Chapter 2
$\mathbb{R}$-Varieties

In the introduction to Chapter 1 we warned the reader that our category of *real algebraic varieties* was insufficient for certain purposes. In this chapter we introduce complex varieties with a conjugation map, which Atiyah [Ati66] calls "real spaces".

In this introduction we will assume for simplicity that our varieties are projective. Let $X \subset \mathbb{P}^n(\mathbb{C})$ be a complex algebraic set defined by *real* homogeneous equations. The set $V \subset \mathbb{P}^n(\mathbb{R})$ of real solutions to these equations, which is simply $X \cap \mathbb{P}^n(\mathbb{R})$, is then a real algebraic set. Both X and V are sometimes called *real* varieties in the literature, depending on the type of problem being studied. It is tempting to distinguish the objects V and X by calling V a real algebraic variety (as in Chapter 1) and X an algebraic variety defined over $\mathbb{R}$. Some authors make this distinction—see [BK99, Hui95] for example—but not all—see [Sil89, DIK00] for example. It is fairly common to consider that a "real algebraic variety" and an "algebraic variety defined over $\mathbb{R}$" are the same thing, namely a complex algebraic variety which has a set of real defining equations, or alternatively, a complex variety stable under conjugaison.

In practice we can mostly specify which point of view we are using on a case by case basis, since many problems require just one point of view or the other. Occasionally, however, we will need to jump between definitions in the course of a single argument. We have chosen to call a pair of a complex algebraic variety and a conjugation map an *algebraic* $\mathbb{R}$*-variety* (see Definition 2.1.10) and reserve the expression *real algebraic variety* for algebraic subsets of $\mathbb{P}^n(\mathbb{R})$. Note that the "real varieties" defined in [Sil89, I.2] and [DIK00] are our $\mathbb{R}$-varieties.

This chapter deals with $\mathbb{R}$-varieties and their relationship with the real algebraic varieties defined in the previous chapter. After defining $\mathbb{R}$-varieties and studying their main properties in Section 2.1, we explain to what extent an $\mathbb{R}$-variety determines a real algebraic variety in Section 2.2. In the subsequent section we will consider the following question: given a real algebraic variety, does it determine an $\mathbb{R}$-variety? We end Section 2.2 with a summary of the logical relations between real algebraic varieties, $\mathbb{R}$-varieties and schemes over $\mathbb{R}$, achieving thereby one of the goals stated in the Introduction. The final part of this chapter deals with refinements and consequences of this theory. Section 2.5, which is technically difficult and can be skipped

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on first reading, deals with sheaves and bundles, Section 2.6 deals with divisors and Section 2.7 deals with ℝ-plane curves.

2.1 Real Structures on Complex Varieties

In this section we introduce complex varieties to the study of real varieties. The following example illustrates their usefulness: further on, Example 2.1.29 illustrates the usefulness of abstract real structures on complex varieties.

Example 2.1.1 (*Continuation of Example* 1.5.20) Let us return to the real irreducible algebraic variety $F := \mathcal{Z}(x^2 + y^2) \subset \mathbb{A}^2(\mathbb{R})$ which is an isolated point $(0, 0)$. Consider the algebraic set $X := \mathcal{Z}_{\mathbb{C}}(x^2 + y^2) \subset \mathbb{A}^2(\mathbb{C})$ which is a reducible complex curve. The restriction of $\sigma : (x, y) \mapsto (\overline{x}, \overline{y})$ to X is an involution sending X to itself: its set of fixed points is $F = X^\sigma = \{(0, 0)\}$. The complex algebraic *curve* X has a unique *real* point. The point $(0, 0)$ is the intersection of the two irreducible components $\mathcal{Z}_{\mathbb{C}}(x - iy)$ and $\mathcal{Z}_{\mathbb{C}}(x + iy)$ and it is the only real point of X. We have $\dim X = 1$ and $\dim F = 0$.

Going further, consider the variety $V := \mathcal{Z}(x^2 + y^2 - z) \subset \mathbb{A}^3(\mathbb{R})$ and the morphism $\pi : V \to \mathbb{A}^1(\mathbb{R})$, $(x, y, z) \mapsto z$. For any $z_0 \in \mathbb{A}^1(\mathbb{R})$ the fibre $\pi^{-1}(z_0)$ is an algebraic subset of the affine plane $\mathcal{Z}((z - z_0)) \simeq \mathbb{A}^2(\mathbb{R})$. If $z_0 > 0$, $\pi^{-1}(z_0)$ is a non singular real curve; $\pi^{-1}(0) \simeq F$ on the other hand is a point and for all $z_0 < 0$, $\pi^{-1}(z_0)$ is empty. Consider $Y := \mathcal{Z}_{\mathbb{C}}(x^2 + y^2 - z) \subset \mathbb{A}^3(\mathbb{C})$ and $\pi_{\mathbb{C}} : Y \to \mathbb{A}^1(\mathbb{C})$, $(x, y, z) \mapsto z$. For any z_0 the preimage $\pi_{\mathbb{C}}^{-1}(z_0)$ is an algebraic subset of the affine plane $\mathcal{Z}((z - z_0)) \simeq \mathbb{A}^2(\mathbb{C})$. Consider a point $z_0 \in \mathbb{A}^1(\mathbb{R}) \subset \mathbb{A}^1(\mathbb{C})$. If $z_0 > 0$ then $\pi_{\mathbb{C}}^{-1}(z_0)$ is a non singular complex curve whose real locus is a non singular real curve. If $z_0 = 0$ then $\pi_{\mathbb{C}}^{-1}(0) \simeq X$ is a singular complex curve whose real locus is a point. If $z_0 < 0$ then $\pi_{\mathbb{C}}^{-1}(z_0)$ is a non singular complex curve whose real locus is empty.

The complex variety Y provides a deeper understanding of this example. The real variety V can be recovered as the set of fixed points of the involution defined by complex conjugation on $\mathbb{C}^3$. More generally, we will seek to imitate the standard conjugation map. On $\mathbb{A}^n(\mathbb{C}) = \mathbb{C}^n$ we denote by $\sigma_{\mathbb{A}} := \sigma_{\mathbb{A}^n}$ the involution

$$\sigma_{\mathbb{A}} : \begin{cases} \mathbb{A}^n(\mathbb{C}) & \longrightarrow & \mathbb{A}^n(\mathbb{C}) \\ (z_1, \dots, z_n) & \longmapsto & (\overline{z_1}, \dots, \overline{z_n}) \end{cases} .$$

In particular, for any $z \in \mathbb{C}$, $\sigma_{\mathbb{A}^1}(z) = \overline{z}$. Similarly, on $\mathbb{P}^n(\mathbb{C})$ we denote by $\sigma_{\mathbb{P}} := \sigma_{\mathbb{P}^n}$ the standard conjugation map

$$\sigma_{\mathbb{P}} : \begin{cases} \mathbb{P}^n(\mathbb{C}) & \longrightarrow & \mathbb{P}^n(\mathbb{C}) \\ (x_0 : x_1 : \cdots : x_n) & \longmapsto & (\overline{x_0} : \overline{x_1} : \cdots : \overline{x_n}) \end{cases} .$$

We can recover $\mathbb{R}^n \subset \mathbb{C}^n$ as the set of fixed points of $\sigma_{\mathbb{A}^n}$ and the real projective plane $\mathbb{P}^n(\mathbb{R}) \subset \mathbb{P}^n(\mathbb{C})$ as the set of fixed points of $\sigma_{\mathbb{P}^n}$. We will generalise this situation to an arbitrary (algebraic or analytic) complex variety. In other words, we will introduce *real structures* (analogues of $\sigma_{\mathbb{A}}$ and $\sigma_{\mathbb{P}}$) on complex varieties: see Definition 2.1.10 for more details. We note immediately that for general $X \subset \mathbb{C}^n$ it is not enough to consider the restriction of $\sigma_{\mathbb{A}}$ to X for two reasons. Firstly, we have to require that this restriction induces a morphism from X to X (i.e. $\sigma_{\mathbb{A}}(X) \subset X$). Secondly, a given complex variety X can have several different *real forms* (see Definition 2.1.13) corresponding to different real structures. In other words, there are pairs of complex varieties X_1 and X_2 defined by real polynomials which are isomorphic as complex varieties but do not have an isomorphism defined over $\mathbb{R}$: see Example 2.1.29 for an example.

Let f be a holomorphic function (such as a polynomial) defined in a neighbourhood of $z_0 = (z_{0,1}, \dots, z_{0,n}) \in \mathbb{C}^n$ by

$$f(z) = \sum_{k \in \mathbb{N}^n} a_k (z_1 - z_{0,1})^{k_1} \dots (z_n - z_{0,n})^{k_n} \ .$$

There is then a *conjugate* holomorphic function of f, denoted ${}^{\sigma}f$, defined in a neighbourhood of $\overline{z_0} = (\overline{z_{0,1}}, \dots, \overline{z_{0,n}}) \in \mathbb{C}^n$ by

$${}^{\sigma}f(z) = \sum_{k \in \mathbb{N}^n} \overline{a_k} (z_1 - \overline{z_{0,1}})^{k_1} \dots (z_n - \overline{z_{0,n}})^{k_n}$$

or in other words ${}^{\sigma}f = \overline{f} \circ \sigma_{\mathbb{A}^n} = \sigma_{\mathbb{A}^1} \circ f \circ \sigma_{\mathbb{A}^n}$. If F is a subset of $\mathbb{C}^n$ defined by the vanishing of the functions $f_1, \dots, f_k$ then

$$\overline{F} := \{z \in \mathbb{C}^n | \sigma_{\mathbb{A}^n}(z) \in F\}$$

is the set of common zeros of the functions ${}^{\sigma}f_1, \dots, {}^{\sigma}f_k$. It follows that if $F \subset \mathbb{A}^n(\mathbb{C})$ is a complex algebraic affine set then $\overline{F} \subset \mathbb{A}^n(\mathbb{C})$ is also a complex algebraic affine set.

Remark 2.1.2 Note that ${}^{\sigma}f$ and $\overline{f}$ are not the same thing. If f is a holomorphic function then ${}^{\sigma}f$ is also holomorphic whereas $\overline{f} = \sigma_{\mathbb{A}} \circ f$ anti-holomorphic. Passing from f to ${}^{\sigma}f$ simply involves conjugating coefficients. If f is a polynomial then ${}^{\sigma}f$ is also a polynomial, unlike $\overline{f}$. The coefficients of the polynomial f are real if and only if ${}^{\sigma}f = f$.

Exercise 2.1.3 (Sheaf on a conjugate algebraic set)

1. *Let $\mathcal{O}$ be the sheaf of regular functions on $\mathbb{A}^n(\mathbb{C})$ (resp. $\mathbb{P}^n(\mathbb{C})$). We define a sheaf ${}^{\sigma}\mathcal{O}$ on $\mathbb{A}^n(\mathbb{C})$ (resp. $\mathbb{P}^n(\mathbb{C})$) by setting*

$${}^{\sigma}\mathcal{O}(U) := \left\{{}^{\sigma}f \mid f \in \mathcal{O}(\overline{U})\right\} \ .$$

for every open set U in $\mathbb{A}^n(\mathbb{C})$ (resp. $\mathbb{P}^n(\mathbb{C})$)
Prove that ${}^{\sigma}\mathcal{O} = \mathcal{O}$.

2. *Let $F \subset \mathbb{A}^n(\mathbb{C})$ be an affine algebraic set. The sheaf of regular functions on F is denoted $\mathcal{O}_F$ and the sheaf of regular functions on $\overline{F}$ is denoted $\mathcal{O}_{\overline{F}}$ (These are sheaves deduced from $\mathcal{O}$: equipped with these sets, F and $\overline{F}$ are sub-varieties of $\mathbb{A}^n(\mathbb{C})$—see Definition 1.3.7 and Example 1.3.8).*
Prove that if $\overline{F} = F$ then ${}^\sigma\mathcal{O}_F := ({}^\sigma\mathcal{O})_F$ is a sheaf on F which is equal to $\mathcal{O}_F$ by the above. We then say that $\mathcal{O}_F$ is an ℝ-sheaf*: see Definition 2.2.1 for more details.*

Proposition 2.1.4 *Let $X \subset \mathbb{A}^n(\mathbb{C})$ be an algebraic set. The restriction of $\sigma_{\mathbb{A}^n}$ to X is an involution of X if and only if X can be defined by real polynomials.*

Let $X \subset \mathbb{P}^n(\mathbb{C})$ be an algebraic set. The restriction of $\sigma_{\mathbb{P}^n}$ to X is an involution of X if and only if X can be defined by real homogeneous polynomials.

Proof If $X = \mathcal{Z}(P_1, \ldots, P_l)$ then by definition we have that $\overline{X} = \mathcal{Z}({}^\sigma P_1, \ldots, {}^\sigma P_l)$ and the restriction $\sigma_{\mathbb{A}}|_X$ is an endomorphism of X if and only if $\overline{X} = X$. Suppose that $\overline{X} = X$. We then have that $\mathcal{Z}(P_1, \ldots, P_l) = \mathcal{Z}({}^\sigma P_1, \ldots, {}^\sigma P_l) = \mathcal{Z}(\frac{1}{2}(P_1 + {}^\sigma P_1), \ldots, \frac{1}{2}(P_l + {}^\sigma P_l), \frac{1}{2i}(P_1 - {}^\sigma P_1), \ldots, \frac{1}{2i}(P_l - {}^\sigma P_l))$. The proposition follows on noting that for any polynomial P with complex coefficients the polynomials $\frac{1}{2}(P + {}^\sigma P)$ and $\frac{1}{2i}(P - {}^\sigma P)$ have real coefficients. The converse is immediate.

Similarly, if X is a projective algebraic variety defined in $\mathbb{P}^n(\mathbb{C})$ by homogeneous polynomial equations

$$P_1(z_0, \ldots, z_n) = \cdots = P_l(z_0, \ldots, z_n) = 0\ ,$$

then the variety $\overline{X}$ defined by ${}^{\sigma_{\mathbb{A}^{n+1}}}P_1(z_0, \ldots, z_n) = \cdots = {}^{\sigma_{\mathbb{A}^{n+1}}}P_l(z_0, \ldots, z_n) = 0$ is an algebraic subvariety of $\mathbb{P}^n(\mathbb{C})$. It is easy to check that if P is a homogeneous polynomial then $\frac{1}{2}(P + {}^\sigma P)$ and $\frac{1}{2i}(P - {}^\sigma P)$ are homogenous polynomials The restriction of $\sigma_{\mathbb{P}}$ to X is therefore an endomorphism of X if and only if X can be defined by real homogeneous polynomials. □

Before generalising the above to abstract varieties we need the following definition.

Definition 2.1.5 Let $\mathcal{L}$ be a sheaf of complex functions over a topological space X. The *anti-sheaf* $\overline{\mathcal{L}}$ of $\mathcal{L}$ is defined over any open set U in X by

$$\overline{\mathcal{L}}(U) := \{\overline{f} := \sigma_{\mathbb{A}} \circ f \mid f \in \mathcal{L}(U)\}\ .$$

More generally, let X be a topological space and let $\mathcal{L}$ be a sheaf of maps to $\mathbb{C}^n$.[1] We define the sheaf $\overline{\mathcal{L}}$ over any open set U of X by

$$\overline{\mathcal{L}}(U) := \{\overline{f} := \sigma_{\mathbb{A}^n} \circ f \mid f \in \mathcal{L}(U)\}\ .$$

[1] Note that $\mathcal{L}$ is no longer a sheaf of rings, but a sheaf of vector spaces.

Definition 2.1.6 Let $(X, \mathcal{O}_X)$ be a complex algebraic variety (resp. a complex analytic space[2]). The *conjugate* variety (resp. the *conjugate* analytic space) of X is defined to be the topological space X equipped with the anti-sheaf of $\mathcal{O}_X$

$$\overline{X} := (X, \overline{\mathcal{O}_X}) \ .$$

Exercise 2.1.7 *If F is the subset of $\mathbb{C}^n$ defined by the vanishing of functions $f_1, \ldots, f_k$ then $\overline{F} := \{z \in \mathbb{C}^n | \sigma_{\mathbb{A}^n}(z) \in F\}$ is the vanishing locus of the functions ${}^\sigma f_1, \ldots, {}^\sigma f_k$. If $F \subset \mathbb{A}^n(\mathbb{C})$ is a complex affine algebraic set then $\overline{F}$ is a complex affine algebraic set and $\sigma_{\mathbb{A}}$ induces an isomorphism of varieties from $(\overline{F}, \mathcal{O}_{\overline{F}})$ to the conjugate variety $(F, \overline{\mathcal{O}_F})$.*

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be complex algebraic varieties (resp. complex analytic spaces). In particular, $\mathcal{O}_X$ and $\mathcal{O}_Y$ are sheaves of complex valued functions. Recall that a map $\varphi \colon X \to Y$ is *regular* (resp. *holomorphic*) if and only if it is continuous and for any function $f \in \mathcal{O}_Y(V)$ the function $f \circ \varphi$ belongs to $\mathcal{O}_X(\varphi^{-1}(V))$. (See Definition 1.3.4).

Definition 2.1.8 A map $\varphi \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is *anti-regular* (resp. *anti-holomorphic*) if and only if it is continuous and for every open set V in Y and every function $f \in \mathcal{O}_Y(V)$ the function $\overline{f} \circ \varphi$ belongs to $\mathcal{O}_X(\varphi^{-1}(V))$.

Remark 2.1.9 If X is a complex algebraic variety (resp. complex analytic space) and $\mathcal{O}_X$ is its sheaf of regular functions (resp. holomorphic functions) the anti-sheaf $\overline{\mathcal{O}_X}$ is the sheaf of anti-regular (resp. anti-holomorphic) functions. A continuous map $\varphi \colon X \to Y$ is anti-regular (or anti-holomorphic) from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ if and only if it is regular (or holomorphic) when considered as a map from $(X, \mathcal{O}_X)$ to the conjugate variety $(Y, \overline{\mathcal{O}_Y})$—see Exercise 2.1.7.

As promised in the introduction, we now generalise the involutions $\sigma_{\mathbb{A}}$ and $\sigma_{\mathbb{P}}$ to complex varieties. (We invite the reader to compare this definition with Atiyah's "real structures on a bundle" in [Ati66].)

Definition 2.1.10 (*Real structure*) A *real structure* on a complex algebraic variety (resp. complex analytic space) X is an anti-regular (resp. anti-holomorphic) global involution σ on X.

Examples 2.1.11 (*Basic examples*)

1. $\sigma_{\mathbb{A}}$ on $\mathbb{A}^n(\mathbb{C})$;
2. $\sigma_{\mathbb{P}}$ on $\mathbb{P}^n(\mathbb{C})$;
3. $(x : y) \mapsto (-\overline{y} : \overline{x})$ on $\mathbb{P}^1(\mathbb{C})$.

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[2]In complex analytic geometry the term *variety* is usually only used for non singular complex analytic spaces see Appendix D.

Definition 2.1.12 (ℝ-*variety*) In short, we will say that a pair (X, σ) is an ℝ-*variety* if X is a complex variety and σ is a real structure on X. If necessary we will specify whether (X, σ) is an *algebraic* ℝ-*variety* or *analytic* ℝ-*variety*. On occasion we will wish to authorise our analytic varieties to be singular: we will then call them *analytic* ℝ-*spaces*.

Definition 2.1.13 An ℝ-variety (X, σ) is also called a *real form* of the complex variety X.

Example 2.1.14 Real forms of Lie groups provide a rich family of examples. See [MT86] for more details.

Remark 2.1.15 Generalising ℝ-varieties to complex analytic varieties is particularly useful when studying real K3 surfaces (Definition 4.5.3), 2-dimensional complex ℝ-toruses (Definition 4.5.22), real elliptic surfaces (Definition 4.6.1) and real Moishezon varieties (Definition 6.1.4).

Remark 2.1.16 Let (X, σ) be an ℝ-variety and let $U \subset X$ be an open affine set. The set $\sigma(U)$ is then also an open affine set, since σ is a homeomorphism. Moreover, if $\varphi \colon U \to \mathbb{A}^n(\mathbb{C})$ is an embedding of U as an affine algebraic variety of ideal $I = (P_1, \ldots, P_l) \subset \mathbb{C}[X_1, \ldots, X_n]$ then $\sigma_{\mathbb{A}} \circ \varphi \circ \sigma \colon \sigma(U) \to \mathbb{A}^n(\mathbb{C})$ is an embedding of $\sigma(U)$ as an affine variety of ideal ${}^{\sigma}I = ({}^{\sigma}P_1, \ldots, {}^{\sigma}P_l) \subset \mathbb{C}[X_1, \ldots, X_n]$.

Definition 2.1.17 A pair (Y, τ) is an ℝ-*subvariety* of (X, σ) if and only if $Y \subset X$ is a complex subvariety of X and $\tau = \sigma|_Y$.

By definition, an ℝ-variety (X, σ) is *quasi-affine* (resp. *affine*, resp. *quasi-projective*, resp. *projective*) if the *complex* variety X has a regular embedding $\varphi \colon X \hookrightarrow \mathbb{A}^n(\mathbb{C})$ (resp. $\varphi \colon X \hookrightarrow \mathbb{A}^n(\mathbb{C})$ with closed image, resp. $\psi \colon X \hookrightarrow \mathbb{P}^n(\mathbb{C})$, resp. $\psi \colon X \hookrightarrow \mathbb{P}^n(\mathbb{C})$ with closed image). The central question is whether there is always a regular embedding such that $\varphi \circ \sigma = \sigma_{\mathbb{A}} \circ \varphi$ (resp. $\psi \circ \sigma = \sigma_{\mathbb{P}} \circ \psi$). In other words, is (X, σ) isomorphic as a ℝ-variety to a ℝ-subvariety of $(\mathbb{A}^n(\mathbb{C}), \sigma_{\mathbb{A}})$ (resp. $(\mathbb{P}^n(\mathbb{C}), \sigma_{\mathbb{P}})$)? The answer to this question is yes: this is one of the main results of the theory. Any quasi-projective ℝ-variety can be defined by equations with real coefficients: see Theorem 2.1.33.

The well known identification (see [Ser56] for more details) of a complex projective algebraic variety with an analytic variety is compatible with its real structure.

Proposition 2.1.18 *Let X be a complex projective algebraic variety. The variety X then has a real structure if and only if there is an anti-holomorphic involution on the analytic space underlying X.*

Proof Let X^h be the underlying analytic space of X, by which we mean that X^h is the set X with its Euclidean topology and the sheaf $\mathcal{O}_X^h$ of holomorphic functions associated to the sheaf $\mathcal{O}_X$. If X is projective then the conjugate variety $\overline{X}$ is also projective. Let $\sigma \colon X^h \to X^h$ be an anti-holomorphic involution and let $\psi \colon X^h \to$

$\overline{X^h}$ be the canonical map induced by the identity on topological spaces. The map $\sigma \circ \psi \colon X^h \to X^h$ is holomorphic and X is projective so by Serre's **GAGA** theorems [Ser56] it is regular for the Zariski topology. In other words, $\sigma \colon X \to X$ is an anti-regular involution. □

Consider $X \subset \mathbb{P}^n(\mathbb{C})$ and let $\psi \colon X \to \mathbb{P}^N(\mathbb{C})$ be a morphism of complex varieties. We denote by ${}^{\sigma}\psi := \sigma_{\mathbb{P}} \circ \psi \circ \sigma_{\mathbb{P}}$.

Proposition 2.1.19 (Conditions for the existence of a real structure) *If a complex quasi-projective variety $X \subset \mathbb{P}^n(\mathbb{C})$ has a real structure then there is an isomorphism $\psi \colon X \to \overline{X}$ satisfying ${}^{\sigma}\psi \circ \psi = \mathrm{id}_X$.*

Proof If σ is a real structure on X then we simply set $\psi := \overline{\sigma} = \sigma_{\mathbb{P}} \circ \sigma$. We then have that $\psi^{-1} = \sigma^{-1} \circ {\sigma_{\mathbb{P}}}^{-1} = \sigma \circ \sigma_{\mathbb{P}}$. Moreover, ${}^{\sigma}\psi = \sigma_{\mathbb{P}} \circ \psi \circ \sigma_{\mathbb{P}} = \sigma_{\mathbb{P}} \circ (\sigma_{\mathbb{P}} \circ \sigma) \circ \sigma_{\mathbb{P}} = \sigma \circ \sigma_{\mathbb{P}}$. □

Remark 2.1.20 We insist on the fact that a real structure σ is an involution (i.e. $\sigma \circ \sigma = \mathrm{id}$). The following example by Shimura [Shi72a, p. 177] (see also [Sil92, p. 152]) shows that a complex variety can be isomorphic to its conjugate without having a real structure! (The variety in question has an anti-isomorphism or order 4 but no anti-isomorphism of order 2.)

Exercise 2.1.21 (Curves without real structures) *Let m be an odd number, let $a_0 \in \mathbb{R}$ be a real number and let $a_k \in \mathbb{C} \setminus \mathbb{R}$, $k = 1, \ldots, m$ be non real complex numbers. Consider the curve $C_{m,a_0,\ldots,a_m}$ which is the projective completion (i.e. the Zariski closure of the image of the affine curve under the inclusion $j \colon \mathbb{A}^2(\mathbb{C}) \hookrightarrow \mathbb{P}^2(\mathbb{C})$—see Lemma 1.2.43 and Exercise 1.2.44) of the affine plane curve of equation*

$$y^2 = a_0 x^m + \sum_{k=1}^{m} \left(a_k x^{m+k} + (-1)^k \overline{a_k} x^{m-k} \right) .$$

1. *Prove that the curve $C_{m,a_0,\ldots,a_m}$ is isomorphic to its conjugate via the map $\varphi \colon (x, y) \mapsto (-\frac{1}{x}, \frac{i}{x^m} y)$ for $(x, y) \neq (0, 0)$ and $\varphi(0, 0) = (0, 0)$.*
2. *Prove that φ induces an anti-isomorphism of $C_{m,a_0,\ldots,a_m}$ of order 4.*
3. *Assume that the number a_0, the numbers a_k and the numbers $\overline{a_k}$ are all algebraically independent over $\mathbb{Q}$.*
   (a) *Prove that the only automorphisms of $C_{m,a_0,\ldots,a_m}$ are the identity and $\rho \colon (x, y) \mapsto (x, -y)$. (Use Exercise 1.2.80(3a).)*
   (b) *Deduce that $C_{m,a_0,\ldots,a_m}$ has no real structure.*

See Section 5.5 and [KK02, Theorem 5.1] for examples of complex surfaces with no real structure, or even with no anti-automorphism.

Definition 2.1.22 The *real locus*, or *real part* of an $\mathbb{R}$-variety (X, σ) is the set of fixed points $X^{\sigma} := \{x \in X \mid \sigma(x) = x\}$ of the real structure. By analogy with the set of real points of a scheme defined over $\mathbb{R}$ the set of fixed points of σ is often denoted

$$X(\mathbb{R}) := X^\sigma$$

when there is no possible confusion.

Remark 2.1.23 Obviously, if (Y, τ) is an $\mathbb{R}$-subvariety of (X, σ) then $Y(\mathbb{R}) \subset X(\mathbb{R})$.

Examples 2.1.24 (*Real loci of Examples* 2.1.11)

1. $\mathbb{A}^n(\mathbb{R})$;
2. $\mathbb{P}^n(\mathbb{R})$;
3. $\varnothing$.

Definition 2.1.25 Let (X, σ) and (Y, τ) be $\mathbb{R}$-varieties. A *morphism of* $\mathbb{R}$*-varieties* (or *regular map of* $\mathbb{R}$*-varieties*) $(X, \sigma) \to (Y, \tau)$ is a morphism of complex varieties $\varphi \colon X \to Y$ which commutes with the real structures

$$\forall x \in X, \quad \varphi(\sigma(x)) = \tau(\varphi(x)) \; .$$

Remark 2.1.26 $\mathbb{R}$-varieties (X, σ) and (Y, τ) are therefore *isomorphic* if and only if there is an isomorphism $X \overset{\varphi}{\simeq} Y$ of complex varieties commuting with the real structures. Indeed, if $\varphi \colon X \to Y$ commutes with the real structures i.e. $\varphi \circ \sigma = \tau \circ \varphi$ then $\varphi^{-1} \colon Y \to X$ is a morphism of $\mathbb{R}$-varieties; for any $y \in Y$ and $x = \varphi^{-1}(y)$ we have that $\varphi(\sigma(\varphi^{-1}(y))) = \varphi(\sigma(x)) = \tau(\varphi(x)) = \tau(y)$ and hence $\sigma(\varphi^{-1}(y)) = \varphi^{-1}(\tau(y))$.

Definition 2.1.27 Let (X, σ) and (Y, τ) be $\mathbb{R}$-varieties. A *rational map of* $\mathbb{R}$*-varieties* $(X, \sigma) \dashrightarrow (Y, \tau)$ is a rational map of complex varieties

$$\varphi \colon X \dashrightarrow Y$$

which commutes with the real structures

$$\forall x \in \mathrm{dom}(\varphi) \subset X, \quad \varphi(\sigma(x)) = \tau(\varphi(x)) \; .$$

Remark 2.1.28 Denoting the Galois group by $G := \mathrm{Gal}(\mathbb{C}|\mathbb{R})$, the involution σ (resp. τ) equips X (resp. Y) with a G-action. A regular map of $\mathbb{R}$-varieties $(X, \sigma) \to (Y, \tau)$ is then by definition a G-equivariant regular map of complex varieties. Similarly, a rational map of $\mathbb{R}$-varieties is a G-equivariant rational map of complex varieties.

If X is a projective algebraic variety defined in some $\mathbb{P}^n(\mathbb{C})$ by homogeneous polynomial equations

$$P_1(z_0, \dots, z_n) = \cdots = P_l(z_0, \dots, z_n) = 0 \; ,$$

then, as we have seen above, the variety X has a real structure induced by $\sigma_{\mathbb{P}}\colon \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^n(\mathbb{C})$ if and only if we can assume the polynomials P_i have *real* coefficients, or in other words if the homogeneous ideal generated by the P_is has a system of generators with real coefficients. If this is the case then the real locus of the $\mathbb{R}$-variety $(X, \sigma_{\mathbb{P}}|_X)$ is simply $X(\mathbb{R}) = X \cap \mathbb{P}^n(\mathbb{R})$. Similarly, if X is an affine algebraic variety defined in $\mathbb{A}^n(\mathbb{C})$ by polynomial equations

$$P_1(z_1, \dots, z_n) = \cdots = P_l(z_1, \dots, z_n) = 0 ,$$

then $\sigma_{\mathbb{A}}\colon \mathbb{A}^n(\mathbb{C}) \to \mathbb{A}^n(\mathbb{C})$ induces a real structure on the complex variety X if and only if we can assume the polynomials P_i have real coefficients and in this case the real locus of the $\mathbb{R}$-variety $(X, \sigma_{\mathbb{A}}|_X)$ is given by

$$X(\mathbb{R}) = X \cap \mathbb{A}^n(\mathbb{R}) .$$

Note that the variety X may however have other real structures than the restriction of $\sigma_{\mathbb{P}}$ or $\sigma_{\mathbb{A}}$.

Example 2.1.29 (*Two distinct real structures on the same complex variety*) Consider the affine algebraic plane curve $C \subset \mathbb{A}^2(\mathbb{C})$ determined by the equation $y^2 = x^3 - x$. As this equation has real coefficients, the conjugation $\sigma_{\mathbb{A}}$ restricted to C yields a real structure. If we set $\sigma_1 := \sigma_{\mathbb{A}}|_C$ then (C, σ_1) is an $\mathbb{R}$-variety whose set of real points $C(\mathbb{R}) = \mathcal{Z}\left(y^2 - x(x-1)(x+1)\right) \cap \mathbb{A}^2(\mathbb{R})$ has two connected components in the Euclidean topology—see Figure 2.1.

Now let us consider σ_2, the restriction to C of the anti-regular involution $\mathbb{A}^2(\mathbb{C}) \to \mathbb{A}^2(\mathbb{C})$, $(x, y) \mapsto (-\overline{x}, i\overline{y})$. We check that $\sigma_2(C) \subset C$ so the pair (C, σ_2) is an $\mathbb{R}$-variety whose real structure is not induced by $\sigma_{\mathbb{A}}$. Let C' be the curve of equation $y^2 = x^3 + x$ in $\mathbb{A}^2(\mathbb{C})$ end let ζ be a square root of $-i$, $\zeta^2 = -i$. The morphism $\varphi\colon \mathbb{A}^2(\mathbb{C}) \to \mathbb{A}^2(\mathbb{C})$, $(x, y) \mapsto (ix, \zeta y)$ is an automorphism of $\mathbb{A}^2(\mathbb{C})$ whose

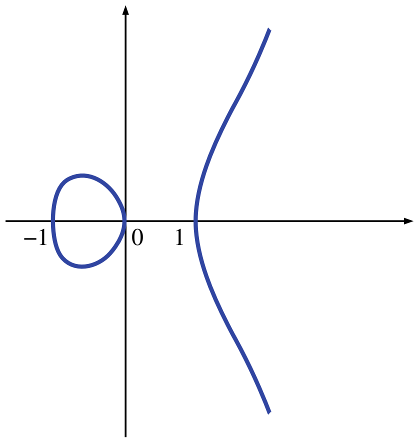


Fig. 2.1 $C : y^2 = x(x-1)(x+1)$

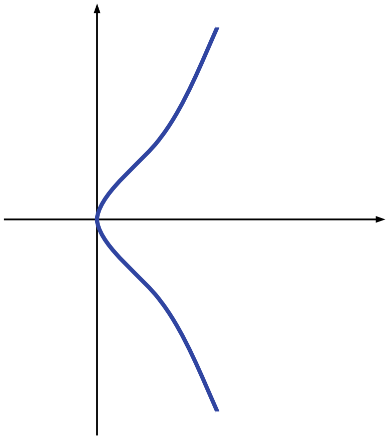


Fig. 2.2 $C' : y^2 = x(x-i)(x+i)$

restriction $\varphi|_C : C \to C'$ is an isomorphism of complex varieties. Set $\sigma' := \varphi|_C \circ \sigma_2 \circ \varphi^{-1}|_{C'}$: the $\mathbb{R}$-curves (C, σ_2) and (C', σ') are then isomorphic. It is easy to check that $\sigma' = \sigma_{\mathbb{A}}|_{C'}$. The set of real points $C'(\mathbb{R}) = \mathcal{Z}\left(y^2 - x(x-i)(x+i)\right) \cap \mathbb{A}^2(\mathbb{R})$ has only one connected components—see Figure 2.2. The $\mathbb{R}$-varieties (C, σ_1) and (C, σ_2) are therefore not isomorphic by Proposition 2.1.38 below.

In the above example, the *abstract* $\mathbb{R}$-variety (C, σ_2) is isomorphic to the $\mathbb{R}$-variety (C', σ') whose real structure is induced by the real structure on the surrounding space. The fact that there is always an $\mathbb{R}$-subvariety of some $\mathbb{A}^n$ isomorphic to a given affine abstract $\mathbb{R}$-variety is guaranteed by the fundamental Theorem 2.1.30 below. We insist on the fact that the isomorphism of complex varieties $C \to C'$ is not always induced by an automorphism of the surrounding space.

Theorem 2.1.30 (Real embedding of an affine $\mathbb{R}$-variety) *Let (X, σ) be an algebraic $\mathbb{R}$-variety. If the complex variety X is affine, $X \hookrightarrow \mathbb{A}^m(\mathbb{C})$ then there is an affine algebraic set $F \subset \mathbb{A}^n(\mathbb{C})$ such that $\sigma_{\mathbb{A}}(F) \subset F$ and there is an isomorphism of $\mathbb{R}$-varieties*

$$(F, \sigma_{\mathbb{A}}|_F) \simeq (X, \sigma) \ .$$

In particular, the ideal $\mathcal{I}(F)$ is generated by real polynomials or in other words there is an ideal $I \subset \mathbb{R}[X_1, \ldots, X_n]$ such that $\mathcal{I}(F) = I_{\mathbb{C}}$ and $\mathcal{A}(X)$ is isomorphic to $\mathcal{A}(F) = (\mathbb{R}[X_1, \ldots, X_n]/I) \otimes_{\mathbb{R}} \mathbb{C}$.

Remark 2.1.31 Note that $n \neq m$ in general.

This theorem is a reformulation—modulo Lemma A.7.3—of the following result.

Lemma 2.1.32 *Let (X, σ) be an affine algebraic $\mathbb{R}$-variety. There is then a real affine algebraic set $V \subset \mathbb{A}^n(\mathbb{R})$ with defining ideal $I = \mathcal{I}(V) \subset \mathbb{R}[X_1, \ldots, X_n]$ such that the $\mathbb{R}$-algebra $\mathcal{A}(V) = \mathbb{R}[X_1, \ldots, X_n]/I$ is isomorphic to the $\mathbb{R}$-algebra of affine invariant coordinates $\mathcal{A}(X)^\sigma = \{f \in \mathcal{A}(X) \mid {}^\sigma f = f\}$ of X.*

Proof The above result is a special case of the scheme-theoretic result stating that there is an equivalence between the data of an affine scheme X over $\mathbb{C}$ with a real structure σ and the data of a real scheme X_0, namely that if $X = \operatorname{Spec} A$ then $X_0 = \operatorname{Spec} A^\sigma$. See Section 2.4 for more details. □

Theorem 2.1.33 (Real embedding of a quasi-projective $\mathbb{R}$-variety) *Let (X, σ) be an algebraic $\mathbb{R}$-variety. If the complex algebraic variety X is projective (resp. quasi-projective), $X \hookrightarrow \mathbb{P}^m(\mathbb{C})$ then there is a projective (resp. quasi-projective) algebraic set $F \subset \mathbb{P}^n(\mathbb{C})$ such that $\sigma_{\mathbb{P}}(F) \subset F$ and there is an isomorphism of $\mathbb{R}$-varieties*

$$(F, \sigma_{\mathbb{P}}|_F) \simeq (X, \sigma) .$$

Remark 2.1.34 We insist on the fact that, as in the affine case, $n \neq m$ in general.

Proof The above statement is a special case of the scheme-theoretic statement that there is an equivalence between the data of a quasi-projective scheme X over $\mathbb{C}$ with a real structure σ and the data of a real scheme X_0 such that $X_0 = X/\langle\sigma\rangle$. See Section 2.4 for more details. □

Like many other authors, Silhol [Sil89] states the above result as a special case of a general result of the Galois descent theory developed first by Weil [Wei56, Theorem 7] then Grothendieck [Gro95, Théorème 3]. See also Borel–Serre [BS64, Proposition 2.6, p. 129]. We give an alternative Proof of Theorem 2.1.33 in Section 2.6, namely Theorem 2.6.44.

In Example 2.1.29, $\sigma_{\mathbb{A}}$ and $\sigma_{\mathbb{A}}{}'\colon (x, y) \mapsto (-\overline{x}, i\overline{y})$ are distinct real structures on $\mathbb{A}^2(\mathbb{C})$. The $\mathbb{R}$-varieties $\big(\mathbb{A}^2(\mathbb{C}), \sigma_{\mathbb{A}}\big)$ and $\big(\mathbb{A}^2(\mathbb{C}), \sigma_{\mathbb{A}}{}'\big)$, however, are isomorphic via the map $\varphi\colon (x, y) \mapsto (ix, \zeta y)$. In this situation we say that the real structures are *equivalent*.

Definition 2.1.35 Two real structures σ and τ on a complex variety X are *equivalent* if they are conjugate under an automorphism of the complex variety X or in other words if there is an automorphism φ of X such that

$$\sigma = \varphi^{-1} \circ \tau \circ \varphi$$

In other words, σ and τ are equivalent if there is an isomorphism of $\mathbb{R}$-varieties, $\varphi\colon (X, \sigma) \to (X, \tau)$.

Remark 2.1.36 Two real forms (see Definition 2.1.13), (X, σ) and (X, τ) of a complex variety X are isomorphic if and only if the real structures σ and τ are equivalent.

Example 2.1.37 It is proved in [Kam75] that all real structures on the affine complex plane are equivalent.

We recall that for any $\mathbb{R}$-variety (X, σ) we define $\#\pi_0(X^\sigma) = \#\pi_0(X(\mathbb{R}))$ to be the number of connected components of the real locus in the Euclidean topology.

Proposition 2.1.38 (Real locus and isomorphism) *An isomorphism of* $\mathbb{R}$*-varieties* $\varphi\colon (X,\sigma)\to(Y,\tau)$ *induces a homeomorphism between* X^σ *and* Y^τ *in the Euclidean topology. In particular*

$$\#\pi_0(X^\sigma)=\#\pi_0(Y^\tau) \quad \textit{or in other words} \quad \#\pi_0(X(\mathbb{R}))=\#\pi_0(Y(\mathbb{R}))\ .$$

Proof Start by noting that for a any given real structure the Euclidean topology on the real locus is simply the topology induced by the Euclidean topology on the complex variety. As φ is a homeomorphism for the Euclidean topology (see Exercise 1.4.4) and commutes with the real structures, it induces a bijection $X^\sigma\to Y^\tau$ between the fixed loci which is a homeomorphism. □

Corollary 2.1.39 (Real locus and equivalence) *Let* σ *and* τ *be real structures on a complex variety* X*. If* σ *and* τ *are equivalent then* X^σ *and* X^τ *are homeomorphic for the Euclidean topology and in particular*

$$\#\pi_0(X^\sigma)=\#\pi_0(X^\tau)\ .$$

Proof The real structures σ and τ are equivalent so there is an isomorphism of $\mathbb{R}$-varieties $\varphi\colon (X,\sigma)\to(X,\tau)$. □

Example 2.1.40 (*Two real forms on the same complex variety*) We return to the two complex algebraic curves C and C' studied in Example 2.1.29 whose equations in $\mathbb{A}^2(\mathbb{C})$ are $y^2=x^3-x$ and $y^2=x^3+x$ respectively. It is easy to check that the set of real points of $C(\mathbb{R})\subset\mathbb{A}^2(\mathbb{R})$ has two connected components, see Figure 2.1, and that the set of real points of $C'(\mathbb{R})\subset\mathbb{A}^2(\mathbb{R})$ has only one connected component, see Figure 2.2. In particular, by Proposition 2.1.38, the $\mathbb{R}$-curves (C,σ_1) and (C,σ_2) are not isomorphic.

The complex variety C therefore has two non-equivalent real structures $\sigma_1=\sigma_{\mathbb{A}}|_C\colon (x,y)\mapsto(\overline{x},\overline{y})$ and $\sigma_2=\varphi^{-1}|_{C'}\circ\sigma_{\mathbb{A}}|_{C'}\circ\varphi|_C\colon (x,y)\mapsto(-\overline{x},i\overline{y})$. It is interesting to note that these non equivalent real structures are restrictions of real structures $\sigma_{\mathbb{A}}$ and $\varphi^{-1}\circ\sigma_{\mathbb{A}}\circ\varphi$ on $\mathbb{A}^2(\mathbb{C})$ which are equivalent by definition.

Remark 2.1.41 (*Non-standard real structure on the projective line*) We have already met the antipodal map on the Riemann sphere:

$$\sigma_{\mathbb{P}}'\colon \mathbb{P}^1(\mathbb{C})\to\mathbb{P}^1(\mathbb{C}),\quad (x_0:x_1)\mapsto(-\overline{x_1}:\overline{x_0})$$

which is a real structure on $\mathbb{P}^1(\mathbb{C})$ whose set of fixed points is empty and which is therefore not equivalent to $\sigma_{\mathbb{P}}$.

Exercise 2.1.42 (Real structures on a complex torus) *Find four pairwise non-equivalent real structures on* $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$*. (There are in fact exactly four classes of real structures on* $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$*.)*

Remark 2.1.43 Until recently it was not known whether the number of equivalence classes of real structures on a given complex variety was finite. See [DIK00, Appendix D] for a review of this question.

In [Les18], John Lesieutre constructs a variety of dimension 6 with a discrete automorphism group which cannot be generated by a finite number of generators and which has infinitely many non-isomorphic real forms. In [DO19], Dinh and Oguiso use different methods to construct examples of projective varieties of any dimension greater than one with non-finitely automorphism generated group. Their work also provides examples of real varieties of any dimension greater than one with infinitely many non-isomorphic real forms. In [DFM18], Dubouloz, Freudenburg and Moser–Jauslin construct affine rational varieties with infinitely many pairwise non-isomorphic real forms in every dimension ≥ 4.

Surprisingly, this finiteness question is still open for rational surfaces. See Benzerga's work [Ben16a, Ben16b, Ben17] for the most recent results on this question.

2.2 $\mathbb{R}$-Varieties and Real Algebraic Varieties

For a given quasi-projective $\mathbb{R}$-variety (X, σ) we seek to define a sheaf of regular functions on $X(\mathbb{R})$ with which $X(\mathbb{R})$ becomes a real algebraic variety as in Definition 1.3.9. By Theorem 2.1.33 and Exercise 2.1.3 the structural sheaf satisfies ${}^{\sigma}\mathcal{O}_X = \mathcal{O}_X$, which justifies the following definition. Recall that a real structure is a Zariski homeomorphism and in particular if U is open in X then so is $\sigma(U)$. Let $\mathcal{L}$ be a sheaf of $\mathbb{C}^n$-valued functions. For any open set U in X and any map $f \in \mathcal{L}(U)$ we denote by ${}^{\sigma}f \colon \sigma(U) \to \mathbb{C}^n$ the map $\overline{f} \circ \sigma = \sigma_{\mathbb{A}} \circ f \circ \sigma$. We then have that ${}^{\sigma}f \in \mathcal{L}(\sigma(U))$ which generalises the notion of conjugate function introduced at the beginning of Section 2.1.

Definition 2.2.1 Let (X, σ) be an $\mathbb{R}$-variety and let $\mathcal{L}$ be a sheaf of $\mathbb{C}^n$-valued functions. The sheaf ${}^{\sigma}\mathcal{L}$ defined on any open set U of X by

$$ {}^{\sigma}\mathcal{L}(U) := \{{}^{\sigma}f \mid f \in \mathcal{L}(\sigma(U))\} \ . $$

is a sheaf on X called the *conjugate sheaf*. We say that $\mathcal{L}$ is an $\mathbb{R}$-*sheaf* if and only if ${}^{\sigma}\mathcal{L} = \mathcal{L}$. Note that this is required to be an equality, not an isomorphism.

From a cohomological point of view, the sheaves $\mathcal{L}$ and ${}^{\sigma}\mathcal{L}$ are similar. (See [Liu02, Section 5.2] for an introduction to sheaf cohomology.) In particular, we have the following proposition.

Proposition 2.2.2 *Let (X, σ) be an $\mathbb{R}$-variety and let $\mathcal{L}$ be a coherent sheaf (Definition C.6.7) of $\mathbb{C}^n$-valued functions. We then have that*

$$ \dim_{\mathbb{C}} H^k(X, {}^{\sigma}\mathcal{L}) = \dim_{\mathbb{C}} H^k(X, \mathcal{L}) \ . $$

Proof See [Sil89, I.(1.9)]. □

Let (X, σ) be a quasi-projective $\mathbb{R}$-variety. We saw above that the sheaf of $\mathbb{C}$-algebras $\mathcal{O}_X$ is an $\mathbb{R}$-sheaf: ${}^\sigma\mathcal{O}_X = \mathcal{O}_X$. In particular, for any open set U in X, the morphism

$$\begin{array}{ccc} \mathcal{O}_X(U) & \longrightarrow & \mathcal{O}_X(\sigma(U)) \\ f & \longmapsto & {}^\sigma f \end{array}$$

is a ring isomorphism.

Remark 2.2.3 We can prove more: this map is an anti-isomorphism of $\mathbb{C}$-algebras. Let us prove anti-linearity: for any $\lambda \in \mathbb{C}$ and for any regular function f on U we have that ${}^\sigma(\lambda f) = \overline{\lambda f} \circ \sigma = \overline{\lambda}(\overline{f} \circ \sigma) = \overline{\lambda}({}^\sigma f)$.

If A is an $\mathbb{R}$-algebra equipped with a G-action, where $G := \mathrm{Gal}(\mathbb{C}|\mathbb{R})$, and σ is the corresponding involution of A then we denote by $A^G := A^\sigma = \{a \in A \mid \sigma(a) = a\}$ the sub-algebra of invariants of A (see Definition A.7.2).

Let (X, σ) be an $\mathbb{R}$-variety. A subset $U \subset X$ is said to be *invariant* if and only if $\sigma(U) = U$. Any such subset inherits a G-action: since σ is a homeomorphism, for any open set U in X the intersection $U \cap \sigma(U)$ is an invariant open set in X. For any invariant open set U we say that a local section f over U is *invariant* if ${}^\sigma f = f$. Let $\mathcal{F}$ be an $\mathbb{R}$-sheaf of functions on X. We denote by $\mathcal{F}_{X(\mathbb{R})}$ the sheaf of its restrictions to $X(\mathbb{R})$, see Definition C.1.6 and by $\mathcal{F}^G_{X(\mathbb{R})}$ its invariant subset. We apply this definition to $\mathcal{O}_X$, which is an $\mathbb{R}$-sheaf of functions on X, and obtain a sheaf

$$(\mathcal{O}_X)^G_{X(\mathbb{R})} := \big((\mathcal{O}_X)_{X(\mathbb{R})}\big)^G$$

of real-valued functions on $X(\mathbb{R})$. It takes some work to prove that these functions are $\mathbb{R}$-valued, since a priori they are $\mathbb{C}$-valued—see below for the proof.

Let us describe the local sections of this new sheaf. Let $\Omega \subset X(\mathbb{R})$ be an open subset in the induced topology. We check first that any $f \in (\mathcal{O}_X)^G_{X(\mathbb{R})}(\Omega)$ is $\mathbb{R}$-valued. As f is invariant, for any $x \in \Omega$ we have that $f(x) = ({}^\sigma f)(x) = \overline{f(\sigma(x))}$ and since x is a point in $X(\mathbb{R})$ we have that $\sigma(x) = x$ so $f(x) \in \mathbb{R}$. By definition of $(\mathcal{O}_X)_{X(\mathbb{R})}$ there is an open neighbourhood $U \subset X$ of x and an element $g \in \mathcal{O}_X(U)$ such that $g|_{U\cap\Omega} = f|_{U\cap\Omega}$. Replacing U by $U \cap \sigma(U)$ and g by $\frac{1}{2}(g + {}^\sigma g)$ we get an element $g \in (\mathcal{O}_X(U))^G$ such that $g|_{U\cap\Omega} = f|_{U\cap\Omega}$. In other words, the local sections of $(\mathcal{O}_X)^G_{X(\mathbb{R})}$ over an open set Ω in $X(\mathbb{R})$ are as follows.

$$\begin{aligned} (\mathcal{O}_X)^G_{X(\mathbb{R})}(\Omega) = \big\{ f\colon \Omega \to \mathbb{R} \mid \forall x \in \Omega,& \\ \exists U \text{ open invariant neighbourhood of } x \text{ in } X \text{ and}& \\ \exists g \in (\mathcal{O}_X(U))^G \mid g|_{U\cap\Omega} = f|_{U\cap\Omega} \big\}&. \end{aligned}$$

We invite the reader to compare the following theorem with Theorem 2.1.30.

Theorem 2.2.4 *Let $F \subset \mathbb{A}^n(\mathbb{C})$ be a complex affine algebraic set such that $\mathcal{I}(F)$ is generated by polynomials with real coefficients. In particular, $F(\mathbb{R}) := F \cap \mathbb{A}^n(\mathbb{R})$ is a real algebraic affine set.*

If $F(\mathbb{R})$ is dense in F with respect to the Zariski topology then

$$\mathcal{O}_{F(\mathbb{R})} \simeq (\mathcal{O}_F)^G_{F(\mathbb{R})} .$$

Proof Let $I \subset \mathbb{R}[X_1, \ldots, X_n]$ be an ideal and let $F = \mathcal{Z}_\mathbb{C}(I) \subset \mathbb{A}^n(\mathbb{C})$ be the complex algebraic set whose ideal is $\mathcal{I}(F) = I_\mathbb{C}$ and whose sheaf of regular functions is $\mathcal{O}_F$. The set $F(\mathbb{R}) = F \cap \mathbb{A}^n(\mathbb{R}) = \mathcal{Z}_\mathbb{R}(I) \subset \mathbb{A}^n(\mathbb{R})$ is then a real algebraic set whose sheaf of regular functions will be denoted by $\mathcal{O}_{F(\mathbb{R})}$. By hypothesis F is stable under $\sigma_\mathbb{A}$. By Proposition C.3.12 these sheaves are isomorphic if and only if their stalks are isomorphic.

Let $\Omega \subset F(\mathbb{R})$ be a Zariski open subset in $\mathbb{A}^n(\mathbb{R})$ and let f be an element of $\mathcal{O}_{F(\mathbb{R})}(\Omega)$. Passing to a smaller open set if necessary, we can assume that on Ω $f = \frac{p}{q}$ where p, q are polynomials with real coefficients and q does not vanish at any point of Ω. There is then an open set U of F in $\mathbb{A}^n(\mathbb{C})$ on which q does not vanish and hence $f \in \mathcal{O}_{F(\mathbb{R})}(\Omega)$ can be extended to a regular function $f_\mathbb{C} \in \mathcal{O}_F(U)$ such that ${}^\sigma f_\mathbb{C} = f_\mathbb{C}$. As $F(\mathbb{R})$ is dense in F, the germ of the extension $f_\mathbb{C}$ of f is uniquely determined by the germ of f. It follows that $\mathcal{O}_{F(\mathbb{R})} \simeq (\mathcal{O}_F)^G_{F(\mathbb{R})}$. □

Theorem 2.2.4 motivates our next definition.

Definition 2.2.5 Let (X, σ) be an $\mathbb{R}$-variety. We say that (X, σ) has *enough real points* if and only if $X(\mathbb{R})$ is Zariski-dense in X.

Exercise 2.2.6 *Let $I \subset \mathbb{R}[X_1, \ldots, X_n]$ be a radical ideal and let $F = \mathcal{Z}_\mathbb{C}(I) \subset \mathbb{A}^n(\mathbb{C})$ be the associated complex algebraic set as in Definition 1.2.12. Let $(F, \sigma_\mathbb{A}|_F)$ be the associated affine $\mathbb{R}$-variety.*

1. *Prove that the $\mathbb{R}$-variety $(F, \sigma_\mathbb{A}|_F)$ has enough real points if and only if $\mathcal{I}(\mathcal{Z}(I)) \subset I$ in $\mathbb{R}[X_1, \ldots, X_n]$.*
2. *Prove that the $\mathbb{R}$-variety $(F, \sigma_\mathbb{A}|_F)$ has enough real points if and only if I is a real ideal, see Definition A.5.14.*

Exercise 2.2.7 *Prove that the $\mathbb{R}$-variety $(F = \mathcal{Z}_\mathbb{C}(x^2 + y^2), \sigma_\mathbb{A}|_F)$—which has a non-empty real locus—does not have enough real points. (See Example 2.1.1). Further prove that $\mathcal{O}_{F(\mathbb{R})} \neq (\mathcal{O}_F)^G_{F(\mathbb{R})}$.*

Theorem 2.2.9 below characterises those $\mathbb{R}$-varieties that have enough real points. In particular, any irreducible non singular $\mathbb{R}$-variety with non-empty real locus has enough real points.

Lemma 2.2.8 *Let (X, σ) be an algebraic $\mathbb{R}$-variety, let $a \in X(\mathbb{R})$ be a real point and let $\mathfrak{m}_a$ be the maximal ideal of the local ring $\mathcal{O}_{X,a}$. We then have that*

$$\dim_\mathbb{C} \mathfrak{m}_a/\mathfrak{m}_a^2 = \dim_\mathbb{R}((\mathfrak{m}_a/\mathfrak{m}_a^2)^G) .$$

Proof As a is real σ induces an anti-linear involution on $\mathcal{O}_{X,a}$ and by Lemma A.7.3 there is a basis of $\mathfrak{m}_a/\mathfrak{m}_a^2$ whose elements are all σ-invariant. □

Theorem 2.2.9 (Density of the real locus in the complex variety)

1. *The space $\mathbb{A}^n(\mathbb{R})$ is dense in $\mathbb{A}^n(\mathbb{C})$ for the Zariski topology.*
2. *Let $V \subset \mathbb{A}^n(\mathbb{C})$ be an irreducible affine algebraic set whose ideal $I = \mathcal{I}(V)$ is generated by polynomials with real coefficients. The real locus $V(\mathbb{R}) = V \cap \mathbb{A}^n(\mathbb{R})$ is Zariski dense in V if and only if it contains at least one non singular point of V.*
3. *Let (X, σ) be an algebraic $\mathbb{R}$-variety. The real locus $X(\mathbb{R})$ is Zariski dense in every irreducible component Z of X containing a non singular real point if and only if $X(\mathbb{R})$ is not contained in the singular locus of X. In other words, $\overline{X(\mathbb{R})}^{Zar} \cap Z = Z$ if and only if $(\operatorname{Reg} Z) \cap X(\mathbb{R})$ is non empty.*

Corollary 2.2.10 *Let (X, σ) be an algebraic $\mathbb{R}$-variety. If the complex variety X is irreducible and non singular and if $X(\mathbb{R}) \neq \emptyset$ then (X, σ) has enough real points, or in other words $\overline{X(\mathbb{R})}^{Zar} = X$.*

The behaviour of the Euclidean topology is very different.

Proposition 2.2.11 *The real locus $X(\mathbb{R})$ of an algebraic $\mathbb{R}$-variety (X, σ) is closed in X for the Euclidean topology.*

Proof The real structure σ is continuous for the Euclidean topology and the real locus $X(\mathbb{R}) = \{x \in X \mid x = \sigma(x)\}$ is therefore closed in X because the Euclidean topology is Hausdorff. □

Proof of Theorem 2.2.9 1. We reuse the argument of Proposition 1.5.29. Assume for the moment that we have proved that if a polynomial function vanishes on all real affine points then it is identically zero—this will be proved below by induction on the dimension. If $\mathcal{Z}(I)$ is a closed subset of $\mathbb{A}^n(\mathbb{C})$ containing $\mathbb{A}^n(\mathbb{R})$ then for any $f \in I$ the function f vanishes on every point of $\mathbb{A}^n(\mathbb{R})$ and by assumption f is the zero polynomial. It follows that $I = (0)$ and $\mathcal{Z}(I) = \mathbb{A}^n(\mathbb{C})$.

Let us now prove that for any n, any polynomial vanishing on all real affine points is identically zero. For $n = 1$, the result is immediate. Suppose that $n > 1$ and the induction hypothesis holds for $n - 1$. Let $f \in \mathbb{C}[X_1, \ldots, X_n]$ be a polynomial function vanishing on $\mathbb{R}^n$. We can write

$$f(X', X_n) = X_n^d f_d(X') + X_n^{d-1} f_{d-1}(X') + \cdots + f_0(X')$$

where $X' = (X_1, \ldots, X_{n-1})$, $d = \deg f$ and $\forall i = 0, \ldots, d$, $f_i \in \mathbb{C}[X_1, \ldots, X_{n-1}]$.

For any $X' \in \mathbb{R}^{n-1}$ the function $X_n \mapsto f(X', X_n)$ vanishes at every real point so $f(X', X_n)$ is the zero polynomial for any fixed X'. It follows that the polynomial functions f_i vanish for every real $X' \in \mathbb{R}^{n-1}$ and are therefore identically zero by the induction hypothesis.

2. As V is irreducible in $\mathbb{A}^n(\mathbb{C})$, $I = \mathcal{I}(V)$ is a prime ideal in $\mathbb{C}[X_1, \dots, X_n]$ and $I_{\mathbb{R}} := I \cap \mathbb{R}[X_1, \dots, X_n]$ is a prime ideal in $\mathbb{R}[X_1, \dots, X_n]$ (Lemma A.2.9). We then have that $V = \mathcal{Z}_{\mathbb{C}}(I_{\mathbb{R}})$ and $V(\mathbb{R}) = \mathcal{Z}(I_{\mathbb{R}})$. Set $d = \dim_{\mathbb{C}} V$: by the Nullstellensatz (Corollary A.5.13), we have that $\dim I = d$ (see Definition 1.5.9) and $\dim I_{\mathbb{R}} = d$ by Lemma 1.5.15. Note that *a priori* $\dim_{\mathbb{R}} V(\mathbb{R})$ is not necessarily equal to d: see Example 1.5.20 or 2.2.15.

We now use the fact that there is a non singular point $a = (a_1, \dots, a_n) \in (\operatorname{Reg} V) \cap \mathbb{A}^n(\mathbb{R})$. By Remark 1.5.28, V is a differentiable submanifold of dimension $2d \leqslant 2n$ at a or in other words there is a Euclidean neighbourhood W of a in $\mathbb{C}^n$ such that $W \cap V$ is a Euclidean neighbourhood of a in V of real dimension $2d$ and $W \cap V(\mathbb{R})$ is a Euclidean neighbourhhood of a in $V(\mathbb{R})$ of real dimension d (take an open chart (W, φ) where $W = \sigma(W)$ and justify that φ can be chosen G-equivariant using Lemma A.7.3 if necessary). The subvariety $V(\mathbb{R})$ is therefore a submanifold of real dimension d at a. The real algebraic set $V(\mathbb{R})$ then has Zariski dimension d by Proposition 1.5.29, or in other words the dimension of the ideal $\mathcal{I}(V(\mathbb{R}))$ is equal to d. There is therefore a length d chain of prime ideals in $\mathbb{R}[X_1, \dots, X_n]$ containing $\mathcal{I}(V(\mathbb{R}))$. As $\mathcal{I}(V(\mathbb{R})) \supset I_{\mathbb{R}}$ by definition if $\mathcal{I}(V(\mathbb{R}))$ were different from $I_{\mathbb{R}}$ we would get a chain of length $d + 1$ of prime ideals containing $I_{\mathbb{R}}$, contradicting the fact that $\dim I_{\mathbb{R}} = d$. It follows that $\mathcal{I}(\mathcal{Z}(I_{\mathbb{R}})) = I_{\mathbb{R}}$ and hence $\overline{V(\mathbb{R})} = V$ by 2.2.6(1).

3. We can assume that X is irreducible. By definition of a algebraic variety, X can be covered by open affine subsets. By hypothesis we can therefore chose an open affine subset U in X such that $U \cap X(\mathbb{R})$ is not contained in the singular locus of X (and in particular, U is not empty and since X is irreducible, U is Zariski-dense). Replacing U by $U \cap \sigma(U)$ if necessary we can assume that U is stable under σ. As U is affine (see Exercise 1.3.15(4)) the $\mathbb{R}$-variety $(U, \sigma|_U)$ is isomorphic to an affine $\mathbb{R}$-variety $(V, \sigma_{\mathbb{A}}|_V) \subset (\mathbb{A}^n(\mathbb{C}), \sigma_{\mathbb{A}})$ by Theorem 2.1.30 so we now simply apply (2) to this affine $\mathbb{R}$-variety. $V \cap \mathbb{A}^n(\mathbb{R})$ is dense in $V \cap \mathbb{A}^n(\mathbb{C}) = V$ and we note that $U \cap X(\mathbb{R}) = \varphi^{-1}(V \cap \mathbb{A}^n(\mathbb{R}))$ for any $\mathbb{R}$-isomorphism $\varphi \colon U \to V$. □

Example 2.2.12 (*Reducible, singular, non empty and non dense*) We return to Example 1.5.20. Consider the reducible affine algebraic $\mathbb{R}$-variety

$$(V, \sigma) := (\mathcal{Z}_{\mathbb{C}}(x^2 + y^2), \sigma_{\mathbb{A}}|_V)$$

whose real locus is the isolated point $a = (0, 0)$. By definition we have that $\mathcal{O}_{V,a} = \left(\frac{\mathbb{C}[x,y]}{(x^2+y^2)}\right)_{(0,0)}$ whence $\dim \mathcal{O}_{V,a} = \dim \mathcal{O}^G_{V,a} = 1$ et $\dim_{\mathbb{C}} \mathfrak{m}_{V,a}/\mathfrak{m}^2_{V,a} = \dim_{\mathbb{R}}((\mathfrak{m}_{V,a}/\mathfrak{m}^2_{V,a})^G) = 2$, illustrating the fact that a is a real singular point of the 1-dimensional complex variety. *A contrario* we have that $\dim \mathcal{O}_{V(\mathbb{R}),a} = \dim_{\mathbb{R}} \mathfrak{m}_{V(\mathbb{R}),a}/\mathfrak{m}^2_{V(\mathbb{R}),a} = 0$ illustrating the fact that the real algebraic variety $\{a\}$ is a zero-dimensional non singular variety.

Example 2.2.13 (*Irreducible, singular, dense*) We return to Example 1.5.21. Consider the affine algebraic $\mathbb{R}$-curve

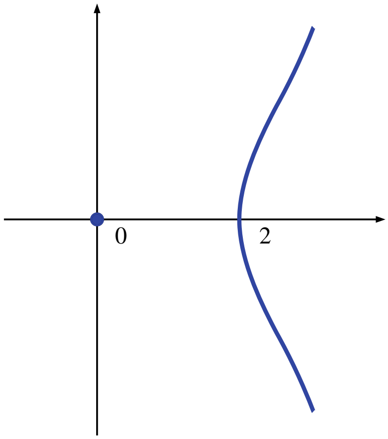


Fig. 2.3 $V(\mathbb{R}) = \{y^2 - x^2(x-2) = 0\} \subset \mathbb{A}^2(\mathbb{R})$

$$(V, \sigma) := (\mathcal{Z}_{\mathbb{C}}(y^2 - x^2(x-2)), \sigma_{\mathbb{A}}|_V)$$

whose real locus is shown in Figure 2.3. The Zariski closure in $\mathbb{A}^2(\mathbb{C})$ of the "branch" $(\operatorname{Reg} V) \cap V(\mathbb{R}) = V(\mathbb{R}) \cap \{x > 1\}$ is V.

Remark 2.2.14 The point (0, 0) is not, however, contained in the Euclidean closure of the branch $V(\mathbb{R}) \cap \{x > 1\}$.

Example 2.2.15 (*Irreducible, singular, non empty and non dense*) This is an example of an irreducible singular algebraic set V whose real locus is neither empty nor Zariski dense in V. Consider

$$P(x, y) = ((x+i)^2 + y^2 - 1)((x-i)^2 + y^2 - 1) + x^2 = \\ x^4 + 2x^2y^2 + y^4 - 4y^2 + 4 + x^2$$

which is a polynomial in $\mathbb{R}[x, y]$. The set $V := \mathcal{Z}_{\mathbb{C}}(P) \subset \mathbb{A}^2(\mathbb{C})$ is an irreducible algebraic set and its real locus contains exactly two points. Indeed, let $P_1(x, y) = P(x, y) - x^2$ and set $V_1 := \mathcal{Z}(P_1)$. If (x, y) is a real point of V_1 then $y^2 = 1 - (x+i)^2$ or $y^2 = 1 - (x-i)^2$. As x and y are real, x must be identically zero so $y = \pm\sqrt{2}$ and $V_1(\mathbb{R}) = \{(0, \sqrt{2}), (0, -\sqrt{2})\}$. We will now prove that we also have that $V(\mathbb{R}) = \{(0, \sqrt{2}), (0, -\sqrt{2})\}$. Note that if $P(x, y) = 0$ for some real x then this implies that $P_1(x, y) = x^4 + 2x^2y^2 + y^4 - 4y^2 + 4$ is a negative or zero real number. Considering P_1 as a degree 2 polynomial in the variable $Y = y^2$ with coefficients in $\mathbb{R}[x]$ we see that its discriminant is equal to $-4x^2$. If x is non zero then this discriminant is strictly negative so for real x and y, $P(x, y) = 0$ if and only if $P_1(x, y) = 0$. We leave it as an exercise for the reader to show that P is irreducible, a long but unsurprising calculation. (We constructed the polynomial P by starting from the polynomial P_1 and looking for a perturbation of P_1 preserving the two real points in V_1, whose existence is guaranteed by Brusotti's Theorem 2.7.10.).

Exercise 2.2.16 *Construct a similar example from the example given in Remark 1.2.31(2).*

Theorem 2.2.17 *Let (X, σ) be a quasi-projective algebraic $\mathbb{R}$-variety. If the variety (X, σ) has* enough real points*, or in other words if $X(\mathbb{R})$ is Zariski dense in X, then the real locus equipped with the restriction of the structural sheaf, $\left(X(\mathbb{R}), (\mathcal{O}_X)^G_{X(\mathbb{R})}\right)$, is a real algebraic variety as in Definition 1.3.9.*

Proof This follows easily from Theorem 2.1.33 and the projective analogue of Theorem 2.2.4. □

Corollary 2.2.18 *Let (X, σ) be a quasi-projective algebraic $\mathbb{R}$-variety. If the complex variety X is irreducible and non singular and $X(\mathbb{R}) \neq \varnothing$ then $\left(X(\mathbb{R}), (\mathcal{O}_X)^G_{X(\mathbb{R})}\right)$ is a real algebraic variety.*

Proof See Corollary 2.2.10. □

The following proposition justifies the introduction of a third type of morphism between $\mathbb{R}$-varieties, somewhere between regular maps Definition 2.1.25 and rational maps Definition 2.1.27.

Proposition 2.2.19 *Let (X, σ) and (Y, τ) be $\mathbb{R}$-varieties with enough real points and let*

$$\psi\colon (X, \sigma) \dashrightarrow (Y, \tau)$$

be a rational map of $\mathbb{R}$-varieties. If the domain of ψ contains the real locus $X(\mathbb{R})$, then ψ induces by restriction a regular map of real algebraic varieties $\left(X(\mathbb{R}), (\mathcal{O}_X)^G_{X(\mathbb{R})}\right) \to \left(Y(\mathbb{R}), (\mathcal{O}_Y)^G_{Y(\mathbb{R})}\right)$.

Proof See Exercise 2.2.26(2). □

Definition 2.2.20 Let (X, σ) and (Y, τ) be $\mathbb{R}$-varieties.
A *rational $\mathbb{R}$-regular map* or *real morphism*

$$\psi\colon (X, \sigma) \dashrightarrow (Y, \tau)$$

is a rational map of $\mathbb{R}$-varieties such that $X(\mathbb{R}) \subset \operatorname{dom}(\psi)$.

Remark 2.2.21 A morphism of $\mathbb{R}$-varieties is of course always a rational $\mathbb{R}$-regular map but the converse is false.

Proposition 2.2.22 *Let (X, σ) and (Y, τ) be quasi-projective $\mathbb{R}$-varieties. Suppose that these varieties have enough real points. The following then hold.*

1. *A rational $\mathbb{R}$-regular map of $\mathbb{R}$-varieties $(X, \sigma) \dashrightarrow (Y, \tau)$ induces a regular map of real algebraic varieties*

$$\left(X(\mathbb{R}), (\mathcal{O}_X)^G_{X(\mathbb{R})}\right) \to \left(Y(\mathbb{R}), (\mathcal{O}_Y)^G_{Y(\mathbb{R})}\right) .$$

2. *Conversely, any regular map of real algebraic varieties*

$$\big(X(\mathbb{R}), (\mathcal{O}_X)^G_{X(\mathbb{R})}\big) \to \big(Y(\mathbb{R}), (\mathcal{O}_Y)^G_{Y(\mathbb{R})}\big)$$

is the restriction of an $\mathbb{R}$*-regular rational map* $\psi\colon (X,\sigma) \dashrightarrow (Y,\tau)$.
3. *Any rational map of* $\mathbb{R}$*-varieties* $(X,\sigma) \dashrightarrow (Y,\tau)$ *induces a rational map of real algebraic varieties*

$$\big(X(\mathbb{R}), (\mathcal{O}_X)^G_{X(\mathbb{R})}\big) \dashrightarrow \big(Y(\mathbb{R}), (\mathcal{O}_Y)^G_{Y(\mathbb{R})}\big)\ .$$

4. *Conversely, any rational map*

$$\big(X(\mathbb{R}), (\mathcal{O}_X)^G_{X(\mathbb{R})}\big) \dashrightarrow \big(Y(\mathbb{R}), (\mathcal{O}_Y)^G_{Y(\mathbb{R})}\big)$$

is the restriction of a rational map $(X,\sigma) \dashrightarrow (Y,\tau)$.

Proof Left for the reader as an exercise. □

Remark 2.2.23 We insist on (2) in the above proposition: the complex extension of a real regular map is not generally regular. The map $(x, y) \mapsto \frac{1}{x^2+y^2+1}$ from $\mathbb{A}^2(\mathbb{R})$ to $\mathbb{A}^1(\mathbb{R})$ is a regular map of real algebraic varieties but does not extend to a morphism of $\mathbb{R}$-varieties.

Remark 2.2.24 The "isomorphisms" corresponding to $\mathbb{R}$-regular rational maps are the $\mathbb{R}$-*biregular birational maps*. Note that it is important the map be both birational and $\mathbb{R}$-biregular: blowing up a real point (or in other words, contracting a (-1)-real curve) on an $\mathbb{R}$-surface (see Definition 4.1.26 for more details) is an $\mathbb{R}$-regular birational map but it is not $\mathbb{R}$-biregular.

Definition 2.2.25 Let (X,σ) and (Y,τ) be $\mathbb{R}$-varieties.
A $\mathbb{R}$-*biregular birational map* or *real isomorphism*

$$\psi\colon (X,\sigma) \dashrightarrow (Y,\tau)$$

is a birational map of $\mathbb{R}$-varieties inducing a biregular map of real algebraic varieties

$$\big(X(\mathbb{R}), (\mathcal{O}_X)^G_{X(\mathbb{R})}\big) \xrightarrow{\simeq} \big(Y(\mathbb{R}), (\mathcal{O}_Y)^G_{Y(\mathbb{R})}\big)\ .$$

Exercise 2.2.26 (Use Exercises 1.2.56 and 1.3.25) *Let* $F_1 \subset \mathbb{A}^n(\mathbb{C})$ *and* $F_2 \subset \mathbb{A}^m(\mathbb{C})$ *be affine algebraic sets stable under* $\sigma_{\mathbb{A}}$ *so that* $(F_1, \sigma_{\mathbb{A}}|_{F_1})$ *and* $(F_2, \sigma_{\mathbb{A}}|_{F_2})$ *are affine* $\mathbb{R}$*-varieties and let* $\varphi\colon F_1 \dashrightarrow F_2$ *be a rational map of complex varieties.*

1. *Prove that* φ *is a morphism of* $\mathbb{R}$*-varieties if and only if there are polynomial functions* $f_1, \dots, f_m \in \mathbb{R}[x_1, \dots, x_n]$ *such that for any point* $(x_1, \dots, x_n) \in F_1$,

$$\varphi(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))\ .$$

In this case, $F_1 \subset \mathrm{dom}(\varphi)$ *and* $\varphi\colon F_1 \to F_2$ *is a morphism of complex varieties.*

2. *Prove that* φ *is an* $\mathbb{R}$*-regular birational map if and only if there are polynomial functions* $g_1, \ldots, g_m \in \mathbb{R}[x_1, \ldots, x_n]$ *and* $h_1, \ldots, h_m \in \mathbb{R}[x_1, \ldots, x_n]$ *such that for every point* $(x_1, \ldots, x_n) \in F_1(\mathbb{R})$*,* $h_1(x_1, \ldots, x_n) \neq 0, \ldots, h_m(x_1, \ldots, x_n) \neq 0$ *and*

$$\varphi(x_1, \ldots, x_n) = \left(\frac{g_1(x_1, \ldots, x_n)}{h_1(x_1, \ldots, x_n)}, \ldots, \frac{g_m(x_1, \ldots, x_n)}{h_m(x_1, \ldots, x_n)} \right) .$$

In this case $F_1(\mathbb{R}) \subset \mathrm{dom}(\varphi)$ *and if* F_1 *and* F_2 *have enough real points then* $\varphi|_{F_1(\mathbb{R})}\colon F_1(\mathbb{R}) \to F_2(\mathbb{R})$ *is a regular map of real algebraic varieties with the induced structure.*

2.2.1 Non Singular $\mathbb{R}$*-Varieties*

A non singular complex variety of complex dimension n is naturally a real differential manifold of dimension $2n$ with the Euclidean topology. For example, for any non singular projective algebraic variety $X \subset \mathbb{P}^N(\mathbb{C})$ we have that X inherits a differential submanifold structure from $\mathbb{P}^N(\mathbb{C})$. If X is stable under $\sigma_{\mathbb{P}}$ and $X(\mathbb{R}) \neq \varnothing$ then $X(\mathbb{R})$ is a real algebraic variety by Corollary 2.2.18. The variety $X(\mathbb{R})$ inherits a Euclidean topology from $\mathbb{P}^N(\mathbb{R})$ (the same as in Definition 1.4.1) and can be thought of as a differential submanifold of $\mathbb{P}^N(\mathbb{R})$.

Proposition 2.2.27 *Let* (X, σ) *be an* $\mathbb{R}$*-variety. If the complex variety* X *is non singular and has complex dimension* n *then the set* X *with its Euclidean topology is a differential manifold of real dimension* $2n$*. If moreover* $X(\mathbb{R}) \neq \varnothing$ *then the set* $X(\mathbb{R})$ *with its euclidean topology is a differentiable manifold of real dimension* n*.*

We invite the reader to compare this result with Remark 1.5.28. We recall that under the hypotheses of the above proposition, $X(\mathbb{R})$ is Euclidean closed but Zariski dense in X. See Corollary 2.2.10 and Proposition 2.2.11 for more details.

Proof As we have seen above, as $\mathcal{O}_X$ is an $\mathbb{R}$-sheaf, the morphism

$$\begin{array}{ccc} \mathcal{O}_X(U) & \longrightarrow & \mathcal{O}_X(\sigma(U)) \\ f & \longmapsto & {}^{\sigma}f \end{array}$$

is a ring isomorphism for any open set U in X. As the variety X is non singular and of dimension n we can find a local system of parameters $\{\varphi_x\}_{x\in X}$—see Definition 1.5.47. Exercise 1.5.48 tells us that in terms of local coordinates we get a set of systems (U_x, φ_x) where $\varphi_x\colon U_x \to \mathbb{C}^n$ is analytic and on refining this open cover using Euclidean open sets we can assume that $\forall x \in X$, $U_{\sigma(x)} = \sigma(U_x)$ and

$$\forall x \in X, \quad {}^{\sigma}(\varphi_x) = \varphi_{\sigma(x)} \ . \tag{2.1}$$

where ${}^{\sigma}(\varphi_x) = \sigma_{\mathbb{A}} \circ \varphi_x \circ \sigma$.

It follows that if $(z_1, \ldots, z_n)_x$ is a system of local coordinates satisfying (2.1), then the system $(\Re(z_1), \Im(z_1), \ldots, \Re(z_n), \Im(z_n))_x$ is a system of real local coordinates for the manifold structure, equivalent to the complex local system of coordinates $(z_1, \overline{z_1}, \ldots, z_n, \overline{z_n})$.

The real structure σ then transforms $(z_1, \overline{z_1}, \ldots, z_n, \overline{z_n})_x$ into

$$(\overline{z_1}, z_1, \ldots, \overline{z_n}, z_n)_{\sigma(x)} \ .$$

In particular, if $x \in X(\mathbb{R})$ is a non singular point of X then by (2.1), ${}^{\sigma}(\varphi_x) = \overline{\varphi_{\sigma(x)}} = \varphi_x$ from which it follows that $\sigma_{\mathbb{A}} \circ \varphi_x = \varphi_x \circ \sigma$ and if $y \in U_x \cap X(\mathbb{R})$ then $\overline{\varphi_x(y)} = \varphi_x(y)$. The local coordinates of a real point are therefore real and the restriction of φ_x to $X(\mathbb{R})$ induces a system of real smooth (and in fact analytic) local coordinates $(\Re(z_1), \ldots, \Re(z_n))$ on $X(\mathbb{R})$ in a neighbourhood of x.

Alternatively, we can bypass the first part of this argument by using Lemma 2.2.8. Let x be a real point of X: there is then a system of local parameters which is invariant under σ. By Exercise 1.5.48 we can derive from this an invariant system of local coordinates. □

The underlying $2n$-dimensional manifold structure on the non singular complex variety X is not only *orientable* (since a holomorphic change of coordinate map has a positive determinant), but also *oriented*. Any isomorphism $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ yields an orientation on $\mathbb{R}^{2n}$ by pull back and the complex structure on X yields such an isomorphism. (See Exercise B.5.11 for more details).

Proposition 2.2.28 *Let (X, σ) be a non-singular $\mathbb{R}$-variety. The real structure σ is a diffeomorphism of the $2n$-dimensional oriented manifold X which preserves the orientation if n is even and reverses it otherwise.*

Proof This follows immediately from the previous proof. The map σ takes $(z_1, \overline{z_1}, \ldots, z_n, \overline{z_n})_x$ to $(\overline{z_1}, z_1, \ldots, \overline{z_n}, z_n)_{\sigma(x)}$, so the determinant of its differential is $(-1)^n$. □

2.2.2 Compatible Atlas

Exercise 2.2.29 *If X is a non singular complex analytic variety of dimension n we can reframe the definition of the conjugate variety using a maximal atlas $(U_i, \varphi_i)_i$ determining the complex structure on X: the complex structure of the conjugate variety $(X, \overline{\mathcal{O}_X})$ is given by the atlas $(U_i, \sigma_{\mathbb{A}^n} \circ \varphi_i)_i$.*

Definition 2.2.30 A *compatible atlas* on a smooth analytic $\mathbb{R}$-variety (X, σ) of dimension n is an atlas $\mathcal{A} = \{(U_i, \varphi_i \colon U_i \to \mathbb{C}^n)\}_i$ on the complex analytic variety X satisfying the following conditions. (Recall that ${}^{\sigma}\varphi_i = \sigma_{\mathbb{A}} \circ \varphi_i \circ \sigma$.)

1. The atlas is globally stable for the real structure, or in other words

$$(U_i, \varphi_i) \in \mathcal{A} \implies (\sigma(U_i), {}^\sigma\varphi_i) \in \mathcal{A} ;$$

2. If $U_i \cap X(\mathbb{R}) \neq \varnothing$ then $U_i = \sigma(U_i)$ and ${}^\sigma\varphi_i = \varphi_i$;
3. If $U_i \cap X(\mathbb{R}) = \varnothing$ then $U_i \cap \sigma(U_i) = \varnothing$.

Exercise 2.2.31 *Give a compatible atlas for* $(\mathbb{P}^1(\mathbb{C}), \sigma_{\mathbb{P}})$.

Proposition 2.2.32 *Every smooth analytic* $\mathbb{R}$*-variety has a compatible atlas.*

Proof This follows from the existence of local systems of parameters satisfying (2.1). □

2.3 Complexification of a Real Variety

We have seen that the real locus of an $\mathbb{R}$-variety is a real algebraic variety whenever it is Zariski dense. In this section we will study the converse: given a real algebraic variety V, is there an $\mathbb{R}$-variety whose real locus is isomorphic to V?

Let K be a field and let $L \supset K$ be an extension of K. The set $\mathbb{A}^n(K)$ is then a subspace of $\mathbb{A}^n(L)$ and $\mathbb{P}^n(K)$ is a subset of $\mathbb{P}^n(L)$.

Definition 2.3.1 (*Revisions of Definition* 1.2.12) Let $F \subset \mathbb{A}^n(K)$ be an algebraic set over K of ideal $I = \mathcal{I}(F) \subset K[X_1, \dots X_n]$. We define the algebraic set F_L over L to be the set $\mathcal{Z}_L(I)$ of zeros of I in $\mathbb{A}^n(L)$:

$$F_L := \mathcal{Z}_L(I) \subset \mathbb{A}^n(L) .$$

Similarly, if $F \subset \mathbb{P}^n(K)$ is a projective algebraic set of homogeneous ideal $I = \mathcal{I}(F) \subset K[X_0, \dots X_n]$ then we define an algebraic set

$$F_L := \mathcal{Z}_L(I) \subset \mathbb{P}^n(L) .$$

More generally, if $U = F \setminus F' \subset \mathbb{A}^n(K)$ is a quasi-affine set and $I = \mathcal{I}(F) \subset K[X_1, \dots, X_n]$ and $I' = \mathcal{I}(F') \subset K[X_1, \dots, X_n]$ are the associated ideals then we can define a quasi-affine set

$$U_L := F_L \setminus F'_L = \mathcal{Z}_L(I) \setminus \mathcal{Z}_L(I') \subset \mathbb{A}^n(L) .$$

And finally if $U = F \setminus F' \subset \mathbb{P}^n(K)$ is a quasi-projective algebraic set and $I = \mathcal{I}(F) \subset K[X_0, \dots, X_n]$ and $I' = \mathcal{I}(F') \subset K[X_0, \dots, X_n]$ are the associated homogeneous ideals then we define a set

$$U_L := F_L \setminus F'_L = \mathcal{Z}_L(I) \setminus \mathcal{Z}_L(I') \subset \mathbb{P}^n(L) .$$

Any real algebraic set (which here will be assumed affine to simplify the notation) $F \subset \mathbb{R}^n$ with vanishing ideal $I := \mathcal{I}(F) \subset \mathbb{R}[X_1, \dots, X_n]$ is therefore naturally associated to a *complexification* $F_{\mathbb{C}} := \mathcal{Z}_{\mathbb{C}}(\mathcal{I}(F)) = \mathcal{Z}_{\mathbb{C}}(I) \subset \mathbb{C}^n$ which is just the set of *complex* common zeros of the real polynomials vanishing on F. Note that the ideal I is made up of polynomials with real coefficients whereas $F_{\mathbb{C}} \subset \mathbb{C}^n$ is a set of complex points. As $F_{\mathbb{C}}$ is defined by polynomials with real coefficients, $\sigma_{\mathbb{A}}(F_{\mathbb{C}}) \subset F_{\mathbb{C}}$ and the restriction σ of the standard real structure $\sigma_{\mathbb{A}} \colon (x_1, \dots, x_n) \mapsto (\overline{x_1}, \dots, \overline{x_n})$ to $F_{\mathbb{C}}$ is a real structure with which $(F_{\mathbb{C}}, \sigma)$ is an $\mathbb{R}$-variety. Our initial real algebraic variety can be recovered as the set of fixed points of $F = (F_{\mathbb{C}})^{\sigma}$.

The above construction depends heavily on the equations defining F. The following definition enables us to consider abstract complexifications, by which we mean complexifications which are independent of a particular embedding into affine or projective space, or alternatively independent of a choice of equations.

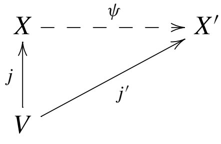
Definition 2.3.2 Let $(V, \mathcal{O}_V)$ be a real algebraic variety. A pair $((X, \sigma), j)$ is a *complexification* of V if (X, σ) is an $\mathbb{R}$-variety with enough real points and $j \colon V \to X$ is an injective map inducing an isomorphism of real algebraic varieties

$$(V, \mathcal{O}_V) \overset{\simeq}{\to} (X(\mathbb{R}), (\mathcal{O}_X)^G_{X(\mathbb{R})}) \ .$$

A complexification $((X, \sigma), j)$ of a real algebraic variety V is *quasi-projective* (resp. *non singular*) if X is a quasi-projective (resp. non singular) complex variety.

Let $((X, \sigma), j)$ be a complexification of a real algebraic variety V and let $\psi \colon (X, \sigma) \dashrightarrow (Y, \tau)$ be an $\mathbb{R}$-biregular birational map. It is easy to check that $((Y, \tau), \psi \circ j)$ is then a complexification of V. Indeed, since $X(\mathbb{R})$ is dense in X and ψ is birational the set $Y(\mathbb{R}) = \psi(X(\mathbb{R}))$ is dense in Y. The following proposition establishes the converse.

Proposition 2.3.3 *Let V be a real algebraic variety and let $((X, \sigma), j)$ be a complexification of V. Then for any complexification $((X', \sigma'), j')$ of V, there is a unique $\mathbb{R}$-biregular birational map $\psi \colon (X, \sigma) \dashrightarrow (X', \sigma')$, $X(\mathbb{R}) \subset \mathrm{dom}(\psi)$ such that the following diagram commutes.*



Proof We start by proving the proposition in the case where V, X and X' are affine. The uniqueness of the map for affine varieties will then enable us to glue complexifications and $\mathbb{R}$-biregular birational maps on open affine subsets of V to prove the general result. By hypothesis the morphism $h = j' \circ j^{-1} \colon X(\mathbb{R}) \to X'(\mathbb{R})$ is an isomorphism of real algebraic varieties. By the solution to Exercise 1.2.56(2), there is a morphism defined on an open neighbourhood of $X(\mathbb{R})$ in X extending $j' \circ j^{-1}$.

As $X(\mathbb{R})$ is dense in X, the rational map $\psi\colon (X,\sigma) \dashrightarrow (X',\sigma')$ induced by this extension is an $\mathbb{R}$-biregular birational map uniquely determined by $j' \circ j^{-1}$. □

Proposition 2.3.4 *Any real affine algebraic set has an affine complexification. Any real projective algebraic set has a projective complexification.*

Proof Let $X \subset \mathbb{A}^n(\mathbb{R})$ be a real affine algebraic set and let $I = \mathcal{I}(X) \subset \mathbb{R}[X_1, \ldots, X_n]$ be its ideal. The set X is then the set of real zeros of $\mathcal{Z}(I) \subset \mathbb{A}^n(\mathbb{R})$ and the Zariski closure, $X_{\mathbb{C}}$ of X in $\mathbb{A}^n(\mathbb{C})$ is the set of complex zeros $\mathcal{Z}_{\mathbb{C}}(I) \subset \mathbb{A}^n(\mathbb{C})$ by Remark 1.2.13. By construction the $\mathbb{R}$-variety $(X_{\mathbb{C}}, \sigma_{\mathbb{A}}|_{X_{\mathbb{C}}})$ has enough real points; denoting by $j\colon X \hookrightarrow X_{\mathbb{C}}$ the inclusion map, the pair $\big((X_{\mathbb{C}}, \sigma_{\mathbb{A}}|_{X_{\mathbb{C}}}), j\big)$ is then an affine complexification of X. Similarly, if $X \subset \mathbb{P}^n(\mathbb{R})$ is a real projective algebraic set and $I = \mathcal{I}(X) \subset \mathbb{R}[X_0, \ldots, X_n]$ is its homogeneous ideal then we take $\mathcal{Z}_{\mathbb{C}}(I) \subset \mathbb{P}^n(\mathbb{C})$, the set of complex zeros of I. □

Remark 2.3.5 We have seen that any real projective variety is also affine, and therefore has an affine complexification.

A complex projective algebraic variety is not generally affine, so a projective $\mathbb{R}$-variety is not typically affine, and neither is a projective complexification.

Certain real affine algebraic varieties also have projective complexifications, and these will be studied in Theorem 2.3.7 below.

Remark 2.3.6 Let X be a quasi-projective real algebraic variety with $X = V \setminus W \subset \mathbb{P}^n(\mathbb{R})$. Let $I_V \subset \mathbb{R}[X_0, \ldots, X_n]$ be the homogeneous ideal of V and let $I_W \subset \mathbb{R}[X_0, \ldots, X_n]$ be the homogeneous ideal of W. The set $V_{\mathbb{C}} = \mathcal{Z}_{\mathbb{C}}(I_V)$ is a projective complexification of V by the above and $W_{\mathbb{C}} = \mathcal{Z}_{\mathbb{C}}(I_W)$ is a projective complexification of W. The variety $X_{\mathbb{C}} = V_{\mathbb{C}} \setminus W_{\mathbb{C}}$ is therefore a quasi-projective complexification of X.

We recall Definition 1.4.11 which states that a real algebraic variety is complete if and only if it is compact for the Euclidean topology.

Theorem 2.3.7 *Any non singular complete real affine algebraic variety has a non singular projective complexification.*

Before proving this theorem we state some very useful lemmas concerning birational morphisms of $\mathbb{R}$-varieties. Let (X, σ) be an $\mathbb{R}$-variety and let $x \in X(\mathbb{R})$ be a real point. We denote by C_x the *connected component* of $X(\mathbb{R})$ containing x. Throughout this section, connected means connected in the Euclidean topology.

Lemma 2.3.8 *Let (X, σ) be an $\mathbb{R}$-variety and let $x \in X(\mathbb{R}) \cap \operatorname{Reg} X$ be a regular real point. The Euclidean connected component $C_x \subset X(\mathbb{R})$ is not then contained in any strict Zariski closed subset of X.*

Proof By Proposition 1.5.29, x has a connected Euclidean open neighbourhood $U \subset X(\mathbb{R})$ homeomorphic to a non empty subset of $\mathbb{R}^n$ where n is the Zariski dimension of X at x. As $U \subset C_x$ and any strict Zariski closed subset of X is of strictly positive codimension the result follows. □

Lemma 2.3.9 *Let $\varphi\colon (Y,\tau) \to (X,\sigma)$ be a birational morphism of $\mathbb{R}$-varieties and let $Z \subset Y$ be the smallest Zariski closed subset such that $\varphi|_{Y\setminus Z}$ is an isomorphism onto its image. Consider a point $y \in Y(\mathbb{R}) \cap \operatorname{Reg} Y$: the connected Euclidean component $C_{\varphi(y)}$ is not then contained in $\varphi(Z)$.*

Proof As codim $Z > 0$, $C_y \cap (Y \setminus Z) \neq \varnothing$ by Lemma 2.3.8. It follows that $\varphi(C_y) \cap (X \setminus \varphi(Z)) \neq \varnothing$ and as the image of a connected subset under a continuous map is still connected, $\varphi(C_y) \subset C_{\varphi(y)}$ and hence $C_{\varphi(y)} \cap (X \setminus \varphi(Z)) \neq \varnothing$. □

Proposition 2.3.10 *Let (X,σ) be an $\mathbb{R}$-variety and let $\varphi\colon (Y,\tau) \to (X,\sigma)$ be a resolution of singularities of X. Suppose that the connected component of a real singular point $x \in X(\mathbb{R})$ is contained in the singular locus $C_x \subset \operatorname{Sing} X$. We then have that $\varphi^{-1}(x) \cap Y(\mathbb{R}) = \varnothing$.*

Proof By Theorem 1.5.51, Sing X is a strict Zariski closed subset of X. The result then follows from Lemma 2.3.9 applied to $Z = \pi^{-1}(\operatorname{Sing} X)$ using Definition 1.5.53. □

Proof of Theorem 2.3.7 Let V be a non singular real affine algebraic variety which is compact for the Euclidean topology. By Proposition 2.3.4, V has an affine complexification $((X,\sigma), j)$. By Theorem 2.2.9, $X(\mathbb{R}) \simeq V$ does not meet Sing X. We consider a projective completion (X', σ') of (X,σ): in particular, X is a subvariety of X' and $\sigma = \sigma'|_X$. By Hironaka's resolution of singularities Theorem 1.5.54 there is a non singular projective $\mathbb{R}$-variety (Y,τ) and a birational morphism $\pi : (Y,\tau) \to (X',\sigma')$ of $\mathbb{R}$-varieties which is an isomorphism on $\pi^{-1}(\operatorname{Reg} X') \to \operatorname{Reg} X'$. As $X(\mathbb{R}) \subset \operatorname{Reg} X$, the restriction of the composition $(Y,\tau) \to (X,\sigma)$ to $X(\mathbb{R})$ is an isomorphism.

As V is compact, $X(\mathbb{R})$ is also compact, so it is closed in $X'(\mathbb{R})$ for the Euclidean topology. It follows that for every $x \in X'(\mathbb{R}) \setminus X(\mathbb{R})$ there is an inclusion $C_x \subset X'(\mathbb{R}) \setminus X(\mathbb{R})$ and Proposition 2.3.10 tells us that $\pi^{-1}(X'(\mathbb{R}) \setminus X(\mathbb{R})) \cap Y(\mathbb{R}) = \varnothing$. We can therefore conclude that $((Y,\tau), (\pi|_{Y(\mathbb{R})})^{-1} \circ j)$ is a non singular projective complexification of V. □

Remark 2.3.11 In the above proof, $X'(\mathbb{R}) \setminus X(\mathbb{R})$ may be non empty. In Example 2.6.38, examined in detail below, we consider the set

$$W := \mathcal{Z}(16(x_1^2 + x_2^2) - (x_1^2 + x_2^2 + x_3^2 + 3)^2) \subset \mathbb{A}^3(\mathbb{R}) \ .$$

and the projective complexification given by

$$\widehat{W}_{\mathbb{C}} := \mathcal{Z}\left(16(x_1^2 + x_2^2) - (x_1^2 + x_2^2 + x_3^2 + 3x_0^2)^2\right) \subset \mathbb{P}^3(\mathbb{C}) \ .$$

The $\mathbb{R}$-variety $(\widehat{W}_{\mathbb{C}}, \sigma_{\mathbb{P}}|_{\widehat{W}_{\mathbb{C}}})$ contains real points that do not belong to the torus of revolution $W_{\mathbb{C}}(\mathbb{R}) = W$. Indeed, if $x_1^2 + x_2^2 \leqslant 16$ then the point $\left(0 : x_1 : x_2 : \sqrt{4\sqrt{(x_1^2 + x_2^2)} - (x_1^2 + x_2^2)}\right)$ belongs to $\widehat{W}_{\mathbb{C}}(\mathbb{R}) \setminus W_{\mathbb{C}}(\mathbb{R})$. The $\mathbb{R}$-morphism $\psi : \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \to \widehat{W}_{\mathbb{C}}$ is a resolution of singularities of $\widehat{W}_{\mathbb{C}}$.

We use the above results to prove Theorem 1.5.55 for $\mathbb{R}$-varieties.

Theorem 2.3.12 *Let $\varphi\colon (Y,\tau)\to(X,\sigma)$ be a birational morphism of non singular $\mathbb{R}$-varieties. If the real loci $X(\mathbb{R})$ et $Y(\mathbb{R})$ are compact for the Euclidean topology then they have the same number of connected components.*

$$\#\pi_0(Y(\mathbb{R})) = \#\pi_0(X(\mathbb{R}))\ .$$

Proof Let $Z\subset Y$ be the smallest Zariski closed subset such that $\varphi|_{Y\setminus Z}$ is an isomorphism onto its image. The map φ is continuous for the Euclidean topology so $\#\pi_0(Y(\mathbb{R}))\geqslant\#\pi_0(X(\mathbb{R}))$. To prove the opposite inequality, assume there are two distinct connected components Y_1 and Y_2 in $Y(\mathbb{R})$ such that $\varphi(Y_1)\cap\varphi(Y_2)$ is non empty. Let U be an open Euclidean neighbourhood of $x\in\varphi(Y_1)\cap\varphi(Y_2)$ in $X(\mathbb{R})$. We then have that $U\cap\varphi(Y_1)\neq\varnothing$ and $U\cap\varphi(Y_2)\neq\varnothing$. Indeed for $i=1,2$, $\varphi^{-1}(U)\cap Y_i$ is a non empty open space in $Y(\mathbb{R})$ and as Y is non singular $\varphi^{-1}(U)\cap Y_i\setminus Z$ is non empty by Lemma 2.3.8. As X is non singular we can assume that U is homeomorphic to a non empty open set in $\mathbb{R}^n$, where n is the dimension of X, which by the above is cut into two disjoint parts by the algebraic subset $\varphi(Z)$. The codimension of $\varphi(Z)$ is at least two because φ is a birational morphism (see [Sha94, II.4.4, Theorem 2] for example) which contradicts the fact that $\varphi(Z)$ disconnects the open set U. This yields a contradiction. □

The behaviour of an $\mathbb{R}$-variety away from its real points is often irrelevant for the study of the real locus $X(\mathbb{R})$—but not always. We saw in Remark 2.3.11 an example where we needed to consider the non real points of the complex variety.

Definition 2.3.13 A quasi-algebraic affine or projective set U over K is said to be *geometrically irreducible* if the set $U_{\overline{K}}$ (see Definition 2.3.1) defined over the algebraic closure $\overline{K}$ of K is irreducible.

A quasi-projective algebraic set V over K, is said to be *geometrically irreducible* if the image U of V under embedding into a projective space over K is geometrically irreducible. Under these circumstances the image under any projective embedding of V is geometrically irreducible by Exercise 2.3.14.

An $\mathbb{R}$-variety (X,σ) is said to be *irreducible* if and only if X is irreducible as a complex variety.

Exercise 2.3.14 *Check that if $\varphi\colon V\to\mathbb{P}^N(K)$ and $\varphi'\colon V\to\mathbb{P}^{N'}(K)$ are two projective embeddings of V then $\varphi(V)_{\overline{K}}$ is irreducible if and only if $\varphi'(V)_{\overline{K}}$ is irreducible.*

Proposition 2.3.15 *Let K be a field.*

1. *An algebraic set over K which is geometrically irreducible is irreducible.*
2. *An algebraic variety over K which is geometrically irreducible is irreducible.*
3. *A real algebraic variety V is geometrically irreducible if and only if it has an irreducible complexification.*

4. *Let (X, σ) be a quasi-projective algebraic ℝ-variety with enough real points. We then have that (X, σ) is irreducible if and only if the real algebraic variety $\left(X(\mathbb{R}), (\mathcal{O}_X)^G_{X(\mathbb{R})}\right)$ is geometrically irreducible.*

Proof Left as an exercise for the reader. □

Remark 2.3.16 Recall that by Corollary 2.2.10 the real locus of a non singular irreducible algebraic ℝ-variety is Zariski dense whenever it is non empty.

Exercise 2.3.17 (Review of Example *2.1.1*)

1. *The real algebraic set $F := \mathcal{Z}(x^2 + y^2) \subset \mathbb{A}^2(\mathbb{R})$ is geometrically irreducible.*
2. *On the other hand, the ℝ-variety (V, σ), where $V := \mathcal{Z}_{\mathbb{C}}(x^2 + y^2) \subset \mathbb{A}^2(\mathbb{C})$ and $\sigma = \sigma_{\mathbb{A}}|_V$, is not irreducible.*
3. *This appears to contradict the fact that $V^\sigma = F$—what is happening?*

2.3.1 Rational Varieties

Definition 2.3.18 (Rational ℝ-varieties)

1. An ℝ-variety (X, σ) of dimension n is *rational* (or ℝ-*rational*) if it is birationally equivalent to the ℝ-variety $(\mathbb{P}^n(\mathbb{C}), \sigma_{\mathbb{P}})$, or in other words if there is a birational map of ℝ-varieties $(X, \sigma) \dashrightarrow (\mathbb{P}^n(\mathbb{C}), \sigma_{\mathbb{P}})$.
2. An ℝ-variety (X, σ) of dimension n is *geometrically rational* (or ℂ-rational) if and only if the complex variety X is rational, or in other words if there is a birational map of complex varieties $X \dashrightarrow \mathbb{P}^n(\mathbb{C})$.

Remark 2.3.19 We invite the reader to compare this definition with Definition 1.3.37 in the first chapter. Note that "geometric" irreducibility and rationality behave differently: a geometrically irreducible variety is irreducible, whereas a rational variety is geometrically rational.

Proposition 2.3.20 *Any ℝ-rational ℝ-variety is ℂ-rational*

Remark 2.3.21 The converse of the above proposition is false, an example being given by $\mathbb{P}^1(\mathbb{C})$ with its anti-holomorphic involution $z \mapsto -\frac{1}{\bar{z}}$. See Remark 2.1.41 for more details. Chapter 4 contains many 2-dimensional examples.

Proposition 2.3.22 *Let (X, σ) be a quasi-projective non singular ℝ-variety. If (X, σ) is ℝ-rational and has non zero dimension then $X(\mathbb{R})$ is connected and non empty.*

Proof This follows from Theorem 2.3.12 since $\mathbb{P}^n(\mathbb{R})$ is connected and non empty for all $n > 0$. □

2.4 ℝ-Varieties, Real Algebraic Varieties and Schemes Over ℝ—a Comparison

This section reviews the various types of ℝ-varieties met so far and the logical relationships between them. We have identified two different types of real variety: real algebraic varieties and ℝ-varieties. In total, there are five different incarnations of real algebraic varieties:

1. The real locus of a set of real equations.
2a. A complex variety defined by equations with real coefficients.
2b. A complex variety with an anti-regular involution.
These last two cases of special complex varieties are equivalent if we make the extra assumption that the variety is *quasi-projective*.
3a. A scheme defined over ℝ.
3b. A scheme defined over ℂ with a real structure.
Once again, these last two cases are equivalent if we make the assumption that the scheme is *quasi-projectif*.

At the end of the day, the last four definitions are all equivalent for quasi-projective varieties and only the first is different. A variety of type (1) can be thought of as the *germ* of a variety of type (2a) in a neighbourhood of the real locus.

Moreover, any such variety has two topologies and two associated structures

- Zariski topology and algebraic variety structure.
- Euclidean topology and analytic variety structure.

There is a dictionary translating algebraic structures into underlying analytic structures. For example, the (anti)-regular maps become (anti)-holomorphic. This "translation" is not however an equivalence unless the variety is *projective*. See Appendix D.5 for more details.

Let us examine these structures in more detail.

1. (Section 1.3) A *real algebraic variety* (resp. *complex algebraic variety*) is a topological space X with a subsheaf $\mathcal{O}_X$ of the sheaf of functions with a finite covering of *affine* open sets U, by which we mean that $(U, \mathcal{O}_X|_U)$ is isomorphic to the zero set $\mathcal{Z}(I) \subset \mathbb{A}^n(\mathbb{R})$ of an ideal $I \subset \mathbb{R}[X_1, \dots, X_n]$ with the sheaf of functions which are locally rational fractions without *real* poles (resp. the set of zeros $\mathcal{Z}(I) \subset \mathbb{A}^n(\mathbb{C})$ of an ideal $I \subset \mathbb{C}[X_1, \dots, X_n]$ with the sheaf of functions which are locally rational functions without poles). Varieties X and Y are isomorphic if and only if there exists a *biregular* map $X \to Y$.
2. (Section 2.1) An ℝ-*variety* (X, σ) is a complex variety X with an anti-regular involution (or in other words a *real structure*) σ. The ℝ-varieties (X, σ) and (Y, τ) are isomorphic if there is a *biregular* isomorphism of complex varieties that commutes with the real structure. The varieties (X, σ) and (Y, τ) are *birationally* ℝ-*biregularly isomorphic* if there is a birational map $\varphi\colon X \dashrightarrow Y$ commuting with real structure such that $X(\mathbb{R}) \subset \mathrm{dom}(\varphi)$ and $Y(\mathbb{R}) \subset \mathrm{dom}(\varphi^{-1})$. (Section 2.3) A *complexification* of a real algebraic variety V is an ℝ-variety

(X, σ) with enough real points whose real locus $X(\mathbb{R})$ is isomorphic to V as a real algebraic variety.

(a) (Section 2.1) Any quasi-projective $\mathbb{R}$-variety can be realised as a variety defined by real coefficients (as can its principal sheaves, see Section 2.5).
(b) (Section 2.2) A quasi-projective $\mathbb{R}$-variety with enough real points induces by restriction a real algebraic variety structure on its real locus. A morphism of quasi-projective $\mathbb{R}$-varieties with enough real points induces a regular map of real algebraic varieties.
(c) (Section 2.3) Conversely, any quasi-projective real algebraic variety has a complexification which is an $\mathbb{R}$-variety with enough real points. Any morphism of quasi-projective real algebraic varieties can be extended to a rational $\mathbb{R}$-regular map of $\mathbb{R}$-varieties.
(d) (Section 2.3) Two $\mathbb{R}$-varieties which are complexifications of isomorphic real algebraic varieties are birationally $\mathbb{R}$-isomorphic but not generally isomorphic.

3. This paragraph requires some knowledge of schemes—see [Duc14] or [Liu02] for more details. See also [Ben16b, Section 3.1] for a more specific discussion of realisations of schemes over $\mathbb{R}$. We leave it is an exercise for the reader to check the claims made below.

   A scheme over a field K (or a K-schema) is a scheme X with a scheme morphism (called the *structural map*) $X \to \operatorname{Spec} K$. Throughout this paragraph, we assume X is of finite type over K (or in other words that X is covered by a finite number of spectra of finitely generated K-algebras). Two $\mathbb{R}$-schemes X and Y are *birationally* $\mathbb{R}$*-biregularly isomorphic* if there is a birational map $\varphi\colon X \dashrightarrow Y$ of $\mathbb{R}$-schemes such that φ is regular at every $\mathbb{R}$-rational point of X and φ^{-1} is regular at every $\mathbb{R}$-rational point of Y. Let X be a scheme over $\mathbb{C}$ equipped with an involution σ lifting complex conjugation $\sigma_{\mathbb{A}}^* = \operatorname{Spec}(z \mapsto \bar{z})\colon \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}$: we call such an involution a *real structure on* X. If X is quasi-projective then by [BS64, Proposition 2.6] there is a scheme $Z = X/\langle\sigma\rangle$ over $\mathbb{R}$ and an isomorphism of $\mathbb{C}$-schemes $\varphi\colon X \to Z \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$ such that $\sigma = \varphi^{-1} \circ (\mathrm{id} \times \sigma_{\mathbb{A}}^*) \circ \varphi$. Moreover, the pair (Z, φ) is uniquely determined by the pair (X, σ) up to $\mathbb{R}$-isomorphism. For example if $X = \operatorname{Spec} A$ is affine then $Z = \operatorname{Spec} A^{\sigma}$.
   Implicitly, most types of algebraic varieties used in this book are different manifestations of $\mathbb{R}$-schemes of finite type.

(a) The set $X(\mathbb{R})$ of $\mathbb{R}$-rational points of a scheme X over $\mathbb{R}$ with the restriction of the structural sheaf is a real algebraic variety. A morphism of $\mathbb{R}$-schemes induces a morphism of real algebraic varieties.
(b) Conversely, any quasi-projective real algebraic variety can be obtained as the set of $\mathbb{R}$-rational points of a scheme X over $\mathbb{R}$. Any morphism of quasi-projective real algebraic varieties can be extended to an $\mathbb{R}$-regular map of schemes over $\mathbb{R}$.

(c) Any two schemes over $\mathbb{R}$ whose real loci are isomorphic as real algebraic varieties are birationally $\mathbb{R}$-biregularly isomorphic.
(d) Let Z be a scheme of finite type over $\mathbb{R}$. We can associate to it the following $\mathbb{R}$-variety: X is the topological space of $\mathbb{C}$-rational points of the $\mathbb{C}$-scheme $Z \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$, The pair (X, σ) is the $\mathbb{R}$-variety obtained on equipping X with the real structure $\sigma := \operatorname{id} \times \operatorname{Spec}(z \mapsto \bar{z})$. We denote by $X(\mathbb{R})$ the set of closed points fixed by σ. If $Z(\mathbb{R})$ is the set of $\mathbb{R}$-rational points of the $\mathbb{R}$-scheme Z then $X(\mathbb{R}) = Z(\mathbb{R})$. A morphism of schemes over $\mathbb{R}$ induces a morphism of $\mathbb{R}$-varieties.
(e) Conversely, if (X, σ) is an $\mathbb{R}$-variety then there is a $\mathbb{C}$-scheme Z such that $Z(\mathbb{C}) = X$, [Har77, II.2.6] and there is an involutive morphism $\sigma_Z \colon Z \to Z$ lifting $\sigma_{\mathbb{A}}^* \colon \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}$ such that $\sigma_Z|_{Z(\mathbb{C})} = \sigma$. As we have seen above, if X is quasi-projective then (Z, σ_Z) corresponds to an $\mathbb{R}$-scheme. A morphism of $\mathbb{R}$-varieties induces a morphism of schemes over $\mathbb{R}$.

2.4.1 *Real Forms of a $\mathbb{C}$-Scheme*

By the above, Definition 2.1.13 can be reformulated scheme theoretically as follows.

Definition 2.4.1 A *real form* of a scheme X over $\mathbb{C}$ is a scheme X_0 over $\mathbb{R}$ whose complexification $X_0 \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$ is isomorphic to X.

2.4.2 *Notations X, $X(\mathbb{R})$, $X(\mathbb{C})$, $X_{\mathbb{C}}$, $X_{\mathbb{R}}$*

We now briefly discuss the various notations the reader may meet in the literature.

As in scheme theory, where by abuse of notation the structural morphism $Z \to \operatorname{Spec}\mathbb{R}$ is often omitted, the abbreviation X for the $\mathbb{R}$-variety (X, σ) is often used. Consequently, the notation $X_{\mathbb{C}}$ for the variety X is often used to emphasise the fact that we are concentrating on the complex variety and "forgetting" σ. Some authors, particularly of the Russian school, use the notation $X_{\mathbb{C}}$ or $\mathbb{C}X$ for the complex locus and $X_{\mathbb{R}}$ or $\mathbb{R}X$ for the real locus of $\mathbb{R}$-varieties.

Remark 2.4.2 In case that wasn't confusing enough, there is another object called $X_{\mathbb{R}}$ in the literature, constructed using extension of scalars. In the embedded case, it simply means separating the real and imaginary parts of the equations of a complex variety. From the scheme point of view this corresponds to taking the scheme morphism $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$ associated to the inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$ and compose maps $X \to \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$ to see that a scheme over $\mathbb{C}$ is *necessarily* a scheme over $\mathbb{R}$. For example, if $X \subset \mathbb{A}^n(\mathbb{C})$ is defined by r equations

$$\left\{P_i(z_1, \dots, z_n) = 0\right\}_{i=1,\dots,r}$$

then $X_{\mathbb{R}} \subset \mathbb{A}^{2n}(\mathbb{R})$ is defined by the $2r$ equations

$$\Big\{\Re(P_i(x_1 + iy_1, \ldots, x_n + iy_n) = 0), \qquad \Im(P_i(x_1 + iy_1, \ldots, x_n + iy_n) = 0\Big\}_{i=1,\ldots,r}.$$

Let X be an algebraic variety defined over $\mathbb{C}$ which for simplicity we will assume to be non singular. Consider the product variety $Z := X \times \overline{X}$ with the anti-regular involution $\sigma_Z\colon (x, y) \mapsto (\overline{y}, \overline{x})$. The set of real points of the $\mathbb{R}$-variety (Z, σ_Z) is then a real algebraic variety as in Definition 1.3.9, homeomorphic in the Euclidean topology to the topological manifold underlying the complex variety X. Some authors use $X_{\mathbb{R}} = Z(\mathbb{R})$ to denote this *underlying* real algebraic variety.

2.5 Coherent Sheaves and Algebraic Bundles

We will now generalise the above constructions to certain sheaves and vector bundles needed in the development of the theory.

2.5.1 Coherent ℝ-Sheaves

Let (X, σ) be an $\mathbb{R}$-variety, let $\mathcal{L}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules (see Theorem C.7.3) and let U be an open affine set in X. The space of sections $M := \mathcal{L}(\sigma(U))$ is then an $\mathcal{O}_X(\sigma(U))$-module. We define an $\mathcal{O}_X(U)$-module ${}^{\sigma}M$ by equipping the group M with the following $\mathcal{O}_X(U)$-twisted action.

$$(f, m) \mapsto {}^{\sigma}f \cdot m \tag{2.2}$$

where

$$(f, m) \mapsto f \cdot m$$

denotes the $\mathcal{O}_X(\sigma(U))$-action on M.

Definition 2.5.1 Let (X, σ) be an $\mathbb{R}$-variety and let $\mathcal{L}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. The *conjugate sheaf* ${}^{\sigma}\mathcal{L}$ is the sheaf of $\mathcal{O}_X$-modules defined over U by declaring ${}^{\sigma}\mathcal{L}(U)$ to be the twisted $\mathcal{O}_X(U)$-module ${}^{\sigma}M$. We say that $\mathcal{L}$ is an $\mathbb{R}$-*sheaf* if and only if $\mathcal{L} = {}^{\sigma}\mathcal{L}$. This is required to be an equality, not simply an isomorphism.

Remark 2.5.2 These definitions generalise Definition 2.2.1. Indeed, for any open set U in X, there is an equality of $\mathcal{O}_X(U)$-modules ${}^{\sigma}\mathcal{L}(U) = \mathcal{L}(\sigma(U))$ provided the right hand side is equipped with the twisted action (2.2). In particular, if $\mathcal{L}$ is a sheaf of $\mathbb{C}^n$-valued functions then ${}^{\sigma}\mathcal{L}(U) = \{{}^{\sigma}f \mid f \in \mathcal{L}(\sigma(U))\}$. Moreover, $\mathcal{L}$ is an $\mathbb{R}$-sheaf if and only if ${}^{\sigma}\mathcal{L}(U) = \mathcal{L}(U)$ for any open set U in X.

Our definition of an $\mathbb{R}-$sheaf is motivated by the following result which explicits the relationship between $\mathbb{R}$-sheaves on an $\mathbb{R}$-variety (X, σ) and sheaves of invariant functions. *A priori* an $\mathbb{R}$-sheaf is only a sheaf which is globally fixed by σ.

Lemma 2.5.3 *Let (X, σ) be a quasi-projective $\mathbb{R}$-variety and let $\mathcal{L}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. If $\mathcal{L}$ is an $\mathbb{R}$-sheaf then there is a quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{L}_0$ such that for any open affine subset $U \subset X$,*

$$\mathcal{L}(U \cap \sigma(U)) \simeq \mathcal{L}_0(U \cap \sigma(U)) \otimes_{\mathbb{R}} \mathbb{C}$$

and $\forall f \in \mathcal{L}_0(U \cap \sigma(U))$, ${}^{\sigma}f = f$. When this is the case we will say that f has real coefficients.

Proof Recall that by definition σ is a homeomorphism for the Zariski topology on X and in particular if U is a Zariski open set in X then the intersection $U \cap \sigma(U)$ is also Zariski open. Moreover, by Exercise 1.3.15(4), the open set $U \cap \sigma(U)$ is affine. It will therefore be enough to prove the result for an affine $\mathbb{R}$-variety so by Theorem 2.1.33 we may assume we are in the case where $X \subset \mathbb{A}^n(\mathbb{C})$ and $\mathcal{I}(X) \subset \mathbb{R}[X_1, \ldots, X_n]$. Under these hypotheses we have that $\sigma = \sigma_{\mathbb{A}}|_X$ and

$$\mathcal{O}_X(X) = \mathcal{A}(X) = (\mathbb{R}[X_1, \ldots, X_n]/\mathcal{I}(X)) \otimes_{\mathbb{R}} \mathbb{C} \ .$$

Let M be the $\mathcal{A}(X)$-module of global sections of the $\mathcal{O}_X$-module $\mathcal{L}(X)$. By hypothesis, σ induces a Galois action on M for which, on equipping the subgroup of fixed points M^G with its natural $\mathcal{A}(X(\mathbb{R}))$-module structure, we have that

$$M = M^G \otimes_{\mathcal{A}(X(\mathbb{R}))} (\mathcal{A}(X(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}) \ .$$

We then simply define $\mathcal{L}_0$ to be the sheaf associated to the $\mathcal{A}(X(\mathbb{R}))$-module M^G. See Definition C.7.2 for more details, □

We will make intensive use of coherent $\mathbb{R}$-sheaves, particularly invertible sheaves, see Definition C.5.8. These are in bijective correspondence with line bundles, see Corollary 2.5.13.

Let $(X, \mathcal{O}_X)$ be an *affine* real or complex algebraic variety and let $\mathcal{F}$ be a quasi-coherent sheaf. The set of global sections $\Gamma(X, \mathcal{F})$ is then a $\Gamma(X, \mathcal{O}_X)$-module. If $\mathcal{F}$ is locally free then this module is *projective*, by which we mean that it is a direct summand of a free $\Gamma(X, \mathcal{O}_X)$-module, see Definition A.4.6.

The next lemma requires us to generalise Definition C.7.2. Let M be a $\Gamma(X, \mathcal{O}_X)$-module and let $\mathcal{O}_X \otimes_{\Gamma(X,\mathcal{O}_X)} M$ be the sheaf of $\mathcal{O}_X$-modules associated to the presheaf $U \mapsto \mathcal{O}_X(U) \otimes_{\Gamma(X,\mathcal{O}_X)} M$. If $(X, \mathcal{O}_X)$ is a *complex* variety then $\mathcal{O}_X(U) = \Gamma(X, \mathcal{O}_X)_f$ for any principal open set $U = \mathcal{D}(f)$ and $\mathcal{O}_X \otimes_{\Gamma(X,\mathcal{O}_X)} M$ can be identified with the sheaf $\widetilde{M}$ of Definition C.7.2. In particular, $\left(\mathcal{O}_X \otimes_{\Gamma(X,\mathcal{O}_X)} M\right)(U) = \widetilde{M}(U) = M_f$ for any principal open set $U = \mathcal{D}(f)$. If $(X, \mathcal{O}_X)$ is a *real* variety then for any open set U in X, $\mathcal{O}_X(U)$ can be identified with the inductive limit $\varinjlim_{\mathcal{D}(f) \supset U} \Gamma(X, \mathcal{O}_X)_f$ of the localisations $\Gamma(X, \mathcal{O}_X)_f$ where f runs over the set of regular functions which do not vanish on any point of U and $\left(\mathcal{O}_X \otimes_{\Gamma(X,\mathcal{O}_X)} M\right)(U) \simeq \varinjlim_{\mathcal{D}(f)\supset U} M_f$.

The special case of locally free finitely generated sheaves leads us directly to vector bundles.

Lemma 2.5.4 *Let $(X, \mathcal{O}_X)$ be a real or complex affine algebraic variety. Let $\mathcal{F}$ be a sheaf of finitely generated locally free $\mathcal{O}_X$-modules. The $\Gamma(X, \mathcal{O}_X)$-module $\Gamma(X, \mathcal{F})$ of global sections of $\mathcal{F}$ is then projective and finitely generated. Conversely, let M be a projective finitely generated $\Gamma(X, \mathcal{O}_X)$-module. The associated $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_{\Gamma(X,\mathcal{O}_X)} M$ is then finitely generated and locally free.*

Proof Left as an exercise for the reader. □

If $(X, \mathcal{O}_X)$ is a *complex* variety then every locally free finitely generated $\mathcal{O}_X$-module $\mathcal{F}$ is equal to the sheaf $\widetilde{\Gamma(X, \mathcal{F})}$ associated to its $\Gamma(X, \mathcal{O}_X)$-module $\Gamma(X, \mathcal{F})$ of global sections.

Proposition 2.5.5 *If $(X, \mathcal{O}_X)$ is a* complex *affine algebraic variety then the map $M \mapsto \widetilde{M}$ yields a bijective correspondence between finitely generated projective $\Gamma(X, \mathcal{O}_X)$-modules and finitely generated locally free $\mathcal{O}_X$-modules.*

Proof See [Har77, Corollary II.5.5]. □

On the other hand, as the following example shows, if $(X, \mathcal{O}_X)$ is a *real* affine variety then there are finitely generated locally free sheaves which are not associated to $\Gamma(X, \mathcal{O}_X)$-modules.

Example 2.5.6 Based on [BCR98, Example 12.1.5], see also [FHMM16, Example 5.35].

Let $P \in \mathbb{R}[x, y]$ be the polynomial defined by

$$P(x, y) = x^2(x-1)^2 + y^2$$

which has exactly two real zeros, $a_0 = (0, 0)$ and $a_1 = (1, 0)$. Set $U_i = \mathbb{R}^2 \setminus \{a_i\}$ for $i = 0, 1$. The Zariski open subsets U_0 and U_1 form an open covering of $\mathbb{A}^2(\mathbb{R})$. We define a locally free coherent rank 1 sheaf $\mathcal{F}$ by gluing the sheaves $\mathcal{O}_{\mathbb{A}^2(\mathbb{R})}|_{U_0}$ and $\mathcal{O}_{\mathbb{A}^2(\mathbb{R})}|_{U_1}$ over $U_0 \cap U_1$ using the transition function $\psi_{01} = P$ on $U_0 \cap U_1$. In other words, two sections $s_0 \in \mathcal{O}_{\mathbb{A}^2(\mathbb{R})}|_{U_0}(V_0)$ and $s_1 \in \mathcal{O}_{\mathbb{A}^2(\mathbb{R})}|_{U_1}(V_1)$ on the Zariski open sets V_0 and V_1 are glued together if and only if $\psi_{01}s_1 = s_0$ over $V_0 \cap V_1$.

The $\mathcal{O}_{\mathbb{A}^2(\mathbb{R})}$-module $\mathcal{F}$ is not generated by its global sections because any global section s of $\mathcal{F}$ vanishes at a_1. Indeed, the restriction s_i of s to U_i is a regular function on U_i for $i = 0, 1$. The gluing condition is $\psi_{01}s_1 = s_0$ on $U_0 \cap U_1$. Set $s_i = g_i/h_i$ where $g_i, h_i \in \mathbb{R}[x, y]$, with $h_i \neq 0$ at every point on U_i and g_i, h_i coprime for $i = 0, 1$. The gluing condition implies that $Ph_0g_1 = g_0h_1$ on $\mathbb{R}^2$. As P is irreducible and $h_1(a_0) \neq 0$ the polynomial P divides g_0 or in other words there is an $\lambda \in \mathbb{R}^*$ such that $g_0 = \lambda P g_1$ and $h_1 = \lambda^{-1}h_0$. In particular $g_0(a_1) = 0$ and hence $s(a_1) = 0$. It follows that the quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{A}^2(\mathbb{R})$ is not generated by global sections. *A fortiori*, there is no $\Gamma(\mathbb{A}^2(\mathbb{R}), \mathcal{O}_{\mathbb{A}^2(\mathbb{R})})$-module whose associated sheaf is $\mathcal{F}$.

Note that the module of global sections $\Gamma(\mathbb{A}^2(\mathbb{R}), \mathcal{F})$ is isomorphic to $\Gamma(\mathbb{A}^2(\mathbb{R}), \mathcal{O}_{\mathbb{A}^2(\mathbb{R})}) = \mathcal{R}(\mathbb{R}^2)$ *via* the map $(s_0, s_1) \mapsto s_1 = \frac{g_1}{h_1}$ since $h_1 = \lambda^{-1} h_0$ does not vanish at any point of $\mathbb{R}^2$.

2.5.2 Algebraic Vector Bundles

Definition 2.5.7 Let $(X, \mathcal{O}_X)$ be an algebraic variety over a field K. A *rank r pre-algebraic vector bundle* over X is a K-vector bundle (E, π), see Definition C.3.5, where E is an algebraic variety over K, $\pi : E \to X$ is a regular map and the homeomorphisms $\psi_i : \pi^{-1}(U_i) \xrightarrow{\simeq} U_i \times K^r$ are biregular maps. More generally, a *pre-algebraic vector bundle* has constant rank on every connected component of X.

Remark 2.5.8 On an affine real algebraic variety the vector bundles defined above are called *pre-algebraic* in [BCR98] but *algebraic* in the previous version [BCR87].

Consider a pre-algebraic (resp. rank r) vector bundle on X. Its sheaf of algebraic local sections is then naturally equipped with a $\mathcal{O}_X$-module structure which is locally free (resp. of rank r).

Proposition 2.5.9 *Let $(X, \mathcal{O}_X)$ be an algebraic variety over a base field K. There is a bijective correspondence between the class of finitely generated locally free (resp. of rank r) coherent sheaves on X and isomorphism classes of pre-algebraic (resp. rank r) vector bundles on X.*

Proof See [BCR98, Proposition 12.1.3]. □

If $(X, \mathcal{O}_X)$ is a *complex* variety, pre-algebraic bundles are well behaved, as we saw in Proposition 2.5.5. If $(X, \mathcal{O}_X)$ is a *real* variety, the pre-algebraic line bundle associated to the sheaf $\mathcal{F}$ of Example 2.5.6 is not generated by its global sections, illustrating the fact that on a real variety the notion of pre-algebraic vector bundles is not particularly useful and motivating thereby the following definition.

Definition 2.5.10 A pre-algebraic vector bundle (E, π) on an affine real algebraic variety X is said to be *algebraic* if it is isomorphic to a pre-algebraic subbundle of a direct sum of structural sheaves. Similarly, a finitely generated locally free sheaf is said to be *algebraic* if its associated vector bundle is algebraic.

Remark 2.5.11 (*Real and complex bundles*)

1. Proposition 2.5.5 implies that any pre-algebraic vector bundle on an affine complex algebraic variety is algebraic.
2. On a real affine algebraic variety the vector bundles defined above were said to be *algebraic* in [BCR98, Definition 12.1.6] but were *strongly algebraic* in [BCR87].

Definition 2.5.12 A rank one algebraic vector bundle is called a *line bundle*.

Corollary 2.5.13 *Let $(X, \mathcal{O}_X)$ be a real or complex algebraic variety. There is a bijective correspondence between isomorphism classes of invertible algebraic sheaves on X and (algebraic) line bundles on X.*

Proof This follows immediately from Proposition 2.5.9. □

Theorem 2.5.14 *Let $(X, \mathcal{O}_X)$ be a real affine algebraic variety and let (E, π) be a pre-algebraic vector bundle on X. The bundle E is then algebraic if and only if there is a finitely generated projective $\Gamma(X, \mathcal{O}_X)$-module M such that the $\Gamma(X, \mathcal{O}_X)$-module of algebraic sections of (E, π) is isomorphic to the $\Gamma(X, \mathcal{O}_X)$-module $\mathcal{O}_X \otimes_{\Gamma(X, \mathcal{O}_X)} M$.*

Proof See [BCR98, Theorem 12.1.7]. □

As in [Hui95], we see that Definition 2.5.10 of "nice" vector bundles on a real algebraic variety V, which may initially seem unnatural, simply says that "nice" vector bundles are precisely those that can be obtained by restricting an $\mathbb{R}$-vector bundle on some complexification (X, σ) of V.

Let (X, σ) be a quasi-projective algebraic $\mathbb{R}$-variety with enough real points (see Definition 2.2.5 and Theorem 2.2.9) and let $\mathcal{L}$ be a finitely generated locally free $\mathbb{R}$-sheaf. It is immediate that the restriction $\mathcal{L}_0|_{X(\mathbb{R})}$ of the sheaf $\mathcal{L}_0$ defined in Lemma 2.5.3 is a finitely generated locally free sheaf on the real algebraic variety $\left(X(\mathbb{R}), (\mathcal{O}_X)^G_{X(\mathbb{R})}\right)$.

Theorem 2.5.15 *Let (X, σ) be a quasi-projective algebraic $\mathbb{R}$-variety with enough real points and let $\mathcal{L}$ be a finitely generated locally free $\mathbb{R}$-sheaf. The finitely generated locally free sheaf $\mathcal{L}_0|_{X(\mathbb{R})}$ on the real algebraic variety $\left(X(\mathbb{R}), (\mathcal{O}_X)^G_{X(\mathbb{R})}\right)$ is then algebraic.*

Corollary 2.5.16 *Let (X, σ) be a quasi-projective algebraic $\mathbb{R}$-variety with enough real points and let (E, π) be a* topological *vector bundle on the real algebraic variety $\left(X(\mathbb{R}), (\mathcal{O}_X)^G_{X(\mathbb{R})}\right)$.*

The vector bundle (E, π) is then algebraic if and only if there is a pre-algebraic $\mathbb{R}$-vector bundle $(\mathcal{E}, \eta)$ on (X, σ) whose restriction $(\mathcal{E}|_{X(\mathbb{R})}, \eta|_{X(\mathbb{R})})$ is isomorphic to $(E \otimes \mathbb{C}, \pi \otimes \mathbb{C})$.

Remark 2.5.17 In other words, a *topological* $\mathbb{R}$-vector space E on a real affine algebraic variety V is *algebraic* if and only if tensoring with $\mathbb{C}$ yields the restriction to V of an algebraic $\mathbb{C}$-vector bundle $\mathcal{E}$ equipped with a real structure on some complexification $V_{\mathbb{C}}$ of V.

2.6 Divisors on a Projective $\mathbb{R}$-Variety

This section draws on [Liu02, Chapter 7], where the interested reader will find all the proofs left out below. A handful of statements and proofs in this section require some knowledge of sheaf cohomology, for which we also refer to [Liu02, Section 5.2].

2.6.1 Weil Divisors

Definition 2.6.1 Let X be a quasi-projective irreducible normal complex algebraic variety (Definition 1.5.37). This is not the weakest possible hypothesis we could make: everything that follows holds on any variety that is *non singular in codimension* 1.

- A *prime divisor* on X is an irreducible closed subvariety of X of codimension 1.
- A *Weil divisor* on X is a *codimension* 1 *cycle*, i.e. a finite formal sum of prime divisors with integer coefficients[3]

$$D = \sum_{\substack{A \text{ prime Weil} \\ \text{divisor on } X}} a_A A \,, \qquad a_A \in \mathbb{Z} \text{ almost all zero.}$$

- Let $D = \sum a_A A$ be a divisor. For any prime divisor A in X, the integer a_A is called the *multiplicity*, denoted $\mathrm{mult}_A(D)$, of D along A.
- The *support* of a divisor is the subvariety

$$\mathrm{Supp}\, D = \bigsqcup_{a_A \neq 0} A \,.$$

- If all the coefficients vanish, i.e. $\mathrm{Supp}\, D = \varnothing$, we write $D = 0$.
- If all the coefficients are positive or zero D is said to be *effective* and we write $D \geqslant 0$.

We denote by $Z^1(X)$ we set of all Weil divisors on X. By definition, $Z^1(X)$ is the free abelian group generated by prime divisors.

Example 2.6.2 1. If X is a curve then the prime divisors on X are the points of X. We define the *degree* of a Weil divisor $\sum_{i=1}^{s} a_i D_i$ to be the sum of the coefficients

$$\deg D = \sum_{i=1}^{s} a_i \,.$$

[3] Or in other words—zero except for a finite number of them.

2. If X is a projective surface then the prime divisors on X are the irreducible curves in X. There is then no intrinsic definition of the degree of a divisor but we can define the degree with respect to a choice of very ample divisor or projective embedding.
3. If $X = \mathbb{P}^n$ then prime divisors are irreducible hypersurfaces. The degree of a hypersurface D_i is then well-defined (it is the degree of a polynomial generating the principal ideal $\mathcal{I}(D_i)$, see [Har77, Chapitre I]) and the *degree* of a Weil divisor $\sum_{i=1}^{s} a_i D_i \in Z^1(\mathbb{P}^n)$ is defined by

$$\deg D = \sum_{i=1}^{s} a_i \deg D_i \ .$$

If $f \in K(X)^* = \mathbb{C}(X)^*$ is a rational function not identically zero (see Definition 1.2.69 and Remark1.2.74) and A is a prime divisor we define the *multiplicity* $\mathrm{mult}_A(f)$ of f along A as follows:

- $\mathrm{mult}_A(f) = k > 0$ if f vanishes along A to order k;
- $\mathrm{mult}_A(f) = -k$ if f has a pole of order k along A (i.e. if $\frac{1}{f}$ vanishes along A to order k;
- $\mathrm{mult}_A(f) = 0$ in all other cases.

We can associate to any rational function $f \in K(X)^*$ a divisor $\mathrm{div}(f) \in Z^1(X)$ defined by

$$\mathrm{div}(f) := \sum_{\substack{A \text{ prime Weil} \\ \text{divisor in } X}} \mathrm{mult}_A(f) A \ .$$

Note that $\mathrm{div}(f) \in Z^1(X)$ since $\mathrm{mult}_A(f)$ vanishes for almost all prime divisors A. Such divisors are called *principal divisors*. Since $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$ the set of such divisors is a subgroup $\mathcal{P}(X)$ in $Z^1(X)$.

Exercise 2.6.3 *Prove that for any rational function f on $\mathbb{P}^n$ we have that*

$$\deg(\mathrm{div}(f)) = 0 \ .$$

Definition 2.6.4 Two divisors D, D' on a variety X are said to be *linearly equivalent* if $D - D'$ is a principal divisor. We denote by $D \sim D'$ the equivalence relation thus defined and by

$$\mathrm{Cl}(X) := Z^1(X)/\mathcal{P}(X) = Z^1(X)/\sim$$

the group of divisors modulo linear equivalence.

Exercise 2.6.5 *Prove that the group $\mathrm{Cl}(\mathbb{P}^n)$ is isomorphic to $\mathbb{Z}$ and it is generated by the linear class of the divisor $1H$ associated to a hyperplane $H \subset \mathbb{P}^n$.*

Example 2.6.6 Let C be a projective plane curve of degree d—see Definition 1.6.1— and let L be a line in $\mathbb{P}^2(\mathbb{C})$. The curve C is then linearly equivalent to d times the line L. In particular, any projective conic (see Exercise 1.2.68) is linearly equivalent to the double line $2L$.

2.6.2 Cartier Divisors

Let X be an algebraic variety, let $U \subset X$ be an open subset and let $f \in K(U)^*$ be a rational function which is not identically zero on U. By definition there is then a dense open subset $V \subset U$ such that $\forall p \in V,\ f(p) = \frac{g(p)}{h(p)}$ for some $g, h \in \mathcal{O}_X(V)$.

Definition 2.6.7 A *Cartier divisor* (or locally principal divisor) on an algebraic variety X is a global section of the quotient sheaf arising from the following exact sequence of multiplicative sheaves

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{M}_X^* \longrightarrow \mathcal{M}_X^*/\mathcal{O}_X^* \longrightarrow 1 \tag{2.3}$$

where $\mathcal{O}_X^*$ is the sheaf of regular functions that do not vanish at any point and $\mathcal{M}_X^*$ is the sheaf of rational functions that are not identically zero[4]. We denote by

$$\mathrm{Div}(X) := \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$$

the group of Cartier divisors. The group law on $\mathrm{Div}(X)$ is abelian and is written additively.

Definition 2.6.8 A Cartier divisor is said to be *principal* if it is associated to a global rational function. We say that two divisors D_1 and D_2 are *linearly equivalent* if $D_1 - D_2$ is principal. We then write $D_1 \sim D_2$ as for Weil divisors. The subgroup of $\mathrm{Div}(X)$ of principal divisors is isomorphic to $\mathcal{P}(X)$ and we denote by

$$\mathrm{CaCl}(X) := \mathrm{Div}(X)/\mathcal{P}(X) = \mathrm{Div}(X)/\!\sim$$

the group of Cartier divisors modulo linear equivalence.

Proposition 2.6.9 *Let X be an algebraic variety. The group* $\mathrm{CaCl}(X)$ *is a subgroup of the cohomology group $H^1(X, \mathcal{O}^*)$.*

Proof We consider the long exact sequence associated to the short exact sequence (2.3). Part of this long exact sequence is given by $H^0(X, \mathcal{M}_X^*) \xrightarrow{f} H^0(X, \mathcal{M}_X^*/\mathcal{O}_X^*) \xrightarrow{g} H^1(X, \mathcal{O}_X^*)$. By definition, the image of $H^0(X, \mathcal{M}_X^*)$ under f is the group of principal divisors so g induces an inclusion

$$\mathrm{CaCl}(X) \hookrightarrow H^1(X, \mathcal{O}^*)\ .$$ □

[4]Of course, $\mathcal{M}_X^*(X) = K(X)^*$. The notation $\mathcal{M}_X$, chosen to emphasise the fact that the corresponding analytic sheaf is the sheaf of meromorphic functions, is used to avoid confusion with the canonical sheaf $\mathcal{K}_X$. See Definition 2.6.26 for more details.

Let $D = (U_i, f_i)_i \in \mathrm{Div}(X)$ be a Cartier divisor described with respect to an open covering $\{U_i\}_i$ of X. There are therefore germs of regular functions $g_i, h_i \in \mathcal{O}_X(U_i)$ such that

$$f_i = \frac{g_i}{h_i} \quad \text{and} \quad \frac{g_i}{h_i} \cdot \left(\frac{g_j}{h_j}\right)^{-1} \in \mathcal{O}_X^*(U_i \cap U_j).$$

Let D be a Cartier divisor on X. For any prime divisor A on X we define the *multiplicity* $\mathrm{mult}_A(D)$ of D on A as follows. If D is represented by $(U_i, f_i)_{i\in I}$ then we set $\mathrm{mult}_A(D) = \mathrm{mult}_A(f_i)$: since by hypothesis $\frac{f_i}{f_j}$ is nowhere vanishing, the value $\mathrm{mult}_A(D)$ does not depend on i. If a Cartier divisor D is represented by data $(U_i, f_i)_{i\in I}$ then we associate to it a Weil divisor

$$[D] := \sum_{\substack{A \text{ prime divisor} \\ \text{on } X}} \mathrm{mult}_A(D)A.$$

The map $\mathrm{Div}(X) \to Z^1(X)$, $D \mapsto [D]$ thus defined is a group morphism.

Proposition 2.6.10 *Let X be an irreducible complex variety.*

1. *If X is normal then the map* $\mathrm{Div}(X) \to Z^1(X)$, $D \mapsto [D]$ *is injective and induces an injective morphism*
$$\mathrm{CaCl}(X) \to \mathrm{Cl}(X)\ .$$
2. *If X is non singular then $D \mapsto [D]$ is an isomorphism*
$$\mathrm{Div}(X) \simeq Z^1(X)$$
and the induced morphism
$$\mathrm{CaCl}(X) \simeq \mathrm{Cl}(X)$$

is an isomorphism.

Proof See [Har77, II.6]. □

2.6.3 Line Bundles

We recall that an (algebraic) complex line bundle is an algebraic vector bundle of fiber $\mathbb{C}$ as in Definition 2.5.7. We further remark that over $\mathbb{C}$, any pre-algebraic vector bundle is algebraic, as in Remark 2.5.11(1). The sheaf of sections of such a bundle is an invertible sheaf, see Definition C.5.8, and the correspondence thus induced between isomorphism classes of line bundles and invertible sheaves is one-to-one, see Proposition 2.5.9.

To any Cartier divisor D represented by $(U_i, f_i)_i$ we can associate the sub-sheaf $\mathcal{O}_X(D) \subset \mathcal{M}_X$ defined by $\mathcal{O}_X(D)|_{U_i} = f_i^{-1}\mathcal{O}_X|_{U_i}$. The sheaf $\mathcal{O}_X(D)$ is an invertible sheaf over X. By abuse of notation we will also denote by $\mathcal{O}_X(D)$ the associated line bundle. More explicitly, the line bundle $\mathcal{O}_X(D)$ is given by the data of the open cover $\{U_i\}_{i\in I}$ of X and the transition functions $f_{ij}\colon U_i \cap U_j \to \mathbb{C}^*$ where $f_{ij} = f_j|_{U_i\cap U_j} \circ f_i^{-1}|_{U_i\cap U_j}$. The total space of the bundle is the quotient of the disjoint union $\sqcup_i (U_i \times \mathbb{C})$ by the equivalence relation $(x, z) \sim (x, f_{jk}(x)z)$ for any pair of open sets U_j, U_k containing x. This quotient is well defined because these functions satisfy the *cocycle condition*:

$$f_{ik} = f_{ij} f_{jk} \quad \text{sur} \quad U_i \cap U_j \cap U_k \quad \forall i, j, k\ .$$

By construction, D is effective if and only if $\mathcal{O}_X(-D) \subset \mathcal{O}_X$. If U is an open subset of X then $\mathcal{O}_X(D)|_U = \mathcal{O}_U(D|_U)$.

Definition 2.6.11 The line bundle $\mathcal{O}_X(D)$ is the line bundle *associated* to D.

We denote by $\mathrm{Pic}(X)$ the *Picard group* of line bundles modulo isomorphism with group operation given by tensor product and by $\rho\colon \mathrm{Div}(X) \to \mathrm{Pic}(X)$ the map associating to a divisor D the isomorphism class of the line bundle $\mathcal{O}_X(D)$.

Proposition 2.6.12 *Let X be a complex algebraic variety. The Picard group* $\mathrm{Pic}(X)$ *is isomorphic to the cohomology group* $H^1(X, \mathcal{O}^*)$.

Proof See [Har77, III, Exercise 4.5] or [GH78, Section 1.1] for an analytic version of this theorem. □

Example 2.6.13 Consider $X = \mathbb{P}^n$. By Exercise 2.6.5, the group $\mathrm{Cl}(\mathbb{P}^n)$ is isomorphic to $\mathbb{Z}$ and it is generated by the class of a hyperplane $H \subset \mathbb{P}^n$. The Picard group $\mathrm{Pic}(\mathbb{P}^n)$ is therefore isomorphic to $\mathbb{Z}$ and has a natural generator, namely the line bundle associated to H. By convention, we denote this line bundle by $\mathcal{O}_{\mathbb{P}^n}(1) := \mathcal{O}_{\mathbb{P}^n}(H)$. The other generator of $\mathrm{Pic}(\mathbb{P}^n)$ is its dual bundle, denoted $\mathcal{O}_{\mathbb{P}^n}(-1) := \mathcal{O}_{\mathbb{P}^n}(1)^\vee$.

By extension, we write $\mathcal{O}_{\mathbb{P}^n}(k) := \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes k}$ and $\mathcal{O}_{\mathbb{P}^n}(-k) := \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes k}$ for any positive integer k. In particular, $\mathcal{O}_{\mathbb{P}^n}(0) = \mathcal{O}_{\mathbb{P}^n}$. It follows that the line bundle associated to the divisor kH is $\mathcal{O}_{\mathbb{P}^n}(k)$ for any $k \in \mathbb{Z}$. See [Ser55a, Chapitre III, Section 2] for the original construction of the sheaves $\mathcal{O}(k)$.

Definition 2.6.14 The line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ is called *Serre's twisting sheaf* and the line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ is called the *tautological bundle*. See Section F.1 for a direct construction of this bundle.

Exercise 2.6.15 *Consider an integer* $d>1$. *Prove that the vector space* $\Gamma(\mathbb{P}^n, \mathcal{O}^n_{\mathbb{P}}(dH))$ *of global sections of the line bundle* $\mathcal{O}_{\mathbb{P}^n}(d)$ *is exactly the space of degree* d *homogeneous polynomials in* $n+1$ *variables. Deduce that* $\dim H^0\left(\mathbb{P}^n, \mathcal{O}^n_{\mathbb{P}}(d)\right) = \binom{n+d}{d}$.

Proposition 2.6.16 *Let X be an irreducible quasi-projective complex algebraic variety.*

1. *For any* $D_1, D_2 \in \operatorname{Div}(X)$ *we have that*

$$\rho(D_1 + D_2) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2) \; .$$

2. *The map* $\rho\colon \operatorname{Div}(X) \to \operatorname{Pic}(X)$ *induces an isomorphism*

$$\operatorname{CaCl}(X) \simeq \operatorname{Pic}(X) \; .$$

Proof See [Har77, II.6]. □

By abuse of notation we will often write $D \in \operatorname{Pic}(X)$ for the linear class of a divisor $D \in \operatorname{Div}(X)$.

Corollary 2.6.17 *Let* X *be a non singular irreducible quasi-projective complex algebraic variety. There are isomorphisms*

$$\operatorname{Cl}(X) \simeq \operatorname{CaCl}(X) \simeq \operatorname{Pic}(X) \simeq \operatorname{Div}(X)/\mathcal{P}(X) \; .$$

Definition 2.6.18 Let D be a divisor on an algebraic variety X. The *linear system* $|D|$ is the set of effective divisors which are linearly equivalent to D. We identify this set with the projectivisation of the complex vector space $H^0(X, \mathcal{O}_X(D))$ of global sections of $\mathcal{O}_X(D)$.

We have that $H^0(X, \mathcal{O}_X(D)) = \{f \in K(X)^* \mid D + (f) \geqslant 0\} \cup \{0\}$. If this complex vector space is of finite dimension then any basis $\{s_0, \ldots, s_N\}$ of $H^0(X, \mathcal{O}_X(D))$ is a set of global rational functions on X which enables us to defined a rational map

$$\varphi_D\colon \begin{cases} X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(D))) = \mathbb{P}^N(\mathbb{C}) \\ x \mapsto (s_0(x) : \cdots : s_N(x)) \; . \end{cases}$$

Remark 2.6.19 The map φ_D depends on a choice of basis for $H^0(X, \mathcal{O}_X(D))$ and is only determined by D up to automorphism of $\mathbb{P}(H^0(X, \mathcal{O}_X(D)))$.

Definition 2.6.20 A divisor D on a variety X is *very ample* if the rational map φ_D is a morphism embedding X in $\mathbb{P}(H^0(X, \mathcal{O}_X(D)))$. A divisor D is *ample* if one of its multiples mD, $m \geqslant 1$, is very ample.

Likewise, an invertible sheaf $\mathcal{L}$ is *very ample* if it is associated to a very ample divisor $\mathcal{L} = \mathcal{O}_X(D)$, and it is *ample* if $\mathcal{L}^{\otimes m}$ is very ample for some $m \geqslant 1$.

Proposition 2.6.21 *An abstract algebraic variety (constructed by "gluing together" affine algebraic varieties as in Definition 1.3.1) is projective if and only if it has an ample divisor.*

Proof Suppose that D is an ample divisor on X. There is then a multiple mD, $m \geqslant 1$, which is very ample and the associated morphism φ_{mD} embeds X as a closed subvariety of projective space. Conversely, let X be a projective algebraic variety and let $\varphi\colon X \to \mathbb{P}^N$ be an embedding. For any hyperplane H in $\mathbb{P}^N$ the divisor $\varphi^*(H)$ is a very ample divisor on X (or in terms of line bundles, $\varphi^*(\mathcal{O}_{\mathbb{P}^N}(1))$ is very ample on X). The divisor $\varphi^*(H)$ is the divisor of the *hyperplane section* of X relative to the embedding φ. □

Definition 2.6.22 A divisor D on an algebraic variety X (which we will assume *complete* in order to be sure that the maps φ_{mD} exist) is *big* if there exists an $m > 0$ for which the dimension of the image of the rational map $\varphi_{mD} : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(mD)))$ is maximal, or in other words, if

$$\dim \varphi_{mD}(X) = \dim X \ .$$

Likewise, a line bundle $\mathcal{L}$ is *big* if for some $m > 0$ we have that

$$\varphi_{\mathcal{L}^{\otimes m}}(X) = \dim X \ .$$

Example 2.6.23 1. Any ample line bundle is of course big.
2. The pull back of an ample line bundle along a generically finite map is a big line bundle. See [Laz04, Section 2.2] for more details.

Theorem 2.6.24 *If X is a normal variety* (*which is the case in particular, for any non singular variety*) *then a line bundle $\mathcal{L}$ is big if and only if there is some $m > 0$ for which the rational map $\varphi_{\mathcal{L}^{\otimes m}} : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(mD)))$ is birational onto its image.*

Proof This result follows from the existence of the Iitaka fibration. See [Laz04, Section 2.2] for more details. □

Remark 2.6.25 The bigness of a line bundle is invariant under birational transformations.

If X is a non singular quasi-projective complex algebraic variety then the sheaf of regular differential forms (see [Liu02, Chapter 6] or [Har77, II.8] for regular differential forms and Definition D.3.2 for holomorphic differential forms) of degree 1 on X, denoted $\Omega_X := \Omega^1_X$, is a locally free finitely generated sheaf. The associated vector bundle, also denoted Ω_X, has rank equal to the dimension of X and its determinant bundle $\det \Omega_X$ is a line bundle.

Definition 2.6.26 Let X be a non singular quasi-projective complex algebraic variety. The *canonical bundle* on X is the complex line bundle defined by

$$\mathcal{K}_X := \det \Omega_X = \bigwedge^n \Omega_X \ .$$

The *canonical divisor* of X denotes any divisor associated to the canonical bundle

$$\mathcal{O}_X(K_X) = \mathcal{K}_X \ .$$

It is customary to talk about “the” canonical divisor, even though such divisors are only defined up to linear equivalence.

Exercise 2.6.27 *Prove that $\mathcal{K}_{\mathbb{P}^n}$ is isomorphic to the line bundle $\mathcal{O}_{\mathbb{P}^n}(-n-1)$.*

Exercise 2.6.28 (See [CM09, Theorem 4.3]) *Let X be a non singular projective variety. Prove that if $H^0(X, \mathcal{O}_X(-K_X)) \neq 0$ and $H^0(X, \Omega^1_X) = 0$ then $H^0(X, \Omega^1_X(K_X)) = 0$.*

Using Serre duality (Theorem D.2.5) deduce that

$$H^2(X, \Theta_X) = 0$$

where Θ_X is the tangent bundle.

Definition 2.6.29 A non singular projective variety X is said to be of *general type* if its canonical bundle $\mathcal{K}_X$ is big.

2.6.4 Galois Group Action on the Picard Group

Let (X, σ) be an $\mathbb{R}$-surface: we denote by σ the involution induced on the divisor group of X. If $D = \sum n_i D_i$ is a Weil divisor on X then $\sigma D := \sum n_i \sigma(D_i)$. If $D = (U_i, f_i)_i$ is a Cartier divisor on X then $\sigma D = (\sigma(U_i), {}^\sigma f_i)_i$. If $\mathcal{L}$ is a line bundle on X with cocycle (U_{ij}, g_{ij}) then the conjugate sheaf (Definition 2.5.1) ${}^\sigma\mathcal{L}$ is the line bundle on X of cocycle $(\sigma(U_{ij}), {}^\sigma g_{ij})$.

Proposition 2.6.30 *Let X be projective. If D is a Cartier divisor and $\mathcal{O}_X(D)$ is the associated invertible sheaf then*

$$\mathcal{O}_X(\sigma D) = {}^\sigma(\mathcal{O}_X(D)).$$

Conversely, if $\mathcal{L}$ is an invertible sheaf on X, D is a divisor associated to $\mathcal{L}$ and D' is a divisor associated to ${}^\sigma\mathcal{L}$ then $D' \sim \sigma D$.

Proof Let $D = (U_i, f_i)_i$ be a Cartier divisor. The sheaf $\mathcal{O}_X(D)$ is determined by the cocycle $(g_{ij})_{ij} = (\frac{f_i}{f_j})_{ij}$. Indeed, $\Gamma(U, \mathcal{O}_X(D)) = \{f \in \mathcal{O}_X(U) \mid (f) + D \geqslant 0\}$. Let $(s_i)_i$ be a family of local sections of $\mathcal{O}_X(D)$. We then have that

$$\forall i, j, s_i = g_{ij} s_j \ . \tag{2.4}$$

By definition of the conjugate sheaf, $({}^\sigma s_i)_i$ is a family of local sections of the sheaf ${}^\sigma(\mathcal{O}_X(D))$ and by (2.4) we have that

$$\forall i, j, {}^\sigma s_i = {}^\sigma g_{ij} {}^\sigma s_j \ . \tag{2.5}$$

The proof follows on noting that $\mathcal{O}_X(\sigma D)$ is determined by the cocycle $({}^\sigma g_{ij})_{ij} = (\frac{{}^\sigma f_i}{{}^\sigma f_j})_{ij}$. □

Proposition 2.6.31 *Let D be a divisor invariant under (X, σ). There is then a basis $\{s_0, \ldots, s_N\}$ of the complex vector space $H^0(X, \mathcal{O}_X(D)) = \{f \in K(X)^* \mid D + (f) \geqslant 0\} \cup \{0\}$ consisting of invariant functions ${}^\sigma s_i = s_i$, $i = 0, \ldots, N$.*

Proof Follows immediately from Lemma A.7.3. □

Theorem 2.6.32 *Let (X, σ) be an irreducible non singular complex projective algebraic ℝ-variety. If $X(\mathbb{R}) \neq \varnothing$ then for any divisor D linearly equivalent to $\sigma(D)$ there is a divisor D' linearly equivalent to D such that $D' = \sigma(D')$. In other words,*[5]

$$\mathrm{Div}(X)^G/\mathcal{P}(X)^G = \mathrm{Pic}(X)^G .$$

Proof See [Sil89, pp. 19–20]. □

Example 2.6.33 $(\mathrm{Div}(X)^G/\mathcal{P}(X)^G \neq \mathrm{Pic}(X)^G)$ The example of the conic X in $\mathbb{P}^2$ of equation $x_0^2 + x_1^2 + x_3^2 = 0$ shows that when $X(\mathbb{R}) = \varnothing$, $\mathrm{Pic}(X)^G$ can be larger than $\mathrm{Div}(X)^G/\mathcal{P}(X)^G$. In this example, $\mathrm{Pic}(X)^G = \mathrm{Pic}(X) = \mathbb{Z}$ which is generated by a point, but all the invariant divisors are of even degree and there is an exact sequence

$$0 \to \mathrm{Div}(X)^G/\mathcal{P}(X)^G \longrightarrow \mathrm{Pic}(X)^G \longrightarrow \mathbb{Z}/2\mathbb{Z} \to 0 .$$

Up till now we have studied the Picard group of linear divisor classes. We now present another group of divisor classes, the *Néron–Severi* group.

Definition 2.6.34 Let X be a non singular complex projective variety and let $\mathrm{Pic}^0(X)$ be the connected component of $\mathrm{Pic}(X)$ containing the identity ($\mathrm{Pic}^0(X)$ is the *Picard variety* of X, see Definition D.6.6). The *Néron–Severi group* $\mathrm{NS}(X)$ is the group of components of $\mathrm{Pic}(X)$:

$$0 \to \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \to 0 .$$

Two divisors in the same class in the Néron Severi group are said to be *algebraically equivalent*.[6]

Theorem 2.6.35 (Néron–Severi theorem) *Let X be a non singular complex projective variety. The group $\mathrm{NS}(X)$ is then finitely generated.*

Proof See [GH78, IV.6, pp. 461–462]. □

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[5] Scheme-theoretically, if X is a scheme defined over $\mathbb{R}$ satisfying the hypotheses of the theorem then $\mathrm{Pic}(X) = \mathrm{Pic}(X_{\mathbb{C}})^G$.

[6] See [GH78, III.5] for an explanation of this term. The term "numerically equivalent" is also common in the literature: see [Ful98, Section 19.3] for more details.

Definition 2.6.36 Let X be a non singular complex projective variety. The rank of the Neron–Severi group $\rho(X) := \operatorname{rk} \operatorname{NS}(X) = \operatorname{rk}(\operatorname{Pic}(X)/\operatorname{Pic}^0(X))$ is called the *Picard number* of X. Let (X, σ) be a non singular projective $\mathbb{R}$-variety. If $X(\mathbb{R})$ is non empty then the *real Picard number* of (X, σ) is the rank of the *real* Néron–Severi group $\rho_{\mathbb{R}}(X) := \operatorname{rk}(\operatorname{Pic}(X)^G/\operatorname{Pic}^0(X)^G)$.

Proposition 2.6.37 *Let X be a non singular complex projective variety such that $q(X) = \dim H^1(X, \mathcal{O}_X) = 0$. We then have that*

$$\operatorname{NS}(X) \simeq \operatorname{Pic}(X) .$$

Proof It follows from the exact sequence (D.3) following Proposition D.6.7 that if $q(X) = 0$ then the group $\operatorname{Pic}^0(X)$ is trivial. □

2.6.5 Projective Embeddings

We have seen that any compact real affine algebraic variety has a projective complexification. The aim of this section is to study these projective models using ample divisors.

Example 2.6.38 ($\mathbb{R}$-*embedding of the product torus*) This example draws on [BCR98, Example 3.2.8]. Let V be the product torus $V := \mathcal{Z}\left(t^2 + u^2 - 1\right) \times \mathcal{Z}\left(v^2 + w^2 - 1\right) \subset \mathbb{A}^2(\mathbb{R}) \times \mathbb{A}^2(\mathbb{R})$ and let W be the quartic torus in $\mathbb{R}^3_{x_1,x_2,x_3}$ obtained by rotating the circle of centre $(2, 0)$ and radius 1 in the (x_1, x_3) plan around the x_3 axis

$$W := \mathcal{Z}(16(x_1^2 + x_2^2) - (x_1^2 + x_2^2 + x_3^2 + 3)^2) \subset \mathbb{A}^3(\mathbb{R}) .$$

Both of these real algebraic sets are diffeomorphic to the torus with the Euclidean topology $V \approx W \approx \mathbb{S}^1 \times \mathbb{S}^1$.

Consider W as a subset of $\mathbb{P}^3(\mathbb{R})$ via the inclusion $\mathbb{R}^3_{x_1,x_2,x_3} \subset \mathbb{P}^3(\mathbb{R})_{x_0:x_1:x_2:x_3}$. The polynomial map

$$\begin{array}{rccc} \varphi\colon & V & \longrightarrow & W \\ & (t, u, v, w) & \longmapsto & (1 : t(2 + v) : u(2 + v) : w) \end{array}$$

is bijective and its inverse $\varphi^{-1}\colon W \to V$,

$$\varphi^{-1}(x_0 : x_1 : x_2 : x_3) = \left(x_1 x_0/\rho, x_2 x_0/\rho, (\rho - 2x_0^2)/x_0^2, x_3/x_0\right)$$

where $\rho = (x_1^2 + x_2^2 + x_3^2 + 3x_0^2)/4$, is a regular map of real algebraic varieties since $W \cap \{x_0 = 0\} = \varnothing$.

The map φ is therefore an isomorphism of real algebraic varieties and the algebras $\mathcal{R}(V)$ and $\mathcal{R}(W)$ are isomorphic by Corollary 1.3.20: the algebras $\mathcal{P}(V)$ and $\mathcal{P}(W)$, however, are different, since the first is regular, unlike the second. Consider the projective complexifications of the toruses V and W: $\overline{V}_{\mathbb{C}} \simeq \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ for the first and the singular quartic hypersurface

$$\widehat{W}_{\mathbb{C}} := \mathcal{Z}(16(x_1^2 + x_2^2) - (x_1^2 + x_2^2 + x_3^2 + 3x_0^2)^2) \subset \mathbb{P}^3(\mathbb{C}) \; .$$

for the second. The map φ is then the restriction of a birational map of $\mathbb{R}$-varieties

$$\psi : \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \to \widehat{W}_{\mathbb{C}}$$

which is a resolution of singularities of $\widehat{W}_{\mathbb{C}}$.

Note that ψ is a morphism of $\mathbb{R}$-varieties but ψ^{-1} is only a rational map. Note also that as $\widehat{W}_{\mathbb{C}}$ is a quartic in $\mathbb{P}^3(\mathbb{C})$ which is birational to $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ it must be singular. Indeed, $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ is a rational surface whereas a non singular quartic in $\mathbb{P}^3$ is a non rational surface (called a *K3 surface*, see Definition 4.5.3). The $\mathbb{R}$-surfaces $(\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}), \sigma_{\mathbb{P}} \times \sigma_{\mathbb{P}})$ and $(\widehat{W}_{\mathbb{C}}, \sigma_{\mathbb{P}}|_{\widehat{W}_{\mathbb{C}}})$ are birationally equivalent but not isomorphic.

2.6.6 Review of Theorem 2.1.33

We have seen that a variety X embedded in $\mathbb{P}^n(\mathbb{C})$ and stable by the conjugation $\sigma_{\mathbb{P}}$ has a natural real structure σ induced by $\sigma_{\mathbb{P}}$. Note that if X is a projective complex variety with a real structure σ then its image under an arbitrary projective embedding is not always stable under $\sigma_{\mathbb{P}}$, but we can always find a real embedding by Theorem 2.6.44 below. We will give a proof of this theorem based on the Nakai–Moishezon criterion. Of course, Theorem 2.6.44 implies Theorem 2.1.33 for which we have only provided a reference for the proof. In what follows, up to and including the proof of Theorem 2.6.44, we will not use Theorem 2.1.33.

The key fact to remember is that if X is a complex projective variety then for any real structure σ on X the $\mathbb{R}$-variety (X, σ) has an equivariant embedding in projective space.

2.6.7 Nakai–Moishezon Criterion

See [Har77, Appendix A, p. 424] for the definition and main properties of intersection theory on varieties of arbitrary dimension. If the global variety has a real structure then this intersection theory is compatible with the real structure. If r is the dimension of a non singular variety Y and $D_1, D_2, \ldots, D_r$ are divisors on Y then their intersection product $(D_1 \cdot D_2 \cdots D_r)$ belongs to $\mathbb{Z}$ and only depends on the linear class of the divisors D_i. In particular, if the D_is are hypersurfaces meeting transversally then $(D_1 \cdot D_2 \cdots D_r)$ is equal to the number of points in the intersection of the D_is.

Theorem 2.6.39 (Nakai–Moishezon criterion) *Let D be a Cartier divisor on a complex projective algebraic variety X. The divisor D is then ample on X if and only if for any irreducible subvariety $Y \subset X$ of dimension r we have that*

$$(D|_Y)^r > 0 \ .$$

Proof See [Har77, Appendix A, Theorem 5.1, p. 434], for example. The above statement also holds for singular X, but requires a modified intersection theory. See [Kle66, Ful98] for more details. □

Corollary 2.6.40 (Nakai–Moishezon criterion for surfaces) *A divisor D on a non singular irreducible complex projective algebraic surface X is ample if and only if $(D)^2 > 0$ and $D \cdot C > 0$ for any irreducible curve C in X.*

Proof Simply set $Y = X$ in the general criterion to obtain $(D)^2 > 0$ and for any irreducible curve $C \subset X$, $D \cdot C > 0$. □

Definition 2.6.41 A divisor D on a variety X is *nef* (for *numerically eventually free*[7]) if for any irreducible subvariety $Y \subset X$ of dimension r we have that

$$(D|_Y)^r \geqslant 0 \ .$$

Similarly, a line bundle $\mathcal{L}$ is *nef* if and only if it is associated to a nef divisor $\mathcal{L} = \mathcal{O}_X(D)$.

Remark 2.6.42 Any ample bundle is of course nef.

Proposition 2.6.43 *Let X be a complex projective variety with a real structure σ. There is then an ample divisor D such that $D = \sigma D$.*

Proof Let H be an ample divisor on X. For any irreducible subvariety $Y \subset X$ of dimension r the conjugate subvariety σY is irreducible and of dimension r and by the Nakai–Moishezon criterion (Theorem 2.6.39) we have that $(H|_{\sigma Y})^r > 0$. Since the real structure is involutive, $(\sigma H)|_Y = \sigma(H|_{\sigma Y})$ and since the real structure is compatible with the intersection product we get that $((\sigma H)|_Y)^r = (H|_{\sigma Y})^r > 0$. By the Nakai–Moishezon criterion, σH is ample, as is

$$D := H + \sigma H \ .$$

□

Theorem 2.6.44 *Let (X, σ) be an algebraic ℝ-variety. If the complex algebraic variety X is quasi-projective then there is an ℝ-embedding*

[7] If the linear system $|mD|$ is free for some $m > 0$ (eventually free), then D is nef. The incorrect interpretation *numerically effective* often appears in the literature, but considering (-1)-curves—see Definition 4.3.2—we see that a divisor can be effective without being either nef or linearly equivalent to a nef divisor.

$$\varphi\colon (X,\sigma) \hookrightarrow (\mathbb{P}^N(\mathbb{C}), \sigma_{\mathbb{P}}) \ .$$

Proof We start by assuming X is projective, so by Proposition 2.6.43, there is an ample divisor D_0 and a positive integer m such that $D = mD_0$ is very ample on X and satisfies $\sigma D = D$. By Proposition 2.6.31, there is a basis $\{s_0, \ldots, s_N\}$ of $H^0(X, \mathcal{O}_X(D))$ such that ${}^{\sigma}s_i = s_i$, $i = 0, \ldots, N$. As the divisor D is very ample, the map

$$\varphi_D\colon \begin{cases} X \dashrightarrow & \mathbb{P}^N(\mathbb{C}) \\ x \mapsto & (s_0(x) : \cdots : s_N(x)) \end{cases}$$

is a morphism which induces an isomorphism of ℝ-varieties

$$(X, \sigma) \simeq (\varphi_D(X), \sigma_{\mathbb{P}}|_{\varphi_D(X)}) \ .$$

Now consider a quasi-projective variety $U = X \setminus Y$, where X is a projective ℝ-variety and $Y \subset X$ is a closed ℝ-subvariety of X. We have just proved the existence of an ℝ-embedding; $\varphi\colon (X, \sigma) \hookrightarrow (\mathbb{P}^N(\mathbb{C}), \sigma_{\mathbb{P}})$: in particular, φ is a homeomorphism onto its image $\varphi(X \setminus Y) = \varphi(X) \setminus \varphi(Y)$ and φ therefore induces an embedding of U as a quasi-projective algebraic set

$$(U, \sigma|_U) \simeq (\varphi(X) \setminus \varphi(Y), \sigma_{\mathbb{P}}|_{\varphi(X)\setminus\varphi(Y)}) \ .$$

□

2.6.8 Degree of a Subvariety of Projective Space

Classically, we define the degree of a subvariety of $\mathbb{P}^N$ using its Hilbert polynomial [Har77, Section I.7] and only subsequently prove that this definition is equivalent to the definition given below.

Definition 2.6.45 (*Degree of a subvariety of projective space*) The *degree* of an n dimensional subvariety X of $\mathbb{P}^N$ is the degree of the 0-cycle $D := (H \cdot X)$ obtained on intersecting X with a general codimension n projective subspace H in $\mathbb{P}^N$.

There is a hidden difficulty in the above definition, namely finding the coefficients of the 0-cycle $D := (H \cdot X)$ for an arbitrary X. See the section preceding [Har77, Theorem 7.7, p. 53] for more details. If X is complex and non singular then by Bertini's Theorem D.9.1 if we choose a sufficiently general H then the 0-cycle D is the sum of all points in $H \cap X$.

Definition 2.6.46 (*Complex degree*) The *complex degree* of a complex projective algebraic variety is the smallest degree of any of its embeddings in a complex projective space $\mathbb{P}^N(\mathbb{C})$.

Definition 2.6.47 (*Real degree*) Let (X, σ) be a projective $\mathbb{R}$-variety. The *real degree* of (X, σ) is the smallest degree of a real embedding in projective space $(\mathbb{P}^N(\mathbb{C}), \sigma_{\mathbb{P}})$.

The real degree exists by Proposition 2.6.43. As any real embedding is also a complex embedding, the real degree is not smaller than the complex degree. The minimal degree of a complex projective embedding is frequently strictly smaller than the minimal degree of a real projective embedding. The simplest example is that of conic without real points, whose complex degree is 1 but whose real degree is 2. Let X be the projective plane curve defined by the equation $x^2 + y^2 + z^2 = 0$ with the restriction of $\sigma_{\mathbb{A}}$. The curve X is isomorphic as an abstract complex curve to the curve $\mathbb{P}^1(\mathbb{C})$ and has degree 1 embeddings—namely lines—in every $\mathbb{P}^n(\mathbb{C})$. None of these embeddings can be real because any embedding as an $\mathbb{R}$-line has real points. The following proposition generalises this principle.

Proposition 2.6.48 *Let $X \subset \mathbb{P}^n(\mathbb{C})$ be a algebraic subvariety, stable under $\sigma_{\mathbb{P}}$. If the degree of X is odd then $X(\mathbb{R}) \neq \varnothing$.*

Proof We can assume that $r := n - \dim X > 0$. Let H be a projective subspace of dimension r in $\mathbb{P}^n$ which is not contained in X. By hypothesis, the degree of the 0-cycle $D := (H \cdot X)$ is odd. In particular, the real part of D has odd degree and its support consists of an odd number of points so it is non empty. □

2.7 $\mathbb{R}$-Plane Curves

We end this chapter by applying the above theory to plane curves. We refer to Section 1.6 of the first chapter for the general definitions. Bézout's theorem on plane curves, given in Chapter 1, is here applied to $\mathbb{R}$-curves. It will be generalised to curves on other surfaces in Chapter 4.

Theorem 2.7.1 (Bézout's theorem for $\mathbb{R}$-plane curves) *Let C_1 and C_2 be projective plane $\mathbb{R}$-curves of degrees d_1 and d_2 respectively*

1. *If C_1 and C_2 have no common component then*

$$(C_1 \cdot C_2) = d_1 d_2 \ .$$

2. *If the intersection $C_1(\mathbb{R}) \cap C_2(\mathbb{R})$ is finite then*

$$(C_1(\mathbb{R}) \cdot C_2(\mathbb{R})) \leqslant d_1 d_2 \ .$$

3. *If moreover the branches of C_1 and C_2 are transverse at every point then the number of intersection points $\#(C_1(\mathbb{R}) \cap C_2(\mathbb{R}))$ is congruent modulo 2 to the product $d_1 d_2$.*

Proof We simply defined the intersection multiplicity modulo 2 at a point $a \in \mathbb{A}^2(\mathbb{R})$ of two affine plane ℝ-curves C_1 and C_2 of equations $P_1(x, y)$ and $P_2(x, y)$ to be

$$(C_1 \cdot C_2)_a^{\mathbb{R}} := \dim_{\mathbb{R}} \mathcal{O}_{\mathbb{A}^2(\mathbb{R}),a}/(P_1, P_2) \mod 2 ;$$

and the intersection number modulo 2 to be

$$(C_1 \cdot C_2)^{\mathbb{R}} := \sum_{a \in C_1(\mathbb{R}) \cap C_2(\mathbb{R})} (C_1 \cdot C_2)_a^{\mathbb{R}} \mod 2 .$$

We then apply Theorem 1.6.16 to the complex curves C_1 and C_2. □

We recall the genus formula proved in Chapter 1, Theorem 1.6.17. If C is a non singular irreducible projective plane curve of genus $g = g(C)$ then

$$g = \frac{(d-1)(d-2)}{2} .$$

The real locus of a non singular projective ℝ-curve is a compact differentiable variety of dimension 1. It is therefore homeomorphic to a finite union of disjoint embedded circles.

Theorem 2.7.2 (Harnack 1876) *Let (C, σ) be a non singular projective plane ℝ-curve of degree d. Let s be the number of connected components of $C(\mathbb{R})$. We then have that*

$$s \leqslant \frac{(d-1)(d-2)}{2} + 1 = g(C) + 1 . \tag{2.6}$$

Remark 2.7.3 Further on we will give an elementary proof of this inequality based on Bézout's theorem. It is useful to note that the number of connected components of a plane curve of degree d is bounded above by $\frac{(d-1)(d-2)}{2} + 1$ even when C is singular. First of all, it is enough to prove the result when C is irreducible. If not, C is defined by a product of polynomials of degrees d_1 and d_2, so that $d = d_1 + d_1$ and

$$\frac{(d_1-1)(d_1-2)}{2} + 1 + \frac{(d_2-1)(d_2-2)}{2} + 1 \leqslant \frac{(d-1)(d-2)}{2} + 1 .$$

We then show that we can assume that $C(\mathbb{R})$ contains at least one component of dimension 1 using Brusotti's Theorem 2.7.10 as in Corollary 3.3.20. The proof then follows the proof for the smooth case given below, see [BR90, Second proof of 5.3.2].

Remark 2.7.4 More generally, for any non singular projective ℝ-curve(C, σ) (note that C is not assumed to be plane), we have that $s \leqslant g(C) + 1$, where $g(C)$ is the genus of the topological surface C. We will give two proofs of this in Chapter 3 and Corollary 3.3.7. We will also see in Chapter 3 that this inequality can be generalised to higher dimension using Smith theory.

Lemma 2.7.5 *There is a real projective curve of degree d which passes through any given set of* $\binom{d+2}{2} - 1 = \frac{1}{2}(d+2)(d+1) - 1$ *points in* $\mathbb{P}^2(\mathbb{R})$.

Proof The number of degree d monomials in three variables is $\binom{d+2}{2}$. We deduce from this a bijection between the set of degree d curves in the real projective plane and a real projective space of dimension $\frac{1}{2}(d+2)(d+1) - 1$. □

Proposition 2.7.6 *For any point* $p \in \mathbb{RP}^2$,

$$\pi_1(\mathbb{RP}^2, p) \simeq \mathbb{Z}_2 \ .$$

Proof Consider $\mathbb{RP}^2$ as the quotient of $\mathbb{S}^2$ by the antipodal map. □

Definition 2.7.7 A simple closed curve in the real projective space is an *oval* if it is homotopic to 0 and a *pseudo-line* if it is not homotopically trivial.

Lemma 2.7.8 (Ovals and pseudo-lines) *Let* (C, σ) *be a non singular projective plane* $\mathbb{R}$*-curve of degree d.*

1. *If d is even all the connected components of* $C(\mathbb{R})$ *are ovals.*
2. *If d is odd then one connected component of* $C(\mathbb{R})$ *is a pseudo-line and all the others are ovals.*
3. *Any curve meets any oval in an even number of intersection points, counted with multiplicity.*

Proof The proof is left as an exercise. Use Bézout's theorem. □

Proof of Theorem 2.7.2 Suppose that $d > 2$. We argue by contradiction: suppose that Γ is a non singular irreducible plane $\mathbb{R}$-curve of degree d whose real locus has at least $g(d) + 1$ connected components. Let $h = g(d) + 1$ and $\Omega_1, \ldots, \Omega_h$ be ovals in $\Gamma(\mathbb{R})$: there is at least one other component in $\Gamma(\mathbb{R})$. Choose $\frac{1}{2}d(d-1) - 1$ points on $\Gamma(\mathbb{R})$. Since $\frac{1}{2}d(d-1) - 1 \geqslant g(d) + 1$ for any $d > 2$ we can choose one point on each of the ovals $\Omega_1, \ldots, \Omega_h$ and the other points on some other connected component of $\Gamma(\mathbb{R})$. Consider an $\mathbb{R}$-curve Δ of degree $d-2$ passing through these $\frac{1}{2}d(d-1) - 1$ points. The curves Γ and Δ have no common components because Γ is irreducible and the degree of Δ is $d-2$. By Bézout's theorem, the number of intersection points of Γ with Δ counted with multiplicity is less than or equal to $d(d-2)$. If Δ meets an oval Ω_i with multiplicity 1 then Δ meets Ω_i at some other point, so that $\Gamma \cdot \Delta \geqslant \frac{1}{2}d(d-1) - 1 + g(d) + 1 = (d-1)^2$ which is larger than $d(d-2)$. The theorem follows. □

The bound (2.6) is optimal: Harnack's bound is realised for any degree d:

Proposition 2.7.9 *For any* $d \in \mathbb{N}^*$ *there is a non singular projective plane* $\mathbb{R}$*-curve* (C, σ) *of degree d whose real locus* $C(\mathbb{R})$ *contains* $s = \frac{(d-1)(d-2)}{2} + 1$ *connected components.*

Proof See [BCR98, pp. 287–288] or [BR90, 5.3.11] for Harnack's construction. □

The constructions of the curves described above often use explicit deformations of reducible curves. We can often prove the existence of configurations of ovals of given degree without explicit constructions using Brusotti's useful theorem.

Theorem 2.7.10 (Brusotti's theorem) *Let $C \subset \mathbb{P}^2(\mathbb{R})$ be a degree d real plane curve whose singularities are ordinary double points. Suppose given a local deformation of each of the ordinary double points. There is then a deformation of the curve C in the space of real curves of degree d which realises each of the local deformations.*

Proof See [BR90, Section 5.5]. □

As well as (2.6) which gives a bound on the number of connected components, we have restrictions on the positions of ovals of plane ℝ-curves.

Definition 2.7.11 The complement $\mathbb{RP}^2 \setminus \Omega$ of a oval in the real projective plane has two connected components. One of these is diffeomorphic to the disc and is called the *interior* of the oval, and the other is diffeomorphic to a Moebius band. We say that another oval is *contained* in Ω if it is contained in its interior. An oval component of a real curve is said to be *empty* if it does not contain any other oval component. A family E is said to be a *nest* of ovals if and only if it is totally ordered by inclusion.

Definition 2.7.12 An oval is said to be *positive* (or *even*) if it is contained in an even number of ovals and *negative* (or *odd*) otherwise.[8]

Theorem 2.7.13 (Petrovskii's inequalities) *Let (C, σ) be a non-singular projective plane ℝ-curve of even degree $d = 2k$. Let p be the number of even ovals of $C(\mathbb{R})$ and let n be the number of negative ovals. We then have that*

$$p - n \leqslant \frac{3}{8}d(d-2) + 1 = \frac{3}{2}k(k-1) + 1 \ ;$$
$$n - p \leqslant \frac{3}{8}d(d-2) = \frac{3}{2}k(k-1) \ .$$

See [Pet33, Pet38] or [Arn71]. In Chapter 3, Theorem 3.3.14 we prove these inequalities using double covers.

Corollary 2.7.14 *Let (C, σ) be a non singular projective plane ℝ-curve of even degree $d = 2k$. Let p be the number of positive ovals of $C(\mathbb{R})$ and n be the number of negative ovals. Then we have that*

$$p \leqslant \frac{7}{4}k^2 - \frac{9}{4}k + \frac{3}{2} \quad ; \qquad n \leqslant \frac{7}{4}k^2 - \frac{9}{4}k + 1 \ .$$

Proof For any curve of even degree $d = 2k$, Harnack's inequality (2.6) gives $p + n \leqslant 2k^2 - 3k + 2$. Adding with the Petrovskii inequalities yields the desired result. □

[8] See [Pet38, p. 190] for a justification of this terminology.

Remark 2.7.15 (*Ragsdale's conjecture*) A famous, but incorrect, conjecture by Ragsdale [Rag06] states that p and n actually satisfy the inequalities $p \leqslant \frac{3}{2}k(k-1)+1$, et $n \leqslant \frac{3}{2}k(k-1)$. We will come back to this conjecture in Chapter 3, at the end of Section 3.5.

When the curve does not have any nest of ovals, all ovals are positive and Petrovskii's first inequality gives us the following.

Corollary 2.7.16 *Let C be a non singular projective plane $\mathbb{R}$-curve of even degree $d = 2k$ without a nest of ovals. The number of ovals $s := \#\pi_0(C(\mathbb{R}))$ is then bounded by*

$$s \leqslant \frac{3}{2}k(k-1) + 1 \ .$$

Corollary 2.7.17 *The maximal even degree d curves, by which we mean the curves with the maximal number of connected components in their real locus, namely $\frac{(d-1)(d-2)}{2} + 1$, (see Definition 3.3.10) have at least one nesting from degree 6 onwards.*

2.8 Solutions to exercises of Chapter 2

2.1.3 1. Let U be an open set in $\mathbb{A}^n(\mathbb{C})$ and consider $f \in {}^\sigma\mathcal{O}(U)$. By definition there is a function $g \in \mathcal{O}(\sigma_\mathbb{A}(U))$ such that $f = {}^\sigma g$ so $f = \overline{g} \circ \sigma_\mathbb{A} : U \to \mathbb{C}$ is regular and hence $f \in \mathcal{O}(U)$. The opposite inclusion $\mathcal{O}(U) \subset {}^\sigma\mathcal{O}(U)$ is proved by a similar argument.

2. Apply Definition 1.3.7 to the sheaf ${}^\sigma\mathcal{O}$ and the subspace F to get the sheaf ${}^\sigma\mathcal{O}_F$. If U is an open subset of F then $\overline{U}$ is an open set of $\overline{F}$ and hence of F by hypothesis. A function $f : U \to \mathbb{C}$ belongs to ${}^\sigma\mathcal{O}_F(U)$ if and only if for any point x in U there is a neighbourhood V of x in $\mathbb{A}^n(\mathbb{C})$ and a function $g \in {}^\sigma\mathcal{O}(V)$ such that $g(y) = f(y)$ for any $y \in V \cap U$. By the previous question $g \in \mathcal{O}(V)$ and hence ${}^\sigma\mathcal{O}_F = \mathcal{O}_F$.

2.1.7 The sets F and $\overline{F}$ are subsets of $\mathbb{A}^n(\mathbb{C})$ and $\mathcal{O}_{\overline{F}} = (\mathcal{O}_{\mathbb{A}^n})_{\overline{F}}$ (see Definition 1.3.7). The restriction $\sigma_\mathbb{A} : \overline{F} \to F$ is clearly bijective. Moreover, $\sigma_\mathbb{A}$ is continuous since if $Z = \mathcal{Z}(I)$ is a Zariski closed subset of F defined by an ideal I in $\mathbb{C}[X_1, \dots, X_n]$ then ${\sigma_\mathbb{A}}^{-1}(Z) = \sigma_\mathbb{A}(Z) = \overline{Z} = \mathcal{Z}({}^\sigma I)$ where ${}^\sigma I := \{{}^\sigma f \mid f \in I\}$. Finally, $\sigma_\mathbb{A}|_{\overline{F}}$ induces an isomorphism of ringed spaces (see Exercise C.5.3) $(\overline{F}, \mathcal{O}_{\overline{F}}) \to (F, \overline{\mathcal{O}_F})$ because if U is an open subset of F then $\sigma_\mathbb{A}(U)$ is an open subset of $\overline{F}$ and if $f \in \overline{\mathcal{O}_F}(U)$ then $f \circ \sigma_\mathbb{A} : \sigma_\mathbb{A}(U) \to \mathbb{C}$ is regular or in other words $f \circ \sigma_\mathbb{A} \in \mathcal{O}_{\overline{F}}(\sigma_\mathbb{A}(U))$. Indeed, as $f \in \overline{\mathcal{O}_F(U)}$ there is a function $f_0 \in \mathcal{O}_F(U)$ such that $f = \overline{f_0}$ and it follows that $f \circ \sigma_\mathbb{A} = \overline{f_0} \circ \sigma_\mathbb{A} = {}^\sigma f_0$. As f_0 is regular on U, ${}^\sigma f_0$ is regular on $\sigma_\mathbb{A}(U)$.

$$\begin{array}{ccccc} & & & & \mathbb{C} \\ & & & \overset{f_0}{\nearrow} & \big\downarrow{\scriptstyle z\mapsto\overline{z}} \\ F & \xrightarrow{\sigma_{\mathbb{A}}} & \overline{F} & \xrightarrow{f} & \mathbb{C} \end{array}$$

2.1.21 1. Recall that if C is the zero locus of a polynomial P then $\overline{C}$ is the zero locus of ${}^{\sigma}P$. A straightforward calculation shows that $(\varphi \circ \varphi)(x, y) = (x, y)$ so φ is an involutive automorphism of $\mathbb{A}^2(\mathbb{C})$ and in particular $\varphi^{-1} = \varphi$. Now consider $P(x, y) = y^2 - a_0 x^m - \sum_{k=1}^{m} \left(a_k x^{m+k} + (-1)^k \overline{a_k} x^{m-k}\right)$. On substituting $P(\varphi(x, y))$ we obtain $-\frac{y^2}{x^{2m}} + a_0 \frac{1}{x^m} + \sum_{k=1}^{m} \left(a_k \frac{1}{x^{m-k}} + (-1)^k \overline{a_k} \frac{1}{x^{m+k}}\right)$ and hence $-x^{2m} P(\varphi(x, y)) = {}^{\sigma}P(x, y)$.

2. Set $\tau = \sigma_{\mathbb{A}} \circ \varphi$. We then have that $\tau(x, y) = (-\frac{1}{\overline{x}}, -\frac{i\overline{y}}{\overline{x}^m})$ et $(\tau \circ \tau)(x, y) = (x, -y)$.

3a. Restricting the projection $(x, y) \mapsto x$ we exhibit the curve $C := C_{m,a_0,\ldots,a_m}$ as a degree 2 covering of $\mathbb{P}^1(\mathbb{C})$. Its function field $\mathbb{C}(C)$ is therefore a degree two extension of $\mathbb{C}(x) = \mathbb{C}(\mathbb{P}^1(\mathbb{C}))$. Moreover, there is a one-to-one correspondence between automorphisms of C and automorphisms of the field $\mathbb{C}(C)$.[9] The two elements of the automorphism group of the extension $\mathbb{C}(C)|\mathbb{C}(x)$ are represented by id_C and ρ. Any automorphism of $\mathbb{C}(C)$ therefore induces an automorphism of Frac $(\mathbb{C}[x, y]/(P))$. If the coefficients of the one-variable polynomial $P(x, y) - y^2$ are independent over $\mathbb{Q}$ then the only non trivial automorphism is represented by ρ.

3b. By Proposition 2.1.19, if C has a real structure then there is an isomorphism between C and $\overline{C}$ satisfying ${}^{\sigma_{\mathbb{A}}}\psi \circ \psi = \mathrm{id}_C$.

Moreover, it follows from 3a that the only isomorphisms between $C_{m,a_0,\ldots,a_m}$ and its conjugate are φ and $\varphi'\colon (x, y) \mapsto (-\frac{1}{x}, -\frac{i}{x^m} y)$, but $\varphi \circ {}^{\sigma_{\mathbb{A}}}\varphi = (\varphi') \circ ({}^{\sigma_{\mathbb{A}}}(\varphi')) = \rho \neq \mathrm{id}_{C_{m,a_0,\ldots,a_m}}$. It follows that if $a_0, a_k, \overline{a_k}$ are independent over $\mathbb{Q}$ then the curve $C_{m,a_0,\ldots,a_m}$ has no real structure.

2.1.42 We have two non-equivalent real structures on $\mathbb{P}^1(\mathbb{C})$:

$$\sigma_{\mathbb{P}}\colon (x_0 : x_1) \mapsto (\overline{x_0} : \overline{x_1})$$

et

$$\sigma_{\mathbb{P}}'\colon (x_0 : x_1) \mapsto (-\overline{x_1} : \overline{x_0})$$

which give rise to three non-equivalent structures on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$: the involution $\sigma_{\mathbb{P}} \times \sigma_{\mathbb{P}}$ whose fixed locus is the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ and the involutions $\sigma_{\mathbb{P}} \times \sigma_{\mathbb{P}}'$ and $\sigma_{\mathbb{P}}' \times \sigma_{\mathbb{P}}'$ whose fixed loci are empty.

The fourth structure is $((x : y), (z : t)) \mapsto \left((\overline{z} : \overline{t}), (\overline{x} : \overline{y})\right)$ whose fixed locus is the sphere $\mathbb{S}^2$.

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[9] As an automorphism of C is also a birational transformation of C we simply apply Theorem 1.3.30 which states there is a one-to-one correspondence between automorphisms of $\mathbb{C}(C)$ and birational transformations of C. The stronger correspondence used in this proof relies on the fact that C is a smooth projective curve.

2.2.6 1. We have that $F(\mathbb{R}) = \mathcal{Z}(I)$ and $\overline{F(\mathbb{R})} = \mathcal{Z}_{\mathbb{C}}(\mathcal{I}(F(\mathbb{R})))$.

If $\mathcal{I}(\mathcal{Z}(I)) \subseteq I$ then $\mathcal{Z}_{\mathbb{C}}(\mathcal{I}(F(\mathbb{R}))) \supseteq \mathcal{Z}_{\mathbb{C}}(I)$ or in other words $\overline{F(\mathbb{R})} \supseteq F$ so $F(\mathbb{R})$ is dense in F.

If $F(\mathbb{R})$ is dense in F then $\mathcal{Z}_{\mathbb{C}}(\mathcal{I}(F(\mathbb{R}))) = F = \mathcal{Z}_{\mathbb{C}}(I)$. As the ideal I is radical the ideal $I_{\mathbb{C}} = I \otimes_{\mathbb{R}[X_1,\ldots,X_n]} \mathbb{C}[X_1, \ldots, X_n]$ is also radical. It follows by the Nullstellensatz that $\mathcal{I}_{\mathbb{C}}(F(\mathbb{R})) \subseteq I_{\mathbb{C}}$ and hence $\mathcal{I}(F(\mathbb{R})) \subseteq I$.

2. This follows immediately from (1) using Theorem A.5.15.

2.2.7 Set $I = (x^2 + y^2)$: we then have that $F = \mathcal{Z}_{\mathbb{C}}(I) = \{x \pm iy = 0\}$ and the real locus is $F(\mathbb{R}) = \mathcal{Z}(I) = \{(0, 0)\}$ and $\mathcal{I}(\mathcal{Z}(I)) = (x, y) \subsetneq I$ in $\mathbb{R}[X_1, \ldots, X_n]$.

We set $a = (0, 0)$. On the one hand, $\mathcal{O}_{F(\mathbb{R}),a} = \left(\frac{\mathbb{R}[x,y]}{(x,y)}\right)_{\mathfrak{m}_{F(\mathbb{R}),a}} = \mathbb{R}$ and on the other hand $\left(\mathcal{O}_F^G|_{F(\mathbb{R})}\right)_a = \mathcal{O}_{F,a}^G = \left(\left(\frac{\mathbb{C}[x,y]}{(x^2+y^2)}\right)_{\mathfrak{m}_{F,a}}\right)^G \supsetneq \mathbb{R}$ since the class of the polynomial x modulo $(x^2 + y^2)$ belongs to $\mathcal{O}_{F,a}^G$ since its coefficients are real.

2.2.26 1. φ is a morphism of $\mathbb{R}$-varieties if and only if

- φ is an morphism of complex varieties and
- $\varphi \circ \sigma_{\mathbb{A}}|_{F_1} = \sigma_{\mathbb{A}}|_{F_2} \circ \varphi$.

By Exercise 1.2.56 the first condition is equivalent to the existence of polynomial functions $f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_n]$ such that for every $(x_1, \ldots, x_n) \in F_1$, $\varphi(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$. The second condition is equivalent to

$$\varphi\left(\overline{x_1}, \ldots, \overline{x_n}\right) = \overline{\varphi(x_1, \ldots, x_n)}\ ,$$

which simply means that for every $(x_1, \ldots, x_n) \in F_1$ and every $i = 1 \ldots m$,

$$f_i\left(\overline{x_1}, \ldots, \overline{x_n}\right) = \overline{f_i(x_1, \ldots, x_n)}\ .$$

i.e. for every $i = 1 \ldots m$, ${}^{\sigma} f_i = f_i$ or in other words f_i has real coefficients.

2. φ is an $\mathbb{R}$-regular rational map if and only if

- φ is a rational map of $\mathbb{R}$-varieties;
- $F_1(\mathbb{R}) \subset \mathrm{dom}(\varphi)$.

In other words, φ is an $\mathbb{R}$-regular rational map if and only if

- φ is a rational map of complex varieties
- $\varphi \circ \sigma_{\mathbb{A}}|_{F_1} = \sigma_{\mathbb{A}}|_{F_2} \circ \varphi$;
- $F_1(\mathbb{R}) \subset \mathrm{dom}(\varphi)$.

By Exercise 1.3.25, the first condition is equivalent to the existence of polynomial functions $g_1, \ldots, g_m \in \mathbb{C}[x_1, \ldots, x_n]$ and $h_1, \ldots, h_m \in \mathbb{C}[x_1, \ldots, x_n]$ such that for any $(x_1, \ldots, x_n) \in \mathrm{dom}(\varphi)$,

$$\varphi(x_1, \ldots, x_n) = \left(\frac{g_1(x_1, \ldots, x_n)}{h_1(x_1, \ldots, x_n)}, \ldots, \frac{g_m(x_1, \ldots, x_n)}{h_m(x_1, \ldots, x_n)}\right)\ .$$

The map φ is therefore an $\mathbb{R}$-regular rational map if and only if g_i and h_i have real coefficients and the functions h_i do not vanish at any point of $F_1(\mathbb{R})$.

2.2.31 The usual atlas is a compatible atlas because the functions defining the open sets have real coefficients. We set

$$U_0 := \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_0 \neq 0\}$$

and

$$\varphi_0 \colon \begin{cases} U_0 & \longrightarrow \mathbb{C} \\ (x_0 : x_1) & \longmapsto \frac{x_1}{x_0} \end{cases} .$$

Similarly, set $U_1 := \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_1 \neq 0\}$ and

$$\varphi_1 \colon \begin{cases} U_1 & \longrightarrow \mathbb{C} \\ (x_0 : x_1) & \longmapsto \frac{x_0}{x_1} \end{cases} .$$

We then have that

$$^{\sigma}\varphi_0 \colon \begin{cases} \sigma(U_0) & \overset{\sigma_{\mathbb{P}}}{\to} & U_0 & \overset{\varphi_0}{\to} & \mathbb{C} & \overset{\sigma_{\mathbb{A}}}{\to} & \mathbb{C} \\ (x_0 : x_1) & \longmapsto & (\overline{x_0} : \overline{x_1}) & \longmapsto & \frac{\overline{x_1}}{\overline{x_0}} & \longmapsto & \frac{x_1}{x_0} \end{cases}$$

and

$$^{\sigma}\varphi_1 \colon \begin{cases} U_1 & \longrightarrow \mathbb{C} \\ (x_0 : x_1) & \longmapsto \frac{x_0}{x_1} \end{cases} .$$

2.3.14 Use Exercise 1.2.56(3) to write the isomorphism

$$\varphi' \circ \varphi^{-1} \colon \varphi'(V) \to \varphi(V)$$

in homogeneous coordinates then check that $\varphi' \circ \varphi^{-1}$ extends to an isomorphism $\varphi(V)_{\overline{K}} \to \varphi'(V)_{\overline{K}}$.

2.3.17 1. $\mathcal{I}(F) = (x, y)$ so $F_{\mathbb{C}} = \{(0, 0\}$ is a complexification of F which is irreducible so F is geometrically irreducible.

2. $V = \mathcal{Z}_{\mathbb{C}}(x + iy) \cup \mathcal{Z}_{\mathbb{C}}(x - iy)$.

3. The $\mathbb{R}$-variety (V, σ) does not have enough real points so it is not a complexification of F.

2.6.15 See [Ser55a, Chapitre III, Section 2] if necessary.

2.6.27 To simplify notation we will prove this result only for $n = 2$. Take a system of linear homogeneous coordinates $(x_0 : x_1 : x_2)$ and let $U_k := \mathbb{P}^2 \setminus \mathcal{Z}(x_k)$ be the standard open affine set defined by $x_k \neq 0$. Consider U_0 with its coordinates u_1, u_2. Sections of $\mathcal{K}_{\mathbb{P}^2}$ on U_0 are all of the form $p(u_1, u_2)\, \mathrm{d}u_1 \wedge \mathrm{d}u_2$. We will calculate the poles and zeros of the section $\mathrm{d}u_1 \wedge \mathrm{d}u_2$ outside of U_0. There is only one divisor outside of U_0, namely $x_0 = 0$, so it is enough to check the multiplicity along this divisor. We will calculate in U_1 with coordinates v_0, v_2 such that $(1 : u_1 : u_2) = (v_0 : 1 : v_2)$. In other words, $u_1 = \frac{1}{v_0}$ and $u_2 = \frac{v_2}{v_0}$, from which we get that

$$\mathrm{d}u_1 \wedge \mathrm{d}u_2 = \left(-\frac{1}{v_0^2}\,\mathrm{d}v_0\right) \wedge \left(\frac{v_0\,\mathrm{d}v_2 - v_2\,\mathrm{d}v_0}{v_0^2}\right) = -\frac{1}{v_0^3}\,\mathrm{d}v_0 \wedge \mathrm{d}v_2\ .$$

This form therefore has a pole of order 3 along $v_0 = 0$ as claimed.

2.6.28 Since $H^0(X, \mathcal{O}_X(-K_X)) \neq 0$, there is an *effective* divisor C linearly equivalent to $-K_X$.

There is an exact sequence

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

which on tensorising with Ω^1_X gives us

$$0 \to \Omega^1_X(K_X) \to \Omega^1_X \to \Omega^1_X|_C \to 0$$

whose initial terms in the long exact sequence are

$$0 \to H^0(X, \Omega^1_X(K_X)) \to H^0(X, \Omega^1_X) \to \cdots$$

and the conclusion follows because $H^0(X, \Omega^1_X) = 0$.

For the second question simply note that Θ_X is the dual of Ω^1_X and apply Theorem D.2.5.