\langle \gamma_j(w), \gamma_i(w)\rangle\big)_{i,j=1}^n, \quad w \in \Omega,$$

where $\gamma(w) = \bigvee\{\gamma_1(w), \cdots, \gamma_n(w)\}$, $w \in \Omega$.

If we let $\bar{\partial}$ denote the complex structure of the vector bundle E_T, then the connection compatible with both the complex structure $\bar{\partial}$ and the metric h_γ is canonically determined and is given by the formula $h_\gamma^{-1}\partial h_\gamma dw$. The curvature of the holomorphic Hermitian vector bundle E_T is then the (1, 1) form

$$\mathsf{K}_T(w) = -\bar{\partial}\big(h_\gamma^{-1}\partial h_\gamma\big)dw \wedge d\bar{w}.$$

We let $K_T(w)$ denote the coefficient of this (1, 1) form, that is,

$$K_T(w) := -\frac{\partial}{\partial \overline{w}}\big(h_\gamma^{-1}(w)\frac{\partial}{\partial w}h_\gamma(w)\big).$$

Note that it is an endomorphism of the fiber $E_T(w)$.

Since the curvature K_T may be thought of as a bundle map, following the definition of the partial derivatives of bundle map, we can define its partial derivatives $K_{T,w^i\overline{w}^j}$, $i, j \in \mathbb{N} \cup \{0\}$ as follows:

(1) $K_{T,w^i\overline{w}^{j+1}} = \frac{\partial}{\partial \overline{w}}(K_{T,w^i\overline{w}^j})$;
(2) $K_{T,w^{i+1}\overline{w}^j} = \frac{\partial}{\partial w}(K_{T,w^i\overline{w}^j}) + [h_\gamma^{-1}\frac{\partial}{\partial w}h_\gamma, K_{T,w^i\overline{w}^j}]$.

The curvature and its derivatives are complete unitary invariants of Cowen-Douglas operators. M. J .Cowen and R. G. Douglas showed the following result on unitarily equivalence.

Theorem 1.3 ([2]) *Let T and $\tilde{T}$ be two Cowen-Douglas operators with index n. Then T and $\tilde{T}$ are unitarily equivalent if and only if there exists an isometric* *holomorphic bundle map* $V : E_T \to E_{\tilde{T}}$ *such that*

$$V\big(K_{T,w^i\overline{w}^j}\big) = \big(K_{\tilde{T},w^i\overline{w}^j}\big)V,\ i, j = 0, 1, \cdots, n-1.$$

In particular, if T and $\tilde{T}$ are Cowen-Douglas operators with index one, then $T \sim_u \tilde{T}$ if and only if $K_T = K_{\tilde{T}}$.

However, in the case of similarity equivalence, the global invariants are not easily detected by the local invariants such as the curvature and its covariant derivatives. This is due to the holomorphic bundle map determined by invertible operators is not rigid. That is, we do not know when a bundle map that is locally isomorphic can be extended to an invertible operator in $\mathcal{B}(\mathcal{H})$. In the absence of a characterization of equivalent classes under an invertible linear transformation, M. J. Cowen and R. G. Douglas gave the following conjecture in [2].

Conjecture Let $T, \tilde{T} \in B_1(\mathbb{D})$ with the spectrums of T and $\tilde{T}$ are just closure of $\mathbb{D}$ (denoted by $\bar{\mathbb{D}}$). Then $T \sim_s \tilde{T}$ if and only if

$$\lim_{w\to\partial\mathbb{D}} \frac{K_T(w)}{K_{\tilde{T}}(w)} = 1.$$

Unfortunately, this conjecture turned out to be false by D. N. Clark and G. Misra (cf. [4, 5]).

Thus, it is very natural to ask how to characterize similarity invariants of Cowen-Douglas operators by using some geometric terms including curvature. For this purpose, we mention some recent results on similarity.

By using curvature to describe similarity invariants of Cowen-Douglas operator are due to some very recent results of R. G. Douglas, H. Kwon, S. Treil [7, 15] and Y. Hou, H. Kwon and K.Ji [9]. Combining their results, we have the following theorem:

Theorem 1.4 ([7, 9]) *Let* $T \in B_n(\mathbb{D})$ *be an* n*-hypercontraction and let* S_z *be the multiplication on the weighted Bergman space. Then* T *is similar to* $\bigoplus_{i=1}^{m} S_z^*$ *if and only if there exists a bounded subharmonic function* ψ *defined on* $\mathbb{D}$ *such that*

$$trace K_T - trace K_{S_z^*} \leq \Delta\psi.$$

We need to point out that even if T is a Cowen-Douglas operator with index one, its spectral picture is very complicated. The following theorem due to D. A. Herreor shows its complexity.

Theorem 1.5 ([10]) *Let* $T \in \mathcal{B}(\mathcal{H})$ *be a quasi-triangular operator with connected spectral picture. If there exists a point* w *in the Fredholm domain of* T *such that* $ind(T - w) = 1$*. Then for any* $\epsilon > 0$*, there exists a compact operator* K *with* $\|K\| < \epsilon$ *such that* $T + K$ *is a Cowen-Douglas operator with index one.*

The structure of Cowen-Douglas operators and the fact that the property of invertible bundle map is not rigid make situation complicated. Therefore, for any two Cowen-Douglas operators T and $\tilde{T}$ with index one, we have to further explore the equivalent relation between K_T and $K_{\tilde{T}}$.

In this paper, we introduce a concept related to curvature named by integral curvature. By using integral curvature, we give a similarity classification theorem for the operator in $B_1(\Omega)$. Moreover, we also describe the $U + K$ similarity of Cowen-Douglas operators with index one by using the difference of curvatures.

2 Integral Curvature and Similarity of Operators in $B_1(\Omega)$

It is an open problem proposed by M. J. Cowen and R. G. Douglas how to describe the similarity for operators in $B_1(\Omega)$ by considering the curvature. M. J. Cowen and R. G. Douglas once had the following conjecture: If T and S are similar, then

$$\lim_{w \to w_0 \in \mathbb{T}} \frac{K_T(w)}{K_S(w)} = 1.$$

However, in [4], a counter example was given by D. N. Clark and G. Misra. Instead of the quotient of the curvatures, they used a_w, the quotient of metrics of E_T and E_S for weighted shift operators. It was proved in [6] that T is similar to some weighted shift operators if and only if a_w is bounded and bounded below by 0. In some sense, this result can be regarded as a geometric version of the classical result for the weighted shifts given by A. L. Shields (See [16]).

In [18], a spanning holomorphic cross-section for the Hermitian holomorphic vector bundle corresponding to a Cowen-Douglas operator was introduced by K. Zhu. For $T \in B_n(\Omega)$, a holomorphic section of vector bundle E_T is a holomorphic function $\gamma : \Omega \to \mathcal{H}$ such that for each $w \in \Omega$, the vector $\gamma(w)$ belongs to the fibre of E_T over w. We say γ is a spanning holomorphic section for E_T if $\bigvee \{\gamma(w) : w \in \Omega\} = \mathcal{H}$. In [18], it is proved that for any Cowen-Douglas operator $T \in B_n(\Omega)$, E_T possesses a spanning holomorphic cross-section. In this case, the unitary equivalence problem of E_T could be attributed to the case of some line bundle. Suppose T and $\widetilde{T}$ belong to $B_n(\Omega)$, then T and $\widetilde{T}$ are unitarily equivalent (or similarity equivalent) if and only if there exist spanning holomorphic cross-sections γ_T and $\gamma_{\widetilde{T}}$ for E_T and $E_{\widetilde{T}}$, respectively, such that $\gamma_T \sim_u \gamma_{\widetilde{T}}$ (or $\gamma_T \sim \gamma_{\widetilde{T}}$).

Let $T \in \mathcal{B}(\mathcal{H})$ and let $\{T\}'$ denote the commutant of T. The operator T is said to be strongly irreducible if $\{T\}'$ contains no nontrivial idempotents. A strongly irreducible operator can be regarded as a natural generalization of a Jordan block matrix on the infinite dimensional case. In [12], the first author proved that for any Cowen-Douglas operator T, $\{T\}'/rad(\{T\}')$ is commutative, where $rad(\{T\}')$ denotes the Jacobson radical of $\{T\}'$. Based on this, a similarity classification theorem of strongly irreducible Cowen-Douglas operators was given using the K_0-group of their commutant algebra as an invariant (See more details in [12]).

In this section, we define a concept related to curvature called integral curvature. By using integral curvature, we give a similarity classification theorem for operators in $B_1(\mathbb{D})$.

Lemma 2.1 ([2]) *Let Ω_1 and Ω_2 be two open bounded connected subsets of $\mathbb{C}$. If $\Omega_1 \subset \Omega_2$, then $B_n(\Omega_1)$ contains $B_n(\Omega_2)$.*

Integral Curvature Let T be a Cowen-Douglas operator with index one associated with a bounded open connected subset Ω, and a holomorphic frame $e(w) \in \ker(T - w)$, $w \in \Omega$. Let $w_0 \in \Omega$. Set $O_\delta = \{w : |w - w_0| < \delta\} \subset \Omega$, we then have the following Poisson equation

$$\begin{cases} \overline{\partial}\partial u(w) = K_T(w), w \in O_\delta, \\ u(w) = \phi(w) = ln(\|e(w)\|^2), w \in \partial O_\delta, \end{cases}$$

Now we assume that G is a Green function on O_δ. Then

$$u(w) = \int\limits_{\partial O_\delta} \phi(y)\frac{\partial G(w,y)}{dV_y}do(y) + \int\limits_{O_\delta} K_T(y)G(w,y)dy,$$

where dV_y is the directional derivative along with “y".

Furthermore, $\int\limits_{\partial O_\delta} \phi(y)\frac{\partial G(w,y)}{dV_y}do(y)$ is harmonic and $\int\limits_{O_\delta} K_T(y)G(w,y)dy = ln(\|e(w)\|^2)$. In this case, $\int\limits_{O_\delta} K_T(y)G(w,y)dy$ is called the integral curvature (denoted by $\widehat{K}_T$) corresponding to K_T.

Let $T \in B_n(\Omega)$. Choose any $w_0 \in \Omega$, then there exists $\delta > 0$ such that $O_\delta := \{w : |w - w_0| < \delta\} \subset \Omega$. By Lemma 2.1, we have that $B_n(\Omega) \subset B_n(O_\delta)$. Without loss of generality, we can also assume that the domain Ω is the form of O_δ.

In this paper, we are concerned with the case of unit disk. The other cases are similar.

Definition 2.2 Let $T, S \in B_1(\mathbb{D})$. We say that $\widehat{K}_T$ and $\widehat{K}_S$ are equivalent (denoted by $\widehat{K}_T \cong \widehat{K}_S$) if there exist $0 < m < M < \infty$ such that

$$\begin{aligned} m\sum_{i,j}\alpha_i\overline{\alpha}_j\partial^i\overline{\partial}^j exp(\widehat{K}_T)(0) &\le \sum_{i,j}\alpha_i\overline{\alpha}_j\partial^i\overline{\partial}^j exp(\widehat{K}_S)(0) \\ &\le M\sum_{i,j}\alpha_i\overline{\alpha}_j\partial^i\overline{\partial}^j exp(\widehat{K}_T)(0) \end{aligned}$$

for any $\alpha_i \in \mathbb{C}, i, j = 1, 2 \cdots$.

Now we will give a geometric characterization of similarity equivalence about Cowen-Douglas operators with index one by using the equivalence of integral curvature.

Lemma 2.3 *Let* $T \in B_1(\mathbb{D})$ *and let* $e(w) \in ker(T-w)$, $w \in \mathbb{D}$. *Then we have that*

$$Span\{e^{(n)}(0) : n = 0, 1, \cdots\} = \mathcal{H}.$$

Proof Since $0 \in \sigma(T)$, T is surjective. Then T has a right inverse operator denoted by B, i.e. $TB = I$. Choose any $e_0 \in kerT$, by a directly computation, we have that $T(\sum\limits_{n=0}^{\infty} B^n(e_0)w^n) = w(\sum\limits_{n=0}^{\infty} B^n(e_0)w^n)$. That means $\sum\limits_{n=0}^{\infty} B^n(e_0)w^n \in ker(T-w)$. Then there exists a holomorphic function ϕ on $\mathbb{D}$ such that

$$e(w) = \phi(w)(\sum_{n=0}^{\infty} B^n(e_0)w^n) = \sum_{n=0}^{\infty}(\sum_{i+j=n} \frac{\phi^{(i)}(0)}{i!} B^j(e_0))w^n.$$

Notice that $\frac{B^n(e_0)}{n!} = \frac{d^n}{dw^n}(\sum\limits_{n=0}^{\infty} B^n(e_0)w^n)|_{w=0}$. By using Lemma 1.22 and formula 1.7.1 in [2], we have that $Span\{B^n(e_0) : n = 0, 1, \cdots\} = \mathcal{H}$. Since $e^{(n)}(0) = \sum\limits_{i+j=n} \frac{\phi^{(i)}(0)}{i!} B^j(e_0)$, this finishes the proof the lemma. □

Theorem 2.4 *Let* $T, S \in B_1(\mathbb{D})$. *Then* $T \sim_s S$ *if and only if* $\widehat{K}_T$ *and* $\widehat{K}_S$ *are equivalent.*

Proof By Lemma 2.1, there exist $e(w) \in ker(T-w)$ and $\bar{e}(w) \in ker(S-w)$ such that $ln(\|e(w)\|^2) = \widehat{K}_T(w)$ and $ln(\|\bar{e}(w)\|^2) = \widehat{K}_S(w)$. Since $e(w)$ is a holomorphic section of E_S, we have

$$\partial^i \overline{\partial}^j \langle e(w), e(w)\rangle = \langle e^{(i)}(w), e^{(j)}(w)\rangle.$$

Thus, it follows that

$$\sum_{i,j} \alpha_i \overline{\alpha}_j \partial^i \overline{\partial}^j exp(\widehat{K}_T)(0) = \sum_{i,j} \alpha_i \overline{\alpha}_j \langle e^{(i)}(0), e^{(j)}(0)\rangle,$$

$$\sum_{i,j} \alpha_i \overline{\alpha}_j \partial^i \overline{\partial}^j exp(\widehat{K}_S)(0) = \sum_{i,j} \alpha_i \overline{\alpha}_j \langle \bar{e}^{(i)}(0), \bar{e}^{(j)}(0)\rangle.$$

Thus, we have that $\{e^{(n)}(0)\}_{n=0}^{\infty}$ and $\{\bar{e}^{(n)}(0)\}_{n=0}^{\infty}$ are equivalent, that is, there exist $0 < m < M$ such that

$$\begin{aligned} m \leq \sum_{i,j} \alpha_i \overline{\alpha}_j \langle e^{(i)}(0), e^{(j)}(0)\rangle &\leq \sum_{i,j} \alpha_i \overline{\alpha}_j \langle \bar{e}^{(i)}(0), \bar{e}^{(j)}(0)\rangle \\ &\leq M \sum_{i,j} \alpha_i \overline{\alpha}_j \langle e^{(i)}(0), e^{(j)}(0)\rangle \end{aligned}$$

Now define an operator X as follows:

$$X(e^{(i)}(0)) = \bar{e}^{(i)}(0), i = 0, 1, \cdots .$$

Then we have that X is invertible. Notice that since

$$e(w) = \sum_{i=0}^{\infty} \frac{e^{(i)}(0)}{i!} w^i, \bar{e}(w) = \sum_{i=0}^{\infty} \frac{\bar{e}^{(i)}(0)}{i!} w^i, w \in \mathbb{D},$$

$X(e(w)) = \bar{e}(w), w \in O_\delta$. It follows that $XT = SX$. This finishes the proof of sufficiency. On the other hand, if T is similar to S, i.e. $XT = SX$ so that $exp(\widehat{K}_T) = \|e(w)\|^2$, then we can choose $\bar{e}(w) = X(e(w))$, and $exp(\widehat{K}_S)(w) = \|X(e(w))\|^2$. Then we have that $X(e^{(i)}(0)) = \bar{e}^{(i)}(0), i = 0, 1, \cdots$. Since X is invertible, there exist $0 < m \leq M < 0$ such that

$$m\|\sum_{i=0}^{\infty} \alpha_i e^{(i)}(0)\|^2 \leq \|\sum_{i=0}^{\infty} \alpha_i \bar{e}^{(i)}(0)\| = \|X(\sum_{i=0}^{\infty} \alpha_i e^{(i)}(0))\|^2 \leq M\|\sum_{i=0}^{\infty} \alpha_i e^{(i)}(0)\|^2.$$

Repeating the calculation above, we have that

$$\begin{aligned} m \sum_{i,j} \alpha_i \overline{\alpha}_j \partial^i \overline{\partial}^j exp(\widehat{K}_T(0)) &\leq \sum_{i,j} \alpha_i \overline{\alpha}_j \partial^i \overline{\partial}^j exp(\widehat{K}_S(0)) \\ &\leq M \sum_{i,j} \alpha_i \overline{\alpha}_j \partial^i \overline{\partial}^j exp(\widehat{K}_T(0)), \end{aligned}$$

and this proves the necessity. □

By using the following lemmas, we will show in Example 2.8 that the equivalence of integral curvature directly implies the similarity equivalence of any chosen operator T in $B_1(\mathbb{D})$ and S_1^*. This example should be compared with the one given by H. Kwon and S. Treil's in the case of contraction (See in [15]).

Lemma 2.5 ([14]) *Let $T \in B_1(\mathbb{D})$. For any $a_0 \in \mathbb{D}$, there exists $\{e_i\}_{i=0}^{\infty}$, an ONB of $\mathcal{H}$ and $r > 0$ such that T admits the upper-triangular matrix representation*

$$T = \begin{pmatrix} a_0 & a_{1,2} & a_{1,3} & \cdots & a_{1,n} & \cdots \\ & a_0 & a_{2,3} & \cdots & a_{2,n} & \cdots \\ & & \ddots & \ddots & \vdots & \cdots \\ & & & a_0 & a_{n-1,n} & \cdots \\ & & & & a_0 & \ddots \\ & & & & & \ddots \end{pmatrix}, \tag{2.1}$$

with respect to $\{e_i\}_{i=0}^{\infty}$ and $|a_{i,i+1}| > r > 0, i \geq 1$.

Lemma 2.6 ([16]) *Let T_1 and T_2 be unilateral shifts with weight sequences $\{\omega_j\}_{j=0}^{\infty}$ and $\{\tilde{\omega}_j\}_{j=0}^{\infty}$, respectively. Then T_1 and T_2 are similar if and only if there exist positive constants C_1 and C_2 such that*

$$0 < C_1 \leq \left|\frac{\omega_k \cdots \omega_j}{\tilde{\omega}_k \cdots \tilde{\omega}_j}\right| \leq C_2,$$

for all $k \leq j$.

Lemma 2.7 *Let $T \in B_1(\mathbb{D})$ and set $a_0 = 0$ in expression (2.1) of Lemma 2.5. Suppose that T_0 denotes the weighted backward shift operator with weight sequence $\{w_n\}_{n=0}^{\infty}$, where $w_n = a_{n+1,n+2}, n = 0, 1, \cdots$. If T is similar to a weighted shift operator S with weight sequence $\{b_n\}_{n=0}^{\infty}$, $b_n \neq 0$, then we have that $T_0 \sim_s S$.*

Proof Let $\{e_n\}_{n=0}^{\infty}$ be the ONB of $\mathcal{H}$. Suppose there exists an invertible operator X satisfies that $XT = SX$. Let $X = ((x_{i,j}))$ relative to $\{e_n\}_{n=0}^{\infty}$. Then we have that

$$\begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} & \cdots \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} & \cdots \\ \ddots & \ddots & \ddots & \vdots & \cdots \\ \ddots & \ddots & x_{n-1,n-1} & x_{n-1,n} & \cdots \\ \ddots & \ddots & \ddots & x_{n,n} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \cdots & a_{1,n} & \cdots \\ & 0 & a_{2,3} & \cdots & a_{2,n} & \cdots \\ & & \ddots & \ddots & \vdots & \cdots \\ & & & 0 & a_{n-1,n} & \cdots \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 0 & b_0 & 0 & \cdots & 0 & \cdots \\ & 0 & b_1 & \cdots & 0 & \cdots \\ & & \ddots & \ddots & \vdots & \cdots \\ & & & 0 & b_{n-1} & \cdots \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} & \cdots \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} & \cdots \\ \ddots & \ddots & \ddots & \vdots & \cdots \\ \ddots & \ddots & x_{n-1,n-1} & x_{n-1,n} & \cdots \\ \ddots & \ddots & \ddots & x_{n,n} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Comparing the entries of the products of these matrices, we have that $b_0 x_{2,1} = 0$. It follows that $x_{2,1} = 0$. Then we have that $b_1 x_{3,1} = b_1 x_{3,2} = 0$ which implies $x_{3,1} = x_{3,2} = 0$. If we go on with this computation, then we have that $x_{i,j} = 0, i > j$. This means that X also has an upper-triangular matrix representation. Thus, we have the following equation

$$\begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} & \cdots \\ & x_{2,2} & \cdots & x_{2,n} & \cdots \\ & & \ddots & \vdots & \cdots \\ & & x_{n-1,n-1} & x_{n-1,n} & \cdots \\ & & & x_{n,n} & \ddots \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \cdots & a_{1,n} & \cdots \\ & 0 & a_{2,3} & \cdots & a_{2,n} & \cdots \\ & & \ddots & \ddots & \vdots & \cdots \\ & & & 0 & a_{n-1,n} & \cdots \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 0 & b_0 & 0 & \cdots & 0 & \cdots \\ & 0 & b_1 & \cdots & 0 & \cdots \\ & & \ddots & \ddots & \vdots & \cdots \\ & & & 0 & b_{n-1} & \cdots \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} & \cdots \\ & x_{2,2} & \cdots & x_{2,n} & \cdots \\ & & \ddots & \vdots & \cdots \\ & & x_{n-1,n-1} & x_{n-1,n} & \cdots \\ & & & x_{n,n} & \ddots \\ & & & & \ddots \end{pmatrix}.$$

Then we have the following equalities

$$\begin{cases} x_{1,1}a_{1,2} = b_0 x_{2,2} \\ x_{2,2}a_{2,3} = b_1 x_{3,3} \\ \vdots \\ x_{i,i}a_{i,i+1} = b_{i-1}x_{i+1,i+1} \\ \vdots \end{cases}.$$

It follows that $x_{i,i} \neq 0, i = 1, 2 \cdots$, and

$$\frac{\prod_{i=1}^{n} a_{i,i+1}}{\prod_{i=0}^{n-1} b_i} = \frac{x_{n+1,n+1}}{x_{1,1}}. \tag{2.2}$$

Now set $Y = X^{-1} = ((y_{i,j}))$. Since $x_{i,i} \neq 0$, we have that $y_{i,j} = 0, i > j$. Thus, we have that

$$\begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \cdots & a_{1,n} & \cdots \\ & 0 & a_{2,3} & \cdots & a_{2,n} & \cdots \\ & & \ddots & \ddots & \vdots & \cdots \\ & & & 0 & a_{n-1,n} & \cdots \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,n} & \cdots \\ & y_{2,2} & \cdots & y_{2,n} & \cdots \\ & & \ddots & \vdots & \cdots \\ & & y_{n-1,n-1} & y_{n-1,n} & \cdots \\ & & & y_{n,n} & \ddots \\ & & & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,n} & \cdots \\ & y_{2,2} & \cdots & y_{2,n} & \cdots \\ & & \ddots & \vdots & \cdots \\ & & y_{n-1,n-1} & y_{n-1,n} & \cdots \\ & & & y_{n,n} & \ddots \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} 0 & b_0 & 0 & \cdots & 0 & \cdots \\ & 0 & b_1 & \cdots & 0 & \cdots \\ & & \ddots & \ddots & \vdots & \cdots \\ & & & 0 & b_{n-1} & \cdots \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix}.$$

By a similar calculation, we have that

$$\frac{\prod_{i=1}^{n} a_{i,i+1}}{\prod_{i=0}^{n-1} b_i} = \frac{y_{1,1}}{y_{n+1,n+1}}. \tag{2.3}$$

Notice that since X and Y are both bounded, there exists some $M > 1$ such that $max\{\frac{x_{n+1,n+1}}{x_{1,1}}, \frac{y_{n+1,n+1}}{y_{1,1}}\} < M$, for each $n \in \mathbb{N}$. That means $0 < \frac{1}{M} < \frac{\prod_{i=1}^{n} a_{i,i+1}}{\prod_{i=0}^{n-1} b_i} < M, n \in \mathbb{N}$. By Lemma 2.6, we have that $T_0 \sim_s S$. This finishes the proof of the lemma. □

Example 2.8 Recall that S_1^* denote the backward Hardy shift. Suppose that $T \in B_1(\mathbb{D})$ and T is similar to a weighted shift operator S with weight sequence $\{b_n\}_{n=0}^{\infty}$. By Lemma 2.7, it suffices to consider the case when the weighted backward shift operator T_0 induced by T is similar to S_1^*. Now we assume that T is a weighted backward shift operator with weight sequences $\{\omega_k\}_{k=0}^{\infty}$ and $\|T\| \leq 1$. Set $\alpha_n = \frac{1}{\prod_{k=0}^{n} \omega_k}, n = 0, 1, \cdots$, and $\{e_n\}_{n=0}^{\infty}$ be ONB of $\mathcal{H}$. Then we have

$$\sum_{n=0}^{\infty} \alpha_n w^n e_n \in ker(T - w) \text{ and } \sum_{n=0}^{\infty} w^n e_n \in ker(S_1^* - w).$$

Thus we have that

$$\widehat{K}_T = ln(\|\sum_{n=0}^{\infty} \alpha_n w^n e_n\|^2) = ln(\sum_{n=0}^{\infty} |\alpha_n|^2 (|w|^2)^n)$$

and

$$\widehat{K}_{S_1^*} = ln(\|\sum_{n=0}^{\infty} w^n e_n\|^2) = ln(\sum_{n=0}^{\infty} (|w|^2)^n) = ln((1 - |w|^2)^{-1}).$$

That means $exp(\widehat{K}_T)(w) = \sum_{n=0}^{\infty} |\alpha_n|^2 (|w|^2)^n$, and $exp(\widehat{K}_{S_1^*})(w) = (1 - |w|^2)^{-1}$. If $\widehat{K}_T$ and $\widehat{K}_{S_1^*}$ are equivalent, then there exist $0 < m < M$ such that

$$m \partial^i \overline{\partial}^j exp(\widehat{K}_{S_1^*})(0) < \partial^i \overline{\partial}^j exp(\widehat{K}_T)(0) < M \partial^i \overline{\partial}^j exp(\widehat{K}_{S_1^*})(0).$$

Thus we have

$$m \leq \frac{exp(\widehat{K}_T)(w)}{exp(\widehat{K}_{S_1^*})(w)} = \frac{\sum_{n=0}^{\infty} |\alpha_n|^2 (|w|^2)^n}{(1 - |w|^2)^{-1}} \leq M, w \in \mathbb{D}.$$

For any $w_0 \in \mathbb{D}$, it follows that $\sum\limits_{n=0}^{\infty} |\alpha_n|^2(|w_0|^2)^n$ is convergent. By direct calculation, we have that

$$(1-|w_0|^2)(\sum_{n=0}^{\infty} |\alpha_n|^2(|w_0|^2)^n) = |\alpha_0|^2 + \sum_{n=1}^{\infty}(|\alpha_n|^2 - |\alpha_{n-1}|^2)(|w_0|^2)^n.$$

Since $\|T\| \leq 1$, we have that $|\omega_k| \leq 1$ and $|\alpha_n|^2 \geq |\alpha_{n-1}|^2$ for any $n > 0$. If we choose any fixed integer N and $\delta_0 < M$, then for $w_0 \in \mathbb{D}$, we can find $N_{\delta_0,w_0} \geq N$ such that

$$|\alpha_0|^2+\sum_{n=1}^{N}(|\alpha_n|^2-|\alpha_{n-1}|^2)(|w_0|^2)^n \leq |\alpha_0|^2+\sum_{n=1}^{N_{\delta_0,w_0}}(|\alpha_n|^2-|\alpha_{n-1}|^2)(|w_0|^2)^n \leq M-\delta_0.$$

Then for the fixed N, the inequality

$$|\alpha_0|^2 + \sum_{n=1}^{N}(|\alpha_n|^2 - |\alpha_{n-1}|^2)(|w_0|^2)^n \leq M - \delta_0$$

holds for any $w_0 \in \mathbb{D}$. When $|w_0| \to 1$, we will have that

$$|\alpha_N|^2 = |\alpha_0|^2 + \sum_{n=1}^{N}(|\alpha_n|^2 - |\alpha_{n-1}|^2) \leq M - \delta_0.$$

Also notice that $|\alpha_N|^2 \geq 1$. By Lemma 2.6, we have that T is similar to S_1^*.

3 $U + K$ Similarity of Operators in $B_1(\Omega)$

Instead of considering the quotient of the curvatures of operators in $B_1(\Omega)$, it is shown that the study on the difference of curvature is a natural choice for the similarity problem of Cowen-Douglas operators in $B_1(\Omega)$. In [17], S. Treil and B. D. Wick gave a sufficient condition for the existence of a bounded analytic projection onto a holomorphic family of generally infinite dimensional subspaces induced by some holomorphic bundle. As a corollary, they also obtained some new results about the Operator Corona Problem.

Let E be a Hilbert space, $P : \mathbb{D} \to B(E)$ be a $\mathbb{C}^2$ projection-valued function and $P\partial P = 0$. In [17], as their main theorem, it was proved that if there exists a bounded non-negative subharmonic function ϕ such that

$$\Delta\phi(w) \geq \|\partial P(w)\|^2, \forall w \in \mathbb{D},$$

then there exists some analytic idempotent valued function $\Pi \in H^{\infty}_{E\to E}$ such that $ran\Pi(w) = ranP(w)$.

By using this result and a model theorem for contractions, H. Kwon and S. Treil gave a very impressive theorem to decide when a contraction operator T be similar to the n copies of M_z^* on the Hardy space. For any contraction operator $T \in B_n(\mathbb{D})$, let $P(w)$ denote the projection onto $ker(T-w)$. It was proved that $T \sim_s \bigoplus\limits^{n} M_z^*$ if and only if

$$\|\frac{\partial P(w)}{\partial w}\|^2_{HS} - \frac{n}{(1-|w|^2)^2} \leq \Delta\phi(w), \forall w \in \mathbb{D},$$

where $\|\partial P(w)\|^2_{HS}$ is pointed out to be the curvature for the Hardy shift (cf. [15]) and ϕ is a bounded subharmonic function. Subsequently, the result was generalized from the Hardy shift case to some weighted Bergman shift cases ($S_n, n \geq 1$) by R. G. Douglas, H. Kwon and S. Treil (see in [7]). In [9, 11], $\|\partial P(w)\|^2_{HS}$ is shown to be the trace of the curvature K_T for any operator $T \in B_n(\Omega)$. Thus, when T is an n-hypercontraction, the difference of the curvatures $K_T - K_S$ (or difference of trace of curvatures $traceK_T - traceK_S$) could be regarded as the similarity invariant of Cowen-Douglas operators of rank one (or rank n). We then have the following important question that rise naturally:

Question *Let $T, S \in B_1(\Omega)$ be two arbitrary Cowen-Douglas operators. Under what kind of assumptions on the function ϕ, do we have that T is similar to S if and only if $K_T - K_S = \Delta\phi$?*

Let T and S be two operators in $\mathcal{B}(\mathcal{H})$. We call T and S are $U+K$ similarity equivalent if there exists a unitary operator U and a compact operator K such that $X := U + K$ is an invertible operator which satisfies $XT = SX$. In the following lemma, we will give a description of the $U+K$ similarity of Cowen-Douglas operators with index one by using the difference of curvatures. Note the difference between this result and the result given by R.G. Douglas, H. Kwon and S. Treil, in that we do not need the operator model theorem given by J. Alger. Thus, we do not need to assume that T or S is an n-hypercontraction operator.

It is well known that any operator $T \in B_1(\Omega)$ can be realized as the adjoint of the multiplication operator on a reproducing kernel Hilbert space of holomorphic functions on Ω^* (see in [3]). Thus, in the following theorem, we will assume that the operator $S \sim_u (M_z^*, \mathcal{H}, \mathcal{K}_S)$, where $\mathcal{H}$ is a functional Hilbert space with reproducing kernel $\mathcal{K}_S$.

Theorem 3.1 *Let $T, S \in B_1(\Omega)$, and $S \sim_u (M_z^*, \mathcal{H}, \mathcal{K}_S)$. Let $\{e_n(z)\}_{n=0}^{\infty}$ denote the canonical ONB of the functional Hilbert space $\mathcal{H}$. Then there exist unitary operator U and a compact operator K ($\|K\| < 1$) such that $T \sim_{U+K} S$ if and only if $K_S - K_T = \Delta ln\phi$, where ϕ is a bounded function defined as*

$$\phi(w) = 1 + \frac{\sum\limits_{i=0}^{m} 2Re\phi_i(w)\overline{\psi_i(w)} + \sum\limits_{i=0}^{m} |\phi_i(w)|^2}{\mathcal{K}_S(w,w)}, w \in \Omega,$$

for some positive integer m, and $\{\phi_i\}_{i=0}^m, \{\psi_i\}_{i=0}^m$ *are both orthogonal with* $1 > \|\phi_i\| \to 0$ *as* $i \to \infty$, *and* $\|\psi_i\| = 1$. *The integer m is in fact the rank of K. Furthermore, if* $K_S \geq K_T$, *then* $ln\phi$ *is a bounded subharmonic function.*

Proof Set $\phi_i(z) = \sum_{n=0}^{\infty} \alpha_n^i \overline{e_n(\bar{z})}$, $\psi_i(z) = \sum_{n=0}^{\infty} \beta_n^i \overline{e_n(\bar{z})}$, where $\{\alpha_n^i\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty} \in l^2$ such that

$$1 > \sum_{n=0}^{\infty} |\alpha_n^i|^2 \to 0, i \to \infty,$$

and

$$\sum_{n=0}^{\infty} |\beta_n^i|^2 = 1, i \geq 0.$$

Then

$$\|\phi_i\|^2 = \sum_{n=0}^{\infty} |\alpha_n^i|^2 \to 0, i \to \infty, \|\psi_i\|^2 = \sum_{n=0}^{\infty} |\beta_n^i|^2 = 1.$$

Now set $e_i = \sum_{n=0}^{\infty} \overline{\alpha_n^i} e_n(z)$ and $f_i = \sum_{n=0}^{\infty} \overline{\beta_n^i} e_n(z)$. Define an operator $\tilde{K}$ as follows:

$$\tilde{K}(f) = \sum_{i=0}^{m} (e_i \otimes f_i)(f), \quad \text{for any } \; f \in \mathcal{H}.$$

Since $\|f_i\|^2 = \sum_{n=0}^{\infty} |\beta_n^i|^2 = 1$, and $1 > \|e_i\|^2 = \sum_{n=0}^{\infty} |\alpha_n^i|^2 \to 0$, as $i \to \infty$, we can see that $\tilde{K}$ is a compact operator with $\|\tilde{K}\| < 1$ and m is the rank of $\tilde{K}$.

Set $X = I + \tilde{K}$ and $e(w) := \mathcal{K}_S(z, \bar{w}) = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(\bar{w})} \in ker(M_z^* - w)$. Then X is invertible and we have

$$\begin{aligned} \|X(e(w))\|^2 &= \langle e(w) + \tilde{K}(e(w)), e(w) + \tilde{K}(e(w))\rangle \\ &= \mathcal{K}_S(w, w) + \langle e(w), \tilde{K}(e(w))\rangle + \langle \tilde{K}(e(w)), e(w)\rangle + \|\tilde{K}(e(w))\|^2 \\ &= \mathcal{K}_S(w, w) + 2Re\langle \sum_{i=0}^{m} \langle e(w), e_i\rangle f_i, e(w)\rangle + \sum_{i=0}^{m} |\langle e(w), e_i\rangle|^2. \end{aligned}$$

Notice that since

$$\begin{aligned}\langle e(w), e_i\rangle &= \langle \sum_{n=0}^{\infty} \overline{e_n(\bar{w})} e_n(z), \sum_{n=0}^{\infty} \overline{a_n^i} e_n(z)\rangle \\ &= \sum_{n=0}^{\infty} a_n^i \overline{e_n(\bar{w})} \\ &= \phi_i(w),\end{aligned}$$

and $\langle e(w), f_i\rangle = \psi_i(w)$, we then have

$$\|X(e(w))\|^2 = \mathcal{K}_S(w, w) + 2Re \sum_{i=0}^{m} \phi_i(w)\overline{\psi_i(w)} + \sum_{i=0}^{m} |\phi_i(w)|^2.$$

Since $X(e(w)) \in ker(XSX^{-1} - w)$, then $K_{XSX^{-1}} = -\Delta ln\|X(e(w))\|^2$. Thus, for any $w \in \Omega$, we have that

$$\begin{aligned}K_S - K_{XSX^{-1}} &= \Delta ln\|X(e(w))\|^2 - \Delta ln\|e(w)\|^2 \\ &= \Delta ln(1 + \frac{\sum\limits_{i=0}^{m} 2Re\phi_i(w)\overline{\psi_i(w)} + \sum\limits_{i=0}^{m} |\phi_i(w)|^2}{\mathcal{K}_S(w, w)}).\end{aligned}$$

By the assumptions of the lemma, we can see that $K_S - K_T = K_S - K_{XSX^{-1}}$. That means $K_T = K_{XSX^{-1}}$. Thus, there exists a unitary operator U such that

$$T = UXSX^{-1}U^*.$$

If we set $K = U\tilde{K}$, then $UX = U(1 + \tilde{K}) = U + K$, and therefore $T = (U + K)S(U + K)^{-1}$. This finishes the proof of the sufficient part.

On the other hand, suppose there exist a unitary operator U and a compact operator K such that $U + K$ is invertible and $(U + K)S(U + K)^{-1} = T$. If we let $e(w) \in ker(S - w)$, $w \in \Omega$, then $(U + K)(e(w)) \in ker(T - w)$, and

$$K_T(w) = -\Delta ln\|(U + K)(e(w))\|^2.$$

Furthermore,

$$\begin{aligned}\|(U + K)(e(w))\|^2 &= \langle (U + K)(e(w)), (U + K)(e(w))\rangle \\ &= \langle U^*(U + K)(e(w)), U^*(U + K)(e(w))\rangle \\ &= \langle (I + U^*K)(e(w)), (I + U^*K)(e(w))\rangle.\end{aligned}$$

Notice that U^*K is a compact operator, there exist orthogonal sets $\{e_i\}_{i=0}^m$, $\{f_i\}_{i=0}^m$, with $\|e_i\| \to 0$, $\|f_i\| = 1$ such that $U^*K = \sum_{i=0}^m e_i \otimes f_i$.

Similar to the proof of the sufficient part, suppose that

$$e_i = \sum_{n=0}^{\infty} \overline{\alpha_n^i} e_n(z),\ f_i = \sum_{n=0}^{\infty} \overline{\beta_n^i} e_n(z).$$

Then

$$\begin{aligned}
U^*K(e(w)) &= \sum_{i=0}^{\infty} (e_i \otimes f_i)(e(w)) = \sum_{i=0}^{m} \langle e(w), e_i \rangle f_i \\
&= \sum_{i=0}^{m} (\langle \sum_{n=0}^{\infty} \overline{e_n(\bar{w})} e_n(z), \sum_{n=0}^{\infty} \overline{\alpha_n^i} e_n(z) \rangle f_i) \\
&= \sum_{i=0}^{m} (\sum_{n=0}^{\infty} \alpha_n^i \overline{e_n(\bar{w})}) f_i.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|(I + U^*K)(e(w))\|^2 \\
&= \|e(w)\|^2 + \langle U^*K(e(w)), e(w) \rangle + \langle e(w), U^*K(e(w)) \rangle + \|U^*K(e(w))\|^2 \\
&= \mathcal{K}_S(w, w) + 2Re \langle \sum_{i=0}^{m} (\sum_{n=0}^{\infty} \alpha_n^i \overline{e_n(\bar{w})}) f_i, \sum_{n=0}^{\infty} \overline{e_n(\bar{w})} e_n(z) \rangle + \sum_{i=0}^{m} \sum_{n=0}^{\infty} |\alpha_n^i \overline{e_n(\bar{w})}|^2 \\
&= \mathcal{K}_S(w, w) + 2Re \langle \sum_{i=0}^{m} (\sum_{i=0}^{\infty} \alpha_n^i \overline{e_n(\bar{w})}) (\sum_{n=0}^{\infty} \overline{\beta_n^i} e_n(z)), \sum_{n=0}^{\infty} \overline{e_n(\bar{w})} e_n(z) \rangle \\
&+ \sum_{i=0}^{m} \sum_{n=0}^{\infty} |\alpha_n^i \overline{e_n(\bar{w})}|^2 \\
&= \mathcal{K}_S(w, w) + 2Re \sum_{i=0}^{m} (\sum_{n=0}^{\infty} (\sum_{n=0}^{\infty} \alpha_n^i \overline{e_n(\bar{w})}) \overline{\beta_n^i} e_n(\bar{w})) + \sum_{i=0}^{m} \sum_{n=0}^{\infty} |\alpha_n^i \overline{e_n(\bar{w})}|^2 \\
&= \mathcal{K}_S(w, w) + 2Re \sum_{i=0}^{m} (\sum_{n=0}^{\infty} \alpha_n^i \overline{e_n(\bar{w})}) \overline{(\sum_{n=0}^{\infty} \beta_n^i e_n(\bar{w}))} + \sum_{i=0}^{m} \sum_{n=0}^{\infty} |\alpha_n^i \overline{e_n(\bar{w})}|^2.
\end{aligned}$$

Now set $\phi_i(w) = \sum_{n=0}^{\infty} \alpha_n^i \overline{e_n(\bar{w})}$, and $\psi_i(w) = \sum_{n=0}^{\infty} \beta_n^i \overline{e_n(\bar{w})}$. Then we have ϕ_i, and $\psi_i \in \mathcal{H}$, for any $i \geq 0$. And $\|\phi_i\| = \|e_i\|, \|\psi_i\| = \|f_i\|$. By the definition of e_i, f_i, we have $\|\phi_i\| \to 0, \|\psi_i\| = 1$, and $\{\psi_i\}_{i=0}^{\infty}, \{\phi_i\}_{i=0}^{\infty}$ are orthogonal sets. Furthermore, for any $w \in \Omega$,

$$\begin{aligned} K_S(w) - K_T(w) &= \Delta ln(\frac{\|(U+K)(e(w))\|^2}{\|e(w)\|^2}) \\ &= \Delta ln(1 + \frac{\sum_{i=0}^{m} 2Re\phi_i(w)\overline{\psi_i(w)} + \sum_{i=0}^{m} |\phi_i(w)|^2}{\mathcal{K}_S(w,w)}) \\ &= \Delta ln\phi(w). \end{aligned}$$

Since $U+K$ is bounded, then we have that ϕ is a bounded function. This finishes the proof of the necessary part. Furthermore, if $K_S \geq K_T$, then we have that $\Delta ln\phi \geq 0$, that means $ln\phi$ is a bounded subharmonic function. □

Acknowledgments The authors appreciate very much the suggestions of the reviewers. The suggestions will benefit the improvement of the paper and our future research.

The authors were supported by the National Natural Science Foundation of China (Grant No. 11831006 and 11922108).

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Singular Subgroups in $\widetilde{A}_2$-Groups and Their von Neumann Algebras

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To the memory of Professor Ronald G. Douglas

Abstract We show that certain amenable subgroups inside $\widetilde{A}_2$-groups are singular in the sense of Boutonnet and Carderi. This gives a new family of examples of singular group von Neumann subalgebras. We also give a geometric proof that if G is an acylindrically hyperbolic group, H is an infinite amenable subgroup containing a loxodromic element, then $H < G$ is singular. Finally, we present (counter)examples to show both situations happen concerning maximal amenability of LH inside LG if H does not contain loxodromic elements.

Keywords $\widetilde{A}_2$-groups · Maximal amenability · Singular subgroups · Acylindrically hyperbolic groups · Loxodromic elements

Mathematics Subject Classification (2010) Primary 46L10; Secondary 43A07, 51E24, 20F67

1 Introduction

Let M be a finite von Neumann algebra, N be a von Neumann subalgebra of M and denote by $\mathbb{E}_N$ the trace-preserving conditional expectation from M onto N. A classical topic in von Neumann algebras is to study the relative position of N inside M. There are two closely related notions to describe the relative position of N inside M. One is singularity and the other one is maximal amenability.

Recall that N is called *singular* in M [34] if the normalizer of N, i.e. $\mathcal{N}(N) := \{u \in \mathcal{U}(M) : uNu^* = N\}$, is contained in N. In general, it is not easy to decide

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_18

whether given subalgebras, e.g. maximal abelian subalgebras (masas), are singular and this prompted Sinclair and Smith to introduce, a priori, stronger notion of singularity, which was called strongly singularity in [26]. Recall that N is said to be *strongly singular* if, for every unitary $u \in M$

$$\sup_{||x||\leq 1} ||(\mathbb{E}_N - \mathbb{E}_{uNu^*})x||_2 \geq ||(Id - \mathbb{E}_N)u||_2,$$

where $||\cdot||_2$ denotes the L^2-norm associated with a prescribed faithful normal trace on M. Although the definition is more involved, it is easier to check, especially for group von Neumann subalgebras. For example, certain subgroups of hyperbolic groups are shown to give rise to strongly singular von Neumann subalgebras in [26]. Moreover, it was shown in [27] that a singular masas is in fact also strongly singular for a separable II_1 factor M.

Besides singularity, one also studies maximal amenability. Recall that N is *maximal amenable* in M if N is amenable and there are no amenable subalgebras in M that strictly contain N.

Clearly, a maximal amenable von Neumann subalgebra is automatically singular. Although every nonamenable von Neumann algebra M contains maximal amenable von Neumann subalgebras by Zorn's lemma, it is rather difficult to construct concrete examples of maximal amenable von Neumann subalgebras.

The first such a concrete example is due to Popa. In [20] Popa proved that the abelian von Neumann subalgebra generated by one of the generators of the non-abelian free group F_n, i.e. the generator masas, is maximal injective in the free group factor $L(F_n)$. One ingredient in his proof is the so-called "asymptotic orthogonality property" for the generator masas inside $L(F_n)$. This method was later applied elsewhere, see e.g. [2, 11].

More recently, new techniques introduced in [1] allowed to obtain more explicit examples of maximal amenable group von Neumann subalgebras that come from infinite maximal amenable subgroups. This strategy is best suited for groups acting, in an appropriately regular way, on geometric objects and includes hyperbolic groups, many semisimple Lie groups of higher rank such as $\mathrm{SL}_3(\mathbb{Z})$.

In [23, 24] for groups acting on geometric objects, e.g. affine buildings, certain subgroups are shown to give rise to strongly singular von Neumann subalgebras. If we regard the homogeneous tree as a one dimensional affine building of type $\widetilde{A}_1$, then the degenerate case of the results in [23, 24] states that the generator masas in $L(F_n)$ are strongly singular. Hence, it is natural to ask whether results in [23, 24] can be strengthened to show the von Neumann subalgebras are actually maximal amenable. The main result of our paper is an affirmative answer to this question, see Theorem 2.5, Corollary 2.6. The proof is based on the geometric approach in [1] and previous work in [23].

We organize this paper as follows. In Sect. 2, we recall necessary background on affine buildings that are used in our main theorem. In Sect. 3, we apply the geometric approach to acylindrically hyperbolic groups, which inspired questions studied in Sect. 4.

2 Preliminaries and Main Theorem

2.1 Affine Buildings

Let us briefly recall several standard facts on affine buildings, we refer the readers to [4, 25] for more details.

Let Δ be an affine building. By Δ^0 we denote its set of vertices. Similarly, let $\mathcal{A}$ be an apartment, then $\mathcal{A}^0$ denotes its vertices. Recall that a *sector* is a simplicial cone based at a special vertex in some apartment; two sectors are *equivalent* (or parallel) if their intersection contains a sector. The *boundary* Ω is defined to be the set of equivalent classes of sectors in Δ. Fix a special vertex $x \in \Delta$, for any $\omega \in \Omega$, there is a unique sector $[x, \omega)$ in the class ω having base vertex x [25, Theorem 9.6, Lemma 9.7]. Ω also has the structure of a spherical building [25, Theorem 9.6] and topologically, Ω is a totally disconnected compact Hausdorff space and a basis for the topology is given by the set of the form $\Omega_x(v) = \{\omega \in \Omega : [x, \omega) \text{ contains } v\}$.

Two boundary points $\omega, \bar{\omega}$ in Ω are said to be *opposite* if the distance between them is the diameter of the spherical building Ω. This is equivalent to the property that they are represented by opposite sectors $S, \bar{S}$ with the same base vertex in some apartment of Δ by [12, Lemma 3.5].

For an ω in Ω, we can define $\mathcal{O}(\omega)$ to be the set of all $\omega' \in \Omega$ such that ω' is opposite to ω. Note that $\mathcal{O}(\omega)$ is an open set. Moreover, if $\omega \in \Omega$ and $\mathcal{A}$ is an apartment in Δ, then there exists a boundary point $\bar{\omega}$ of $\mathcal{A}$ such that $\bar{\omega}$ is opposite ω [12, Lemma 3.2]. As a corollary, if $\omega_1, \ldots, \omega_n$ are the boundary points of an apartment, then $\Omega = \mathcal{O}(\omega_1) \cup \cdots \cup \mathcal{O}(\omega_n)$ [12, Corollary 3.3].

Motivated by [23, Section 5] we fix a group G of automorphisms of an affine building Δ with boundary Ω satisfying the following properties.

- (B1) G acts freely on the vertex set Δ^0 with finitely many vertex orbits (i.e. cocompactly).
- (B2') There is an apartment $\mathcal{A}$ in Δ and an amenable subgroup H of G such that H preserves $\mathcal{A}$ and $H \setminus \mathcal{A}$ is compact, i.e. $\mathcal{A}$ is a *periodic* apartment. In particular, $H \setminus \mathcal{A}^0$ is finite, where $\mathcal{A}^0$ is the vertex set of $\mathcal{A}$.
- (B3) The natural mapping $H \setminus \mathcal{A}^0 \to G \setminus \Delta^0$ is injective.

Remark 2.1

(1) In [23, Section 5], an almost identical set of conditions is introduced. The only difference is condition (B2) therein, which was stated as follows (using our notations): There is an apartment $\mathcal{A}$ in Δ and an abelian subgroup H of G such that $H \setminus \mathcal{A}^0$ is finite, where $\mathcal{A}^0$ is the vertex set of $\mathcal{A}$. It was made clear in [23, Remark 5.1(b)] that the sole reason for assuming that H is abelian in condition (B2) is to obtain an abelian von Neumann algebra LH. Everything else works equally well without this assumption. Here, we use notation (B2') to distinguish it from condition (B2). Note that both (B2) and (B2') are checked and applied for the same type of examples.

(2) As observed in [23, Lemma 5.2], since G acts freely on Δ^0, condition (B3) guarantees that for any $g \in G$, $g\mathcal{A}^0 \cap \mathcal{A}^0 \neq \emptyset$ implies $g \in H$.
(3) The above notion of periodic apartments was called "doubly periodic apartments" in [21, 24]; while in [12, 23], it was simply called periodic apartments.

Let $\mathcal{A}$ be the periodic apartment appeared in condition (B2'), and fix a special vertex z in $\mathcal{A}$. As explained in [12, P. 207], we choose a pair of opposite sectors W^+, W^- in $\mathcal{A}$ based at z and denote by $\omega^\pm$ the boundary points represented by $W^\pm$, respectively. By periodicity of $\mathcal{A}$, there is a periodic direction represented by a line L in the sector direction of W^+. This means that there exists some element $u \in G$ which leaves L invariant and translates the apartment $\mathcal{A}$ in the direction of L. Then $u^n\omega^+ = \omega^+, u^n\omega^- = \omega^-$ for all $n \in \mathbb{Z}$.

One ingredient for our proof is [12, Proposition 3.7], which shows that ω^- is an attracting fixed point for u^{-1}. We state it below (using our notations) for readers' convenience.

Proposition 2.2 (Proposition 3.7 in [12]) *Let G acts properly and cocompactly on an affine building Δ with boundary Ω. Let $\mathcal{A}$ be a periodic apartment and choose a pair of opposite boundary points $\omega^\pm$. Let $u \in G$ be an element which translates the apartment $\mathcal{A}$ in the direction of ω^+. Then u^{-1} attracts $\mathcal{O}(\omega^+)$ towards ω^-; that is, for each compact subset K of $\mathcal{O}(\omega^+)$ we have* $\lim_{n\to\infty} u^{-n}(K) = \{\omega^-\}$.

We note that during the proof of this proposition, the authors introduced an increasing family of compact open sets $K_0 \subset K_1 \subset K_2 \subset \cdots$ such that $\cup_{N=0}^{\infty} K_N = \mathcal{O}(\omega^+)$ and they actually proved that $\lim_{n\to\infty} u^{-n}(K_N) = \{\omega^-\}$ for each $N \geq 0$.

2.2 $\widetilde{A}_2$-Groups

An $\widetilde{A}_2$-group acts simply transitively on the vertices of an affine building of type $\widetilde{A}_2$. Such groups were studied in [5, 6] through a combinatorial description, i.e. the so-called *triangle presentation*. We would not recall the definition here, but refer the readers to [24, Introduction] for a clean presentation and the above papers for details.

$\widetilde{A}_2$-groups have Kazhdan's property (T) by [7, Theorem 4.6] and operator algebras associated with $\widetilde{A}_2$-groups were studied extensively, see e.g. [12, 21–24].

2.3 Singular Subgroups

Let us recall the notion of singular subgroups as introduced in [1].

Definition 2.3 Consider an amenable subgroup H of a discrete countable group G. Suppose that G acts by homeomorphisms on the compact space X. We say that H is singular in G (with respect to X) if for any H-invariant probability measure μ on X and $g \in G \setminus H$ we have $g \cdot \mu \perp \mu$.

For convenience we will denote by $\mathrm{Prob}_H(X)$ the space of H-invariant probability measures on X.

It turns out that with the presence of singularity, an amenable subgroup is automatically maximal amenable [1, Lemma 2.2]. More importantly, this fact is also witnessed at the level of von Neumann algebras as shown in the following theorem.

Theorem 2.4 (Theorem 2.4 in [1]) *Suppose G is a discrete countable group admitting an amenable, singular subgroup H. Then for any trace preserving action $G \curvearrowright (Q, \tau)$ on a finite amenable von Neumann algebra, $Q \rtimes H$ is maximal amenable inside $Q \rtimes G$.*

2.4 Main Theorem and Its Proof

Now, we are ready to state our main theorem, which is a strengthening of [23, Theorem 5.8].

Theorem 2.5 *Let G be a group of automorphisms of a locally finite affine building Δ with boundary Ω. Assume that (B1), (B2'), (B3) hold and $H \subseteq G$ is as described in condition (B2'). Then H is a singular subgroup in G.*

Proof According to Definition 2.3, we need to show that for any $\mu \in \mathrm{Prob}_H(\Omega)$ and every $g \in G \setminus H$, we have $g \cdot \mu \perp \mu$. Consider such a $\mu \in \mathrm{Prob}_H(\Omega)$ and denote by $\{w_1, \ldots, w_k\}$ the boundary points of the apartment $\mathcal{A}$ appeared in condition (B2'). Note that Ω is a spherical building and k equals the cardinality of the spherical Weyl group, which is finite.

We claim that $\mathrm{supp}(\mu) \subseteq \{w_1, \ldots, w_k\}$. Indeed, assume the contrary and take any $w \in \mathrm{supp}(\mu) \setminus \{w_1, \ldots, w_k\}$. Since $w \in \mathrm{supp}(\mu)$, we may take a small closed neighborhood of w, say N_w, such that $N_w \cap \{w_1, \ldots, w_k\} = \emptyset$ and $\mu(N_w) > 0$. Since the boundary points of the apartment $\mathcal{A}$ are exactly $w_1, \ldots, w_k$, we may apply [12, Corollary 3.3] to deduce $\Omega = \mathcal{O}(w_1) \cup \ldots \cup \mathcal{O}(w_k)$, where $\mathcal{O}(w)$ is the set of all $w' \in \Omega$ such that w' is opposite to w.

Without loss of generality, we may assume $\mu(N_w \cap \mathcal{O}(w_1)) > 0$ and that w_k is the opposite boundary point of w_1. $N_w \cap \mathcal{O}(w_1)$ may not be a compact subset of $\mathcal{O}(w_1)$, but one may replace it with the intersection with some K_n defined in the proof of [12, Proposition 3.7]. See the paragraph after Proposition 2.2 for a quick explanation.

Hence we obtain a compact subset with $\mu(N_w \cap \mathcal{O}(w_1) \cap K_n) > 0$. Note that [12, Proposition 3.7] applies since condition (B1) guarantees that the action of G on the vertex set Δ^0 is proper and cocompact and $\mathcal{A}$ is periodic by condition (B2').

Without loss of generality, we assume $N_w \cap \mathcal{O}(w_1)$ is a compact subset of $\mathcal{O}(w_1)$. Then by [23, Proposition 3.7], we know that $\lim_{n\to\infty} u^{-n}(N_w \cap \mathcal{O}(w_1)) = \{w_k\}$, where $u \in G$ is an element which translates the apartment $\mathcal{A}$ in the direction of w_1.

Note that u satisfies the extra property $u\omega_1 = \omega_1$, this implies that $u \in H$. Indeed, since we have sector representatives for $u\omega_1$ and ω_1 in the apartment $u\mathcal{A}$ and $\mathcal{A}$ respectively, $u\mathcal{A}^0 \cap \mathcal{A}^0$ contains a subsector; in particular, $u\mathcal{A}^0 \cap \mathcal{A}^0 \neq \emptyset$. Then by [12, Lemma 5.2], condition (B3) implies $u \in H$.

Since all w_i are fixed points under H, we deduce for any $n \geq 1$, $w_k \notin u^{-n}(N_w)$, which implies $w_k \notin \overline{u^{-n}(N_w \cap \mathcal{O}(w_1))}$, since N_w is closed in Ω. Therefore, we may find an increasing sequence $n_i \to \infty$ such that $u^{-n_i}(N_w \cap \mathcal{O}(w_1)) \cap u^{-n_j}(N_w \cap \mathcal{O}(w_1)) = \emptyset$ for all $i \neq j$. Hence, we deduce that $1 = \mu(\Omega) \geq \mu(\sqcup_{i=1}^{\infty} u^{-n_i}(N_w \cap \mathcal{O}(w_1))) = \sum_{i=1}^{\infty} \mu(u^{-n_i}(N_w \cap \mathcal{O}(w_1))) = \sum_{i=1}^{\infty} \mu(N_w \cap \mathcal{O}(w_1)) = \infty$, a contradiction.

We now claim that $g \cdot \mathrm{supp}(\mu) \cap \mathrm{supp}(\mu) = \emptyset$ for any $g \in G \setminus H$. To see this, assume the contrary. Then $gw_i = w_j$ for some $i, j \in \{1, \ldots, k\}$ by the above claim. Since we have sector representatives for gw_i and w_j in the apartment $g\mathcal{A}$ and $\mathcal{A}$ respectively, $g\mathcal{A}^0 \cap \mathcal{A}^0$ contains a subsector; in particular, $g\mathcal{A}^0 \cap \mathcal{A}^0 \neq \emptyset$. Then by [23, Lemma 5.2], condition (B3) implies $g \in H$, a contradiction.

Then, combining the above two claims, we deduce that $g \cdot \mu \perp \mu$ for all $g \notin H$. □

Applying Theorem 2.5 to $\widetilde{A}_2$-buildings, we have the following corollary.

Corollary 2.6 *Let G be an $\widetilde{A}_2$ group acting on an $\widetilde{A}_2$-building Δ and $H < G$ be an abelian subgroup which acts simply transitively on the vertex set of an apartment $\mathcal{A}$ in Δ. Then H is singular in G.*

The above result is a strengthening of [24, Theorem 2.8].

Indeed, in the above example, $H \cong \mathbb{Z}^2$ by [24, P. 6] and the apartment $\mathcal{A}$ is (doubly) periodic [24, p. 6–7]. By [23, Example 5.9], we know all conditions (B1), (B2'), (B3) are satisfied.

As explained in [23, Example 5.9] or [24, Remark 1.5], we can apply the above corollary to G being the groups (4.1), (5.1), (6.1), (9.2), (13.1) and (28.1) in the table of the end of [6].

Note that $\widetilde{A}_2$ groups have Kazhdan's property (T) by [7, Theorem 4.6] and they give rise to II_1 factors by [24, Lemma 0.2] or [23, Lemma 5.6]. So we have more examples of higher rank abelian, maximal amenable subalgebras in II_1 factors with property (T). See [1, 2] for more examples.

3 Acylindrically Hyperbolic Groups

In [2, p. 1201], it was mentioned that if $H < G$ is an infinite amenable subgroup which is hyperbolically embedded then LH is maximal amenable inside LG. Since the proof was based on Popa's asymptotic orthogonality approach and was omitted, we take this opportunity to include a proof of a somewhat different version (see Remark 3.6) of this result using the geometric approach in [1]. The proof is similar to the proof of [1, Lemma 3.2], but uses more recent work on acylindrically hyperbolic groups.

Let us first briefly recall the standard terminology related to acylindrically hyperbolic groups, we refer the readers to [8, 17] for details.

An action of a group G on a metric space S is called *acylindrical* if for every $\epsilon > 0$ there exist $R, N > 0$ such that for every two points x, y with $d(x, y) > R$, there are at most N elements $g \in G$ satisfying $d(x, gx) \leq \epsilon$ and $d(y, gy) \leq \epsilon$. From now on, we assume the space S is hyperbolic and G acts on S isometrically, this action extends to an action on its Gromov boundary $X := \partial S$ by homeomorphisms. We say an element $g \in G$ is *loxodromic* if the map $\mathbb{Z} \to S$ defined by $n \mapsto g^n s$ is a quasi-isometry for some (equivalently, any) $s \in S$. Every loxodromic element $g \in G$ has exactly two limit points $g^{\pm\infty}$ on ∂S. Loxodromic elements $g, h \in G$ are called *independent* if the sets $\{g^{\pm\infty}\}$ and $\{h^{\pm\infty}\}$ are disjoint.

We say the action $G \curvearrowright S$ is *elementary* if the limit set of G on ∂S contains at most two points. Here, the limit set of G is just the set of accumulation points of a G-orbits on ∂S. In fact, this definition does not depend on the choice of G-orbits.

G is called an *acylindrically hyperbolic group* if it admits a non-elementary acylindrical action on a hyperbolic space S. Typical examples of acylindrically hyperbolic groups include non-elementary hyperbolic groups, certain non-virtually-cyclic relatively hyperbolic groups, mapping class groups and $Out(F_n)$ for $n \geq 2$ etc.

A useful tool used later is the following theorem of Osin on classification of groups acting acylindrically on hyperbolic spaces. Note that for an acylindrically hyperbolic group G (w.r.t. $G \curvearrowright S$), condition (3) below holds.

Theorem 3.1 (Theorem 1.1 in [17]) *Let G be a group acting acylindrically on a hyperbolic space S (isometrically). Then G satisfies exactly one of the following three conditions.*

1. *G has bounded orbits.*
2. *G is virtually cyclic and contains a loxodromic element.*
3. *G contains infinitely many independent loxodromic elements.*

We are now in the position to state the following result.

Theorem 3.2 *Let G be an acylindrically hyperbolic group (say w.r.t the action $G \curvearrowright S$) and H be any maximal amenable subgroup containing a loxodromic element h (w.r.t. $G \curvearrowright S$). Then $LH < LG$ is maximal injective.*

This is a direct corollary of the following proposition which proves that H is singular in G.

Proposition 3.3 *Let $H < G$ be an infinite maximal amenable subgroup containing a loxodromic element h. Let $X = \partial S$. Then the following statements hold.*

1. *There exist two points $a, b \in X$ such that H is the stabilizer of the set $\{a, b\}$, that is* $H = \mathrm{Stab}_G(\{a, b\}) := \{g \in G : g \cdot \{a, b\} = \{a, b\}\}$.
2. *Any H-invariant probability measure on X is of the form $t\delta_a + (1-t)\delta_b$ for some* $t \in [0, 1]$.
3. *Any element $g \in G \setminus H$ is such that $g \cdot \{a, b\} \cap \{a, b\} = \emptyset$.*

For the proof, we record the following lemma.

Lemma 3.4 *Let a, b be the two fixed points of the loxodromic element h in $X = \partial S$. Then* $\mathrm{Stab}_G(a) = \mathrm{Stab}_G(b)$.

Proof Assume not, then $\mathrm{Stab}_G(a) \Delta \mathrm{Stab}_G(b) \neq \emptyset$. If $g \in \mathrm{Stab}_G(a) \setminus \mathrm{Stab}_G(b)$. Then ghg^{-1} is also loxodromic by definition. And note that $Fix(h) = \{a, b\}$, but $b \notin Fix(ghg^{-1}) \ni a$. Hence, for each $t \in \langle ghg^{-1}, h\rangle$, $a \in Fix(t)$. Then by [17, Theorem 1.1], $\langle ghg^{-1}, h\rangle$ is virtually cyclic and contains a loxodromic element t. Then $e \neq gh^n g^{-1} = h^{n'} \in \langle t\rangle$ for some nonzero integers n, n'. Then $Fix(ghg^{-1}) = Fix(gh^n g^{-1}) = Fix(h^{n'}) = Fix(h)$, a contradiction. The other case is proved similarly. □

Proof of Proposition 3.3 By [10, Proposition 3.4], h acts on X with a north-south dynamics. Denote by a, b the two fixed points of h in $X = \partial S$, and let us assume a is the attracting point.

(i) Let $s \in H$. Then shs^{-1} is a loxodromic element with fixed points $s \cdot a$ and $s \cdot b$. If $\{a, b\} \cap \{s \cdot a, s \cdot b\} = \emptyset$, then by the ping-pong lemma, $H \supseteq \langle h, shs^{-1}\rangle$ contains a free group, which is impossible since H is amenable. Then by Lemma 3.4, $\{s \cdot a, s \cdot b\} = \{a, b\}$ since shs^{-1} and h fix a common point and hence the other point. Hence $H \subseteq \mathrm{Stab}_G(\{a, b\})$. To show that equality holds we note that $\mathrm{Stab}_G(\{a, b\})$ is amenable since $[\mathrm{Stab}_G(\{a, b\}) : \mathrm{Stab}_G(a) \cap \mathrm{Stab}_G(b)] \leq 2$ and $\mathrm{Stab}_G(a)$ is virtually cyclic by [17, Theorem 1.1].

(ii) We only need to show the support of any H-invariant probability measure is contained in $\{a, b\}$. This is a consequence of the north-south dynamics action of h. We sketch the proof for completeness. Assume there exists $p \in \mathrm{supp}(\mu) \setminus \{a, b\}$, then since X is complete Hausdorff (i.e. for any two distinct points $u, v \in X$, there are open sets U, V containing u, v respectively, such that $\bar{U} \cap \bar{V} = \emptyset$, see [28]), we may find a closed neighborhood $\mathcal{O}_p$ of p such that $\mathcal{O}_p \cap \{a, b\} = \emptyset$ and $\mu(\mathcal{O}_p) > 0$. Then there exists an increasing sequence n_i such that $h^{n_i}\mathcal{O}_p \to a$ and the family of sets $\{h^{n_i}\mathcal{O}_p\}_i$ is pairwise disjoint, hence we get a contradiction since $\mu(h^{n_i}\mathcal{O}_p) = \mu(\mathcal{O}_p)$.

(iii) By Lemma 3.4 and (1), we know that $\mathrm{Stab}_G(a) = \mathrm{Stab}_G(b) \subseteq H$. Then the proof goes similarly as in [1]. We include it for completeness. Take $g \in G$ such that $g \cdot a = b$. If there exists some $s \in H$ which exchanges a and b. Then sg fixes a and so $g \in H$. Otherwise, all elements in H fix a and b, then gsg^{-1} fixes b and $g^{-1}sg$ fixes a, for all $s \in H$. Hence g normalizes H so $g \in H$ by maximal amenability. □

Remark 3.5 H always exists since every group element $g \in G$ is contained in a maximal amenable subgroup by Zorn's lemma. And such an H is virtually cyclic by [17, Theorem 1.1].

Remark 3.6 Note that in Theorem 3.2, the subgroup H is hyperbolically embedded by [8, Theorem 6.8]. It is not clear to us whether one can find a loxodromic element inside any infinite hyperbolically embedded amenable subgroup H. It is known that if H does not contain any loxodromic elements, then H is elliptic by [17, Theorem 1.1]. On the one hand, there do exist elliptic subgroups that are not hyperbolically embedded, see [15, Corollary 7.8]; on the other hand, if a hyperbolic embedded subgroup H is virtually cyclic, then it contains a loxodromic element by the proof of $(L_4) \Rightarrow (L_1)$ in the proof of [17, Theorem 1.4].

4 The Case of No Loxodromic Elements in *H*

Motivated by Theorem 3.2, it is natural to ask whether we can drop the assumption that H contains loxodromic elements or more generally H is hyperbolically embedded. Moreover, by [16], we know that many non-amenable groups with positive first ℓ^2-Betti number are acylindrically hyperbolic, then it is natural to ask whether LH is maximal amenable in LG if $H < G$ is infinite maximal amenable and $\beta_1^{(2)}(G) > 0$. Modifying the example in [1, p. 1697], we show that both questions have negative answers.

Proposition 4.1 *Let* $K = BS(m,n) = \langle a, t | ta^m t^{-1} = a^n \rangle$, $H = \langle a \rangle < K$ *and* $G = K * F_2$, *where* F_2 *denotes the non-abelian free group on two generators. Then the following statements hold.*

1. $\beta_1^{(2)}(G) > 0$, *and if* $|m|, |n| \geq 3$, *then* H *is maximal amenable in* G *but* LH *is not maximal amenable in* LG.
2. K *is not acylindrically hyperbolic if* $|m|, |n| \geq 3$, *while* G *is acylindrically hyperbolic and* $H < G$ *is not hyperbolically embedded.*
3. *If* $c : G \to \mathcal{H}$ *is any cocycle with* $c(a) = 0$, *then* $c(t) = 0$, *i.e.* $ker(c) \neq H$, *where* $G \curvearrowright \mathcal{H}$ *is a mixing unitary representation.*

Proof

(1) By [19, Proposition 3.1], $\beta_1^{(2)}(G) \geq \beta_1^{(2)}(K) + \beta_1^{(2)}(F_2) \geq 1$. As explained in [1, p. 1697], H is maximal amenable in K if $|m|, |n| \geq 3$ (see Proposition 4.3 below for a different proof) but LH is not maximal amenable LK since $x := \sum_{k=0}^{n-1} a^k tat^{-1} a^{-k} \in \mathbb{C}K$ commutes with LH, hence LH is not maximal amenable in LG either. Then we can apply Proposition 4.2 below to see H is still maximal amenable in G since K is torsion free if $mn \neq 0$ by [13].

(2) If $|m|, |n| \geq 3$, then K is not acylindrically hyperbolic by [17, Example 7.4]. While G is acylindrically hyperbolic by [15] or [16, Corollary 1.3 or Theorem 1.1], and observe that $H < G$ is not almost malnormal since $tHt^{-1} \cap H$ is

infinite, hence it is not hyperbolically embedded by [17, Lemma 7.1] or [8, Proposition 2.8].

(3) From $c(ta^nt^{-1}) = c(a^m) = 0$, we deduce that $c(t) = ta^nt^{-1}c(t)$. Then since ta^nt^{-1} has infinite order, we get $||c(t)||^2 = \langle c(t), (ta^nt^{-1})^k c(t)\rangle \to 0$ as $k \to \infty$. □

Proposition 4.2 *Let H, K and L be countable discrete groups. If H is maximal amenable in K, and both K and L are torsion free. Then H is also maximal amenable in $K * L$.*

Proof First, we observe that it suffices to show K is free from g for every $g \in K * L \setminus K$, i.e. $\langle K, g\rangle = K * \langle g\rangle$.

To see this, one just check that for all $g \in K * L \setminus H$, $\langle H, g\rangle$ is not amenable. If $g \in K * L \setminus K$, then $\langle H, g\rangle = H * \langle g\rangle \geq F_2$ by assumption. If $g \in K \setminus H$, then $\langle H, g\rangle \subseteq K$ is not amenable since $H < K$ is maximal amenable.

We are left to show for all $g \in K * L \setminus K$, K is free from g.

Claim 1 for every $e \neq k \in K$ and every $g \in K * L \setminus K$, K is free from g if and only if K is free from kg.

Proof of Claim 1 By symmetry, it suffices to show ⇒ holds.

Suppose $k_1(kg)^{m_1} \cdots k_i(kg)^{m_i} = e$ for some $k_2, \ldots k_i \in K \setminus \{e\}, k_1 \in K$, $m_1 \cdots m_{i-1} \neq 0$ and $m_i \in \mathbb{Z}$. Then, since K is free from g, we deduce $|m_1|, \ldots, |m_{i-1}| = 1$; otherwise, by looking at the middle word pieces between any two successive $g^{\pm}$, we deduce $k = e$, a contradiction.

Then, we divide the argument into four cases.

Case 1: $m_j = 1$ for all $j \in \{1, \ldots, i-1\}$. By freeness, we deduce $k_jk = e$ for all $j \in \{1, \ldots, i-1\}$ and hence $g^{i-1}k_i(kg)^{m_i} = e$. If $m_i = 0$ or $m_i = -1$, then $k_i = e$, a contradiction. If $m_i = 1$, then $k_ik = e$ and $g^i = e$, this is a contradiction since $K * L$ is torsion free. If $|m_i| \geq 2$, then $k = e$, a contradiction.

Case 2: $m_j = -1$ for all $j \in \{1, \ldots, i-1\}$. The proof is similar to the proof of case 1.

Case 3: $m_1 = 1$ and there exists the smallest $j \in \{1, \ldots, i-1\}$ such that $m_j = -1$. Then by freeness, we must have $k_j = e$, a contradiction.

Case 4: $m_1 = -1$ and there exists the smallest $j \in \{1, \ldots, i-1\}$ such that $m_j = 1$. Then by freeness, we must have $k^{-1}k_jk = e$, i.e. $k_j = e$ a contradiction. □

By Claim 1 and taking inverses it is also clear that for any $e \neq k \in K$, K is free from g if and only if K is free from gk. Hence, to prove g is free from K, we may assume when written in reduced form, either $g = x$ or $g = xty$, where $x, y \in L \setminus \{e\}$ and $e \neq t$ is a reduced word in G with head and tail come from K. Then clearly, g is free from K. □

Proposition 4.3 *Let $G = BS(m, n) = \langle a, t \mid ta^mt^{-1} = a^n\rangle$ and $H = \langle a\rangle$. Then H is maximal amenable in G if $|m|, |n| \geq 3$.*

Proof It suffices to prove $K := \langle g, a\rangle$ contains free group F_2 for every $g \in G \setminus H$.

By the normal form theorem for HNN extension [14, Page 182], for every $e \neq g \in G$, we may write g in reduced normal form, i.e.

$$g = a^{i_0} t^{\epsilon_1} a^{i_1} t^{\epsilon_2} \cdots t^{\epsilon_k} a^{i_k},$$

where $\epsilon_i \in \{\pm 1\}$ and no substrings of the form $ta^{m*}t^{-1}$ or $t^{-1}a^{n*}t$ appear, where $m*$ (respectively, $n*$) denotes any integer divisible by m (respectively, n). Moreover, if $\epsilon_j = 1$ for some $1 \leq j \leq k$, then $0 \leq i_j < m$; similarly, if $\epsilon_j = -1$ for some $1 \leq j \leq k$, then $0 \leq i_j < n$.

Notice that $K = \langle a^{-i_0} g a^{-i_k}, a\rangle$ and $a^{-i_0} g a^{-i_k} \in G \setminus H$, so without loss of generality, we may assume that $g = t^{\epsilon_1} a^{i_1} t^{\epsilon_2} \cdots t^{\epsilon_k}$ in reduced normal form.

Then, using Britton's lemma (see [3] or [14, Page 181]), one can check that $gag^{-1}a$ is free from $agag^{-1}$ and both have infinite order if $|m|, |n| \geq 3$; in other words, $F_2 \cong \langle gag^{-1}a, agag^{-1}\rangle \subseteq K$. □

Despite the existence of the above examples, we also have examples showing that some maximal amenable but not hyperbolically embedded subgroups may give rise to maximal amenable group von Neumann algebras. Indeed, let $G = (\mathbb{Z} \times F_2) * F_2 = (\langle a\rangle \times \langle b, c\rangle) * F_2$, $K = \mathbb{Z} \times F_2 = \langle a\rangle \times \langle b, c\rangle$ and $H = \mathbb{Z}^2 = \langle a, b\rangle$. Since $\langle a\rangle \subseteq cHc^{-1} \cap H$ is infinite, H is not almost malnormal; therefore it is not hyperbolically embedded in the acylindrically hyperbolic group G. While LH is maximal amenable in LK by [1, Theorem 2.4], hence LH is still maximal amenable in $LG = LK * LF_2$ since any amenable subalgebra (in LG) containing LH is contained in LK by [11] or [18].

Acknowledgments Y. Jiang was partially supported by the NCN (National Centre of Science) grant 2014/14/E/ST1/00525.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement no. 677120-INDEX).

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A K-Theoretic Selberg Trace Formula

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In memory of Ronald G. Douglas

Abstract Let G be a semisimple Lie group and Γ a uniform lattice in G. The Selberg trace formula is an equality arising from computing in two different ways the traces of convolution operators on the Hilbert space $L^2(\Gamma\backslash G)$ associated to test functions $f \in C_c(G)$.

In this paper we present a cohomological interpretation of the trace formula involving the K-theory of the maximal group C^*-algebras of G and Γ. As an application, we exploit the role of group C^*-algebras as recipients of "higher indices" of elliptic differential operators and we obtain the index theoretic version of the Selberg trace formula developed by Barbasch and Moscovici from ours.

Keywords Trace formula · K-theory · Group C^*-algebra · Uniform lattice

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B. Mesland and M. H. Şengün gratefully acknowledge support from the Max Planck Institute for Mathematics in Bonn, Germany. H. Wang is supported by NSFC-11801178 and Shanghai Rising-Star Program 19QA1403200.

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_19

Mathematics Subject Classification (2010) Primary 58B34; Secondary: 46L80, 19K56

1 Introduction

Let G be a semisimple real Lie group and Γ a cocompact lattice in G. A compactly supported smooth function on G (a.k.a. a test function) acts on the Hilbert space $L^2(G/\Gamma)$ via convolution and it is well-known the bounded operator obtained from this action is of trace class. By computing this trace in two different ways, by "summing the eigenvalues" and by "summing the diagonal entries", we arrive at Selberg's trace formula which has the following rough shape (see Sect. 2 for the precise formula):

$$\sum_{\pi} m_\Gamma(\pi)\, \mathrm{tr}_\pi(f) = \sum_{(\gamma)} \mathcal{O}_\gamma(f).$$

The sum on the left hand side (the *spectral side*) runs over the irreducible unitary representations of G and we sum, counting multiplicities, the trace of our operator on "eigenspaces". The sum on the right hand side (the *geometric side*) runs over the conjugacy classes of Γ and we sum over the *orbital integrals* which arise from the "diagonal entries" of the kernel function associated to our operator.

The trace formula has fundamental applications in number theory. For example, the multiplicities $m_\Gamma(\pi)$ give dimensions of various spaces of automorphic forms. If π is integrable, by plugging a carefully chosen test function (a *pseudo-coefficient*) into the trace formula, one can isolate $m_\Gamma(\pi)$ on the spectral side and thus get an explicit formula for it via the geometric side. Another important application is to the functoriality principle in the Langlands programme. For this application, one compares trace formulas on different groups.

In this paper, we present a cohomological carnation of the trace formula. We do this by considering the K-theory groups of the group C^*-algebras associated to G and Γ, and interpreting the terms that appear in the trace formula in K-theoretic terms. Our beginning point is a simple commutative diagramme that is a K-theoretic manifestation of the fact that the quasi-regular representation R^Γ of G on $L^2(G/\Gamma)$ is the induction to G of the trivial representation $\mathbf{1}$ of Γ:

$$\begin{array}{ccc} K_0(C^*(G)) & \xrightarrow{R^\Gamma_*} & \mathbf{Z} \\ \Big\downarrow{\scriptstyle \mathrm{res}} & \nearrow{\scriptstyle \mathbf{1}_*} & \\ K_0(C^*(\Gamma)) & & \end{array}$$

The homomorphism R_*^Γ (resp. $\mathbf{1}_*$) arises via functoriality from R_Γ (resp. $\mathbf{1}$) and the vertical arrow is the restriction homomorphism given by the Mackey–Rieffel theory of imprimitivity bimodules. We then use the spectral decomposition of R^Γ into irreducible unitary representations π to decompose R_*^Γ into homomorphisms π_*, thus obtaining the spectral side of our K-theoretic trace formula. For the geometric side, we decompose 1_* over the conjugacy classes of Γ. The decomposition starts at the level of the convolution algebra $L^1(\Gamma)$ of integrable functions on Γ. At this level, the partitioning of Γ into its conjugacy classes (γ) gives rise to a decomposition of the *-homomorphism $\mathbf{1} : L^1(\Gamma) \to \mathbf{C}$ into maps τ_γ satisfying the trace property (i.e. degree 0 cyclic cohains). Just for this introduction, let us assume that Γ has "controlled growth" (e.g. Gromov hyperbolic). Then it is possible to continuously extend the maps τ_γ to a holomorphically closed dense subalgebra of $C_r^*(\Gamma)$ and thus obtain a decomposition of $\mathbf{1}_*$ into $\tau_{\gamma,*}$ at the level of $K_0(C^*(\Gamma))$.

We arrive at the following K-theoretic trace formula:

$$\sum_{\pi \in \widehat{G}} m_\Gamma(\pi)\pi_*(x) = \sum_{(\gamma)\in\langle\Gamma\rangle} \tau_{\gamma,*}(\mathrm{res}(x)). \tag{1}$$

for every $x \in K_0(C^*(G))$. The general result, which does not make any assumptions on Γ unlike we did above, is given in Theorem 4.13.

An important feature of group C^*-algebras is that they are the recipients of the so-called *higher indices*. Let K denote a maximal compact subgroup of G. Let D denote the Dirac operator on the symmetric space G/K, possibly twisted by a finite dimensional representation of K and denote by $\mathrm{Ind}_G D$ the higher index of D over $C^*(G)$. We show that if one takes $x = [\mathrm{Ind}_G D]$ in Eq. (1), then we obtain, crucially using results of [29], the index theoretic version of the Selberg trace formula developed by Barbasch and Moscovici [5]:

$$\sum_{\pi \in \widehat{G}} m_\Gamma(\pi)\mathrm{Tr}_s\pi(k_t) = \sum_{(\gamma)\in\langle\Gamma\rangle} \int_{\Gamma_\gamma\backslash G} \mathrm{Tr}_s k_t(x^{-1}\gamma x)dx. \tag{2}$$

Here $t > 0$ and k_t is the kernel associated to the heat operator e^{-tD^2} viewed as a matrix valued smooth function on G with Tr_s denoting the supertrace.

From a larger perspective, our result allows for the use of techniques from the representation theory of the semisimple Lie group G in the study of the K-theory of the lattice Γ. It is part of a program that explores the use of operator algebras and K-theory in the theory of automorphic forms. See [19, 20] for other such results.

2 Statement of Selberg Trace Formula

We will now review the trace formula of Selberg in the setting of cocompact lattices in Lie groups (see, for example, [1] for a detailed general account). Let G be a semisimple Lie group, K a maximal compact subgroup and Γ a uniform lattice in G. Let $f \in C_c^\infty(G)$ and denote by $R^\Gamma : G \to \mathcal{U}(L^2(\Gamma\backslash G))$ the right regular representation which extends to $R^\Gamma : C_c^\infty(G) \to \mathcal{L}(L^2(\Gamma\backslash G))$ via

$$R^\Gamma(f) = \int_G f(y) R^\Gamma(y) dy \qquad f \in C_c^\infty(G).$$

The operator $R^\Gamma(f)$ is trace class (see below) and the Selberg trace formula is an equality arising from computing the trace of $R^\Gamma(f)$ in two different ways.

On the "geometric side", regarding $\phi \in L^2(\Gamma\backslash G)$ as a Γ-invariant function on G, and from

$$\begin{aligned}[R^\Gamma(f)\phi](x) &= \int_G f(y)[R^\Gamma(y)\phi](x) dy = \int_G f(y)\phi(xy) dy \\ &= \int_G f(x^{-1}y)\phi(y) dy = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\phi(y) dy,\end{aligned}$$

we obtain the Schwartz kernel

$$K(x, y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y),$$

associated to f. The fact that f is compactly supported and smooth ensures that the sum is locally finite and thus K is a smooth kernel. Coupled with the fact that $\Gamma\backslash G$ is compact, we deduce that $R^\Gamma(f)$ is trace class. We have

$$\begin{aligned}\mathrm{Tr} R^\Gamma(f) &= \int_{\Gamma\backslash G} K(x, x) dx = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x) dx \\ &= \int_{\Gamma\backslash G} \sum_{(\gamma)\in\langle\Gamma\rangle} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x) dx \\ &= \sum_{(\gamma)\in\langle\Gamma\rangle} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(x^{-1}\gamma x) dx.\end{aligned}$$

As usual, in the above $\langle\Gamma\rangle$ denotes the conjugacy classes of Γ and Γ_γ (resp. G_γ) is the centralizer of γ in Γ (resp. G).

On the "spectral side", the Hilbert space $L^2(\Gamma\backslash G)$ splits as a direct sum

$$L^2(\Gamma\backslash G) \simeq \bigoplus_{\pi\in\widehat{G}} H_\pi^{\oplus m_\Gamma(\pi)} \tag{3}$$

of irreducible unitary representations $(\pi, H_\pi) \in \widehat{G}$ of G with each appearing with finite multiplicity $m_\Gamma(\pi)$. By restricting $R^\Gamma(f)$ to the irreducible subspaces of

$L^2(\Gamma\backslash G)$, we obtain the following from (3),

$$\mathrm{Tr} R^{\Gamma}(f) = \sum_{\pi \in \widehat{G}} m_{\Gamma}(\pi)\mathrm{Tr}(\pi(f)).$$

where $\pi(f) = \int_G f(y)\pi(y)dy$ (which can be regarded as the Fourier transform of f at π). The Selberg trace formula is the equality

$$\sum_{\pi \in \widehat{G}} m_{\Gamma}(\pi)\mathrm{Tr}(\pi(f)) = \sum_{(\gamma)\in\langle\Gamma\rangle} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx. \tag{4}$$

Remark 2.1 If $\Gamma\backslash G$ has finite volume but is noncompact, the spectral decomposition of $L^2(\Gamma\backslash G)$ also involves a continuous component and $R^{\Gamma}(f)$ is not necessarily a trace class operator. A truncated version of traces need to be introduced in the Selberg trace formula in this setting (see [1]).

3 Index Theoretic Trace Formula

In this section we formulate an analogue of the Selberg trace formula in the context of index theory, which will be proved using the framework of K-theory in Sect. 4.

We **assume** in addition that the rank of G equals that of K. Note that as a consequence, the symmetric space G/K is of even dimension. Assume that G/K admits a G-equivariant spin^c structure and let $S = S^+ \oplus S^-$ denote the associated spinor bundle on G/K. Let V be a finite dimensional representation of K and $E = G \times_K V$ be the associated G-equivariant vector bundle on G/K. Let D be the G-invariant Dirac operator on the bundle $G \times_K (V \otimes S) \to G/K$ with $\mathbf{Z}_2$-grading, i.e., D has the form $\begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$. Consider the heat operator e^{-tD^2}, which admits a smooth kernel $K_t(x, y)$. The kernel $K_t(x, y)$ satisfies

$$K_t(gx, gy) = K_t(x, y) \qquad x, y \in G/K,\ g \in G,$$

which is equivalent to the G-invariance of D. Regarding $K_t(x, y)$ as a matrix-valued function on $G \times G$, invariant under the $K \times K$-action

$$K_t(x, y) = aK_t(xa, yb)b^{-1} \in \mathrm{End}(V \otimes S) \qquad a, b \in K.$$

We define

$$k_t(x^{-1}y) := K_t(x, y) \qquad x, y \in G.$$

Then k_t is a matrix-valued function on G that is invariant under the action of $K \times K$:

$$k_t \in [C^\infty(G) \otimes \mathrm{End}(V \otimes S)]^{K \times K}, \qquad k_t(x) = a k_t(a^{-1} x b) b^{-1}, \quad a, b \in K.$$

Moreover, k_t is smooth and of Schwartz type. To be more precise,

$$k_t \in [\mathcal{S}(G) \otimes \mathrm{End}(V \otimes S)]^{K \times K}.$$

where $\mathcal{S}(G)$ is Harish-Chandra's Schwartz algebra of G. See [7, Section 1] and [5, Section 2] for more details.

We now replace the test function $f \in C_c^\infty(G)$ acting on $L^2(\Gamma \backslash G)$ in the previous section by the heat kernel k_t acting on the $\mathbf{Z}_2$-graded space $(L^2(\Gamma \backslash G) \otimes V \otimes S)^K$ and formally state the Selberg trace formula associated to the heat operator. We write $\pi(k_t)$ for the operator obtained from the action of k_t on $(H_\pi \otimes (V \otimes S))^K$.

Choose an invariant Haar measure on G so that it is compactible with G-invariant measure on G/K. This in particular means that the volume of the maximal compact subgroup K of G is assumed to be 1.

Proposition 3.1 *Let* Tr_s *be the supertrace of* $\mathbf{Z}_2$*-graded vector spaces. Then*

$$\sum_{\pi \in \widehat{G}} m_\Gamma(\pi) \mathrm{Tr}_s \pi(k_t) = \sum_{(\gamma) \in \langle \Gamma \rangle} \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} \mathrm{Tr}_s[k_t(x^{-1} \gamma x) \gamma] dx. \tag{5}$$

The heat kernel k_t is not compactly supported, however it is of L^p-Schwartz class for all $p > 0$ (see [5, Prop. 2.4.]). Using this, one can prove that each summand on the right hand side of equality (5) converges, see [29, Thm. 6.2]. In any case, the equality (5) will follow from our main result Theorem 4.16 below.

In the rest of this section, we reformulate the equality (5) in terms of an equality of indices of Dirac operators.

3.1 Geometric Side

In this subsection we will see that every term on the geometric side can be identified as the L^2-Lefschetz number associated to a conjugacy class of Γ using results from [29], and that the sum of all L^2-Lefschetz numbers over all conjugacy classes $\langle \Gamma \rangle$ can be identified with the Kawasaki index theorem [14] for the orbifold $\Gamma \backslash G/K$.

Because Γ acts properly and cocompactly on G/K, there exists a cutoff function c on G/K with respect to the action of Γ, i.e., $c : G/K \to [0, 1]$ such that

$$\sum_{\gamma \in \Gamma} c(\gamma x) = 1 \qquad \forall x \in G/K.$$

Define the L^2-Lefschetz number associated to the conjugacy class (γ) of $\gamma \in \Gamma$ by

$$\operatorname{ind}_\gamma D := \operatorname{tr}_s^{(\gamma)} e^{-tD^2} = \operatorname{tr}^{(\gamma)} e^{-tD^-D^+} - \operatorname{tr}^{(\gamma)} e^{-tD^+D^-}$$

where

$$\operatorname{tr}^{(\gamma)} S := \sum_{h \in (\gamma)} \int_{G/K} c(x) \operatorname{Tr}[h^{-1} K_S(hx, x)] dx,$$

for any (γ)-trace class operator S with smooth Schwartz kernel K_S, see (3.22) of [29]. By Theorems 3.23 and 6.1 of [29], $\operatorname{ind}_\gamma D$ admits a fixed point formula: for $h \in (\gamma)$ the kernel $h^{-1}K_t(hx, h)$ localizes to the submanifold $(G/K)^\gamma$ consisting of points in G/K fixed by γ. By [29, Theorem 6.2], the L^2-Lefschetz number for G/K admits the following expression analogous to the orbital integral on the geometric side of (4):

$$\operatorname{ind}_\gamma D = \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} \operatorname{Tr}_s[k_t(x^{-1}\gamma x)\gamma] dx. \tag{6}$$

Example 3.2 If Γ acts freely on G/K, then $(G/K)^\gamma$ is empty and all orbital integrals vanish except for the term where γ is the group identity. The geometric side of (5) in this special case is equal to $\operatorname{vol}(\Gamma \backslash G) \cdot \operatorname{ind}_e D$. Here $\operatorname{ind}_e D$ is a topological index called L^2-index (denoted $\operatorname{ind}_{L^2} D$) for the symmetric space G/K introduced in [7].

Denote by D_Γ the operator on $\Gamma\backslash G/K$ descending from the one on G/K. It is a Dirac operator on the compact orbifold $\Gamma\backslash G/K$. Recall that by the Kawasaki orbifold index formula, the Fredholm index $\operatorname{ind} D_\Gamma$ can be decomposed as a sum over conjugacy classes of Γ and the summands are exactly the L^2-Lefschetz numbers:

$$\operatorname{ind} D_\Gamma = \sum_{(\gamma) \in \langle \Gamma \rangle} \operatorname{ind}_\gamma D. \tag{7}$$

See [14] and Section 2.2 and Theorem 6.2 of [29]. See also [9] for the relationship between heat kernels on X and the orbifold $\Gamma\backslash X$.

Remark 3.3 In the special case of a free action, $\operatorname{ind} D_\Gamma$ is the Fredholm index of D_Γ on the closed manifold $\Gamma\backslash G/K$. By Atiyah's L^2-index theorem [2], we obtain

$$\operatorname{vol}(\Gamma\backslash G/K) \operatorname{ind}_{L^2} D = \operatorname{ind} D_\Gamma,$$

which is a special case of (7). Noting that K is assumed to have volume 1, this formula is compatible with Example 3.2.

3.2 Spectral Side

Recall the Plancherel decomposition of $L^2(G)$:

$$L^2(G) = \int_{\widehat{G}}^{\oplus} (H_\pi^* \otimes H_\pi) d\mu(\pi),$$

Accordingly, the Dirac operator D on $[L^2(G) \otimes V \otimes S]^K$ has a decomposition into a family of Dirac type operators D_π on $[H_\pi \otimes V \otimes S]^K$, parameterized by irreducible unitary representations $(\pi, H_\pi) \in \widehat{G}$.

We recall the definition and properties of D_π introduced in [21] and Section 7 of [7]. See also [13] for a comprehensive treatment. Let H_π^∞ be the space of C^∞-vectors for the G-representation π. The G-invariant operator D can be written as a finite sum $D = \sum_i R^\Gamma(X_i) \otimes A_i$ where X_i belongs to the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ and $A_i^\pm \in \mathrm{Hom}(V \otimes S^\pm, V \otimes S^\mp)$. Here R^Γ stands for the right regular representation. Then D_π is given by

$$D_\pi : [H_\pi^\infty \otimes V \otimes S]^K \to [H_\pi^\infty \otimes V \otimes S]^K, \quad D_\pi = \sum_i \pi(X_i) \otimes A_i.$$

It is proved in [7, 21] that D_π is essentially self-adjoint on the dense domain $[H_\pi^\infty \otimes V \otimes S]^K$. Its closure (also denoted D_π) is a Fredholm operator whose Fredholm index

$$\mathrm{ind}\, D_\pi = \dim(\ker D_\pi^+) - \dim(\ker D_\pi^-)$$

is equal to $\mathrm{Tr}_s \pi(k_t)$ on the spectral side of (5). Indeed, by Proposition 2.1 of [5], one has

$$\mathrm{Tr}(\pi(k_t^\pm)) = \mathrm{Tr}(e^{-tD_\pi^\mp D_\pi^\pm}).$$

Then by the McKean–Singer formula, we obtain

$$\begin{aligned} \mathrm{ind}\, D_\pi &= \mathrm{Tr}(e^{-tD_\pi^- D_\pi^+}) - \mathrm{Tr}(e^{-tD_\pi^+ D_\pi^-}) \\ &= \mathrm{Tr}(\pi(k_t^+)) - \mathrm{Tr}(\pi(k_t^-)) = \mathrm{Tr}_s(\pi(k_t)). \end{aligned} \tag{8}$$

Remark 3.4 For an irreducible representation V of K, the dimensions of $[H_\pi \otimes V \otimes S^+]^K$ and $[H_\pi \otimes V \otimes S^-]^K$ are finite (see [3, 5, 7]). The index of the Dirac operator D_π corresponding to π can be calculated as the difference

$$\mathrm{ind}\, D_\pi = \mathrm{Tr}_s \pi(k_t) = \dim[H_\pi \otimes V \otimes S^+]^K - \dim[H_\pi \otimes V \otimes S^-]^K.$$

In view of (6) and (8), we conclude the following equivalent statement of the index theoretic Selberg trace formula.

Proposition 3.5 *The equality (5) is equivalent to the equality of indices:*

$$\sum_{\pi \in \widehat{G}} m_\Gamma(\pi)\,\mathrm{ind}\, D_\pi = \sum_{(\gamma)\in\langle\Gamma\rangle} \mathrm{ind}_\gamma D.$$

The left hand side is finite sum (see Corollary 4.3 below) and the convergence of the right hand side is determined by (7).

Remark 3.6 Let (η, H_η) be a discrete series representation, and $\mu \in \mathfrak{t}^*_{\mathbf{C}}$ the Harish-Chandra parameter of η. There is a standard way, due to Atiyah and Schmid [3], to realize H_η as the L^2-kernel of a Dirac type operator on G/K. For an irreducible representation $V_\nu \in R(K)$ of K with highest weight $\nu \in \mathfrak{t}^*_{\mathbf{C}}$, let D^ν denote the associated Dirac operator defined on the homogeneous vector bundle

$$G \times_K (V_\nu \otimes S) \to G/K.$$

Denoting by ρ_c the half sum of compact positive roots, then H_η can be identified with the L^2-kernel of $D^{\mu+\rho_c}$. It is also proved in [3, Section 4] that

$$\dim(H^*_\eta \otimes V_\nu \otimes S^\pm)^K = 0,$$

unless $\mu = \nu + \rho_c$. From this, one can derive Proposition 3.5 when $D = D^\eta$. See also [4] for some concrete examples.

Example 3.7 Let $(\eta, H_\eta) \in \widehat{G}$ be a discrete series with Harish-Chandra parameter μ. Let π be an arbitrary discrete series. It follows from [3] that

$$\mathrm{ind}(D^{\mu+\rho_c})_\pi = \dim[H^*_\pi \otimes V_{\mu+\rho_c} \otimes S^+]^K - \dim[H^*_\pi \otimes V_{\mu+\rho_c} \otimes S^-]^K = \begin{cases} 1 & \eta = \pi \\ 0 & \eta \neq \pi. \end{cases}$$

Assume further that Γ is a torsion-free. Then the index theoretic Selberg trace formula (5) reduces to

$$m_\Gamma(\eta) = \mathrm{vol}(\Gamma\backslash G/K)\mathrm{ind}_{L^2} D^{\mu+\rho_c},$$

recovering a main result of Pierrot [22], which is a special case of [18].

4 K-Theoretic Selberg Trace Formula

4.1 Decomposition of Right Regular Representation of G

Let us consider again the right regular representation R^Γ of G on $L^2(\Gamma\backslash G)$. As mentioned earlier, $C_c^\infty(G)$ acts on $L^2(\Gamma\backslash G)$ by compact operators. We use the standard notation $\mathbb{K}(X)$ for the algebra of compact operators on a Hilbert C^*-module X. As $C_c^\infty(G)$ is a dense subalgebra of the maximal group C^*-algebra $C^*(G)$ of G, we obtain a $*$-homomorphism

$$R^\Gamma : C^*(G) \to \mathbb{K}(L^2(\Gamma\backslash G)).$$

This induces a homomorphism on K-theory:

$$R_*^\Gamma : K_0(C^*(G)) \to K_0(\mathbb{K}(L^2(\Gamma\backslash G)) \simeq \mathbf{Z}.$$

As G is liminal, the above observation applies to any irreducible unitary representation as well. Indeed, for any $(\pi, H_\pi) \in \widehat{G}$, we have homomorphisms

$$\pi : C^*(G) \to \mathbb{K}(H_\pi), \qquad \pi_* : K_0(C^*(G)) \to \mathbf{Z}.$$

The morphism π_* determines an element in $KK(C^*(G), \mathbf{C})$ which we will denote by $[\pi]$. Observe that we have

$$\pi_*(x) = x \otimes_{C^*(G)} [\pi],$$

for every $x \in K_0(C^*(G))$, where $\otimes_{C^*(G)}$ denotes the Kasparov product. We remark that the maps π_* feature[1] heavily in Chapter 2 of [16].

Proposition 4.1 *For every $x \in K_0(C^*(G))$ there exists a finite subset $S_x \subset \widehat{G}$ such that*

1. $\pi_*(x) = 0$ *for all* $\pi \in \widehat{G} \setminus S_x$,
2. $R_*^\Gamma(x) = \sum_{\pi \in S_x} \pi_*(x)$.

Proof Let $x \in K_0(C^*(G))$ be given by the even $(\mathbf{C}, C^*(G))$ Kasparov module (X, F). Since $1 \in \mathbf{C}$ acts on X by a projection, say p, after replacing X by pX and F by pFp, we may without loss of generality assume that $\mathbf{C}$ acts unitally on X and $F^2 - 1 \in \mathbb{K}(X)$. Since $C^*(G)$ acts on $L^2(\Gamma \setminus G)$ by compact operators, we have that $\mathbb{K}(X) \otimes 1 \subset \mathbb{K}(X \otimes_{R^\Gamma} L^2(\Gamma \setminus G))$ (see [17, Proposition 4.7]). The operator

$$F \otimes 1 : X \otimes_{R^\Gamma} L^2(\Gamma \setminus G) \to X \otimes_{R^\Gamma} L^2(\Gamma \setminus G),$$

[1] Lafforgue uses the notation $\langle H, x\rangle$ for our $\pi_*(x)$, where H is the representation space of π.

satisfies $(F\otimes 1)^2-1=(F^2-1)\otimes 1\in\mathbb{K}(X)\otimes 1\subset\mathbb{K}(X\otimes L^2(\Gamma\backslash G))$. Thus $F\otimes 1$ is a self-adjoint unitary modulo compact operators, and therefore it is Fredholm. The integer $R_*^\Gamma(x)\in\mathbb{Z}$ is thus given by the index of $F\otimes 1$. Since there is a direct sum splitting

$$X\otimes_{R^\Gamma} L^2(\Gamma\setminus G)\simeq\bigoplus_{\pi\in\widehat{G}}(X\otimes_\pi H_\pi)^{\oplus m_\Gamma(\pi)},$$

it follows that $F\otimes 1=\bigoplus_{\pi\in\widehat{G}}F_\pi$ where the operators are defined to be

$$F_\pi:=F\otimes_\pi 1:(X\otimes_\pi H_\pi)^{\oplus m_\Gamma(\pi)}\to(X\otimes_\pi H_\pi)^{\oplus m_\Gamma(\pi)}.$$

Since $F\otimes 1$ is Fredholm, its kernel and cokernel are finite dimensional and thus it follows that F_π is unitary for all but finitely many $\pi\in\widehat{G}$. Therefore the set

$$S_x:=\{\pi\in\widehat{G}:\pi_*(x)=\text{ind}\,(F_\pi)\neq 0\},$$

is finite and the restriction

$$F_{\widehat{G}\setminus S_x}:=F\otimes 1:\bigoplus_{\pi\in\widehat{G}\setminus S_x}(X\otimes_\pi H_\pi)^{\oplus m_\Gamma(\pi)}\to\bigoplus_{\pi\in\widehat{G}\setminus S_x}(X\otimes_\pi H_\pi)^{\oplus m_\Gamma(\pi)},$$

is Fredholm of index 0. Moreover

$$F_{S_x}:\bigoplus_{\pi\in S_x}(X\otimes_\pi H_\pi)^{\oplus m_\Gamma(\pi)}\to\bigoplus_{\pi\in S_x}(X\otimes_\pi H_\pi)^{\oplus m_\Gamma(\pi)},$$

obviously satisfies $\text{Ind}\,(F_{S_x})=\sum_{\pi\in S_x}\pi_*(x)$. Since

$$R_*^\Gamma(x)=\text{ind}\,(F\otimes 1)=\text{ind}\,(F_{S_x})=\sum_{\pi\in S_x}\pi_*(x),$$

the result follows. □

For the quasi-regular representation, the above proposition gives a decomposition of R_*^Γ.

Corollary 4.2 *For $x\in K_0(C^*(G))$, we have*

$$R_*^\Gamma(x)=\sum_{\pi\in\widehat{G}}m_\Gamma(\pi)\,\pi_*(x),$$

understood as a sum with only finitely many nonzero terms.

As is well-known, $K_0(C^*(G))$ is the recipient of the so-called higher indices which we recall now. A Dirac type operator D on G/K has a G-index

$$\mathrm{Ind}_G(D) := [p] \otimes_{C_0(G/K)\rtimes G} j^G([D]) \in K_0(C^*(G))$$

defined by the descent map $j^G : K_G^*(C_0(X)) \to KK(C_0(G/K) \rtimes G, C^*(G))$ followed by a compression on the left by the projection $p \in C_0(G/K) \rtimes G$ given by

$$p(x, g) := c(gx)c(x), \quad x \in G/K, g \in G$$

where $c : G/K \to [0, \infty)$ is a continuous compactly supported function satisfying $\int_G c(s^{-1}x)^2 ds = 1$ for all $x \in G/K$. See, for example, [10, Section 4.2] for details.

Corollary 4.3 *Let D be a G-invariant Dirac type operator on G/K with index* $\mathrm{Ind}_G D \in K_0(C^*(G))$. *Then*

(i) *Given $\pi \in \widehat{G}$, we have* $\pi_*(\mathrm{Ind}_G D) = \mathrm{ind}\, D_\pi$,
(ii) $\mathrm{ind}\, D_\pi \neq 0$ *for at most finitely many $\pi \in \widehat{G}$.*

Remark 4.4 In Fox and Haskell [12], the example of $G = Spin(4, 1)$, its unitary dual and K-theory is explicitly studied. The paper showed that for a generator $[f] \in K_0(C^*(G))$, discrete series, reducible principle series, trivial representations and endpoint representations could appear in the support of f. Moreover, if $\pi \in \hat{G}$ appears in the support of f, then $\pi_*([f]) = \pm 1$ which is nonvanishing.

Remark 4.5 If G has property (T), then the trivial representation $\mathbf{1}$ gives rise to a generator $[\mathbf{1}] = [(\mathbf{C}, 0)] \in K_0(C^*(G))$. Then we have $R_*^\Gamma([\mathbf{1}]) = \mathbf{1}_*([\mathbf{1}]) = 1$ since

$$\dim(\mathbf{C} \otimes_{C^*(G)} L^2(\Gamma\backslash G)) = \dim(L^2(G)^G) = 1.$$

It is possible to obtain refined information about the localized indices using representation theory. We have already remarked in Example 3.7 that the index of D_π can be nonzero when π is in the discrete series. The next proposition is Prop. 3 of [15] (see also [7, Prop. 7.3] and [8, Lem. 1.1] for related results).

Proposition 4.6 *Let $\pi \in \widehat{G}$ be tempered. If π is not in the discrete series nor a limit of discrete series, then the index of D_π is zero.*

Remark 4.7 Recall that there is a surjective morphism $\lambda : C^*(G) \to C_r^*(G)$ and the Connes–Kasparov isomorphism $R(K) \simeq K_0(C_r^*(G))$ assuming existence of G-equivariant spin^c-structure of G/K. The Connes–Kasparov isomorphism is defined through Dirac induction

$$R(K) \simeq K_0(C_r^*(G)), \qquad [V_\mu] \mapsto \mathrm{Ind}_{G,r}(D^{V_\mu}).$$

Here V_μ is an irreducible unitary representation of K with highest weight μ and D^{V_μ} is the Dirac operator on G/K twisted by the associated bundle $G \times_K V_\mu \to G/K$.

The twisted Dirac operator D^{V_μ} has an equivariant index $\mathrm{Ind}_G(D^{V_\mu}) \in K_0(C^*(G))$ so that $\lambda_*(\mathrm{Ind}_G(D^{V_\mu})) = \mathrm{Ind}_{G,r}(D^{V_\mu})$. That is, there is a commutative diagram

$$\begin{array}{ccc} R(K) & \xrightarrow{\mathrm{Ind}_G} & K_0(C^*(G)) \\ & \underset{\mathrm{Ind}_{G,r}}{\searrow} & \downarrow \lambda_* \\ & & K_0(C_r^*(G)). \end{array}$$

If G is not K-amenable, i.e., λ_* is not an isomorphism, then not all $x \in K_0(C^*(G))$ are represented by the equivariant index of some elliptic operator. For example, if G has property (T), the Kazhdan projection defines a nontrivial element $[p] \in K_0(C^*(G))$ but $\lambda_*[p] = 0 \in K_0(C_r^*(G))$.

We will discuss a special case of Corollary 4.2 where

$$x = \mathrm{Ind}_G(D^{V_\mu}) \in K_*(C^*(G)).$$

Recall that

$$R_*(\mathrm{Ind}_G(D^{V_\mu})) = \mathbf{1}_*(\mathrm{Ind}_\Gamma(D^{V_\mu})) = \mathrm{ind} D_\Gamma^{V_\mu}$$

where $D_\Gamma^{V_\mu}$ is the operator D^{V_μ} descended to $\Gamma\backslash G/K$. Then by [21, Section 3] and [5, (1.2.4)]

$$\mathrm{ind} D_\Gamma^{V_\mu} = \sum_{\pi \in \widehat{G}} m_\Gamma(\pi)\mathrm{ind}(D^{V_\mu})_\pi.$$

Furthermore, the summand on the right hand side is nonvanishing only when the infinitesimal character χ_π of $\pi \in \widehat{G}$ coincides with that of $\mu + \rho_c$, hence it is a finite sum. See [3] and [5, (1.3.7)–(1.3.8)]. In summary, we obtain

$$R_*(x) = R_*(\mathrm{Ind}_G(D^{V_\mu})) = \sum_{\pi \in \widehat{G}, \chi_\pi = \chi_{\mu+\rho_c}} m_\Gamma(\pi)\mathrm{ind}(D^{V_\mu})_\pi.$$

A typical example of such x is represented by the integrable discrete series. The integrable discrete series representations of G appear as isolated points in $\widehat{G}$ (see [28]). As a result, they give rise to generators $[p_\pi]$ for $K_0(C^*(G))$. Recall that every discrete series representation of G gives rise to a generator of $K_0(C_r^*(G))$ (see [25, Thm. 4.6]). So $\lambda_*[p_\pi] = [p_\pi] \in K_0(C_r^*(G))$. If π is an integrable representation, it follows from Example 3.7 and Corollary 4.2 that

$$R_*^\Gamma([p_\pi]) = m_\Gamma(\pi).$$

Let $x \in K_0(C^*(G))$ now be arbitrary. Then there exists finitely many $\lambda_\mu \in \widehat{K}$ with multiplicity m_{λ_μ} such that

$$\lambda_*(x) = \sum_{\lambda_\mu \in \widehat{K}} m_{\lambda_\mu} \mathrm{Ind}_{G,r} D^{V_\mu}.$$

Let $x_0 = \sum_{\lambda_\mu \in \widehat{K}} m_{\lambda_\mu} \mathrm{Ind}_G D^{V_\mu}$. By definition

$$\lambda_*(x - x_0) = 0, \quad \text{and} \quad R_*(x) = R_*(x_0) + R_*(x - x_0),$$

where $R_*(x - x_0)$ does not come from index theory and $R_*(x_0)$ can be calculated by

$$R_*(x_0) = \sum_{\lambda_\mu \in \widehat{K}} R_*(\mathrm{Ind}_G D^{V_\mu}) = \sum_{\pi \in \widehat{G}} m_\Gamma(\pi) \sum_{\substack{\lambda_\mu \in \widehat{K}, \\ \chi_\pi = \chi_{\mu+\rho_c}}} m_{\lambda_\mu} \mathrm{ind}(D^{V_\mu})_\pi.$$

Thus if we were to consider x_0 in $K_0(C_r^*(G))$, then x_0 can be decomposed into a finite sum involving elliptic operators on G/K. Note however that we cannot work with $C_r^*(G)$ in this paper, as one knows that non-tempered representations of G may enter R^Γ [27, p. 177].

4.2 Decomposition of the Trivial Representation of Γ

Consider a uniform lattice Γ in G and denote by $\mathbf{1}$ the trivial representation of Γ. The corresponding representation of $C^*(\Gamma)$ is a $*$-homomorphism and hence a trace

$$\mathbf{1} : C^*(\Gamma) \to \mathbf{C} \qquad \sum_{\gamma \in \Gamma} a_\gamma \gamma \mapsto \sum_{\gamma \in \Gamma} a_\gamma.$$

Thus there is a morphism of K-theory:

$$\mathbf{1}_* : K_0(C^*(\Gamma)) \to \mathbf{Z}.$$

Let $\gamma \in \Gamma$ with conjugacy class (γ). Then we have a well-defined localized trace on the Banach subalgebra $L^1(\Gamma) \subset C^*(\Gamma)$,

$$\tau_\gamma : L^1(\Gamma) \to \mathbf{C}, \qquad \sum_{g \in \Gamma} a_g g \mapsto \sum_{g \in (\gamma)} a_g.$$

For $a, b \in L^1(\Gamma)$ we have the tracial property $\tau_\gamma(a * b) = \tau_\gamma(b * a)$ with respect to the convolution product $*$. This implies the existence of the morphism

$$\tau_{\gamma,*} : K_0(L^1(\Gamma)) \to \mathbf{C}.$$

Let $\iota : L^1(\Gamma) \to C^*(\Gamma)$ be the inclusion map. It is straightforward to observe that

$$\sum_{(\gamma)\in\langle\Gamma\rangle} \tau_{\gamma,*} = \mathbf{1}_* \iota_* : K_0(L^1(\Gamma)) \to \mathbf{Z}. \tag{9}$$

Given a Dirac type operator D on G/K, we can form its Γ-index $\mathrm{Ind}_\Gamma(D)$ which lands in $K_0(C^*(\Gamma))$ following the recipe we outlined right before Corollary 4.3. If we choose a properly supported parametrix, the index class can be presented in a smaller algebra $\mathbf{C}\Gamma \otimes \mathcal{R}$ where $\mathcal{R}$ is the algebra of operators with smooth kernels and compact support. In particular, the Γ-index homomorphism Ind_Γ factors through the L^1-index homomorphism $K_0^\Gamma(G/K) \xrightarrow{\mathrm{Ind}_{\Gamma,L^1}} K_*(L^1(\Gamma))$. That is, there is $\mathrm{Ind}_{\Gamma,L^1}(D) \in K_0(L^1(\Gamma))$ such that

$$\iota_*(\mathrm{Ind}_{\Gamma,L^1}(D)) = \mathrm{Ind}_\Gamma(D).$$

The following lemma is proved in Definition 5.7 and Proposition 5.9 in [29].

Lemma 4.8

$$\tau_{\gamma,*}(\mathrm{Ind}_{\Gamma,L^1}(D)) = \mathrm{ind}_\gamma(D).$$

Then together with (9) we have

$$\mathbf{1}_*(\mathrm{Ind}_\Gamma(D)) = \sum_{(\gamma)\in\langle\Gamma\rangle} \mathrm{ind}_\gamma(D).$$

Let D_Γ be the operator on the quotient orbifold $\Gamma\backslash G/K$ induced by D. Note that $\mathbf{1}_*(\mathrm{Ind}_\Gamma(D))$ is equal to the Fredholm index of D_Γ (see [6]). Therefore, we have an equality of indices obtained from an identity on K-theory:

$$\mathrm{ind}(D_\Gamma) = \sum_{(\gamma)\in\langle\Gamma\rangle} \mathrm{ind}_\gamma(D).$$

The reader should compare this to (7).

4.3 The Restriction Map

Let $\Gamma \subset G$ be a discrete subgroup and consider the restriction map

$$\rho : C_c(G) \to C_c(\Gamma) \subset C^*(\Gamma),$$

defined by $\rho(f)(\gamma) := f(\gamma)$. The map ρ defines a positive definite $C^*(\Gamma)$ valued inner product on $C_c(G)$ by

$$\langle f, h\rangle(\gamma) := \rho(f^* * h)(\gamma) = \int_G f^*(\xi)g(\xi^{-1}\gamma)d\mu_G(\xi).$$

Upon completion, we obtain a $(C^*(G), C^*(\Gamma))$-C^*-bimodule, denoted Res, where the action of $C^*(G)$ is induced by convolution on $C_c(G)$. Denote by $[g]$ the class of g in G/Γ. The commutative C^*-algebra $C_0(G/\Gamma)$ acts on $C_c(G)$ from the left via

$$(f \cdot \phi)(g) := f([g])\phi(g),$$

and this extends to an action by adjointable endomorphisms of Res. Elements of $C_0(G/\Gamma)$ do not act compactly unless G is discrete.

By Rieffel's imprimitivity theorem (see [24]) the algebra of compact endomorphisms of Res is

$$\mathbb{K}(\mathsf{Res}) = C^*(G \ltimes G/\Gamma).$$

The dense subalgebra $C_c(G \ltimes G/\Gamma)$ acts on the dense submodule $C_c(G) \subset \mathsf{Res}$ via

$$(K \cdot \phi)(g) := \int_G K(\xi, [\xi^{-1}g])\phi(\xi^{-1}g)d\mu_G\xi.$$

Lemma 4.9 *Let $f \in C_c(G)$ and $h \in C_0(G/\Gamma)$. Then $(f \cdot h)(g, x) := f(g)h(x)$ is an element of $C_c(G \ltimes G/\Gamma)$ whose action on $C_c(G)$ is given by*

$$(f \cdot h)\phi(g) = \int_G f(\xi)h([\xi^{-1}g])\phi(\xi^{-1}g)d\mu_G\xi = \int_G f(\xi)(h \cdot \phi)(\xi^{-1}g)d\mu_G\xi. \tag{10}$$

Consequently we have that $C^*(G) \cdot C_0(G/\Gamma) \subset \mathbb{K}(\mathsf{Res})$.

Proof The function $f \cdot h$ satisfies supp $(f \cdot h) \subset$ supp $f \times$ supp h and therefore has compact support, that is it is an element of $C_c(G \ltimes G/\Gamma)$. The remaining claims now follow by direct calculation. □

Proposition 4.10 *Let $\Gamma \subset G$ be a discrete cocompact group and* Res *the associated $(C^*(G), C^*(\Gamma))$-bimodule defined above. Then $C^*(G)$ acts on* Res

by compact module endomorphisms. Hence the bimodule Res *defines an element* [Res] $\in KK_0(C^*(G), C^*(\Gamma))$.

Proof As G/Γ is compact we have that $C_0(G/\Gamma) = C(G/\Gamma)$ and the constant function $1 \in C_0(G/\Gamma)$. By Lemma 4.9 it follows that $f \cdot 1 \in \mathbb{K}(\mathsf{Res})$. It follows from Eq. (4.9) that the action of f on $C_c(G)$ coincides with that of $f \cdot 1$ on $C_c(G)$ and thus $C_c(G)$ acts by compact endomorphisms. Hence by continuity all of $C^*(G)$ acts compactly. The remaining claims now follow directly. □

Definition 4.11 The restriction map

$$\mathrm{res} : K_0(C^*(G)) \to K_0(C^*(\Gamma))$$

is given by the KK-product with $\mathsf{Res} \in KK_0(C^*(G), C^*(\Gamma))$ in Proposition 4.10:

$$\mathrm{res}([x]) := [x] \otimes_{C^*(G)} [\mathsf{Res}].$$

Denote by $[\mathbf{1}] \in KK_0(C^*(\Gamma), \mathbf{C})$ the K-homology class associated to the trivial representation of Γ.

Proposition 4.11 *The following diagram commutes.*

$$\begin{array}{ccc} K_*(C^*(G)) & \xrightarrow{-\otimes[R^\Gamma]} & \mathbf{Z} \\ \Big\downarrow \mathrm{res} & \nearrow_{-\otimes[\mathbf{1}]} & \\ K_*(C^*(\Gamma)) & & \end{array} \tag{11}$$

Here $[R^\Gamma]$ *is the element in* $K^0(C^*(G))$ *determined by the morphism* R_*^Γ.

Proof Let (X, F) be a $(\mathbf{C}, C^*(G))$ Kasparov module with $\mathbf{C}$ acting unitally (as in the proof of Proposition 4.1). Then by definition

$$[(X, F)] \otimes_{C^*(G)} [R^\Gamma] = R_*^\Gamma([(X, F)]) \in \mathbf{Z},$$

is given by the index of the Fredholm operator

$$F \otimes 1 : X \otimes_{R^\Gamma} L^2(\Gamma \setminus G) \to X \otimes_{R^\Gamma} L^2(\Gamma \setminus G).$$

On the other hand,

$$\begin{aligned} \mathrm{res}([(X, F)]) \otimes_{C^*(\Gamma)} [\mathbf{1}] &= [(X, F)] \otimes_{C^*(G)} [\mathsf{Res}] \otimes_{C^*(\Gamma)} [\mathbf{1}] \\ &= [(X \otimes_{C^*(G)} \mathsf{Res} \otimes_{C^*(\Gamma)} \mathbf{C}, F \otimes 1 \otimes 1)]. \end{aligned}$$

By Proposition 4.10 we have $C^*(G) \to \mathbb{K}(\mathsf{Res})$ and the trivial representation $\mathbf{1}$ is finite dimensional. Thus a successive application of [17, Proposition 4.7] shows that

$$(F \otimes 1 \otimes 1)^2 - 1 = (F^2 - 1) \otimes 1 \otimes 1 \in \mathbb{K}(X) \otimes 1 \subset \mathbb{K}(X \otimes_{C^*(G)} \mathsf{Res} \otimes_{C^*(\Gamma)} \mathbf{C}),$$

proving that $F \otimes 1 \otimes 1$ is Fredholm. Therefore $\mathrm{res}([(X, F)]) \otimes_{C^*(\Gamma)} [\mathbf{1}] \in \mathbf{Z}$ equals the index of $F \otimes 1 \otimes 1$ and to show the diagram commutes, it is sufficient to show that there is a unitary isomorphism

$$\Phi : \mathsf{Res} \otimes_{C^*(\Gamma)} \mathbf{C} \xrightarrow{\sim} L^2(\Gamma \setminus G), \tag{12}$$

of Hilbert spaces intertwining the $C^*(G)$-representations. Define a map

$$\Phi : C_c(G) \otimes_{C_c(\Gamma)} \mathbf{C} \to C(\Gamma \setminus G), \qquad \Phi(f)(\Gamma g) = \sum_{\gamma \in \Gamma} f(g^{-1}\gamma).$$

Note that the above map is well defined, i.e., the right hand sides coincide for $\Gamma g = \Gamma g'$. Then on one hand

$$\begin{aligned}\langle \Phi(f), \Phi(h) \rangle &= \int_{G/\Gamma} \overline{\Phi(f)(\Gamma g)} \Phi(h)(\Gamma g) \mathrm{d}(\Gamma g) \\ &= \int_{G/\Gamma} \sum_{\gamma \in \Gamma} \overline{f(g^{-1}\gamma)} \sum_{\delta \in \Gamma} h(g^{-1}\delta) \mathrm{d}(\Gamma g) \\ &= \sum_{\delta \in \Gamma} \int_G \overline{f(g^{-1})} h(g^{-1}\delta) \mathrm{d}g,\end{aligned}$$

whereas on the other hand

$$\langle f \otimes 1, g \otimes 1 \rangle = \sum_{\delta \in \Gamma} \langle f, h \rangle_{C^*(\Gamma)}(\delta) = \sum_{\delta \in \Gamma} \int_G \overline{f(g^{-1})} h(g^{-1}\delta) \mathrm{d}g = \langle \Phi(f), \Phi(h) \rangle.$$

Hence the map Φ is an isometry and extends to an injection. Since the action of Γ on G is properly discontinuous, functions of small support on $\Gamma \setminus G$ are in the image of Φ, and a partition of unity argument shows that Φ is surjective, so it extends to a unitary isomorphism. For $\xi \in G$ we have

$$\xi(\Phi(f))(\Gamma g) = \Phi(f)(\Gamma g \xi) = \sum_{\gamma \in \Gamma} f(\xi^{-1} g^{-1} \gamma) = \sum_{\gamma \in \Gamma} (\xi f)(g^{-1}\gamma) = \Phi(\xi f)(\Gamma g),$$

hence Φ intertwines the G-representations and therefore the $C^*(G)$- representations. Thus we proved the isomorphism (12). □

4.4 Selberg Trace Formula in K-Theory

We are ready to formulate the K-theoretic Selberg trace formula. Let $\iota : L^1(\Gamma) \to C^*(\Gamma)$ be the natural inclusion and $\iota_* : K_0(L^1(\Gamma)) \to K_0(C^*(\Gamma))$ be the associated map. Putting the preceding discussions together, we obtain the following.

Theorem 4.13 *Let $x \in K_0(C^*(G))$. Assume that there exists $y \in K_0(L^1(\Gamma))$ such that $\iota_*(y) = \mathrm{res}(x) \in K_0(C^*(\Gamma))$. Then*

$$\boxed{\sum_{\pi \in \widehat{G}} m_\Gamma(\pi)\pi_*(x) = \sum_{(\gamma)\in\langle\Gamma\rangle} \tau_{\gamma,*}(y).}$$

Remark 4.14 We can drop the hypothesis on x by putting conditions on Γ. For example, if Γ admits the polynomial growth condition on each of its conjugacy classes (see [26]), or if Γ is a word hyperbolic group (see [23]), then the trace $\tau_\gamma : L^1(\Gamma) \to \mathbf{C}$ can be extended to a subalgebra of $C^*_r(\Gamma)$ that is stable under holomorphic functional calculus and thus gives rise to a well-defined morphism

$$\tau_{\gamma,*} : K_0(C^*(\Gamma)) \to \mathbf{C}.$$

It then follows that

$$\sum_{\pi \in \widehat{G}} m_\Gamma(\pi)\pi_*(x) = \sum_{(\gamma)\in\langle\Gamma\rangle} \tau_{\gamma,*}(\mathrm{res}\,(x))$$

for *all* $x \in K_0(C^*(\Gamma))$.

With a little more work, we can derive a general index theoretic trace formula from the above. For this, we need to bring the equivariant K-homology groups into the picture.

Lemma 4.15 *The following diagram commutes:*

$$\begin{array}{ccc} K_0^G(G/K) & \xrightarrow{\mathrm{Ind}_G} & K_*(C^*(G)) \\ \Big\downarrow{\scriptstyle \mathfrak{r}} & & \Big\downarrow{\scriptstyle \mathrm{res}} \\ K_0^\Gamma(G/K) & \xrightarrow{\mathrm{Ind}_\Gamma} & K_*(C^*(\Gamma)) \end{array}$$

where the left vertical arrow $\mathfrak{r}$ is the restriction map in equivariant KK-theory.

Proof Every cycle representing an element of $K_0^G(G/K)$ is given by $([L^2(G) \otimes V]^K, F)$ where V is a $\mathbf{Z}_2$-graded finite dimensional representation of K, and F is a bounded properly supported G-invariant operator on the Hilbert space $H = [L^2(G) \otimes V]^K$. Its images under Ind_G and under $\mathrm{Ind}_\Gamma \circ \mathfrak{r}$ are represented by cycles of the form $(\mathcal{E}_G, F)$, $(\mathcal{E}_\Gamma, F)$. Here the Hilbert $C^*(G)$-module $\mathcal{E}_G$ (resp. $C^*(\Gamma)$-module $\mathcal{E}_\Gamma$) is given by completion of $[C_c(G) \otimes V]^K$ with respect to the $C_c(G)$-valued ($C_c(\Gamma)$-valued) inner product determined by

$$\langle f_1, f_2\rangle(t) = \int_G \langle f_1(s), tf_2(t^{-1}s)\rangle_H ds \qquad f_1, f_2 \in H$$

for $t \in G$ ($t \in \Gamma$). Recall that Res is the closure of $C_c(G)$. To show that

$$[(\mathcal{E}_G, F)] \otimes_{C^*(G)} [\mathsf{Res}] = [(\mathcal{E}_\Gamma, F)],$$

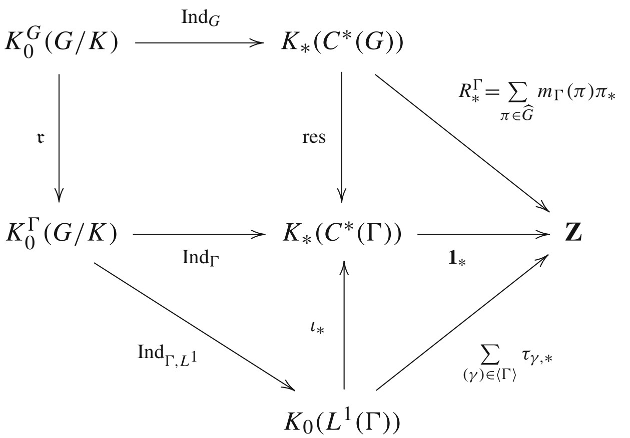
we only need to observe that $\mathcal{E}_\Gamma = \mathcal{E}_G \otimes_{C^*(G)} \mathsf{Res}$. This follows directly from the isomorphism of $C_c(\Gamma)$-modules

$$[C_c(G) \otimes V]^K \otimes_{C_c(G)} C_c(G)_{C_c(\Gamma)} \simeq [C_c(G)_{C_c(\Gamma)} \otimes V]^K,$$

which is compatible with the inner products. □

By Theorem 4.13 and Lemmas 4.3, 4.8 and 4.15 we obtain the following theorem, giving the Selberg trace formula in its index theory form.

Theorem 4.16 *The following diagram commutes*



.

In particular, if D is a G-invariant Dirac type operator on G/K, the above commutative diagram gives rise to the equality

$$\sum_{\pi\in\widehat{G}} m_\Gamma(\pi)\pi_*(\mathrm{Ind}_G D) = \sum_{(\gamma)\in\langle\Gamma\rangle} \tau_{\gamma,*}(\mathrm{Ind}_{\Gamma,L^1} D),$$

which reduces to the index theoretic Selberg trace formula:

$$\boxed{\sum_{\pi \in \widehat{G}} m_\Gamma(\pi) \mathrm{ind} D_\pi = \sum_{(\gamma) \in \langle \Gamma \rangle} \mathrm{ind}_\gamma D.} \tag{13}$$

Remark 4.17 Denote by D_Γ the Dirac type operator on $\Gamma \backslash G/K$ whose lift to G/K is D. Then

$$\mathrm{ind}\, D_\Gamma = \sum_{(\gamma) \in \langle \Gamma \rangle} \mathrm{ind}_\gamma D.$$

Also $[\mathrm{Ind}_G D] \otimes_{C^*(G)} [R^\Gamma] = R^\Gamma_*(\mathrm{Ind}_G D) = \sum_{\pi \in \widehat{G}} m_\Gamma(\pi) \mathrm{ind} D_\pi$. Then (13) is reduced to

$$[\mathrm{Ind}_G D] \otimes_{C^*(G)} [R^\Gamma] = [D_\Gamma] \in KK(\mathbf{C}, \mathbf{C}) = \mathbf{Z}.$$

This recovers Theorem 2.3 in [12].

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Singular Hilbert Modules on Jordan–Kepler Varieties

Gadadhar Misra and Harald Upmeier

This paper is dedicated to the memory of Ronald G. Douglas

Abstract We study submodules of analytic Hilbert modules defined over certain algebraic varieties in bounded symmetric domains, the so-called Jordan–Kepler varieties V_ℓ of arbitrary rank ℓ. For $\ell > 1$, the singular set of V_ℓ is not a complete intersection. Hence the usual monoidal transformations do not suffice for the resolution of the singularities. Instead, we describe a new higher rank version of the blow-up process, defined in terms of Jordan algebraic determinants, and apply this resolution to obtain the rigidity of the submodules vanishing on the singular set.

Keywords Analytic Hilbert module · Algebraic variety · Symmetric domain · Reproducing kernel · Curvature · Rigidity

Mathematics Subject Classification (2010) Primary 32M15, 46E22; Secondary 14M12, 17C36, 47B35

The first-named author was supported by the J. C. Bose National Fellowship and the MATRICS grant of the SERB as well as the CAS II of the UGC. The second-named author was supported by an Infosys Visiting Chair Professorship at the Indian Institute of Science.

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_20

1 Introduction

R. G. Douglas introduced the notion of Hilbert module $\mathcal{M}$ over a function algebra $\mathcal{A}$ and reformulated several questions of multi-variable operator theory in the language of Hilbert modules. Having done this, it is possible to use techniques from commutative algebra and algebraic geometry to answer some of these questions. One of the very interesting examples is the proof of the Rigidity Theorem for Hilbert modules [19, Section 3], which we discuss below.

A Hilbert module is a complex separable Hilbert space $\mathcal{M}$ equipped with a multiplication

$$\mathfrak{m} : \mathcal{A} \to \mathcal{B}(\mathcal{M}),\ \ \mathfrak{m}_p(f) = p\cdot f,\ \ f \in \mathcal{M},\ \ p \in \mathcal{A},$$

which is a continuous algebra homomorphism. Here $\mathcal{B}(\mathcal{M})$ denotes the algebra of all bounded linear operators on $\mathcal{M}$. The continuity of the module multiplication means

$$\|\mathfrak{m}_p f\| \le C\,\|p\|\,\|f\|,\ f \in \mathcal{M},\ p \in \mathcal{A}$$

for some $C > 0$. Familiar examples are the Hardy and Bergman spaces defined on bounded domains in $\mathbf{C}^d$. Sometimes, it is convenient to consider the module multiplication over the polynomial ring $\mathbf{C}[z]$ in d variables rather than a function algebra. In this case, we require that

$$\|\mathfrak{m}_p f\| \le C_p\,\|f\|,\ f \in \mathcal{M},\ p \in \mathbf{C}[z]$$

for some $C_p > 0$. We make this "weak" continuity assumption throughout the paper.

In what follows, we will consider a natural class of Hilbert modules consisting of holomorphic functions, taking values in $\mathbf{C}^n$, defined on a bounded domain $\Omega \subseteq \mathbf{C}^d$. Thus (i) we assume $\mathcal{M} \subseteq \mathrm{Hol}(\Omega, \mathbf{C}^n)$. A second assumption (ii) is to require that the evaluation functional

$$\mathrm{ev}_z : \mathcal{M} \to \mathbf{C}^n,\quad \mathrm{ev}_z(f) := f(z),$$

is continuous and surjective, see [2, Definition 2.5]. Set

$$\mathcal{K}(z, w) := \mathrm{ev}_z \mathrm{ev}_w^* : \Omega \times \Omega \to \mathbf{C}^{n\times n}.$$

The function $\mathcal{K}$, which is holomorphic in the first variable and anti-holomorphic in the second variable is called the **reproducing kernel** of the Hilbert module $\mathcal{M}$. A further assumption (iii) is that $\mathbf{C}[z] \subseteq \mathcal{M}$ is dense in $\mathcal{M}$. A Hilbert module with these properties is said to be an **analytic Hilbert module**. In this paper, we study a class of Hilbert modules which are submodules of analytic Hilbert modules.

From the closed graph theorem, it follows that $\mathfrak{m}_p f \in \mathcal{M}$ for any $f \in \mathcal{M}$ and $p \in \mathbf{C}[z]$. Also, the density of the polynomials implies that the eigenspace $\ker(\mathfrak{m}_p - p(w))^*$ is spanned by the vectors

$$\mathcal{K}_w(\cdot)\zeta := \mathcal{K}(\cdot, w)\zeta$$

for $\zeta \in \mathbf{C}^n$, i.e.,

$$\ker(\mathfrak{m}_p - p(w))^* = \operatorname{Ran} \mathcal{K}_w,$$

see [15, Remark, p. 285]. Since the matrix $\mathcal{K}(w, w)$ is invertible by our assumption, it follows that the dimension of the kernel $\{\mathcal{K}_w(\cdot)\zeta : \zeta \in \mathbf{C}^n\}$ is exactly n for all $w \in \Omega$. Clearly, the map $w \mapsto \mathcal{K}_w(\cdot)\zeta,\ \zeta \in \mathbf{C}^n$ is a holomorphic map on $\Omega^* := \{w \in \mathbf{C}^d : \overline{w} \in \Omega\}$. It serves as a holomorphic section of the trivial vector bundle

$$\mathcal{E} := \{(w, v) : \ w \in \Omega^*, \ v \in \ker(\mathfrak{m}_p - p(w))^*\} \subseteq \Omega^* \times \mathcal{M}$$

with fibre

$$\mathcal{E}_w = \ker(\mathfrak{m}_p - p(w))^* = \operatorname{Ran} \mathcal{K}_w, \ w \in \Omega^*.$$

A refinement of the argument given in [2] (which, in turn, is an adaptation of ideas from [12]), then shows that the isomorphism class of the module $\mathcal{M}$ and the equivalence class of the holomorphic Hermitian bundle $\mathcal{E}$ determine each other. The case $d = 1$, originally considered in [12], corresponds to Hilbert modules over the polynomial ring in one variable. The proof in [12], in this particular case, has a slightly different set of hypotheses. In the paper [12], among other things, a complete set of invariants for the equivalence class of $\mathcal{E}$ is given. If $n = 1$, as is well known, this is just the curvature of the holomorphic line bundle $\mathcal{E}$.

There is a natural notion of module isomorphism, namely, the existence of a unitary linear map $U : \mathcal{M} \to \widetilde{\mathcal{M}}$, which intertwines the module multiplications $\mathfrak{m}_p$ and $\tilde{\mathfrak{m}}_p$, that is,

$$U\mathfrak{m}_p = \tilde{\mathfrak{m}}_p U.$$

Clearly, a Hilbert module $\mathcal{M}$ over the polynomial ring $\mathbf{C}[z]$ is determined by the commuting tuple of multiplication by the coordinate functions on $\mathcal{M}$ and vice-versa. Thus the notion of module isomorphism corresponds to the usual notion of unitary equivalence of two such d-tuples of multiplication operators by a fixed unitary. If $\Gamma : \mathcal{M}_1 \to \mathcal{M}_2$ is a module map, then it maps the eigenspace of $\mathcal{M}_1$ at w into that of $\mathcal{M}_2$ at w. Thus $\Gamma(\mathcal{K}^1(\cdot, w)\zeta) \subseteq \{\mathcal{K}^2(\cdot, w)\xi : \xi \in \mathbf{C}^n\}$, where $\mathcal{K}^i$ are the reproducing kernels of the Hilbert modules $\mathcal{M}_i$, $i = 1, 2$, respectively. Hence we obtain a holomorphic map $\Phi_\Gamma : \Omega \to \mathbf{C}^{n\times n}$ with the property

$$\Gamma\mathcal{K}^1(z, w) = \Phi_\Gamma(w)^*\mathcal{K}^2(z, w)$$

for any fixed but arbitrary w. Thus any module map between two analytic Hilbert modules is induced by a holomorphic matrix-valued function $\Phi_\Gamma : \Omega \to \mathbf{C}^{n\times n}$, see [14, Theorem 3.7]. Moreover, if the module map is invertible, then $\Phi_\Gamma(z)$ must be invertible. Finally, if the module map is assumed to be unitary, then

$$\mathcal{K}^1(z,w) = \Phi_\Gamma(z)\,\mathcal{K}^2(z,w)\,\Phi_\Gamma^*(w)$$

for all $z, w \in \Omega$.

Let us describe, following [17], an instance of the Sz.-Nagy–Foias theory in the language of Hilbert modules. Let T be a contraction on some Hilbert space $\mathcal{M}$. The module multiplication determined by this operator is the map $\mathfrak{m}_p(f) = p(T)f$, $p \in \mathbf{C}[z]$, $f \in \mathcal{M}$. From the contractivity of T, it follows that $\|\mathfrak{m}_p\| \le \|p\| := \sup\{|p(z)| : z \in \mathbb{D}\}$ and in this case, the Hilbert module $\mathcal{M}$ is said to be contractive. Now, assume that $T^{*n} \to 0$ as $n \to \infty$. Then Sz.-Nagy–Foias show that there exists an isometry R and a co-isometry R' such that, for the unit disk $\mathbf{D}$, the sequence

$$0 \longrightarrow H^2_{\mathcal{E}}(\mathbf{D}) \xrightarrow{R} H^2_{\mathcal{E}'}(\mathbf{D}) \xrightarrow{R'} \mathcal{M} \longrightarrow 0\,,$$

where $\mathcal{E}$ and $\mathcal{E}'$ are a pair of (not necessarily finite dimensional) Hilbert spaces, is exact. The map R is essentially the characteristic function of the contraction T and serves to identify the contractive module $\mathcal{M}$ as a quotient module of $H^2_{\mathcal{E}'}(\mathbf{D})$ by the image of $H^2_{\mathcal{E}}(\mathbf{D})$ under the isometric map R.

For any planar domain Ω, a model theory for completely contractive Hilbert modules over the function algebra $\mathrm{Rat}(\Omega)$, consisting of rational functions with poles off the closure $\overline{\Omega}$, has been developed by Abrahamse and Douglas in the paper [1]. However, the situation is much more complicated for Hilbert modules over the polynomial ring in d variables, $d > 1$.

1.1 The Normalized Kernel

We begin by recalling some notions from complex geometry. Let $\mathcal{L}$ be a holomorphic Hermitian line bundle over a complex manifold Ω. The Hermitian metric of $\mathcal{L}$ is given by some smooth choice of an inner product $\|\cdot\|_w^2$ on the fibre $\mathcal{L}_w$. There is a canonical (Chern) connection on $\mathcal{L}$ which is compatible with both the Hermitian metric and the complex structure of $\mathcal{L}$. The curvature κ of the line bundle $\mathcal{L}$ on any fixed but arbitrary coordinate chart, with respect to the canonical connection, is given by the formula

$$\kappa(w) := -\partial\overline{\partial}\log\|\gamma(w)\|^2 = -\sum_{i,j}\partial_i\overline{\partial}_j\log\|\gamma(w)\|^2 dw_i \wedge d\overline{w}_j,$$

where γ is any non-vanishing holomorphic section of $\mathcal{L}$. Since any two such sections differ by multiplication by a non-vanishing holomorphic function, it is clear that the definition of the curvature is independent of the choice of the holomorphic section γ. Indeed, it is well known that two such line bundles are locally equivalent if and only if their curvatures are equal. For holomorphic Hermitian vector bundles (rank > 1) the local equivalence involves not only the curvature but also its covariant derivatives, see [12].

In general, Lemma 2.3 of [32] singles out a frame $\gamma^{(0)}$ such that the metric has the form: $\|\gamma^{(0)}(w)\|^2 = I + O(|w|^2)$ and it follows that

$$\kappa(0) = -\sum_{i,j} \left(\partial_i \overline{\partial}_j \|\gamma^{(0)}(w)\|^2\right)_{|w=0} dw_i \wedge d\overline{w}_j.$$

In a slightly different language, fixing $w_0 \in \Omega$, a **normalized kernel** $\mathcal{K}^{(0)}$ at w_0 is defined in [14, Remark 4.7(b)] by requiring that $\mathcal{K}^{(0)}(z, w_0) \equiv I$. Setting $\gamma^{(0)}(w) = \mathcal{K}^{(0)}_w$, we see that the normalized kernel $\mathcal{K}^{(0)}$ has no linear terms. There is a neighborhood, say Ω_0, of w_0 on which $\mathcal{K}(z, w_0)$ doesn't vanish (for $n = 1$) or is an invertible $n \times n$-matrix (for $n > 1$). Set

$$\Phi^{(0)}_\Gamma(z) = \mathcal{K}(w_0, w_0)^{1/2}\, \mathcal{K}(z, w_0)^{-1},\ z \in \Omega_0.$$

Then

$$\mathcal{K}^{(0)}(z, w) := \Phi^{(0)}_\Gamma(z)\, \mathcal{K}(z, w)\, \Phi^{(0)}_\Gamma(w)^*$$

is a normalized kernel on Ω_0. Thus starting with an analytic Hilbert module $\mathcal{M}$ possessing a reproducing kernel $\mathcal{K}$, there is a Hilbert module $\mathcal{M}^{(0)}$ possessing a normalized reproducing kernel $\mathcal{K}^{(0)}$, isomorphic to $\mathcal{M}$. Now, it is evident that two Hilbert modules are isomorphic if and only if there is a unitary U such that

$$\mathcal{K}^{(0)}_1(z, w) = U\, \mathcal{K}^{(0)}_2(z, w)\, U^*.$$

In other words, the normalized kernel is uniquely determined up to a fixed unitary. In particular, if $n = 1$, then the two Hilbert modules are isomorphic if and only if the normalized kernels are equal. We gather all this information in the following proposition.

Proposition 1.1 *The following conditions on any pair of (scalar) analytic Hilbert modules over the polynomial ring are equivalent.*

1. *Two analytic Hilbert modules $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic.*
2. *The holomorphic line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ determined by the eigenspaces of the analytic Hilbert modules $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, are locally equivalent as Hermitian holomorphic bundles.*

3. *The curvature of the two line bundles* $\mathcal{L}_i$, $i = 1, 2$, *are equal.*
4. *The normalized kernels* $\mathcal{K}_i^{(0)}$, $i = 1, 2$, *at any fixed but arbitrary point* w_0 *are equal.*

2 Invariants for Submodules

In the paper [13], Cowen and Douglas pointed out that all submodules of the Hardy module $H^2(\mathbf{D})$ are isomorphic. They used this observation to give a new proof of Beurling's theorem describing all invariant subspaces of $H^2(\mathbf{D})$. Although all submodules of the Hardy module $H^2(\mathbf{D})$ are isomorphic, the quotient modules are not. Surprisingly enough, this phenomenon distinguishes the multi-variable situation from the one variable case. Consider for instance the submodule $H^2_{(0,0)}(\mathbf{D}^2)$ of all functions vanishing at $(0, 0)$ in the Hardy space $H^2(\mathbf{D}^2)$ over the bidisk $\mathbf{D}^2$. Then the module tensor product of $H^2_{(0,0)}(\mathbf{D}^2)$ over the polynomial ring $\mathbf{C}[z]$ in two variables with the one dimensional module $\mathbf{C}_w$, $(p, w) \mapsto p(w)$, is easily seen to be

$$H^2_{(0,0)}(\mathbf{D}^2) \otimes_{\mathbf{C}[z]} \mathbf{C}_w = \begin{cases} \mathbf{C} \oplus \mathbf{C} & \text{if } w = (0, 0) \\ \mathbf{C} & \text{if } w \neq (0, 0) \end{cases} \tag{2.1}$$

while $H^2(\mathbf{D}^2) \otimes_{\mathbf{C}[z]} \mathbf{C}_w = \mathbf{C}$. It follows that the submodule $H^2_{(0,0)}(\mathbf{D}^2)$ is not isomorphic to the module $H^2(\mathbf{D}^2)$, in stark contrast to the case of one variable.

The existence of non-isomorphic submodules of the Hardy module $H^2(\mathbf{D}^2)$ indicates that inner functions alone may not suffice to characterize submodules in this case. It is therefore important to determine when two submodules of the Hardy module, and also more general analytic Hilbert modules, are isomorphic. This question was considered in [10] for the closure of some ideals $\mathcal{I} \subseteq \mathbf{C}[z]$ in the Hardy module $H^2(\mathbf{D}^2)$ with the common zero set $\{(0, 0)\}$. It was extended to a much larger class of ideals in the paper [3]. A systematic study in a general setting culminated in the paper [19] describing a **rigidity phenomenon** for submodules of analytic Hilbert modules in more than one variable. A different proof of the Rigidity Theorem using the sheaf model was given in [9]. A slightly different approach to obtaining invariants by resolving the singularity at $(0, 0)$ was initiated in [16], and considerably expanded in [9]. We describe this approach briefly.

A systematic study of Hilbert submodules of analytic Hilbert modules was initiated in the papers [8, 9]. If $\mathcal{I}$ is an ideal in $\mathbf{C}[z]$, consider the submodule $\widetilde{\mathcal{M}} = [\mathcal{I}]$ in an analytic Hilbert module $\mathcal{M} \subseteq \mathrm{Hol}(\Omega, \mathbf{C})$ obtained by taking the closure of $\mathcal{I}$. Let

$$\Omega_{\mathcal{I}} := \{z \in \Omega : \ f(z) = 0 \ \forall \ f \in \mathcal{I}\}$$

denote the algebraic subvariety of Ω determined by $\mathcal{I}$. For the reproducing kernel $\mathcal{K}(z,w)$ of $\mathcal{M}$, the vectors $\mathcal{K}_w \in \mathcal{M}$ will in general not belong to the submodule $\widetilde{\mathcal{M}}$. However, one has a **truncated kernel** $\widetilde{\mathcal{K}}(z,w) = \widetilde{\mathcal{K}}_w(z)$ such that $\widetilde{\mathcal{K}}_w \in \widetilde{\mathcal{M}}$ for all $w \in \Omega$, which induces a holomorphic Hermitian line bundle $\tilde{\mathcal{L}}$ defined on $\Omega \setminus \Omega_{\mathcal{I}}$, with fibre

$$\tilde{\mathcal{L}}_w = \operatorname{Ran} \widetilde{\mathcal{K}}_w, \ \ w \in \Omega \setminus \Omega_{\mathcal{I}},$$

and positive definite metric $\widetilde{\mathcal{K}}(w,w)$. This line bundle $\tilde{\mathcal{L}}$ does not necessarily extend to all of Ω. In fact, on the singular set $\Omega_{\mathcal{I}}$ the eigenspace of the submodule $\widetilde{\mathcal{M}}$ will in general be higher dimensional. However, in the paper [9], using the monoidal transform, a line bundle $\hat{\mathcal{L}}$ was constructed on a certain blow-up space $\hat{\Omega}$, with a holomorphic map $\pi : \hat{\Omega} \to \Omega$. (Actually, this construction holds locally, near any given point $w_0 \in \Omega_{\mathcal{I}}$.) The restriction of this line bundle to the exceptional set $\pi^{-1}(\Omega_{\mathcal{I}})$ in the blow-up space was shown to be an invariant for the submodule $\widetilde{\mathcal{M}}$.

For the submodule $\widetilde{\mathcal{M}} = H^2_{(0,0)}(\mathbf{D}^2) \subseteq H^2(\mathbf{D}^2)$ of the Hardy module, corresponding to the point singularity $(0,0) \in \Omega := \mathbf{D}^2$, the above construction can be made very explicit: The eigenspace of $\widetilde{\mathcal{M}}$ at $w := (w_1, w_2) \neq (0,0)$ is the one dimensional space spanned by the truncated kernel vector

$$\widetilde{\mathcal{K}}_w(z) := \frac{1}{(1-\overline{w}_1 z_1)(1-\overline{w}_2 z_2)} - 1 = \frac{\overline{w}_1 z_1 + \overline{w}_2 z_2 - \overline{w}_1 z_1 \overline{w}_2 z_2}{(1-\overline{w}_1 z_1)(1-\overline{w}_2 z_2)}. \tag{2.2}$$

At $(0,0)$, this vector is the zero vector while the eigenspace of $\widetilde{\mathcal{M}}$ is two dimensional, spanned by the vectors z_1 and z_2. We observe, however, that for $j = 1, 2$ the limit $\frac{\widetilde{\mathcal{K}}_w(z)}{w_j}$, along lines through the origin as $w \to 0$, exists and is non-zero. Parametrizing the lines through $(0,0)$ in $\mathbf{D}^2$ by $w_2 = \vartheta_1 w_1$ or $w_1 = \vartheta_2 w_2$, we obtain the coordinate charts for the projective space $\mathbf{P}^1(\mathbf{C})$. On these, we have

$$\lim_{\overline{w}_2 = \vartheta_1 \overline{w}_1,\, \overline{w} \to 0} \frac{\widetilde{\mathcal{K}}_w(z)}{\overline{w}_1} = z_1 + \vartheta_1 z_2.$$

Similarly, we have

$$\lim_{\overline{w}_1 = \vartheta_2 \overline{w}_2,\, \overline{w} \to 0} \frac{\widetilde{\mathcal{K}}_w(z)}{\overline{w}_2} = z_2 + \vartheta_2 z_1.$$

Setting $s(\vartheta_1) := z_1 + \vartheta_1 z_2$ and $s(\vartheta_2) = z_2 + \vartheta_2 z_1$ taking values in $H^2_{(0,0)}(\mathbf{D}^2)$, we obtain a holomorphic Hermitian line bundle $\hat{\mathcal{L}}$ over projective space $\mathbf{P}^1(\mathbf{C})$. The metric of this line bundle is given by the formula

$$\|s(\vartheta_j)\|^2_{\widetilde{\mathcal{M}}} = 1 + |\vartheta_j|^2$$

for $j = 1, 2$. It is shown in [16, Theorem 5.1], see also [9, Theorem 3.4], that for many submodules of analytic Hilbert modules, the class of this holomorphic Hermitian line bundle on the projective space is an invariant for the submodule. Since the curvature is a complete invariant, it follows that in our case the curvature

$$\kappa(\vartheta_j) = (1 - |\vartheta_j|^2)^{-2} d\vartheta_j \wedge d\overline{\vartheta}_j$$

for the coordinate ϑ_j $(j = 1, 2)$ is an invariant for the submodule $H^2_{(0,0)}(\mathbf{D}^2)$.

Often it is possible to determine when two submodules of an analytic Hilbert module are isomorphic without explicitly computing a set of invariants. A particular case is the class of submodules in an analytic Hilbert module which are obtained by taking the closure of an ideal in the polynomial ring. Here the surprising discovery is that many of these submodules are isomorphic if and only if the ideals are equal. Of course, one must impose some mild condition on the nature of the ideal. For instance, principal ideals have to be excluded. Several different hypotheses that make this "rigidity phenomenon" possible are discussed in Section 3 of [19]. One of these is the theorem of [19, Theorem 3.6]. A slightly different formulation given below is Theorem 3.1 of [9].

Let $\Omega \subset \mathbf{C}^d$ be a bounded domain. For $k = 1, 2$, let $[\mathcal{I}_k]$ be the closure in an analytic Hilbert module $\mathcal{M} \subseteq \mathrm{Hol}(\Omega)$ of the ideal $\mathcal{I}_k \subseteq \mathbf{C}[z]$.

Theorem 2.1 (Theorem 3.1, [9]) *Assume that the dimension of $[\mathcal{I}_k]/[\mathcal{I}_k]_w$ is finite and that the dimension of the zero set of these modules is at most $d - 2$. Also, assume that every algebraic component of $V(\mathcal{I}_k)$ intersects Ω. Then $[\mathcal{I}_1]$ and $[\mathcal{I}_2]$ are isomorphic if and only if $\mathcal{I}_1 = \mathcal{I}_2$.*

In this paper we study submodules of (scalar valued) analytic Hilbert modules $(n = 1)$ which are related to higher-dimensional singularities. Starting with the weighted Bergman spaces defined on a bounded symmetric domain, the submodules are determined by a vanishing condition on a certain "Kepler variety". The new feature is that the singularity set is not a complete intersection (in the sense of algebraic geometry) which means that the usual projectivization involving monoidal transforms (blow-up process) is not sufficient for the resolution of singularities. We will replace it by a higher-rank blow-up process, having as exceptional fibres compact Hermitian symmetric spaces of higher rank instead of projective spaces. The charts and analytic continuation we use are adapted to the geometry of the Kepler variety. The simplest case of rank 1 reduces to the usual blow-up process.

In this setting we again obtain a rigidity theorem which is not a special case of Theorem 2.1, since we do not consider different ideals (i.e. different subvarieties) for the singular modules, but we consider a fixed subvariety and vary the underlying "big" Hilbert module, by choosing an arbitrary coefficient sequence or, as a special case, a K-invariant probability measure. This situation is most interesting in the symmetric case, where one has a full scale of different Hilbert modules like the weighted Bergman spaces. Then we show that the "truncated" kernel of the submodule can be recovered from the reduction to the blow-up space. This is a kind of rigidity in the parameter space instead of selecting different ideals.

3 Jordan–Kepler Varieties

Hilbert modules and submodules defined by analytic varieties have been mostly studied for domains Ω which are strongly pseudoconvex with smooth boundary, or a product of such domains. From an operator-theoretic point of view, this is natural since for strongly pseudoconvex (bounded) domains, Toeplitz operators with continuous symbols (in particular, with symbols given by the coordinate functions) are essentially normal, so that the Toeplitz C^*-algebra generated by such operators is essentially commutative and has a classical Fredholm and index theory. There are, however, interesting classes of bounded domains which are only weakly pseudoconvex (and are therefore domains of holomorphy, by the Cartan—Thullen theorem) with a non-smooth boundary. A prominent class of such domains are the **bounded symmetric domains** of arbitrary rank r, which generalize the (strongly pseudoconvex) unit ball, having rank $r = 1$. The Hardy space and the weighted Bergman spaces of holomorphic functions on bounded symmetric domains have been extensively studied from various points of view (see, e.g., [6, 21, 30]). More recently, irreducible subvarieties of symmetric domains, given by certain determinant type equations, have been studied in [20] under the name of “Jordan–Kepler varieties”. This terminology is used since the rank $r = 2$ case corresponds to the classical Kepler variety in the cotangent bundle of spheres [11].

In order to describe bounded symmetric domains and their determinantal subvarieties, we will use the **Jordan theoretic** approach to bounded symmetric domains which is best suited for harmonic and holomorphic analysis on symmetric domains. For background and details concerning the Jordan theoretic approach, we refer to [22, 26, 30].

Let V be an irreducible Hermitian Jordan triple of rank r, with Jordan triple product denoted by $\{u; v; w\}$. The so-called **spectral unit ball** $\Omega \subset V$ is a bounded symmetric domain. Conversely, every (irreducible) bounded symmetric domain can be realized in this way. An example is the matrix space $V = \mathbf{C}^{r\times s}$ with triple product

$$\{u; v; w\} := uv^*w + wv^*u,$$

giving rise to the matrix ball

$$\Omega = \{z \in \mathbf{C}^{r\times s} : I_r - zz^* > 0\}.$$

In particular, for rank $r = 1$ we obtain the triple product

$$\{u; v; w\} := (u|v)w + (w|v)u$$

on $V = \mathbf{C}^d$, with inner product $(u|v)$, giving rise to the unit ball

$$\Omega = \{z \in \mathbf{C}^d : (z|z) < 1\}.$$

Let G denote the identity component of the full holomorphic automorphism group of Ω. Its maximal compact subgroup

$$K := \{k \in G : \ k(0) = 0\}$$

consists of linear transformations preserving the Jordan triple product. For $z, w \in V$ define the **Bergman operator** $B_{z,w}$ acting on V by

$$B_{z,w}v = v - \{z; w; v\} + \frac{1}{4}\{z\{w; v; w\}z\}.$$

We can also write

$$B_{z,w} = I - D(z, w) + Q_z Q_w, \tag{3.1}$$

where

$$D(z, w)v = \{z; w; v\},$$

and

$$Q_z w := \frac{1}{2}\{z; w; z\}$$

denotes the so-called quadratic representation (conjugate linear in w). For matrices, we have $D(z, w)v = zw^*v + vw^*z, \ Q_z w = zw^*z$ and hence

$$B_{z,w}v = (1_r - zw^*)v(1_s - w^*z). \tag{3.2}$$

An element $c \in V$ satisfying $c = Q_c c$ is called a **tripotent**. For matrices these are the partial isometries. Any tripotent c induces a **Peirce decomposition**

$$V = V_2^c \oplus V_1^c \oplus V_0^c.$$

Now we introduce certain K-invariant varieties. Every Hermitian Jordan triple V has a natural notion of **rank** defined via spectral theory. For fixed $\ell \leq r$ let

$$\mathring{V}_\ell = \{z \in V : \ \text{rank}(z) = \ell\}$$

denote the **Jordan–Kepler manifold** studied in [20]. It is a $K^{\mathbf{C}}$-homogeneous manifold whose closure is the **Jordan–Kepler variety**

$$V_\ell = \{z \in V : \ \text{rank}(z) \leq \ell\}.$$

We have

$$d_\ell := \dim \mathring{V}_\ell = d_2^c + d_1^c,$$

where

$$d_2^c = \dim V_2^c = \ell(1 + \frac{a}{2}(\ell - 1)),$$

$$d_1^c = \dim V_1^c = \ell(a(r - \ell) + b).$$

Here a, b are the so-called characteristic multiplicities defined in terms of a joint Peirce decomposition [26]. Moreover,

$$\frac{2d_2^c + d_1^c}{\ell} = 2(1 + \frac{a}{2}(\ell - 1)) + a(r - \ell) + b = 2 + a(r - 1) + b = p$$

is the genus. As a fundamental property, there exists a **Jordan triple determinant**

$$\Delta : V \times V \to \mathbf{C}, \tag{3.3}$$

which is a (non-homogeneous) sesqui-polynomial satisfying

$$\det B_{z,w} = \Delta(z, w)^p.$$

For $(r \times s)$-matrices, we have $p = r + s$ and

$$\Delta(z, w) = \det(1_r - zw^*)$$

as a consequence of (3.2). In particular, $\Delta(z, w) = 1 - (z|w)$ in the rank 1 case $V = \mathbf{C}^d$. A Hermitian Jordan triple U is called **unital** if it contains a (non-unique) tripotent u such that $D(u, u) = 2{\cdot}I$. In this case U becomes a Jordan *-algebra with unit element u under the multiplication

$$z \circ w := \frac{1}{2}\{z; u; w\}$$

and involution

$$z^* := Q_u z = \frac{1}{2}\{u; z; u\}.$$

This Jordan algebra has a homogeneous **determinant polynomial** $N : U \to \mathbf{C}$ defined in analogy to Cramer's rule for square matrices. Every Peirce 2-space V_2^c is a unital Jordan triple with unit c.

One can show that the smooth part of V_ℓ (in the sense of algebraic geometry) is precisely given by $\mathring{V}_\ell$. Thus the **singular points** of V_ℓ form the closed subvariety $V_{\ell-1}$, which has codimension > 1, unless we have the case $\ell = r$ for tube domains ($b = 0$). This case will be excluded in the sequel. The **center** $S_\ell \subset \mathring{V}_\ell$ consists of all tripotents of rank ℓ.

4 Hilbert Modules on Kepler Varieties

Combining the Kepler variety and the spectral unit ball, we define the **Kepler ball**

$$\Omega_\ell := \Omega \cap V_\ell$$

for any $0 \le \ell \le r$. The Kepler ball Ω_ℓ has singularities exactly at $\Omega_{\ell-1}$, so that the smooth part of Ω_ℓ is given by

$$\mathring{\Omega}_\ell := \mathring{V}_\ell \cap \Omega_\ell = \Omega_\ell \setminus \Omega_{\ell-1}.$$

Apart from the case $\ell = r$ on tube type domains, which we exclude here, the singular set $\Omega_{\ell-1} \subset \Omega_\ell$ has codimension > 1. Combining this with the fact that V_ℓ is a normal variety (so that the second Riemann extension theorem holds) it follows that every holomorphic function on $\mathring{\Omega}_\ell$ has a unique holomorphic extension to Ω_ℓ. Henceforth we will identify holomorphic functions on $\mathring{\Omega}_\ell$ with their unique holomorphic extension to Ω_ℓ. For any K-invariant measure ρ on $\mathring{V}_\ell$ we have a **polar integration formula**

$$\int\limits_{\mathring{V}_\ell} d\rho(z)\ f(z) = \int\limits_{\Lambda_2^c} d\rho^c(t) \int\limits_K dk\ f(k\sqrt{t})$$

where ρ^c is a measure on the symmetric cone Λ_2^c of V_2^c [22] called the **radial part** of ρ. Here $\sqrt{t}$ denotes the Jordan algebraic square root in Λ_2^c. As a special case, consider the **Riemann measure** $\lambda_\ell(dz)$ on $\mathring{V}_\ell$ which is induced by the normalized inner product on V. Denoting by Γ_ℓ the Koecher–Gindikin Gamma function of Λ_2^c [22], its polar decomposition is

$$\int\limits_{\mathring{V}_\ell} \frac{\lambda_\ell(dz)}{\pi^{d_\ell}}\ f(z) = \frac{\Gamma_\ell(\frac{a\ell}{2})}{\Gamma_\ell(\frac{d}{r})\Gamma_\ell(\frac{ar}{2})} \int\limits_{\Lambda_2^c} dt\ N_c(t)^{d_1^c/\ell} \int\limits_K dk\ f(k\sqrt{t}). \tag{4.1}$$

Here N_c is the Jordan algebra determinant on V_2^c normalized by $N_c(c) = 1$. For $\ell = r$ the Riemann measure on the open dense subset $\mathring{V}_r = \mathring{V} \subset V$ agrees with the

Lebesgue measure, and (4.1) gives the well-known formula

$$\int_V \frac{dz}{\pi^d} f(z) = \frac{1}{\Gamma(\frac{d}{r})} \int_{\Lambda_2^e} dt\ N_e(t)^b \int_K dk\ f(k\sqrt{t})$$

for any maximal tripotent $e \in S = S_r$. As a consequence of (4.1) we have for the Kepler ball

$$\int_{\mathring{\Omega}_\ell} \frac{\lambda_\ell(dz)}{\pi^{d_\ell}} \Delta(z,z)^{\nu-p} f(z)$$

$$= \frac{\Gamma_\ell(\frac{a\ell}{2})}{\Gamma_\ell(\frac{d}{r})\Gamma_\ell(\frac{ar}{2})} \int_{\Lambda_2^c \cap (c-\Lambda_2^c)} dt\ N_c(t)^{d_1^c/\ell}\ N_c(c-t)^{\nu-p} \int_K dk\ f(k\sqrt{t}) \tag{4.2}$$

since $\Delta(k\sqrt{t}, k\sqrt{t}) = \Delta(\sqrt{t}, \sqrt{t}) = N_c(c-t)$ for all $t \in \Lambda_2^c \cap (c - \Lambda_2^c)$.

As a fundamental fact [22, 30] of harmonic analysis on Jordan algebras and Jordan triples, the Fischer–Fock reproducing kernel $e^{(z|w)}$, for the normalized K-invariant inner product $(z|w)$ on V, has a "Taylor expansion"

$$e^{(z|w)} = \sum_{\boldsymbol{m}} E^{\boldsymbol{m}}(z,w)$$

over all integer partitions $\boldsymbol{m} = m_1 \geq m_2 \geq \ldots \geq m_r \geq 0$, where $E^{\boldsymbol{m}}(z,w) = E_w^{\boldsymbol{m}}(z)$ are sesqui-polynomials which are K-invariant such that the finite-dimensional vector space

$$\mathcal{P}_{\boldsymbol{m}}(V) = \{E_w^{\boldsymbol{m}} :\ w \in V\}$$

is an irreducible K-module. These K-modules are pairwise inequivalent and span the polynomial algebra $\mathcal{P}(V)$. Let

$$(\nu)_{\boldsymbol{m}} = \prod_{j=1}^r (\nu - \frac{a}{2}(j-1))_{m_j}$$

denote the multi-variable **Pochhammer symbol**. Let $\mathbf{N}_+^r$ denote the set of all partitions of length $\leq r$. Restricted to the Kepler variety we only consider partitions in $\mathbf{N}_+^\ell$ of length $\leq \ell$, completed by zeroes at the end.

Lemma 4.1 *For any partition $\boldsymbol{m} \in \mathbf{N}_+^\ell$ of length $\leq \ell$ we have*

$$\int_{\Lambda_2^c \cap (c-\Lambda_2^c)} dt\ N_c(t)^{d_1^c/\ell}\ N_c(c-t)^{\nu-p}\ N_{\boldsymbol{m}}(t) = \frac{\Gamma_\ell(\frac{d_\ell}{\ell})\ \Gamma_\ell(\nu - \frac{d_\ell}{\ell})}{\Gamma_\ell(\nu)} \frac{(d_\ell/\ell)_{\boldsymbol{m}}}{(\nu)_{\boldsymbol{m}}}. \tag{4.3}$$

Proof Applying [22, Theorem VII.1.7] to Λ_2^c yields

$$\int\limits_{\Lambda_2^c\cap(c-\Lambda_2^c)} dt\ N_c(t)^{d_1^c/\ell}\ N_c(c-t)^{\nu-p}\ N_{\boldsymbol{m}}(t) = \frac{\Gamma_\ell(\boldsymbol{m}+\frac{d_1^c}{\ell}+\frac{d_2^c}{\ell})\ \Gamma_\ell(\nu-p+\frac{d_2^c}{\ell})}{\Gamma_\ell(\boldsymbol{m}+\nu-p+\frac{d_1^c+2d_2^c}{\ell})}$$

$$= \frac{\Gamma_\ell(\boldsymbol{m}+\frac{d_\ell}{\ell})\ \Gamma_\ell(\nu-\frac{d_\ell}{\ell})}{\Gamma_\ell(\boldsymbol{m}+\nu)} = \frac{\Gamma_\ell(\frac{d_\ell}{\ell})\ \Gamma_\ell(\nu-\frac{d_\ell}{\ell})}{\Gamma_\ell(\nu)}\ \frac{(d_\ell/\ell)_{\boldsymbol{m}}}{(\nu)_{\boldsymbol{m}}}. \qquad \square$$

Let du be the K-invariant probability measure on S_ℓ and put

$$(f|g)_{S_\ell} = \int\limits_{S_\ell} du\ \overline{f(u)}\ g(u) = \int\limits_K dk\ \overline{f(kc)}\ g(kc). \tag{4.4}$$

Definition 4.2 Consider a coefficient sequence $(\rho_{\boldsymbol{m}})_{\boldsymbol{m}\in\mathbf{N}_+^\ell}$ normalized by $\rho_0 = 1$. Define a Hilbert space $\mathcal{M} = \mathcal{M}_\rho$ of holomorphic functions on Ω_ℓ by imposing the K-invariant inner product

$$(f|g)_\rho := \sum_{\boldsymbol{m}\in\mathbf{N}_+^\ell} \rho_{\boldsymbol{m}}(f_{\boldsymbol{m}}|g_{\boldsymbol{m}})_{S_\ell}. \tag{4.5}$$

where $f_{\boldsymbol{m}} \in \mathcal{P}_{\boldsymbol{m}}(V)$ denotes the $\boldsymbol{m}$-th component of f.

The **subnormal case** arises when the inner product (4.5) has the form

$$(f|g)_\rho = \int d\rho(z)\ \overline{f(z)}g(z),$$

where ρ is a K-invariant probability measure on the closure of Ω_ℓ or a suitable K-invariant subset which is a set of uniqueness for holomorphic functions. For the case $\ell = r$, this was studied in detail for the tube type domains in [7] and completed for all bounded symmetric domains in [5]. By [20, Proposition 4.4] the Hilbert space

$$\mathcal{M} = \mathcal{M}_\rho := \{\phi \in L^2(d\rho) :\ \phi \text{ holomorphic on } \Omega_\ell\}$$

has the coefficient sequence

$$\rho_{\boldsymbol{m}} = \int\limits_{\Lambda_2^c} d\rho^c(t)\ N_{\boldsymbol{m}}(t)$$

given by the **moments** of the radial part ρ^c, which is a probability measure on Λ_2^c (not necessarily of full support). As a special case the Hardy type inner product (4.4), corresponding to the K-invariant probability measure du on S_ℓ, has the point mass at c as its radial part, showing that all radial moments $\rho_{\boldsymbol{m}} = 1$.

It is clear that the Hilbert spaces $\mathcal{M}_\rho$ defined by K-invariant measures are analytic Hilbert modules as defined above (however, consisting of holomorphic functions on a manifold $\mathring{\Omega}_\ell$ instead of a domain). For more general coefficient sequences $\rho_{\boldsymbol{m}}$, one could in principle determine whether multiplication operators by polynomials are bounded (using certain growth conditions on the coefficient sequence), and whether the other requirements for analytic Hilbert modules hold. Important examples are listed below where the reproducing kernels are given by hypergeometric series. For the classical case $\ell = r$, the well-understood analytic continuation of the scalar holomorphic discrete series of weighted Bergman spaces on $\Omega = \Omega_r$ [21] shows that the Hilbert module property extends beyond the subnormal case.

Proposition 4.3 *For a given coefficient sequence $\rho_{\boldsymbol{m}}$, $\mathcal{M}$ has the* ***reproducing kernel***

$$\mathcal{K}(z, w) = \sum_{\boldsymbol{m} \in \mathbf{N}_+^\ell} \frac{(d/r)_{\boldsymbol{m}}}{\rho_{\boldsymbol{m}}} \frac{(ra/2)_{\boldsymbol{m}}}{(\ell a/2)_{\boldsymbol{m}}} E^{\boldsymbol{m}}(z, w). \tag{4.6}$$

Proof This follows from [20, Proposition 4.3] and the formula

$$\frac{d_{\boldsymbol{m}}}{d_{\boldsymbol{m}}^c} = \frac{(d/r)_{\boldsymbol{m}}}{(d_2^c/\ell)_{\boldsymbol{m}}} \frac{(ra/2)_{\boldsymbol{m}}}{(\ell a/2)_{\boldsymbol{m}}}$$

obtained in [20, equation (5.5) in the proof of Theorem 5.1]. □

We will now present some examples, where the reproducing kernel (4.6) can be expressed in closed form as a multivariate hypergeometric series defined in general by

$${}_p\begin{pmatrix}\alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q\end{pmatrix}_q (z, w) = \sum_{\boldsymbol{m}} \frac{(\alpha_1)_{\boldsymbol{m}} \cdots (\alpha_p)_{\boldsymbol{m}}}{(\beta_1)_{\boldsymbol{m}} \cdots (\beta_q)_{\boldsymbol{m}}} E^{\boldsymbol{m}}(z, w).$$

Applying (4.3) to $\boldsymbol{m} = 0$ it follows that

$$\rho^\nu(dz) = \frac{\Gamma_\ell(\frac{d}{r})}{\Gamma_\ell(\frac{d_\ell}{\ell})} \frac{\Gamma_\ell(\frac{ra}{2})}{\Gamma_\ell(\frac{\ell a}{2})} \frac{\Gamma_\ell(\nu)}{\Gamma_\ell(\nu - \frac{d_\ell}{\ell})} \frac{\lambda_\ell(dz)}{\pi^{d_\ell}} \Delta(z, z)^{\nu - p}$$

is a probability measure on $\mathring{\Omega}_\ell$. Moreover, applying (4.3) to any $\boldsymbol{m} \in \mathbf{N}_+^\ell$ it follows that the measure ρ^ν has the coefficient sequence

$$\rho_{\boldsymbol{m}}^\nu = \frac{(d_\ell/\ell)_{\boldsymbol{m}}}{(\nu)_{\boldsymbol{m}}}.$$

Thus the Hilbert space

$$\mathcal{M}_\nu := \{\phi \in L^2(d\rho^\nu) : \ \phi \text{ holomorphic on } \Omega_\ell\}$$

of holomorphic functions on Ω_ℓ has the reproducing kernel

$$\mathcal{K}(z,w) = \sum_{\boldsymbol{m}\in\mathbf{N}_+^\ell} \frac{(d/r)_{\boldsymbol{m}}}{(d_\ell/\ell)_{\boldsymbol{m}}} \frac{(ra/2)_{\boldsymbol{m}}}{(\ell a/2)_{\boldsymbol{m}}} (\nu)_{\boldsymbol{m}}\ E^{\boldsymbol{m}}(z,w) = {}_3\binom{\frac{d}{r},\ \frac{ra}{2},\ \nu}{\frac{d_\ell}{\ell},\ \frac{\ell a}{2}}_2 (z,w).$$

In the classical case $\ell = r$ we have the probability measure

$$d\rho^\nu(z) = \frac{\Gamma(\nu)}{\Gamma(\nu - \frac{d}{r})} \frac{dz}{\pi^d} \Delta(z,z)^{\nu-p}$$

on Ω, whose reproducing kernel is given by

$$\mathcal{K}(z,w) = \sum_{\boldsymbol{m}\in\mathbf{N}_+^r} (\nu)_{\boldsymbol{m}}\ E^{\boldsymbol{m}}(z,w) = {}_1\binom{\nu}{\ }_0 (z,w) = \Delta(z,w)^{-\nu}$$

according to the Faraut–Korányi formula [21].

5 The Singular Set and Its Resolution

The only strongly pseudoconvex symmetric domains are the unit balls of rank $r = 1$. Here the singularity Ω_0 consists of a single point $\{0\}$. The classical procedure to resolve this singularity is the monoidal transformation (blow-up process) where a point is replaced by a projective space of appropriate dimension. As the main geometric result in this paper, we obtain a generalization of the blow-up process for higher dimensional Kepler varieties and domains of arbitrary rank. The Jordan theoretic approach leads to quite explicit formulas which generalize the equations of the classical blow-up process of a point.

The general procedure outlined in Sect. 2 using monoidal transformations works in the case where the singularity is given by a regular sequence $g_1, \ldots, g_m$ of polynomials generating the vanishing ideal $\mathcal{I}$. In this case the variety is a smooth complete intersection. If $m = d$ equals the dimension, this variety reduces to a single point. The usual blow-up process around a point $0 \in \mathbf{C}^d$ is the proper holomorphic map

$$\pi : \hat{\mathbf{C}}^d \to \mathbf{C}^d$$

where

$$\hat{\mathbf{C}}^d := \{(w, U) : \ w \in \mathbf{C}^d, \ U \in \mathbf{P}^{d-1}, \ w \in U\}$$

is the tautological bundle over $\mathbf{P}^{d-1}$, with "collapsing map" $\pi(w, U) := w$. The map π is biholomorphic outside the exceptional fibre $\pi^{-1}(0) = \mathbf{P}^{d-1}$. For the Kepler varieties studied here the singular set $\Omega_{\ell-1}$ has higher dimension and is not a complete intersection (unless $\ell = 1$). Thus a regular generating sequence of polynomials does not exist. Instead, we use the harmonic analysis of polynomials provided by the Jordan theoretic approach to study the singular set. The main idea is to replace the projective space (a compact Hermitian symmetric space of rank 1) by a compact Hermitian symmetric space of higher rank, namely the **Peirce manifold**

$$M_\ell = \{V_2^c : \ c \in S_\ell\}$$

of all Peirce 2-spaces of rank ℓ in V. This can also be realized as the conformal compactification of the Peirce 1-space V_1^c, for any rank ℓ tripotent c. For example, in the full matrix triple $V = \mathbf{C}^{r \times s}$ the Peirce 1-space of $c = \begin{pmatrix} 1_\ell & 0 \\ 0 & 0 \end{pmatrix} \in S_\ell$ is given by

$$V_1^c = \begin{pmatrix} 0 & \mathbf{C}^{\ell \times (s-\ell)} \\ \mathbf{C}^{(r-\ell) \times \ell} & 0 \end{pmatrix}.$$

Hence, in this case, the Peirce manifold M_ℓ is the direct product of two Grassmann manifolds

$$M_\ell = \mathrm{Grass}_\ell(\mathbf{C}^r) \times \mathrm{Grass}_\ell(\mathbf{C}^s).$$

In the simplest case $r = 1$ we have $V = \mathbf{C}^d$ and for the tripotent $c = (1, 0^{d-1})$ we have $V_1^c = (0, \mathbf{C}^{d-1})$. Its conformal compactification is $\hat{V}_1^c = \mathbf{P}^{d-1}$, which is the exceptional fibre of the usual blow-up process for $0 \in \mathbf{C}^d$. More generally, for any non-zero tripotent c we have $V_2^c = \mathbf{C} \cdot c$ and hence V_1^c becomes the orthogonal complement $c^\perp = \mathbf{C}^{d-1}$, with conformal compactification $\hat{V}_1^c = \mathbf{P}^{d-1}$.

The standard charts of projective space $\mathbf{P}^{d-1}$ have the form

$$\tau_i : \mathbf{C}^{d-1} \to \mathbf{P}^{d-1}, \quad \tau_i(t_1, \ldots, \hat{t}_i, \ldots, t_d) := [t_1 : \ldots : 1_i : \ldots : t_d]$$

using homogeneous coordinates on $\mathbf{P}^{d-1}$. Note that for $1 \leq i \leq d$, the rank 1 tripotent $c_i := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbf{C}^d$ has the Peirce 1-space

$$V_1^{c_i} := \{(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_d) : \ (t_1, \ldots, \hat{t}_i, \ldots, t_d) \in \mathbf{C}^{d-1}\}.$$

In the higher rank setting, the Bergman operators (3.1) serve to define canonical charts for the Peirce manifolds. For each tripotent $c \in S_\ell$ and every $t \in V_1^c$ the transformation $B_{t,-c} \in K^{\mathbf{C}}$ preserves the rank. It follows that $B_{t,-c}c \in \mathring{V}_\ell$ has a Peirce 2-space denoted by $[B_{t,-c}c]$. As shown in [29] the map

$$\tau_c : V_1^c \to M_\ell, \ \tau_c(t) := [B_{t,-c}c] \tag{5.1}$$

is a holomorphic chart of M_ℓ. The range of the chart τ_c is

$$M_c := \{U \in M_\ell : \ N_U(c) \neq 0\}.$$

Here $N_U : U \to \mathbf{C}$ denotes a Jordan algebra determinant of the Jordan triple U which, as a Peirce 2-space, is of tube type. The Jordan determinant is only defined after choosing a maximal tripotent in U as a unit element, but any two such determinant functions differ by a non-zero multiple. It is shown in [29] that the local charts τ_c of M_ℓ, for different tripotents $c, c' \in S_\ell$, are compatible and hence form a holomorphic atlas on M_ℓ.

One can make the passage $z \mapsto [z]$ to the Peirce 2-space more explicit by introducing the so-called (Moore-Penrose) pseudo-inverse. Every element $z \in \mathring{V}_\ell$ has a **pseudo-inverse** $\tilde{z} \in \mathring{V}_\ell$ determined by the properties

$$Q_z\tilde{z} = z, \ Q_{\tilde{z}}z = \tilde{z}, \ Q_z\, Q_{\tilde{z}} = Q_{\tilde{z}}\, Q_z.$$

Using the pseudo-inverse, the orthogonal projection onto the Peirce 2-space of V_2^z can be explicitly written down.

Lemma 5.1 *The pseudo-inverse of $z := B_{t,-c}c$ is given by*

$$\tilde{z} = B_{t,-c}B_{t,-t}^{-1}c.$$

Combining these remarks, the chart (5.1) can be written down explicitly. It is also instructive to embed M_ℓ into the **conformal compactification** $\hat{V}$ of the underlying Jordan triple V (the compact Hermitian symmetric space that is dual to the spectral unit ball Ω). According to [26] $\hat{V}$ can elegantly be described using a certain equivalence relation $[z; w]$ for pairs $z, w \in Z$. As shown in [29], one may identify the Peirce 2-space V_2^z with the equivalence class $[z; \tilde{z}] \in \hat{V}$. Thus the local chart (5.1) associated to a tripotent $c \in S_\ell$ can also be expressed via the embedding

$$\tau_c : V_1^c \to M_\ell \subset \hat{V}$$

given by

$$\tau_c(t) = [z; \tilde{z}],$$

where $z := B_{t,-c}c \in \overset{\circ}{V}_\ell$ and $\tilde{z}$ is computed via Lemma 5.1. In the sequel this more refined description of the local charts will not be needed.

Having found the exceptional fibre M_ℓ for the higher-rank blow-up process, we now consider the **tautological bundle**

$$\hat{V}_\ell = \{(w, U) \in V \times M_\ell : \ w \in U\} \subset V_\ell \times M_\ell$$

over M_ℓ, together with the **collapsing map**

$$\pi : \hat{V}_\ell \to V_\ell, \quad \pi(w, U) := w$$

whose range is V_ℓ. In [20] this map is used to show that V_ℓ is a **normal variety**. This property implies the so-called second Riemann extension theorem for holomorphic functions, of crucial importance in the following. For each $s \in V_2^c$ the rank ℓ element

$$\sigma_c(s, t) := B_{t,-c}s \tag{5.2}$$

has the same Peirce 2-space $\tau_c(t)$ as $B_{t,-c}c$. We define a local chart

$$\rho_c : V_2^c \times V_1^c \to \hat{V}_\ell$$

by

$$\rho_c(s, t) := (\sigma_c(s, t), \tau_c(t)). \tag{5.3}$$

By (5.2) the range of the chart ρ_c is

$$\hat{V}_\ell^c := \{(w, U) \in \hat{V}_\ell : \ U \in \text{Ran}\ \tau_c\} = \{(w, U) \in \hat{V}_\ell : \ N_U(c) \neq 0\}.$$

One shows that the charts ρ_c, for $c \in S_\ell$, define a holomorphic atlas on $\hat{V}_\ell$, such that the collapsing map $\pi : \hat{V}_\ell \to V_\ell$ is holomorphic and is biholomorphic outside the singular set. We call $\hat{V}_\ell$, together with the collapsing map the (higher rank) **blow-up** of V_ℓ.

Proposition 5.2 *For rank 1, let $c := (1, 0)$. Then*

$$\rho_c(s, t) := ((s, st), [1 : t]) = ((s, st), \mathbf{C}(1, t)),$$

where $s \in \mathbf{C}$ and $t \in \mathbf{C}^{d-1}$. Here $[s : t] = [s : t_1 : \ldots : t_{d-1}]$ denotes the homogeneous coordinates in $\mathbf{P}^{d-1}$.

Proof Clearly, $V_2^c = \mathbf{C}\cdot c = (\mathbf{C}, 0) = [1 : 0]$ and $V_1^c = (0, \mathbf{C}^{d-1})$. Then

$$\sigma_c(s,t) = B_{t,-c}s = \left(1 + (0,t)\begin{pmatrix}1\\0\end{pmatrix}\right)(s,0)\left(\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0&t\end{pmatrix}\right)$$
$$= (s,0)\begin{pmatrix}1&t\\0&1\end{pmatrix} = (s, st).$$

In particular, $\sigma_c(1,t) = (1,t)$ has the Peirce 2-space $\tau_c(t) = \mathbf{C}\cdot(1,t) = [1 : t]$. It follows that

$$\rho_c(s,t) = (\sigma_c(s,t), \tau_c(t)) = ((s,st), \mathbf{C}\cdot(1,t)) = ((s,st), [1:t]).$$

□

More generally, taking for $c = e_i$ the i-th basis unit vector $(1 \le i \le d)$ we obtain local charts

$$\rho_i(\zeta^i, \zeta') = ((\zeta^i, \zeta^i\zeta'), \mathbf{C}(1^i, \zeta')) = ((\zeta^i, \zeta^i\zeta'), [1^i : \zeta'])$$

where $\zeta' = (\zeta^j)_{j\ne i}$. The finitely many charts ρ_i $(1 \le i \le d)$ form already a covering. Using the grid approach to Jordan triples one can similarly choose finitely many charts in the general case. However, for many arguments using K-invariance it is more convenient to take the continuous family of charts $(\rho_c)_{c\in S_\ell}$.

Since the analytic Hilbert modules considered here are supported on the Kepler ball $\Omega_\ell = \Omega \cap V_\ell$ we restrict the tautological bundle to the open subset

$$\hat\Omega_\ell := \{(w, U) \in \hat V_\ell : \ w \in \Omega_\ell\}$$

and obtain a collapsing map $\pi : \hat\Omega_\ell \to \Omega_\ell$ by restriction. The main idea to study singular submodules $\widetilde{\mathcal{M}}$ is now to construct a Hermitian holomorphic line bundle $\hat{\mathcal{L}}$ over $\hat\Omega_\ell$, whose curvature will be the crucial invariant of $\widetilde{\mathcal{M}}$.

Proposition 5.3 *There exists a holomorphic line bundle $\hat{\mathcal{L}}$ on $\hat\Omega_\ell$ consisting of all equivalence classes*

$$[s, t, \lambda\, \overline{N_c(s)}]_c = \left[s', t', \lambda\, \overline{N_{c'}(s')}\right]_{c'} \tag{5.4}$$

with $\lambda \in \mathbf{C}$. Here $c, c' \in S_\ell$ are tripotents such that

$$\rho_c(s,t) = \rho_{c'}(s', t') \tag{5.5}$$

for $(s,t) \in V_2^c \times V_1^c$ and $(s', t') \in V_2^{c'} \times V_1^{c'}$.

Proof The condition (5.5) implies $\sigma_c(s,t) = \sigma_{c'}(s',t')$ and $[\sigma_c(1,t)] = \tau_c(t) = \tau_{c'}(t') = [\sigma_{c'}(1,t')]$. This implies that $N_c(s)$ and $N_{c'}(s')$ do not vanish. Since the quotient maps $\frac{\overline{N_{c'}(s')}}{\overline{N_c(s)}}$ satisfy a cocycle property, it follows that

$$[s,t,\lambda]_c = \left[s', t', \lambda \frac{\overline{N_{c'}(s')}}{\overline{N_c(s)}}\right]_{c'}$$

defines an equivalence relation yielding a holomorphic line bundle. □

At this point we do not fix a Hermitian metric on the line bundle $\hat{\mathcal{L}}$ over $\hat{\Omega}_\ell$. The metric depends on the choice of singular submodules $\widetilde{\mathcal{M}}$ which will be defined below.

6 Singular Hilbert Submodules

Consider the partition

$$\mathbf{1} := (1, \ldots, 1, 0, \ldots, 0)$$

of length ℓ, with 1 repeated ℓ times. Given the Hilbert module $\mathcal{M} = \mathcal{M}_\rho$ as above, consider the K-invariant Hilbert submodule

$$\widetilde{\mathcal{M}} = \{\psi \in \mathcal{M} : \ \psi|_{V_{\ell-1}} = 0\}.$$

The formula (4.6) yields the **truncated kernel** in the form

$$\widetilde{\mathcal{K}}(z,w) = \sum_{\boldsymbol{m} \in \mathbf{N}_+^\ell} \frac{(d/r)_{\boldsymbol{m}+\mathbf{1}}}{\rho_{\boldsymbol{m}+\mathbf{1}}} \frac{(ra/2)_{\boldsymbol{m}+1}}{(\ell a/2)_{\boldsymbol{m}+\mathbf{1}}} E^{\boldsymbol{m}+\mathbf{1}}(z,w), \tag{6.1}$$

corresponding to vanishing of order ≥ 1 on $V_{\ell-1}$. Using the identity

$$(\nu)_{\boldsymbol{m}+\mathbf{1}} = (\nu+1)_{\boldsymbol{m}} \ (\nu)_{\mathbf{1}}$$

one can also express this using Pochhammer symbols for $\boldsymbol{m}$ instead of $\boldsymbol{m}+\mathbf{1}$.

Lemma 6.1 *Let V be a unital Jordan triple, with Jordan algebra determinant N. Then we have*

$$E^{\boldsymbol{m}+\mathbf{1}}(z,w) = \frac{(d/r)_{\boldsymbol{m}}}{(d/r)_{\boldsymbol{m}+\mathbf{1}}} N(z)\overline{N(w)} \ E^{\boldsymbol{m}}(z,w).$$

Proof For tube type we have

$$E^{\boldsymbol{m}}(e,e) = \frac{d_{\boldsymbol{m}}}{(d/r)_{\boldsymbol{m}}}.$$

Writing

$$E^{\mathbf{m}+\mathbf{1}}(z, w) = c_{\mathbf{m}}\ N(z)\overline{N(w)}\ E^{\mathbf{m}}(z, w)$$

it follows that

$$\frac{d_{\mathbf{m}+\mathbf{1}}}{(d/r)_{\mathbf{m}+\mathbf{1}}} = E^{\mathbf{m}+\mathbf{1}}(e, e) = c_{\mathbf{m}}\ E^{\mathbf{m}}(e, e) = c_{\mathbf{m}}\ \frac{d_{\mathbf{m}}}{(d/r)_{\mathbf{m}}}.$$

Since $d_{\mathbf{m}+\mathbf{1}} = d_{\mathbf{m}}$ in the unital case, it follows that

$$c_{\mathbf{m}} = \frac{(d/r)_{\mathbf{m}}}{(d/r)_{\mathbf{m}+\mathbf{1}}}.$$

□

Lemma 6.2 *For* $\mathbf{m} \in \mathbf{N}_+^\ell$ *we have for* $s \in V_2^c$ *and* $t \in V_1^c$

$$E^{\mathbf{m}+\mathbf{1}}(z, B_{t,-c}s) = \frac{(d_2^c/\ell)_{\mathbf{m}}}{(d_2^c/\ell)_{\mathbf{m}+\mathbf{1}}}\ N_c(P_c B_{t,-c}^* z)\ \overline{N_c(s)}\ E^{\mathbf{m}}(z, B_{t,-c}s).$$

Proof Applying Lemma 6.1 to the tube type Peirce 2-space V_2^c of rank ℓ implies

$$E^{\mathbf{m}+\mathbf{1}}(z, B_{t,-c}s) = E^{\mathbf{m}+\mathbf{1}}(B_{t,-c}^* z, s) = E_c^{\mathbf{m}+\mathbf{1}}(P_c B_{t,-c}^* z, s)$$

$$= \frac{(d_2^c/\ell)_{\mathbf{m}}}{(d_2^c/\ell)_{\mathbf{m}+\mathbf{1}}}\ N_c(P_c B_{t,-c}^* z)\ \overline{N_c(s)}\ E_c^{\mathbf{m}}(P_c B_{t,-c}^* z, s).$$

Since $E_c^{\mathbf{m}}(P_c B_{t,-c}^* z, s) = E^{\mathbf{m}}(B_{t,-c}^* z, s) = E^{\mathbf{m}}(z, B_{t,-c}s)$, the assertion follows.

□

Since the truncated kernel $\widetilde{\mathcal{K}}$ of $\widetilde{\mathcal{M}}$ vanishes on the singular set $V_{\ell-1}$ it cannot be used directly to define a Hermitian line bundle over $V_{\ell-1}$. Instead, we first consider the module tensor product of $H_0^2(\Omega_\ell)$ over the polynomial ring $\mathcal{P}(V)$ with the one dimensional module $\mathbf{C}_w$, $(p, w) \mapsto p(w)$. Similar as in (2.1) we have, as a consequence of (6.1)

$$H_0^2(\Omega_\ell) \otimes_{\mathcal{P}(V)} \mathbf{C}_w = \begin{cases} \mathbf{C} & \text{if } w \in \mathring{\Omega}_\ell \\ \mathcal{P}_{\mathbf{1}}(V) & \text{if } w \in \Omega_{\ell-1} \end{cases}.$$

Here $\mathcal{P}_{\mathbf{1}}(V)$ is the finite-dimensional K-module belonging to the partition $\mathbf{1}$. The K-module $\mathcal{P}_{\mathbf{1}}(V)$ has dimension > 1 (since we exclude the case $\ell = r$ for tube type, where $\mathcal{P}_{\mathbf{1}}(V)$ is spanned by the Jordan algebra determinant N). The ideal $\mathcal{I}$ associated to the variety $V_{\ell-1}$ is generated by $\mathcal{P}_{\mathbf{1}}(V)$. For each $w \in \Omega_\ell$ there is a "cross-section" $\mathcal{P}_{\mathbf{1}}(V) \to H_0^2(\Omega_\ell)$ given by

$$p(z) \mapsto p(z)\cdot\Psi_w(z)$$

where

$$\Psi(z, w) = \hat{\mathcal{K}}_w(z) = \sum_{\boldsymbol{m}\in\mathbf{N}_+^\ell} \frac{(d/r)_{\boldsymbol{m}+\mathbf{1}}}{\rho_{\boldsymbol{m}+\mathbf{1}}} \frac{(ra/2)_{\boldsymbol{m}+1}}{(\ell a/2)_{\boldsymbol{m}+\mathbf{1}}} \frac{(d_2^c/\ell)_{\boldsymbol{m}}}{(d_2^c/\ell)_{\boldsymbol{m}+\mathbf{1}}} E^{\boldsymbol{m}}(z, w). \tag{6.2}$$

Then $\Psi_w(z) \in \mathcal{M}$ for each $w \in \Omega_\ell$. Let $N_i,\ i \in I$ be an orthonormal basis of $\mathcal{P}_{\mathbf{1}}(V)$. Then there is a holomorphic vector subbundle $\mathcal{E} \subset \Omega_\ell \times \mathcal{M}$ over the Kepler ball Ω_ℓ, whose fibre at $w \in V_\ell$ is the span

$$\mathcal{E}_w := \langle N_i(z)\ \Psi_w(z) :\ i \in I\rangle = \mathcal{P}_{\mathbf{1}}(V)\cdot\Psi_w \subset \mathcal{M}.$$

The vector bundle $\mathcal{E}$ is independent of the choice of orthonormal basis N_i. Consider the pull-back vector bundle

$$\begin{array}{ccc} \pi^*\mathcal{E} & & \mathcal{E} \\ \downarrow & & \downarrow \\ \hat{\Omega}_\ell & \xrightarrow[\pi]{} & \Omega_\ell \end{array}$$

over $\hat{\Omega}_\ell$, under the collapsing map π. We note that the "canonical" choice of higher rank vector bundle $\mathcal{E}$ over Ω_ℓ, with typical fibre $\mathcal{P}_{\mathbf{1}}(V)$ associated with the quotient module, is only possible for irreducible domains. In the reducible case (2.2) of the bidisk there is no natural choice of a rank 2 vector bundle having the fibre $< z_1, z_2 >$ at the origin.

Proposition 6.3 *For all* $(s, t) \in V_2^c \oplus V_1^c$ *we have*

$$\widetilde{\mathcal{K}}(z, B_{t,-c}s) = N_c(P_c B_{t,-c}^* z)\ \overline{N_c(s)}\ \Psi(z, B_{t,-c}s).$$

Proof This follows from the computation

$$\widetilde{\mathcal{K}}(z, B_{t,-c}s) = \sum_{\boldsymbol{m}\in\mathbf{N}_+^\ell} \frac{(d/r)_{\boldsymbol{m}+\mathbf{1}}}{\rho_{\boldsymbol{m}+\mathbf{1}}} \frac{(ra/2)_{\boldsymbol{m}+1}}{(\ell a/2)_{\boldsymbol{m}+\mathbf{1}}} E^{\boldsymbol{m}+\mathbf{1}}(z, B_{t,-c}s)$$

$$= \sum_{\boldsymbol{m}\in\mathbf{N}_+^\ell} \frac{(d/r)_{\boldsymbol{m}+\mathbf{1}}}{\rho_{\boldsymbol{m}+\mathbf{1}}} \frac{(ra/2)_{\boldsymbol{m}+1}}{(\ell a/2)_{\boldsymbol{m}+\mathbf{1}}} \frac{(d_2^c/\ell)_{\boldsymbol{m}}}{(d_2^c/\ell)_{\boldsymbol{m}+\mathbf{1}}} N_c(P_c B_{t,-c}^* z)\ \overline{N_c(s)}\ E^{\boldsymbol{m}}(z, B_{t,-c}s)$$

$$= N_c(P_c B_{t,-c}^* z)\ \overline{N_c(s)} \sum_{\boldsymbol{m}\in\mathbf{N}_+^\ell} \frac{(d/r)_{\boldsymbol{m}+\mathbf{1}}}{\rho_{\boldsymbol{m}+\mathbf{1}}} \frac{(ra/2)_{\boldsymbol{m}+1}}{(\ell a/2)_{\boldsymbol{m}+\mathbf{1}}} \frac{(d_2^c/\ell)_{\boldsymbol{m}}}{(d_2^c/\ell)_{\boldsymbol{m}+\mathbf{1}}} E^{\boldsymbol{m}}(z, B_{t,-c}s). \qquad \square$$

Now consider the holomorphic line bundle $\hat{\mathcal{L}}$ over the blow-up space $\hat{\Omega}_\ell$ defined in Proposition 5.3.

Theorem 6.4 *There exists an anti-holomorphic embedding $\hat{\mathcal{L}} \subset \pi^*\mathcal{E}$, defined on each fibre $\hat{\mathcal{L}}_{w,U} \subset (\pi^*\mathcal{E})_{w,U} = \mathcal{E}_w$ by*

$$[s,t,1]_c \mapsto N_c(B^*_{t,-c}z)\ \Psi_{B_{t,-c}s}(z). \tag{6.3}$$

In short,

$$[s,t,1]_c \mapsto N_c \circ B^*_{t,-c}\ \Psi_{B_{t,-c}s}.$$

Proof First we show that the map (6.3) is well-defined via the local charts (5.3). Suppose that $c, c' \in S_\ell$ satisfy

$$\rho_c(s,t) = \rho_{c'}(s',t'),$$

where $(s,t) \in V_2^c \times V_1^c$ and $(s',t') \in V_2^{c'} \times V_1^{c'}$. Then we have

$$B_{t,-c}s = \sigma_c(s,t) = \sigma_{c'}(s',t') = B_{t',-c'}s'.$$

It follows that $\widetilde{\mathcal{K}}_{B_{t,-c}s} = \widetilde{\mathcal{K}}_{B_{t',-c'}s'}$ and Proposition 6.3 implies

$$\overline{N_c(s)}\ [s,t,1]_c = \widetilde{\mathcal{K}}_{B_{t,-c}s} = \widetilde{\mathcal{K}}_{B_{t',-c'}s'} = \overline{N_{c'}(s')}\ [s',t',1]_{c'}.$$

Since $N_c(s)$ and $N_{c'}(s')$ don't vanish on the overlap of the charts, it follows that

$$[s,t,1]_c = \frac{\overline{N_{c'}(s')}}{\overline{N_c(s)}}\ [s',t',1]_{c'} = \left[s',t',\frac{\overline{N_{c'}(s')}}{\overline{N_c(s)}}\right]_{c'}.$$

Thus the map (6.3) respects the equivalence relation (5.4). Moreover, the map (6.3) is anti-holomorphic in (s,t), with values in $\mathcal{M}$. In order to see that the range belongs to the span of $N_i(z)\ \Psi_w(z)$, where $w = B_{t,-c}s$, choose holomorphic functions $c_i(t)$ such that

$$N_c(B^*_{t,-c}z) = \sum_{i\in I} \overline{c_i(t)}\ N_i(z)$$

for all $t \in V_1^c$. It follows that

$$N_c(B^*_{t,-c}z)\ \Psi_{B_{t,-c}s}(z) = \sum_i N_i(z)\ \overline{c_i(t)}\ \Psi(z, B_{t,-c}s) \in \mathcal{E}_{B_{t,-c}s}.$$

□

We are now able to define a **Hermitian metric** on the line bundle $\hat{\mathcal{L}}$ over $\hat{\Omega}_\ell$. A Jordan theoretic argument yields

Lemma 6.5 *For $t \in V_1^c$ we have*

$$P_c B^*_{t,-c} B_{t,-c} c = P_c B_{t,-t} c$$

and hence

$$N_c(B^*_{t,-c} B_{t,-c} c) = \Delta(t,t).$$

Here Δ denotes the Jordan triple determinant (3.3).

Proposition 6.6 *For all $(s,t) \in V_2^c \oplus V_1^c$ we have*

$$\widetilde{\mathcal{K}}(B_{t,-c}s, B_{t,-c}s) = \Delta(t,t)\ |N_c(s)|^2\ \Psi(B_{t,-c}s, B_{t,-c}s).$$

Proof Since $P_c B_{t,-c} B^*_{t,-c} P_c$ belongs to the structure group of V_c^2 it follows from Lemma 6.5 that

$$N_c(B^*_{t,-c} B_{t,-c} s) = N_c(B^*_{t,-c} B_{t,-c} c)\ N_c(s) = \Delta(t,t)\ N_c(s).$$

Now apply Proposition 6.3. □

Proposition 6.7 *For each submodule $\widetilde{\mathcal{M}} \subset \mathcal{M}$, with truncated kernel (6.1), there exists a Hermitian metric on the line bundle $\hat{\mathcal{L}}$ over $\hat{\Omega}_\ell$, given by the local representatives*

$$([s,t,1]_c | [s,t,1]_c) := \Delta(t,t)\ \Psi(B_{t,-c}s, B_{t,-c}s).$$

For this metric, the embedding (6.3) is isometric.

Proof Since Proposition 6.6 implies

$$\begin{aligned}\|N_c(B^*_{t,-c}z)\ \Psi_{B_{t,-c}s}(z)\|^2 &= \left\|\frac{\widetilde{\mathcal{K}}_{B_{t,-c}s}}{\overline{N_c(s)}}\right\|^2 = \frac{1}{|N_c(s)|^2}\,\widetilde{\mathcal{K}}(B_{t,-c}s, B_{t,-c}s)\\ &= \Delta(t,t)\ \Psi(B_{t,-c}s, B_{t,-c}s)\end{aligned}$$

it follows that the embedding (6.3) is isometric. □

Definition 6.8 The Hilbert module over $\hat{\Omega}_\ell$ associated with the Hermitian holomorphic line bundle $\hat{\mathcal{L}}$ will be called the **reduction** of $\widetilde{\mathcal{M}}$, and denoted by $\hat{\mathcal{M}}$. Note that this is different from the pull-back $\pi^*\mathcal{E}$ which is a vector bundle containing $\hat{\mathcal{L}}$ as a subbundle.

The following *rigidity theorem for singular submodules on Kepler varieties* is our main analytic result.

Theorem 6.9 *Consider two K-invariant Hilbert modules $\widetilde{\mathcal{M}}_\rho$ and $\widetilde{\mathcal{M}}_{\rho'}$ on Ω_ℓ, for given coefficient sequences $\rho_{\boldsymbol{m}}$ and $\rho'_{\boldsymbol{m}}$, respectively. Suppose that the reduced Hilbert modules $\hat{\mathcal{M}}_\rho$ and $\hat{\mathcal{M}}_{\rho'}$ on the blow-up space $\hat{\Omega}_\ell$ are equivalent. Then we have equality $\widetilde{\mathcal{M}}_\rho = \widetilde{\mathcal{M}}_{\rho'}$.*

Proof The proof is an application of the 'normalized kernel argument' summarized in Proposition 1.1. Consider the reproducing kernels $\hat{\mathcal{K}}^\rho$ and $\hat{\mathcal{K}}^{\rho'}$ of the reduced Hilbert modules. It suffices to consider a local chart $V_2^c \times V_1^c$ of $\hat{\Omega}_\ell$ for a given tripotent $c \in S_\ell$ defined in (5.3). As a consequence of module equivalence for line bundles, there exists a non-vanishing holomorphic function ϕ on the local chart $V_2^c \times V_1^c$ of $\hat{\Omega}_\ell$ such that

$$\hat{\mathcal{K}}^{\rho'}(x, y) = \phi(x)\, \hat{\mathcal{K}}^\rho(x, y)\, \overline{\phi(y)}. \tag{6.4}$$

Putting $y = 0$ we obtain

$$1 = \hat{\mathcal{K}}^{\rho'}(x, 0) = \phi(x)\, \hat{\mathcal{K}}^\rho(x, 0)\, \overline{\phi(0)} = \phi(x)\, \overline{\phi(0)}.$$

Therefore ϕ is constant. After normalization, we may assume $\phi = 1$. Then (6.4) implies

$$\hat{\mathcal{K}}^{\rho'}(x, y) = \hat{\mathcal{K}}^\rho(x, y)$$

for all x, y. In view of (6.2), this implies $\rho_{\boldsymbol{m}+\boldsymbol{1}} = \rho'_{\boldsymbol{m}+\boldsymbol{1}}$ for all $\boldsymbol{m} \in \mathbf{N}_+^\ell$. By (6.1), the singular submodules $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}'$ have the same truncated kernel $\widetilde{\mathcal{K}}(z, w) = \widetilde{\mathcal{K}}'(z, w)$. □

7 Outlook and Concluding Remarks

For the Hardy module $H^2(\mathbf{D}^d)$ it is evident that not all submodules are of the form $[\mathcal{I}]$, for some ideal $\mathcal{I}$ of the polynomial ring. (Here $[\mathcal{I}]$ is the closure of $\mathcal{I}$ in $H^2(\mathbf{D}^d)$). Ahern and Clark [4] show that all submodules (of the Hardy module) of finite codimension are of this form. In general, if a submodule $\widetilde{\mathcal{M}} \subseteq \mathcal{M}$ is not of the form $[\mathcal{I}]$, then it is not covered by the known Rigidity theorems with only one exception, namely [18, Theorem, pp. 70]. However, the geometric invariants constructed in [9] and in the current paper, it is hoped, might be useful in studying a much larger class of submodules. Recall that a submodule of an analytic Hilbert module $\mathcal{M}$ based on the domain Ω defines a coherent analytic sheaf [8, 9]. It possesses a Hermitian structure away from the zero variety and on this smaller open set, we have a holomorphic Hermitian vector bundle, which determines the class of the submodule. What we have shown here is that it has an analytic Hermitian continuation to the blow-up space. This interesting phenomenon naturally leads to the notion of, what one may call a Hermitian sheaf and eventually determine the equivalence class of these in terms of the geometric data already implicit in the definition, as in the examples we have discussed here.

We conclude this paper with several remarks concerning interesting directions for future research.

Remark 7.1 In [28] we consider more general Hilbert modules related to Kepler varieties, where the integration does not take place on the Kepler ball Ω_ℓ but on certain boundary strata, including the Hardy type inner product (4.4). These Hilbert modules, and their submodules defined by a vanishing condition on $\Omega_{\ell-1}$ provide a wider class of natural examples to which the above treatment is applicable.

Remark 7.2 It is easy to generalize the singular Hilbert modules treated in this paper, defined by a vanishing condition of order 1 on the singular set, to vanishing conditions of higher order. In this case the truncated kernel, generalizing (6.1), has the form

$$\widetilde{\mathcal{K}}(z, w) = \sum_{\boldsymbol{m}\in\mathbf{N}_+^\ell} \frac{(d/r)_{\boldsymbol{m}+\boldsymbol{k}}}{\rho_{\boldsymbol{m}+\boldsymbol{k}}} \frac{(ra/2)_{\boldsymbol{m}+\boldsymbol{k}}}{(\ell a/2)_{\boldsymbol{m}+\boldsymbol{k}}} E^{\boldsymbol{m}+\boldsymbol{k}}(z, w),$$

corresponding to vanishing of order $\geq k$ on $V_{\ell-1}$. Here $\boldsymbol{k} = (k, \ldots, k, 0, \ldots, 0)$ with k repeated ℓ times. In principle, one could also start with an arbitrary partition $\mu > 0$ of length ℓ and consider truncations such as

$$\widetilde{\mathcal{K}}(z, w) = \sum_{\boldsymbol{n}\in\mathbf{N}_+^\ell,\ \boldsymbol{n}\geq\mu} \frac{(d/r)_{\boldsymbol{n}}}{\rho_{\boldsymbol{n}}} \frac{(ra/2)_{\boldsymbol{n}}}{(\ell a/2)_{\boldsymbol{n}}} E^{\boldsymbol{n}}(z, w).$$

In this case one expects to have the finite-dimensional K-module $\mathcal{P}_\mu(V)$ occurring as a quotient module. On the other hand, treating singularities where the rank decreases by more than 1, for example $V_{\ell-2} \subset V_\ell$, or the origin $V_0 = \{0\}$ as a singularity in $\Omega = \Omega_r$, seems to be more difficult.

Remark 7.3 In the maximal rank case $\ell = r$ the ball $\Omega_r = \Omega$ is invariant under the full non-linear group G. For tube type domains, the singular set Ω_{r-1} has codimension 1, defined by vanishing of the Jordan algebra determinant. This case formally resembles the one-dimensional situation and is not covered by our approach (it was excluded to begin with). On the other hand, let V be a Hermitian Jordan triple not of tube type. There are three cases

- The rectangular matrices $V = \mathbf{C}^{r\times s}$ with $s > r$.
- The skew-symmetric matrices $V = \mathbf{C}^{N\times N}_{asym}$ of odd order $N = 2r + 1$
- The exceptional Jordan triple $V = \mathbf{O}^{1\times 2}_{\mathbf{C}}$ of rank $r = 2$ and dimension 16.

For these cases the singular set

$$V_{r-1} = \{z \in V : \operatorname{rank}(z) < r\}$$

has codimension > 1. The intersection

$$\Omega_{r-1} := V_{r-1} \cap \Omega$$

with the unit ball $\Omega \subset V$ is an analytic subvariety of Ω. For any automorphism $g \in G = \mathrm{Aut}(\Omega)$ we obtain another subvariety $g(\Omega_{r-1}) \subset \Omega$. Since G acts on the weighted Bergman spaces $\mathcal{M}_\nu = H^2_\nu(\Omega)$ one can consider submodules of $\mathcal{M}_\nu$ defined by vanishing on Ω_{r-1} and $g(\Omega_{r-1})$, respectively, where $g \in G$ does not belong to K.

A similar situation arises for the so-called **Mok embeddings**

$$\iota_c : B \to \Omega$$

of the unit ball $B = \mathbf{B}_n$ into a symmetric domain Ω of higher rank, constructed in [31]. Here $c \in S_1$ is any rank 1 tripotent. These embeddings have the property that the respective Bergman kernels satisfy

$$\mathcal{K}_B(x, y) = \mathcal{K}_\Omega(\iota_c(x), \iota_c(y))$$

for all $x, y \in B$. Let $B_c := \iota_c(B) \subset \Omega$ be the image variety (whose defining equations are explicitly known [31]) and consider, for $g \in G$, the subvariety $g(B_c)$ with associated Hilbert submodule $\widetilde{\mathcal{M}}_\nu \subseteq \mathcal{M}_\nu$ defined by a vanishing condition on $g(B_c)$.

It would be of interest to study the reduced modules and rigidity problems for singular submodules in such a G-equivariant setting.

Remark 7.4 Beyond the scalar case treated in this paper, analytic Hilbert modules for higher rank vector bundles ($n > 1$) have recently attracted much attention [23–25, 27] and should give rise to interesting singular submodules as well.

Acknowledgement We thank the referee for going through the first draft of this paper carefully and making several very useful suggestions.

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A Survey of Ron Douglas's Contributions to the Index Theory of Toeplitz Operators

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To the memory of Ron Douglas

Abstract In this survey article, we look back on some of the important contributions that Ron Douglas made to the study of index theory for various generalizations of Toeplitz operators.

Keywords Toeplitz operators · Index

Mathematics Subject Classification (2010) Primary 47B35; Secondary 47A53, 58J22

1 Introduction

On February 27, 2018, the world lost a great mathematician and an even better person, Ron Douglas. Ron made fundamental contributions in many areas of operator theory, operator algebras, and index theory. Ron was most famously known for Brown-Douglas-Fillmore (BDF) theory and its connections to K-homology, and younger operator algebraists may know him best for his papers on the interface between operator theory and algebraic geometry. However, in the 70s, 80s, and 90s, Ron published several important and interesting papers that looked at the index theory of generalizations of Toeplitz operators. In this article, I will highlight some of Ron's work in these areas.

My intended audience is mathematicians and graduate students who are (or are becoming) operator algebraists. I will assume the reader is familiar with Fredholm operators, C^*-algebras, and von Neumann algebras. In the penultimate section, I

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_21

will need to assume some familiarity with vector bundles, elliptic operator theory, and characteristic classes, and in the final section I will have to assume some knowledge about foliations. However, my aim in these last two sections is to present this material in such a way that even a reader with only a vague notion of what these objects are will still be able to appreciate the main results.

Let's begin by reviewing the classical situation. If we endow the circle group $\mathbb{T}$ with Haar measure, the monomials $\{z^n : n \in \mathbb{Z}\}$ constitute an orthonormal basis of $L^2(\mathbb{T})$. The algebra $C(\mathbb{T})$ of complex-valued continuous functions on $\mathbb{T}$ acts on $L^2(\mathbb{T})$ via pointwise multiplication: $M_\phi(f) = \phi f$. The Hardy space $H^2(\mathbb{T})$ can be taken to be the Hilbert subspace of $L^2(\mathbb{T})$ generated by the set $\{z^n : n \geq 0\}$; let P denote the orthogonal projection operator from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. Then, for each ϕ in $C(\mathbb{T})$, we define the Toeplitz operator $T_\phi : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ as the composition PM_ϕ. Let $\mathcal{T}$ be the C^*-subalgebra of $\mathcal{B}(H^2(\mathbb{T}))$ generated by the set $\{T_\phi : \phi \in C(\mathbb{T})\}$. It is straightforward to show that commutators $[T_\phi, T_\psi]$ of Toeplitz operators are compact. Furthermore, the fact that T_z, the unilateral shift, has no nontrivial reducing subspaces (this is essentially Beurling's theorem) implies that $\mathcal{T}$ acts irreducibly on $H^2(\mathbb{T})$, whence the commutator ideal of $\mathcal{T}$ is the entire ideal $\mathcal{K}$ of compact operators on $H^2(\mathbb{T})$. A bit more work allows one to establish a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(\mathbb{T}) \longrightarrow 0$$

with a linear splitting $\xi : C(\mathbb{T}) \to \mathcal{T}$ given by the formula $\xi(\phi) = T_\phi$. From this exact sequence, we can easily deduce that each element of $\mathcal{T}$ can be written uniquely in the form $T_\phi + K$ for some compact operator K and some continuous function ϕ; furthermore, the algebra homomorphism $\sigma : \mathcal{T} \to C(\mathbb{T})$ has the explicit formula $\sigma(T_\phi + K) = \phi$.

We can immediately infer from this exact sequence that T in $\mathcal{T}$ is Fredholm if and only if its symbol $\sigma(T)$ is a nowhere-vanishing function on $\mathbb{T}$. Furthermore, we have the following classical result, due independently to many researchers, including F. Noether, Kreĭn, Widom, and Devinatz:

Theorem 1.1 *Suppose that $T \in \mathcal{T}$ has the property that $\sigma(T)$ is invertible. Then the index of T equals the negative of the winding number of $\sigma(T)$.*

This theorem asserts the equality of two seemingly very different quantities: the *analytic index* $\dim\ker T - \dim\ker T^*$ of T and a *topological index* associated to T.

A theorem this elegant cries out to be generalized! The remaining sections of this paper highlight generalizations due to Douglas and various coauthors.

2 Toeplitz Operators Associated to a Semigroup

We begin by looking at an general approach considered by Coburn and Douglas [5]. Note that as a C^*-subalgebra of $\mathcal{B}(L^2(\mathbb{T}))$, the Toeplitz algebra $\mathcal{T}$ is generated by the set $\{T_{z^n} : n \in \mathbb{Z}^+\}$. This suggests the following generalization. Let G a locally compact abelian group. The (Pontryagin) dual $\widehat{G}$ of G is the group of continuous homomorphisms from G into $\mathbb{T}$ endowed with the compact-open topology. Fix a subsemigroup S of $\widehat{G}$ that is also a Borel subset. If we equip G and $\widehat{G}$ with Haar measure, then the Fourier transform defines a Hilbert space isomorphism from $L^2(G)$ to $L^2(\widehat{G})$. Define $H^2(S)$ to be the Hilbert subspace of $L^2(G)$ consisting of functions whose Fourier transform is supported on S, and let P be the orthogonal projection from $L^2(G)$ onto $H^2(S)$. Pointwise multiplication by a bounded measurable function ϕ on G defines a bounded linear operator $M_\phi : L^2(G) \to L^2(G)$, and we define T_ϕ on $H^2(S)$ by the formula $T_\phi = PM_\phi$. Note that if γ is an element of S (and thus a bounded measurable function on G), then M_γ maps $H^2(S)$ to itself and the projection P is unnecessary. Define $\mathcal{T}(G, S)$ to be the C^*-subalgebra of $\mathcal{B}(H^2(S))$ generated by the set $\{T_\gamma : \gamma \in S\}$. Observe that if we take $G = \mathbb{T}$ and $S = \mathbb{Z}^+$, we recover the classic Toeplitz algebra $\mathcal{T}$.

In order to avoid pathologies, we need to ensure that our subsemigroup S is not too small. We shall henceforth assume that S has strictly positive Haar measure in $\widehat{G}$, and also that the group generated by S is all of $\widehat{G}$. If $\widehat{G}$ is connected, then our first condition on S implies the second. On the other hand, if G is discrete, then the two conditions on S force $S = \widehat{G}$, which of course is not very interesting.

Let $\mathcal{C}(G, S)$ be the closed $*$-ideal of $\mathcal{T}(G, S)$ generated by commutators of elements in $\mathcal{T}(G, S)$. To identify the quotient algebra $\mathcal{T}(G, S)/\mathcal{C}(G, S)$, we need a definition from harmonic analysis. The C^*-algebra $\mathrm{AP}(G)$ of *almost periodic functions* on G is the closure in the supremum norm of the algebra of finite complex linear combinations of elements of $\widehat{G}$. It is not hard to show that if G is compact, then $\mathrm{AP}(G) = C(G)$.

Theorem 2.1 ([5], Theorem 2) *Every element of $\mathcal{T}(G, S)$ can be uniquely written in the form $T_\phi + C$ for some $\phi \in \mathrm{AP}(G)$ and $C \in \mathcal{C}(G, S)$, and there is a short exact sequence of C^*-algebras*

$$0 \longrightarrow \mathcal{C}(G, S) \longrightarrow \mathcal{T}(G, S) \xrightarrow{\ \sigma\ } \mathrm{AP}(G) \longrightarrow 0$$

with $\sigma(T_\phi + C) = \phi$.

While you could attempt to produce some sort of index theory associated to this short exact sequence, it is very unlikely that you could say much at this level of generality. In the next two sections we consider special cases of this result.

3 Toeplitz Operators on the Real Line

For the setup described in the previous section, take $G = \mathbb{R}$. The dual group is isomorphic and homeomorphic to $\mathbb{R}$; take $S = \mathbb{R}^+$.

Unlike the situation we saw with the circle, the ideal $\mathcal{C}(\mathbb{R}, \mathbb{R}^+)$ is not the ideal of compact operators; in fact, $\mathcal{C}(\mathbb{R}, \mathbb{R}^+)$ contains no nonzero compact operator. Thus, to have an interesting index problem here, we need a different notion of the analytical index, due to M. Breuer [1, 2]. Let $\mathcal{H}$ be a Hilbert space, and suppose that $\mathcal{M}$ is a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ that is also a II_∞ factor. (Breuer worked in greater generality than we will discuss here, but this will suffice for our purposes.) Let $\mathrm{Proj}(\mathcal{M})$ be the set of projections in $\mathcal{M}$. For X in $\mathcal{M}$, define the *null projection* N_X by the formula

$$N_X = \sup\{E \in \mathrm{Proj}(\mathcal{M}) : XE = 0\}.$$

We say that X is *Fredholm* (*relative to* $\mathcal{M}$) if

1. N_X is finite;
2. there is a finite projection E in $\mathcal{M}$ such that $(1 - E)(\mathcal{H}) \subseteq X(\mathcal{H})$.

Condition 2 is an analogue, but not the same as, the classical requirement that X have closed range and finite dimensional cokernel. If X satisfies both conditions 1 and 2, then N_{X^*} is finite, and we can define the analytic index of X relative to $\mathcal{M}$ by the formula

$$\mathrm{index}_{\mathcal{M}}\, X = \dim_{\mathcal{M}} N_X - \dim_{\mathcal{M}} N_{X^*}.$$

The Breuer index satisfies similar properties to the features that the ordinary Fredholm index enjoys: for X and Y Fredholm relative to $\mathcal{M}$, we have

- $\mathrm{index}_{\mathcal{M}}\, X^* = -\,\mathrm{index}_{\mathcal{M}}\, X$;
- $\mathrm{index}_{\mathcal{M}}(XY) = \mathrm{index}_{\mathcal{M}}\, X + \mathrm{index}_{\mathcal{M}}\, Y$;
- $\mathrm{index}_{\mathcal{M}}(X + F) = \mathrm{index}_{\mathcal{M}}\, X$ for all finite projections F in $\mathcal{M}$;
- the Breuer index is locally constant on the subspace of operators that are Fredholm relative to $\mathcal{M}$.

Let $\mathbb{R}_d$ denote the set of real numbers equipped with the discrete topology. In [7], Coburn, Douglas, Schaeffer, and Singer constructed a faithful representation ρ of $\mathcal{T}(\mathbb{R}, \mathbb{R}^+)$ onto a II_∞ factor $\mathcal{M}$ in $\mathcal{B}(L^2(\mathbb{R}) \otimes \ell^2(\mathbb{R}_d))$ and showed that $\rho(T)$ in $\mathcal{T}(\mathbb{R}, \mathbb{R}^+)$ is Fredholm relative to $\mathcal{M}$ if and only if $\sigma(T)$ is invertible in $\mathrm{AP}(\mathbb{R})$. In this case, we can define $\mathrm{index}_{\mathcal{M}}\, T$ to be the Breuer index of $\rho(T)$. The main theorem is that $\mathrm{index}_{\mathcal{M}}\, T$ can be computed as the average winding number of $\sigma(T)$.

Theorem 3.1 ([7], Theorem 2.2) *Suppose that T in $\mathcal{T}(\mathbb{R}, \mathbb{R}^+)$ has the property that $\sigma(T)$ is invertible in* $\mathrm{AP}(\mathbb{R})$. *Then*

$$\mathrm{index}_{\mathcal{M}}\, T = \lim_{x \to \infty} \frac{\arg(\sigma(T)(x)) - \arg(\sigma(-T)(x)}{x}.$$

4 Quarter-Plane Toeplitz Operators

Suppose that G is $\mathbb{T}^2$ and take S to be $\mathbb{Z}_+^2 = \{(m,n) : m,n \geq 0\}$ in $\widehat{G} = \mathbb{Z}^2$. We call the C^*-algebra $\mathcal{T}(\mathbb{T}^2, \mathbb{Z}_+^2)$ the *quarter-plane Toeplitz algebra*. In contrast with what we saw in $\mathcal{T}(\mathbb{R}, \mathbb{R}^+)$, the commutator ideal $\mathcal{C}(\mathbb{T}^2, \mathbb{Z}_+^2)$ contains nonzero compact operators—in fact, it contains all of them—but also contains noncompact operators as well. To analyze the Fredholm index theory of $\mathcal{T}(\mathbb{T}^2, \mathbb{Z}_+^2)$, Douglas and Howe ([8]) observed that $\mathcal{T}(\mathbb{T}^2, \mathbb{Z}_+^2)$ is naturally isomorphic to the tensor product $\mathcal{T} \otimes \mathcal{T}$ of the Toeplitz algebra on the circle with itself. We can therefore write down the following nine-term (or 21-term, depending on how you count) commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K} \otimes \mathcal{K} & \longrightarrow & \mathcal{K} \otimes \mathcal{T} & \longrightarrow & \mathcal{K} \otimes C(\mathbb{T}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T} \otimes \mathcal{K} & \longrightarrow & \mathcal{T} \otimes \mathcal{T} & \longrightarrow & \mathcal{T} \otimes C(\mathbb{T}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C(\mathbb{T}) \otimes \mathcal{K} & \longrightarrow & C(\mathbb{T}) \otimes \mathcal{T} & \longrightarrow & C(\mathbb{T}) \otimes C(\mathbb{T}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Let $\mathcal{S}$ be the pullback of $\mathcal{T} \otimes C(\mathbb{T})$ and $C(\mathbb{T}) \otimes \mathcal{T}$ along $C(\mathbb{T}) \otimes C(\mathbb{T})$; i.e.,

$$\mathcal{S} = \{(X, Y) \in (\mathcal{T} \otimes C(\mathbb{T})) \oplus (C(\mathbb{T}) \otimes \mathcal{T}) : (\sigma \otimes \mathrm{id})(X) = (\mathrm{id} \otimes \sigma)(Y)\}.$$

Then an easy diagram chase yields the following short exact sequence

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{K} \longrightarrow \mathcal{T} \otimes \mathcal{T} \xrightarrow{\widetilde{\sigma}} \mathcal{S} \longrightarrow 0,$$

where $\widetilde{\sigma}$ has the property that $\widetilde{\sigma}(T_1 \otimes T_2) = (T_1 \otimes \sigma(T_2), \sigma(T_1) \otimes T_2)$. Thus an operator T in $\mathcal{T}(\mathbb{T}^2, \mathbb{Z}_+^2) \cong \mathcal{T} \otimes \mathcal{T}$ is Fredholm if and only if its symbol in the noncommutative symbol space $\mathcal{S}$ is invertible.

Using the standard isomorphism of $C(\mathbb{T}) \otimes \mathcal{T}$ with $C(\mathbb{T}, \mathcal{T})$, we can alternately describe $\mathcal{S}$ as

$$\mathcal{S} = \{(\Phi_1, \Phi_2) \in C(\mathbb{T}, \mathcal{T}) \oplus C(\mathbb{T}, \mathcal{T}) : \\ \sigma(\Phi_1(z))(w) = \sigma(\Phi_2(w))(z) \text{ for all } w, z \in \mathbb{T}\}.$$

Coburn, Douglas, and Singer proved the following.

Theorem 4.1 ([6], Section 3) *Suppose that T is a Fredholm operator in $\mathcal{T}(\mathbb{T}^2, \mathbb{Z}_+^2)$ whose symbol in $\mathcal{S}$ is (Φ_1, Φ_2). Then there is a homotopy in the subspace of invertible elements of $\mathcal{S}$ from (Φ_1, Φ_2) to $(z^m I, z^n I)$ for some integers m and n, and* $\operatorname{index}(T) = -(m+n)$.

While this theorem does indeed give a topological formula for the index of Fredholm operators in the quarter-place Toeplitz algebra, one might not view this result as being completely satisfactory, because the proof does not give a procedure for finding the necessary homotopy. An alternate index theorem that applies to a dense set of Fredholm operators in $\mathcal{T}(\mathbb{T}^2, \mathbb{Z}_+^2)$ can be obtained using cyclic cohomology. The reader who is unfamiliar with cyclic cohomology can consult Connes' book on the subject ([4], Section III.3), but fortunately one does not need to know cyclic cohomology to understand the final result of this section.

Let $\mathcal{L}^1$ denote the ideal of trace class operators on $H^2(\mathbb{T})$ and define

$$\mathcal{T}^\infty = \{T_\phi + L : \phi \in C^\infty(\mathbb{T}), L \in \mathcal{L}^1\}.$$

It is not too difficult to show that $\mathcal{T}^\infty$ is a dense $*$-subalgebra of $\mathcal{T}$, and we have a short exact sequence

$$0 \longrightarrow \mathcal{L}^1 \longrightarrow \mathcal{T}^\infty \xrightarrow{\sigma^\infty} C^\infty(\mathbb{T}) \longrightarrow 0.$$

Define $\xi : C^\infty(\mathbb{T}) \to \mathcal{T}^\infty$ by $\xi(\phi) = T_\phi$. The map ξ is a linear splitting to the symbol map σ^∞. Next, using the projective tensor product, we can produce a "smooth" version of our short exact sequence on the preceding page:

$$0 \longrightarrow \mathcal{L}^1 \otimes \mathcal{L}^1 \longrightarrow \mathcal{T}^\infty \otimes \mathcal{T}^\infty \xrightarrow{\widetilde{\sigma}^\infty} \mathcal{S}^\infty \longrightarrow 0.$$

This short exact sequence also has a linear splitting $\rho : \mathcal{S}^\infty \to \mathcal{T}^\infty \otimes \mathcal{T}^\infty$ defined by

$$\begin{aligned}\rho(\Phi_1, \Phi_2) &= (\mathrm{id} \otimes \xi)(\Phi_1) + (\xi \otimes \mathrm{id})(\Phi_2) - (\xi \otimes \xi)(\sigma^\infty \otimes \mathrm{id})(\Phi_1) \\ &= (\mathrm{id} \otimes \xi)(\Phi_1) + (\xi \otimes \mathrm{id})(\Phi_2) - (\xi \otimes \xi)(\mathrm{id} \otimes \sigma^\infty)(\Phi_2),\end{aligned}$$

and ρ defines a 1-cyclic cocycle $[\rho]$ on $\mathcal{S}^\infty$. Using the pairing of cyclic cohomology with K-theory, we obtain the following index theorem.

Theorem 4.2 ([11], Theorem 4.4) *Let T be a Fredholm operator in $\mathcal{T}^\infty \otimes \mathcal{T}^\infty$. Then*

$$\operatorname{index} T = \operatorname{Trace}\big(\rho(\widetilde{\sigma}^\infty(T))\rho(\widetilde{\sigma}^\infty(T)^{-1}) - \rho(\widetilde{\sigma}^\infty(T)^{-1})\rho(\widetilde{\sigma}^\infty(T))\big).$$

5 Toeplitz Operators Associated to Elliptic Operators

We now generalize Toeplitz operators on the circle in a very different direction from the generalizations we have been considering so far. To motivate what we will do in this section, consider the differential operator $-i\frac{d}{d\theta}$ applied to smooth functions on the circle. We can extend $-i\frac{d}{d\theta}$ to an unbounded self-adjoint operator D on $L^2(\mathbb{T})$. The collection $\{e^{in\theta} : n \in \mathbb{Z}\}$ is a complete set of eigenvectors for D, and we can view $H^2(\mathbb{T})$ as the positive spectral subspace of D; that is, the closed linear span of the eigenvectors of D that are associated to nonnegative eigenvalues. The spectral theorem functional calculus allows us to write the projection $P : L^2(\mathbb{T}) \to H^2(\mathbb{T})$ as the positive spectral projection $\chi_{[0,\infty)}(D)$ of D. Furthermore, if we choose a smooth real-valued function f with the property that $f(k) = \chi_{[0,\infty)}(k)$ for every integer k, then P can also be expressed as $f(D)$. The advantage of viewing P in this fashion is that it shows that P is a zero-order pseudo-differential operator. We can use the powerful theory of such operators to deduce that $[P, M_\phi]$ is compact for all $\phi \in C^\infty(\mathbb{T})$, and an approximation argument shows that such commutators are compact for all $\phi \in C(\mathbb{T})$. The ordinary circle Toeplitz algebra extension follows fairly easily.

Now let X be an oriented smooth compact manifold, let V be a smooth complex vector bundle over X, and equip V with a Hermitian structure; this is a smoothly-varying choice of complex inner product for each fiber of V. Let π denote the projection of vectors in V down to X. A section of V is a map $s : X \to V$ such that πs is the identity map on X; let $L^2(V)$ be the Hilbert space of square-summable measurable sections of V. Given a complex-valued function ϕ on X, it acts on $L^2(V)$ by pointwise scalar multiplication and therefore we have a bounded linear operator M_ϕ on $L^2(V)$.

Next, a differential operator on $L^2(V)$ is a densely-defined linear map D that locally can be written as a matrix of partial derivatives. We are interested in the case when D is symmetric as an unbounded operator and can be extended to a true self-adjoint operator. We also require that D be elliptic. Very roughly speaking, this means that D differentiates in all directions. In any event, for what follows, it is not necessary to know exactly what an elliptic operator is in order to understand the setup or the results.

So suppose D is a self-adjoint elliptic differential operator on L^2-sections of a Hermitian vector bundle V. The ellipticity of D implies that the spectrum of D

consists entirely of eigenvalues (necessarily real, because D is self-adjoint) with finite multiplicity, and each eigenspace has a basis of smooth sections. Because D is self-adjoint, we can apply the spectral theorem functional calculus to it. Choose a smooth function $f : \mathbb{R} \to \mathbb{R}$ with the property that $f(a) = \chi_{[0,\infty)}(a)$ for each eigenvalue a of D. Then $f(D)$ equals the positive spectral projection of D. As was true in the circle case, the commutators $[P, M_\phi]$ are compact operators. Let $H^2(D)$ denote the positive spectral subspace of D; note that $H^2(D)$ is the range of P. Consider Toeplitz operators $T_\phi = PM_\phi$ on $H^2(D)$, and define $\mathcal{T}(D)$ to be the C^*-subalgebra of $H^2(D)$ generated by $\mathcal{K}$ and $\{T_\phi : \phi \in C(X)\}$. We have a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}(D) \xrightarrow{\sigma} C(X) \longrightarrow 0$$

with $\sigma(T_\phi + K) = \phi$ for all ϕ in $C(X)$ and K in $\mathcal{K}$.

In [3], Baum and Douglas used the Atiyah-Singer index theorem to give a topological index formula for Fredholm operators in $\mathcal{T}(D)$. We will use the rest of section to look at the various elements that make up this formula.

Our generalized Toeplitz algebras typically only admit an interesting index problem when X is odd-dimensional, so we shall henceforth assume that. It will also be convenient to assume that X is connected. We will restrict our attention to what are called spin^c manifolds. Such manifolds admit an elliptic first order differential operator, called the Dirac operator, that implements a Poincaré duality isomorphism from the K-theory of X to the K-homology of X, and this fact simplifies the topological index formula. Many orientable manifolds are spin^c; for example, every orientable three-manifold has this property. Finally, to produce interesting examples with nonzero index, we allow Toeplitz operators with matrix-valued symbols. For each natural number n, we can tensor our short exact sequence by $\mathrm{M}(n, \mathbb{C})$, and, identifying $\mathcal{K} \otimes \mathrm{M}(n, \mathbb{C}) \cong \mathrm{M}(n, \mathcal{K})$ with $\mathcal{K}$, we obtain the short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathrm{M}(n, \mathcal{T}(D)) \xrightarrow{\sigma} \mathrm{M}(n, C(X)) \longrightarrow 0.$$

Then we see that for any continuous map $\Phi : X \to \mathrm{GL}(n, C(X))$, the Toeplitz operator T_Φ is Fredholm. We can approximate Φ by a smooth map from X into $\mathrm{GL}(n, C^\infty(X))$ without changing the index of our Toeplitz operator, so let's assume that Φ is smooth. Let $d\Phi$ denote the matrix of one-forms obtained by taking the exterior derivative of each matrix entry of Φ and define

$$\omega_\Phi = \sum_{k=0}^{\infty} \frac{k\,!}{(2k+1)\,!} \operatorname{Trace}\left((\Phi^{-1} d\Phi)^{2k+1}\right).$$

Note that $(\Phi^{-1} d\Phi)^{2k+1} = 0$ for $2k + 1 > \dim X$, so the sum on the right-hand side of the equation is finite. The differential form ω_Φ is closed and therefore determines a class $\mathrm{Ch}(\Phi)$ in the odd-degree cohomology of X; this class is called the *Chern character* of Φ ([12], Section 4.4). Let T^*X be the cotangent bundle of X and let $\mathrm{Th} : H^*(X) \to H^*(T^*X)$ be the Thom isomorphism in cohomology. Finally, let $\mathrm{Td}(T^*X)$ denote the Todd class of T^*X; the Todd class is an even-degree cohomology class on X that serves as a "correction factor" between the Thom isomorphisms in cohomology and K-theory. The desired index formula comes from the pairing between homology and cohomology.

Theorem 5.1 ([3], Theorem 4)

$$\operatorname{index} T_\Phi = \left\langle \mathrm{Ch}(\Phi) \cup \mathrm{Th}^{-1}(\mathrm{Td}(T^*X)), [X] \right\rangle.$$

If we use Chern-Weil theory to represent $\mathrm{Th}^{-1}(\mathrm{Td}(T^*X))$ as a closed differential form, then the right-hand side can be computed by evaluating an integral over X. When $X = \mathbb{T}$ and $D = -i\frac{d}{dx}$, we recover the winding number formula for the index of Toeplitz operators on the circle.

6 Toeplitz Operators Associated to a Foliation

Suppose X is a compact manifold that admits a foliation $\mathcal{F}$. This is not the venue for an extensive discussion of the theory of foliations, but we will note that if $\mathcal{F}$ is a foliation of X, then for each point x in X there exists a neighborhood U of x that is locally diffeomorphic to a product manifold $L \times T$. The sets $L \times \{t\}$ are restrictions of the leaves of $\mathcal{F}$ to U. A good example to keep in mind throughout this section is the following: let $X = [0, 1] \times [0, 1]$ with opposite sides identified, and foliate X by parallel lines with a fixed slope.

Given a foliated manifold $(X, \mathcal{F})$, we can consider the leaf space $X/\mathcal{F}$ obtained by identifying each leaf of $\mathcal{F}$ to a point and endowing the resulting set with the quotient topology. This quotient space can be very badly behaved topologically; consider our torus example when the foliating lines have an irrational slope. A more productive approach to understanding the leaf space of a foliation comes from one of the central tenets of noncommutative geometry and topology: it is easier to understand noncommutative C^*-algebras than it is to understand non-Hausdorff topological spaces. We take our foliation $\mathcal{F}$ and look at its *holonomy groupoid* $\mathcal{G}$. This topological groupoid $\mathcal{G}$ is related to $\mathcal{F}$, but takes into account the fact that when going around loops in a leaf, the nearby leaves may be twisted; roughly speaking, the groupoid $\mathcal{G}$ "undoes" this twisting. The set of smooth functions on $\mathcal{G}$ becomes a $*$-algebra under convolution, and by completing this $*$-algebra in an appropriate norm we obtain the *(reduced) foliation algebra* $C^*(X, \mathcal{F})$. More generally, given a foliated vector bundle E over X, we can form a C^*-algebra $C^*(X, \mathcal{F}, E)$.

Next, let D be a self-adjoint differential operator on sections of E, and suppose that D is elliptic along the leaves of $\mathcal{F}$; in analogy with our discussion of (fully) elliptic operators in the previous section, this very roughly means that D

differentiates in all the leaf directions. Let $P = \chi_{[0,\infty)}(D)$, and for each continuous function ϕ on X, form the Toeplitz operator T_ϕ. Let $\mathcal{T}(D)$ be the C^*-subalgebra of $\mathcal{B}(L^2(\mathcal{G}, E))$ generated by the T_ϕ and $C^*(X, \mathcal{F}, E)$. Then we have a short exact sequence

$$0 \longrightarrow \mathcal{C}_S(D) \longrightarrow \mathcal{T}(D) \xrightarrow{\sigma} C(X) \longrightarrow 0,$$

where $\mathcal{C}_S(D)$ is the *semicommutator ideal* of $\mathcal{T}(D)$ generated by elements of the form $T_{\phi\psi} - T_\phi T_\psi$.

To use this short exact sequence to produce an index theorem, we need to know more about the semicommutator ideal $\mathcal{C}_S(D)$. Both this ideal and the foliation algebra $C^*(X, \mathcal{F}, E)$ are contained in the von Neumann algebra $W^*(X, \mathcal{F}, E)$ generated by $C^*(X, \mathcal{F}, E)$, but are not generally equal. The problem is that, unlike the situation with the elliptic operators we considered in the previous section, self-adjoint leafwise elliptic operators do not generally have a “gap” in their spectrum near zero, and therefore the positive spectral projection $P = \chi_{[0,\infty)}(D)$ is typically not a leafwise elliptic pseudodifferential operator.

Douglas, Hurder, and Kaminker [9] got around this problem in the following way. For each $\epsilon > 0$, define $f_\epsilon : \mathbb{R} \to \mathbb{R}$ by the formula $f_\epsilon(x) = x(\epsilon + x^2)^{-1/2}$. Using Roe’s leafwise functional calculus [13], we can define $P_\epsilon = f_\epsilon(D)$ and look at “approximate” leafwise Toeplitz operators $T_\phi^\epsilon = P_\epsilon M_\phi P_\epsilon$ for every continuous function ϕ on X. This gives a family of short exact sequences

$$0 \longrightarrow C^*(X, \mathcal{F}) \longrightarrow \mathcal{T}^\epsilon(D) \xrightarrow{\sigma} C(X) \longrightarrow 0.$$

The significance of these exact sequences is that if there exists a smooth measure μ on X that is compatible with the foliation $\mathcal{F}$ (specifically, we require that μ be invariant under the holonomy group of $\mathcal{F}$), then just as in Sect. 3, we can embed $C^*(X, \mathcal{F})$ into a II_∞ factor and obtain a Breuer index for elements in $\mathcal{T}^\epsilon(D)$ that are invertible modulo $C^*(X, \mathcal{F})$. By taking a limit as ϵ goes to zero, we obtain an analytic index for Toeplitz operators in $\mathcal{T}(D)$ that have invertible symbol. We can extend this analytic index to Toeplitz operators with matrix-valued symbols, just as we did in the previous section.

We now seek a topological formula for this analytic index. Unfortunately, but necessarily, the formula is rather involved. Like the index formula in the previous section, it involves pairing a cohomology class with a homology class.

Theorem 6.1 ([9], Proposition 4.5)

$$\operatorname{index}_{\mathcal{M}} T_\Phi = \Big\langle \operatorname{Ch}(\Phi) \cup \operatorname{Th}^{-1}(\operatorname{symb}(D)) \cup \operatorname{Td}(T\mathcal{F} \otimes_{\mathbb{R}} \mathbb{C}), [C_\mu] \Big\rangle.$$

In this formula, we have the inverse of the Thom isomorphism Th in cohomology applied to the symbol symb(D) of our leafwise elliptic differential operator D. The homology class $[C_\mu]$ comes from the *Ruelle-Sullivan current*; as the notation C_μ suggests, the Ruelle-Sullivan current is defined in terms of the measure μ. For the case of a foliation of the torus by irrational lines, we essentially recover the index problem and formula that we discussed in Sect. 2.

The Toeplitz index theorem provided here is interesting in its own right, but represents a small part of a much more ambitious project. Douglas, Hurder, and Kaminker used these ideas to study spectral flow and computations of secondary index invariants, such as the eta invariant, for elliptic operators. The interested reader should consult [9] and [10] for more information.

Acknowledgments Thanks to José Carrión and the referee for their suggestions and constructive criticism.

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Differential Subalgebras and Norm-Controlled Inversion

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In memory of Ronald G. Douglas

Abstract In this introduction article, we consider the norm-controlled inversion for differential $*$-subalgebras of a symmetric $*$-algebra with common identity and involution.

Keywords Banach algebra · Differential subalgebra · Norm-controlled inversion · Wiener's lemma

Mathematics Subject Classification (2010) 47G10, 45P05, 47B38, 31B10, 46E30

1 Introduction

In [49, Lemma IIe], it states that "*If* $f(x)$ *is a function with an absolutely convergent Fourier series, which nowhere vanishes for real arguments,* $1/f(x)$ *has an absolutely convergent Fourier series.*" The above statement is now known as the classical Wiener's lemma.

We say that a Banach space $\mathcal{A}$ with norm $\|\cdot\|_{\mathcal{A}}$ is a *Banach algebra* if it has operation of multiplications possessing the usual algebraic properties, and

$$\|AB\|_{\mathcal{A}} \le K\|A\|_{\mathcal{A}}\|B\|_{\mathcal{A}} \text{ for all } A, B \in \mathcal{A}, \tag{1.1}$$

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_22

where K is a positive constant. Given two Banach algebras $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}$ is a Banach subalgebra of $\mathcal{B}$, we say that $\mathcal{A}$ is *inverse-closed* in $\mathcal{B}$ if $A \in \mathcal{A}$ and $A^{-1} \in \mathcal{B}$ implies $A^{-1} \in \mathcal{A}$. Inverse-closedness is also known as spectral invariance, Wiener pair, local subalgebra, etc. [13, 16, 30, 46]. Let $\mathcal{C}$ be the algebra of all periodic continuous functions under multiplication, and $\mathcal{W}$ be its Banach subalgebra of all periodic functions with absolutely convergent Fourier series,

$$\mathcal{W} = \Big\{ f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx}, \ \ \|f\|_{\mathcal{W}} := \sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty \Big\}. \tag{1.2}$$

Then the classical Wiener's lemma can be reformulated as that $\mathcal{W}$ is an inverse-closed subalgebra of $\mathcal{C}$. Due to the above interpretation, we also call the inverse-closed property for a Banach subalgebra $\mathcal{A}$ as Wiener's lemma for that subalgebra. Wiener's lemma for Banach algebras of infinite matrices and integral operators with certain off-diagonal decay can be informally interpreted as localization preservation under inversion. Such a localization preservation is of great importance in applied harmonic analysis, numerical analysis, optimization and many mathematical and engineering fields [2, 10, 11, 23, 28, 44]. The readers may refer to the survey papers [18, 27, 37], the recent papers [14, 34, 36] and references therein for historical remarks and recent advances.

Given an element A in a Banach algebra $\mathcal{A}$ with the identity I, we define its *spectral set* $\sigma_{\mathcal{A}}(A)$ and *spectral radius* $\rho_{\mathcal{A}}(A)$ by

$$\sigma_{\mathcal{A}}(A) := \big\{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible in } \mathcal{A}\big\}$$

and

$$\rho_{\mathcal{A}}(A) := \max \big\{ |\lambda| : \lambda \in \sigma_{\mathcal{A}}(A)\big\}$$

respectively. Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with common identity I and $\mathcal{A}$ be a Banach subalgebra of $\mathcal{B}$. Then an equivalent condition for the inverse-closedness of $\mathcal{A}$ in $\mathcal{B}$ is that the spectral set of any $A \in \mathcal{A}$ in Banach algebras $\mathcal{A}$ and $\mathcal{B}$ are the same, i.e.,

$$\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{B}}(A).$$

By the above equivalence, a necessary condition for the inverse-closedness of $\mathcal{A}$ in $\mathcal{B}$ is that the spectral radius of any $A \in \mathcal{A}$ in the Banach algebras $\mathcal{A}$ and $\mathcal{B}$ are the same, i.e.,

$$\rho_{\mathcal{A}}(A) = \rho_{\mathcal{B}}(A). \tag{1.3}$$

The above necessary condition is shown by Hulanicki [24] to be sufficient if we further assume that $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras with common identity and involution, and that $\mathcal{B}$ is symmetric. Here we say that a Banach algebra $\mathcal{B}$ is a $*$-algebra if there is a continuous linear *involution* $*$ on $\mathcal{B}$ with the properties that

$$(AB)^* = B^* A^* \text{ and } A^{**} = A \text{ for all } A, B \in \mathcal{B},$$

and that a $*$-algebra $\mathcal{B}$ is *symmetric* if

$$\sigma_{\mathcal{A}}(A^* A) \subset [0, \infty) \text{ for all } A \in \mathcal{B}.$$

The spectral radii approach (1.3), known as the Hulanicki's spectral method, has been used to establish the inverse-closedness of symmetric $*$-algebras [9, 20, 21, 41, 43, 45], however the above approach does not provide a norm estimate for the inversion, which is crucial for many mathematical and engineering applications.

To consider norm estimate for the inversion, we recall the concept of norm-controlled inversion of a Banach subalgebra $\mathcal{A}$ of a symmetric $*$-algebra $\mathcal{B}$, which was initiated by Nikolski [31] and coined by Gröchenig and Klotz [20]. Here we say that a Banach subalgebra $\mathcal{A}$ of $\mathcal{B}$ admits *norm-controlled inversion* in $\mathcal{B}$ if there exists a continuous function h from $[0, \infty) \times [0, \infty)$ to $[0, \infty)$ such that

$$\|A^{-1}\|_{\mathcal{A}} \le h\big(\|A\|_{\mathcal{A}}, \|A^{-1}\|_{\mathcal{B}}\big) \tag{1.4}$$

for all $A \in \mathcal{A}$ being invertible in $\mathcal{B}$ [19, 20, 34, 36].

The norm-controlled inversion is a strong version of Wiener's lemma. The classical Banach algebra $\mathcal{W}$ in (1.2) is inverse-closed in the algebra $\mathcal{C}$ of all periodic continuous functions [49], however it does not have norm-controlled inversion in $\mathcal{C}$ [5, 31]. To establish Wiener's lemma, there are several methods, including the Wiener's localization [49], the Gelfand's technique [16], the Brandenburg's trick [9], the Hulanicki's spectral method [24], the Jaffard's boot-strap argument [25], the derivation technique [21], and the Sjöstrand's commutator estimates [36, 39]. In this paper, we will use the Brandenburg's trick to establish norm-controlled inversion of a differential $*$-subalgebra $\mathcal{A}$ of a symmetric $*$-algebra $\mathcal{B}$.

This introduction article is organized as follows. In Sect. 2, we recall the concept of differential subalgebras and present some differential subalgebras of infinite matrices with polynomial off-diagonal decay. In Sect. 3, we introduce the concept of generalized differential subalgebras and present some generalized differential subalgebras of integral operators with kernels being Hölder continuous and having polynomial off-diagonal decay. In Sect. 4, we use the Brandenburg's trick to establish norm-controlled inversion of a differential $*$-subalgebra of a symmetric $*$-algebra, and we conclude the section with two remarks on the norm-controlled inversion with the norm control function bounded by a polynomial and the norm-controlled inversion of nonsymmetric Banach algebras.

2 Differential Subalgebras

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras such that $\mathcal{A}$ is a Banach subalgebra of $\mathcal{B}$. We say that $\mathcal{A}$ is a *differential subalgebra of order* $\theta \in (0, 1]$ in $\mathcal{B}$ if there exists a positive constant $D_0 := D_0(\mathcal{A}, \mathcal{B}, \theta)$ such that

$$\|AB\|_{\mathcal{A}} \leq D_0\|A\|_{\mathcal{A}}\|B\|_{\mathcal{A}}\Big(\Big(\frac{\|A\|_{\mathcal{B}}}{\|A\|_{\mathcal{A}}}\Big)^{\theta} + \Big(\frac{\|B\|_{\mathcal{B}}}{\|B\|_{\mathcal{A}}}\Big)^{\theta}\Big) \quad \text{for all } A, B \in \mathcal{A}. \tag{2.1}$$

The concept of differential subalgebras of order θ was introduced in [7, 26, 32] for $\theta = 1$ and [12, 20, 36] for $\theta \in (0, 1)$. We also refer the reader to [3, 15, 19–21, 25, 33, 34, 41–43, 45] for various differential subalgebras of infinite matrices, convolution operators, and integral operators with certain off-diagonal decay.

For $\theta = 1$, the requirement (2.1) can be reformulated as

$$\|AB\|_{\mathcal{A}} \leq D_0\|A\|_{\mathcal{A}}\|B\|_{\mathcal{B}} + D_0\|A\|_{\mathcal{B}}\|B\|_{\mathcal{A}} \quad \text{for all } A, B \in \mathcal{A}. \tag{2.2}$$

So the norm $\|\cdot\|_{\mathcal{A}}$ satisfying (2.1) is also referred as a Leibniz norm on $\mathcal{A}$.

Let $C[a, b]$ be the space of all continuous functions on the interval $[a, b]$ with its norm defined by

$$\|f\|_{C[a,b]} = \sup_{t\in[a,b]} |f(t)|, \quad f \in C[a, b],$$

and $C^k[a, b]$, $k \geq 1$, be the space of all continuously differentiable functions on the interval $[a, b]$ up to order k with its norm defined by

$$\|h\|_{C^k[a,b]} = \sum_{j=0}^{k} \|h^{(j)}\|_{C[a,b]} \text{ for } h \in C^k[a, b].$$

Clearly, $C[a, b]$ and $C^k[a, b]$ are Banach algebras under function multiplication. Moreover

$$\begin{aligned}
\|h_1h_2\|_{C^1[a,b]} &= \|(h_1h_2)'\|_{C[a,b]} + \|h_1h_2\|_{C[a,b]} \\
&\leq \|h_1'\|_{C[a,b]}\|h_2\|_{C[a,b]} + \|h_1\|_{C[a,b]}\|h_2'\|_{C[a,b]} \\
&\qquad + \|h_1\|_{C[a,b]}\|h_2\|_{C[a,b]} \\
&\leq \|h_1\|_{C^1[a,b]}\|h_2\|_{C[a,b]} \\
&\qquad + \|h_1\|_{C[a,b]}\|h_2\|_{C^1[a,b]} \text{ for all } h_1, h_2 \in C^1[a, b],
\end{aligned} \tag{2.3}$$

where the second inequality follows from the Leibniz rule. Therefore we have

Theorem 2.1 *$C^1[a, b]$ is a differential subalgebra of order one in $C[a, b]$.*

Due to the above illustrative example of differential subalgebras of order one, the norm $\|\cdot\|_{\mathcal{A}}$ satisfying (2.1) is also used to describe smoothness in abstract Banach algebra [7].

Let $\mathcal{W}^1$ be the Banach algebra of all periodic functions such that both f and its derivative f' belong to the Wiener algebra $\mathcal{W}$, and define the norm on $\mathcal{W}^1$ by

$$\|f\|_{\mathcal{W}^1} = \|f\|_{\mathcal{W}} + \|f'\|_{\mathcal{W}} = \sum_{n\in\mathbb{Z}} (|n|+1)|\hat{f}(n)| \tag{2.4}$$

for $f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx} \in \mathcal{W}^1$. Following the argument used in the proof of Theorem 2.1, we have

Theorem 2.2 *$\mathcal{W}^1$ is a differential subalgebra of order one in $\mathcal{W}$.*

Recall from the classical Wiener's lemma that $\mathcal{W}$ is an inverse-closed subalgebra of $\mathcal{C}$, the algebra of all periodic continuous functions under multiplication. This leads to the following natural question:

Question 2.3 *Is $\mathcal{W}^1$ a differential subalgebra of $\mathcal{C}$?*

Let ℓ^p, $1 \le p \le \infty$, be the space of all p-summable sequences on $\mathbb{Z}$ with norm denoted by $\|\cdot\|_p$. To answer the above question, we consider Banach algebras $\mathcal{C}$, $\mathcal{W}$ and $\mathcal{W}^1$ in the "frequency domain". Let $\mathcal{B}(\ell^p)$ be the algebra of all bounded linear operators on ℓ^p, $1 \le p \le \infty$, and let

$$\tilde{\mathcal{W}} = \Big\{A := (a(i-j))_{i,j\in\mathbb{Z}},\ \|A\|_{\tilde{\mathcal{W}}} = \sum_{k\in\mathbb{Z}} |a(k)| < \infty\Big\} \tag{2.5}$$

and

$$\tilde{\mathcal{W}}^1 = \Big\{A := (a(i-j))_{i,j\in\mathbb{Z}},\ \|A\|_{\tilde{\mathcal{W}}^1} = \sum_{k\in\mathbb{Z}} |k||a(k)| < \infty\Big\} \tag{2.6}$$

be Banach algebras of Laurent matrices with symbols in $\mathcal{W}$ and $\mathcal{W}^1$ respectively. Then the classical Wiener's lemma can be reformulated as that $\tilde{\mathcal{W}}$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^2)$, and an equivalent statement of Theorem 2.2 is that $\tilde{\mathcal{W}}^1$ is a differential subalgebra of order one in $\tilde{\mathcal{W}}$. Due to the above equivalence, Question 2.3 in the "frequency domain" becomes whether $\mathcal{W}^1$ is a differential subalgebra of order $\theta \in (0,1]$ in $\mathcal{C}$. In [45], the first example of differential subalgebra of infinite matrices with order $\theta \in (0,1)$ was discovered.

Theorem 2.4 *$\mathcal{W}^1$ is a differential subalgebra of $\mathcal{C}$ with order $2/3$.*

To consider differential subalgebras of infinite matrices in the noncommutative setting, we introduce three noncommutative Banach algebras of infinite matrices with certain off-diagonal decay. Given $1 \le p \le \infty$ and $\alpha \ge 0$, we define the

Gröchenig-Schur family of infinite matrices by

$$\mathcal{A}_{p,\alpha} = \Big\{A = (a(i,j))_{i,j\in\mathbb{Z}},\ \|A\|_{\mathcal{A}_{p,\alpha}} < \infty\Big\} \tag{2.7}$$

[22, 25, 29, 35, 43, 45], the Baskakov-Gohberg-Sjöstrand family of infinite matrices by

$$\mathcal{C}_{p,\alpha} = \Big\{A = (a(i,j))_{i,j\in\mathbb{Z}},\ \|A\|_{\mathcal{C}_{p,\alpha}} < \infty\Big\} \tag{2.8}$$

[4, 17, 22, 39, 43], and the Beurling family of infinite matrices

$$\mathcal{B}_{p,\alpha} = \Big\{A = (a(i,j))_{i,j\in\mathbb{Z}},\ \|B\|_{\mathcal{A}_{p,\alpha}} < \infty\Big\} \tag{2.9}$$

[6, 36, 41], where $u_\alpha(i,j) = (1+|i-j|)^\alpha$, $\alpha \ge 0$, are polynomial weights on $\mathbb{Z}^2$,

$$\|A\|_{\mathcal{A}_{p,\alpha}} = \max\Big\{\sup_{i\in\mathbb{Z}} \big\|\big(a(i,j)u_\alpha(i,j)\big)_{j\in\mathbb{Z}}\big\|_p,\ \sup_{j\in\mathbb{Z}} \big\|\big(a(i,j)u_\alpha(i,j)\big)_{i\in\mathbb{Z}}\big\|_p\Big\}, \tag{2.10}$$

$$\|A\|_{\mathcal{C}_{p,\alpha}} = \Big\|\Big(\sup_{i-j=k} |a(i,j)|u_\alpha(i,j)\Big)_{k\in\mathbb{Z}}\Big\|_p, \tag{2.11}$$

and

$$\|A\|_{\mathcal{B}_{p,\alpha}} = \Big\|\Big(\sup_{|i-j|\ge|k|} |a(i,j)|u_\alpha(i,j)\Big)_{k\in\mathbb{Z}}\Big\|_p. \tag{2.12}$$

Clearly, we have

$$\mathcal{B}_{p,\alpha} \subset \mathcal{C}_{p,\alpha} \subset \mathcal{A}_{p,\alpha} \text{ for all } 1 \le p \le \infty \text{ and } \alpha \ge 0. \tag{2.13}$$

The above inclusion is proper for $1 \le p < \infty$, while the above three families of infinite matrices coincide for $p = \infty$,

$$\mathcal{B}_{\infty,\alpha} = \mathcal{C}_{\infty,\alpha} = \mathcal{A}_{\infty,\alpha} \text{ for all } \alpha \ge 0, \tag{2.14}$$

which is also known as the Jaffard family of infinite matrices [25],

$$\mathcal{J}_\alpha = \Big\{A = (a(i,j))_{i,j\in\mathbb{Z}},\ \|A\|_{\mathcal{J}_\alpha} = \sup_{i,j\in\mathbb{Z}} |a(i,j)|u_\alpha(i-j) < \infty\Big\}. \tag{2.15}$$

Observe that $\|A\|_{\mathcal{A}_{p,\alpha}} = \|A\|_{\mathcal{C}_{p,\alpha}}$ for a Laurent matrix $A = (a(i-j))_{i,j\in\mathbb{Z}}$. Then Banach algebras $\tilde{\mathcal{W}}$ and $\tilde{\mathcal{W}}^1$ in (2.5) and (2.6) are the commutative subalgebra of the Gröchenig-Schur algebra $\mathcal{A}_{1,\alpha}$ and the Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_{1,\alpha}$ for $\alpha = 0, 1$ respectively,

$$\tilde{\mathcal{W}} = \mathcal{A}_{1,0} \cap \mathcal{L} = \mathcal{C}_{1,0} \cap \mathcal{L} \tag{2.16}$$

and

$$\tilde{\mathcal{W}}^1 = \mathcal{A}_{1,1} \cap \mathcal{L} = \mathcal{C}_{1,1} \cap \mathcal{L}, \tag{2.17}$$

where $\mathcal{L}$ is the set of all Laurent matrices $A = (a(i-j))_{i,j\in\mathbb{Z}}$. The sets $\mathcal{A}_{p,\alpha}, \mathcal{C}_{p,\alpha}, \mathcal{B}_{p,\alpha}$ with $p = 1$ and $\alpha = 0$ are noncommutative Banach algebras under matrix multiplication, the Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_{1,0}$ and the Beurling algebra $\mathcal{B}_{1,0}$ are inverse-closed subalgebras of $\mathcal{B}(\ell^2)$ [4, 8, 17, 39, 41], however the Schur algebra $\mathcal{A}_{1,0}$ is not inverse-closed in $\mathcal{B}(\ell^2)$ [47]. We remark that the inverse-closedness of the Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_{1,0}$ in $\mathcal{B}(\ell^2)$ can be understood as a noncommutative extension of the classical Wiener's lemma for the commutative subalgebra $\tilde{\mathcal{W}}$ of Laurent matrices in $\mathcal{B}(\ell^2)$.

For $1 \le p \le \infty$ and $\alpha > 1 - 1/p$, one may verify that the Gröchenig-Schur family $\mathcal{A}_{p,\alpha}$, the Baskakov-Gohberg-Sjöstrand family $\mathcal{C}_{p,\alpha}$ and the Beurling family $\mathcal{B}_{p,\alpha}$ of infinite matrices form Banach algebras under matrix multiplication and they are inverse-closed subalgebras of $\mathcal{B}(\ell^2)$ [22, 25, 41, 43, 45]. In [41, 43, 45], their differentiability in $\mathcal{B}(\ell^2)$ is established.

Theorem 2.5 *Let $1 \le p \le \infty$ and $\alpha > 1 - 1/p$. Then $\mathcal{A}_{p,\alpha}, \mathcal{C}_{p,\alpha}$ and $\mathcal{B}_{p,\alpha}$ are differential subalgebras of order $\theta_0 = (\alpha + 1/p - 1)/(\alpha + 1/p - 1/2) \in (0, 1)$ in $\mathcal{B}(\ell^2)$.*

Proof The following argument about differential subalgebra property for the Gröchenig-Schur algebra $\mathcal{A}_{p,\alpha}$, $1 < p < \infty$, is adapted from [45]. The reader may refer to [41, 43, 45] for the detailed proof to the differential subalgebra property for the Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_{p,\alpha}$ and the Beurling algebra $\mathcal{B}_{p,\alpha}$. Take $A = (a(i,j))_{i,j\in\mathbb{Z}}$ and $B = (b(i,j))_{i,j\in\mathbb{Z}} \in \mathcal{A}_{p,\alpha}$, and write $C = AB = (c(i,j))_{i,j\in\mathbb{Z}}$. Then

$$\begin{aligned}
\|C\|_{\mathcal{A}_{p,\alpha}} &= \max\Big\{\sup_{i\in\mathbb{Z}} \big\|\big(c(i,j)u_\alpha(i,j)\big)_{j\in\mathbb{Z}}\big\|_p,\ \sup_{j\in\mathbb{Z}} \big\|\big(c(i,j)u_\alpha(i,j)\big)_{i\in\mathbb{Z}}\big\|_p\Big\}\\
&\le 2^\alpha \max\Big\{\sup_{i\in\mathbb{Z}} \Big\|\Big(\sum_{k\in\mathbb{Z}} |a(i,k)||b(k,j)|\big(u_\alpha(i,k)+u_\alpha(k,j)\big)\Big)_{j\in\mathbb{Z}}\Big\|_p,\\
&\qquad \sup_{j\in\mathbb{Z}} \Big\|\Big(\sum_{k\in\mathbb{Z}} |a(i,k)||b(k,j)|\big(u_\alpha(i,k)+u_\alpha(k,j)\big)\Big)_{i\in\mathbb{Z}}\Big\|_p\Big\}\\
&\le 2^\alpha \|A\|_{\mathcal{A}_{p,\alpha}} \|B\|_{\mathcal{A}_{1,0}} + 2^\alpha \|A\|_{\mathcal{A}_{1,0}} \|B\|_{\mathcal{A}_{p,\alpha}},
\end{aligned} \tag{2.18}$$

where the first inequality follows from the inequality

$$u_\alpha(i,j) \le 2^\alpha\big(u_\alpha(i,k)+u_\alpha(k,j)\big),\ \ i,j,k\in\mathbb{Z}.$$

Let $1/p' = 1 - 1/p$, and define

$$\tau_0 = \left\lfloor \left(\left(\frac{\alpha p'+1}{\alpha p'-1} \right)^{1/p'} \frac{\|A\|_{\mathcal{A}_{p,\alpha}}}{\|A\|_{\mathcal{B}(\ell^2)}} \right)^{1/(\alpha+1/2-1/p')} \right\rfloor, \tag{2.19}$$

where $\lfloor t \rfloor$ denotes the integer part of a real number t. Then for $i \in \mathbb{Z}$, we have

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} |a(i,j)| &= \Big(\sum_{|j-i|\le\tau_0} + \sum_{|j-i|>\tau_0} \Big) |a(i,j)| \\
&\le \Big(\sum_{|j-i|\le\tau_0} |a(i,j)|^2 \Big)^{1/2} \Big(\sum_{|j-i|\le\tau_0} 1 \Big)^{1/2} \\
&\quad + \Big(\sum_{|j-i|\ge\tau_0+1} |a(i,j)|^p (u_\alpha(i,j))^p \Big)^{1/p} \Big(\sum_{|j-i|\ge\tau_0+1} (u_\alpha(i,j))^{-p'} \Big)^{1/p'} \\
&\le \|A\|_{\mathcal{B}(\ell^2)} (2\tau_0+1)^{1/2} + 2^{1/p'} (\alpha p'-1)^{-1/p'} \|A\|_{\mathcal{A}_{p,\alpha}} (\tau_0+1)^{-\alpha+1/p'} \\
&\le D \|A\|_{\mathcal{A}_{p,\alpha}}^{1-\theta_0} \|A\|_{\mathcal{B}(\ell^2)}^{\theta_0},
\end{aligned} \tag{2.20}$$

where D is an absolute constant depending on p, α only, and the last inequality follows from (2.19) and the following estimate

$$\|A\|_{\mathcal{B}(\ell^2)} \le \|A\|_{\mathcal{A}_{1,0}} \le \Big(\sum_{k\in\mathbb{Z}} (|k|+1)^{-\alpha p'} \Big)^{1/p'} \|A\|_{\mathcal{A}_{p,\alpha}} \le \Big(\frac{\alpha p'+1}{\alpha p'-1} \Big)^{1/p'} \|A\|_{\mathcal{A}_{p,\alpha}}.$$

Similarly we can prove that

$$\sup_{j\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} |a(i,j)| \le D \|A\|_{\mathcal{A}_{p,\alpha}}^{1-\theta_0} \|A\|_{\mathcal{B}(\ell^2)}^{\theta_0}. \tag{2.21}$$

Combining (2.20) and (2.21) leads to

$$\|A\|_{\mathcal{A}_{1,0}} \le D \|A\|_{\mathcal{A}_{p,\alpha}}^{1-\theta_0} \|A\|_{\mathcal{B}(\ell^2)}^{\theta_0}. \tag{2.22}$$

Replacing the matrix A in (2.22) by the matrix B gives

$$\|B\|_{\mathcal{A}_{1,0}} \le D \|B\|_{\mathcal{A}_{p,\alpha}}^{1-\theta_0} \|B\|_{\mathcal{B}(\ell^2)}^{\theta_0}. \tag{2.23}$$

Therefore it follows from (2.18), (2.22) and (2.23) that

$$\|C\|_{\mathcal{A}_{p,\alpha}} \le 2^\alpha D\|A\|_{\mathcal{A}_{p,\alpha}}\|B\|_{\mathcal{A}_{p,\alpha}}^{1-\theta_0}\|B\|_{\mathcal{B}(\ell^2)}^{\theta_0} + 2^\alpha D\|B\|_{\mathcal{A}_{p,\alpha}}\|A\|_{\mathcal{A}_{p,\alpha}}^{1-\theta_0}\|A\|_{\mathcal{B}(\ell^2)}^{\theta_0}, \tag{2.24}$$

which proves the differential subalgebra property for Banach algebras $\mathcal{A}_{p,\alpha}$ with $1 < p < \infty$ and $\alpha > 1 - 1/p$. □

The argument used in the proof of Theorem 2.5 involves a triplet of Banach algebras $\mathcal{A}_{p,\alpha}$, $\mathcal{A}_{1,0}$ and $\mathcal{B}^2$ satisfying (2.18) and (2.22). In the following theorem, we extend the above observation to general Banach algebra triplets $(\mathcal{A}, \mathcal{M}, \mathcal{B})$.

Theorem 2.6 *Let $\mathcal{A}, \mathcal{M}$ and $\mathcal{B}$ be Banach algebras such that $\mathcal{A}$ is a Banach subalgebra of $\mathcal{M}$ and $\mathcal{M}$ is a Banach subalgebra of $\mathcal{B}$. If there exist positive exponents $\theta_0, \theta_1 \in (0, 1]$ and absolute constants D_0, D_1 such that*

$$\|AB\|_{\mathcal{A}} \le D_0\|A\|_{\mathcal{A}}\|B\|_{\mathcal{A}}\Big(\Big(\frac{\|A\|_{\mathcal{M}}}{\|A\|_{\mathcal{A}}}\Big)^{\theta_0} + \Big(\frac{\|B\|_{\mathcal{M}}}{\|B\|_{\mathcal{A}}}\Big)^{\theta_0}\Big) \quad \text{for all } A, B \in \mathcal{A}, \tag{2.25}$$

and

$$\|A\|_{\mathcal{M}} \le D_1\|A\|_{\mathcal{A}}^{1-\theta_1}\|A\|_{\mathcal{B}}^{\theta_1} \ \text{ for all } \ A \in \mathcal{A}, \tag{2.26}$$

then $\mathcal{A}$ is a differential subalgebra of order $\theta_0\theta_1$ in $\mathcal{B}$.

Proof For any $A, B \in \mathcal{A}$, we obtain from (2.25) and (2.26) that

$$\begin{aligned}\|AB\|_{\mathcal{A}} &\le D_0\|A\|_{\mathcal{A}}\|B\|_{\mathcal{A}}\Big(\Big(\frac{D_1\|A\|_{\mathcal{A}}^{1-\theta_1}\|A\|_{\mathcal{B}}^{\theta_1}}{\|A\|_{\mathcal{A}}}\Big)^{\theta_0} + \Big(\frac{D_1\|B\|_{\mathcal{A}}^{1-\theta_1}\|B\|_{\mathcal{B}}^{\theta_1}}{\|B\|_{\mathcal{A}}}\Big)^{\theta_0}\Big)\\ &\le D_0 D_1^{\theta_0}\|A\|_{\mathcal{A}}\|B\|_{\mathcal{A}}\Big(\Big(\frac{\|A\|_{\mathcal{B}}}{\|A\|_{\mathcal{A}}}\Big)^{\theta_0\theta_1} + \Big(\frac{\|B\|_{\mathcal{B}}}{\|B\|_{\mathcal{A}}}\Big)^{\theta_0\theta_1}\Big),\end{aligned}$$

which completes the proof. □

Following the argument used in (2.3), we can show that $C^2[a, b]$ is a differential subalgebra of $C^1[a, b]$. For any distinct $x, y \in [a, b]$ and $f \in C^2[a, b]$, observe that

$$|f'(x)| = \frac{|f(y) - f(x) - f''(\xi)(y-x)^2/2|}{|y-x|} \le 2\|f\|_{C[a,b]}|y-x|^{-1} + \frac{1}{2}\|f''\|_{C[a,b]}|y-x|$$

for some $\xi \in [a, b]$, which implies that

$$\|f'\|_{C[a,b]} \le \max\big(4\|f\|_{C[a,b]}^{1/2}\|f''\|_{C[a,b]}^{1/2}, 8(b-a)^{-1}\|f\|_{C[a,b]}\big). \tag{2.27}$$

Therefore there exists a positive constant D such that

$$\|f\|_{C^1[a,b]} \le D\|f\|_{C^2[a,b]}^{1/2}\|f\|_{C[a,b]}^{1/2} \text{ for all } f \in C^2[a,b]. \tag{2.28}$$

As an application of Theorem 2.6, we conclude that $C^2[a,b]$ is a differential subalgebra of order $1/2$ in $C[a,b]$.

We finish the section with the proof of Theorem 2.4.

Proof of Theorem 2.4 The conclusion follows from (2.17) and Theorem 2.5 with $p = 1$ and $\alpha = 1$. □

3 Generalized Differential Subalgebras

By (2.1), a differential subalgebra $\mathcal{A}$ satisfies the Brandenburg's requirement:

$$\|A^2\|_{\mathcal{A}} \le 2D_0\|A\|_{\mathcal{A}}^{2-\theta}\|A\|_{\mathcal{B}}^{\theta}, \ A \in \mathcal{A}. \tag{3.1}$$

To consider the norm-controlled inversion of a Banach subalgebra $\mathcal{A}$ of $\mathcal{B}$, the above requirement (3.1) could be relaxed to the existence of an integer $m \ge 2$ such that the m-th power of elements in $\mathcal{A}$ satisfies

$$\|A^m\|_{\mathcal{A}} \le D\|A\|_{\mathcal{A}}^{m-\theta}\|A\|_{\mathcal{B}}^{\theta}, \ \ A \in \mathcal{A}, \tag{3.2}$$

where $\theta \in (0, m-1]$ and $D = D(\mathcal{A}, \mathcal{B}, m, \theta)$ is an absolute positive constant, see Theorem 4.1 in the next section. For $h \in C^1[a,b]$ and $m \ge 2$, we have

$$\|h^m\|_{C^1[a,b]} = m\|h^{m-1}h'\|_{C[a,b]} + \|h^m\|_{C[a,b]} \le m\|h\|_{C^1[a,b]}\|h\|_{C[a,b]}^{m-1},$$

and hence the differential subalgebra $C^1[a,b]$ of $C[a,b]$ satisfies (3.2) with $\theta = m-1$. In this section, we introduce some sufficient conditions so that (3.2) holds for some integer $m \ge 2$.

Theorem 3.1 *Let $\mathcal{A}, \mathcal{M}$ and $\mathcal{B}$ be Banach algebras such that $\mathcal{A}$ is a Banach subalgebra of $\mathcal{M}$ and $\mathcal{M}$ is a Banach subalgebra of $\mathcal{B}$. If there exist an integer $k \ge 2$, positive exponents θ_0, θ_1, and absolute constants E_0, E_1 such that*

$$\|A_1A_2\cdots A_k\|_{\mathcal{A}} \le E_0\Big(\prod_{i=1}^{k}\|A_i\|_{\mathcal{A}}\Big)\sum_{j=1}^{k}\Big(\frac{\|A_i\|_{\mathcal{M}}}{\|A_i\|_{\mathcal{A}}}\Big)^{\theta_0}, \ \ A_1,\ldots,A_k \in \mathcal{A} \tag{3.3}$$

and

$$\|A^2\|_{\mathcal{M}} \le E_1\|A\|_{\mathcal{A}}^{2-\theta_1}\|A\|_{\mathcal{B}}^{\theta_1}, \ A \in \mathcal{A}, \tag{3.4}$$

then (3.2) *holds for* $m = 2k$ *and* $\theta = \theta_0\theta_1$.

Proof By (1.1), (3.3) and (3.4), we have

$$\|A^{2k}\|_{\mathcal{A}} \le kE_0\|A^2\|_{\mathcal{A}}^{k-\theta_0}\|A^2\|_{\mathcal{M}}^{\theta_0} \le kE_0E_1^{\theta_0}K^{k-\theta_0}\|A\|_{\mathcal{A}}^{2k-\theta_0\theta_1}\|A\|_{\mathcal{B}}^{\theta_0\theta_1}, \quad A \in \mathcal{A}, \tag{3.5}$$

which completes the proof. □

For a Banach algebra triplet $(\mathcal{A}, \mathcal{M}, \mathcal{B})$ in Theorem 2.6, we obtain from (2.25) and (2.26) that

$$\|A_1A_2\cdots A_k\|_{\mathcal{A}} \le D_0\|A_1\|_{\mathcal{A}}\|A_2\cdots A_k\|_{\mathcal{A}}\Big(\Big(\frac{\|A_1\|_{\mathcal{M}}}{\|A_1\|_{\mathcal{A}}}\Big)^{\theta_0} + \Big(\frac{\|A_2\cdots A_k\|_{\mathcal{M}}}{\|A_2\cdots A_k\|_{\mathcal{A}}}\Big)^{\theta_0}\Big)$$
$$\le \tilde{D}_0\Big(\prod_{i=1}^{k}\|A_i\|_{\mathcal{A}}\Big)\sum_{j=1}^{k}\Big(\frac{\|A_j\|_{\mathcal{M}}}{\|A_j\|_{\mathcal{A}}}\Big)^{\theta_0}, \quad A_1, \ldots, A_k \in \mathcal{A}, \tag{3.6}$$

and

$$\|A^2\|_{\mathcal{M}} \le \tilde{K}\|A\|_{\mathcal{M}}^2 \le D_1^2\tilde{K}\|A\|_{\mathcal{A}}^{2-2\theta_1}\|A\|_{\mathcal{B}}^{2\theta_1}, \quad A \in \mathcal{A}, \tag{3.7}$$

where $\tilde{D}_0$ is an absolute constant and $\tilde{K}$ is the constant K in (1.1) for the Banach algebra $\mathcal{M}$. Therefore the assumptions (3.3) and (3.4) in Theorem 3.1 are satisfied for the Banach algebra triplet $(\mathcal{A}, \mathcal{M}, \mathcal{B})$ in Theorem 2.6.

For a differential subalgebra $\mathcal{A}$ of order θ_0 in $\mathcal{B}$, we observe that the requirements (3.3) and (3.4) with $\mathcal{M} = \mathcal{B}, k = 2$ and $\theta_1 = 2$ are met, and hence (3.2) holds for $m = 4$ and $\theta = 2\theta_0$. Recall that $\mathcal{B}$ is a trivial differential subalgebra of $\mathcal{B}$. In the following corollary, we can extend the above conclusion to arbitrary differential subalgebras $\mathcal{M}$ of $\mathcal{B}$.

Corollary 3.2 *Let* $\mathcal{A}, \mathcal{M}$ *and* $\mathcal{B}$ *be Banach algebras such that* $\mathcal{A}$ *is a differential subalgebra of order* θ_0 *in* $\mathcal{M}$ *and* $\mathcal{M}$ *is a differential subalgebra of order* θ_1 *in* $\mathcal{B}$. *Then* (3.2) *holds for* $m = 4$ *and* $\theta = \theta_0\theta_1$.

Following the argument used in the proof of Theorem 3.1, we can show that (3.2) holds for $m = 4$ if the requirement (3.3) with $k = 3$ is replaced by the following strong version

$$\|ABC\|_{\mathcal{A}} \le E_0\|A\|_{\mathcal{A}}\|C\|_{\mathcal{A}}\|B\|_{\mathcal{A}}^{1-\theta_0}\|B\|_{\mathcal{M}}^{\theta_0}, \quad A, B, C \in \mathcal{A}. \tag{3.8}$$

Theorem 3.3 *Let* $\mathcal{A}, \mathcal{M}$ *and* $\mathcal{B}$ *be Banach algebras such that* $\mathcal{A}$ *is a Banach subalgebra of* $\mathcal{M}$ *and* $\mathcal{M}$ *is a Banach subalgebra of* $\mathcal{B}$. *If there exist positive exponents* $\theta_0, \theta_1 \in (0, 1]$ *and absolute constants* E_0, E_1 *such that* (3.4) *and* (3.8) *hold, then* (3.2) *holds for* $m = 4$ *and* $\theta = \theta_0\theta_1$.

Let $L^p := L^p(\mathbb{R})$, $1 \le p \le \infty$, be the space of all p-integrable functions on $\mathbb{R}$ with standard norm $\|\cdot\|_p$, and $\mathcal{B}(L^p)$ be the algebra of bounded linear operators on L^p with the norm $\|\cdot\|_{\mathcal{B}(L^p)}$. For $1 \le p \le \infty$, $\alpha \ge 0$ and $\gamma \in [0, 1)$, we define the norm of a kernel K on $\mathbb{R} \times \mathbb{R}$ by

$$\|K\|_{\mathcal{W}^\gamma_{p,\alpha}} = \begin{cases} \max\Big(\sup_{x\in\mathbb{R}} \big\|K(x,\cdot)u_\alpha(x,\cdot)\big\|_p,\ \sup_{y\in\mathbb{R}} \big\|K(\cdot,y)u_\alpha(\cdot,y)\big\|_p\Big) & \text{if } \gamma = 0 \\ \|K\|_{\mathcal{W}^0_{p,\alpha}} + \sup_{0<\delta\le 1} \delta^{-\gamma} \|\omega_\delta(K)\|_{\mathcal{W}^0_{p,\alpha}} & \text{if } 0 < \gamma < 1, \end{cases} \tag{3.9}$$

where the modulus of continuity of the kernel K is defined by

$$\omega_\delta(K)(x,y) := \sup_{|x'|\le\delta, |y'|\le\delta} |K(x+x', y+y') - K(x,y)|,\ x, y \in \mathbb{R}, \tag{3.10}$$

and $u_\alpha(x, y) = (1+|x-y|)^\alpha$, $x, y \in \mathbb{R}$ are polynomial weights on $\mathbb{R}\times\mathbb{R}$. Consider the set $\mathcal{W}^\gamma_{p,\alpha}$ of integral operators

$$Tf(x) = \int_{\mathbb{R}} K_T(x,y) f(y) dy, \quad f \in L^p, \tag{3.11}$$

whose integral kernels K_T satisfy $\|K_T\|_{\mathcal{W}^\gamma_{p,\alpha}} < \infty$, and define

$$\|T\|_{\mathcal{W}^\gamma_{p,\alpha}} := \|K_T\|_{\mathcal{W}^\gamma_{p,\alpha}},\ T \in \mathcal{W}^\gamma_{p,\alpha}.$$

Integral operators in $\mathcal{W}^\gamma_{p,\alpha}$ have their kernels being Hölder continuous of order γ and having off-diagonal polynomial decay of order α. For $1 \le p \le \infty$ and $\alpha > 1 - 1/p$, one may verify that $\mathcal{W}^\gamma_{p,\alpha}, 0 \le \gamma < 1$, are Banach subalgebras of $\mathcal{B}(L^2)$ under operator composition. The Banach algebras $\mathcal{W}^\gamma_{p,\alpha}, 0 < \gamma < 1$, of integral operators may not form a differential subalgebra of $\mathcal{B}(L^2)$, however the triple $(\mathcal{W}^\gamma_{p,\alpha}, \mathcal{W}^0_{p,\alpha}, \mathcal{B}(L^2))$ is proved in [42] to satisfy the following

$$\|T_0\|_{\mathcal{B}} \le D\|T_0\|_{\mathcal{W}^0_{p,\alpha}} \le D\|T_0\|_{\mathcal{W}^\gamma_{p,\alpha}}, \tag{3.12}$$

$$\|T_0^2\|_{\mathcal{W}^0_{p,\alpha}} \le D\|T_0\|^{1+\theta}_{\mathcal{W}^\gamma_{p,\alpha}} \|T_0\|^{1-\theta}_{\mathcal{B}(L^2)} \tag{3.13}$$

and

$$\|T_1T_2T_3\|_{\mathcal{W}^\gamma_{p,\alpha}} \le D\|T_1\|_{\mathcal{W}^\gamma_{p,\alpha}} \|T_2\|_{\mathcal{W}^0_{p,\alpha}} \|T_3\|_{\mathcal{W}^\gamma_{p,\alpha}} \tag{3.14}$$

holds for all $T_i \in \mathcal{W}^\gamma_{p,\alpha}, 0 \le i \le 3$, where D is an absolute constant and

$$\theta = \frac{\alpha + \gamma + 1/p}{(1+\gamma)(\alpha + 1/p)}.$$

Then the requirements (3.4) and (3.8) in Theorem 3.3 are met for the triplet $(\mathcal{W}^{\gamma}_{p,\alpha}, \mathcal{W}^{0}_{p,\alpha}, \mathcal{B}(L^2))$, and hence the Banach space pair $(\mathcal{W}^{\gamma}_{p,\alpha}, \mathcal{B}(L^2))$ satisfies the Brandenburg's condition (3.2) with $m = 4$ [15, 42].

4 Brandenburg Trick and Norm-Controlled Inversion

Let $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras with common identity and involution, and let $\mathcal{B}$ be symmetric. In this section, we show that $\mathcal{A}$ has norm-controlled inversion in $\mathcal{B}$ if it meets the Brandenburg requirement (3.2).

Theorem 4.1 *Let $\mathcal{B}$ be a symmetric $*$-algebra with its norm $\|\cdot\|_{\mathcal{B}}$ being normalized in the sense that (1.1) holds with $K = 1$,*

$$\|\tilde{A}\tilde{B}\|_{\mathcal{B}} \le \|\tilde{A}\|_{\mathcal{B}}\|\tilde{B}\|_{\mathcal{B}},\ \ \tilde{A}, \tilde{B} \in \mathcal{B}, \tag{4.1}$$

and $\mathcal{A}$ be a $*$-algebra with its norm $\|\cdot\|_{\mathcal{A}}$ being normalized too,

$$\|AB\|_{\mathcal{A}} \le \|A\|_{\mathcal{A}}\|B\|_{\mathcal{A}},\ \ A, B \in \mathcal{A}. \tag{4.2}$$

If $\mathcal{A}$ is a $*$-subalgebra of $\mathcal{B}$ with common identity I and involution $*$, and the pair $(\mathcal{A}, \mathcal{B})$ satisfies the Brandenburg requirement (3.2), then $\mathcal{A}$ has norm-controlled inversion in $\mathcal{B}$. Moreover, for any $A \in \mathcal{A}$ being invertible in $\mathcal{B}$ we have

$$\begin{aligned}\|A^{-1}\|_{\mathcal{A}} &\le \|A^*A\|_{\mathcal{B}}^{-1}\|A^*\|_{\mathcal{A}}\\ &\times \begin{cases} \Big(2t_0 + (1-2^{\log_m(1-\theta/m)})^{-1}(\ln a)^{-1}\Big) a \exp\Big(\frac{\ln m - \ln(m-\theta)}{\ln(m-\theta)} t_0 \ln a\Big) & \text{if } \theta < m-1\\ a^2(\ln a)^{-1}(Db)^{m-1}\Gamma\Big(\frac{(m-1)\ln(Db)}{\ln m \ln a}+1\Big) & \text{if } \theta = m-1,\end{cases}\end{aligned} \tag{4.3}$$

where $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ is the Gamma function, $m \ge 2$ and $0 < \theta \le m-1$ are the constants in (3.2), $\kappa(A^*A) = \|A^*A\|_{\mathcal{B}}\|(A^*A)^{-1}\|_{\mathcal{B}}, a = \big(1 - (\kappa(A^*A))^{-1}\big)^{-1} > 1$,

$$b = \frac{\|I\|_{\mathcal{A}} + \|A^*A\|_{\mathcal{B}}^{-1}\|A^*A\|_{\mathcal{A}}}{1-(\kappa(A^*A))^{-1}} \ge a > 1,$$

and

$$t_0 = \Big(\frac{(m-1)(m-\theta)\log_m(m-\theta)\ln(Db)}{(m-1-\theta)\ln a}\Big)^{\ln m/(\ln m - \ln(m-\theta))} \text{ for } 0 < \theta < m-1. \tag{4.4}$$

Proof Obviously it suffices to prove (4.3). In this paper, we follow the argument in [36] to give a sketch proof. Let $A \in \mathcal{A}$ so that $A^{-1} \in \mathcal{B}$. As $\mathcal{B}$ is a symmetric $*$-algebra, the spectrum of A^*A in $\mathcal{B}$ lies in an interval on the positive real axis,

$$\sigma(A^*A) \subset \big[\|(A^*A)^{-1}\|_{\mathcal{B}}^{-1},\ \|A^*A\|_{\mathcal{B}}\big]. \tag{4.5}$$

Therefore $B := I - \|A^*A\|_{\mathcal{B}}^{-1} A^*A \in \mathcal{A}$ satisfies

$$\|B\|_{\mathcal{B}} \le 1 - (\kappa(A^*A))^{-1} = a^{-1} < 1 \tag{4.6}$$

and

$$\|B\|_{\mathcal{A}} \le \|I\|_{\mathcal{A}} + \|A^*A\|_{\mathcal{B}}^{-1}\|A^*A\|_{\mathcal{A}} = ba^{-1}. \tag{4.7}$$

For a positive integer $n = \sum_{i=0}^{l} \varepsilon_i m^i$, define $n_0 = n$ and $n_k, 1 \le k \le l$, inductively by

$$n_k = \frac{n_{k-1} - \varepsilon_{k-1}}{m} = \sum_{i=k}^{l} \varepsilon_i m^{i-k}, 1 \le k \le l, \tag{4.8}$$

where $\varepsilon_i \in \{0, 1, \dots, m-1\}, 1 \le i \le l-1$ and $\varepsilon_l \in \{1, \dots, m-1\}$. By (3.2) and (4.1), we have

$$\|B^{mn_k}\|_{\mathcal{A}} \le D\|B^{n_k}\|_{\mathcal{A}}^{m-\theta}\|B^{n_k}\|_{\mathcal{B}}^{\theta} \le D\|B^{n_k}\|_{\mathcal{A}}^{m-\theta}\|B\|_{\mathcal{B}}^{n_k\theta}, \quad k = 1, \cdots, l-1. \tag{4.9}$$

By (4.1), (4.2), (4.6), (4.7), (4.8) and (4.9), we obtain

$$\begin{aligned}
\|B^n\|_{\mathcal{A}} &= \|B^{n_0}\|_{\mathcal{A}} \le \|B^{mn_1}\|_{\mathcal{A}}\|B\|_{\mathcal{A}}^{\varepsilon_0} \le D\|B^{n_1}\|_{\mathcal{A}}^{m-\theta}\|B\|_{\mathcal{A}}^{\varepsilon_0}\|B\|_{\mathcal{B}}^{n_1\theta} \\
&\le D^{1+(m-\theta)}\|B^{n_2}\|_{\mathcal{A}}^{(m-\theta)^2}\|B\|_{\mathcal{A}}^{\varepsilon_0+\varepsilon_1(m-\theta)}\|B\|_{\mathcal{B}}^{n_1\theta+n_2\theta(m-\theta)} \\
&\le \cdots \\
&\le D^{\sum_{k=0}^{l-1}(m-\theta)^k}\|B\|_{\mathcal{A}}^{\sum_{k=0}^{l}\varepsilon_k(m-\theta)^k}\|B\|_{\mathcal{B}}^{\theta\sum_{k=1}^{l} n_k(m-\theta)^{k-1}} \\
&= D^{\sum_{k=0}^{l-1}(m-\theta)^k}\|B\|_{\mathcal{A}}^{\sum_{k=0}^{l}\varepsilon_k(m-\theta)^k}\|B\|_{\mathcal{B}}^{n-\sum_{k=0}^{l}\varepsilon_k(m-\theta)^k} \\
&\le D^{\sum_{k=0}^{l-1}(m-\theta)^k} b^{\sum_{k=0}^{l}\varepsilon_k(m-\theta)^k} a^{-n} \\
&\le \begin{cases} (Db)^{\frac{(m-1)(m-\theta)}{m-1-\theta} n^{\log_m(m-\theta)}} a^{-n} & \text{if } \theta < m-1 \\[2ex] (Db)^{(m-1)\log_m(mn+1)} a^{-n} & \text{if } \theta = m-1, \end{cases}
\end{aligned} \tag{4.10}$$

where the last inequality holds since

$$\sum_{k=0}^{l} \varepsilon_k (m-\theta)^k \le (m-1)\sum_{k=0}^{l}(m-\theta)^k \le (m-1)\begin{cases} \frac{(m-\theta)^{l+1}-1}{m-1-\theta} & \text{if } \theta < m-1 \\ l+1 & \text{if } \theta = m-1 \end{cases}$$
$$\le (m-1)\begin{cases} \frac{m-\theta}{m-1-\theta} n^{\log_m(m-\theta)} & \text{if } \theta < m-1 \\ \log_m(mn+1) & \text{if } \theta = m-1. \end{cases}$$

Observe that $A^*A = \|A^*A\|_{\mathcal{B}}(I-B)$. Hence

$$A^{-1} = (A^*A)^{-1}A^* = \|A^*A\|_{\mathcal{B}}^{-1}\Big(\sum_{n=0}^{\infty} B^n\Big)A^*.$$

This together with (4.2), (4.10) and (4.11) implies that

$$\|A^{-1}\|_{\mathcal{A}} \le \|A^*A\|_{\mathcal{B}}^{-1}\|A^*\|_{\mathcal{A}}\sum_{n=0}^{\infty}\|B^n\|_{\mathcal{A}}$$
$$\le \|A^*A\|_{\mathcal{B}}^{-1}\|A^*\|_{\mathcal{A}} \times \begin{cases} \sum_{n=0}^{\infty}(Db)^{\frac{(m-1)(m-\theta)}{m-1-\theta} n^{\log_m(m-\theta)}} a^{-n} & \text{if } \theta < m-1 \\ \sum_{n=0}^{\infty}(Db)^{(m-1)\log_m(mn+1)}a^{-n} & \text{if } \theta = m-1. \end{cases} \tag{4.11}$$

By direct calculation, we have

$$\sum_{n=0}^{\infty}(Db)^{(m-1)\log_m(mn+1)}a^{-n} \le a\sum_{n=0}^{\infty}\int_n^{n+1}(Db)^{(m-1)\log_m(mt+1)}a^{-t}dt$$
$$\le a^2(Db)^{m-1}\int_0^{\infty}(t+1)^{(m-1)\log_m(Db)}e^{-(t+1)\ln a}dt$$
$$\le a^2(Db)^{m-1}(\ln a)^{-1}\Gamma\Big(\frac{(m-1)\ln(Db)}{\ln m\ln a}+1\Big). \tag{4.12}$$

This together with (4.11) proves (4.3) for $\theta = m-1$.

For $0 < \theta < m-1$, set

$$s(t) = t - \frac{(m-1)(m-\theta)\ln(Db)}{(m-1-\theta)\ln a}t^{\log_m(m-\theta)}.$$

Observe that

$$s'(t) = 1 - \frac{(m-1)(m-\theta)\ln(Db)}{(m-1-\theta)\ln a}\log_m(m-\theta)t^{\log_m(1-\theta/m)}.$$

Therefore

$$\min_{t\geq 0} s(t) = s(t_0) = -\frac{\ln m - \ln(m-\theta)}{\ln(m-\theta)} t_0 < 0 \tag{4.13}$$

and

$$1 \geq s'(t) \geq s'(2t_0) = 1 - 2^{\log_m(1-\theta/m)} \qquad \text{for all } \ t \geq 2t_0, \tag{4.14}$$

where t_0 is given in (4.4). By (4.13) and (4.14), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} (Db)^{\frac{(m-1)(m-\theta)}{m-1-\theta} n^{\log_m(m-\theta)}} a^{-n} \leq a \sum_{n=0}^{\infty} \int_n^{n+1} (Db)^{\frac{(m-1)(m-\theta)}{m-1-\theta} t^{\log_m(m-\theta)}} a^{-t} dt \\
&= a \Big(\int_0^{2t_0} + \int_{2t_0}^{\infty} \Big) \exp(-s(t)\ln a) dt \\
&\leq 2at_0 \exp(-s(t_0)\ln a) + (1 - 2^{\log_m(1-\theta/m)})^{-1} a \int_{s(2t_0)}^{\infty} \exp(-u \ln a) du \\
&\leq \Big(2t_0 + (1 - 2^{\log_m(1-\theta/m)})^{-1} (\ln a)^{-1}\Big) a \exp\Big(\frac{\ln m - \ln(m-\theta)}{\ln(m-\theta)} t_0 \ln a\Big).
\end{aligned} \tag{4.15}$$

Combining the above estimate with (4.11) proves (4.3) for $\theta < m-1$. □

For $m = 2$, the estimate (4.3) to the inverse $A^{-1} \in \mathcal{A}$ is essentially established in [19, 20] for $\theta = 1$ and [36, 40] for $\theta \in (0, 1)$, and a similar estimate is given in [34]. The reader may refer to [15, 21, 42, 43, 45] for norm estimation of the inverse of elements in Banach algebras of infinite matrices and integral operators with certain off-diagonal decay.

Remark 4.2 A good estimate for the norm control function h in the norm-controlled inversion (1.4) is important for some mathematical and engineering applications. For an element $A \in \mathcal{A}$ with $A^{-1} \in \mathcal{B}$, we obtain the following estimate from Theorem 4.1:

$$\|(A^*A)^{-1}\|_{\mathcal{A}} \leq C \|A^*A\|_{\mathcal{B}}^{-1} a (\ln a)^{-1} \times \begin{cases} t_1 \exp(C t_1) & \text{if } \theta < m-1 \\ ab^{m-1} \exp\Big(C \frac{\ln b}{\ln a} \ln\big(\frac{\ln b}{\ln a}\big)\Big) & \text{if } \theta = m-1, \end{cases} \tag{4.16}$$

where C is an absolute constant independent of A and

$$t_1 = (\ln b)^{\ln m/(\ln m - \ln(m-\theta))} (\ln a)^{-\ln(m-\theta)/(\ln m - \ln(m-\theta))}.$$

We remark that the above norm estimate to the inversion is far away from the optimal estimation for our illustrative differential subalgebra $C^1[a, b]$. In fact, give any $f \in$

$C^1[a,b]$ being invertible in $C[a,b]$, we have

$$\|1/f\|_{C^1[a,b]} \le \|f'\|_{C[a,b]}\|f^{-1}\|^2_{C[a,b]} + \|1/f\|_{C[a,b]} \le \|1/f\|^2_{C[a,b]}\|f\|_{C^1[a,b]}.$$

Therefore $C^1[a,b]$ has norm-controlled inversion in $C[a,b]$ with the norm control function $h(s,t)$ in (1.4) being $h(s,t) = st^2$. Gröchenig and Klotz first considered norm-controlled inversion with the norm control function h having polynomial growth, and they show in [19] that the Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_{1,\alpha}, \alpha > 0$ and the Jaffard algebra $\mathcal{J}_\alpha, \alpha > 1$ have norm-controlled inversion in $\mathcal{B}(\ell^2)$ with the norm control function h bounded by a polynomial. In [36], we proved that the Beurling algebras $\mathcal{B}_{p,\alpha}$ with $1 \le p \le \infty$ and $\alpha > 1 - 1/p$ admit norm-controlled inversion in $\mathcal{B}(\ell^2)$ with the norm control function bounded by some polynomials. Following the commutator technique used in [36, 39], we can establish a similar result for the Baskakov-Gohberg-Sjöstrand algebras $\mathcal{C}_{p,\alpha}$ with $1 \le p \le \infty$ and $\alpha > 1 - 1/p$.

Theorem 4.3 *Let $1 \le p \le \infty$ and $\alpha > 1 - 1/p$. Then the Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_{p,\alpha}$ and the Beurling algebra $\mathcal{B}_{p,\alpha}$ admit norm-controlled inversion in $\mathcal{B}(\ell^2)$ with the norm control function bounded by a polynomial.*

It is still unknown whether Gröchenig-Schur algebras $\mathcal{A}_{p,\alpha}, 1 \le p < \infty, \alpha > 1 - 1/p$, admit norm-controlled inversion in $\mathcal{B}(\ell^q), 1 \le q < \infty$, with the norm control function bounded by a polynomial. In [19], Gröchenig and Klotz introduce a differential operator $\mathcal{D}$ on a Banach algebra and use it to define a differential $*$-algebra $\mathcal{A}$ of a symmetric $*$-algebra $\mathcal{B}$, which admits norm-controlled inversion with the norm control function bounded by a polynomial. However, the differential algebra in [19] does not include the Gröchenig-Schur algebras $\mathcal{A}_{p,\alpha}$, the Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_{p,\alpha}$ and the Beurling algebra $\mathcal{B}_{p,\alpha}$ with $1 \le p < \infty$ and $\alpha > 1 - 1/p$. It could be an interesting problem to extend the conclusions in Theorem 4.3 to general Banach algebras such that the norm control functions in the norm-controlled inversion have polynomial growth.

Remark 4.4 A crucial step in the proof of Theorem 4.1 is to introduce $B := I - \|A^*A\|^{-1}_{\mathcal{B}} A^*A \in \mathcal{A}$, whose spectrum is contained in an interval on the positive real axis. The above reduction depends on the requirements that $\mathcal{B}$ is symmetric and both $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras with common identity and involution. For the applications to some mathematical and engineering fields, the widely-used algebras $\mathcal{B}$ of infinite matrices and integral operators are the operator algebras $\mathcal{B}(\ell^p)$ and $\mathcal{B}(L^p), 1 \le p \le \infty$, which are symmetric only when $p = 2$. In [1, 15, 36, 38, 42, 48], inverse-closedness of localized matrices and integral operators in $\mathcal{B}(\ell^p)$ and $\mathcal{B}(L^p), 1 \le p \le \infty$, are discussed, and in [14], Beurling algebras $\mathcal{B}_{p,\alpha}$ with $1 \le p < \infty$ and $\alpha > d(1 - 1/p)$ are shown to admit polynomial norm-controlled inversion in nonsymmetric algebras $\mathcal{B}(\ell^p), 1 \le p < \infty$. It is still widely open to discuss Wiener's lemma and norm-controlled inversion when $\mathcal{B}$ and $\mathcal{A}$ are not $*$-algebras and $\mathcal{B}$ is not a symmetric algebra.

Acknowledgments The project is partially supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (NRF-2016R1D1A1B03930571) and the National Science Foundation (DMS-1816313).

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Hermitian Metrics on the Resolvent Set and Extremal Arc Length

Mai Tran and Rongwei Yang

Abstract For a bounded linear operator A on a complex Hilbert space $\mathcal{H}$, the functions $g_x(z) = \|(A - z)^{-1}x\|^2$, where $x \in \mathcal{H}$ with $\|x\| = 1$, defines a family of Hermitian metrics on the resolvent set $\rho(A)$. Thus the arc length of a fixed circle $C \subset \rho(A)$ with respect to the metric g_x is dependent on the choice of x. This paper derives an integral equation for the extremal values of the arc length. Solution x of the equation, if exists, has particular properties as related to A. In the case A is the unilateral shift operator on the Hardy space $H^2(\mathbb{D})$, the paper proves that the arc length of C is maximal if and only if x is an inner function.

Keywords Resolvent set · Hermitian metric · Nilpotent operator · Extremal equation · Unilateral shift · Inner function

Mathematics Subject Classification (2010) Primary: 47A13, Secondary: 53A35

1 Introduction

For a bounded linear operator A on a complex Hilbert space $\mathcal{H}$, its spectrum

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid A - \lambda \ \text{ is not invertible}\},$$

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_23

and its complement $\rho(A) = \mathbb{C} \setminus \sigma(A)$ is called the resolvent set of A. As an open subset of $\mathbb{C}$, $\rho(A)$ is naturally equipped with the Euclidean metric which clearly is indifferent to the operator A. Is there a natural A-dependent metric g on $\rho(A)$ under which the geometric properties of $\rho(A)$ may reveal new information about A? This question is studied in [5, 13] where a family of operator-and-vector-dependent metrics g_x on $\rho(A)$ is defined as follows. Consider the operator-valued 1-form

$$\omega_A = \big(A - z\mathrm{I}\big)^{-1} d\big(A - z\mathrm{I}\big) = -\big(A - z\mathrm{I}\big)^{-1} dz.$$

Then the wedge product

$$-\omega_A^* \wedge \omega_A = \big(A^* - \bar{z}\mathrm{I}\big)^{-1}\big(A - z\mathrm{I}\big)^{-1}\, dz \wedge d\bar{z}$$

is an operator-valued $(1, 1)$-form. For an arbitrary $x \in \mathcal{H}$ such that $\|x\| = 1$ the vector state ϕ_x on the C^*-algebra $B(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$ is defined by

$$\phi_x(a) = \langle ax, x\rangle, \quad a \in B(\mathcal{H}).$$

Then we have

$$\phi_x\big(-\omega_A^* \wedge \omega_A\big) = \|\big(A - z\mathrm{I}\big)^{-1} x\|^2\, dz \wedge d\bar{z},$$

which defines a Hermitian metric on $\rho(A)$ with metric function $g_x(z) = \|\big(A - z\mathrm{I}\big)^{-1} x\|^2$. In other words, the infinitesimal arc length ds under this metric is given by

$$ds^2 = g_x(z)|dz|^2 = g_x(z)(du^2 + dv^2),$$

where $z = u + iv,\ u, v \in \mathbb{R}$. The family of metrics $g_x,\ x \in \mathcal{H}$, makes it possible to study A by geometric means. A motivating example studied in [5] is when A is quasinilpotent, i.e., when $\rho(A) = \mathbb{C} \setminus \{0\}$. In this case, the singular behaviors of g_x at $z = 0$ turn out to reveal much information about A. The connection between the geometry of $(\rho(A),\ g_x)$ and properties of A is further investigated in [9] and [10], where the classical Volterra operators on $L^2[0, 1]$ and $H^2(\mathbb{D})$ are considered in details. This paper follows the same line, with a focus on extremal value problem for the arc length with respect to the choice of x and the example of the unilateral shift operator T_w on the Hardy space $H^2(\mathbb{D})$. In Sect. 2, we will derive an integral equation for the extremal values of the length of circles with respect to g_x. In Sect. 3, we will prove that for the unilateral shift operator T_w on $H^2(\mathbb{D})$, the length of the circle $C_r = \{z \in \mathbb{C} : |z| = r > 1\}$ is maximal with respect to the change of x in the metric g_x if and only if x is an inner function. Further, the infimum of the arc length of C_r is shown to be unattainable.

2 Extremal Equation

Consider a smooth manifold M with a Riemannian metric g defined on its tangent bundle. Then for a piece-wise smooth path $\gamma(t), t \in [0, 1]$ in M connecting two fixed points $p = \gamma(0)$ and $q = \gamma(1)$, its arc length is given by

$$L(\gamma) = \int_0^1 \sqrt{g(\gamma'(t), \gamma'(t))}dt.$$

A path γ that minimizes $L(\gamma)$ is called a geodesic. In this section we will study the extremal value problem for the arc length with respect to the family of metrics g_x but with a different point of view. We will study, given a fixed circle, which choice of x will give a metric g_x that maximizes or minimizes the arc length of the circle. We believe that such x, if exists, may reveal particular information about A.

Consider a bounded linear operator $V \in B(\mathcal{H})$ and fix a $x \in \mathcal{H}$ with $\|x\| = 1$. We pick any circle C_r with a fixed radius r big enough such that $\sigma(V)$ lies properly inside C_r, for instance we may let r be bigger than the spectral radius $r(V)$. Let us parametrize the path $C_r = \{z(t) = re^{2\pi it}, 0 \leq t \leq 1\}$. Then the arc length of C_r with respect to the metric g_x is

$$L_x(C_r) = \int_0^1 \sqrt{g_x(z(t))}|z'(t)|dt = 2\pi r \int_0^1 \left\| \left(V - re^{2\pi it}\mathrm{I}\right)^{-1} x \right\| \, dt. \tag{2.1}$$

As the choice of vector x varies the metric function g_x will also vary, thus the arc length of C_r will change correspondingly. Since $g_x(re^{2\pi it}) \leq \|(V - re^{2\pi it}\mathrm{I})^{-1}\|$ which is a bounded function with respect to t, the set

$$S_r := \{L_x(C_r) \mid x \in \mathcal{H}, \|x\| = 1\}$$

is a bounded subset of $[0, \infty)$. It is then a natural question whether $\sup S_r$ or $\inf S_r$ is obtainable at some x. First, we use calculus of variations to give a necessary condition for this situation. For simplicity, we set $\theta = 2\pi t$ and let $V(\theta) = (V - re^{i\theta}\mathrm{I})^{-1}$. Then (2.1) becomes

$$L_x(C_r) = \int_0^1 \sqrt{g_x(z(t))}|z'(t)|dt = r \int_0^{2\pi} \|V(\theta)x\| \, d\theta.$$

Theorem 2.1 *Let V be a bounded linear operator on the Hilbert space $\mathcal{H}$ and $r > r(V)$. Suppose there exists $x \in \mathcal{H}$ such that $\|x\| = 1$ and $L_x(C_r)$ is extremal in S_r, then x satisfies the extremal equation*

$$r \int_0^{2\pi} \frac{V^*(\theta)V(\theta)x}{\|V(\theta)x\|} d\theta = L_x(C_r)x.$$

Proof Let x be a vector such that either $L_x(C_r) = \sup S_r$ or $L_x(C_r) = \inf S_r$. Without loss of generality we assume the former holds. Let $y \in \mathcal{H}$ be any non-zero vector with $\|y\| < 1$. We set

$$h(t) = \frac{x + ty}{\|x + ty\|},\ t \in \mathbb{C},$$

and let

$$F(t) = \frac{1}{r} L_{h(t)}(C_r) = \int_0^{2\pi} \|V(\theta)h(t)\|\ d\theta.$$

Since $F(0)$ is extremal, we must have

$$\frac{\partial F(t)}{\partial t}\Big|_{t=0} = 0 = \frac{\partial F(t)}{\partial \bar{t}}\Big|_{t=0}. \tag{2.2}$$

Since F is real, the second equality follows from the first. We use several steps to compute $\frac{\partial F(t)}{\partial t}\Big|_{t=0}$. First let's compute $\frac{\partial}{\partial t}\big[\, \|x + ty\|\, \big]\Big|_{t=0}$. We have

$$\begin{aligned}
&\frac{\partial}{\partial t}\big[\, \|x + ty\|\, \big] = \frac{\partial}{\partial t}\left[\sqrt{\langle x + ty, x + ty\rangle}\right] \\
&= \frac{1}{2}\langle x + ty, x + ty\rangle^{-1/2}\left[\frac{\partial}{\partial t}\Big(\langle x, x\rangle + \langle x, ty\rangle + \langle ty, x\rangle + |t|^2\langle y, y\rangle\Big)\right] \\
&= \frac{\langle y, x\rangle + \bar{t}\langle y, y\rangle}{2\sqrt{\langle x + ty, x + ty\rangle}}.
\end{aligned}$$

Setting $t = 0$ and using the fact that $\|x\| = 1$, we have

$$\frac{\partial}{\partial t}\big[\, \|x + ty\|\, \big]\Big|_{t=0} = \frac{1}{2}\langle y, x\rangle. \tag{2.3}$$

Second, we set

$$L(x + ty) = \int_0^{2\pi} \|V(\theta)(x + ty)\|\, d\theta$$

and compute $\frac{\partial}{\partial t}\big[L(x + ty)\big]\Big|_{t=0}$. Through similar calculation,

$$\frac{\partial}{\partial t}\big[L(x + ty)\big] = \int_0^{2\pi} \frac{\partial}{\partial t}\, \|V(\theta)(x + ty)\|\, d\theta$$

$$= \int_0^{2\pi} \frac{\partial}{\partial t}\left[\sqrt{\left\langle V(\theta)(x+ty), V(\theta)(x+ty)\right\rangle}\right] d\theta$$

$$= \int_0^{2\pi} \frac{\left\langle V(\theta)y, V(\theta)x\right\rangle + \bar{t}\|V(\theta)y\|^2}{2\sqrt{\langle V(\theta)(x+ty), V(\theta)(x+ty)\rangle}} d\theta.$$

Setting $t = 0$, we have

$$\frac{\partial}{\partial t}\big[L(x+ty)\big]\Big|_{t=0} = \int_0^{2\pi} \frac{\left\langle V(\theta)y, V(\theta)x\right\rangle}{2\,\|V(\theta)x\|} d\theta.$$

Now since

$$\frac{\partial F(t)}{\partial t} = \frac{\partial}{\partial t}\left[\frac{L(x+ty)}{\|x+ty\|}\right]$$

$$= \frac{\|x+ty\| \frac{\partial}{\partial t}\big[L(x+ty)\big] - L(x+ty)\frac{\partial}{\partial t}\|x+ty\|}{\|x+ty\|^2},$$

Setting $t = 0$ and using (2.3) and the computations above, we have

$$\int_0^{2\pi} \frac{\langle V(\theta)y, V(\theta)x\rangle}{\|V(\theta)x\|} d\theta - \frac{1}{r} L_x(C_r)\langle y, x\rangle = 0, \quad \forall y \in \mathcal{H}, \tag{2.4}$$

which implies

$$r\int_0^{2\pi} \frac{V^*(\theta)V(\theta)x}{\|V(\theta)x\|} d\theta = L_x(C_r)x.$$ □

Observe that if we set

$$T_r = r\int_0^{2\pi} \frac{V^*(\theta)V(\theta)}{\|V(\theta)x\|} d\theta$$

then T_r is a positive linear operator, and Theorem 2.1 indicates that when $L_x(C_r)$ is extremal the vector x is an eigenvector of T_r with eigenvalue equal to $L_x(C_r)$. The following two examples further illustrate this theorem.

Example 2.2 It is shown in [13] that if $\|x\| = 1$ such that $Vx = 0$, then $L_x(C_r) = 2\pi$, which is independent of r and is minimal. One verifies easily that in this case $V(\theta)x = -\frac{1}{r}e^{-i\theta}x$, and hence $\|V(\theta)x\| = \frac{1}{r}$ is constant. Further,

$$T_r x = r\int_0^{2\pi} -V^*(\theta)e^{-i\theta}x\,d\theta = 2\pi\left(\frac{1}{2\pi i}\int_{C_r}(V - zI)^{-1}dz\right)^* x.$$

By functional calculus the integral is equal to I, thus $T_r x = 2\pi x$ which satisfies Theorem 2.1.

Example 2.3 Now we take a look at an elementary example. Consider the nilpotent operator V on $\mathbb{C}^2$, and the vectors x_1 and x_2 as follows:

$$V = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Clearly, in this case we have $V(\theta) = (V - re^{i\theta})^{-1} = \frac{-e^{-i\theta}}{r}(I + \frac{e^{-i\theta}}{r}V)$, and it follows that

$$T_r = r\int_0^{2\pi} \frac{V^*(\theta)V(\theta)}{\|V(\theta)x\|} d\theta = 2\pi \frac{I + \frac{1}{r^2}V^*V}{\sqrt{1 + \frac{1}{r^2}}}.$$

Since $Vx_1 = 0$, we have $L_{x_1}(C_r) = 2\pi$ which is the minimal arc length of C_r as mentioned in Example 2.2. By direct computation one can verify that x_2 also satisfies the extremal equation, and the arc length $L_{x_2}(C_r) = 2\pi\sqrt{1 + \frac{1}{r^2}}$ is maximal.

3 The Unilateral Shift Operator

In the finite dimension case, since the closed unit ball $\mathcal{H}_1$ of $\mathcal{H}$ is compact and the metric function g_x is norm continuous in x, the values $\sup S_r$ and $\inf S_r$ are both obtainable. In the infinite dimension case, although $\mathcal{H}_1$ is *weakly* compact by Alaoglu's theorem, the metric function g_x is not *weakly* continuous in x, hence the values $\sup S_r$ and $\inf S_r$ may not be obtainable. For example, consider the Volterra operator V on $H^2(\mathbb{D})$ defined by

$$Vf(w) = \int_0^w f(t)dt, \quad f \in H^2(\mathbb{D}).$$

It is well-known that V is quasinilpotent, and by [1, 4] its invariant subspaces are of the form $w^k H^2(\mathbb{D})$ for some $k \geq 0$. Let $C = C_1$ be the unit circle. Then in [10] it is shown that $\sup S_1$ is obtainable at constant function 1, and $\inf S_r = 2\pi$ which it is not obtainable because V has no nontrivial kernel. In this section we take a look at the extremal value problem for the unilateral shift operator T_w on the Hardy space over the unit disk which is defined as

$$H^2(\mathbb{D}) = \left\{ f \in hol(\mathbb{D}) : \|f\|^2 := \sup_{0<r<1} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^2 \frac{d\theta}{2\pi} < \infty \right\}.$$

Its inner product is defined by

$$\langle f, g\rangle = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, \frac{d\theta}{2\pi}.$$

It is well-known that the reproducing kernel of $H^2(\mathbb{D})$ is

$$K(\lambda, w) = \frac{1}{1-\overline{\lambda} w}, \quad |\lambda| < 1,\ |w| \le 1.$$

The unilateral shift operator T_w on the Hardy space is defined as $T_w f = wf$, where $f \in H^2(\mathbb{D})$. It is well-known that the spectrum $\sigma(T_w) = \overline{\mathbb{D}}$ (cf. [3, 12]). We let $\mathbb{T}$ stand for the unit circle $\{w \in \mathbb{C} : |w| = 1\}$. A function $f \in H^2(\mathbb{D})$ is said to be inner if $|f(w)| = 1$ almost everywhere on $\mathbb{T}$. A classical theorem due to Beurling states that M is an invariant subspace for T_w if and only if $M = fH^2(\mathbb{D})$ for some inner function f (cf. [2, 6]). Given $f \in H^2(\mathbb{D})$ with $\|f\| = 1$, consider the metric defined on $\rho(T_w) = \{z \in \mathbb{C} : |z| > 1\}$ through the metric function

$$g_f(z) = \left\|(T_w - z)^{-1} f\right\|^2 = \int_{\mathbb{T}} \frac{|f(w)|^2}{|w-z|^2}\, dm(w), \quad |z| > 1,$$

where $dm(w) = \dfrac{d\theta}{2\pi}$ for $w = e^{i\theta}$. Pick any $r > 1$ and let $C_r = \{z(t) = re^{2\pi i t} \mid 0 \le t \le 1\}$ be the circle as before. Then by (2.1) its arc length with respect to the metric g_f is thus

$$L_f(C_r) = 2\pi r \int_0^1 \left[\int_{\mathbb{T}} \frac{|f(w)|^2}{\left|w - re^{2\pi i t}\right|^2}\, dm(w)\right]^{1/2} dt. \tag{3.1}$$

Theorem 3.1 *For the unilateral shift T_w on $H^2(\mathbb{D})$ and any $r > 1$, we have $\sup S_r = \dfrac{2\pi r}{\sqrt{r^2-1}}$, and $L_f(C_r) = \sup S_r$ if and only if f is an inner function.*

Proof First, using the Cauchy-Schwarz inequality for the outside integral we have

$$L_f(C_r) \le 2\pi r \left[\int_{\mathbb{T}} \int_0^1 \frac{|f(w)|^2}{\left|w - re^{2\pi i t}\right|^2}\, dm(w) dt\right]^{1/2}. \tag{3.2}$$

Fourier series and Parseval's identity give

$$\int_0^1 \frac{1}{\left|w - re^{2\pi i t}\right|^2}\, dt = \frac{1}{r^2-1} = \int_{\mathbb{T}} \frac{1}{\left|w - re^{2\pi i t}\right|^2}\, dm(w).$$

Therefore, we have

$$L_f(C_r) \le \frac{2\pi r}{\sqrt{r^2-1}}\|f\| = \frac{2\pi r}{\sqrt{r^2-1}}, \tag{3.3}$$

and the first equality in (3.3) holds if f is inner. Now suppose for some $f \in H^2(\mathbb{D})$ with $\|f\| = 1$ the first equality in (3.3) holds. Then by the Cauchy-Schwarz inequality the inside integral in (3.1) (which is $g_f(re^{2\pi it})$) is constant with respect to t. Write

$$\begin{aligned} g_f(re^{2\pi it}) &= \int_{\mathbb{T}} \frac{|f(w)|^2}{\left|w - re^{2\pi it}\right|^2}\, dm(w) \\ &= \int_{\mathbb{T}} \frac{|f(w)|^2}{\left(w - re^{2\pi it}\right)\left(\overline{w} - re^{-2\pi it}\right)}\, dm(w). \end{aligned}$$

Expanding $g_f(re^{2\pi it})$ out as a Fourier series $\sum_{k\in\mathbb{Z}} C_k e^{2\pi kit}$, one verifies that

$$C_k = \frac{1}{r^{|k|}(r^2-1)} \int_{\mathbb{T}} |f(w)|^2\, w^{-k} dm(w), \quad k \in \mathbb{Z}.$$

Since $g_f(re^{2\pi it})$ is constant, we must have $C_k = 0$ for all $k \neq 0$. This implies that $|f(w)|$ is constant a.e. on $\mathbb{T}$ and hence it is inner since $\|f\| = 1$. □

Theorem 3.1 shows that sup S_r is obtainable precisely at inner functions. Observe that this fact is independent of the choice of r. More interestingly, inner functions are rediscovered here without resorting to the concept of invariant subspaces (of T_w). In view of Theorem 3.1, a natural question then is whether inf S_r is obtainable, and if so by what type of functions in $H^2(\mathbb{D})$. The following theorem addresses this question.

Theorem 3.2 *For the unilateral shift T_w on $H^2(\mathbb{D})$ and any $r > 1$, we have*

$$\inf S_r = 2\pi \int_0^1 \frac{dt}{\left|e^{2\pi it} - \frac{1}{r}\right|},$$

and it is unattainable.

Proof For $f \in H^2(\mathbb{D})$ with $\|f\| = 1$, the measure $dm_f(w) := |f(w)|^2 dm(w)$ is a probability measure on the unit circle. Hence by (3.1) and the Cauchy-Schwarz inequality one has

$$L_f(C_r) = 2\pi r \int_0^1 \left[\int_{\mathbb{T}} |w - re^{2\pi it}|^{-2} dm_f(w)\right]^{1/2} dt$$

$$\geq 2\pi r \int_0^1 \left[\int_{\mathbb{T}} \frac{1}{|w - re^{2\pi it}|} dm_f(w) \right] dt$$
$$= 2\pi \int_{\mathbb{T}} |f(w)|^2 \left[\int_0^1 \frac{1}{|\bar{w}e^{2\pi it} - \frac{1}{r}|} dt \right] dm(w)$$

Since the inside integral in the line above is constant with respect to w and $\|f\| = 1$, one readily obtains the inequality

$$L_f(C_r) \geq 2\pi \int_0^1 \frac{1}{|e^{2\pi it} - \frac{1}{r}|} dt$$

for every f. Hence

$$\inf S_r \geq 2\pi \int_0^1 \frac{dt}{\left|e^{2\pi it} - \frac{1}{r}\right|}.$$

The above argument is communicated to us by C. Zu [14].

For the inequality in the other direction, in view of Theorem 3.1 a nonconstant outer function will give an arc length of C_r that is less than $\sup S_r$. To gauge the value of $\inf S_r$ we consider a particular type of outer functions, namely the normalized reproducing kernel

$$k_\lambda(w) = \frac{\sqrt{1 - |\lambda|^2}}{1 - \bar{\lambda} w}, \quad |\lambda| < 1.$$

In this case the metric function

$$g_{k_\lambda}(z) = \|(T_w - z)^{-1} k_\lambda\|^2 = \int_{\mathbb{T}} \frac{1 - |\lambda|^2}{|1 - \bar{\lambda} w|^2 |w - z|^2} \, dm(w), \quad |z| > 1. \tag{3.4}$$

Letting $\beta = \dfrac{1}{z}$ and using the power series of $\dfrac{1}{(1 - \bar{\lambda} w)(w - z)}$ with respect to w, one computes that

$$\begin{aligned} g_{k_\lambda}(z) &= \|(T_w - z)^{-1} k_\lambda\|^2 \\ &= (1 - |\lambda|^2)|\beta|^2 \left[1 + \left|\bar{\lambda} + \beta\right|^2 + \left|\bar{\lambda}^2 + \bar{\lambda}\beta + \beta^2\right|^2 + \dots \right] \\ &= (1 - |\lambda|^2)|\beta|^2 \sum_{k=1}^{\infty} \left| \frac{\bar{\lambda}^k - \beta^k}{\bar{\lambda} - \beta} \right|^2 = \frac{(1 - |\lambda|^2)|\beta|^2}{|\bar{\lambda} - \beta|^2} \sum_{k=1}^{\infty} |\bar{\lambda}^k - \beta^k|^2. \end{aligned}$$

The infinite sum can be computed as

$$\sum_{k=1}^{\infty}\left|\overline{\lambda}^k-\beta^k\right|^2=\frac{|\lambda|^2}{1-|\lambda|^2}-\frac{\overline{\lambda\beta}}{1-\overline{\lambda\beta}}-\frac{\lambda\beta}{1-\lambda\beta}+\frac{|\beta|^2}{1-|\beta|^2}$$
$$=\frac{\left|\overline{\lambda}-\beta\right|^2\left[1-|\lambda\beta|^2\right]}{(1-|\lambda|^2)|1-\overline{\lambda\beta}|^2(1-|\beta|^2)}.$$

Therefore,

$$g_{k_\lambda}(z)=\frac{|\beta|^2\left[1-|\lambda\beta|^2\right]}{|1-\overline{\lambda\beta}|^2(1-|\beta|^2)}=\frac{|z|^2-|\lambda|^2}{(|z|^2-1)|z-\lambda|^2}. \tag{3.5}$$

Clearly, the metric g_{k_λ} has singularities on the unit circle. By (3.1), we have

$$L_{k_\lambda}(C_r)=\frac{2\pi r}{\sqrt{r^2-1}}\int_0^1\frac{\sqrt{r^2-\lambda^2}}{\left|re^{2\pi it}-\lambda\right|}\,dt=\frac{2\pi r}{\sqrt{r^2-1}}\int_0^1\frac{\sqrt{1-|\Delta|^2}}{\left|e^{2\pi it}-\Delta\right|}\,dt, \tag{3.6}$$

where $\Delta=\dfrac{\lambda}{r}\in\mathbb{D}$. Observe that the integrand in (3.6) is the square root of the Poisson kernel on the unit disc. Hence by the Cauchy-Schwarz inequality we have

$$L_{k_\lambda}(C_r)<\frac{2\pi r}{\sqrt{r^2-1}}\int_0^1\frac{1-|\Delta|^2}{\left|e^{2\pi it}-\Delta\right|^2}\,dt=\frac{2\pi r}{\sqrt{r^2-1}}$$

when $\lambda\neq 0$. This is consistent with Theorem 3.1. The integral in (3.6), which we denote by $I(\Delta)$, is related to the elliptic integral of the first kind. The following chart shows the values for $I(\Delta)$ based on various fixed values for $r>1$ and $\lambda\in\mathbb{D}$.

$I(\Delta)=\int_0^1\frac{\sqrt{1-\lvert\Delta\rvert^2}}{\lvert e^{2\pi it}-\Delta\rvert}\,dt,\quad \Delta=\lambda/r$							
$r=1.1$	$I(\Delta)$	$r=1.25$	$I(\Delta)$	$r=5$	$I(\Delta)$	$r=13$	$I(\Delta)$
$\lambda=1/8$	0.996	$\lambda=1/8$	0.997	$\lambda=1/8$	0.999	$\lambda=1/8$	0.999
$\lambda=1/4$	0.986	$\lambda=1/4$	0.989	$\lambda=1/4$	0.999	$\lambda=1/4$	0.999
$\lambda=1/3$	0.978	$\lambda=1/3$	0.981	$\lambda=1/3$	0.998	$\lambda=1/3$	0.999
$\lambda=1/2$	0.942	$\lambda=1/2$	0.956	$\lambda=1/2$	0.997	$\lambda=1/2$	0.999
$\lambda=2/3$	0.888	$\lambda=2/3$	0.918	$\lambda=2/3$	0.995	$\lambda=2/3$	0.999
$\lambda=3/4$	0.850	$\lambda=3/4$	0.891	$\lambda=3/4$	0.994	$\lambda=3/4$	0.999
$\lambda=8/9$	0.754	$\lambda=8/9$	0.832	$\lambda=8/9$	0.991	$\lambda=8/9$	0.998
$\lambda=9/10$	0.743	$\lambda=9/10$	0.826	$\lambda=9/10$	0.991	$\lambda=9/10$	0.998
$\lambda=1$	0.615	$\lambda=1$	0.762	$\lambda=1$	0.989	$\lambda=1$	0.998

This chart shows that when r is fixed and $|\lambda|$ approaches 1 the value of $I(\Delta)$ is decreasing. Hence by (3.6) we have

$$\begin{aligned}\inf S_r \leq \lim_{|\lambda|\to 1} L_{k_\lambda}(C_r) &= \frac{2\pi r}{\sqrt{r^2-1}} \int_0^1 \frac{\sqrt{1-\frac{1}{r^2}}}{\left|e^{2\pi i t}-\frac{1}{r}\right|}\, dt \\ &= 2\pi \int_0^1 \frac{dt}{\left|e^{2\pi i t}-\frac{1}{r}\right|},\end{aligned}$$

and this completes the proof. □

4 Energy Functional

In the remaining part of the paper we make a brief remark about the energy functional. Consider a smooth manifold M with a Riemannian metric g defined on its tangent bundle. Then for a piece-wise smooth path $\gamma(t), t \in [0, 1]$ in M that connects two fixed points $\gamma(0) = p$ and $\gamma(1) = q$ in M, the associated energy functional is

$$E(\gamma) = \frac{1}{2} \int_0^1 g(\gamma'(t), \gamma'(t))dt,$$

where $g(\gamma'(t), \gamma'(t))$ is the square of the length of $\gamma'(t)$ in the tangent space with respect to metric g. Recall that the arc length of γ is given by

$$L(\gamma) = \int_0^1 \sqrt{g(\gamma'(t), \gamma'(t))}dt.$$

Hence it follows easily from the Cauchy-Schwarz inequality that

$$L^2(\gamma) \leq 2E(\gamma).$$

If γ is the path of a moving particle in a field defined by the metric g, then the length of the vector $\gamma'(t)$ is the speed of the particle, hence $E(\gamma)$ is the integral of the kinetic energy of a particle of unit mass along path γ. In particular, it is known that γ is a geodesic if and only if the energy functional $E(\gamma)$ is minimal with respect to variation of γ. This reflects the natural phenomenon that a particle in a field travels between two points with minimal work. For more information about energy functional we refer readers to [7, 8]. In this section we use the unilateral shift operator T_w as an example to show that this phenomenon does not occur in the extremal value problem with respect to the variation of x in the metric g_x.

As before, we let V be a bounded linear operator in the Hilbert space $\mathcal{H}$ and let $x \in \mathcal{H}$ such that $\|x\| = 1$. Assume the radius $r > r(V)$. Then using parametrization $\gamma(t) = re^{2\pi it},\ 0 \le t \le 1$, the energy functional

$$E_x(C_r) = \frac{1}{2}\int_0^1 g_x(\gamma'(t), \gamma'(t))dt = \frac{1}{2}\int_0^1 \|(V - re^{i\theta})^{-1}x\|^2|\gamma'(t)|^2 dt$$
$$= \pi r^2 \int_0^{2\pi} \|(V - re^{i\theta})^{-1}x\|^2 d\theta.$$

Since

$$V(\theta) = (V - re^{i\theta})^{-1}$$
$$= -\frac{e^{-i\theta}}{r}(I - \frac{e^{-i\theta}}{r}V)^{-1} = -\frac{e^{-i\theta}}{r}\sum_{k=0}^{\infty}\left(\frac{e^{-i\theta}}{r}V\right)^k, \tag{4.1}$$

direct computation shows that

$$T(r) := \pi r^2 \int_0^{2\pi} V^*(\theta)V(\theta)d\theta = 2\pi^2 \sum_{k=0}^{\infty}\frac{V^{*k}V^k}{r^{2k}},$$

which is a positive operator. Hence $E_x(C_r) = \langle T(r)x, x\rangle$. The energy functional E_x was defined and studied in [10, 11], where among other things it shows that if $E_x(C_r)$ is extremal, then x is an eigenvector of $T(r)$ with corresponding eigenvalue equal to $E_x(C_r)$. A natural question is whether a unital vector $x \in \mathcal{H}$ that minimizes $L_x(C_r)$ if and only if it minimizes $E_x(C_r)$. When $V = T_w$ is the unilateral shift operator on the Hardy space, we have $V^*V = I$, and hence $T(r) = \frac{2\pi^2 r^2}{r^2-1}I$. Thus

$$E_x(C_r) = \langle T(r)x, x\rangle = \frac{2\pi^2 r^2}{r^2 - 1},$$

which is independent of the choice of unital vector $x \in H^2(\mathbb{D})$. Therefore, if $x = k_\lambda$ for some nonzero $\lambda \in \mathbb{D}$, then $E_x(C_r)$ is minimal (and also maximal), but $L_x(C_r)$ is not minimal in view of Theorem 3.1.

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Hybrid Normed Ideal Perturbations of n-Tuples of Operators II: Weak Wave Operators

Dan-Virgil Voiculescu

Dedicated to the memory of Ronald G. Douglas.

Abstract We prove a general weak existence theorem for wave operators for hybrid normed ideal perturbations. We then use this result to prove the invariance of Lebesgue absolutely continuous parts of n-tuples of commuting hermitian operators under hybrid normed ideal perturbations from a class studied in the first paper of this series.

Keywords Hybrid normed ideal perturbation · Weak wave operators · Invariance of n-dimensional absolutely continuous spectrum

2010 Mathematics Subject Classification Primary: 47L30; Secondary: 47L20, 47A13

1 Introduction

In [4] we adapted basics of normed ideal perturbations to the hybrid setting and then turned to hybrid perturbations of n-tuples of commuting hermitian operators and found an unexpected result which also required a substantial amount of technical work. We showed that up to a factor of proportionality the modulus of quasicentral approximation with respect to the normed ideal $\mathcal{C}_n^-$ [2] is the same as the hybrid one with respect to $(\mathcal{C}_{p_1}^-, \dots, \mathcal{C}_{p_n}^-)$ when $p_1^{-1} + \cdots + p_n^{-1} = 1$. In particular this implies that under such a hybrid perturbation the existence or the absence of n-dimensional Lebesgue absolutely continuous spectrum is preserved. To get in full generality that the Lebesgue absolutely continuous parts are actually preserved up to unitary

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_24

equivalence requires some existence results for weak wave operators, which is the aim of the present paper (the particular case of integrable multiplicity functions could have been deduced directly from the formula for the modulus of quasicentral approximation). Thus our aim here will be to extend one of the main results of [3] showing that certain weak limits for the quantities which are considered in order to get wave operators, give rise to intertwiners with vanishing kernels and kernels of adjoints. In essence the extension is not far from the earlier result in [3], however the argument in [3] is already rather intricate and having also to make a few technical improvements we felt the reader may not be too happy to get to fill in all these details as one of the so-called "exercises left to the reader". So we opted for a more detailed presentation of the proofs.

This paper has two more sections besides the introduction and references. Section 2 gives the hybrid existence result for weak wave operators. Section 3 is devoted to consequences, especially the invariance of the n-dimensional Lebesgue absolutely continuous parts.

This paper being the second one on hybrid normed ideal perturbations we use consistently the notation and definitions introduced in the first paper of the series.

2 Existence of Generalized Wave Operators

The hybrid setting which we will use in this section involves a separable C^*-algebra $\mathcal{A}$, $1 \in \mathcal{B} \subset \mathcal{A}$ a dense $*$-subalgebra with a countable basis as a vector space and $1 \in \mathcal{B}_k \subset \mathcal{B}$, $1 \leq k \leq n$, $*$-subalgebras of $\mathcal{B}$, so that $\mathcal{B}$ is generated by $\cup_{1\leq k\leq n}\mathcal{B}_k$ as an algebra. Let also $\varphi \in \mathcal{F}([n])$. Here, we recall, $[n]$ is the set $\{1, 2, \ldots, n\}$ and $\mathcal{F}$ the set of norming functions for normed ideals, so that φ is an n-tuple of norming functions indexed by $[n]$.

If ρ is a non-degenerate $*$-representation of $\mathcal{A}$ on $\mathcal{H}$, we define the φ-singular and φ-absolutely continuous projections $E^0_\varphi(\rho)$ and $E_\varphi(\rho)$ and the corresponding subspace $\mathcal{H}^0_\varphi(\rho)$ and $\mathcal{H}_\varphi(\rho)$ as follows. We consider all p-tuples, $p \in \mathbb{N}$, τ of operators in $\tilde{\beta} = \amalg_{1\leq j\leq n}\mathcal{B}_j$ and denote this by $\tau \subset \tilde{\beta}$ (all $p \in \mathbb{N}$ are considered). Then we form

$$E^0_\varphi(\rho) = \bigwedge_{\tau\subset\tilde{\beta}} E^0_{\varphi_\tau}(\rho(\tau))$$

where $\varphi_\tau(h) = \varphi(j)$ if $\tau(h) \in \mathcal{B}_j, 1 \leq h \leq p, 1 \leq j \leq n$ (concerning $E^0_\varphi(\rho(\tau))$, $E_\varphi(\rho(\tau))$ see the definitions in section 6 of [4]). Remark that

$$E^0_{\varphi_{\tau_1\amalg\tau_2}}(\rho(\tau_1 \amalg \tau_2)) \subset E^0_{\varphi_{\tau_1}}(\rho(\tau_1)) \wedge E^0_{\varphi_{\tau_2}}(\rho(\tau_2))$$

so that for $E_{\varphi_\tau}(\rho(\tau)) = I - E^0_{\varphi_\tau}(\rho(\tau))$ we have

$$E_{\varphi_{\tau_1 \amalg \tau_2}}(\rho(\tau_1 \amalg \tau_2)) \supset E_{\varphi_{\tau_1}}(\rho(\tau_1)) \vee E_{\varphi_{\tau_2}}(\rho(\tau_2)).$$

We also define $E_\varphi(\rho) = I - E^0_\varphi(\rho)$ so that

$$E_\varphi(\rho) = \bigvee_{\tau \subset \tilde{\beta}} E_\varphi(\rho(\tau)).$$

We shall also use the notation $\mathcal{H}^0_\varphi(\rho) = E^0_\varphi(\rho)\mathcal{H}$, $\mathcal{H}_\varphi(\rho) = E_\varphi(\rho)\mathcal{H}$.

It is easy to infer the following extension of Proposition 6.1 [4].

Proposition 2.1 *If $A_m = A_m^* \in \mathcal{K}$, $m \in \mathbb{N}$ are so that*

$$\sup_{m \in \mathbb{N}} \|A_m\| < \infty \text{ and}$$

$$\lim_{m \to \infty} |[\rho(b), A_m]|_{\varphi(j)} = 0$$

when $b \in \mathcal{B}_j$, $1 \le j \le m$, then we have

$$s - \lim_{m \to \infty} A_m E_\varphi(\rho) = 0.$$

Moreover $\mathcal{H}^0_\varphi(\rho)$ and $\mathcal{H}_\varphi(\rho)$ are $\rho(\mathcal{A})$-invariant and the restrictions $\rho \mid \mathcal{H}^0_\varphi(\rho)$ and $\rho \mid \mathcal{H}_\varphi(\rho)$ are disjoint representations of $\mathcal{A}$.

Proof For all p-tuples $\tau \subset \tilde{\beta}$ by Prop. 6.1 [4] we have that

$$s - \lim_{m \to \infty} A_m E_\varphi(\rho(\tau)) = 0.$$

Since the union of the $\mathcal{H}_{\varphi(\tau)}(\rho(\tau)) = E_{\varphi(\tau)}(\rho(\tau))\mathcal{H}$ is dense in $\mathcal{H}_\varphi(\rho)$ we get that

$$s - \lim_{m \to \infty} A_m E_\varphi(\rho) = 0.$$

Clearly $\mathcal{H}^0_{\varphi_\tau}(\rho(\tau)) = E^0_{\varphi_\tau}(\rho(\tau))\mathcal{H}$ is invariant under $\rho(\tau)$ and hence their intersection over all $\tau \subset \tilde{\beta}$, $\mathcal{H}^0_\varphi(\rho) = E^0_\varphi(\rho)\mathcal{H}$ is invariant under $\rho(\tilde{\beta})$, that is under $\rho(\mathcal{A})$. Also, since there are no non-zero $\rho(\tau)$-intertwiners between $E^0_{\varphi_\tau}(\rho(\tau))\mathcal{H}$ and $E_{\varphi_\tau}(\rho(\tau))\mathcal{H}$, there are no non-zero $\rho(\tau)$-intertwiners between $E^0_\varphi(\rho)\mathcal{H}$ and $E_{\varphi_\tau}(\rho(\tau))\mathcal{H}$. It follows also that there are no non-zero $\rho(\tilde{\beta})$-intertwiners between $E^0_\varphi(\rho)\mathcal{H}$ and $E_\varphi(\rho)\mathcal{H}$. □

One of the facts which will be used in the proof of the main result of this section, a theorem which improves and extends Theorem 1.4 of [3], is a fact also used in the proof of the earlier result. If $\mathcal{G}_\Phi^{(0)} \neq \mathcal{C}_1$ then for every $X \in \mathcal{G}_\Phi^{(0)}$ we have

$$\lim_{j\to\infty} 1/j\ |\underbrace{X \oplus \cdots \oplus X}_{j\text{-times}}|_\Phi = 0.$$

When X is a rank one orthogonal projection this is due to Kuroda (see [1, ch. X, §2, the proof of Theorem 2.3]). For general $X \in \mathcal{G}_\Phi^{(0)}$ this then follows immediately from the fact that rank one projections are total in $\mathcal{G}_\Phi^{(0)}$.

Theorem 2.1 *Let φ be such that $\varphi(j) \neq \Phi_1$, $1 \leq j \leq n$ and let ρ_1, ρ_2 be unital $*$-representations of $\mathcal{A}$ on $\mathcal{H}$ such that $\rho_1(b) - \rho_2(b) \in \mathcal{G}_{\varphi(j)}^{(0)}$ if $b \in \mathcal{B}_j$, $1 \leq j \leq n$. Assume moreover that there is a sequence of unitary elements $u_m \in Z(\mathcal{A})$, $m \in \mathbb{N}$, where $Z(\mathcal{A})$ is the center of $\mathcal{A}$, such that*

$$w - \lim_{m\to\infty} \rho_1(u_m) = w - \lim_{m\to\infty} \rho_2(u_m) = 0$$

and that the weak limit

$$W = w - \lim_{m\to\infty} \rho_2(u_m^*)\rho_1(u_m)E_\varphi(\rho_1)$$

exists. Then W intertwines ρ_1 and ρ_2 and Ker $W = E_\varphi^0(\rho_1)$, Ker $W^* = E_\varphi^0(\rho_2)$. *Moreover we have*

$$W^* = w - \lim_{m\to\infty} \rho_1(u_m^*)\rho_2(u_m)E_\varphi(\rho_2)$$

and the representations $\rho_1|\mathcal{H}_\varphi(\rho_1)$, $\rho_2|\mathcal{H}_\varphi(\rho_2)$ of $\mathcal{A}$, are unitarily equivalent.

Proof Since the $\mathcal{B}_j$'s generate $\mathcal{B}$, we have $\rho_2(b) - \rho_1(b) \in \mathcal{K}$ for all $b \in \mathcal{B}$ and hence since $\mathcal{B}$ is dense in $\mathcal{A}$ it follows that $\rho_1(a) - \rho_2(a) \in \mathcal{K}$ for all $a \in \mathcal{A}$. If $a \in \mathcal{A}$ we have

$$W\rho_1(a) - \rho_2(a)W = w - \lim_{m\to\infty} \rho_2(u_m^*)(\rho_1(a) - \rho_2(a))\rho_1(u_m)E_\varphi(\rho_1)$$

and since $\rho_1(a) - \rho_2(a) \in \mathcal{K}$ and $w - \lim_{m\to\infty} \rho_1(u_m) = 0$ we infer that

$$s - \lim_{m\to\infty} \rho_2(u_m^*)(\rho_1(a) - \rho_2(a))\rho_1(u_m)E_\varphi(\rho_1) = 0$$

and hence that $W\rho_1(a) = \rho_2(a)W$. This implies that $\rho_2 \mid \overline{W\mathcal{H}}$ is unitarily equivalent to a subrepresentation of $\rho_1 \mid \overline{W^*\mathcal{H}}$ and since $W = WE_\varphi(\rho_1)$ this is a subrepresentation of $\rho_1 \mid \mathcal{H}_\varphi(\rho_1)$. We then must have $\overline{W\mathcal{H}} \perp \mathcal{H}_\varphi^0(\rho_2)$, that is $\overline{W\mathcal{H}} \subset \mathcal{H}_\varphi(\rho_2)$ or equivalently $W = E_\varphi(\rho_2)W$. Let $\tilde{W} = \mathcal{H}_\varphi(\rho_2) \mid W \mid \mathcal{H}_\varphi(\rho_1)$, that is the operator from $\mathcal{H}_\varphi(\rho_1)$ to $\mathcal{H}_\varphi(\rho_2)$ one gets from W. The main fact to be proved will be that Ker $\tilde{W} = 0$ and Ker $\tilde{W}^* = 0$. Before taking up this task, we shall prove a certain symmetry between W and W^*. More precisely the symmetry is between ρ_1, ρ_2, u_m, W and ρ_2, ρ_1, u_m, W^*. For this, we must show that the weak limit

$$V = w - \lim_{m\to\infty} \rho_1(u_m^*)\rho_2(u_m)E_\varphi(\rho_2)$$

exists and that $V = W^*$. Without assuming the existence of this weak limit, we can pass to a subsequence so that the weak limit defining V exists and it will suffice to show that $V = W^*$ in this case, since then the operator V we get will not depend on the chosen subsequence. Repeating for V the argument with which we began the proof of the theorem, we find that V intertwines ρ_2 and ρ_1 and hence that $E_\varphi(\rho_1)VE_\varphi(\rho_2) = V$. But then $V = w - \lim_{m\to\infty} E_\varphi(\rho_1)\rho_1(u_m^*)\rho_2(u_m)E_\varphi(\rho_2)$ and since $W = w - \lim_{m\to\infty} E_\varphi(\rho_2)\rho_2(u_m^*)\rho_1(u_m)E_\varphi(\rho)$ it follows that $V = W^*$.

To prove that Ker $\tilde{W} = 0$ and Ker $\tilde{W}^* = 0$, we shall assume the contrary and show that this leads to a contradiction. Let P, Q be the orthogonal projections onto Ker $\tilde{W}$ and Ker $\tilde{W}^*$ respectively and remark that

$$P \in ((\rho_1 \mid \mathcal{H}_\varphi(\rho_1))(\mathcal{A}))',\ Q \in ((\rho_2 \mid \mathcal{H}_\varphi(\rho_2)))(\mathcal{A}))'.$$

In view of the symmetry we can assume that $P \neq 0$. Note also that rank P must be infinite since otherwise $P \leq E_\varphi^0(\rho_1)$.

The assumption $P \neq 0$ means there is $\xi \in \mathcal{H}_\varphi(\rho_1)$, $\|\xi\| = 1$ so that

$$w - \lim_{m\to\infty} W_m\xi = 0$$

where $W_m = \rho_2(u_m^*)\rho_1(u_m)$. Replacing the u_m's by a subsequence we may assume $p \neq q \Rightarrow \|W_p\xi - W_q\xi\| > 1$. Let then $A_k = W_{m_{k+1}}^* W_{m_k} - I$ for a sequence $m_1 < m_2 < \ldots$ which we shall define recurrently. Since P has infinite rank let $(\xi_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $P\mathcal{H}$ so that $\xi_1 = \xi$. Let further β_j be a basis of the vector space $\mathcal{B}_j$ and let $(b_r)_{r\in\mathbb{N}}$ be an enumeration of $\beta_1 \amalg \cdots \amalg \beta_n$. In particular there is a map $\gamma : \mathbb{N} \to [n]$ so that $b_r \in \beta_{\gamma(r)}$ and $\beta_j = \{b_r \mid r \in \gamma^{-1}(j)\}$ for $1 \leq j \leq n$. We take $m_1 = 1$. Suppose $m_1 < \cdots < m_k$ have been chosen. Then we can find $m_{k+1} > m_k$ so that

$$\|(\rho_1(b_l) - \rho_2(b_l))\rho_2(u_{m_{k+1}})\rho_2(u_{m_k}^*)\rho_1(u_{m_k})\xi_i\| < 1/k$$

$$\|(\rho_1(b_l) - \rho_2(b_l))\rho_2(u_{m_k})\rho_2(u_{m_{k+1}}^*)\rho_1(u_{m_{k+1}})\xi_i\| < 1/k$$

for $1 \leq i, l \leq k+1$. This is indeed possible because $\rho_1(b_l) - \rho_2(b_l) \in \mathcal{K}$ and

$$w - \lim_{m\to\infty} \rho_2(u_m) = 0,\ w - \lim_{m\to\infty} \rho_2(u_m^*)\rho_1(u_m)P = 0$$

which implies that

$$\lim_{m\to\infty} \|(\rho_1(b_l) - \rho_2(b_l))\rho_2(u_m)\rho_2(u^*_{m_k})\rho_1(u_{m_k})\xi_i\| = 0$$

$$\lim_{m\to\infty} \|(\rho_1(b_l) - \rho_2(b_l))\rho_2(u_{m_k})\rho_2(u^*_m)\rho_1(u_m)\xi_i\| = 0$$

for all $i, l \in \mathbb{N}$.

For the above choice of the sequence $m_1 < m_2 < \dots$ we shall prove that

$$s - \lim_{k\to\infty} [PA_kP, \rho_1(b)] = 0 \text{ and}$$

$$s - \lim_{k\to\infty} [PA^*_kP, \rho_1(b)] = 0$$

for all $b \in \mathcal{B}$. Since $\beta_1 \cup \cdots \cup \beta_n$ generates $\mathcal{B}$ as an algebra it will suffice to prove this when $b \in \beta_1 \cup \cdots \cup \beta_n = \{b_r \mid r \in \mathbb{N}\}$.

We have $[PA_kP, \rho_1(b_r)] = P[A_k, \rho_1(b_r)]P$ and $[PA^*_kP, \rho_1(b_r)] = P[A^*_k, \rho_1(b_r)]P$ so that it will suffice to show that for all $i, r \in \mathbb{N}$ we have

$$\lim_{k\to\infty} \|[A_k, \rho_1(b_r)]\xi_i\| = 0$$

$$\lim_{k\to\infty} \|[A^*_k, \rho_1(b_r)]\xi_i\| = 0.$$

We have

$$\begin{aligned}
[A_k, \rho_1(b_r)] &= [W^*_{m_{k+1}} W_{m_k}, \rho_1(b_r)] \\
&= [\rho_1(u^*_{m_{k+1}})\rho_2(u_{m_{k+1}})\rho_2(u^*_{m_k})\rho_1(u_{m_k}), \rho_1(b_r)] \\
&= \rho_1(u^*_{m_{k+1}})\rho_2(u_{m_{k+1}}u^*_{m_k})(\rho_1(b_r) - \rho_2(b_r))\rho_1(u_{m_k}) \\
&\quad - \rho_1(u^*_{m_{k+1}})(\rho_1(b_r) - \rho_2(b_r))\rho_2(u_{m_{k+1}})\rho_2(u^*_{m_k})\rho_1(u_{m_k}).
\end{aligned}$$

This gives

$$\begin{aligned}
&\limsup_{k\to\infty} \|[A_k, \rho_1(b_r)]\xi_i\| \\
&\quad \le \limsup_{k\to\infty} (\|(\rho_1(b_r) - \rho_2(b_r))\rho_1(u_{m_k})\xi_i\| \\
&\quad + \|(\rho_1(b_r) - \rho_2(b_r))\rho_2(u_{m_{k+1}})\rho_2(u^*_{m_k})\rho_1(u_{m_k})\xi_i\|) = 0
\end{aligned}$$

because $\rho_1(b_r) - \rho_2(b_r) \in \mathcal{K}$, $w - \lim_{k\to\infty} \rho_1(u_{m_k}) = 0$ and because of the choice of m_k's we made. Similarly we have

$$\begin{aligned}
[A^*_k, \rho_1(b_r)] &= \rho_1(u^*_{m_k})\rho_2(u_{m_k}u^*_{m_{k+1}})(\rho_1(b_r) - \rho_2(b_r))\rho_1(u_{m_{k+1}}) \\
&\quad - \rho_1(u^*_{m_k})(\rho_1(b_r) - \rho_2(b_r))\rho_2(u_{m_k})\rho_2(u^*_{m_{k+1}})\rho_1(u_{m_{k+1}})
\end{aligned}$$

so that

$$\begin{aligned}\limsup_{k\to\infty}\|[A_k^*, \rho_1(b_r)]\xi_i\| &\le \limsup_{k\to\infty}(\|(\rho_1(b_r) - \rho_2(b_r))\rho_1(u_{m_{k+1}})\xi_i\| \\ &\quad + \|(\rho_1(b_r) - \rho_2(b_r))\rho_2(u_{m_k})\rho_2(u_{m_{k+1}}^*)\rho_1(u_{m_{k+1}})\xi_1\|) \\ &= 0\end{aligned}$$

again because $\rho_1(b_r) - \rho_2(b_r) \in \mathcal{K}$, $w - \lim_{k\to\infty} \rho_1(u_{m_{k+1}}) = 0$ and of the choice of the m_k's.

Remark now that $A_k + I$ being unitary we have

$$A_k^* A_k = A_k A_k^* = -A_k - A_k^*.$$

This then gives

$$s - \lim_{k\to\infty}[P A_k^* A_k P, \rho_1(b_r)] = s - \lim_{k\to\infty}[\rho_1(b_r), P A_k P + P A_k^* P] = 0$$

for all $r \in \mathbb{N}$. Since $\mathcal{B}$ is self-adjoint this also gives

$$s - \lim_{k\to\infty}([P A_k^* A_k P, \rho_1(b_r)])^* = 0$$

for all $r \in \mathbb{N}$. Further, since $u_n \in \mathcal{A}$, we have $\rho_1(u_n) - \rho_2(u_n) \in \mathcal{K}$ so that $W_n \in I + \mathcal{K}$ and $A_k \in \mathcal{K}, k \in \mathbb{N}$. The computations of $[A_k, \rho_1(b_r)]$ and $[A_k^*, \rho_1(b_r)]$ we did earlier in this proof, show that there are unitary operators $V_k, V_k', V_k'', V_k''', \widetilde{V}_k, \widetilde{V}_k', \widetilde{V}_k'', \widetilde{V}_k'''$ so that

$$[\rho_1(b_r), A_k] = V_k(\rho_1(b_r) - \rho_2(b_r))V_k' + V_k''(\rho_1(b_r) - \rho_2(b_r))V_k'''$$

and

$$[\rho_1(b_r), A_k^*] = \widetilde{V}_k(\rho_1(b_r) - \rho_2(b_r))\widetilde{V}_k' + \widetilde{V}_k''(\rho_1(b_r) - \rho_2(b_r))\widetilde{V}_k'''.$$

It follows that

$$\begin{aligned}[P A_k^* A_k P, \rho_1(b_r)] &= P[\rho_1(b_r), A_k + A_k^*]P \\ &= P(V_k(\rho_1(b_r) - \rho_2(b_r))V_k' + V_k''(\rho_1(b_r) - \rho_2(b_r))V_k''' \\ &\quad + \widetilde{V}_k'(\rho_1(b_r) - \rho_2(b_r))\widetilde{V}_k' + \widetilde{V}_k''(\rho_1(b_r) - \rho_2(b_r))\widetilde{V}_k''')P \\ &\in \mathcal{G}_{\rho(\gamma(r))}^{(0)}.\end{aligned}$$

On the other hand the $*$-strong convergence of $[PA_k^*A_kP, \rho_1(b_r)]$ to 0, easily gives that there are $k_1 < k_2 < \dots$ so that

$$\lim_{j\to\infty} \Big| j^{-1}|[PA_{k_1}^*A_{k_1}P, \rho_1(b_r)] + \cdots + [PA_{k_j}^*A_{k_j}P, \rho_1(b_r)]|_{\varphi(\gamma(r))}$$
$$- j^{-1}\ |([PA_{k_1}^*A_{k_1}P, \rho_1(b_r)]) \oplus \cdots \oplus ([PA_{k_j}^*A_{k_j}P, \rho_1(b_r)])|_{\varphi(\gamma(r))}\Big|$$
$$= 0$$

for all $r \in \mathbb{N}$.

In view of the result of the computation of $[PA_k^*A_kP, \rho_1(b_r)]$ we have

$$|([PA_{k_1}^*A_{k_1}P_1, \rho_1(b_r)]) \oplus \cdots \oplus ([PA_{k_j}^*A_{k_j}P, \rho_1(b_r)])|_{\varphi(\gamma(r))}$$
$$\leq 4|\underbrace{(\rho_1(b_r) - \rho_2(b_r)) \oplus \cdots \oplus (\rho_1(b_r) - \rho_2(b_r))}_{j\text{-times}}|_{\varphi(\gamma(r))}.$$

Since $\mathcal{G}_{\varphi(\gamma(r))}^{(0)} \neq \mathcal{C}_1$ we have

$$\lim_{j\to\infty} j^{-1}|\underbrace{(\rho_1(b_r) - \rho_2(b_r)) \oplus \cdots \oplus (\rho_1(b_r) - \rho_2(b_r))}_{j\text{-times}}|_{\varphi(\gamma(r))} = 0.$$

Hence, if $B_j = j^{-1}(PA_{k_1}^*A_{k_1}P + \cdots + PA_{k_j}^*A_{k_j}P)$ we have

$$\lim_{j\to\infty} |[B_j, \rho_1(b_r)]|_{\varphi(\gamma(r))} = 0$$

for all $r \in \mathbb{N}$, which then implies

$$\lim_{j\to\infty} |[B_j, \rho_1(b)]|_{\varphi(\ell)} = 0$$

for all $b \in \mathcal{B}_\ell$.

Since $\|B_j\| \leq 4$, $B_j \in \mathcal{K}$ and $0 \leq B_j \leq P \leq E_\varphi(\rho_1)$ it follows from Proposition 2.1 that

$$s - \lim_{j\to\infty} B_j = 0.$$

Recall now that $\xi_1 = \xi \in P\mathcal{H}_\varphi(\rho_1)$ had the property that $p \neq q \Rightarrow \|W_p\xi - W_q\xi\| > 1$ which implies $\|A_kP\xi\| = \|A_k\xi\| = \|W_{m_{k+1}}^*W_{m_k}\xi - \xi\| > 1$ or equivalently $\langle PA_k^*A_kP\xi, \xi\rangle > 1$ for all $k \in \mathbb{N}$. This in turn implies $\langle B_j\xi, \xi\rangle > 1$, for all $j \in \mathbb{N}$ which is a contradiction. □

In the statement of Theorem 2.1 if we leave out the assumption that the weak limit

$$w - \lim_{m\to\infty} \rho_2(u_m^*)\rho_1(u_m)E_\varphi(\rho_1)$$

exists, it is always possible to find a subsequence of the u_m's for which this weak limit exists and draw the conclusion that $\rho_1 \mid E_\varphi(\rho_1)$ and $\rho_2 \mid E_\varphi(\rho_2)$ are unitarily equivalent. Thus we have the following corollary.

Corollary 2.1 *Let φ be such that $\varphi(j) \neq \Phi_1, 1 \le j \le n$ and let ρ_1, ρ_2 be unital $*$-representations of $\mathcal{A}$ and $\mathcal{H}$ such that $\rho_1(b) - \rho_2(b) \in \mathcal{G}^{(0)}_{\varphi(j)}$ if $b \in \mathcal{B}_j, 1 \le j \le n$. Assume moreover that there is a sequence of unitary elements $u_m \in Z(\mathcal{A}), m \in \mathbb{N}$, where $Z(\mathcal{A})$ is the center of $\mathcal{A}$, such that*

$$w - \lim_{m\to\infty} \rho_1(u_m) = w - \lim_{m\to\infty} \rho_2(u_m) = 0.$$

Then the representations $\rho_1 \mid \mathcal{H}_\varphi(\rho_1)$ and $\rho_2 \mid \mathcal{H}_\varphi(\rho_2)$ of $\mathcal{A}$ are unitarily equivalent.

3 Invariance of Lebesgue Absolutely Continuous Parts Under Perturbations

Theorem 3.1 *Let $\varphi \in \mathcal{F}([n]), \varphi(j) = \Phi^-_{p_j}, p_j > 1, 1 \le j \le n, n > 1$ be so that $p_1^{-1} + \cdots + p_n^{-1} = 1$. Let τ and τ' be two n-tuples of commuting hermitian operators on $\mathcal{H}$ so that $\tau(j) - \tau'(j) \in \mathcal{C}^-_{p_j}, 1 \le j \le n$. Then the Lebesgue absolutely continuous parts τ_{ac} and τ'_{ac} of τ and τ', are unitarily equivalent.*

Proof Consider the decompositions $\tau = \tau_{ac} \oplus \tau_s, \tau' = \tau'_{ac} \oplus \tau'_s$ with respect to n-dimensional Lebesgue measure and let $L > 0$ be such that $[-L, L]^n \supset \sigma(\tau) \cup \sigma(\tau')$. Recall also that by section 10 of [4] these decompositions coincide with those into φ-singular and φ-absolutely continuous subspaces, in particular we have $k_\varphi(\tau_s) = k_\varphi(\tau'_s) = 0$. Consider also δ and n-tuple of multiplication operators by the coordinate functions in $L^2([-L, L]^n, d\lambda)$, where λ is Lebesgue measure. If $\mathcal{A} = C([-L, L]^n)$ is the C^*-algebra of continuous functions and $\mathcal{B} \subset \mathcal{A}$, the subalgebra of polynomial functions, with generator $\beta = (b_1, \ldots, b_n)$ the n coordinate functions we may form the representations of $\mathcal{A}$ arising from functional calculus. Using Theorem 5.1 [4] the adaptation of our non-commutative Weyl–von Neumann type theorem, we find that $\tau_s \oplus \delta$ is unitarily equivalent mod the hybrid n-tuple $\mathcal{G}^{(0)}_\varphi$ with δ and similarly $\tau'_s \oplus \delta$ is also unitarily equivalent with δ mod $\mathcal{G}^{(0)}_\varphi$. This implies the existence of a unitary operator U so that $\tau_{ac} \oplus \delta - U(\tau'_{ac} \oplus \delta)U^*$ is in $\mathcal{G}^{(0)}_\varphi$. Let ρ_1 and ρ_2 be the representations of $\mathcal{A}$ defined by $f \to f(\tau_{ac} \oplus \delta)$ and $f \to f(U(\tau'_{ac} \oplus \delta)U^*)$. Denoting by $\mathcal{B}_j$ the subalgebra of $\mathcal{B}$ consisting of polynomials in the j-th coordinate function we will have $\rho_1(b) - \rho_2(b) \in \mathcal{G}^{(0)}_{\varphi(j)} = \mathcal{C}^-_{p_j}$ if $b \in \mathcal{B}_j$. Let further $u_m \in \mathcal{A}$ be the function $u_m(x_1, \dots, x_n) = \exp(imx_1)$. Since the spectral measures of $\tau_{ac} \oplus \delta$ and $U(\tau'_{ac} \oplus \delta)U^*$ are absolutely continuous with respect to Lebesgue measure it is easily seen that $w - \lim_{m\to\infty} \rho_1(u_m) = w - \lim_{m\to\infty} \rho_2(u_m) = 0$. Thus the assumptions of Corollary 2.1 are satisfied and we get that ρ_1 and ρ_2 are unitarily equivalent (the singular parts being zero). This is in turn the same as the unitary equivalence of $\tau_{ac} \oplus \delta$ and $U(\tau'_{ac} \oplus \delta)U^*$ or $\tau'_{ac} \oplus \delta$. If m_{ac} and m'_{ac} are the multiplicity functions of τ_{ac} and τ'_{ac} we have proved that $m_{ac} + \chi_{[-L,L]^n}$ and $m'_{ac} + \chi_{[-L,L]^n}$ are equal almost everywhere with respect to Lebesgue measure. Clearly this implies $m_{ac} = m'_{ac}$ a.e. which is the unitary equivalence of τ_{ac} and τ'_{ac}. □

Acknowledgement Research supported in part by NSF Grant DMS-1665534.

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An Introduce to Curvature Inequalities for Operators in the Cowen–Douglas Class

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In Memory of Professor Ronald Douglas

Abstract We give a short survey of some recent results on the curvature inequalities for operators in the Cowen–Douglas class.

Keywords Cowen–Douglas class · Curvature inequality

Mathematics Subject Classification (2010) Primary 47B32; Secondary 47B35

1 Vector Bundle and Curvature Relevant to Operator

Let $\mathbf{T} = (T_1, \cdots, T_m)$ be a commuting tuple of operators on a separable infinite dimensional complex Hilbert space H, that is, $T_iT_j = T_jT_i, 1 \le i, j \le m$. We write $\mathbf{T} \in B_n(\Omega)$ over an open domain Ω in $\mathbb{C}^m$ if the following conditions are satisfied:

1. The operator $D_{\mathbf{T}-w} = (T_1 - w_1, \cdots, T_m - w_m)$, for all $w = (w_1, \cdots, w_m)$ in Ω, has closed range;
2. span$\{\ker D_{\mathbf{T}-w} = \cap_{i=1}^m \ker(T_i - w_i) : \ w \in \Omega\}$ is dense in H;
3. $\dim \cap_{i=1}^m \ker(T_i - w_i) = n$ for all $w \in \Omega$.

These operators, called Cowen–Douglas operators, were introduced in [5, 6] and surprising connection with complex geometry was established.

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_25

Associated to each $\mathbf{T} \in B_n(\Omega)$ is a family of eigenspaces $\ker(\mathbf{T} - w)$, $w \in \Omega$ of dimension n, and it defines a vector bundle Ω

$$E_{\mathbf{T}} = \{(w, x) :\ w \in \Omega,\ x \in E_{\mathbf{T}}(w) := \bigcap_i \ker(T_i - w_i)\}.$$

Note that $E_{\mathbf{T}}$, as a sub-bundle of the trivial bundle $\Omega \otimes H$, has a natural Hermitian holomorphic structure. Cowen and Doulgas proved the following very deep theorem.

Theorem 1.1 (Cowen–Douglas) *The operator tuples* $\mathbf{T}$ *and* $\tilde{\mathbf{T}}$ *in* $B_n(\Omega)$ *are unitarily equivalent if and only if the corresponding Hermitian bundle* $E_{\mathbf{T}}$ *and* $E_{\tilde{\mathbf{T}}}$ *are equivalent.*

This theorem provides a bridge between the study of unitary invariant classes of $\mathbf{T}$ in $B_n(\Omega)$ with the complex geometry of the Hermitian bundles $E_{\mathbf{T}}$ leading to new methods for the unitary classifications of operators in $B_n(\Omega)$. In complex geometry, a significant tool in the study of Hermitian bundles is the curvature. Let E be a Hermitian vector bundles of rank n over the domain Ω. For a given point z, choose a holomorphic frame $\gamma_1, \cdots, \gamma_n$ nearby z. Then the curvature matrix is given by the formula

$$\mathcal{K}_{\mathbf{T}}(z) =: -\overline{\partial}(G_\gamma^{-1}\partial G_\gamma),$$

where G_γ is Gram matrix whose (i, j) component $G_\gamma(i, j) = \langle \gamma_j, \gamma_i \rangle$. In case of $\Omega \subseteq \mathbb{C}$ and $T \in B_1(\Omega)$, we have, in particular,

$$\mathcal{K}_{\mathbf{T}}(w) = -\partial_w \overline{\partial}_w \log \|\gamma_1(w)\|^2.$$

Since the curvature is a complete invariant for a line bundle, we obtain the following theorem:

Theorem 1.2 (Cowen–Douglas) *For* $\mathbf{T} \in B_1(\Omega)$*, the curvature* $\mathcal{K}_{\mathbf{T}}$ *is a complete invariant for the unitarily equivalence of* $\mathbf{T}$.

The fundamental work of Cowen and Douglas has inspired sustained research to find operator theoretic properties for the tuple $\mathbf{T}$ from the geometry of $E_{\mathbf{T}}$ and its curvature $\mathcal{K}_{\mathbf{T}}$. Specifically, much progress has been made on the similarly and unitary equivalence problems of operators in the Cowen–Douglas class. We refer the reader to a comprehensive references [13].

In a recent paper [20], refining the construction of Cowen–Douglas, Zhang and the author of this paper, associate a class of holomorphic Hermitian vector bundles $E^l_{\mathbf{T}}$ to commuting tuples of operators $\mathbf{T}$ in $B_n(\Omega)$. Here $E^l_{\mathbf{T}}$ is the vector bundle of rank l defined by

$$E^l_{\mathbf{T}} = \{(w, x) \in \Omega \times H : x \in \bigcap_{|\alpha|=l} \ker(\mathbf{T} - w)^\alpha\}$$

with the projection $\pi(w, x) = w$, where α is a multi-index and

$$(\mathbf{T} - w)^{\alpha} = (T_1 - w_1)^{\alpha_1} \cdots (T_m - w_m)^{\alpha_m}.$$

As above, $E^l_{\mathbf{T}}$ is also a Hermitian sub-bundle of $\Omega \times H$. Obviously, the vector bundle $E^l_{\mathbf{T}}$ and the curvature are also invariants for the operator tuple $\mathbf{T}$.

We introduce an equivalent definition of the curvature. Consider $E^l_{\mathbf{T}}$ as a sub-bundle of the trivial bundle $\Omega \times H$. Then we have that the $D = P\partial$ is the canonical Chern connection on $E^l_{\mathbf{T}}$, where P is the fiberwise orthogonal projection from $\Omega \times H$ onto the fiber $E^1_{\mathbf{T}}$ and ∂ is the ordinary derivative. Therefore, for any holomorphic section u in $E^l_{\mathbf{T}}$, the curvature tensor $R_{\mathbf{T}}$ satisfies that

$$(R_{\mathbf{T}}(\partial_i, \overline{\partial}_j)u = -\overline{\partial}_j P \partial_i u.$$

It is well known that two definitions for curvature coincide [3]. Moreover, a direct computation show that if u is a holomorphic section of $E^{l-1}_{\mathbf{T}}$, which is a sub-bundle of $E^l_{\mathbf{T}}$, then

$$R_{\mathbf{T}}(\partial_i, \overline{\partial}_j)u = 0$$

since ∂u is a section of $E^l_{\mathbf{T}}$.

Let $W^l(w)$ be the orthogonal complement of $E^{l-1}_{\mathbf{T}}(w)$ in $E^l_{\mathbf{T}}(w)$ with respect to the Hermitian inner product inherited from H, that is, $E^l_{\mathbf{T}}(w) = E^{l-1}_{\mathbf{T}}(w) \oplus W^l(w)$. We have the following:

Theorem 1.3 (Wang-Zhang) *On the Hermitian vector bundle $E^l_{\mathbf{T}}$ with $l > 1$ induced from $\mathbf{T} \in B_n(\Omega)$, we have the decomposition*

$$R_{\mathbf{T},w}(\partial_i, \overline{\partial}_j) = \begin{pmatrix} 0 & 0 \\ 0 & R^l_{\mathbf{T},w}(\partial_i, \overline{\partial}_j) \end{pmatrix}$$

with respect to $E^l_{\mathbf{T}}(w) = E^{l-1}_{\mathbf{T}}(w) \oplus W^l(w)$.

Note $R^l_{\mathbf{T},w}(\partial_i, \overline{\partial}_j)$ is also a unitary invariant for the Cowen–Douglas class $B_n(\Omega)$. We will see that they contain some new information for Cowen–Douglas class.

2 Curvature Inequities for Cowen–Douglas Operator

The Cowen–Douglas theory shows how the properties of the operator $\mathbf{T}$ in $B_n(\Omega)$ are completely determined by the corresponding vector bundle $E_{\mathbf{T}}$. However, it seems an intractable problem to obtain properties of $\mathbf{T}$ from the curvature alone. We discuss below what is known.

2.1 Curvature Inequalities of Order 1

In [11] Misra started to attack this type problem and investigate the relation of operator inequalities versus curvature inequalities. He first considered the case of $T \in B_1(\Omega)$ on a planar domain $\Omega \subseteq \mathbb{C}$.

Recall that the closed set $cl\Omega$ is said to be a spectral set for T if $cl\Omega \supseteq \sigma(T)$ and $\|f(T)\| \leq \|f\|_\infty$ for all $f \in Rat(cl\Omega)$.

Theorem 2.1 (Misra) *If $T \in B_1(\Omega)$ and $cl\Omega$ is a spectral set of T, then*

$$\mathcal{K}_T(w) \leq -\sup\{|f'(w)|^2 : f \in Hol_w(\Omega, D)\}.$$

where $Hol_w(\Omega, D) = \{f : \Omega \to \mathbb{D}\ analytic,\ f(w) = 0\}$.

The main idea of the proof is that the local operator $N_w = T|_{\ker(T-w)^2}$ is a 2×2 matrix and $cl\Omega$ is also a spectral set for it. Let $H^2(\partial\Omega, dm_w(z))$ be the Hardy space with respect to the measure $dm_w(z)$, where $dm_w(z)$ be the harmonic measure for the point $w \in \Omega$. Then the supremum on the right hand side of the curvature inequality equals $-\hat{K}_\Omega(w, \bar{w})^2$, where $\hat{K}_\Omega(w, \bar{w})$ is the reproducing kernel at point w for $H^2(\partial\Omega, dm_w)$.

In the special case of $\Omega = \mathbb{D}$, by the von Neumann inequality, $cl\mathbb{D}$ is a spectral set for an operator T if and only if T is contractive, that is, $\|T\| \leq 1$. Recall that $\mathcal{K}_{S^*}(z) = -K_{\mathbb{D}}(w, \bar{w}) = -\frac{1}{(1-|w|^2)^2}$, where S is the Hardy shift on the classical Hardy space $H^2(\mathbb{D})$.

Theorem 2.2 (Misra) *For $T \in B_1(\mathbb{D})$, if $\|T\| \leq 1$, then*

$$K_T(w) \leq K_{S^*}(w) = -\frac{1}{(1-|w|^2)^2}.$$

This result provides one way to check whether an operator is contractive or not from a numerical inequality involving the curvature function on $\mathbb{D}$.

Uchiyama [19] extended the above result to the case of $B_n(\Omega)$ for a planar domain Ω. He observed the connection between the inequality and the Sz.-Nagy and Foias dilation theory[18], and prove that

Theorem 2.3 (Uchiyama) *If a contraction $T \in B_n(\Omega)$ for $\Omega \subset \mathbb{D}$, then*

$$\mathcal{K}_T(w) \leq -\frac{1}{(1-|w|^2)^2} I_n.$$

For a general planar domain $\Omega \subseteq \mathbb{C}$, using the Ahlfors map from Ω to the disc, he also prove that

Theorem 2.4 (Uchiyama) *Let Ω be a p-ply connected Jordan region, and $T \in B_n(\Omega)$. Suppose $cl\Omega$ is a spectral set of T. Then we have*

$$\mathcal{K}_T(w) \leq -\hat{K}(w, \bar{w})^2 I_n.$$

To describe the case of domains which are not necessarily planar, we use the Hilbert framework of Hilbert modules for $\mathbf{T} \in B_n(\Omega)$, which was introduced by Douglas and Paulsen[7].

Definition 2.5 (Douglas and Paulsen) The Hilbert space H is called a Hilbert module over an algebra A provided that H is equipped with a mapping $A \times H \to H$, which we denote $(f, h) \to f \cdot h$, satisfying: for $f, g \in A$, $h, k \in H$, $\alpha, \beta \in \mathbb{C}$.

(1) $1 \cdot h = h$,
(2) $(fg) \cdot h = f \cdot (g \cdot h)$,
(3) $(f + g) \cdot h = f \cdot h + g \cdot h$,
(4) $f \cdot (\alpha h + \beta k) = \alpha(f \cdot h) + \beta(f \cdot h)$,
(5) There exists $K_f (> 0)$ such that $\|f \cdot h\| \leq K_f \|f\| \|h\|$.

For a tuple operators $\mathbf{T}$ on a Hilbert space H, we endows H with a Hilbert module structure over the polynomial ring $\mathbb{C}[z_1, \cdots, z_n]$ by

$$p \cdot \xi = p(T_1, \cdots, T_n)\xi, \quad p \in \mathbb{C}[z_1, \cdots, z_n], \ \xi \in H.$$

In such a framework, the central theme is to understand the structure of the Hilbert module. For example, we call a Hilbert module H contractive if for any polynomial p, $\|p(T)\| \leq \|p\|_\Omega := \sup\{|p(z)| : z \in \Omega\}$.

In [15], Misra and Sastry proved the following inequality.

Theorem 2.6 (Misra and Sastry) *Suppose $H_\mathbf{T}$ is a contractive Hilbert module determined by $\mathbf{T} \in B_1(\Omega)$. For a fixed but arbitrary point $w \in \Omega$, let θ_w is a bi-holomorphic automorphism on Ω such that $\theta_w(w) = 0$. Then*

$$\|D\theta_w(w) K_\mathbf{T}(w)^{-1} \overline{D\theta_w(w)}^t\|_{\Omega^*}^{\Omega} \leq 1.$$

For the case of domains like the Euclidean ball, the polydisc and bounded symmetric domains, we refer the reader to the careful computation [2, 12–14, 16]. Another curvature inequality involving the Caratheodory metric appeared in [17] by Misra and Reza. The technique is also heavily depended on the local operator.

In the recent past, Misra and his collaborators obtained the curvature inequality without using the local operators but directly using the metric of the holomorphic vector bundle[4, 8]. This also inspired us to consider curvature inequalities of higher order.

2.2 Curvature Inequalities of Higher Order

We first recall the definition of curvature negative in complex geometry. The vector bundle is said to be positive in the sense of Griffiths if for any section u and vector $v \in \mathbb{C}^m$ with $u \neq 0$ and $v \neq 0$,

$$\sum (R(\partial_i, \overline{\partial}_j)u, u)v_i\overline{v}_j > 0.$$

It is said to be positive in the sense of Nakano if for any nonzero m-tuple $u_1, \cdots, u_m$ of sections

$$\sum (R(\partial_i, \overline{\partial}_j)u_i, u_j) > 0.$$

Clearly Nakano positivity implies Griffiths positivity in general.

In the case $l = 1$, the theorem stated below shows that the bundle is negative in the sense of Nakano.

Theorem 2.7 (Wang and Zhang) *Let* $\mathbf{T} \in B_n(\Omega)$. *The bundle* $E^1_{\mathbf{T}}$ *over* Ω *is Nakano negative.*

In the case, where the rank $l > 1$, we have only the following fact:

Theorem 2.8 (Wang and Zhang) *For* $l > 1$, *the restriction* $R^l_{\mathbf{T},\mathbf{w}}$ *is negative in the sense of Griffiths: For a holomorphic section u with* $0 \neq u(w) \in W^l(w)$ *and the coefficients* $0 \neq (a_1, \cdots, a_m)$,

$$\sum_{1\leq i,j\leq m} a_i\overline{a_j}\langle R^l_{\mathbf{T},w}(\partial_i, \overline{\partial_j})u\ , u\rangle < 0.$$

In particular the Ricci curvature $Ric^l(\partial_i, \overline{\partial_j})$ *of* E^l, $Ricl(\partial_i, \overline{\partial_j}) = tr R^l_{\mathbf{T},\mathbf{w}}(\partial_i, \overline{\partial_j})$ *is always negative.*

Such a result would yields the corresponding inequality. Unfortunately, we can only solve the case of planar domain.

Theorem 2.9 (Wang and Zhang) *If* $T \in B_n(\mathbb{D})$ *and* T *is contractive, then* $R^l_T \leq \mathcal{K}^l_{S^*} I_n = -\frac{l^2}{(1-|w|^2)^2} I_n$.

For a general multi-connected planar domain, we also prove that

Theorem 2.10 (Wang and Zhang) *Suppose* $T \in B_n(\Omega)$ *and* $cl\Omega$ *is a spectral set for the operator* T. *Then* $R^l_{T,w} \leq -l^2 \widehat{K}_\Omega(w, \overline{w})^2 I_n$ *for* $w \in \Omega$.

It might be interesting to find the converse to these results.

Proposition 2.11 (Wang and Zhang) *If $T \in B_1(\mathbb{D})$ and the adjoint T^* is a unilateral weighted shift operator with no zero weights, then T is contractive if and only if $\mathcal{K}^l_{T,0} \leq \mathcal{K}^l_{S^*,0} = -l^2, \forall l \geq 1$.*

The new curvature inequalities of higher order is necessary for the converse direction. Note that [4] Biswas, Keshari and Misra have constructed a backward shift T which is not contractive, but satisfies the curvature inequality of order 1.

2.3 Further Discussion

The most interesting problem would be the converse of the curvature inequalities.

Problem 2.12 *Suppose $T \in B_1(\mathbb{D})$ and $R^l_{T,w} \leq R^l_{S^*,w}$ for all integer $l \geq 1$, then does it follow that T is contractive?*

For unilateral weighted shifts, by Proposition 2.11, the answer is affirmative. However, to prove or disprove the general case appears to be too ambitious. As suggested by Rongwei Yang, we may consider the following question first.

Problem 2.13 *Suppose $T \in B_n(\mathbb{D})$ and T is a homogeneous operator in the sense of Misra, prove or disprove that $R^l_{T,w} \leq \mathcal{K}^l_{S^*,w} I_n$ for all integer $l \geq 1$ imply that T is contractive.*

The homogeneous operator on the unit disc $\mathbb{D}$ has been completely classified by Koranyi and Misra [9, 10]. By the homogeneous property, we only need to check the inequalities at one point. However, this problem doesn't seem so simple at first glance, due to the lack of computational technique for curvature of higher order.

Another direction to continue is to establish the inequalities for domains of higher dimension.

Problem 2.14 *What is the right generalization of Theorems 2.9 and 2.10 on the domain $\Omega \subseteq \mathbb{C}^n$.*

Theorem 2.8 prompts the possibility of considering the Ricci curvature. In the case of the unit ball $\mathbb{B}^d$. Thus, we may compare the Ricci curvature of Cowen–Douglas operator with the backward shifts over the Drury–Arveson space H^2_d.

Example The Drury–Arveson space H^2_d, by the computation in [1], has an orthonormal basis

$$\{e^\alpha = \sqrt{\frac{|\alpha|!}{\alpha!}} z^\alpha : \alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d_+\}$$

and the reproducing kernel is

$$K(z, w) = \sum_\alpha \frac{|\alpha|!}{\alpha!} z^\alpha \overline{w^\alpha} = \frac{1}{1 - \langle z, w \rangle}.$$

Therefore, at $z = 0$, the space

$$E^l = \{\partial^\alpha K_0(z) : |\alpha| \leq l - 1\}$$

with

$$\partial^\alpha K_0(z) = |\alpha|! z^\alpha.$$

Write ϵ_i be the multi-index with 1 in position i and 0 in others. By the computation in [20], for $|\alpha| = |\beta| = l - 1$,

$$\langle R_0(\partial_i, \overline{\partial_i})\partial^\alpha K_0, \partial^\beta K_0\rangle = -\langle \partial^{\alpha+\epsilon_i} K_0, \partial^{\beta+\epsilon_i} K_0\rangle.$$

Therefore, we have that for $|\alpha| = |\beta| = l - 1$,

$$\sum_i \langle R_0^l(\partial_i, \overline{\partial_i})e^\alpha, e^\alpha\rangle = -(l+1)(l+n).$$

and

$$\sum_i \langle R_0^l(\partial_i, \overline{\partial_i})e^\alpha, e^\beta\rangle = 0, if \ \alpha \neq \beta.$$

Using the transitivity of the automorphism group for the unit ball $\mathbb{B}^d$, we have that at a point $w \in \mathbb{B}^d$

$$Ric_w^l = -\frac{(l+1)(l+n)}{(1-|w|^2)^2} I.$$

Problem 2.15 *For an operator tuple* $\mathbf{T} = (T_1, \cdots, T_d) \in B_1(\mathbb{B}^d)$*, if* $\mathbf{T}$ *is a row contractive, can we have*

$$Ric_{\mathbf{T},w}^l \leq -\frac{(l+1)(l+n)}{(1-|w|^2)^2} I?$$

This also shows that for domains other than the Euclidean ball, the nature of the curvature inequalities may be different.

Acknowledgments The author would like to thank the anonymous reviewers to improve the overall quality of the paper. This work was partially supported by NSFC (11722102,11420101001).

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Reproducing Kernel of the Space $R^t(K, \mu)$

Liming Yang

Dedicated to the memory of Ronald G. Douglas

Abstract For $1 \leq t < \infty$, a compact subset K of the complex plane $\mathbb{C}$, and a finite positive measure μ supported on K, $R^t(K, \mu)$ denotes the closure in $L^t(\mu)$ of rational functions with poles off K. Let Ω be a connected component of the set of analytic bounded point evaluations for $R^t(K, \mu)$. In this paper, we examine the behavior of the reproducing kernel of $R^t(K, \mu)$ near the boundary $\partial\Omega \cap \mathbb{T}$, assuming that $\mu(\partial\Omega \cap \mathbb{T}) > 0$, where $\mathbb{T}$ is the unit circle.

Keywords Reproducing kernel · Cauchy transform · Analytic capacity · Analytic bounded point evaluations

Mathematics Subject Classification (2010) Primary 47A15; Secondary 30C85, 31A15, 46E15, 47B38

1 Introduction

Throughout this paper, let $\mathbb{D}$ denote the unit disk $\{z : |z| < 1\}$ in the complex plane $\mathbb{C}$, let $\mathbb{T}$ denote the unit circle $\{z : |z| = 1\}$, let m denote normalized Lebesgue measure on $\mathbb{T}$. Let μ be a finite, positive Borel measure that is compactly supported in $\mathbb{C}$. We require that the support of μ be contained in some compact set K and we indicate this by $\text{spt}(\mu) \subseteq K$. Under these circumstances and for $1 \leq t < \infty$ and $t' = \frac{t}{t-1}$, functions in $\mathcal{P}$(the set of analytic polynomials) and $\text{Rat}(K) := \{q : q$ is a rational function with poles off $K\}$ are members of $L^t(\mu)$.

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R. E. Curto et al. (eds.), *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, Operator Theory: Advances and Applications 278, https://doi.org/10.1007/978-3-030-43380-2_26

We let $P^t(\mu)$ denote the closure of $\mathcal{P}$ in $L^t(\mu)$ and let $R^t(K, \mu)$ denote the closure of Rat(K) in $L^t(\mu)$. A point z_0 in $\mathbb{C}$ is called a *bounded point evaluation* for $P^t(\mu)$ (resp., $R^t(K, \mu)$) if $f \mapsto f(z_0)$ defines a bounded linear functional for functions in $\mathcal{P}$ (resp., Rat(K)) with respect to the $L^t(\mu)$ norm. The norm of the bounded linear functional is denoted by M_{z_0}. The collection of all such points is denoted bpe$(P^t(\mu))$ (resp., bpe$(R^t(K, \mu))$). If z_0 is in the interior of bpe$(P^t(\mu))$ (resp., bpe$(R^t(K, \mu))$) and there exist positive constants r and M such that $|f(z)| \le M\|f\|_{L^t(\mu)}$, whenever $|z - z_0| \le r$ and $f \in \mathcal{P}$ (resp., $f \in$ Rat(K)), then we say that z_0 is an *analytic bounded point evaluation* for $P^t(\mu)$ (resp., $R^t(K, \mu)$). The collection of all such points is denoted abpe$(P^t(\mu))$ (resp., abpe$(R^t(K, \mu))$). Actually, it follows from Thomson's Theorem [10] (or see Theorem 1.1, below) that abpe$(P^t(\mu))$ is the interior of bpe$(P^t(\mu))$. This also holds in the context of $R^t(K, \mu)$ as was shown by J. Conway and N. Elias in [6]. Now, abpe$(P^t(\mu))$ is the largest open subset of $\mathbb{C}$ to which every function in $P^t(\mu)$ has an analytic continuation under these point evaluation functionals, and similarly in the context of $R^t(K, \mu)$. Let S_μ denote the multiplication by z on $R^t(K, \mu)$. It is well known that $R^t(K, \mu) = R^t(\sigma(S_\mu), \mu)$ and $\sigma(S_\mu) \subset K$, where $\sigma(S_\mu)$ denotes the spectrum of S_μ (see, for example, Proposition 1.1 in [6]). Throughout this paper, we assume $K = \sigma(S_\mu)$.

Our story begins with celebrated results of J. Thomson, in [10].

Theorem 1.1 (Thomson [10]) *Let μ be a finite, positive Borel measure that is compactly supported in $\mathbb{C}$ and suppose that $1 \le t < \infty$. There is a Borel partition $\{\Delta_i\}_{i=0}^\infty$ of spt(μ) such that the space $P^t(\mu|_{\Delta_i})$ contains no nontrivial characteristic function (i.e., $P^t(\mu|_{\Delta_i})$ is irreducible) and*

$$P^t(\mu) = L^t(\mu|_{\Delta_0}) \oplus \left\{\oplus_{i=1}^\infty P^t(\mu|_{\Delta_i})\right\}.$$

Furthermore, if $U_i :=$ abpe$(P^t(\mu|_{\Delta_i}))$ for $i \ge 1$, then U_i is a simply connected region and $\Delta_i \subseteq \overline{U_i}$.

We mention a remarkable result of A. Aleman, S. Richter and C. Sunberg. It's proof involves a modification of Thomson's scheme along with results of X. Tolsa on analytic capacity.

Theorem 1.2 (Aleman et al. [2]) *Suppose that μ is supported in $\overline{\mathbb{D}}$, abpe$(P^t(\mu))$ $= \mathbb{D}$, $P^t(\mu)$ is irreducible, and that $\mu(\mathbb{T}) > 0$.*

(a) If $f \in P^t(\mu)$, then the nontangential limit $f^*(\zeta)$ of f at ζ exists a.e. $\mu|_{\mathbb{T}}$ and $f^* = f|_{\mathbb{T}}$ as elements of $L^t(\mu|_{\mathbb{T}})$.
(b) Every nontrivial, closed invariant subspace $\mathcal{M}$ for the shift S_μ on $P^t(\mu)$ has index 1; that is, the dimension of $\mathcal{M}/z\mathcal{M}$ is one.
(c) If $t > 1$, then

$$\lim_{\lambda \to z} (1 - |\lambda|^2)^{\frac{1}{t}} M_\lambda = \frac{1}{h(z)^{\frac{1}{t}}}$$

nontangentially for m-a.a. $z \in \mathbb{T}$, where $\mu|_{\mathbb{T}} = hm$.

J. Thomson's proof of the existence of bounded point evaluations for $P^t(\mu)$ uses Davie's deep estimation of analytic capacity, S. Brown's technique, and Vitushkin's localization for uniform rational approximation. The proof is excellent but complicated, and it does not really lend itself to showing the existence of nontangential boundary values in the case that $\mathrm{spt}(\mu) \subseteq \overline{\mathbb{D}}$, $P^t(\mu)$ is irreducible and $\mu(\mathbb{T}) > 0$. X. Tolsa's remarkable results on analytic capacity opened the door for a new view of things, through the works of [1–3] and [4], etc.

In this paper, we assume that $R^t(K,\mu)$ is irreducible and Ω is a connected region satisfying:

$$abpe(R^t(K,\mu)) = \Omega,\ K = \overline{\Omega},\ \Omega \subset \mathbb{D},\ \mathbb{T} \subset \partial\Omega. \tag{1.1}$$

It is well known that, in this case, $\mu|_{\mathbb{T}} << m$. So we assume $\mu|_{\mathbb{T}} = hm$.

For $\delta > 0$ and $\lambda \in \mathbb{C}$, set $B(\lambda,\delta) = \{z : |z-\lambda| < \delta\}$. For $0 < \sigma < 1$, let $\Gamma_\sigma(e^{i\theta})$ denote the polynomial convex hull of $\{e^{i\theta}\}$ and $B(0,\sigma)$. Define $\Gamma_\sigma^\delta(e^{i\theta}) = \Gamma_\sigma(e^{i\theta}) \cap B(e^{i\theta},\delta)$. In order to define a nontangential limit of a function in $R^t(K,\mu)$ at $e^{i\theta} \in \partial\Omega$, one needs $\Gamma_\sigma^\delta(e^{i\theta}) \subset \Omega$ for some δ. Therefore, we define the strong outer boundary of Ω as the following:

$$\partial_{so,\sigma}\Omega = \{e^{i\theta} \in \partial\Omega :\ \exists 0 < \delta < 1,\ \Gamma_\sigma^\delta(e^{i\theta}) \subset \Omega\}. \tag{1.2}$$

It is known that $\partial_{so,\sigma}\Omega$ is a Borel set (i.e., see Lemma 4 in [9]) and $m(\partial_{so,\sigma_1}\Omega \setminus \partial_{so,\sigma_2}\Omega) = 0$ for $\sigma_1 \neq \sigma_2$. Therefore, we set $\partial_{so}\Omega = \partial_{so,\frac{1}{2}}\Omega$.

The paper [1] presents an alternate and simpler route to prove Theorem 1.2 (a) and (b) that has extension to the context of mean rational approximation as in Theorem 1.3 below. It also uses the results of X. Tolsa on analytic capacity.

Theorem 1.3 (Akeroyd et al. [1]) *Let Ω be a bounded connected open set satisfying (1.1). Suppose that μ is a finite positive measure supported in K, $abpe(R^t(K,\mu)) = \Omega$, $R^t(K,\mu)$ is irreducible, $\mu|_{\mathbb{T}} = hm$, and $\mu(\partial_{so}\Omega) > 0$. Then:*

(a) If $f \in R^t(K,\mu)$ then the nontangential limit $f^*(z)$ of f exists for $\mu|_{\partial_{so}\Omega}$ almost all z, and $f^* = f|_{\partial_{so}\Omega}$ as elements of $L^t(\mu|_{\partial_{so}\Omega})$.
(b) Every nonzero rationally invariant subspace $\mathcal{M}$ of $R^t(K,\mu)$ has index 1, that is, $dim(\mathcal{M}/(S_\mu - \lambda_0)\mathcal{M}) = 1$, for $\lambda_0 \in \Omega$.

Theorem 1.3 is a direct application of Theorem 3.6 in [1], which proves a generalized Plemelj's formula for a compactly supported finite complex-valued measure. In fact, the generalized Plemelj's formula holds for rectifiable curve (other than $\mathbb{T}$), so Theorem 1.3 is valid if $\partial\Omega$ is a certain rectifiable curve.

In this paper, we continue the work of section 3 in [1] to generalize Theorem 1.2 (c). We refine the estimates of Cauchy transform of a finite measure in [1] and provide an alternate proof of Theorem 1.2 (c) that can extend the result to the context of certain mean rational approximation space $R^t(K,\mu)$.

By Riesz representation theorem, there exists $k_\lambda \in L^{t'}(\mu)$ for $\lambda \in abpe(R^t(K,\mu))$ such that $M_\lambda = \|k_\lambda\|_{L^{t'}(\mu)}$ and

$$f(\lambda) = \int f(z)\bar{k}_\lambda(z)d\mu(z), \ \ f \in \text{Rat}(K).$$

The function k_λ is called the reproducing kernel for $R^t(K,\mu)$.

Theorem 1.4 (Main Theorem) *Let Ω be a bounded connected open set satisfying (1.1). Suppose that μ is a finite positive measure supported in K, $abpe(R^t(K,\mu)) = \Omega$, $R^t(K,\mu)$ is irreducible, $\mu|_{\mathbb{T}} = hm$, and $\mu(\partial_{so}\Omega) > 0$. If $t > 1$, then*

$$\lim_{\Gamma_{\frac{1}{4}}(e^{i\theta})\ni\lambda\to e^{i\theta}} (1-|\lambda|^2)^{\frac{1}{t}} M_\lambda = \lim_{\Gamma_{\frac{1}{4}}(e^{i\theta})\ni\lambda\to e^{i\theta}} (1-|\lambda|^2)^{\frac{1}{t}} \|k_\lambda\|_{L^{t'}(\mu)} = \frac{1}{h(e^{i\theta})^{\frac{1}{t}}}$$

for μ-almost all $e^{i\theta} \in \partial_{so}\Omega$.

2 Proof of Main Theorem

Let ν be a finite complex-valued Borel measure that is compactly supported in $\mathbb{C}$. For $\epsilon > 0$, $\mathcal{C}_\epsilon(\nu)$ is defined by

$$\mathcal{C}_\epsilon(\nu)(z) = \int_{|w-z|>\epsilon} \frac{1}{w-z} d\nu(w). \tag{2.1}$$

The (principal value) Cauchy transform of ν is defined by

$$\mathcal{C}(\nu)(z) = \lim_{\epsilon\to 0} \mathcal{C}_\epsilon(\nu)(z) \tag{2.2}$$

for all $z \in \mathbb{C}$ for which the limit exists. If $\lambda \in \mathbb{C}$ and $\int \frac{d|\nu|}{|z-\lambda|} < \infty$, then $\lim_{r\to 0} \frac{|\nu|(B(\lambda,r))}{r} = 0$ and $\lim_{\epsilon\to 0} \mathcal{C}_\epsilon(\nu)(\lambda)$ exists. Therefore, a standard application of Fubini's Theorem shows that $\mathcal{C}(\nu) \in L^s_{\text{loc}}(\mathbb{C})$, for $0 < s < 2$. In particular, it is defined for almost all z with respect to area measure on $\mathbb{C}$, and clearly $\mathcal{C}(\nu)$ is analytic in $\mathbb{C}_\infty \setminus \text{spt}(\nu)$, where $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. In fact, from Corollary 3.1 in [1], we see that (2.2) is defined for all z except for a set of zero analytic capacity. Thoughout this section, the Cauchy transform of a measure always means the principal value of the transform.

The maximal Cauchy transform is defined by

$$\mathcal{C}_*(\nu)(z) = \sup_{\epsilon>0} |\mathcal{C}_\epsilon(\nu)(z)|.$$

If $K \subset \mathbb{C}$ is a compact subset, then we define the analytic capacity of K by

$$\gamma(K) = \sup |f'(\infty)|,$$

where the supremum is taken over all those functions f that are analytic in $\mathbb{C}_\infty \setminus K$ such that $|f(z)| \le 1$ for all $z \in \mathbb{C}_\infty \setminus K$; and $f'(\infty) := \lim_{z\to\infty} z(f(z) - f(\infty))$. The analytic capacity of a general subset E of $\mathbb{C}$ is given by:

$$\gamma(E) = \sup\{\gamma(K) : K \subset\subset E\}.$$

Good sources for basic information about analytic capacity are Chapter VIII of [7], Chapter V of [5], and [13].

A related capacity, γ_+, is defined for subsets E of $\mathbb{C}$ by:

$$\gamma_+(E) = \sup \|\mu\|,$$

where the supremum is taken over positive measures μ with compact support contained in E for which $\|\mathcal{C}(\mu)\|_{L^\infty(\mathbb{C})} \le 1$. Since $\mathcal{C}\mu$ is analytic in $\mathbb{C}_\infty \setminus \text{spt}(\mu)$ and $(\mathcal{C}(\mu)'(\infty) = \|\mu\|$, we have:

$$\gamma_+(E) \le \gamma(E)$$

for all subsets E of $\mathbb{C}$. X. Tolsa has established the following astounding results.

Theorem 2.1 (Tolsa [12])

(1) *γ_+ and γ are actually equivalent. That is, there is an absolute constant A_T such that*

$$\gamma(E) \le A_T \gamma_+(E) \tag{2.3}$$

for all $E \subset \mathbb{C}$.

(2) *Semiadditivity of analytic capacity:*

$$\gamma\left(\bigcup_{i=1}^{m} E_i\right) \le A_T \sum_{i=1}^{m} \gamma(E_i) \tag{2.4}$$

where $E_1, E_2, \dots, E_m \subset \mathbb{C}$.

(3) *There is an absolute positive constant C_T such that, for any $a > 0$, we have:*

$$\gamma(\{\mathcal{C}_*(\nu) \ge a\}) \le \frac{C_T}{a}\|\nu\|. \tag{2.5}$$

Proof (1) and (2) are from [12] (also see Theorem 6.1 and Corollary 6.3 in [13]).

(3) follows from Proposition 2.1 of [11] (also see [13] Proposition 4.16). □

The following lemma is a modification of Lemma 3.2 of [1].

Lemma 2.2 *Let ν be a finite measure supported in $\bar{\mathbb{D}}$ and $|\nu|(\mathbb{T}) = 0$. Let $1 < p \le \infty$, $q = \frac{p}{p-1}$, $f \in C(\bar{\mathbb{D}})$, and $g \in L^q(|\nu|)$. Assume that for some $e^{i\theta} \in \mathbb{T}$ we have:*

$$\lim_{r \to 0} \frac{\int_{B(e^{i\theta},r)} |g|^q d|\nu|}{r} = 0 \tag{2.6}$$

Then, for any $a > 0$, there exists δ_a, $0 < \delta_a < \frac{1}{4}$, such that whenever $0 < \delta < \delta_a$, there is a subset E_δ^f of $B(e^{i\theta}, \delta)$ and $\epsilon(\delta) > 0$ satisfying:

$$\lim_{\delta \to 0} \epsilon(\delta) = 0, \tag{2.7}$$

$$\gamma(E_\delta^f) < \epsilon(\delta)\delta, \tag{2.8}$$

for all $\lambda \in B(e^{i\theta}, \delta) \setminus E_\delta^f$, $|\lambda_0 - e^{i\theta}| = \frac{\delta}{2}$ and $\lambda_0 \in \Gamma_{\frac{1}{2}}(e^{i\theta})$,

$$\lim_{\epsilon \to 0} \mathcal{C}_\epsilon \left((1 - \bar{\lambda}_0 z)^{\frac{2}{p}} \delta^{-\frac{1}{p}} f g \nu \right)(\lambda) \tag{2.9}$$

exists, and

$$\left| \mathcal{C} \left((1 - \bar{\lambda}_0 z)^{\frac{2}{p}} \delta^{-\frac{1}{p}} f g \nu \right)(\lambda) - \mathcal{C} \left((1 - \bar{\lambda}_0 z)^{\frac{2}{p}} \delta^{-\frac{1}{p}} f g \nu \right)\left(\frac{1}{\bar{\lambda}_0}\right) \right| \le a \|f\|_{L^p(|\nu|)}. \tag{2.10}$$

Notice that the set E_δ^f depends on f and all other parameters are independent of f.

Proof Let

$$M = \sup_{r > 0} \frac{\int_{B(e^{i\theta},r)} |g|^q d|\nu|}{r}.$$

Then, by (2.6), $M < \infty$. For $a > 0$, choose N and δ_a, $0 < \delta_a < \frac{1}{4}$, satisfying:

$$N = 6 + \left(\frac{256}{a} \sum_{k=0}^{\infty} 2^{\frac{-k}{q}} \right)^q M,$$

$$\left(\frac{\int_{B(\lambda_0, N\delta)} |g|^q d|\nu|}{\delta} \right)^{\frac{1}{q}} < \frac{a}{4^{3+\frac{2}{q}}}$$

for $0 < \delta < \delta_a$. We now fix δ, $0 < \delta < \delta_a$, and let

$$\nu_\delta = \frac{\chi_{B(e^{i\theta}, N\delta)}}{(1-\bar{\lambda}_0 z)^{1-\frac{2}{p}}\delta^{\frac{1}{p}}} fg\nu,$$

where χ_A denotes the characteristic function of the set A. For $0 < \epsilon < \delta$ and $\lambda \in B(e^{i\theta}, \delta)$, we get:

$$2(1-|\lambda_0|) \le \delta \le 4(1-|\lambda_0|),$$

$$\overline{B(\lambda, \epsilon)} \subset B(e^{i\theta}, 2\delta) \subset B(e^{i\theta}, N\delta),$$

and

$$\begin{aligned}
&\left|\mathcal{C}_\epsilon\left((1-\bar{\lambda}_0 z)^{\frac{2}{p}}\delta^{-\frac{1}{p}} fg\nu\right)(\lambda) - \mathcal{C}\left((1-\bar{\lambda}_0 z)^{\frac{2}{p}}\delta^{-\frac{1}{p}} fg\nu\right)\left(\frac{1}{\bar{\lambda}_0}\right)\right| \\
\le& \frac{|1-\bar{\lambda}_0\lambda|}{\delta^{\frac{1}{p}}}\left|\int_{|z-\lambda|>\epsilon}\frac{fgd\nu}{(z-\lambda)(1-\bar{\lambda}_0 z)^{1-\frac{2}{p}}}\right| \\
&+\left|\mathcal{C}\left(\chi_{\bar{B}(\lambda,\epsilon)}\frac{(1-\bar{\lambda}_0 z)^{\frac{2}{p}}}{\delta^{\frac{1}{p}}} fg\nu\right)\left(\frac{1}{\bar{\lambda}_0}\right)\right| \\
\le& 2\delta^{\frac{1}{q}}\left|\int_{B(e^{i\theta},N\delta)^c}\frac{fgd\nu}{(z-\lambda)(1-\bar{\lambda}_0 z)^{1-\frac{2}{p}}}\right| + 2\delta\left|\int_{|z-\lambda|>\epsilon}\frac{d\nu_\delta}{(z-\lambda)}\right| \\
&+\int_{\bar{B}(\lambda,\epsilon)}\frac{\delta^{-\frac{1}{p}}}{|1-\bar{\lambda}_0 z|^{1-\frac{2}{p}}}|fg|d|\nu| \\
\le& 2\delta^{\frac{1}{q}}\sum_{k=0}^{\infty}\int_{2^k N\delta\le|z-e^{i\theta}|<2^{k+1}N\delta}\frac{|f||g|d|\nu|}{|z-\lambda||1-\bar{\lambda}_0 z|^{1-\frac{2}{p}}} + 2\delta|\mathcal{C}_\epsilon\nu_\delta(\lambda)| \\
&+\int_{B(e^{i\theta},2\delta)}\frac{|1-\bar{\lambda}_0 z|\delta^{-\frac{1}{p}}}{|1-\bar{\lambda}_0 z|^{\frac{2}{q}}}|fg|d|\nu| \\
\le& 2\delta^{\frac{1}{q}}\sum_{k=0}^{\infty}\frac{(2^{k+1}N\delta)^{\frac{1}{q}}(2^k N\delta+2\delta)^{\frac{2}{p}}}{(2^k N\delta-\delta)(2^k N\delta-2\delta)}\left(\frac{\int_{B(e^{i\theta},2^{k+1}N\delta)}|g|^q d|\nu|}{2^{k+1}N\delta}\right)^{\frac{1}{q}}\|f\|_{L^p(|\nu|)} \\
&+2\delta\mathcal{C}_*\nu_\delta(\lambda) + 4\int_{B(e^{i\theta},2\delta)}\frac{\delta^{\frac{1}{q}}}{|1-\bar{\lambda}_0 z|^{\frac{2}{q}}}|fg|d|\nu|
\end{aligned}$$

$$\leq \frac{4(N+2)^{1+\frac{1}{p}} \sum_{k=0}^{\infty} 2^{\frac{-k}{q}} M^{\frac{1}{q}}}{(N-1)(N-2)} \|f\|_{L^p(|\nu|)} + 2\delta \mathcal{C}_* \nu_\delta(\lambda)$$

$$+ 4^{1+\frac{2}{q}} \|f\|_{L^p(|\nu|)} \left(\frac{\int_{B(e^{i\theta}, 2\delta)} |g|^q d|\nu|}{\delta} \right)^{\frac{1}{q}}$$

$$\leq \frac{a}{4} \|f\|_{L^p(|\nu|)} + 2\delta \mathcal{C}_* \nu_\delta(\lambda).$$

Let

$$\mathcal{E}_\delta = \{\lambda : C_*(\nu_\delta)(\lambda) \geq \frac{a\|f\|_{L^p(|\nu|)}}{8\delta}\} \cap B(\lambda_0, \delta).$$

Then

$$\{\lambda : |\mathcal{C}_\epsilon \left((1-\bar{\lambda}_0 z)^{\frac{2}{p}} \delta^{-\frac{1}{p}} f g \nu \right)(\lambda) - \mathcal{C}\left((1-\bar{\lambda}_0 z)^{\frac{2}{p}} \delta^{-\frac{1}{p}} f g \nu \right)(\frac{1}{\bar{\lambda}_0})|$$

$$\geq a\|f\|_{L^p(|\nu|)}\} \cap B(\lambda_0, \delta) \subset \mathcal{E}_\delta.$$

From Theorem 2.1 (3), we get

$$\gamma(\mathcal{E}_\delta) \leq \frac{8C_T \delta}{a\|f\|_{L^p(|\nu|)}} \|\nu_\delta\| \leq \frac{32 C_T \delta}{a} \left(\frac{\int_{B(e^{i\theta}, N\delta)} |g|^q d|\nu|}{\delta} \right)^{\frac{1}{q}}.$$

Let E be the set of $\lambda \in \mathbb{C}$ such that $\lim_{\epsilon \to 0} \mathcal{C}_\epsilon (fg\nu)(\lambda)$ does not exist. By Corollary 3.1 in [1], we see that $\gamma(E) = 0$. It is clear that (2.9) exists for $\lambda \in \mathbb{D} \setminus E$ because

$$\lim_{\epsilon \to 0} \mathcal{C}_\epsilon \left((1-\bar{\lambda}_0 z)^{\frac{2}{p}} \delta^{-\frac{1}{p}} f g \nu \right)(\lambda) - (1-\bar{\lambda}_0 \lambda)^{\frac{2}{p}} \delta^{-\frac{1}{p}} \lim_{\epsilon \to 0} \mathcal{C}_\epsilon (fg\nu)(\lambda)$$

$$= \int \frac{(1-\bar{\lambda}_0 z)^{\frac{2}{p}} \delta^{-\frac{1}{p}} - (1-\bar{\lambda}_0 \lambda)^{\frac{2}{p}} \delta^{-\frac{1}{p}}}{z - \lambda} f g d\nu$$

exists for all $\lambda \in \mathbb{D} \setminus E$.

Now let $E_\delta^f = \mathcal{E}_\delta \cup E$. Applying Theorem 2.1 (2) we find that

$$\gamma(E_\delta) \leq A_T(\gamma(\mathcal{E}_\delta) + \gamma(E)) < \frac{32 A_T C_T}{a} \left(\frac{\int_{B(e^{i\theta}, N\delta)} |g|^q d|\nu|}{\delta} \right)^{\frac{1}{q}} \delta.$$

Letting

$$\epsilon(\delta) = \frac{32A_T C_T}{a}\left(\frac{\int_{B(e^{i\theta},N\delta)}|g|^q d|\nu|}{\delta}\right)^{\frac{1}{q}},$$

we conclude that (2.7) and (2.8) hold. On $B(\lambda_0,\delta)\setminus E_\delta$ and for $\epsilon < \delta$, we conclude that

$$\left|\mathcal{C}_\epsilon\left((1-\bar{\lambda}_0 z)^{\frac{2}{p}}\delta^{-\frac{1}{p}}fg\nu\right)(\lambda) - \mathcal{C}\left((1-\bar{\lambda}_0 z)^{\frac{2}{p}}\delta^{-\frac{1}{p}}fg\nu\right)(\frac{1}{\bar{\lambda}_0})\right| < a\|f\|_{L^p(|\nu|)}.$$

Therefore, (2.10) follows since

$$\lim_{\epsilon\to 0}\mathcal{C}_\epsilon\left((1-\bar{\lambda}_0 z)^{\frac{2}{p}}\delta^{-\frac{1}{p}}fg\nu\right)(\lambda) = \mathcal{C}\left((1-\bar{\lambda}_0 z)^{\frac{2}{p}}\delta^{-\frac{1}{p}}fg\nu\right)(\lambda).$$

□

Proposition 2.3 *Let μ be a finite positive measure with support in $K \subset \bar{\mathbb{D}}$ and $\mu|_{\mathbb{T}} = hm$. Let $1 < p < \infty$, $q = \frac{p}{p-1}$, $f \in C(\bar{\mathbb{D}})$, $g \in L^q(\mu)$, and $fg\mu \perp Rat(K)$. Then for $0 < \beta < \frac{1}{16}$, $b > 0$, and m-almost all $e^{i\theta} \in \mathbb{T}$, there exist $0 < \delta_a < \frac{1}{4}$, $E_\delta^f \subset B(e^{i\theta},\delta)$, and $\epsilon(\delta) > 0$, where $0 < \delta < \delta_a$, such that $\lim_{\delta\to 0}\epsilon(\delta) = 0$, $\gamma(E_\delta^f) < \epsilon(\delta)\delta$, and for $\lambda_0 \in (\partial B(e^{i\theta},\frac{\delta}{2}))\cap\Gamma_{\frac{1}{4}}(e^{i\theta})$,*

$$\left|\mathcal{C}\left(\frac{(1-\bar{\lambda}_0 z)^{\frac{2}{p}}}{(1-|\lambda_0|^2)^{\frac{1}{p}}}fg\mu\right)(\lambda)\right| \le \left(b + \frac{1+4\beta}{1-4\beta}\left(\int_{\mathbb{T}}\frac{1-|\lambda_0|^2}{|1-\bar{\lambda}_0 z|^2}|g|^q d\mu\right)^{\frac{1}{q}}\right)\|f\|_{L^p(\mu)}$$

for all $\lambda \in B(\lambda_0,\beta\delta)\setminus E_\delta^f$.

Proof Let $\nu = \mu|_{\mathbb{D}}$. We now apply Lemma 2.2 for $p,\ q,\ f,\ g$, and $a = \frac{1}{2^{\frac{1}{p}}}b$. From Lemma 3.5 in [1], there exists E with $\gamma(E) = 0$ such that for $e^{i\theta} \in \mathbb{T}\setminus E$, $|g|^q d|\nu|$ satisfies (2.6). There exist $0 < \delta_a < \frac{1}{4}$, $E_\delta^f \subset B(e^{i\theta},\delta)$, and $\epsilon(\delta) > 0$, where $0 < \delta < \delta_a$, such that $\lim_{\delta\to 0}\epsilon(\delta) = 0$, $\gamma(E_\delta^f) < \epsilon(\delta)\delta$, and for $\lambda_0 \in (\partial B(e^{i\theta},\frac{\delta}{2}))\cap\Gamma_{\frac{1}{4}}(e^{i\theta})$,

$$\left|\mathcal{C}\left(\frac{(1-\bar{\lambda}_0 z)^{\frac{2}{p}}}{(1-|\lambda_0|^2)^{\frac{1}{p}}}fg\nu\right)(\lambda) - \mathcal{C}\left(\frac{(1-\bar{\lambda}_0 z)^{\frac{2}{p}}}{(1-|\lambda_0|^2)^{\frac{1}{p}}}fg\nu\right)(\frac{1}{\bar{\lambda}_0})\right| \le b\|f\|_{L^p(\mu)}$$

for all $\lambda \in B(e^{i\theta}, \delta) \setminus E_\delta^f$.

$$\mathcal{C}\left(\frac{(1-\bar{\lambda}_0 z)^{\frac{2}{p}}}{(1-|\lambda_0|^2)^{\frac{1}{p}}} fg\mu\right)\left(\frac{1}{\bar{\lambda}_0}\right) = 0$$

since $fg\mu \perp Rat(K)$. Therefore, for all $\lambda \in B(\lambda_0, \beta\delta) \setminus E_\delta^f$, we get

$$\begin{aligned}
&\left|\mathcal{C}\left(\frac{(1-\bar{\lambda}_0 z)^{\frac{2}{p}}}{(1-|\lambda_0|^2)^{\frac{1}{p}}} fg\mu\right)(\lambda)\right| \\
\le &\left|\mathcal{C}\left(\frac{(1-\bar{\lambda}_0 z)^{\frac{2}{p}}}{(1-|\lambda_0|^2)^{\frac{1}{p}}} fg\nu\right)(\lambda) - \mathcal{C}\left(\frac{(1-\bar{\lambda}_0 z)^{\frac{2}{p}}}{(1-|\lambda_0|^2)^{\frac{1}{p}}} fg\nu\right)\left(\frac{1}{\bar{\lambda}_0}\right)\right| \\
&+ \left|\int_{\mathbb{T}} \left(\frac{1}{z-\lambda} - \frac{1}{z-\frac{1}{\bar{\lambda}_0}}\right) \frac{(1-\bar{\lambda}_0 z)^{\frac{2}{p}}}{(1-|\lambda_0|^2)^{\frac{1}{p}}} fg\mu\right| \\
\le & b\|f\|_{L^p(\mu)} + \int_{\mathbb{T}} \frac{|1-\lambda\bar{\lambda}_0|}{|z-\lambda|} \frac{(1-|\lambda_0|^2)^{-\frac{1}{p}}}{|1-\bar{\lambda}_0 z|^{1-\frac{2}{p}}} |fg| d\mu \\
\le & b\|f\|_{L^p(\mu)} + \frac{1+4\beta}{1-4\beta} \int_{\mathbb{T}} \frac{(1-|\lambda_0|^2)^{\frac{1}{q}}}{|1-\bar{\lambda}_0 z|^{\frac{2}{q}}} |fg| d\mu
\end{aligned}$$

where the last step follows from

$$\begin{aligned}
\frac{|1-\lambda\bar{\lambda}_0|}{|z-\lambda|} &\le \frac{1-|\lambda_0|^2 + |\lambda_0||\lambda-\lambda_0|}{|z-\lambda_0| - |\lambda-\lambda_0|} \\
&\le \frac{(1+4\beta)(1-|\lambda_0|^2)}{|z-\lambda_0| - 4\beta(1-|\lambda_0|)} \\
&\le \frac{(1+4\beta)(1-|\lambda_0|^2)}{(1-4\beta)|1-\bar{\lambda}_0 z|}
\end{aligned}$$

for $z \in \mathbb{T}$. The proposition now follows from Holder's inequality. □

Let $R = \{z : |Re(z)| < \frac{1}{2} \text{ and } |Im(z)| < \frac{1}{2}\}$ and $Q = \bar{\mathbb{D}} \setminus R$. For a bounded Borel set $E \subset \mathbb{C}$ and $1 \le p \le \infty$, $L^p(E)$ denotes the L^p space with respect to the area measure dA restricted to E. The following Lemma is a simple application of Thomson's coloring scheme.

Lemma 2.4 *There is an absolute constant* $\epsilon_1 > 0$ *with the following property. If* $\gamma(\mathbb{D} \setminus K) < \epsilon_1$, *then*

$$|f(\lambda)| \le \|f\|_{L^\infty(Q \cap K)}$$

for $\lambda \in R$ *and* $f \in A(\mathbb{D})$, *the uniform closure of* $\mathcal{P}$ *in* $C(\bar{\mathbb{D}})$.

Proof We use Thomson's coloring scheme that is described at the beginning of section 2 of [14]. Let ϵ_1 be chosen as in Lemma 2 of [14]. By our assumption $\gamma(\mathbb{D} \setminus K) < \epsilon_1$ and Lemma 2 of [14], we conclude that Case II on Page 225 of [14] holds, that is, $scheme(Q, \epsilon, m, \gamma_n, \Gamma_n, n \ge m)$ ($\epsilon < 10^{-3}$) does not terminate. In this case, one has a sequence of heavy ϵ barriers inside Q, that is, $\{\gamma_n\}_{n \ge m}$ and $\{\Gamma_n\}_{n \ge m}$ are infinite.

Let $f \in A(\mathbb{D})$, by the maximal modulus principle, we can find $z_n \in \gamma_n$ such that $|f(\lambda)| \le |f(z_n)|$ for $\lambda \in R$. By the definition of γ_n, we can find a heavy ϵ square S_n with $z_n \in S_n \cap \gamma_n$. Since $\gamma(Int(S_n) \setminus K) \le \epsilon d_{S_n}$ (see (2.2) in [14]), we must have $Area(S_n \cap K) > 0$. We can choose $w_n \in S_n \cap K$ with $|f(w_n)| = \|f\|_{L^\infty(S_n \cap K)}$. $\frac{f(w) - f(z_n)}{w - z_n}$ is analytic in $\mathbb{D}$, therefore, by the maximal modulus principle again, we get

$$\left| \frac{f(w_n) - f(z_n)}{w_n - z_n} \right| \le \sup_{w \in \gamma_{n+1}} \left| \frac{f(w) - f(z_n)}{w - z_n} \right| \le \frac{2\|f\|_{L^\infty(\mathbb{D})}}{dist(z_n, \gamma_{n+1})}.$$

By Lemma 2.1 in [10] (there is a buffer zone of yellow squares between γ_n and γ_{n+1}), we have $dist(z_n, \gamma_{n+1}) \ge n^2 2^{-n}$. Therefore,

$$\begin{aligned} |f(\lambda)| \le & |f(z_n)| \le |f(w_n)| + \frac{2|z_n - w_n| \|f\|_{L^\infty(\mathbb{D})}}{dist(z_n, \gamma_{n+1})} \\ \le & \|f\|_{L^\infty(Q \cap K)} + \frac{2\sqrt{2} 2^{-n} \|f\|_{L^\infty(\mathbb{D})}}{n^2 2^{-n}} \end{aligned}$$

for $\lambda \in R$. The lemma follows by taking $n \to \infty$. □

Corollary 2.5 *There is an absolute constant* $\epsilon_1 > 0$ *with the following property. If* $\lambda_0 \in \mathbb{C}$, $\delta > 0$, *and* $\gamma(B(\lambda_0, \delta) \setminus K) < \epsilon_1 \delta$, *then*

$$|f(\lambda)| \le \|f\|_{L^\infty(B(\lambda_0, \delta) \cap K)}$$

for $\lambda \in B(\lambda_0, \frac{\delta}{2})$ *and* $f \in A(B(\lambda_0, \delta))$, *the uniform closure of* $\mathcal{P}$ *in* $C(\overline{B(\lambda_0, \delta)})$.

Proof (Main Theorem) From Lemma VII.1.7 in [5], we find a function $G \in R^t(K, \mu)^\perp$ such that $G(z) \ne 0$ for μ-almost every z. There exists $Z_1 \subset \mathbb{T}$ with $m(Z_1) = 0$ such that $G(e^{i\theta}) h(e^{i\theta}) \ne 0$ for $e^{i\theta} \in \partial_{so}\Omega \cap \mathcal{N}(h) \setminus Z_1$, where $\mathcal{N}(h) = \{e^{i\theta} : h(e^{i\theta}) > 0\}$.

By Theorem 3.6 (Plemelj's Formula for an arbitrary measure) in [1], for $e^{i\theta} \in \partial_{so}\Omega \cap \mathcal{N}(h) \setminus Z_1 \setminus Z_2$ with $m(Z_2) = 0$, $\Gamma_{\frac{1}{2}}^{r_0}(e^{i\theta}) \subset \Omega$, and $b > 0$, there exist $0 < \delta_a^0 < 1 - max(\frac{3}{4}, r_0)$, $E_\delta \subset B(e^{i\theta}, \delta)$, and $\epsilon^0(\delta) > 0$, where $0 < \delta < \delta_a^0$, such that $\lim_{\delta\to 0}\epsilon^0(\delta) = 0$, $\gamma(E_\delta) < \epsilon^0(\delta)\delta$,

$$\left|\mathcal{C}(G\mu)(\lambda) - \mathcal{C}(G\mu)(e^{i\theta}) - \frac{1}{2}e^{-i\theta}G(e^{i\theta})h(e^{i\theta})\right| \le \frac{b}{2}$$

and

$$\left|\mathcal{C}(G\mu)(\frac{1}{\bar{\lambda}}) - \mathcal{C}(G\mu)(e^{i\theta}) + \frac{1}{2}e^{-i\theta}G(e^{i\theta})h(e^{i\theta})\right| \le \frac{b}{2},$$

hence,

$$\left|\mathcal{C}(G\mu)(\lambda) - e^{-i\theta}G(e^{i\theta})h(e^{i\theta})\right| \le b \tag{2.11}$$

since $\mathcal{C}(G\mu)(\frac{1}{\bar{\lambda}}) = 0$ for all $\lambda \in B(e^{i\theta}, \delta) \cap \Gamma_{\frac{1}{2}}(e^{i\theta}) \setminus E_\delta$.

By Proposition 2.3 for $p = t, q = t', f \in Rat(K) \subset C(\overline{\mathbb{D}})$, and $g = G$, for $e^{i\theta} \in \partial_{so}\Omega \cap \mathcal{N}(h) \setminus Z_1 \setminus Z_3$ with $m(Z_3) = 0$, $\Gamma_{\frac{1}{2}}^{r_1}(e^{i\theta}) \subset \Omega$, $0 < \beta < \frac{1}{16}$, and $b > 0$, there exist $0 < \delta_a^1 < 1 - max(\frac{3}{4}, r_1)$, $E_\delta^f \subset B(e^{i\theta}, \delta)$, and $\epsilon^1(\delta) > 0$, where $0 < \delta < \delta_a^1$, such that $\lim_{\delta\to 0}\epsilon^1(\delta) = 0$, $\gamma(E_\delta^f) < \epsilon^1(\delta)\delta$,

$$\begin{aligned} &\left|\mathcal{C}\left(\frac{(1-\bar{\lambda}_0 z)^{\frac{2}{t}}}{(1-|\lambda_0|^2)^{\frac{1}{t}}} f G\mu\right)(\lambda)\right| \\ &\le \left(b + \frac{1+4\beta}{1-4\beta}\left(\int_{\mathbb{T}} \frac{1-|\lambda_0|^2}{|1-\bar{\lambda}_0 z|^2}|G|^{t'}d\mu\right)^{\frac{1}{t'}}\right)\|f\|_{L^t(\mu)} \end{aligned} \tag{2.12}$$

for $\lambda_0 \in \partial B(e^{i\theta}, \frac{\delta}{2}) \cap \Gamma_{\frac{1}{4}}(e^{i\theta})$ and all $\lambda \in B(\lambda_0, \beta\delta) \setminus E_\delta^f$.

Set $Z = Z_1 \cup Z_2 \cup Z_3$. For $e^{i\theta} \in \partial_{so}\Omega \cap \mathcal{N}(h) \setminus Z$, set $\delta_a = min(\delta_a^0, \delta_a^1)$ and $\epsilon(\delta) = min(\epsilon^0(\delta), \epsilon^1(\delta))$. Then for $e^{i\theta} \in \partial_{so}\Omega \cap \mathcal{N}(h) \setminus Z$ and $0 < \delta < \delta_a$, (2.11) and (2.12) hold. From semi-additivity of Theorem 2.1 (2), we get

$$\gamma(E_\delta \cup E_\delta^f) \le A_T(\gamma(E_\delta) + \gamma(E_\delta^f)) \le 2A_T\epsilon(\delta)\delta.$$

Let δ be small enough so that $\epsilon(\delta) < \frac{\beta}{2A_T}\epsilon_1$, where ϵ_1 is as in Corollary 2.5. For $\lambda_0 \in \partial B(e^{i\theta}, \frac{\delta}{2}) \cap \Gamma_{\frac{1}{4}}(e^{i\theta})$ and all $\lambda \in B(\lambda_0, \beta\delta) \setminus (E_\delta \cup E_\delta^f)$, it is clear that

$$f(\lambda)\mathcal{C}(G\mu)(\lambda) = \int \frac{f(z)}{z-\lambda}G(z)d\mu(z) = \mathcal{C}(fG\mu)(\lambda)$$

since $\frac{f(z)-f(\lambda}{z-\lambda} \in R^t(K,\mu)$. Together with (2.11) and (2.12), we have the following calculation:

$$|1-\bar{\lambda}_0\lambda| \ge 1-|\bar{\lambda}_0|^2 - |\lambda-\lambda_0||\bar{\lambda}_0| \ge 1-|\bar{\lambda}_0|^2 - \beta\delta|\lambda_0|$$

and

$$\begin{aligned}(1-|\lambda_0|^2)^{\frac{1}{t}}|f(\lambda)| \le & \frac{|(1-\bar{\lambda}_0\lambda)^{\frac{2}{t}}(1-|\lambda_0|^2)^{-\frac{1}{t}}f(\lambda)|}{(1-\beta\frac{\delta|\lambda_0|}{1-|\lambda_0|^2})^{\frac{2}{t}}} \\ = & \frac{1}{(1-\beta\frac{\delta|\lambda_0|}{1-|\lambda_0|^2})^{\frac{2}{t}}}\left|\frac{\mathcal{C}\left(\frac{(1-\bar{\lambda}_0 z)^{\frac{2}{t}}}{(1-|\lambda_0|^2)^{\frac{1}{t}}}fG\mu\right)(\lambda)}{\mathcal{C}(G\mu)(\lambda)}\right| \\ \le & \frac{b+\frac{1+4\beta}{1-4\beta}\left(\int_{\mathbb{T}}\frac{1-|\lambda_0|^2}{|1-\bar{\lambda}_0 z|^2}|G|^{t'}d\mu\right)^{\frac{1}{t'}}}{(1-4\beta)^{\frac{2}{t}}(|G(e^{i\theta})|h(e^{i\theta})-b)}\|f\|_{L^t(\mu)}.\end{aligned}$$

Since $\gamma(E_\delta \cup E_\delta^f) < \epsilon_1\beta\delta$, from Corollary 2.5, we conclude

$$M_{\lambda_0} \le \sup_{\substack{f\in Rat(K)\\ \|f\|_{L^t(\mu)}=1}} |f(\lambda_0)| \le \sup_{\substack{f\in Rat(K)\\ \|f\|_{L^t(\mu)}=1}} \|f\|_{L^\infty(B(\lambda_0,\beta\delta)\setminus(E_\delta\cup E_\delta^f))}$$

for $\lambda_0 \in \partial B(e^{i\theta},\frac{\delta}{2}) \cap \Gamma_{\frac{1}{4}}(e^{i\theta})$. Hence,

$$\overline{\lim_{\Gamma_{\frac{1}{4}}(e^{i\theta})\ni\lambda_0\to e^{i\theta}}}(1-|\lambda_0|^2)^{\frac{1}{t}}M_{\lambda_0} \le \frac{b+\frac{1+4\beta}{1-4\beta}|G(e^{i\theta})|(h(e^{i\theta}))^{\frac{1}{t'}}}{(1-4\beta)^{\frac{2}{t}}(|G(e^{i\theta})|h(e^{i\theta})-b)}$$

since $\frac{1-|\lambda_0|^2}{|1-\bar{\lambda}_0 z|^2}$ is the Poisson kernel. Taking $b\to 0$ and $\beta\to 0$, we get

$$\lim_{\Gamma_{\frac{1}{4}}(e^{i\theta})\ni\lambda\to e^{i\theta}}(1-|\lambda|^2)^{\frac{1}{t}}M_\lambda \le \frac{1}{h(e^{i\theta})^{\frac{1}{t}}}.$$

The reverse inequality is from Lemma 1 in [8] (applying the lemma to testing function $(1-\bar{\lambda}_0 z)^{-\frac{2}{t}}$). This completes the proof. □

Acknowledgments The author would like to thank the referee for providing helpful comments.

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