Chapter 13
Do We Really Need Pantographic Structures?

Mario Spagnuolo and Emilio Barchiesi

Abstract This chapter attempts to provide a comprehensive answer to the challenging question: do we really need pantographic structures? This question may arise spontaneously given the recent proliferation of contributions on this type of metamaterial in the literature. A reasoned answer to this question may be crucial for the future development and orientation of research concerning this metamaterial. More generally, we show the context from which the studies that led to the development of pantographic structures originated and observe how an excessively orthodox view of Continuum Mechanics may prevent interesting developments. Within the framework of generalised theories and second-gradient models, pantographic structures assume an important role.

Keywords Pantographic structures · Metamaterials · Continuum mechanics · Second gradient theory · Microstructure · Variational principles

13.1 Introduction

Recently, the literature has seen a proliferation of scientific contributions concerning so-called pantographic structures (dell'Isola et al. 2016; Boutin et al. 2017; Eremeyev et al. 2018, 2019; Giorgio et al. 2019; Rahali et al. 2015; Andreaus et al. 2018; Barchiesi et al. 2019a; Scerrato and Giorgio 2019). The mechanical properties of such objects are reported to be exotic and very high-performance. In this short chapter, we

M. Spagnuolo (✉)
Dipartimento di Ingegneria Civile, Ambientale e Architettura (DICAAR), Universitá degli Studi di Cagliari, Cagliari, Italy
e-mail: mario.spagnuolo@unica.it

M. Spagnuolo · E. Barchiesi
International Research Center for the Mathematics and Mechanics of Complex Systems, University of L'Aquila, L'Aquila, Italy

E. Barchiesi
École Nationale d'Ingénieurs de Brest, ENIB, UMR CNRS 6027, IRDL, 29200 Brest, France

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would like to attempt to answer the insistent question that is asked by some scholars when topics related to pantographic structures are presented at scientific congresses: do we really need pantographic structures?

In order to answer this question, we will need to retrace in an organic manner the path that led to the formulation of the theory governing these objects. We will have to give a clear and agreeable definition of what a metamaterial is Barchiesi et al. (2019b), Abali and Yang (2019), Carcaterra et al. (2015), dell'Isola and Steigmann (2020), Eugster et al. (2019), Giorgio et al. (2020), Yang et al. (2018). In the end, we will have to show, as in any worthwhile theory, what is the real theoretical necessity behind the existence of such devices.

As we will see specifically, the environment in which pantographic structures have been developed has to be researched back to the split that originated in the nineteenth century, when the school of Continuum Mechanics related to Cauchy, Navier and Poisson prevailed over that of Lagrangian, D'Alembertian and Piolan inspiration (Lagrange 1853, 1806; dell'Isola et al. 2014, 2019, 2015a).

In order to be able to address this study in a consistent and self-contained manner, it is necessary to specify, at least in brief, what a pantographic structure is. A pantographic structure consists of a planar grid made up of two families of continuous fibres oriented orthogonally and interconnected by hinges located at the intersections. From a purely theoretical point of view, the mechanical behaviour of pantographic structures is treated in the formal context of higher gradient continua, i.e. continua whose deformation energies depend on higher gradients of the displacement field, as opposed to the Cauchy continuum in which the deformation energy is only a function of the first gradient of displacement.

13.2 Metamaterials Are (Natural) Materials *on Demand*

In its formulation due mainly to Lagrange, Continuum Mechanics studies how the equilibrium shapes of a continuum body are modified by external interactions. A given body is assumed to consist, at each of its material points, of a specific material. The actual shape of such a body is mathematically modelled by means of a placement function and, in its elastic deformation range, by the corresponding deformation energy density, which objectively depends on the placement gradient. Further constitutive functions and kinematic descriptors can be introduced for the modelling of damage (Cuomo 2019; Misra and Poorsolhjouy 2020; Placidi 2015; Placidi et al. 2018; Spagnuolo et al. 2017) and plastic phenomena (Altenbach and Eremeyev 2014; Bertram 2015). The adoption of the above mathematical context is important in order to generalise the concepts of Continuum Mechanics to non-standard materials generally referred to as artificial or non-natural. Such a change of perspective, in fact, makes the expression "natural material" completely meaningless. In fact, from a purely modelling point of view, we can only speak of materials with a simple microstructure and materials with a complex microstructure. In the context of Continuum Mechanics based on the definition of the deformation energy density associated with a given material, every material that can exist is natural by definition and what changes is only its mechanical behaviour, which is questioned if it can be described by certain constitutive functions.

To answer the question that underlies this chapter, we are interested in placing pantographic structures, or pantographic material as it is often referred to, in the proper context of generalised theories in Continuum Mechanics. For this reason, we are interested in discussing the positioning of theories concerning metamaterials in Continuum Mechanics. To this end, the definition given of a metamaterial is of fundamental importance. A definition that is often found in the literature and that, for what we have said so far, seems paradoxical consists in defining the theory of metamaterials as the theory of those materials that are not natural: but, we have already pointed out, all materials are by definition natural. Another possibility also often found in the literature (Seppecher et al. 2011) is to define metamaterials as those materials whose mechanical behaviour is "exotic". This definition, however, is also easily attacked. Indeed, it is necessary to specify what we mean by exotic mechanical behaviour. It seems reasonable to argue that exotic mechanical behaviour is a type of behaviour that has not yet been experimentally observed. Of course, what is exotic at one point in time may become standard at another. For example, Lamé, Navier, Cauchy, Poisson, all regarded a material with a negative Poisson's ratio as very exotic, and some scholars of their time even believed that such a material was not physical and could not exist. However, auxetic metamaterials do exist and play an important role in modern engineering (Evans 1991; Evans and Alderson 2000).

The cited approach to the theory of metamaterials produces several misunderstandings that can lead the scientific investigation to results completely detached from reality. Indeed, claiming that a given mechanical behaviour is standard as opposed to another implies that one is confusing a mathematical model for a material with the physical material itself, and that one is implicitly assuming that particular assumptions accepted to describe particular phenomena are universally valid in every physical situation. This attitude, as well as being unscientific, does not lead to any advancement in scientific research and is, therefore, to be avoided. The confusion between model and modelled object has led over time to conceptual statements that are completely cacophonous. The example we want to give here consists of the assumption that "materials described by second-gradient models do not exist because the materials used in engineering do not show their properties and the standard theoretical framework does not envisage them". Those who agree with the above statement are clearly confusing first gradient models (model belonging to a symbolic-mathematical description) with materials existing in nature (physical object describable by a model). Exaggerating this non-scientific attitude to its extreme consequences, one may come to believe that, without having a theory to describe it, one cannot use a material even though he has it in his hands, and may even believe that a certain material does not exist. This is maybe the main reason for which we need a metamaterial as the pantographic one: its existence demonstrates by itself that the first gradient continua do not model every existing material.

In view of what has been said so far, we can now try to give an operational definition of what a metamaterial is. A metamaterial is a material on demand: we establish a priori the mechanical behaviour we wish to observe in such a material, and only secondarily we search for a microstructure that, following a homogenisation procedure, can present the required mechanical behaviour at a macroscopic level.

13.3 Second Gradient Theories

As we have observed, in the self-proclaimed standard school of Continuum Mechanics there is no place for theories other than those studied in the tradition of Caucy, Navier and Poisson. Gabrio Piola introduced in 1848 a generalised continuum model by means of deformation energies dependent on the nth gradient of the placement (Piola 1846). However, this type of model did not fit into the orthodox formulation of Continuum Mechanics à la Cauchy. Paradigmatic, in this context, is the unanimous agreement by Cauchy and his acolytes of the so-called "Cauchy postulate", which asserts that the contact forces, within continua, can only be forces per unit area depending only on the normal to the Cauchy cuts.

In Cauchy's version of Continuum Mechanics some ad hoc restrictions are included, among which is the fact that the deformation energy of a continuum medium can only depend on the first gradient of the displacement field. A priori, nothing would prevent a dependence on higher order gradients, but the simplest choice, coherent with the phenomenology shown by Cauchy's continuum model, is to restrict to the first gradient of the displacement. Piola, on the other hand, introduces, for a mere rational demand, the higher gradients of displacement in the calculation of the deformation energy, arguing for characterising those microstructures for which the homogenised models must be of this more general type. In Alibert et al. (2003), Seppecher et al. (2011), dell'Isola and Seppecher (1995, 1997), Pideri and Seppecher (1997), Seppecher (1989, 2000) it is shown that models in which the second gradient of displacement takes on a non-negligible role, at the macroscopic level, are obtained by homogenisation from a microstructure, or architecture, at a lower scale in a continuum medium where high stiffness contrasts are present.

From what we have said, it seems clear that in order to be able to evaluate and observe experimentally effects that can be assimilated to a description by means of the second gradient of the displacement field, it is necessary to have a technology capable of producing a microstructured material (Spagnuolo et al. 2019; Altenbach and Eremeyev 2014; Eremeyev et al. 2012) and, above all, a material whose microstructure shows the appropriate highly contrasted stiffness fields, so that, at the macroscopic level, the terms used by Piola appear in the deformation energy. In the following we will shortly show how Paul Germain demonstrates that the presence of a microstructure can determine, at the macroscopic level, the necessity of using a second-gradient model (Germain 1973, 2020; Epstein and Smelser 2020).

The technological capacity of an era can also block its scientific development. As long as technology does not reach a sufficient level to test the results introduced in the new theories, the new theories will remain blocked, ignored and, certainly, unusable. The absurdity of the contemporary situation consists in the fact that regardless of the technological ability to produce materials whose behaviour is described by Piola's theory (and cannot be described within the framework of Cauchy's models), there are actually still scholars who insist on denying its usefulness.

Stressing a concept we have already mentioned before, this last remark partly answers the question we asked at the beginning of this chapter: do we really need pantographic structures? From a purely theoretical point of view, the answer is clearly affirmative, because pantographic structures have been expressly designed as a material that can be described fundamentally in terms of a second-gradient theory: if we can fabricate a material that can be described by means of a given theory and not by means of the theory generally adopted in the description of materials, then this theory becomes indispensable and it is absolutely inconceivable that it should not be applicable.

13.4 Microstructure in Continuum Mechanics

At present, with recently developed and improved techniques, the fabrication of materials with complex microstructures is not as implausible as before. As we have specified above, due to the advancements in the field of additive manufacturing, it is now possible to produce microstructured metamaterials exhibiting mechanical properties that cannot be described in the context of Cauchy's Continuum Mechanics.

Here we also want to address an issue concerning the terminology adopted in the field of metamaterials. Some scholars claim that the term microstructure cannot be used because it refers to a scale of micrometres, whereas it is more appropriate to refer to architecture. However, in our opinion, this associates the metamaterial with an artefact connotation that is not of fundamental interest in the theory. More specifically, the relevant point is to understand what determines the mechanical properties of the so-called standard or natural materials: clearly the difference between a cubic material and a tetragonal one, just to give an example, is in the geometry according to which the "particles" that compose it are arranged, i.e. in its microstructure. Now some people refer to architecture in the case of metamaterials because the present technological capabilities do not yet allow manufacture on scales comparable to those currently existing in nature. But the idea is the same as that underlying the differences between the mechanical behaviour of cubic and tetragonal materials: the mechanical behaviour of a metamaterial is determined by the geometric arrangement of its elementary constituents. This is why, in our opinion, distinguishing architecture from microstructure is only misleading.

As we have seen above, Piola was the first to study continuum models in which the deformation energy depends on higher gradients of the displacement field. Later, in the twentieth century, this kind of models was also studied and reformulated by various authors. Among others, we mention in particular two pioneering studies presented by Mindlin (1965) and by Germain (1973, 2020); Epstein and Smelser (2020). A very interesting aspect that emerges from these studies, and of course from others, is that they show how the existence of the microstructure in some cases could induce higher order terms in the equilibrium equations of the material under consideration. Unlike classical homogenisation techniques, equations containing terms dependent on second or higher order derivatives of the displacement are obtained in this case, thus introducing higher gradient theories in a logical way.

Germain shows in general that, by applying the Principle of Virtual Power (analogous to the Principle of Virtual Work), the classical equations of the Continuum Mechanics are easily obtained. When considering a microstructured continuum, these equations have terms that depend on the second gradient of the displacement field. In this context, it is crucial to assign the right kinematics. Therefore, in the case of a usual continuum, this is considered to consist of a continuum distribution of particles geometrically represented by a material point and its velocity components. When considering the microstructure, from a macroscopic point of view each particle is still represented by a material point, but its kinematics must be defined more precisely. The main feature of the method explained by Germain is that, having assigned the required kinematics, the associated continuum theory can be deduced immediately through the Principle of Virtual Work. He shows that the kinematics due to the presence of the microstructure generates a second-gradient continuum at the macroscopic level. We refer to the original work by Germain for the technical details (Germain 1973, 2020; Epstein and Smelser 2020).

This can be considered the starting point in the study of pantographic structures. A certain microstructure is chosen to get a second-gradient continuum as simple as possible and then homogenisation techniques are used to determine this appropriate macroscopic continuum model.

13.4.1 The Synthesis Problem

The real mathematical challenge facing us today, therefore, is to design metamaterials that can be described within the framework of a generalised theory (Maugin 2011; Altenbach et al. 2010; Altenbach and Eremeyev 2010, 2013; Altenbach et al. 2013; Auffray 2015; dell'Isola et al. 2009; Eremeyev and dell'Isola 2018). Thus, the fundamental problem in the theory of metamaterials consists in the problem of the synthesis of microstructures that produce a certain desired macro-behaviour (Rahali et al. 2015; Placidi et al. 2020; Khakalo and Niiranen 2020; Abdoul-Anziz et al. 2019). As we have seen briefly, in this context, the most complex problem to address from a mathematical point of view is to link microstructures and macro-behaviours. This is done in terms of mathematical procedures called homogenisation: starting from the elementary constituents and the basic cell of the chosen microstructure, one must link this microstructure to the given macroscopic theory (appropriate action functions and consequent stationarity conditions) chosen a priori, and this, in order to be of some general use, must be approached in an algorithmic manner, thus allowing generalisation.

The basic ideas in the field of metamaterial synthesis can be deduced by analogy from the theory of analogue circuit synthesis. In this theory, it can be shown that any passive linear element can be synthesised algorithmically using inductors, capacitors, resistors and transformers (Bloch 1944, 1945; Kron 1945; Mablekos and Weidmann 1968). The main challenge is to conjecture that this method can also be applied to the synthesis of non-linear (and multiphysical) mechanical systems (Spagnuolo 2020; Spagnuolo and Scerrato 2020).

13.5 Why We Really Need Pantographic Structures

Up to now we have tried to exhibit the background needed to answer a question often repeated as a result of the enormous development of literature on pantographic structures: do we really need pantographic structures? For what purpose do they serve? There are many arguments advanced on this topic which are commonly referred against second and higher gradient models: some scholars observe that the extremely formal mathematical investigation required to formulate the theory underlying this type of metamaterial is unnecessarily over-discussed and studied, and that there are no practical applications of the object studied; others point out that, on the contrary, from a theoretical point of view it is not necessary to introduce second-gradient theories, because they are useless mathematical complications, but that the classical Cauchy theory on its own is able to explain and represent all the phenomenology observed in Continuum Mechanics; finally, some argue that there are many other metamaterials much more interesting than the pantographic one. We limit ourselves to observe that one of the most cited works in the field of metamaterials (Bertoldi et al. 2010), the work of researchers belonging to the Harvard intelligentsia, seems to us to be rather weak and basically a patchwork of experimental results obtained for a microstructure that produces a material with auxetic mechanical behaviour.

But it is not our purpose to comment on the null and void scientific contribution of the various power groups of the moment. Instead, we are interested in precisely defining the reasons why pantographic structures are actually worth studying. Basically, we can divide the positive scientific contributions resulting from the development of this type of metamaterials into four areas: theoretical, practical, methodological and multiphysical.

13.5.1 The Existence of Pantographic Metamaterial Motivates the Need of Second Gradient Theories

From a theoretical and methodological point of view, the importance of pantographic structures seems indisputable. In fact, consider the fundamental objection that has historically been made to second-gradient theories: they are not necessary, since the materials can be very well described within the framework of the classical Cauchy first gradient theory, possibly by adding *ad hoc* corrections. If, then, we are able to show a material not describable by means of a first gradient theory, but only by invoking a second-gradient one, then this is enough to motivate the necessity of such theories. We want, accordingly, to postulate the existence of such a material by writing its governing equations, i.e. its deformation energy, which should be dependent of the second gradient of displacement. We subsequently ask which microstructure can produce this macroscopic deformation energy after homogenisation. This approach is methodologically the reverse of that used in the majority of works on metamaterials, including the aforementioned (Bertoldi et al. 2010). We therefore ask what characteristics a microstructure must have in order to produce a second-gradient continuum at the macroscopic level.

Methodologically we start from the following observation, which is certainly superficial, but definitely indicative to try to define a microstructure suitable to produce a second-gradient mechanical behaviour: if we consider an Euler-Bernoulli beam, even a linear one, its deformation energy can be separated into two components, elongation and bending. The elongation energy depends on the first derivative of the longitudinal displacement, while the bending energy depends on the second derivative of the transverse displacement. In a sense, we can say that the Euler-Bernoulli beam is first gradient in extension and second gradient in bending. We want to produce a second-gradient material at extension. We can therefore conceive an assembly of beams such that a macroscopic extension action corresponds to a bending action from a microscopic point of view. In other words, by extending or compressing the metamaterial we are flexing the fibres that make up its microstructure. In this way, the deformation energy of the metamaterial subjected to extensional load should correspond to a "microscopic" bending energy and consequently to a second-gradient energy. This is how the pantographic microstructure has been originated.

13.5.2 A Mechanical Diode

From a practical point of view, we would like to highlight just one aspect that seems extremely promising with respect to the possible applications of the pantographic metamaterial. Consider that, due to the peculiar microstructure chosen, from the point of view of mechanical behaviour, we observe a phenomenology similar to that presented by the diode in the analysis of electrical circuits. The diode is a circuit element that exhibits an extremely particular voltage-current response. In various applications of interest, the voltage-current behaviour of an ideal diode, under static conditions, can be approximated by a linear piecewise function. In this approximation, the current can be considered to be zero if the voltage between anode and cathode is less than or equal to a certain threshold value V_γ ; if, on the contrary, the voltage is higher, the diode can be approximated to a voltage generator, whose current is imposed by the circuit to which it is subordinated. In the field of mechanics, a response formally identical to that exhibited by the diode in electrical circuits is shown by the pantographic metamaterial. If the hinges connecting the fibres of the two families are perfect, and therefore no deformation energy is associated with them, then considering a bias-extension test of a pantographic structure the following response will be observed in terms of a force-displacement measurement: up to a certain imposed displacement a very low value of the reaction force will be measured; after a threshold value of the displacement, the force will begin to increase considerably. The shape of the observed curve is reminiscent of the voltage-current diagram of the diode. In this sense, one can refer to the pantographic metamaterial as a kind of mechanical diode.

This mechanical behaviour can be useful when one wants to insert a mechanical element into a structure which does not function directly as a spring, but only exhibits linear elastic behaviour after a certain elongation threshold value (at least for deformations not too far beyond the threshold value).

The explanation for this unusual mechanical behaviour lies in the deformation of the microstructure: in a bias-extension test, in a first phase the predominantly observed deformation corresponds to the bending of the fibres clamped at the ends of the pantographic structure (whereas the unclamped fibres are simply free to rotate). This bending is concentrated in very precise areas of the pantographic structure and gives rise, from a theoretical point of view, to the bending term modelled by means of the second gradient of placement. It is common experience that a beam is easier to bend than to stretch, so it is easy to agree that the bending energy of the pantographic structure (beam assembly) is certainly lower than the elongation energy. However, since the extension test is conducted along a direction in which the fibres are biased, then the first mechanism to occur corresponds to the bending of the fibres, whereas their elongation only begins when the rotating fibres touch (which corresponds to the threshold value above which the measured force begins to increase significantly). One can refer to Spagnuolo (2020), Spagnuolo and Scerrato (2020) for more details.

13.5.3 An Iterative Algorithm for Synthesising Metamaterials

From a methodological-theoretical point of view, we want to underline an important aspect recalled in Alibert et al. (2003), Seppecher et al. (2011). Once the pantographic microstructure has been obtained, which at the macroscopic level produces the required second-gradient behaviour by means of a suitable homogenisation, one can think of generalising this procedure to the production of metamaterials described by energies dependent on higher displacement gradients. In Alibert et al. (2003), Seppecher et al. (2011) it is shown how, by using a Warren bridge microstructure in association with a pantographic microstructure, it is possible to obtain a third gradient-in-bending material. With an iterative procedure it is suggested that one can proceed from $(2n + 1)$-th gradient-in-bending materials to $(2n + 2)$-th gradient-in-extension materials. This problem, only mentioned in Alibert et al. (2003), Seppecher et al. (2011), turns out to be of utmost importance in the synthesis of new metamaterials.

13.6 Conclusion

In this chapter, we have briefly discussed some of the fundamental motivations behind the development of pantographic metamaterial. We have established that studies of this metamaterial are strongly motivated from several points of view: from a theoretical point of view, it demonstrates the necessity of the introduction of second-gradient models; from a practical point of view, this metamaterial possesses extremely peculiar characteristics and, therefore, is worthy of detailed study; from a methodological point of view, it offers a useful example in the field of procedures for the synthesis of new metamaterials.

These reasons that we have listed and discussed lead us to believe that pantographic structures may provide a new class of materials worth studying in depth. In fact, there are several indications that the phenomenology of this metamaterial is very variegated. Here we limit ourselves to a few cases of extreme interest: the observation of Poynting reversal effects for torsion tests conducted on pantographic structures (Misra et al. 2018; Auger et al. 2020); the observation of wave phenomena in the case of very dense mesh structures of sufficiently large length (dell'Isola et al. 2015b); the study of the distribution of displacement and velocity fields for high-frequency vibratory phenomena (Laudato et al. 2018; Barchiesi et al. 2018; Laudato and Barchiesi 2019; Laudato et al. 2020); the extremely non-standard phenomenology in the case of three-point bending tests (Yildizdag et al. 2020).

The enormous amount of phenomenology and experimental observations obtained on pantographic structures require the development of precise methods of analysis, such as those based on Digital Image Correlation, already applied to the case of this metamaterial with good results (Hild et al. 2020; Barchiesi et al. 2020b, a), and of numerical implementation of theoretical models in order to conduct precise model validation. This also requires the development of numerical methods capable of simulating the cases of interest (Cazzani and Atluri 1993; Cazzani and Lovadina 1997; Cazzani et al. 2016c, b, a, 2020; Cuomo et al. 2014; Greco and Cuomo 2013, 2014; Maurin et al. 2019; Capobianco et al. 2018; Turco et al. 2017, 2018, 2019b, a; Turco and Barchiesi 2019; Schulte et al. 2020; Hesch et al. 2017; Capobianco and Eugster 2018; Eugster and Glocker 2013; Huang et al. 2020, 2021; Barchiesi et al. 2020c; Yang et al. 2019; Jafarzadeh et al. 2020; Namnabat et al. 2020; Rahbar et al. 2020). The methods and reasoning developped for pantographic structures may also be interesting for other applications, such as civil engineering (Vaiana et al. 2021, 2019; Serpieri et al. 2018; Sessa et al. 2019a, 2017, 2018a, 2019b, 2018b, 2015; Cricri et al. 2015; Greco et al. 2018; Perricone et al. 2020; Marmo et al. 2018a, b, 2019; Vaiana et al. 2017; Paradiso et al. 2019).

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