On Parametric Border Bases

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Abstract. We study several properties of border bases of parametric polynomial ideals and introduce a notion of a minimal parametric border basis. It is especially important for improving the quantifier elimination algorithm based on the computation of comprehensive Gröbner systems.

Keywords: Parametric border basis · Comprehensive Gröbner system · Quantifier elimination

1 Introduction

We study properties of border bases of zero-dimensional parametric polynomial ideals. Main motivation of our work is to improve the CGS-QE algorithm introduced in [1]. It is a special type of a quantifier elimination (QE) algorithm which has a great effect on QE of a first order formula containing many equalities. The most essential part of the algorithm is to eliminate all existential quantifiers $\exists \bar{X}$ from the following basic first order formula:

$$\phi(\bar{A}) \wedge \exists \bar{X} \; (\bigwedge_{1 \leq i \leq s} f_i(\bar{A}, \bar{X}) = 0 \wedge \bigwedge_{1 \leq i \leq t} h_i(\bar{A}, \bar{X}) \geq 0) \tag{1}$$

with polynomials $f_1, \dots, f_s, h_1, \dots, h_t$ in $\mathbb{Q}[\bar{A}, \bar{X}]$ such that the parametric ideal $I = \langle f_1, \dots, f_s \rangle$ is zero-dimensional in $\mathbb{C}[\bar{X}]$ for any specialization of the parameters $\bar{A} = A_1, \dots, A_m$ satisfying $\phi(\bar{A})$, where $\phi(\bar{A})$ is a quantifier free formula consisting only of equality $=$ and disequality $\neq$. The algorithm computes a reduced comprehensive Gröbner system (CGS) $\mathcal{G} = \{(\mathcal{S}_1, G_1), \dots, (\mathcal{S}_r, G_r)\}$ of the parametric ideal I on the algebraically constructible set $\mathcal{S} = \{\bar{a} \in \mathbb{C}^m | \phi(\bar{a})\}$, then applies the method of [9] with several improvements of [2–4,7]. One of the most important properties of the reduced CGS is that $\mathbb{C}[\bar{X}]/\langle f_1(\bar{X}, \bar{a}) \dots, f_s(\bar{X}, \bar{a})\rangle$ has an invariant basis $\{t \in T(\bar{X}) : t \nmid LT(g) \text{ for any } g \in G_i\}$ as a $\mathbb{C}$-vector space for every $\bar{a} \in \mathcal{S}_i$. It enables us to perform several uniform computations with parameters $\bar{A}$ for every $\bar{a} \in \mathcal{S}_i$. (More detailed descriptions can be found in [1].) In order to obtain a simple quantifier free formula, a compact representation

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of a reduced CGS of I is desirable, minimizing the number r of the partition $\mathcal{S}_1, \ldots, \mathcal{S}_r$ of $\mathcal{S}$ is particularly important. Border bases are alternative tools for handling zero-dimensional ideals [5]. We have observed that the reduced CGS can be replaced with a parametric border basis in our algorithm. Since border bases have several nice properties which Gröbner bases do not possess, we can obtain a simpler quantifier free formula using a parametric border basis.

In this paper, we study border bases in parametric polynomial rings. We give a formal definition of a parametric border basis and show several properties which are important for improving the CGS-QE algorithm. Since our work is still on going and the paper is a short paper, we do not get deeply involved in the application of parametric border bases to QE.

The paper is organized as follows. In Sect. 2, we first give a quick review of a CGS for understanding the merit of our work, then give a formal definition of a parametric border basis. In Sect. 3, we introduce our main results together with a rather simple example for understanding our work. Numerical stability is one of the most important properties of border bases. In Sect. 4, we study this property in our setting. We follow the book [5] for the terminologies and notations concerning border bases.

2 Preliminary

In the rest of the paper, let $\mathbb{Q}$ and $\mathbb{C}$ denote the field of rational numbers and complex numbers, $\bar{X}$ and $\bar{A}$ denote some variables $X_1, \ldots, X_n$ and $A_1, \ldots, A_m$, $T(\bar{X})$ denote a set of terms in $\bar{X}$. For $t_1, t_2 \in T(\bar{X})$, $t_1 \mid t_2$ and $t_1 \nmid t_2$ denote that "t_2 is divisible by t_1" and "t_2 is not divisible by t_1" respectively. For a polynomial $f \in \mathbb{C}[\bar{A}, \bar{X}]$, regarding f as a member of a polynomial ring $\mathbb{C}[\bar{A}][\bar{X}]$ over the coefficient ring $\mathbb{C}[\bar{A}]$, its leading term and coefficient w.r.t. an admissible term order $\succ$ of $T(\bar{X})$ are denoted by $LT_{\succ}(f)$ and $LC_{\succ}(f)$ respectively. When $\succ$ is clear from context, they are simply denoted by $LT(f)$ and $LC(f)$.

2.1 Comprehensive Gröbner System

Definition 1. *For an algebraically constructible subset (ACS in short) $\mathcal{S}$ of $\mathbb{C}^m$, a finite set $\{\mathcal{S}_1, \ldots, \mathcal{S}_r\}$ of ACSs of $\mathbb{C}^m$ which satisfies $\cup_{i=1}^r \mathcal{S}_i = \mathcal{S}$ and $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset (i \neq j)$ is called an algebraic partition of $\mathcal{S}$. Each $\mathcal{S}_i$ is called a segment.*

Definition 2. *Fix an admissible term order on $T(\bar{X})$. For a finite set $F \subset \mathbb{Q}[\bar{A}, \bar{X}]$ and an ACS $\mathcal{S}$ of $\mathbb{C}^m$, a finite set of pairs $\mathcal{G} = \{(G_1, \mathcal{S}_1), \ldots, (G_r, \mathcal{S}_r)\}$ with finite sets $G_1, \ldots, G_r$ of $\mathbb{Q}[\bar{A}, \bar{X}]$ satisfying the following properties is called a reduced comprehensive Gröbner system (CGS) of $\langle F \rangle$ on $\mathcal{S}$ with parameters $\bar{A}$. (When $\mathcal{S}$ is the whole space $\mathbb{C}^m$, "on $\mathbb{C}^m$" is usually omitted.)*

1. *$\{\mathcal{S}_1, \ldots, \mathcal{S}_r\}$ is an algebraic partition of $\mathcal{S}$.*
2. *For each i and $\bar{a} \in \mathcal{S}_i$, $G_i(\bar{a})$ is a reduced Gröbner basis of $\langle F(\bar{a}) \rangle \subset \mathbb{C}[\bar{X}]$, where $G_i(\bar{a}) = \{g(\bar{a}, \bar{X}) | g(\bar{A}, \bar{X}) \in G_i\}$ and $F(\bar{a}) = \{f(\bar{a}, \bar{X}) | f(\bar{A}, \bar{X}) \in F\}$.*
3. *For each i, $\mathrm{LC}(g)(\bar{a}) \neq 0$ for every $g \in G_i$ and $\bar{a} \in \mathcal{S}_i$.*

Remark 3. *The set of leading terms of all polynomials of $G_i(\bar{a})$ is invariant for each $\bar{a} \in \mathcal{S}_i$. Hence, not only the dimension of the ideal $\langle G_i(\bar{a})\rangle$ is invariant but also the $\mathbb{C}$-vector space $\mathbb{C}[\bar{X}]/\langle F(\bar{a})\rangle$ has the same finite basis $\{t \in T(\bar{X}) : t \nmid LT(g) \text{ for any } g \in G_i\}$ for every $\bar{a} \in \mathcal{S}_i$ when $\langle F(\bar{a})\rangle$ is zero-dimensional.*

2.2 Border Bases in Parametric Polynomial Rings

Definition 4. *For a finite set $F \subset \mathbb{Q}[\bar{A}, \bar{X}]$ and an ACS $\mathcal{S}$ of $\mathbb{C}^m$ such that the ideal $\langle F(\bar{a})\rangle$ is zero-dimensional for each $\bar{a} \in \mathcal{S}$, a finite set of triples $\mathcal{B} = \{(B_1, \mathcal{S}_1, \mathcal{O}_1), \ldots, (B_r, \mathcal{S}_r, \mathcal{O}_r)\}$ with a finite set B_i of $\mathbb{Q}(\bar{A})[\bar{X}]$ and an order ideal $\mathcal{O}_i$ of $T(\bar{X})$ for each i satisfying the following properties is called a parametric border basis (PBB) of $\langle F\rangle$ on $\mathcal{S}$ with parameters $\bar{A}$.*
(When $\mathcal{S}$ is the whole space $\mathbb{C}^m$, "on $\mathbb{C}^m$" is usually omitted.)

1. *$\{\mathcal{S}_1, \ldots, \mathcal{S}_r\}$ is an algebraic partition of $\mathcal{S}$.*
2. *For each i, any denominator of a coefficient of an element of B_i does not vanish on $\mathcal{S}_i$.*
3. *For each i and $\bar{a} \in \mathcal{S}_i$, $B_i(\bar{a})$ is a $\mathcal{O}_i$-border basis of $\langle F(\bar{a})\rangle \subset \mathbb{C}[\bar{X}]$.*

3 Properties of Parametric Border Bases

Consider the set $F = \{X^2 + \frac{1}{4}Y^2 - AXY + B - 1, \frac{1}{4}X^2 + Y^2 - BXY + A - 1\}$ of parametric polynomials in $\mathbb{Q}[A, B, X, Y]$ with parameters A and B, which is a similar but a little bit more complicated example than the one discussed in the book [5]. $\langle F(a, b)\rangle$ is zero-dimensional for every $(a, b) \in \mathbb{C}^2$. It has the following reduced CGS $\mathcal{G} = \{(G_1, \mathcal{S}_1), \ldots, (G_7, \mathcal{S}_7)\}$ w.r.t. the lexicographic term order such that $X \succ Y$.

$G_1 = \{-5X^2 + 20BYX + 4, -5Y^2 - 20B + 4\}, \mathcal{S}_1 = \mathbb{V}(A - 4B),$
$G_2 = \{5X^2 - 4YX + 5B - 5, (5B - 1)YX - 5Y^2, (20B - 29)Y^3 + (-25B^3 + 35B^2 - 11B + 1)Y\},$
$\mathcal{S}_2 = \mathbb{V}(4A - B - 3) \setminus \{(\frac{4}{5}, \frac{1}{5}), (\frac{89}{80}, \frac{29}{20})\},$
$G_3 = \{16(A - 4B)(4A - B - 3)X + (-64A^2 + 272AB - 64B^2 - 225)Y^3 + (-64A^3 + (256B + 64)A^2 +$
$(64B^2 - 320B - 240)A - 256B^3 + 256B^2 + 60B + 180)Y, (-64A^2 + 272AB - 64B^2 - 225)Y^4 + (-64A^3 +$
$(256B + 64)A^2 + (64B^2 - 320B - 480)A - 256B^3 + 256B^2 + 120B + 360)Y^2 - 16(4A - B - 3)^2\},$
$S_3 = \mathbb{C}^2 \setminus \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_4 \cup \cdots \cup \mathcal{S}_7 = \mathbb{C}^2 \setminus \mathbb{V}((A - 4B)(4A - B - 3)(64A^2 - 272AB + 64B^2 + 225)),$
$G_4 = \{20X^2 + 9, Y\}, \mathcal{S}_4 = \{(\frac{89}{80}, \frac{29}{20})\},$
$G_5 = \{58Y^2 + 245, 35X - Y\}, \mathcal{S}_5 = \{(\frac{101}{20}, \frac{29}{20})\},$
$G_6 = \{60(20B - 29)X + ((400B^2 - 400B - 36)A - 1600B^3 + 1600B^2 + 519B - 375)Y, 15((400B-$
$64)A - 64B - 425)Y^2 + 128((200B^2 - 80B - 42)A - 50B^3 + 20B^2 - 177B + 75)\},$
$\mathcal{S}_6 = \mathbb{V}(64A^2 - 272AB + 64B^2 + 225) \setminus \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5,$
$G_7 = \{1\}, \mathcal{S}_7 = \mathbb{V}(-10881A - 10000B^3 + 8400B^2 + 9744B + 3925, 25A^2 - 17A + 25B^2 - 17B-$
$25, (400B - 64)A - 64B - 425) = \{(\alpha_1 + \beta_1 \mathrm{i}, \alpha_1 - \beta_1 \mathrm{i}), (\alpha_1 - \beta_1 \mathrm{i}, \alpha_1 + \beta_1 \mathrm{i}), (-\alpha_2 - \beta_2 \mathrm{i}, -\alpha_2+$
$\beta_2 \mathrm{i}), (\alpha_2 + \beta_2 \mathrm{i}, -\alpha_2 - \beta_2 \mathrm{i})\} with \alpha_1 \risingdotseq 1.16856, \beta_1 \risingdotseq 0.266288, \alpha_2 \risingdotseq 0.668559, \beta_2 \risingdotseq 0.633712.$

Note that the $\mathbb{C}$-vector space $\mathbb{C}[X, Y]/\langle F(a, b)\rangle$ has dimension 4, 4, 4, 2, 2, 2 and 1 for $(a, b) \in \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5, \mathcal{S}_6$ and $\mathcal{S}_7$ respectively. Even though $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_3$ are connected and the $\mathbb{C}$-vector space $\mathbb{C}[X, Y]/\langle F(a, b)\rangle$ has the same dimension 4 on $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$, we cannot glue them into a single segment as long as we use a reduced CGS. On the other hand, we can glue them into a single segment with the following PBB $\mathcal{B} = \{(B_1, \mathcal{S}'_1, \mathcal{O}_1), \ldots, (B_5, \mathcal{S}'_5, \mathcal{O}_5)\}$.

$$\begin{aligned}
&B_1 = Y^2 + \tfrac{4(A-4B)}{15}XY + \tfrac{4}{15}(4A-B-3), XY^2 + \tfrac{16(A-4B)(A-4B+3)}{64A^2-272AB+64B^2+225}Y \\
&\quad + \tfrac{60(4A-B-3)}{64A^2-272AB+64B^2+225}X, \\
&X^2 + \tfrac{4(B-4A)}{15}XY + \tfrac{4}{15}(4B-A-3), X^2Y + \tfrac{16(B-4A)(B-4A+3)}{64A^2-272AB+64B^2+225}X + \tfrac{60(4B-A-3)}{64A^2-272AB+64B^2+225}Y, \\
&\mathcal{S}_1' = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 = \mathbb{C}^2 \setminus \mathbb{V}(64A^2 - 272AB + 64B^2 + 225), \mathcal{O}_1 = \{1, X, Y, XY\}, \\
&B_2 = \{X^2 + \tfrac{9}{20}, Y, XY\}, \mathcal{S}_2' = \mathcal{S}_4, \mathcal{O}_2 = \{1, X\}, \\
&B_3 = \{X - \tfrac{1}{35}Y, XY + \tfrac{7}{58}, Y^2 + \tfrac{245}{58}, \}, \mathcal{S}_3' = \mathcal{S}_5 \mathcal{O}_3 = \{1, Y\}, \\
&B_4 = \{X + \tfrac{(400B^2-400B-36)A-1600B^3+1600B^2+519B-375}{60(20B-29)}Y, \\
&XY - \tfrac{32((400B^2-400B-36)A-1600B^3+1600B^2+519B-375)((200B^2-80B-42)A-50B^3+20B^2-177B+75)}{225(20B-29)((400B-64)A-64B-425)}, \\
&Y^2 + \tfrac{128((200B^2-80B-42)A-50B^3+20B^2-177B+75)}{15((400B-64)A-64B-425)}\}, \mathcal{S}_4' = \mathcal{S}_6, \mathcal{O}_4 = \{1, Y\}, \\
&B_5 = \{1\}, \mathcal{S}_5' = \mathcal{S}_7, \mathcal{O}_5 = \emptyset.
\end{aligned}$$

Note also that $\mathbb{C}[X,Y]/\langle F(a,b)\rangle$ has the same dimension 2 on $\mathcal{S}_2', \mathcal{S}_3'$ and $\mathcal{S}_4'$. Even though $\mathcal{S}_2'$, $\mathcal{S}_4'$ and $\mathcal{S}_3'$, $\mathcal{S}_4'$ are connected, however, we cannot glue them into a single segment for both of them. The reason for $\mathcal{S}_2'$, $\mathcal{S}_4'$ is that $\langle F(a,b)\rangle$ has the only one order ideal $\mathcal{O}_2$ on $(a,b) \in S_2'$ (i.e., $(a,b) = (\frac{89}{80}, \frac{29}{20})$), while $\mathcal{S}_4'$ contains a point $(\frac{29}{20}, \frac{89}{80})$ such that $\langle F(\frac{29}{20}, \frac{89}{80})\rangle$ has the only one order ideal $\mathcal{O}_4$ different from $\mathcal{O}_2$. The reason for $\mathcal{S}_3'$, $\mathcal{S}_4'$ is rather subtle. We cannot have a uniform parametric representation for both of B_3 and B_4. Those observations lead us to the following definition of a *minimal* PBB.

Definition 5. *A PBB $\mathcal{B} = \{(B_1, \mathcal{S}_1, \mathcal{O}_1), \ldots, (B_r, \mathcal{S}_r, \mathcal{O}_r)\}$ of $\langle F\rangle$ is said to be minimal if for any pair $(\mathcal{S}_i, \mathcal{S}_j)$ of connected segments such that $\mathbb{C}[\bar{X}]/\langle F(\bar{a}, \bar{X})\rangle$ has the same dimension on them it satisfies either of the following:*

1. *$\mathcal{O}_i \neq \mathcal{O}_j$, but also $\langle F(\bar{a})\rangle$ does not possess a common order ideal on $\mathcal{S}_i \cup \mathcal{S}_j$.*
2. *$\mathcal{O}_i = \mathcal{O}_j$ and there exist no uniform parametric representation for both of B_i and B_j on $\mathcal{S}_i \cup \mathcal{S}_j$.*

Where "$\mathcal{S}_i$ and $\mathcal{S}_j$ are connected" means that $\overline{S_i} \cap \overline{S_j} \cap (S_i \cup S_j) \neq \emptyset$, $\overline{X}$ denotes the Zariski closure of X. Intuitively, $\mathcal{S}_i$ and $\mathcal{S}_j$ are connected if and only if there exist two points $\bar{a}_i \in \mathcal{S}_i$ and $\bar{a}_j \in \mathcal{S}_j$ which are connected by a continuous path in $\mathcal{S}_i \cup \mathcal{S}_j$.

Note that a Gröbner basis can be considered as a border basis with the naturally induced order ideal, we can convert a reduced CGS into a PBB using uniform parametric monomial reductions on each segment. Hence, we can compute a PBB of any given $\langle F\rangle$. Existence of a minimal PBB is also obvious, however, we have not obtained an effective algorithm yet. The reason is that we do not have an algorithm to decide whether the property 2 holds yet, while it is easy to check the property 1 using the (parametric) border division algorithm by B_i on $\mathcal{S}_i$ and by B_j on $\mathcal{S}_j$. At this time, we have obtained the following results.

Lemma 6. *Let $(B, \mathcal{S}, \mathcal{O})$ be a member of a PBB $\mathcal{B}$ of $\langle F\rangle$ such that $\mathcal{S} = \mathbb{C}^m \setminus \mathbb{V}(I)$ for some ideal $I \subset \mathbb{Q}[\bar{A}]$. If there are other members $(B_{n_1}, \mathcal{S}_{n_1}, \mathcal{O}_{n_1}), \ldots, (B_{n_k}, \mathcal{S}_{n_k}, \mathcal{O}_{n_k})$ of $\mathcal{B}$ such that $\mathbb{C}[\bar{X}]/\langle F(\bar{a}, \bar{X})\rangle$ has the same dimension on $\mathcal{S} \cup \mathcal{S}_{n_1} \cup \cdots \cup \mathcal{S}_{n_k}$ and $\langle F(\bar{a}, \bar{X})\rangle$ also has a unique order ideal $\mathcal{O}'$ on every $\bar{a} \in \mathcal{S} \cup \mathcal{S}_{n_1} \cup \cdots \cup \mathcal{S}_{n_k}$, then we can compute a finite subset B' of $\mathbb{Q}(\bar{A})[\bar{X}]$ such that $B'(\bar{a})$ is a $\mathcal{O}'$-border basis of $\langle F(\bar{a})\rangle$ on $\mathcal{S} \cup \mathcal{S}_{n_1} \cup \cdots \cup \mathcal{S}_{n_k}$.*

In the above example, by this lemma, we can glue $(G_1, S_1), (G_2, S_2), (G_3, S_3)$ into $(B_1, \mathcal{S}'_1, \mathcal{O}_1)$ with $\mathcal{S}'_1 = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ and the order ideal $\mathcal{O}_1$ induced from (G_1, S_1).

Lemma 7. *Let $(B_i, \mathcal{S}_i, \mathcal{O})$ and $(B_j, \mathcal{S}_j, \mathcal{O})$ be members of a PBB. If there exists $\bar{a} \in \mathcal{S}_i \cap \overline{\mathcal{S}_j}$ such that we cannot specialize some $t + h(\bar{A}, \bar{X}) \in B_j$ with $t \in \partial\mathcal{O}$ and $\bar{A} = \bar{a}$, then there exists no uniform parametric representation for B_i and B_j on $\mathcal{S}_i \cup \mathcal{S}_j$.*

In the above example, B_3 and B_4 do not have a uniform parametric representation since the denominator $60(20B - 29)$ of a coefficient of a polynomial in B_4 vanishes for $(A, B) = (\frac{101}{20}, \frac{29}{20}) \in \mathcal{S}'_3 \cap \overline{\mathcal{S}'_4}$.

4 Stability of Parametric Border Basis

Numerical stability is one of the most important properties of border bases. We give a precise definition of the stability of a border basis of a parametric ideal as follows.

Definition 8. *Let F be a finite subset of $\mathbb{Q}[\bar{A}, \bar{X}]$ and $\mathcal{S}$ be a subset (not necessary to be algebraically constructible) of $\mathbb{C}^m$ such that the $\mathbb{C}$-vector space $\mathbb{C}[\bar{X}]/\langle F(\bar{a}, \bar{X})\rangle$ has an invariant finite dimension for every $\bar{a} \in \mathcal{S}$. For $\bar{a} \in \mathcal{S}$ which is not an isolated point of $\mathcal{S}$, let $\langle F(\bar{a}, \bar{X})\rangle$ have a $\mathcal{O}$-border basis $B = \{t_1 + g_1, \ldots, t_l + g_l\}$ with $\{t_1, \ldots, t_l\} = \partial\mathcal{O}$ and $g_1, \ldots, g_l \in \mathbb{C}[\bar{X}]$ for some order ideal $\mathcal{O} = \{s_1, \ldots, s_k\}$. If there exists an open neighborhood $\mathcal{S}' \subset \mathcal{S}$ of $\bar{a}$ such that $\langle F(\bar{c}, \bar{X})\rangle$ has an invariant order ideal $\mathcal{O}$ together with a $\mathcal{O}$-border basis $\{t_1 + \phi_1^1(\bar{c})s_1 + \cdots + \phi_k^1(\bar{c})s_k, \ldots, t_l + \phi_1^l(\bar{c})s_1 + \cdots + \phi_k^l(\bar{c})s_k\}$ for each $\bar{c} \in \mathcal{S}'$ with mappings ϕ_j^i from $\mathcal{S}'$ to $\mathbb{C}$. (Note that it is uniquely determined.) In addition, if these mappings are continuous at $\bar{A} = \bar{a}$ that is $\lim_{\bar{c}\to\bar{a}} \phi_1^i(\bar{c})s_1 + \cdots + \phi_k^i(\bar{c})s_k = g_i$ for each $i = 1, \ldots, l$, then we say B is stable at $\bar{A} = \bar{a}$ in $\mathcal{S}$.*

Unfortunately, the stability property does not hold for some parametric ideal $\langle F(\bar{A}, \bar{X})\rangle$.

Example 9. *Let $F = \{A(X - Y), AX^4 + X^2 + A - 1, AY^4 + Y^2 + A - 1\}$. $\mathbb{C}[X, Y]/\langle F(a)\rangle$ has dimension 4 for any $a \in \mathcal{S} = \mathbb{C}$. Possible order ideals of $\langle F(a)\rangle$ are $\{1, X, X^2, X^3\}$ and $\{1, Y, Y^2, Y^3\}$ for $a \neq 0$ but only $\{1, X, Y, XY\}$ for $a = 0$. Hence, the $\{1, X, Y, XY\}$-border basis B of $\langle F(0)\rangle$ is not stable at $A = 0$ in $\mathcal{S}$.*

In case a parametric ideal has an invariant order ideal in some connected region $\mathcal{S}$ its border basis seems to be stable at any point of $\mathcal{S}$, although we have not proved it yet.

Example 10. *For the example of the previous section, $\langle F(a, b, X, Y)\rangle$ has an order ideal $\{1, Y\}$ for every $(a, b) \in \mathcal{S}'_3 \cup \mathcal{S}'_4$. As is mentioned at the end of previous section, we do not have a uniform parametric representation of the $\{1, Y\}$-border basis of $\langle F(a, b, X, Y)\rangle$ for every $(a, b) \in \mathcal{S}'_3 \cup \mathcal{S}'_4$. It seems that the $\{1, Y\}$-border* *basis of* $\langle F(a,b,X,Y)\rangle$ *is not stable at* $(a,b) = (\frac{101}{20}, \frac{29}{20})$*. But it is actually stable at* $(A,B) = (\frac{101}{20}, \frac{29}{20})$ *in* $\mathcal{S}'_3 \cup \mathcal{S}'_4$*. That is* $\frac{(400B^2-400B-36)A-1600B^3+1600B^2+519B-375}{60(20B-29)}$, $\frac{32((400B^2-400B-36)A-1600B^3+1600B^2+519B-375)((200B^2-80B-42)A-50B^3+20B^2-177B+75)}{225(20B-29)((400B-64)A-64B-425)}$ *and* $\frac{128((200B^2-80B-42)A-50B^3+20B^2-177B+75)}{15((400B-64)A-64B-425)}$ *converge to* $-\frac{1}{35}, -\frac{7}{58}$ *and* $\frac{245}{58}$ *as* $(A,B) \rightarrow (\frac{101}{20}, \frac{29}{20})$ *in* $\mathcal{S}'_3 \cup \mathcal{S}'_4$*.*

5 Conclusion and Remarks

A *terrace* introduced in [8] is an ideal algebraic structure for a canonical representation of a comprehensive Gröbner system. It is the smallest commutative von Neumann regular ring extending $\mathbb{Q}[\bar{A}]$, meanwhile $\mathbb{Q}(\bar{A})$ is the smallest field extending $\mathbb{Q}[\bar{A}]$. If we are allowed to use this structure to represent coefficients of parametric polynomials, we can also similarly define a PBB and a minimal PBB. For the definition of a minimal PBB, we do not need the property 2, that is we always have $\mathcal{O}_i \neq \mathcal{O}_j$. Furthermore the better thing is that we can always compute it, though we have not tried to use it yet since the implementation of the structure of terrace is not very straightforward.

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