9 Direct Treatment of Special Optimization Problems

Many of the results derived in this book are concerned with a generally formulated optimization problem. But if a concrete problem is given which has a rich mathematical structure, then solutions or characterizations of solutions can be derived sometimes in a direct way. In this case one takes advantage of the special structure of the optimization problem and can achieve the desired results very quickly.

In this final chapter we present two special optimal control problems and show how they can be treated without the use of general theoretical optimization results. The first problem is a so-called linear quadratic optimal control problem. For the given quadratic objective functional one gets a minimal solution with the aid of a simple quadratic completion without using necessary optimality conditions. The second problem is a time-minimal optimal control problem which can be solved directly by the application of a separation theorem.

9.1 Linear Quadratic Optimal Control Problems

In this section we consider a system of autonomous linear differential equations

$$\dot{x}(t) = Ax(t) + Bu(t) \text{ almost everywhere on } [0, \hat{T}] \tag{9.1}$$

and an initial condition

$$x(0) = x^0 \tag{9.2}$$

(where $\hat{T} > 0$ and $x^0 \in \mathbb{R}^n$ are arbitrarily given). Let A and B be (n, n) and (n, m) matrices with real coefficients, respectively. Let every control $u \in L_\infty^m([0, \hat{T}])$ be feasible (i.e. the controls are unconstrained). It is our aim to steer the system (9.1), (9.2) as close to a state of rest as possible at the terminal time $\hat{T}$. In other

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words: For a given positive definite symmetric (n, n) matrix G with real coefficients the quadratic form $x(\hat{T})^T G x(\hat{T})$ should be minimal. Since we want to reach our goal with a minimal steering effort, for a given positive definite symmetric (m, m) matrix R with real coefficients the expression $\int_0^{\hat{T}} u(t)^T R u(t)\, dt$ should be minimized as well. These two goals are used for the definition of the objective functional $J : L_\infty^m([0, \hat{T}]) \to \mathbb{R}$ with

$$J(u) = x(\hat{T})^T G x(\hat{T}) + \int_0^{\hat{T}} u(t)^T R u(t)\, dt \text{ for all } u \in L_\infty^m([0, \hat{T}]).$$

Under these assumptions the considered linear quadratic optimal control problem then reads as follows:

> Minimize the objective functional J with respect to all controls $u \in L_\infty^m([0, \hat{T}])$ for which the resulting trajectory is given by the system (9.1) of differential equations and the initial condition (9.2). (9.3)

In order to be able to present an optimal control for the problem (9.3) we need two technical lemmas.

Lemma 9.1 (relationship between control and trajectory).

Let $P(\cdot)$ be a real (n, n) matrix function which is symmetric and differentiable on $[0, \hat{T}]$. Then it follows for an arbitrary control $u \in L_\infty^m([0, \hat{T}])$ and a trajectory x of the initial value problem (9.1), (9.2):

$$0 = x^{0^T} P(0) x^0 - x(\hat{T})^T P(\hat{T}) x(\hat{T}) + \int_0^{\hat{T}} \Big[2u(t)^T B^T P(t) x(t) + x(t)^T \left(\dot{P}(t) + A^T P(t) + P(t) A \right) x(t) \Big] dt.$$

Proof For an arbitrary control $u \in L_\infty^m([0, \hat{T}])$ and a corresponding trajectory x of the initial value problem (9.1), (9.2) and an arbitrary real matrix function $P(\cdot)$ defined on $[0, \hat{T}]$ and being symmetric and differentiable it follows:

$$\begin{aligned} \frac{d}{dt}\Big[x(t)^T P(t) x(t) \Big] &= \dot{x}(t)^T P(t) x(t) + x(t)^T \left(\dot{P}(t) x(t) + P(t) \dot{x}(t) \right) \\ &= \left(A x(t) + B u(t) \right)^T P(t) x(t) \end{aligned}$$

$$+x(t)^T \left(\dot{P}(t)x(t) + P(t)\left(Ax(t) + Bu(t)\right)\right)$$
$$= x(t)^T \left(\dot{P}(t) + A^T P(t) + P(t)A\right) x(t)$$
$$+2u(t)^T B^T P(t)x(t) \text{ almost everywhere on } [0, \hat{T}].$$

With the initial condition (9.2) we get immediately by integration

$$x(\hat{T})^T P(\hat{T})x(\hat{T}) - x^{0^T} P(0)x^0$$
$$= \int_0^{\hat{T}} \left[2u(t)^T B^T P(t)x(t) + x(t)^T \left(\dot{P}(t) + A^T P(t) + P(t)A\right) x(t)\right] dt$$

which implies the assertion. □

Lemma 9.2 (Bernoulli matrix differential equation).

The (n, n) matrix function $P(\cdot)$ with

$$P(t) = \left[e^{A(t-\hat{T})} G^{-1} e^{A^T(t-\hat{T})} + \int_t^{\hat{T}} e^{A(t-s)} B R^{-1} B^T e^{A^T(t-s)}\, ds\right]^{-1}$$
$$\text{for all } t \in [0, \hat{T}] \qquad (9.4)$$

is a solution of the Bernoulli matrix differential equation

$$\dot{P}(t) + A^T P(t) + P(t)A - P(t)BR^{-1}B^T P(t) = 0_{(n,n)} \text{ for all } t \in [0, \hat{T}] \qquad (9.5)$$

with the terminal condition

$$P(\hat{T}) = G. \qquad (9.6)$$

The matrix function $P(\cdot)$ defined by (9.4) is symmetric.

Proof First we define the (n, n) matrix function $Q(\cdot)$ by

$$Q(t) = e^{A(t-\hat{T})} G^{-1} e^{A^T(t-\hat{T})} + \int_t^{\hat{T}} e^{A(t-s)} B R^{-1} B^T e^{A^T(t-s)}\, ds \text{ for all } t \in [0, \hat{T}]$$

(notice that the matrix exponential function is defined as a matrix series). It is evident that $Q(\cdot)$ is a symmetric matrix function. For an arbitrary $z \in \mathbb{R}^n$, $z \neq 0_{\mathbb{R}^n}$, we obtain

$$\begin{aligned} z^T Q(t) z &= \underbrace{z^T e^{A(t-\hat{T})} G^{-1} e^{A^T(t-\hat{T})} z}_{>0} + \int_t^{\hat{T}} \underbrace{z^T e^{A(t-s)} B R^{-1} B^T e^{A^T(t-s)} z}_{\geq 0} \, ds \\ &> 0 \text{ for all } t \in [0, \hat{T}]. \end{aligned}$$

Consequently, for every $t \in [0, \hat{T}]$ the matrix $Q(t)$ is positive definite and therefore invertible, i.e. the matrix function $P(\cdot)$ with

$$P(t) = Q(t)^{-1} \text{ for all } t \in [0, \hat{T}]$$

is well-defined. Since $Q(\cdot)$ is symmetric, $P(\cdot)$ is also symmetric.

It is obvious that $P(\cdot)$ satisfies the terminal condition (9.6). Hence, it remains to be shown that $P(\cdot)$ is a solution of the Bernoulli matrix differential equation (9.5). For this proof we calculate the derivative (notice the implications for arbitrary $t \in [0, \hat{T}]$: $Q(t) \cdot Q(t)^{-1} = I \implies \dot{Q}(t) Q(t)^{-1} + Q(t) \frac{d}{dt}\left(Q(t)^{-1}\right) = 0_{(n,n)} \implies \frac{d}{dt}\left(Q(t)^{-1}\right) = -Q(t)^{-1} \dot{Q}(t)\, Q(t)^{-1}$)

$$\begin{aligned} \dot{P}(t) &= \frac{d}{dt}\left(Q(t)^{-1}\right) \\ &= -Q(t)^{-1} \dot{Q}(t) Q(t)^{-1} \\ &= -Q(t)^{-1} \Bigg[A e^{A(t-\hat{T})} G^{-1} e^{A^T(t-\hat{T})} + e^{A(t-\hat{T})} G^{-1} e^{A^T(t-\hat{T})} A^T \\ &\quad + \int_t^{\hat{T}} \Big(A e^{A(t-s)} B R^{-1} B^T e^{A^T(t-s)} \\ &\quad + e^{A(t-s)} B R^{-1} B^T e^{A^T(t-s)} A^T \Big) \, ds - B R^{-1} B^T \Bigg] Q(t)^{-1} \\ &= -Q(t)^{-1} \left[A Q(t) + Q(t) A^T - B R^{-1} B^T \right] Q(t)^{-1} \\ &= -Q(t)^{-1} A - A^T Q(t)^{-1} + Q(t)^{-1} B R^{-1} B^T Q(t)^{-1} \\ &= -P(t) A - A^T P(t) + P(t) B R^{-1} B^T P(t) \text{ for all } t \in [0, \hat{T}]. \end{aligned}$$

Consequently, $P(\cdot)$ satisfies the Bernoulli matrix differential equation (9.5). □

With the aid of the two preceding lemmas it is now possible to present the optimal control of the linear quadratic problem (9.3).

Theorem 9.3 (feedback control).

The so-called feedback control $\bar{u}$ given by

$$\bar{u}(t) = -R^{-1}B^T P(t)x(t) \text{ almost everywhere on } [0, \hat{T}]$$

is the only optimal control of the linear quadratic control problem (9.3) where the matrix function $P(\cdot)$ is given by (9.4).

Proof In the following let $P(\cdot)$ be the matrix function defined by (9.4). Then we have with Lemmas 9.1 and 9.2 for every control $u \in L_\infty^m([0, \hat{T}])$ with $u \neq \bar{u}$:

$$\begin{aligned}
J(u) &= x(\hat{T})^T Gx(\hat{T}) + \int_0^{\hat{T}} u(t)^T Ru(t)\, dt \\
&= {x^0}^T P(0)x^0 + x(\hat{T})^T [G - P(\hat{T})]x(\hat{T}) \\
&\quad + \int_0^{\hat{T}} \Big[u(t)^T Ru(t) + 2u(t)^T B^T P(t)x(t) \\
&\qquad + x(t)^T \Big(\dot{P}(t) + A^T P(t) + P(t)A \Big) x(t) \Big]\, dt \\
&\qquad\qquad \text{(from Lemma 9.1)} \\
&= {x^0}^T P(0)x^0 + \int_0^{\hat{T}} \Big[u(t)^T Ru(t) + 2u(t)^T B^T P(t)x(t) \\
&\qquad + x(t)^T P(t)BR^{-1}B^T P(t)x(t) \Big]\, dt \\
&\qquad\qquad \text{(from Lemma 9.2)} \\
&= {x^0}^T P(0)x^0 + \int_0^{\hat{T}} \Big(u(t) + R^{-1}B^T P(t)x(t) \Big)^T R \\
&\qquad \Big(u(t) + R^{-1}B^T P(t)x(t) \Big)\, dt \\
&> {x^0}^T P(0)x^0 \\
&= J(\bar{u}).
\end{aligned}$$

Hence $\bar{u}$ is the only minimal point of the functional J. □

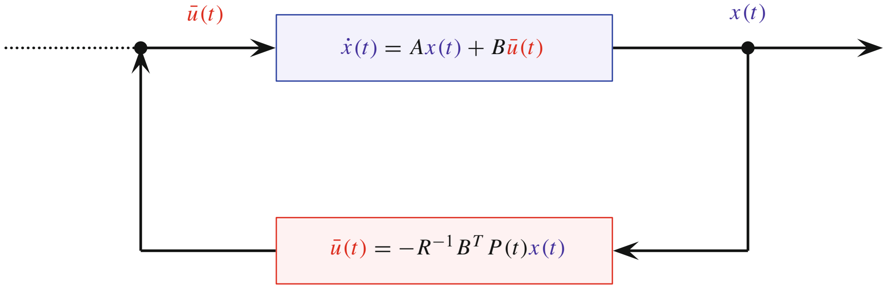

Fig. 9.1 Feedback control of Theorem 9.3

The optimal control presented in Theorem 9.3 depends on the time variable t and the current state $x(t)$. Such a control is called a *feedback* or a *closed loop control* (see Fig. 9.1).

If the control function depends only on t and not on the state $x(t)$, then it is called an *open loop control*. Feedback controls are of special importance for applications. Although feedback controls are also derived from the mathematical model, they make use of the real state of the system which is described mathematically only in an approximate way. Hence, in the case of perturbations which are not included in the mathematical model, feedback controls are often more realistic for the regulation of the system.

Since the matrix function P is analytic and the trajectory x is absolutely continuous, the optimal control $\bar{u}$ in Theorem 9.3 is an absolutely continuous vector function. In fact, a solution of the linear quadratic optimal control problem lies in a smaller subspace of $L_\infty^m([0, \hat{T}])$.

Notice that the proof of Theorem 9.3 could be done with the aid of an optimality condition. Instead of this we use a quadratic completion with Lemmas 9.1 and 9.2 which is simpler from a mathematical point of view.

The linear quadratic control problem (9.3) can be formulated more generally. If one defines the objective functional J by

$$J(u) = x(\hat{T})^T G x(\hat{T}) + \int_0^{\hat{T}} \Big(x(t)^T Q x(t) + u(t)^T R u(t) \Big)\, dt \quad \text{for all } u \in L_\infty^m([0, \hat{T}])$$

where Q is a positive definite symmetric (n, n) matrix with real coefficients, then the result of Theorem 9.3 remains almost true for the modified control problem.

The only difference is that then the matrix function $P(\cdot)$ is a solution of the Riccati matrix differential equation

$$\dot{P}(t) + A^T P(t) + P(t)A + Q - P(t)BR^{-1}B^T P(t) = 0_{(n,n)} \text{ for all } t \in [0, \hat{T}]$$

with the terminal condition $P(\hat{T}) = G$.

Example 9.4 (feedback control).

As a simple model we consider the differential equation

$$\dot{x}(t) = 3x(t) + u(t) \text{ almost everywhere on } [0, 1]$$

with the initial condition

$$x(0) = x^0$$

where $x^0 \in \mathbb{R}$ is arbitrarily chosen. The objective functional J reads as follows:

$$J(u) = x(1)^2 + \frac{1}{5}\int_0^1 u(t)^2\, dt \text{ for all } u \in L_\infty([0, 1]).$$

Then we obtain the function P as

$$\begin{aligned} P(t) &= \left[e^{3(t-1)}e^{3(t-1)} + 5\int_t^1 e^{3(t-s)}e^{3(t-s)}\, ds \right]^{-1} \\ &= \left[e^{6(t-1)} + 5\int_t^1 e^{6(t-s)}\, ds \right]^{-1} \\ &= \left[e^{6(t-1)} - \frac{5}{6}e^{6(t-1)} + \frac{5}{6} \right]^{-1} \\ &= \frac{6}{5 + e^{6(t-1)}} \text{ for all } t \in [0, 1]. \end{aligned}$$

Hence, the optimal control $\bar{u}$ is given by

$$\begin{aligned}\bar{u}(t) &= -5\,\frac{6}{5+e^{6(t-1)}}\,x(t)\\ &= -\frac{30}{5+e^{6(t-1)}}\,x(t) \text{ almost everywhere on } [0,1]. \end{aligned} \tag{9.7}$$

If we plug the feedback control $\bar{u}$ in the differential equation, we can determine the trajectory x:

$$\begin{aligned}\dot{x}(t) &= 3x(t) + \bar{u}(t)\\ &= 3x(t) - \frac{30}{5+e^{6(t-1)}}x(t)\\ &= \left(3 - \frac{30}{5+e^{6(t-1)}}\right)x(t).\end{aligned}$$

Then we obtain the trajectory x as

$$\begin{aligned}x(t) &= x^0\, e^{\int_0^t \left(3-\frac{30}{5+e^{6(s-1)}}\right)ds}\\ &= x^0\, e^{\left(3s-6(s-1)+\ln(e^{6(s-1)}+5)\right)\big|_0^t}\\ &= x^0\, e^{-3t+\ln(e^{6(t-1)}+5)-\ln(e^{-6}+5)}\\ &= \frac{x^0}{e^{-6}+5}\, e^{-3t}\, \left(e^{6(t-1)}+5\right) \text{ for all } t \in [0,1]. \end{aligned} \tag{9.8}$$

If we plug the equation (9.8) in the equation (9.7), we get the optimal control $\bar{u}$ in the open loop form

$$\bar{u}(t) = -\frac{30x^0}{e^{-6}+5}\, e^{-3t} \text{ almost everywhere on } [0,1]$$

(compare Fig. 9.2). This optimal control is even a smooth function.

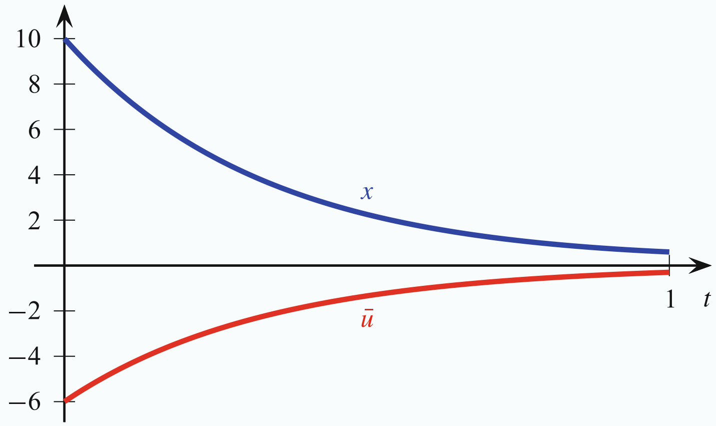

Fig. 9.2 Illustration of the optimal control $\bar{u}$ and the trajectory x

9.2 Time Minimal Control Problems

An important problem in control theory is the problem of steering a linear system with the aid of a bounded control from its initial state to a desired terminal point in minimal time. In this section we answer the questions concerning the existence and the characterization of such a time minimal control. As a necessary condition for such an optimal control we derive a so-called weak bang-bang principle. Moreover, we investigate a condition under which a time minimal control is unique.

In this section we consider the system of linear differential equations

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \text{ almost everywhere on } [0, \hat{T}] \tag{9.9}$$

with the initial condition

$$x(0) = x^0 \tag{9.10}$$

and the terminal condition

$$x(\hat{T}) = x^1 \tag{9.11}$$

where $\hat{T} > 0$, $x^0, x^1 \in \mathbb{R}^n$, A and B are (n, n) and (n, m) matrix functions with real coefficients, respectively, which are assumed to be continuous on $[0, \hat{T}]$, and controls u are chosen from $L_\infty^m([0, \hat{T}])$ with $\|u_i\|_{L_\infty([0,\hat{T}])} \leq 1$ for all $i \in \{1, \ldots, m\}$. Then we ask for a minimal time $\bar{T} \in [0, \hat{T}]$ so that the linear system (9.9) can be steered from x^0 to x^1 on the time interval $[0, \bar{T}]$.

If we consider the linear system (9.9) on a time interval $[0, T]$ with $T \in [0, \hat{T}]$ we use the abbreviation

$$U(T) := \{u \in L_\infty^m([0,T]) \mid \text{for every } k \in \{1,\dots,m\} \text{ we have } |u_k(t)| \le 1 \text{ almost everywhere on } [0,T]\} \quad \text{for all } T \in [0,\hat{T}] \tag{9.12}$$

for the set of all feasible controls with terminal time T.

Definition 9.5 (set of attainability).

For any $T \in [0, \hat{T}]$ consider the linear system (9.9) on $[0, T]$ with the initial condition (9.10). The set

$$K(T) := \{x(T) \in \mathbb{R}^n \mid u \in U(T) \text{ and } x \text{ satisfies the linear system (9.9) on } [0,T] \text{ and the initial condition (9.10)}\}$$

(with $U(T)$ given in (9.12)) is called the *set of attainability*.

The set of attainability consists of all terminal points to which the system can be steered from x^0 at the time T. Since we assume by (9.11) that the system can be steered to x^1 we have $x^1 \in K(\hat{T})$. Hence, the problem of finding a time minimal control for the linear system (9.9) satisfying the conditions (9.10), (9.11) can be transformed to a problem of the following type: Determine a minimal time $\bar{T} \in [0, \hat{T}]$ for which $x^1 \in K(\bar{T})$ (see Fig. 9.3).

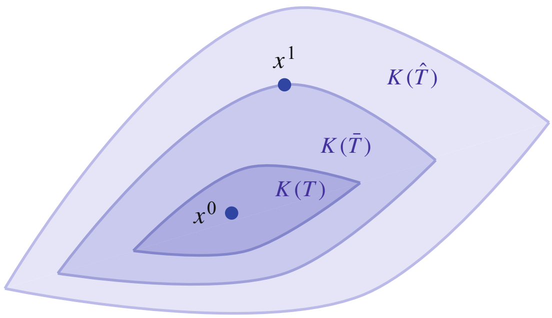

Fig. 9.3 Illustration of the set of attainability with $T \in (0, \bar{T})$

Before going further we recall that for an arbitrary $u \in L^m_\infty([0,T])$ the solution of the initial value problem (9.9), (9.10) with respect to the time interval $[0,T]$, $T \in [0,\hat{T}]$, can be written as

$$x(t) = \Phi(t)x^0 + \Phi(t)\int_0^t \Phi(s)^{-1}B(s)u(s)\,ds \text{ for all } t \in [0,\bar{T}]$$

where Φ is the fundamental matrix with

$$\dot{\Phi}(t) = A(t)\Phi(t) \text{ for all } t \in [0,T],$$

$$\Phi(0) = I \text{ (identity matrix)}[14].$$

Notice that in the case of a time independent matrix A, the fundamental matrix Φ is given as

$$\Phi(t) = e^{At} = \sum_{i=0}^{\infty} A^i \frac{t^i}{i!} \text{ for all } t \in [0,T].$$

In the following, for reasons of simplicity, we use the abbreviations

$$Y(t) := \Phi^{-1}(t)B(t) \text{ for all } t \in [0,T]$$

and

$$R(T) := \left\{ \int_0^T Y(t)u(t)dt \;\middle|\; u \in U(T) \right\} \text{ for all } T \in [0,\hat{T}].$$

The set $R(T)$ is sometimes called the *reachable set*. A connection between K and R is given by

$$\begin{aligned} K(T) &= \Phi(T)\left(x^0 + R(T)\right) \\ &= \{\Phi(T)x^0 + \Phi(T)y \mid y \in R(T)\} \text{ for all } T \in [0,\hat{T}]. \end{aligned} \tag{9.13}$$

First we investigate properties of the set of attainability.

[14] A proof of this existence result can be found e.g. in [212, p. 121–122].

Lemma 9.6 (properties of the set of attainability).

For every $T \in [0, \hat{T}]$ *the set* $K(T)$ *of attainability for the initial value problem (9.9), (9.10) with respect to the time interval* $[0, T]$ *is nonempty, convex and compact.*

Proof We present a proof of this lemma only in a short form. Let some $T \in [0, \hat{T}]$ be arbitrarily given. Because of the initial condition (9.10) it is obvious that $R(T) \neq \emptyset$. Next we show that the reachable set

$$R(T) = \left\{ \int_0^T Y(t)u(t)\,dt \;\middle|\; u \in U(T) \right\}$$

is convex and compact. $U(T)$ is the closed unit ball in $L_\infty^m([0, T])$ and therefore weak*-compact. Next we define the linear mapping $L : L_\infty^m([0, T]) \to \mathbb{R}^n$ with

$$L(u) = \int_0^T Y(t)u(t)\,dt \text{ for all } u \in L_\infty^m([0, T]).$$

L is continuous with respect to the norm topology in $L_\infty^m([0, T])$, and therefore it is also continuous with respect to the weak*-topology in $L_\infty^m([0, T])$. Since L is continuous and linear and the set $U(T)$ is weak*-compact and convex, the image $R(T) = L(U(T))$ is compact and convex. Because of the equation (9.13) the set $K(T)$ is also compact and convex. □

As a first important result we present an existence theorem for time minimal controls.

Theorem 9.7 (existence of a time minimal control).

If there is a control which steers the linear system (9.9) with the initial condition (9.10) to a terminal state x^1 *within a time* $\tilde{T} \in [0, \hat{T}]$, *then there is also a time minimal control with this property.*

Proof We assume that $x^1 \in K(\tilde{T})$. Next we set

$$\bar{T} := \inf\{T \in [0, \hat{T}] \mid x^1 \in K(T)\}.$$

Then we have $\bar{T} \leq \tilde{T}$, and there is a monotonically decreasing sequence $(T_i)_{i\in\mathbb{N}}$ with the limit $\bar{T}$ and a sequence $(u^i)_{i\in\mathbb{N}}$ of feasible controls with

$$x^1 =: x(T_i, u^i) \in K(T_i)$$

(let $x(T_i, u^i)$ denote the terminal state at the time T_i with the control u^i). Then it follows

$$\begin{aligned}
&\|x^1 - x(\bar{T}, u^i)\| \\
&= \|x(T_i, u^i) - x(\bar{T}, u^i)\| \\
&= \left\| \Phi(T_i)x^0 + \Phi(T_i)\int_0^{T_i} Y(t)u^i(t)\,dt - \Phi(\bar{T})\int_0^{T_i} Y(t)u^i(t)\,dt \right. \\
&\qquad \left. -\Phi(\bar{T})x^0 - \Phi(\bar{T})\int_0^{\bar{T}} Y(t)u^i(t)\,dt + \Phi(\bar{T})\int_0^{T_i} Y(t)u^i(t)\,dt \right\| \\
&\leq \left\|(\Phi(T_i) - \Phi(\bar{T}))x^0\right\| + \left\|(\Phi(T_i) - \Phi(\bar{T}))\int_0^{T_i} Y(t)u^i(t)\,dt\right\| \\
&\qquad + \left\|\Phi(\bar{T})\int_{\bar{T}}^{T_i} Y(t)u^i(t)\,dt\right\|
\end{aligned}$$

which implies because of the continuity of Φ

$$x_1 = \lim_{i\to\infty} x(\bar{T}, u^i).$$

Since $x(\bar{T}, u^i) \in K(\bar{T})$ for all $i \in \mathbb{N}$ and the set $K(\bar{T})$ is closed, we get $x^1 \in K(\bar{T})$ which completes the proof. □

In our problem formulation we assume that the terminal condition (9.11) is satisfied. Therefore Theorem 9.7 ensures that a time minimal control exists without additional assumptions. For the presentation of a necessary condition for such a time minimal control we need some lemmas given in the following.

Lemma 9.8 (property of the set of attainability).

Let the linear system (9.9) with the initial condition (9.10) be given. Then the set-valued mapping $K : [0, \hat{T}] \to 2^{\mathbb{R}^n}$ (where $K(\cdot)$ denotes the set of attainability) is continuous (with respect to the Hausdorff distance).

Proof First we prove the continuity of the mapping R. For that proof let $\bar{T}, T \in [0, \hat{T}]$, with $\bar{T} \neq T$, be arbitrarily chosen. Without loss of generality we assume $\bar{T} < T$. Then for an arbitrary $\bar{y} \in R(\bar{T})$ there is a feasible control $\bar{u}$ with

$$\bar{y} = \int_0^{\bar{T}} Y(t)\bar{u}(t)\,dt.$$

For the feasible control u given by

$$u(t) = \left\{ \begin{array}{l} \bar{u}(t) \text{ almost everywhere on } [0, \bar{T}] \\ (1, \ldots, 1)^T \text{ for all } t \in (\bar{T}, T] \end{array} \right\}$$

we have

$$\int_0^T Y(t)u(t)\,dt \in R(T).$$

Consequently we get

$$\begin{aligned} d(\bar{y}, R(T)) &:= \min_{y \in R(T)} \|\bar{y} - y\| \\ &\leq \left\| \bar{y} - \int_0^T Y(t)u(t)\,dt \right\| \\ &= \left\| \int_{\bar{T}}^T Y(t)(1, \ldots, 1)^T dt \right\| \\ &\leq \sqrt{m} \int_{\bar{T}}^T |||Y(t)|||\,dt \end{aligned}$$

and

$$\max_{\bar{y} \in R(\bar{T})} d(\bar{y}, R(T)) \leq \sqrt{m} \int_{\bar{T}}^T |||Y(t)|||\,dt$$

(here $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$ and $|||\cdot|||$ denotes the spectral norm). Similarly one can show

$$\max_{y\in R(T)} d(R(\bar{T}), y) \leq \sqrt{m} \int_{\bar{T}}^{T} |||Y(t)|||\, dt.$$

Hence, we obtain for the metric ϱ:

$$\begin{aligned}\varrho(R(\bar{T}), R(T)) &:= \max_{\bar{y}\in R(\bar{T})} \min_{y\in R(T)} \|\bar{y}-y\| + \max_{y\in R(T)} \min_{\bar{y}\in R(\bar{T})} \|\bar{y}-y\| \\ &\leq 2\sqrt{m} \int_{\bar{T}}^{T} |||Y(t)|||\, dt.\end{aligned}$$

Since the matrix function Y is continuous, there is a constant $\alpha > 0$ with

$$|||Y(t)||| \leq \alpha \text{ for all } t \in [0, \hat{T}].$$

Then we get

$$\varrho(R(\bar{T}), R(T)) \leq 2\alpha\sqrt{m}(T - \bar{T}).$$

Consequently, the set-valued mapping R is continuous. Since the fundamental matrix Φ is continuous and the images of the set-valued mapping R are bounded sets, we obtain with the equation (9.13) (notice for $\bar{T}, T \in [0, \hat{T}]$ and a constant $\beta > 0$ the inequality $\varrho(K(\bar{T}), K(T)) \leq \beta|||\Phi(\bar{T}) - \Phi(T)||| + |||\Phi(\bar{T})|||\, \varrho(R(\bar{T}), R(T)))$ that the mapping K is continuous. □

Lemma 9.9 (property of the set of attainability).

Let the linear system (9.9) with the initial condition (9.10) and some $\bar{T} \in [0, \hat{T}]$ be given. Let $\bar{y}$ be a point in the interior of the set $K(\bar{T})$ of attainability, then there is a time $T \in (0, \bar{T})$ so that $\bar{y}$ is also an interior point of $K(T)$.

Proof Let $\bar{y}$ be an interior point of the set $K(\bar{T})$ (this implies $\bar{T} > 0$). Then there is an $\varepsilon > 0$ so that $B(\bar{y}, \varepsilon) \subset K(\bar{T})$ for the closed ball $B(\bar{y}, \varepsilon)$ around $\bar{y}$ with radius ε. Now we assume that for all $T \in (0, \bar{T})$ $\bar{y}$ is not an interior point of the set $K(T)$. For every $T \in (0, \bar{T})$ the set $K(T) \subset \mathbb{R}^n$ is closed and convex. Then for every $T \in (0, \bar{T})$ there is a hyperplane separating the set $K(T)$ and the point $\bar{y}$ (compare Theorems C.5 and C.3). Consequently, for every $T \in (0, \bar{T})$ there is a point $y_T \in B(\bar{y}, \varepsilon)$ whose distance to the set $K(T)$ is at least ε. But this contradicts the continuity of the set-valued mapping K. □

The next lemma is the key for the proof of a necessary condition for time minimal controls. For the formulation of this result we use the function sgn : $\mathbb{R} \to \mathbb{R}$ given by

$$\operatorname{sgn}(y) = \left\{ \begin{array}{r} 1 \text{ for } y > 0 \\ 0 \text{ for } y = 0 \\ -1 \text{ for } y < 0 \end{array} \right\}.$$

Lemma 9.10 (property of the set of attainability).

Let the linear system (9.9) with the initial condition (9.10) and some $\bar{T} \in (0, \hat{T}]$ be given. If $\bar{x}(\bar{T}, \bar{u}) \in \partial K(\bar{T})$ for some $\bar{u} \in U(\bar{T})$, then there is a vector $\eta \neq 0_{\mathbb{R}^n}$ so that for all $k \in \{1, \ldots, m\}$:

$$\bar{u}_k(t) = sgn[\eta^T Y_k(t)] \text{ almost everywhere on } \{t \in [0, \bar{T}] \mid \eta^T Y_k(t) \neq 0\}$$

($\bar{x}(\bar{T}, \bar{u})$ denotes the state at the time $\bar{T}$ with respect to the control $\bar{u}$; $Y_k(t)$ denotes the k-th column of the matrix $Y(t)$).

Proof Let an arbitrary point $\bar{y} := \bar{x}(\bar{T}, \bar{u}) \in \partial K(\bar{T})$ be given. Since the set $K(\bar{T})$ is a convex and closed subset of $\mathbb{R}^n$, by a separation theorem (see Theorem C.5) there is a vector $\bar{\eta} \neq 0_{\mathbb{R}^n}$ with the property

$$\bar{\eta}^T \bar{y} \geq \bar{\eta}^T y \text{ for all } y \in K(\bar{T}).$$

Because of

$$\bar{\eta}^T \bar{y} = \bar{\eta}^T \Phi(\bar{T}) x^0 + \bar{\eta}^T \Phi(\bar{T}) \int_0^{\bar{T}} Y(t) \bar{u}(t)\, dt$$

and

$$\bar{\eta}^T y = \bar{\eta}^T \Phi(\bar{T}) x^0 + \bar{\eta}^T \Phi(\bar{T}) \int_0^{\bar{T}} Y(t) u(t)\, dt \text{ for all } y \in K(\bar{T})$$

we obtain for $\eta^T := \bar{\eta}^T \Phi(\bar{T})$

$$\eta^T \int_0^{\bar{T}} Y(t)\bar{u}(t)\, dt \geq \eta^T \int_0^{\bar{T}} Y(t)u(t)\, dt \tag{9.14}$$

for all feasible controls steering the linear system (9.9) with the initial condition (9.10) to a state in the set $K(\bar{T})$ of attainability. From the inequality (9.14) we conclude

$$\eta^T Y(t)\bar{u}(t) \geq \eta^T Y(t)u(t) \text{ almost everywhere on } [0, \bar{T}]. \tag{9.15}$$

For the proof of the implication "(9.14) $\Longrightarrow$ (9.15)" we assume that the inequality (9.15) is not true. Then there is a feasible control u and a set $M \subset [0, \bar{T}]$ with positive measure so that

$$\eta^T Y(t)\bar{u}(t) < \eta^T Y(t)u(t) \text{ almost everywhere on } M.$$

If one defines the feasible control u^* by

$$u^*(t) = \left\{ \begin{array}{ll} \bar{u}(t) & \text{almost everywhere on } [0, \bar{T}] \setminus M \\ u(t) & \text{almost everywhere on } M \end{array} \right\},$$

then it follows

$$\begin{aligned} \eta^T \int_0^{\bar{T}} Y(t)u^*(t)\, dt &= \eta^T \int_M Y(t)u(t)\, dt + \eta^T \int_{[0,\bar{T}]\setminus M} Y(t)\bar{u}(t)\, dt \\ &> \eta^T \int_M Y(t)\bar{u}(t)\, dt + \eta^T \int_{[0,\bar{T}]\setminus M} Y(t)\bar{u}(t)\, dt \\ &= \eta^T \int_0^{\bar{T}} Y(t)\bar{u}(t)\, dt \end{aligned}$$

which contradicts the inequality (9.14). Hence, the inequality (9.15) is true.

From the inequality (9.15) we get for all $k \in \{1, \ldots, m\}$

$$\bar{u}_k(t) = \operatorname{sgn}\, [\eta^T Y_k(t)] \text{ almost everywhere on } \{t \in [0, \bar{T}] \mid \eta^T Y_k(t) \neq 0\}. \qquad \square$$

Now we present the afore-mentioned necessary condition for time minimal controls.

Theorem 9.11 (necessary condition for time minimal controls).

Let the linear system (9.9) with the initial condition (9.10) and the terminal condition (9.11) be given. If $\bar{u}$ is a time minimal control with respect to the minimal terminal time $\bar{T} \in [0, \hat{T}]$, then there is a vector $\eta \neq 0_{\mathbb{R}^n}$ so that for all $k \in \{1, \ldots, m\}$:

$$\bar{u}_k(t) = sgn[\eta^T Y_k(t)] \text{ almost everywhere on } \{t \in [0, \bar{T}] \mid \eta^T Y_k(t) \neq 0\}. \tag{9.16}$$

Proof The assertion is obvious for $\bar{T} = 0$. Therefore we assume $\bar{T} > 0$ for the following. We want to show that

$$\bar{y} := \Phi(\bar{T})x^0 + \Phi(\bar{T}) \int_0^{\bar{T}} Y(t)\bar{u}(t)\, dt \ \in \ \partial K(\bar{T}). \tag{9.17}$$

Suppose that $\bar{y}$ were an interior point of the set $K(\bar{T})$ of attainability. Then by Lemma 9.9 there is a time $T \in (0, \bar{T})$ so that $\bar{y}$ is also an interior point of the set $K(T)$. But this contradicts the fact that $\bar{T}$ is the minimal time. Hence, the condition (9.17) is true. An application of Lemma 9.10 completes the proof. □

The statement (9.16) is also called a *weak bang-bang principle*. If the measure of the set $\{t \in [0, \bar{T}] \mid \eta^T Y_k(t) = 0\}$ equals 0 for every $k \in \{1, \ldots, m\}$, the statement (9.16) is called a *strong bang-bang principle*. Theorem 9.11 can also be formulated as follows:

For every time minimal control $\bar{u}$ there is a vector $\eta \neq 0_{\mathbb{R}^n}$
so that $\bar{u}$ satisfies the weak bang-bang principle (9.16).

The next example illustrates the applicability of Theorem 9.11.

Example 9.12 (necessary condition for time minimal controls).

We consider the harmonic oscillator mathematically formalized by

$$\ddot{y}(t) + y(t) = u(t) \text{ almost everywhere on } [0, \hat{T}],$$

$$\|u\|_{L_\infty([0,\hat{T}])} \leq 1$$

where $\hat{T} > 0$ is sufficiently large. An initial condition is not given explicitly. The corresponding linear system of first order reads

$$\dot{x}(t) = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{=:\,A} x(t) + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{=:\,B} u(t).$$

We have

$$\Phi(t) = e^{At} = \sum_{i=0}^{\infty} A^i \frac{t^i}{i!} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

and

$$Y(t) = \Phi(t)^{-1} B = e^{-At} B = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}.$$

Then we obtain for an arbitrary vector $\eta \neq 0_{\mathbb{R}^n}$

$$\eta^T Y(t) = -\eta_1 \sin t + \eta_2 \cos t.$$

Consequently, we get for a number $\alpha \in \mathbb{R}$ and a number $\delta \in [-\pi, \pi]$

$$\eta^T Y(t) = \alpha \sin(t + \delta)$$

and therefore

$$\text{sgn}[\eta^T Y(t)] = \text{sgn}[\alpha \sin(t + \delta)]$$

(see Fig. 9.4).

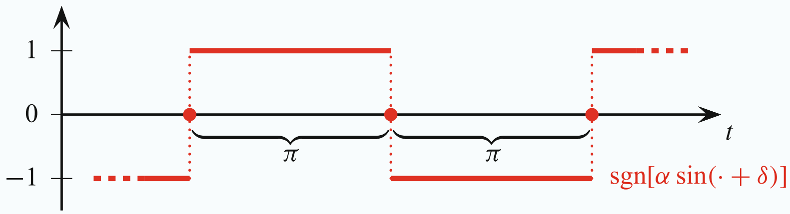

Fig. 9.4 Illustration of the time optimal control

Conclusion: If there is a time minimal control $\bar{u}$, then it fulfills the strong bang-bang principle, and therefore it is unique. After π time units one always gets a change of the sign of $\bar{u}$.
With a standard result from control theory one can see that the considered linear system is null controllable (i.e., it can be steered to the origin in a finite time). Hence, by Theorem 9.7 there is also a time minimal control $\bar{u}$ which steers this system into a state of rest, and therefore the preceding results are applicable.

Now we present an example for which the necessary condition for time minimal controls does not give any information.

Example 9.13 (necessary condition for time minimal controls).

We investigate the simple linear system

$$\left.\begin{aligned}\dot{x}_1(t) &= x_1(t) + u(t)\\ \dot{x}_2(t) &= x_2(t) + u(t)\end{aligned}\right\} \text{ almost everywhere on } [0, \hat{T}]$$

with

$$\|u\|_{L_\infty[0,\hat{T}]} \leq 1$$

and $\hat{T} > 0$. Here we set

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \ \text{ and } \ B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then we obtain

$$Y(t) = e^{-At}B = e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and for any vector $\eta \neq 0_{\mathbb{R}^2}$ we get

$$\eta^T Y(t) = (\eta_1 + \eta_2)e^{-t}.$$

For example, for $\eta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ we conclude

$$\eta^T Y(t) = 0 \text{ for all } t \in [0, \hat{T}],$$

and Theorem 9.11 does not give a suitable necessary condition for time minimal controls.

Next we investigate the question under which conditions time minimal controls are unique. For this investigation we introduce the notion of normality.

Definition 9.14 (normal linear system).

(a) The linear system (9.9) is called *normal on* $[0, T]$ (with $T \in [0, \hat{T}]$), if for every vector $\eta \neq 0_{\mathbb{R}^n}$ the sets

$$G_k(\eta) = \{t \in [0, T] \mid \eta^T Y_k(t) = 0\} \text{ with } k \in \{1, \ldots, m\}$$

have the measure 0. $Y_k(t)$ denotes again the k-th column of the matrix $Y(t)$.

(b) The linear system (9.9) is called *normal*, if for every $T \in [0, \hat{T}]$ this system is normal on $[0, T]$.

Theorem 9.15 (uniqueness of a time minimal control).

Let the linear system (9.9) with the initial condition (9.10) and the terminal condition (9.11) be given. If $\bar{u}$ is a time minimal control with respect to the minimal terminal time $\bar{T} \in [0, \hat{T}]$ and if the linear system (9.9) is normal on $[0, \bar{T}]$, then $\bar{u}$ is the unique time minimal control.

Proof By Theorem 9.11 for every time minimal control $\bar{u}$ there is a vector $\eta \neq 0_{\mathbb{R}^n}$ so that for all $k \in \{1, \ldots, m\}$:

$$\bar{u}_k(t) = \operatorname{sgn}[\eta^T Y_k(t)] \text{ almost everywhere on } [0, \bar{T}] \setminus G_k(\eta).$$

Then the assertion follows from the normality assumption (notice that in the proof of Lemma 9.10 the vector η depends on the terminal state and not on the control). □

A control $\bar{u}$ which satisfies the assumptions of Theorem 9.15 fulfills the strong bang-bang principle

$$\bar{u}(t) = \operatorname{sgn}[\eta^T Y_k(t)] \text{ almost everywhere on } [0, \bar{T}].$$

One obtains an interesting characterization of the concept of normality in the case of an autonomous linear system (9.9) with constant matrix functions A and B.

Theorem 9.16 (characterization of normality).

The autonomous linear system (9.9) with constant matrix functions A and B is normal if and only if for every $k \in \{1, \ldots, m\}$ either

$$\text{rank}\,(B_k, AB_k, \ldots, A^{n-1}B_k) = n \tag{9.18}$$

or

$$\text{rank}\,(A - \lambda I, B_k) = n \text{ for all eigenvalues } \lambda \text{ of } A. \tag{9.19}$$

Here B_k denotes the k-th column of the matrix B.

Proof We fix an arbitrary terminal time $T \in [0, \hat{T}]$. First notice that for every $k \in \{1, \ldots, m\}$ and every $\eta \in \mathbb{R}^n$

$$\eta^T Y_k(t) = \eta^T e^{-At} B_k \text{ for all } t \in [0, T].$$

Consequently, the real-valued analytical function $\eta^T Y_k(\cdot)$ on $[0, T]$ is either identical to 0 or it has a finite number of zeros on this interval. Therefore, the autonomous linear system (9.9) is normal on $[0, T]$ if and only if the following implication is satisfied:

$$\eta^T e^{-At} B_k = 0 \text{ for all } t \in [0, T] \text{ and some } k \in \{1, \ldots, m\} \Rightarrow \eta = 0_{\mathbb{R}^n}. \tag{9.20}$$

Next we show that the implication (9.20) is equivalent to the condition (9.18). For this proof we assume that the condition (9.18) is satisfied. Let a vector $\eta \in \mathbb{R}^n$ with

$$\eta^T e^{-At} B_k = 0 \text{ for all } t \in [0, T] \text{ and some } k \in \{1, \ldots, m\}$$

be arbitrarily given. By repeated differentiation and setting "$t = 0$" we get

$$\eta^T (B_k, AB_k, \ldots, A^{n-1}B_k) = 0_{\mathbb{R}^n}^T \text{ for some } k \in \{1, \ldots, m\}.$$

By assumption the system of row vectors of the matrix $(B_k, AB_k, \ldots, A^{n-1}B_k)$ is linear independent, and therefore we get $\eta = 0_{\mathbb{R}^n}$. Hence, the implication (9.20) is satisfied, i.e. the autonomous linear system (9.9) is normal on $[0, T]$.

Now we assume that the condition (9.18) is not satisfied. This means that for some $k \in \{1, \ldots, m\}$ the system of row vectors of the matrix $(B_k, AB_k, \ldots, A^{n-1}B_k)$ is linear dependent. Then there is a vector $\eta \neq 0_{\mathbb{R}^n}$ with

$$\eta^T (B_k, AB_k, \ldots, A^{n-1}B_k) = 0_{\mathbb{R}^n}^T$$

which implies

$$\eta^T B_k = \eta^T A B_k = \cdots = \eta^T A^{n-1} B_k = 0. \tag{9.21}$$

The Cayley-Hamilton theorem states that the matrix A satisfies its characteristic equation, i.e.

$$A^n = \alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1}$$

with appropriate coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$. Then we obtain with (9.21)

$$\eta^T A^n B_k = \alpha_0 \eta^T B_k + \alpha_1 \eta^T A B_k + \cdots + \alpha_{n-1} \eta^T A^{n-1} B_k = 0$$

and by induction

$$\eta^T A^l B_k = 0 \text{ for all } l \geq n. \tag{9.22}$$

Equations (9.21) and (9.22) imply

$$\eta^T A^l B_k = 0 \text{ for all } l \geq 0$$

which leads to

$$\eta^T e^{-At} B_k = \eta^T \left(\sum_{i=0}^{\infty} A^i \frac{(-t)^i}{i!} \right) B_k = 0 \text{ for all } t \in [0, T].$$

Consequently, the implication (9.20) is not satisfied, i.e. the autonomous linear system (9.9) is not normal on $[0, T]$.

Finally we show the equivalence of the two rank conditions (9.18) and (9.19). Let $k \in \{1, \ldots, m\}$ be arbitrarily chosen.

Assume that the condition (9.19) is not satisfied, i.e. for some possibly complex eigenvalue λ of A we have

$$\text{rank } (A - \lambda I, B_k) \neq n.$$

Then there is a vector $z \in \mathbb{R}^n$ with $z \neq 0_{\mathbb{R}^n}$ and

$$z^T (A - \lambda I, B_k) = 0^T_{\mathbb{R}^{n+1}},$$

i.e.

$$z^T A = \lambda z^T \tag{9.23}$$

and

$$z^T B_k = 0. \tag{9.24}$$

With the equations (9.23) and (9.24) we conclude

$$z^T A B_k = \lambda z^T B_k = 0,$$

and by induction we get

$$z^T A^l B_k = 0 \text{ for all } l \geq 0.$$

Hence we have

$$\text{rank } (B_k, AB_k, \ldots, A^{n-1} B_k) \neq n.$$

Conversely, we assume now that the equation (9.18) is not satisfied. Then there is a $z \neq 0_{\mathbb{R}^n}$ with

$$z^T B_k = 0, \; z^T A B_k = 0, \ldots, \; z^T A^{n-1} B_k = 0.$$

Again with the Cayley-Hamilton theorem we conclude immediately

$$z^T A^l B_k = 0 \text{ for all } l \geq 0.$$

Consequently, the linear subspace

$$S := \{\tilde{z} \in \mathbb{R}^n \mid \tilde{z}^T A^l B_k = 0 \text{ for all } l \geq 0\}$$

has the dimension ≥ 1. Since the set S is invariant under A^T (i.e. $A^T S \subset S$), one eigenvector $\bar{z}$ of A^T belongs to S. Hence, there is an eigenvalue λ of A^T which is also an eigenvalue of A so that

$$A^T \bar{z} = \lambda \bar{z}$$

or alternatively

$$\bar{z}^T (A - \lambda I) = 0_{\mathbb{R}^n}^T. \tag{9.25}$$

Because of $\bar{z} \in S$ we obtain with $l = 0$

$$\bar{z}^T B_k = 0. \tag{9.26}$$

Equations (9.25) and (9.26) imply

$$\text{rank}\,(A - \lambda I, B_k) \neq n \text{ for some eigenvalue } \lambda \text{ of } A.$$

This completes the proof. □

In control theory the condition

$$\text{rank}\,(B, AB, \ldots, A^{n-1}B) = n$$

is called the *Kalman condition*. It is obvious that the condition

$$\text{rank}\,(B_k, AB_k, \ldots, A^{n-1}B_k) = n \text{ for all } k \in \{1, \ldots, m\}$$

which is given in Theorem 9.16 implies the Kalman condition. Moreover, in control theory the condition

$$\text{rank}\,(A - \lambda I, B) = n \text{ for all eigenvalues } \lambda \text{ of } A$$

is called the *Hautus condition* which is implied by the condition

$$\text{rank}\,(A - \lambda I, B_k) = n \text{ for all } k \in \{1, \ldots, m\} \text{ and all eigenvalues } \lambda \text{ of } A.$$

One can show with the same arguments as in the proof of Theorem 9.16 that the Kalman and Hautus conditions are equivalent. In control theory one proves that the Kalman condition (or the Hautus condition) characterizes the controllability of an autonomous linear system, i.e. in this case there is an unconstrained control which steers the autonomous linear system from an arbitrary initial state to an arbitrary terminal state in finite time.

The following example shows that the Kalman condition (or the Hautus condition) does not imply the condition (9.18) (and (9.19), respectively).

Example 9.17 (Kalman condition).

The following autonomous linear system satisfies the Kalman condition but it is not normal:

$$\left.\begin{aligned} \dot{x}_1(t) &= -x_1(t) + u_1(t) \\ \dot{x}_2(t) &= -2x_2(t) + u_1(t) + u_2(t) \end{aligned}\right\} \text{ almost everywhere on } [0, \hat{T}]$$

with some $\hat{T} > 0$. Here we set

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then we have

$$B_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ AB_1 = \begin{pmatrix} -1 \\ -2 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ AB_2 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

The matrix (B_2, AB_2) has the rank 1, and therefore the linear system is not normal. On the other hand we have

$$\text{rank } (B, AB) = 2,$$

i.e. the Kalman condition is satisfied.

Exercises

(9.1) Consider the differential equation

$$\dot{x}(t) = 2x(t) - 3u(t) \text{ almost everywhere on } [0, 2]$$

with the initial condition

$$x(0) = x^0$$

for an arbitrarily chosen $x^0 \in \mathbb{R}$. Determine an optimal control $\bar{u} \in L_\infty([0, 2])$ as a minimal point of the objective functional $J : L_\infty([0, 2]) \to \mathbb{R}$ with

$$J(u) = \frac{1}{2}x(1)^2 + 2\int_0^2 u(t)^2\,dt \text{ for all } u \in L_\infty([0, 2]).$$

(9.2) ([51, p. 132–133]) Let the initial value problem

$$\dot{x}(t) = u(t) \text{ almost everywhere on } [0, 1],$$

$$x(0) = 1$$

be given. Determine an optimal control $u \in L_\infty([0,1])$ for which the objective functional $J : L_\infty([0,1]) \to \mathbb{R}$ with

$$J(u) = \int_0^1 \left(u(t)^2 + x(t)^2\right) dt \text{ for all } u \in L_\infty([0,1])$$

becomes minimal.

(9.3) Consider the linear differential equation of n-th order

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_0 y(t) = u(t)$$
$$\text{almost everywhere on } [0, \hat{T}]$$

where $\hat{T} > 0$ and $a_0, \ldots, a_{n-1} \in \mathbb{R}$ are given constants. The control u is assumed to be an $L_\infty([0,\hat{T}])$ function. Show that the system of linear differential equations of first order which is equivalent to this differential equation of n-th order satisfies the Kalman condition.

(9.4) ([216, p. 22–24]) Let the system of linear differential equations

$$\dot{x}(t) = Ax(t) + Bu(t) \text{ almost everywhere on } [0, \hat{T}]$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ -\beta \\ 0 \\ \gamma \end{pmatrix}$$

be given where $\hat{T} > 0$, $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ are constants. It is assumed that $u \in L_\infty([0,\hat{T}])$. Show that this system satisfies the Hautus condition.

(9.5) For the linear system in exercise (9.4) assume in addition that the terminal time $\hat{T}$ is sufficiently large. Moreover, let the initial condition

$$x(0) = x^0$$

with $x^0 \in \mathbb{R}^4$ and the terminal condition

$$x(\hat{T}) = 0_{\mathbb{R}^4}$$

be given. For the control u we assume

$$\|u\|_{L_\infty([0,\hat{T}])} \leq 1.$$

It can be proved with a known result from control theory that this system can be steered from x^0 to $0_{\mathbb{R}^4}$ in finite time. Show then that a time minimal control exists which is unique, and give a characterization of this time minimal control.