Some q-Exponential Formulas Involving the Double Lowering Operator ψ for a Tridiagonal Pair (Research)

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1 Introduction

Throughout this paper, $\mathbb{K}$ denotes an algebraically closed field. We begin by recalling the notion of a tridiagonal pair. We will use the following terms. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. For a linear transformation $A : V \to V$ and a subspace $W \subseteq V$, we say that W is an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{K}$ such that $W = \{v \in V | Av = \theta v\}$. In this case, θ is called the *eigenvalue* of A associated with W. We say that A is *diagonalizable* whenever V is spanned by the eigenspaces of A.

Definition 1 ([9, Definition 1.1]) Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *tridiagonal pair* (or *TD pair*) on V we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions.

(i) Each of A, A^* is diagonalizable.
(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d), \tag{1}$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.
(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq \delta), \tag{2}$$

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where $V^*_{-1} = 0$ and $V^*_{\delta+1} = 0$.

(iv) There does not exist a subspace W of V such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

We say the pair A, A^* is *over* $\mathbb{K}$.

Note 1 According to a common notational convention A^* denotes the conjugate-transpose of A. We are not using this convention. In a TD pair A, A^* the linear transformations A and A^* are arbitrary subject to (i)–(iv) above.

Referring to the TD pair in Definition 1, by [9, Lemma 4.5] the scalars d and δ are equal. We call this common value the *diameter* of A, A^*. To avoid trivialities, throughout this paper we assume that the diameter is at least one.

TD pairs first arose in the study of Q-polynomial distance-regular graphs and provided a way to study the irreducible modules of the Terwilliger algebra associated with such a graph. Since their introduction, TD pairs have been found to appear naturally in a variety of other contexts including representation theory [1, 7, 10–12, 14, 15, 25], orthogonal polynomials [23, 24], partially ordered sets [22], statistical mechanical models [3, 6, 19], and other areas of physics [16, 18]. As a result, TD pairs have become an area of interest in their own right. Among the above papers on representation theory, there are several works that connect TD pairs to quantum groups [1, 5, 7, 11, 12]. These papers consider certain special classes of TD pairs. We call particular attention to [5], in which the present author describes a new relationship between TD pairs in the q-Racah class and quantum groups. The present paper builds off of this work.

In the present paper, we give a new relationship between the maps $\Delta, \psi : V \to V$ introduced in [4], as well as describe a new decomposition of the underlying vector space that, in some sense, lies between the first and second split decompositions associated with a TD pair. In order to motivate our results, we now recall some basic facts concerning TD pairs. For the rest of this section, let A, A^* denote a TD pair on V, as in Definition 1. Fix an ordering $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) of the eigenspaces of A (resp. A^*) which satisfies (1) (resp. (2)). For $0 \leq i \leq d$ let θ_i (resp. θ^*_i) denote the eigenvalue of A (resp. A^*) corresponding to V_i (resp. V^*_i). By [9, Theorem 11.1] the ratios

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are equal and independent of i for $2 \leq i \leq d-1$. This gives two recurrence relations, whose solutions can be written in closed form. There are several cases [9, Theorem 11.2]. The most general case is called the q-Racah case [12, Section 1]. We will discuss this case shortly.

We now recall the split decompositions of V [9]. For $0 \leq i \leq d$ define

$$U_i = (V^*_0 + V^*_1 + \cdots + V^*_i) \cap (V_i + V_{i+1} + \cdots + V_d),$$

$$U_i^{\Downarrow} = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_0 + V_1 + \cdots + V_{d-i}).$$

By [9, Theorem 4.6], both the sums $V = \sum_{i=0}^{d} U_i$ and $V = \sum_{i=0}^{d} U_i^{\Downarrow}$ are direct. We call $\{U_i\}_{i=0}^{d}$ (resp. $\{U_i^{\Downarrow}\}_{i=0}^{d}$) the first split decomposition (resp. second split decomposition) of V. In [9], the authors showed that A, A^* act on the first and second split decomposition in a particularly attractive way. This will be described in more detail in Sect. 3.

We now describe the q-Racah case. We say that the TD pair A, A^* has *q-Racah type* whenever there exist nonzero scalars $q, a, b \in \mathbb{K}$ such that $q^4 \neq 1$ and

$$\theta_i = aq^{d-2i} + a^{-1}q^{2i-d}, \qquad \theta_i^* = bq^{d-2i} + b^{-1}q^{2i-d}$$

for $0 \leq i \leq d$. For the rest of this section assume that A, A^* has q-Racah type.

We recall the maps K and B [13, Section 1.1]. Let $K : V \to V$ denote the linear transformation such that for $0 \leq i \leq d$, U_i is an eigenspace of K with eigenvalue q^{d-2i}. Let $B : V \to V$ denote the linear transformation such that for $0 \leq i \leq d$, $U_i^{\Downarrow}$ is an eigenspace of B with eigenvalue q^{d-2i}. The relationship between K and B is discussed in considerable detail in [5].

We now bring in the linear transformation $\Psi : V \to V$ [4, Lemma 11.1]. As in [5], we work with the normalization $\psi = (q - q^{-1})(q^d - q^{-d})\Psi$. A key feature of ψ is that by [4, Lemma 11.2, Corollary 15.3],

$$\psi U_i \subseteq U_{i-1}, \qquad \psi U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow}$$

for $1 \leq i \leq d$ and both $\psi U_0 = 0$ and $\psi U_0^{\Downarrow} = 0$. In [5], it is shown how ψ is related to several maps, including the maps K, B, as well as the map Δ which we now recall. By [4, Lemma 9.5], there exists a unique linear transformation $\Delta : V \to V$ such that

$$\begin{aligned} \Delta U_i &\subseteq U_i^{\Downarrow} && (0 \leq i \leq d), \\ (\Delta - I)U_i &\subseteq U_0 + U_1 + \cdots + U_{i-1} && (0 \leq i \leq d). \end{aligned}$$

In [4, Theorem 17.1], the present author showed that both

$$\Delta = \sum_{i=0}^{d} \left(\prod_{j=1}^{i} \frac{aq^{j-1} - a^{-1}q^{1-j}}{q^j - q^{-j}} \right) \psi^i, \quad \Delta^{-1} = \sum_{i=0}^{d} \left(\prod_{j=1}^{i} \frac{a^{-1}q^{j-1} - aq^{1-j}}{q^j - q^{-j}} \right) \psi^i.$$

The primary goal of this paper is to provide factorizations of these power series in ψ and to investigate the consequences of these factorizations. We accomplish this goal using a linear transformation $\mathcal{M} : V \to V$ given by

$$\mathcal{M} = \frac{aK - a^{-1}B}{a - a^{-1}}.$$

By construction, $\mathcal{M}^{\Downarrow} = \mathcal{M}$. One can quickly check that $\mathcal{M}$ is invertible. We show that the map $\mathcal{M}$ is equal to each of

$$(I - a^{-1}q\psi)^{-1}K, \quad K(I - a^{-1}q^{-1}\psi)^{-1}, \quad (I - aq\psi)^{-1}B, \quad B(I - aq^{-1}\psi)^{-1}.$$

We give a number of different relations involving the maps $\mathcal{M}$, K, B, ψ, the most significant of which are the following:

$$K \exp_q\left(\tfrac{a^{-1}}{q-q^{-1}}\psi\right) = \exp_q\left(\tfrac{a^{-1}}{q-q^{-1}}\psi\right)\mathcal{M},$$
$$B \exp_q\left(\tfrac{a}{q-q^{-1}}\psi\right) = \exp_q\left(\tfrac{a}{q-q^{-1}}\psi\right)\mathcal{M}.$$

Using these equations, we obtain our main result which is that both

$$\Delta = \exp_q\left(\tfrac{a}{q-q^{-1}}\psi\right)\exp_{q^{-1}}\left(-\tfrac{a^{-1}}{q-q^{-1}}\psi\right),$$
$$\Delta^{-1} = \exp_q\left(\tfrac{a^{-1}}{q-q^{-1}}\psi\right)\exp_{q^{-1}}\left(-\tfrac{a}{q-q^{-1}}\psi\right).$$

Due to its important role in the factorization of Δ, we explore the map $\mathcal{M}$ further. We show that $\mathcal{M}$ is diagonalizable with eigenvalues $q^d, q^{d-2}, q^{d-4}, \ldots, q^{-d}$. For $0 \le i \le d$, let W_i denote the eigenspace of $\mathcal{M}$ corresponding to the eigenvalue q^{d-2i}. We show that for $0 \le i \le d$,

$$U_i = \exp_q\left(\tfrac{a^{-1}}{q-q^{-1}}\psi\right)W_i, \qquad U_i^{\Downarrow} = \exp_q\left(\frac{a}{q-q^{-1}}\psi\right)W_i,$$

$$W_i = \exp_{q^{-1}}\left(-\tfrac{a^{-1}}{q-q^{-1}}\psi\right)U_i, \qquad W_i = \exp_{q^{-1}}\left(-\frac{a}{q-q^{-1}}\psi\right)U_i^{\Downarrow}.$$

In light of this result, we interpret the decomposition $\{W_i\}_{i=0}^d$ as a sort of halfway point between the first and second split decompositions. We explore this decomposition further and give the actions of ψ, K, B, Δ, A, A^* on $\{W_i\}_{i=0}^d$. We then give the actions of $\mathcal{M}^{\pm 1}$ on $\{U_i\}_{i=0}^d$, $\{U_i^{\Downarrow}\}_{i=0}^d$, $\{V_i\}_{i=0}^d$, $\{V_i^*\}_{i=0}^d$. We conclude the paper with a discussion of the special case when A, A^* is a Leonard pair.

The present paper is organized as follows. In Sect. 2 we discuss some preliminary facts concerning TD pairs and TD systems. In Sect. 3 we discuss the split decompositions of V as well as the maps K and B. In Sect. 4 we discuss the map ψ. In Sect. 5 we recall the map Δ and give Δ as a power series in ψ. In Sect. 6 we introduce the map $\mathcal{M}$ and describe its relationship with A, K, B, ψ. In Sect. 7 we express Δ as a product of two linear transformations; one is a q-exponential in ψ and the other is a q^{-1}-exponential in ψ. In Sect. 8 we describe the eigenvalues and eigenspaces of $\mathcal{M}$ and discuss how the eigenspace decomposition of $\mathcal{M}$ is related to the first and second split decompositions. In Sect. 9 we discuss the actions of ψ, K, B, Δ, A, A^* on the eigenspace decomposition of $\mathcal{M}$. In Sect. 10 we describe the action of $\mathcal{M}$ on the first and second split decompositions of V, as well as on the eigenspace decompositions of A, A^*. In Sect. 11 we consider the case when A, A^* is a Leonard pair.

2 Preliminaries

When working with a tridiagonal pair, it is useful to consider a closely related object called a tridiagonal system. In order to define this object, we first recall some facts from elementary linear algebra [9, Section 2].

We use the following conventions. When we discuss an algebra, we mean a unital associative algebra. When we discuss a subalgebra, we assume that it has the same unit as the parent algebra.

Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *decomposition* of V, we mean a sequence of nonzero subspaces whose direct sum is V. Let $\text{End}(V)$ denote the $\mathbb{K}$-algebra consisting of all linear transformations from V to V. Let A denote a diagonalizable element in $\text{End}(V)$. Let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A. For $0 \leq i \leq d$ let θ_i be the eigenvalue of A corresponding to V_i. Define $E_i \in \text{End}(V)$ by $(E_i - I)V_i = 0$ and $E_i V_j = 0$ if $j \neq i$ $(0 \leq j \leq d)$. In other words, E_i is the projection map from V onto V_i. We refer to E_i as the *primitive idempotent* of A associated with θ_i. By elementary linear algebra, (i) $AE_i = E_i A = \theta_i E_i$ $(0 \leq i \leq d)$; (ii) $E_i E_j = \delta_{ij} E_i$ $(0 \leq i, j \leq d)$; (iii) $V_i = E_i V$ $(0 \leq i \leq d)$; (iv) $I = \sum_{i=0}^d E_i$. Moreover

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \qquad (0 \leq i \leq d).$$

Let M denote the subalgebra of $\text{End}(V)$ generated by A. Note that each of $\{A^i\}_{i=0}^d$, $\{E_i\}_{i=0}^d$ is a basis for the $\mathbb{K}$-vector space M.

Let A, A^* denote a TD pair on V. An ordering of the eigenspaces of A (resp. A^*) is said to be *standard* whenever it satisfies (1) (resp. (2)). Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of A. By [9, Lemma 2.4], the ordering $\{V_{d-i}\}_{i=0}^d$ is standard and no further ordering of the eigenspaces of A is standard. A similar result holds for the eigenspaces of A^*. An ordering of the primitive idempotents of A (resp. A^*) is said to be *standard* whenever the corresponding ordering of the eigenspaces of A (resp. A^*) is standard.

Definition 2 ([17, Definition 2.1]) Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *tridiagonal system* (or *TD system*) on V, we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

that satisfies (i)–(iii) below.

(i) A, A^* is a tridiagonal pair on V.
(ii) $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A.
(iii) $\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A^*.

We call d the *diameter* of Φ, and say Φ is *over* $\mathbb{K}$. For notational convenience, set $E_{-1} = 0$, $E_{d+1} = 0$, $E_{-1}^* = 0$, $E_{d+1}^* = 0$.

In Definition 2 we do not assume that the primitive idempotents $\{E_i\}_{i=0}^d$, $\{E_i^*\}_{i=0}^d$ all have rank 1. A TD system for which each of these primitive idempotents has rank 1 is called a Leonard system [20]. The Leonard systems are classified up to isomorphism [20, Theorem 1.9].

For the rest of this paper, fix a TD system Φ on V as in Definition 2. Our TD system Φ can be modified in a number of ways to get a new TD system [9, Section 3]. For example, the sequence

$$\Phi^{\Downarrow} = (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

is a TD system on V. Following [9, Section 3], we call $\Phi^{\Downarrow}$ the *second inversion* of Φ. When discussing $\Phi^{\Downarrow}$, we use the following notational convention. For any object f associated with Φ, let $f^{\Downarrow}$ denote the corresponding object associated with $\Phi^{\Downarrow}$.

Definition 3 For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E_i^*). We refer to $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) as the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ.

By construction $\{\theta_i\}_{i=0}^d$ are mutually distinct and $\{\theta_i^*\}_{i=0}^d$ are mutually distinct. By [9, Theorem 11.1], the scalars

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \leq i \leq d-1$. For this restriction, the solutions have been found in closed form [9, Theorem 11.2]. The most general solution is called q-Racah [12, Section 1]. This solution is described as follows.

Definition 4 Let Φ denote a TD system on V as in Definition 2. We say that Φ has *q-Racah type* whenever there exist nonzero scalars $q, a, b \in \mathbb{K}$ such that such that $q^4 \neq 1$ and

$$\theta_i = aq^{d-2i} + a^{-1}q^{2i-d}, \qquad \theta_i^* = bq^{d-2i} + b^{-1}q^{2i-d} \tag{3}$$

for $0 \leq i \leq d$.

Note 2 Referring to Definition 4, the scalars q, a, b are not uniquely defined by Φ. If q, a, b is one solution, then their inverses give another solution.

For the rest of the paper, we make the following assumption.

Assumption 1 *We assume that our TD system Φ has q-Racah type. We fix q, a, b as in Definition 4.*

Lemma 1 ([5, Lemma 2.4]) *With reference to Assumption 1, the following hold.*

(i) *Neither of a^2, b^2 is among $q^{2d-2}, q^{2d-4}, \ldots, q^{2-2d}$.*
(ii) $q^{2i} \neq 1$ *for* $1 \leq i \leq d$.

Proof The result follows from the comment below Definition 3. □

3 The First and Second Split Decomposition of *V*

Recall the TD system Φ from Assumption 1. In this section we consider two decompositions of V associated with Φ, called the first and second split decomposition.

For $0 \leq i \leq d$ define

$$U_i = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_iV + E_{i+1}V + \cdots + E_dV).$$

For notational convenience, define $U_{-1} = 0$ and $U_{d+1} = 0$. Note that for $0 \leq i \leq d$,

$$U_i^{\Downarrow} = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_0V + E_1V + \cdots + E_{d-i}V).$$

By [9, Theorem 4.6], the sequence $\{U_i\}_{i=0}^d$ (resp. $\{U_i^{\Downarrow}\}_{i=0}^d$) is a decomposition of V. Following [9], we refer to $\{U_i\}_{i=0}^d$ (resp. $\{U_i^{\Downarrow}\}_{i=0}^d$) as the *first split decomposition* (resp. *second split decomposition*) of V with respect to Φ. By [9, Corollary 5.7], for $0 \leq i \leq d$ the dimensions of E_iV, E_i^*V, U_i, $U_i^{\Downarrow}$ coincide; we denote the common dimension by ρ_i. By [9, Theorem 4.6],

$$E_iV + E_{i+1}V + \cdots + E_dV = U_i + U_{i+1} + \cdots + U_d, \quad (4)$$

$$E_0V + E_1V + \cdots + E_iV = U_{d-i}^{\Downarrow} + U_{d-i+1}^{\Downarrow} + \cdots + U_d^{\Downarrow}, \quad (5)$$

$$E_0^*V + E_1^*V + \cdots E_i^*V = U_0 + U_1 + \cdots + U_i = U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_i^{\Downarrow}. \quad (6)$$

By [9, Theorem 4.6], A and A^* act on the first split decomposition in the following way:

$$(A - \theta_i I)U_i \subseteq U_{i+1} \qquad (0 \leq i \leq d-1), \qquad (A - \theta_d I)U_d = 0,$$

$$(A^* - \theta_i^* I)U_i \subseteq U_{i-1} \qquad (1 \leq i \leq d), \qquad (A^* - \theta_0^* I)U_0 = 0.$$

By [9, Theorem 4.6], A and A^* act on the second split decomposition in the following way:

$$(A-\theta_{d-i}I)U_i^{\Downarrow} \subseteq U_{i+1}^{\Downarrow} \qquad (0 \le i \le d-1), \qquad (A-\theta_0 I)U_d^{\Downarrow} = 0,$$
$$(A^*-\theta_i^* I)U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow} \qquad (1 \le i \le d), \qquad (A^*-\theta_0^* I)U_0^{\Downarrow} = 0.$$

Definition 5 ([5, Definitions 3.1 and 3.2]) Define $K, B \in \mathrm{End}(V)$ such that for $0 \le i \le d$, U_i (resp. $U_i^{\Downarrow}$) is the eigenspace of K (resp. B) with eigenvalue q^{d-2i}. In other words,

$$(K-q^{d-2i}I)U_i = 0, \qquad (B-q^{d-2i}I)U_i^{\Downarrow} = 0 \qquad (0 \le i \le d). \tag{7}$$

Observe that $B = K^{\Downarrow}$.

By construction each of K, B is invertible and diagonalizable on V.

We now describe how K and B act on the eigenspaces of the other one.

Lemma 2 ([5, Lemma 3.3]) *For* $0 \le i \le d$,

$$(B-q^{d-2i}I)U_i \subseteq U_0 + U_1 + \cdots + U_{i-1}, \tag{8}$$
$$(K-q^{d-2i}I)U_i^{\Downarrow} \subseteq U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_{i-1}^{\Downarrow}. \tag{9}$$

Next we describe how A, K, B are related.

Lemma 3 ([13, Section 1.1]) *Both*

$$\frac{qKA - q^{-1}AK}{q-q^{-1}} = aK^2 + a^{-1}I, \qquad \frac{qBA - q^{-1}AB}{q-q^{-1}} = a^{-1}B^2 + aI. \tag{10}$$

Lemma 4 ([5, Theorem 9.9]) *We have*

$$aK^2 - \frac{a^{-1}q - aq^{-1}}{q-q^{-1}}KB - \frac{aq - a^{-1}q^{-1}}{q-q^{-1}}BK + a^{-1}B^2 = 0. \tag{11}$$

4 The Linear Transformation ψ

We continue to discuss the situation of Assumption 1. In [4, Section 11] we introduced an element $\Psi \in \mathrm{End}(V)$. In [5] we used the normalization $\psi = (q-q^{-1})(q^d - q^{-d})\Psi$. In [5, Theorem 9.8], we showed that ψ is equal to some rational expressions involving K, B. We now recall this result. We start with a comment.

Lemma 5 ([5, Lemma 9.7]) *Each of the following is invertible:*

$$aI - a^{-1}BK^{-1}, \qquad a^{-1}I - aKB^{-1}, \tag{12}$$

$$aI - a^{-1}K^{-1}B, \qquad a^{-1}I - aB^{-1}K. \tag{13}$$

Lemma 6 ([5, Theorem 9.8]) *The following four expressions coincide:*

$$\frac{I - BK^{-1}}{q(aI - a^{-1}BK^{-1})}, \qquad \frac{I - KB^{-1}}{q(a^{-1}I - aKB^{-1})}, \tag{14}$$

$$\frac{q(I - K^{-1}B)}{aI - a^{-1}K^{-1}B}, \qquad \frac{q(I - B^{-1}K)}{a^{-1}I - aB^{-1}K}. \tag{15}$$

In (14), (15) *the denominators are invertible by Lemma 5.*

Definition 6 Define $\psi \in \mathrm{End}(V)$ to be the common value of the four expressions in Lemma 6.

We now recall some facts concerning ψ.

Lemma 7 ([5, Lemma 5.4]) *Both*

$$K\psi = q^2\psi K, \qquad B\psi = q^2\psi B. \tag{16}$$

Lemma 8 ([4, Lemma 11.2, Corollary 15.3]) *We have*

$$\psi U_i \subseteq U_{i-1}, \qquad \psi U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow} \qquad (1 \le i \le d) \tag{17}$$

and also $\psi U_0 = 0$ *and* $\psi U_0^{\Downarrow} = 0$. *Moreover* $\psi^{d+1} = 0$.

In Lemma 6 we obtained ψ as a rational expression in BK^{-1} or $K^{-1}B$. Next we solve for BK^{-1} and $K^{-1}B$ as a rational function in ψ. In order to state the answer, we will need the following result.

Lemma 9 ([5, Lemma 9.2]) *Each of the following is invertible:*

$$I - aq\psi, \qquad I - a^{-1}q\psi, \qquad I - aq^{-1}\psi, \qquad I - a^{-1}q^{-1}\psi. \tag{18}$$

Their inverses are as follows:

$$(I - aq\psi)^{-1} = \sum_{i=0}^{d} a^i q^i \psi^i, \qquad (I - a^{-1}q\psi)^{-1} = \sum_{i=0}^{d} a^{-i} q^i \psi^i, \tag{19}$$

$$(I - aq^{-1}\psi)^{-1} = \sum_{i=0}^{d} a^i q^{-i} \psi^i, \qquad (I - a^{-1}q^{-1}\psi)^{-1} = \sum_{i=0}^{d} a^{-i} q^{-i} \psi^i. \tag{20}$$

The next result is an immediate consequence of Lemma 6, Definition 6, and Lemma 9.

Theorem 1 ([5, Theorem 9.4]) *The following hold:*

$$BK^{-1} = \frac{I - aq\psi}{I - a^{-1}q\psi}, \qquad KB^{-1} = \frac{I - a^{-1}q\psi}{I - aq\psi}, \tag{21}$$

$$K^{-1}B = \frac{I - aq^{-1}\psi}{I - a^{-1}q^{-1}\psi}, \qquad B^{-1}K = \frac{I - a^{-1}q^{-1}\psi}{I - aq^{-1}\psi}. \tag{22}$$

In (21), (22) *the denominators are invertible by Lemma 9.*

Lemma 10 ([5, Equation (22)]) *We have*

$$\frac{\psi A - A\psi}{q - q^{-1}} = (I - aq\psi)\, K - \left(I - a^{-1}q^{-1}\psi\right) K^{-1}. \tag{23}$$

Proof This result is a reformulation of [5, Equation (22)] using [5, Equation (14)]. □

5 The Linear Transformation Δ

We continue to discuss the situation of Assumption 1. In [4, Section 9] we introduced an invertible element $\Delta \in \mathrm{End}(V)$. In [4] we showed that Δ, ψ commute and in fact both Δ, Δ^{-1} are power series in ψ. These power series will be the central focus of this paper. We will show that each of those power series factors as a product of two power series, each of which is a quantum exponential in ψ.

Lemma 11 ([4, Lemma 9.5]) *There exists a unique* $\Delta \in \mathrm{End}(V)$ *such that*

$$\Delta U_i \subseteq U_i^{\Downarrow} \qquad (0 \le i \le d), \tag{24}$$

$$(\Delta - I)U_i \subseteq U_0 + U_1 + \cdots + U_{i-1} \qquad (0 \le i \le d). \tag{25}$$

Lemma 12 ([4, Lemmas 9.3 and 9.6]) *The map* Δ *is invertible. Moreover* $\Delta^{-1} = \Delta^{\Downarrow}$ *and*

$$(\Delta^{-1} - I)U_i \subseteq U_0 + U_1 + \cdots + U_{i-1} \qquad (0 \le i \le d). \tag{26}$$

Lemma 13 *The map* $\Delta - I$ *is nilpotent. Moreover* $\Delta K = B\Delta$.

Proof The first assertion follows from (25). The last assertion follows from (24) and Definition 5. □

The map Δ is characterized as follows.

Lemma 14 ([4, Lemma 9.8]) *The map* Δ *is the unique element of* $\mathrm{End}(V)$ *such that*

$$(\Delta - I)E_i^*V \subseteq E_0^*V + E_1^*V + \cdots + E_{i-1}^*V \qquad (0 \le i \le d), \tag{27}$$

$$\Delta(E_iV + E_{i+1}V + \cdots + E_dV) = E_0V + E_1V + \cdots + E_{d-i}V \qquad (0 \le i \le d). \tag{28}$$

Theorem 2 ([4, Theorem 17.1]) *Both*

$$\Delta = \sum_{i=0}^{d} \left(\prod_{j=1}^{i} \frac{aq^{j-1} - a^{-1}q^{1-j}}{q^j - q^{-j}} \right) \psi^i, \tag{29}$$

$$\Delta^{-1} = \sum_{i=0}^{d} \left(\prod_{j=1}^{i} \frac{a^{-1}q^{j-1} - aq^{1-j}}{q^j - q^{-j}} \right) \psi^i. \tag{30}$$

In (29) and (30), the elements Δ, Δ^{-1} are expressed as a power series in ψ. In the present paper, we factor these power series and interpret the results. This interpretation will involve a linear transformation $\mathcal{M}$. We introduce $\mathcal{M}$ in the next section.

6 The Linear Transformation $\mathcal{M}$

We continue to discuss the situation of Assumption 1. In this section we introduce an element $\mathcal{M} \in \mathrm{End}(V)$. We explain how $\mathcal{M}$ is related to K, B, ψ, A.

Definition 7 Define $\mathcal{M} \in \mathrm{End}(V)$ by

$$\mathcal{M} = \frac{aK - a^{-1}B}{a - a^{-1}}. \tag{31}$$

By construction, $\mathcal{M}^{\Downarrow} = \mathcal{M}$. Evaluating (31) using Lemma 5, we see that $\mathcal{M}$ is invertible.

Lemma 15 *The map $\mathcal{M}$ is equal to each of:*

$$(I - a^{-1}q\psi)^{-1}K, \qquad K(I - a^{-1}q^{-1}\psi)^{-1}, \qquad (I - aq\psi)^{-1}B, \qquad B(I - aq^{-1}\psi)^{-1}.$$

Proof We first show that $\mathcal{M} = (I - a^{-1}q\psi)^{-1}K$. By Definition 7,

$$(a - a^{-1})\mathcal{M}K^{-1} = aI - a^{-1}BK^{-1}.$$

The result follows from this fact along with the equation on the left in (21).

The remaining assertions follow from Theorem 1. □

Lemma 15 can be reformulated as follows.

Lemma 16 *We have*

$$K = \left(I - a^{-1}q\psi\right)\mathcal{M}, \qquad K = \mathcal{M}\left(I - a^{-1}q^{-1}\psi\right), \tag{32}$$

$$B = (I - aq\psi)\,\mathcal{M}, \qquad B = \mathcal{M}\left(I - aq^{-1}\psi\right). \tag{33}$$

For later use, we give several descriptions of $\mathcal{M}^{\pm 1}$.

Lemma 17 *The map* $\mathcal{M}^{-1}$ *is equal to each of:*

$$K^{-1}(I - a^{-1}q\psi), \quad (I - a^{-1}q^{-1}\psi)K^{-1}, \quad B^{-1}(I - aq\psi), \quad (I - aq^{-1}\psi)B^{-1}.$$

Proof Immediate from Lemma 15. □

Lemma 18 *The map* $\mathcal{M}$ *is equal to each of:*

$$K\sum_{n=0}^{d} a^{-n}q^{-n}\psi^n, \qquad \sum_{n=0}^{d} a^{-n}q^{n}\psi^n K, \qquad B\sum_{n=0}^{d} a^{n}q^{-n}\psi^n, \qquad \sum_{n=0}^{d} a^{n}q^{n}\psi^n B \tag{34}$$

Proof Use Lemmas 9 and 15. □

We now give some attractive equations that show how $\mathcal{M}$ is related to ψ, K, B, A.

Lemma 19 *We have*

$$\mathcal{M}\psi = q^2\psi\mathcal{M}. \tag{35}$$

Proof Use Lemma 7 and Definition 7. □

Lemma 20 *We have*

$$\frac{q\mathcal{M}^{-1}K - q^{-1}K\mathcal{M}^{-1}}{q - q^{-1}} = I, \qquad \frac{q\mathcal{M}^{-1}B - q^{-1}B\mathcal{M}^{-1}}{q - q^{-1}} = I. \tag{36}$$

Proof Use Lemma 17. □

Lemma 21 *We have*

$$\frac{qA\mathcal{M}^{-1} - q^{-1}\mathcal{M}^{-1}A}{q - q^{-1}} = (a + a^{-1})I - (q + q^{-1})\psi. \tag{37}$$

Proof Use Lemmas 3, 7, 10, and 17. □

Lemma 22 *We have*

$$\mathcal{M}^{-2}A-(q^2+q^{-2})\mathcal{M}^{-1}A\mathcal{M}^{-1}+A\mathcal{M}^{-2}=-(q-q^{-1})^2(a+a^{-1})\mathcal{M}^{-1}. \tag{38}$$

Proof Use Lemmas 19 and 21. □

7 A Factorization of Δ

We continue to discuss the situation of Assumption 1. We now bring in the q-exponential function [8]. In [4, Theorem 17.1] we expressed Δ as a power series in ψ. In this section we strengthen this result in the following way. We express Δ as a product of two linear transformations; one is a q-exponential in ψ and the other is a q^{-1}-exponential in ψ.

For an integer n, define

$$[n]_q=\frac{q^n-q^{-n}}{q-q^{-1}} \tag{39}$$

and for $n \geq 0$, define

$$[n]_q^!=[n]_q[n-1]_q\cdots[1]_q. \tag{40}$$

We interpret $[0]_q^!=1$.

We now recall the q-exponential function [8]. For a nilpotent $T\in \mathrm{End}(V)$,

$$\exp_q(T)=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{[n]_q^!}T^n. \tag{41}$$

The map $\exp_q(T)$ is invertible. Its inverse is given by

$$\exp_{q^{-1}}(-T)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}}{[n]_q^!}T^n. \tag{42}$$

Using (41) we obtain

$$(I-(q^2-1)T)\exp_q(q^2T)=\exp_q(T). \tag{43}$$

For $S\in\mathrm{End}(V)$ such that $ST=q^2TS$, we have

$$S\exp_q(T)S^{-1}=\exp_q(STS^{-1})=\exp_q(q^2T).$$

Consequently

$$S\exp_q(T) = \exp_q(q^2T)S. \tag{44}$$

Combining (43) and (44),

$$(I - (q^2-1)T)S\exp_q(T) = \exp_q(T)S. \tag{45}$$

We return our attention to K, B, ψ, $\mathcal{M}$.

Proposition 1 *Both*

$$K\exp_q\left(\frac{a^{-1}}{q-q^{-1}}\psi\right) = \exp_q\left(\frac{a^{-1}}{q-q^{-1}}\psi\right)\mathcal{M}, \tag{46}$$

$$B\exp_q\left(\frac{a}{q-q^{-1}}\psi\right) = \exp_q\left(\frac{a}{q-q^{-1}}\psi\right)\mathcal{M}. \tag{47}$$

Proof Recall from Lemma 19 that $\mathcal{M}\psi = q^2\psi\mathcal{M}$. We first obtain (46). To do this, in (45) take $S = \mathcal{M}$ and $T = \frac{a^{-1}}{q-q^{-1}}\psi$. Evaluate the result using the equation $\mathcal{M} = (I - a^{-1}q\psi)^{-1}K$ from Lemma 15.

Next we obtain (47). To do this, in (45) take $S = \mathcal{M}$ and $T = \frac{a}{q-q^{-1}}\psi$. Evaluate the result using the equation $\mathcal{M} = (I - aq\psi)^{-1}B$ from Lemma 15. □

The following is our main result.

Theorem 3 *Both*

$$\Delta = \exp_q\left(\frac{a}{q-q^{-1}}\psi\right)\exp_{q^{-1}}\left(-\frac{a^{-1}}{q-q^{-1}}\psi\right), \tag{48}$$

$$\Delta^{-1} = \exp_q\left(\frac{a^{-1}}{q-q^{-1}}\psi\right)\exp_{q^{-1}}\left(-\frac{a}{q-q^{-1}}\psi\right). \tag{49}$$

Proof We first show (48). Let $\tilde{\Delta}$ denote the expression on the right in (48). Combining (46) and (47), we see that $\tilde{\Delta}K = B\tilde{\Delta}$. Therefore $\tilde{\Delta}U_i = U_i^{\Downarrow}$ for $0 \le i \le d$. Observe that $\tilde{\Delta} - I$ is a polynomial in ψ with zero constant term. By Lemma 8, $(\tilde{\Delta} - I)U_i \subseteq U_0 + U_1 + \cdots + U_{i-1}$ for $0 \le i \le d$. By Lemma 11, $\tilde{\Delta} = \Delta$.

To obtain (49) from (48), use (42). □

Corollary 1 *We have*

$$\exp_q\left(\frac{a}{q-q^{-1}}\psi\right)\exp_{q^{-1}}\left(-\frac{a^{-1}}{q-q^{-1}}\psi\right) = \sum_{i=0}^{d}\left(\prod_{j=1}^{i}\frac{aq^{j-1}-a^{-1}q^{1-j}}{q^j-q^{-j}}\right)\psi^i,$$

$$\exp_q\left(\frac{a^{-1}}{q-q^{-1}}\psi\right)\exp_{q^{-1}}\left(-\frac{a}{q-q^{-1}}\psi\right)=\sum_{i=0}^{d}\left(\prod_{j=1}^{i}\frac{a^{-1}q^{j-1}-aq^{1-j}}{q^j-q^{-j}}\right)\psi^i.$$

Proof Combine Theorems 2 and 3. The equations can also be obtained directly by expanding their left-hand sides using (41) and (42), and evaluating the results using the q-binomial theorem [2, Theorem 10.2.1]. □

8 The Eigenvalues and Eigenspaces of $\mathcal{M}$

We continue to discuss the situation of Assumption 1. In Sect. 6 we introduced the linear transformation $\mathcal{M}$. Proposition 1 indicates the role of $\mathcal{M}$ in the factorization of Δ in Theorem 3. In this section we show that $\mathcal{M}$ is diagonalizable. We describe the eigenvalues and eigenspaces of $\mathcal{M}$. We also explain how the eigenspace decomposition for $\mathcal{M}$ is related to the first and second split decompositions.

Lemma 23 *The map $\mathcal{M}$ is diagonalizable with eigenvalues $q^d, q^{d-2}, q^{d-4}, \dots, q^{-d}$.*

Proof Let $E = \exp_q\left(\frac{a^{-1}}{q-q^{-1}}\psi\right)$. By (46), $\mathcal{M} = E^{-1}KE$. By construction K is diagonalizable with eigenvalues $q^d, q^{d-2}, q^{d-4}, \dots, q^{-d}$. The result follows. □

Definition 8 For $0 \le i \le d$, let W_i denote the eigenspace of $\mathcal{M}$ corresponding to the eigenvalue q^{d-2i}. Note that $\{W_i\}_{i=0}^d$ is a decomposition of V, and that $W_i^{\Downarrow} = W_i$ for $0 \le i \le d$. For notational convenience, let $W_{-1} = 0$ and $W_{d+1} = 0$.

Proposition 2 *For $0 \le i \le d$,*

$$U_i = \exp_q\left(\frac{a^{-1}}{q-q^{-1}}\psi\right)W_i, \qquad U_i^{\Downarrow} = \exp_q\left(\frac{a}{q-q^{-1}}\psi\right)W_i, \tag{50}$$

$$W_i = \exp_{q^{-1}}\left(-\frac{a^{-1}}{q-q^{-1}}\psi\right)U_i, \qquad W_i = \exp_{q^{-1}}\left(-\frac{a}{q-q^{-1}}\psi\right)U_i^{\Downarrow}. \tag{51}$$

Proof Define E as in the proof of Lemma 23. We show that $U_i = EW_i$. By (46), $KE = E\mathcal{M}$. Recall that U_i (resp. W_i) is the eigenspace of K (resp. $\mathcal{M}$) corresponding to the eigenvalue q^{d-2i}. By these comments $U_i = EW_i$.

Define $F = \exp_q(\frac{a}{q-q^{-1}}\psi)$. We show $U_i^{\Downarrow} = FW_i$. By (47), $BF = F\mathcal{M}$. Recall that $U_i^{\Downarrow}$ (resp. W_i) is the eigenspace of B (resp. $\mathcal{M}$) corresponding to the eigenvalue q^{d-2i}. By these comments $U_i^{\Downarrow} = FW_i$.

To obtain (51) from (50), use (42). □

Lemma 24 *For $0 \le i \le d$, the dimension of W_i is ρ_i.*

Proof This follows from Proposition 2 and the fact that $U_i, U_i^{\Downarrow}$ have dimension ρ_i. □

Recall from (6) that

$$\sum_{h=0}^{i} E_h^* V = \sum_{h=0}^{i} U_h = \sum_{h=0}^{i} U_h^{\Downarrow} \qquad (52)$$

for $0 \leq i \leq d$.

Lemma 25 *For $0 \leq i \leq d$, the sum $\sum_{h=0}^{i} W_h$ is equal to the common value of (52).*

Proof Define $W = \sum_{h=0}^{i} W_h$ and let U denote the common value of (52). We show that $W = U$. By Lemma 8 and the equation on the left in (51), $W \subseteq U$. By Lemma 24, W and U have the same dimension. Thus $W = U$. □

9 The Actions of ψ, K, B, Δ, A, A^* on $\{W_i\}_{i=0}^{d}$

We continue to discuss the situation of Assumption 1. Recall the eigenspace decomposition $\{W_i\}_{i=0}^{d}$ for $\mathcal{M}$. In this section, we discuss the actions of ψ, K, B, Δ, A, A^* on $\{W_i\}_{i=0}^{d}$.

Lemma 26 *For $0 \leq i \leq d$,*

$$\psi W_i \subseteq W_{i-1}. \qquad (53)$$

Proof Use Lemma 19. □

Lemma 27 *For $0 \leq i \leq d$,*

$$(K - q^{d-2i} I) W_i \subseteq W_{i-1}, \qquad (B - q^{d-2i} I) W_i \subseteq W_{i-1}. \qquad (54)$$

Proof Use Lemmas 16 and 26. □

Lemma 28 *For $0 \leq i \leq d$,*

$$(\Delta - I) W_i \subseteq W_0 + W_1 + \cdots + W_{i-1}, \qquad (55)$$

$$(\Delta^{-1} - I) W_i \subseteq W_0 + W_1 + \cdots + W_{i-1}. \qquad (56)$$

Proof To show (55), use (25) and Lemma 25.
To show (56), use (26) and Lemma 25. □

Lemma 29 *For $0 \leq i \leq d$,*

$$(A - (a + a^{-1}) q^{d-2i} I) W_i \subseteq W_{i-1} + W_{i+1}. \qquad (57)$$

Proof By Lemma 22, the expression

$$(\mathcal{M}^{-1} - q^{2i+2-d}I)(\mathcal{M}^{-1} - q^{2i-2-d}I)(A - (a + a^{-1})q^{d-2i}I)$$

vanishes on W_i. Therefore $(\mathcal{M}^{-1} - q^{2i+2-d}I)(\mathcal{M}^{-1} - q^{2i-2-d}I)$ vanishes on $(A - (a + a^{-1})q^{d-2i}I)W_i$. The result follows. □

Lemma 30 *For* $0 \le i \le d$,

$$(A^* - \theta_i^* I)W_i \subseteq W_0 + W_1 + \cdots + W_{i-1}. \tag{58}$$

Proof Use $(A^* - \theta_i^* I)E_i^* V = 0$ together with (25) and Lemma 25. □

10 The Actions of $\mathcal{M}^{\pm 1}$ on $\{U_i\}_{i=0}^d$, $\{U_i^{\Downarrow}\}_{i=0}^d$, $\{E_i V\}_{i=0}^d$, $\{E_i^* V\}_{i=0}^d$

We continue to discuss the situation of Assumption 1. In Sect. 8 we saw how various operators act on the decomposition $\{W_i\}_{i=0}^d$. In this section we investigate the action of $\mathcal{M}$ on the first and second split decompositions of V, as well as on the eigenspace decompositions of A, A^*.

Lemma 31 *For* $0 \le i \le d$,

$$(\mathcal{M} - q^{d-2i}I)U_i \subseteq U_0 + U_1 + \cdots + U_{i-1}, \tag{59}$$

$$(\mathcal{M} - q^{d-2i}I)U_i^{\Downarrow} \subseteq U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_{i-1}^{\Downarrow}. \tag{60}$$

Proof To show (59), use Definition 5, Lemma 2, and Definition 7.

To show (60), use (59) applied to $\Phi^{\Downarrow}$, along with $\mathcal{M}^{\Downarrow} = \mathcal{M}$. □

Lemma 32 *For* $0 \le i \le d$,

$$(\mathcal{M}^{-1} - q^{2i-d}I)U_i \subseteq U_{i-1}, \qquad (\mathcal{M}^{-1} - q^{2i-d}I)U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow}. \tag{61}$$

Proof We first show the equation on the left in (61). By Lemma 17,

$$\mathcal{M}^{-1} = (I - a^{-1}q^{-1}\psi)K^{-1}. \tag{62}$$

From this and Definition 5, it follows that on U_i,

$$\mathcal{M}^{-1} - q^{2i-d}I = a^{-1}q^{2i-d-1}\psi. \tag{63}$$

The result follows from this along with Lemma 8.

The proof of the equation on the right in (61) follows from the equation on the left in (61) applied to $\Phi^{\Downarrow}$, along with the fact that $\mathcal{M}^{\Downarrow} = \mathcal{M}$. □

Lemma 33 *For* $0 \le i \le d$,

$$\mathcal{M}^{-1} E_i V \subseteq E_{i-1}V + E_i V + E_{i+1}V. \qquad (64)$$

Proof We first show that $\mathcal{M}^{-1} E_i V \subseteq \sum_{h=0}^{i+1} E_h V$. Recall from (5) that $E_i V \subseteq \sum_{h=d-i}^{d} U_h^{\Downarrow}$. By this, Lemma 32, and (5), we obtain $\mathcal{M}^{-1} E_i V \subseteq \sum_{h=0}^{i+1} E_h V$.

We now show that $\mathcal{M}^{-1} E_i V \subseteq \sum_{h=i-1}^{d} E_h V$. Recall from (4) that $E_i V \subseteq \sum_{h=i}^{d} U_h$. By this, Lemma 32, and (4), we obtain $\mathcal{M}^{-1} E_i V \subseteq \sum_{h=i-1}^{d} E_h V$.

Thus $\mathcal{M}^{-1} E_i V$ is contained in the intersection of $\sum_{h=0}^{i+1} E_h V$ and $\sum_{h=i-1}^{d} E_h V$, which is $E_{i-1}V + E_i V + E_{i+1}V$. □

Lemma 34 *For* $0 \le i \le d$,

$$(\mathcal{M} - q^{d-2i} I) E_i^* V \subseteq E_0^* V + E_1^* V + \cdots + E_{i-1}^* V,$$
$$(\mathcal{M}^{-1} - q^{2i-d} I) E_i^* V \subseteq E_0^* V + E_1^* V + \cdots + E_{i-1}^* V.$$

Proof Note that $E_i^* V \subseteq E_0^* V + E_1^* V + \cdots + E_i^* V = W_0 + W_1 + \cdots + W_i$ by Lemma 25. The result follows from this fact along with Definition 8. □

11 When Φ Is a Leonard System

We continue to discuss the situation of Assumption 1. For the rest of the paper we assume $\rho_i = 1$ for $0 \le i \le d$. In this case Φ is called a Leonard system.

We use the following notational convention. Let $\{v_i\}_{i=0}^{d}$ denote a basis for V. The sequence of subspaces $\{\mathbb{K} v_i\}_{i=0}^{d}$ is a decomposition of V said, to be *induced* by the basis $\{v_i\}_{i=0}^{d}$.

We display a basis $\{u_i\}_{i=0}^{d}$ (resp. $\{u_i^{\Downarrow}\}_{i=0}^{d}$) (resp. $\{w_i\}_{i=0}^{d}$) that induces the decomposition $\{U_i\}_{i=0}^{d}$ (resp. $\{U_i^{\Downarrow}\}_{i=0}^{d}$) (resp. $\{W_i\}_{i=0}^{d}$). We find the actions of $\psi, K, B, \Delta^{\pm 1}, A$ on these bases. We also display the transition matrices between these bases.

For the rest of this section fix $0 \ne u_0 \in U_0$. Let M denote the subalgebra of $\mathrm{End}(V)$ generated by A. By [21, Lemma 5.1], the map $M \to V, X \mapsto X u_0$ is an isomorphism of vector spaces. Consequently, the vectors $\{A^i u_0\}_{i=0}^{d}$ form a basis for V.

We now define a basis $\{u_i\}_{i=0}^{d}$ of V that induces $\{U_i\}_{i=0}^{d}$. For $0 \le i \le d$, define

$$u_i = \left(\prod_{j=0}^{i-1} \left(A - \theta_j I\right) \right) u_0. \qquad (65)$$

Observe that $u_i \ne 0$. By [9, Theorem 4.6], $u_i \in U_i$. So u_i is a basis for U_i. Consequently, $\{u_i\}_{i=0}^{d}$ is a basis for V that induces $\{U_i\}_{i=0}^{d}$.

Next we define a basis $\{u_i^{\Downarrow}\}_{i=0}^d$ of V that induces $\{U_i^{\Downarrow}\}_{i=0}^d$. For $0 \le i \le d$, define

$$u_i^{\Downarrow} = \left(\prod_{j=0}^{i-1} \left(A - \theta_{d-j} I\right)\right) u_0. \tag{66}$$

Observe that $u_i^{\Downarrow} \neq 0$. By Lemma 11, $u_i^{\Downarrow} \in U_i^{\Downarrow}$. So $u_i^{\Downarrow}$ is a basis for $U_i^{\Downarrow}$. Consequently, $\{u_i^{\Downarrow}\}_{i=0}^d$ is a basis for V that induces $\{U_i^{\Downarrow}\}_{i=0}^d$.

Lemma 35 *For* $0 \le i \le d$,

$$u_i^{\Downarrow} = \Delta u_i. \tag{67}$$

Proof By Lemma 11, $\Delta U_i = U_i^{\Downarrow}$. So there exists $0 \neq \lambda \in \mathbb{K}$ such that $\Delta u_i = \lambda u_i^{\Downarrow}$. We show that $\lambda = 1$. By [4, Lemma 7.3] and (25), $\Delta u_i - A^i u$ is a linear combination of $\{A^j u\}_{j=0}^{i-1}$. Also, $u_i^{\Downarrow} - A^i u$ is a linear combination of $\{A^j u\}_{j=0}^{i-1}$. The vectors $\{A^j u\}_{j=0}^{i-1}$ are linearly independent. By these comments $\lambda = 1$. □

We next define a basis $\{w_i\}_{i=0}^d$ of V that induces $\{W_i\}_{i=0}^d$. For $0 \le i \le d$, define

$$w_i = \exp_{q^{-1}}\left(-\frac{a^{-1}}{q - q^{-1}}\,\psi\right) u_i. \tag{68}$$

Since $\{u_i\}_{i=0}^d$ is a basis of V and $\exp_{q^{-1}}(-\frac{a^{-1}}{q-q^{-1}}\,\psi)$ is invertible, w_i is a basis for W_i. Consequently, $\{w_i\}_{i=0}^d$ is a basis for V that induces $\{W_i\}_{i=0}^d$.

Lemma 36 *For* $0 \le i \le d$,

$$u_i = \exp_q\left(\frac{a^{-1}}{q-q^{-1}}\psi\right) w_i, \qquad u_i^{\Downarrow} = \exp_q\left(\frac{a}{q - q^{-1}}\psi\right) w_i, \tag{69}$$

$$w_i = \exp_{q^{-1}}\left(-\frac{a^{-1}}{q-q^{-1}}\psi\right) u_i, \qquad w_i = \exp_{q^{-1}}\left(-\frac{a}{q - q^{-1}}\psi\right) u_i^{\Downarrow}. \tag{70}$$

Proof Use (68) to obtain the equations on the left in (69),(70). To obtain the equations on the right in (69),(70), use Theorem 3, Lemma 35, and (68). □

We now describe the actions of ψ, K, B, $\mathcal{M}$, Δ, A on the bases $\{u_i\}_{i=0}^d$, $\{u_i^{\Downarrow}\}_{i=0}^d$, $\{w_i\}_{i=0}^d$. First we recall a notion from linear algebra. Let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra of $(d+1) \times (d+1)$ matrices that have all entries in $\mathbb{K}$. We index the rows and columns by $0, 1, \ldots, d$. Let $\{v_i\}_{i=0}^d$ denote a basis of V. For $T \in \mathrm{End}(V)$ and $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$, we say that X *represents* T with respect to $\{v_i\}_{i=0}^d$ whenever $T v_j = \sum_{i=0}^d X_{ij} v_i$ for $0 \le j \le d$.

By (65) and (66), the matrices that represent A with respect to $\{u_i\}_{i=0}^d$ and $\{u_i^{\Downarrow}\}_{i=0}^d$ are, respectively,

$$\begin{pmatrix} \theta_0 & & & \mathbf{0} \\ 1 & \theta_1 & & \\ & \ddots & \ddots & \\ \mathbf{0} & & 1 & \theta_d \end{pmatrix}, \qquad \begin{pmatrix} \theta_d & & & \mathbf{0} \\ 1 & \theta_{d-1} & & \\ & \ddots & \ddots & \\ \mathbf{0} & & 1 & \theta_0 \end{pmatrix}. \tag{71}$$

By construction, the matrix $\mathrm{diag}(q^d, q^{d-2}, \ldots, q^{-d})$ represents K with respect to $\{u_i\}_{i=0}^d$, and B with respect to $\{u_i^{\Downarrow}\}_{i=0}^d$, and $\mathcal{M}$ with respect to $\{w_i\}_{i=0}^d$.

Definition 9 We define a matrix $\widehat{\psi} \in \mathrm{Mat}_{d+1}(\mathbb{K})$. For $1 \le i \le d$, the $(i-1, i)$-entry is $(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})$. All other entries are 0.

Proposition 3 *The matrix $\widehat{\psi}$ represents ψ with respect to each of the bases $\{u_i\}_{i=0}^d$, $\{u_i^{\Downarrow}\}_{i=0}^d$, $\{w_i\}_{i=0}^d$.*

Proof By [5, Line (23)], $\widehat{\psi}$ represents ψ with respect to $\{u_i\}_{i=0}^d$. The remaining assertions follow from Lemma 36. □

Next we give the matrices that represent $\mathcal{M}^{\pm 1}$ with respect to the bases $\{u_i\}_{i=0}^d$, $\{u_i^{\Downarrow}\}_{i=0}^d$.

Lemma 37 *We give the matrix in* $\mathrm{Mat}_{d+1}(\mathbb{K})$ *that represents $\mathcal{M}$ with respect to $\{u_i\}_{i=0}^d$. This matrix is upper triangular. For $0 \le i \le j \le d$, the (i, j)-entry is*

$$a^{i-j} q^{d-j-i} \left(q - q^{-1}\right)^{2(j-i)} \frac{[j]_q^! [d-i]_q^!}{[i]_q^! [d-j]_q^!}. \tag{72}$$

Proof The matrix $\mathrm{diag}(q^d, q^{d-2}, \ldots, q^{-d})$ represents K with respect to $\{u_i\}_{i=0}^d$. Use this fact along with Lemma 18 and Proposition 3. □

Lemma 38 *We give the matrix in* $\mathrm{Mat}_{d+1}(\mathbb{K})$ *that represents $\mathcal{M}^{-1}$ with respect to $\{u_i\}_{i=0}^d$. For $0 \le i \le d$, the (i, i)-entry is q^{2i-d}. For $1 \le i \le d$, the $(i-1, i)$-entry is*

$$-a^{-1} q^{2i-d-1} \left(q^i - q^{-i}\right) \left(q^{d-i+1} - q^{i-d-1}\right).$$

All other entries are zero.

Proof The matrix $\mathrm{diag}(q^{-d}, q^{2-d}, \ldots, q^d)$ represents K^{-1} with respect to $\{u_i\}_{i=0}^d$. Use this fact along with Lemma 17 and Proposition 3. □

Lemma 39 *We give the matrix in* $\mathrm{Mat}_{d+1}(\mathbb{K})$ *that represents $\mathcal{M}$ with respect to $\{u_i^{\Downarrow}\}_{i=0}^d$. This matrix is upper triangular. For $0 \le i \le j \le d$, the (i, j)-entry is*

$$a^{j-i} q^{d-j-i} \left(q - q^{-1}\right)^{2(j-i)} \frac{[j]_q^! [d-i]_q^!}{[i]_q^! [d-j]_q^!}. \tag{73}$$

Proof The matrix $\operatorname{diag}(q^d, q^{d-2}, \ldots, q^{-d})$ represents B with respect to $\{u_i^{\Downarrow}\}_{i=0}^d$. Use this fact along with Lemma 18 and Proposition 3. □

Lemma 40 *We give the matrix in* $\mathrm{Mat}_{d+1}(\mathbb{K})$ *that represents* $\mathcal{M}^{-1}$ *with respect to* $\{u_i^{\Downarrow}\}_{i=0}^d$. *For* $0 \le i \le d$, *the* (i,i)*-entry is* q^{2i-d}. *For* $1 \le i \le d$, *the* $(i-1,i)$*-entry is*

$$-aq^{2i-d-1}\left(q^i - q^{-i}\right)\left(q^{d-i+1} - q^{i-d-1}\right).$$

All other entries are zero.

Proof The matrix $\operatorname{diag}(q^{-d}, q^{2-d}, \ldots, q^d)$ represents B^{-1} with respect to $\{u_i^{\Downarrow}\}_{i=0}^d$. Use this fact along with Lemma 17 and Proposition 3. □

Next we give the matrices that represent K with respect to the bases $\{u_i^{\Downarrow}\}_{i=0}^d$, $\{w_i\}_{i=0}^d$.

Lemma 41 *We give the matrix in* $\mathrm{Mat}_{d+1}(\mathbb{K})$ *that represents* K *with respect to* $\{u_i^{\Downarrow}\}_{i=0}^d$. *For* $0 \le i \le d$, *the* (i,i)*-entry is* q^{d-2i}. *For* $0 \le i < j \le d$, *the* (i,j)*-entry is*

$$\left(1 - a^{-2}\right) a^{j-i} q^{d-j-i}\left(q - q^{-1}\right)^{2(j-i)} \frac{[j]_q^![d-i]_q^!}{[i]_q^![d-j]_q^!}. \tag{74}$$

All other entries are zero.

Proof Evaluating the equation on the right in (14) using the equation on the left in (12) we get

$$K = \left(a^{-2}I + (1 - a^{-2})\sum_{n=0}^{d} a^n q^n \psi^n\right) B. \tag{75}$$

The result follows from this along with Proposition 3 and the fact that the matrix $\operatorname{diag}(q^d, q^{d-2}, \ldots, q^{-d})$ represents B with respect to $\{u_i^{\Downarrow}\}_{i=0}^d$. □

Lemma 42 *We give the matrix in* $\mathrm{Mat}_{d+1}(\mathbb{K})$ *that represents* K *with respect to* $\{w_i\}_{i=0}^d$. *For* $0 \le i \le d$, *the* (i,i)*-entry is* q^{d-2i}. *For* $1 \le i \le d$, *the* $(i-1,i)$*-entry is*

$$-a^{-1}q^{d-2i+1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}).$$

All other entries are zero.

Proof The matrix $\operatorname{diag}(q^d, q^{d-2}, \ldots, q^{-d})$ represents $\mathcal{M}$ with respect to $\{w_i\}_{i=0}^d$. Use this fact along with Proposition 3 and the equation on the left in (32). □

Next we give the matrices that represent B with respect to the bases $\{u_i\}_{i=0}^d$, $\{w_i\}_{i=0}^d$.

Lemma 43 *We give the matrix in* $\mathrm{Mat}_{d+1}(\mathbb{K})$ *that represents B with respect to $\{u_i\}_{i=0}^d$. For $0 \le i \le d$, the (i, i)-entry is q^{d-2i}. For $0 \le i < j \le d$, the (i, j)-entry is*

$$\left(1 - a^2\right) a^{i-j} q^{d-j-i} \left(q - q^{-1}\right)^{2(j-i)} \frac{[j]_q^! [d-i]_q^!}{[i]_q^! [d-j]_q^!}. \tag{76}$$

All other entries are zero.

Proof Evaluating the equation on the left in (14) using the equation on the right in (12) we get

$$B = \left(a^2 I + (1 - a^2) \sum_{n=0}^{d} a^{-n} q^n \psi^n\right) K. \tag{77}$$

The result follows from this along with Proposition 3 and the fact that the matrix $\mathrm{diag}(q^d, q^{d-2}, \ldots, q^{-d})$ represents K with respect to $\{u_i\}_{i=0}^d$. □

Lemma 44 *We give the matrix in* $\mathrm{Mat}_{d+1}(\mathbb{K})$ *that represents B with respect to $\{w_i\}_{i=0}^d$. For $0 \le i \le d$, the (i, i)-entry is q^{d-2i}. For $1 \le i \le d$, the $(i-1, i)$-entry is*

$$-aq^{d-2i+1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}).$$

All other entries are zero.

Proof The matrix $\mathrm{diag}(q^d, q^{d-2}, \ldots, q^{-d})$ represents $\mathcal{M}$ with respect to $\{w_i\}_{i=0}^d$. Use this fact along with Proposition 3 and the equation on the left in (33). □

Next we consider the matrices

$$\exp_q\left(\frac{a}{q - q^{-1}} \widehat{\psi}\right), \qquad \exp_q\left(\frac{a^{-1}}{q - q^{-1}} \widehat{\psi}\right). \tag{78}$$

Their inverses are

$$\exp_{q^{-1}}\left(-\frac{a}{q - q^{-1}} \widehat{\psi}\right), \qquad \exp_{q^{-1}}\left(-\frac{a^{-1}}{q - q^{-1}} \widehat{\psi}\right) \tag{79}$$

respectively. The matrices in (78), (79) are upper triangular. We now consider the entries of (78), (79).

Lemma 45 *For $0 \ne x \in \mathbb{K}$, the matrix $\exp_q(x\widehat{\psi})$ is upper triangular. For $0 \le i \le j \le d$, the (i, j)-entry is*

$$x^{j-i}q^{\binom{j-i}{2}}\left(q-q^{-1}\right)^{2(j-i)}\cdot\frac{[j]_q^![d-i]_q^!}{[i]_q^![j-i]_q^![d-j]_q^!}. \tag{80}$$

The matrix $\exp_{q^{-1}}(x\widehat{\psi})$ *is upper triangular. For* $0\le i\le j\le d$, *the* (i,j)*-entry is*

$$x^{j-i}q^{-\binom{j-i}{2}}\left(q-q^{-1}\right)^{2(j-i)}\cdot\frac{[j]_q^![d-i]_q^!}{[i]_q^![j-i]_q^![d-j]_q^!}. \tag{81}$$

Lemma 46 *The transition matrices between the basis* $\{w_i\}_{i=0}^d$ *and the bases* $\{u_i\}_{i=0}^d$, $\{u_i^{\Downarrow}\}_{i=0}^d$ *are given in the table below.*

From	To	Transition matrix
$\{u_i\}_{i=0}^d$	$\{w_i\}_{i=0}^d$	$\exp_{q^{-1}}\left(-\frac{a^{-1}}{q-q^{-1}}\widehat{\psi}\right)$
$\{w_i\}_{i=0}^d$	$\{u_i\}_{i=0}^d$	$\exp_{q}\left(\frac{a^{-1}}{q-q^{-1}}\widehat{\psi}\right)$
$\{u_i^{\Downarrow}\}_{i=0}^d$	$\{w_i\}_{i=0}^d$	$\exp_{q^{-1}}\left(-\frac{a}{q-q^{-1}}\widehat{\psi}\right)$
$\{w_i\}_{i=0}^d$	$\{u_i^{\Downarrow}\}_{i=0}^d$	$\exp_{q}\left(\frac{a}{q-q^{-1}}\widehat{\psi}\right)$

Proof Use Lemma 36 and Proposition 3. □

We next consider the product

$$\exp_q\left(\frac{a}{q-q^{-1}}\widehat{\psi}\right)\exp_{q^{-1}}\left(-\frac{a^{-1}}{q-q^{-1}}\widehat{\psi}\right). \tag{82}$$

The inverse of (82) is

$$\exp_q\left(\frac{a^{-1}}{q-q^{-1}}\widehat{\psi}\right)\exp_{q^{-1}}\left(-\frac{a}{q-q^{-1}}\widehat{\psi}\right). \tag{83}$$

The matrices in (82), (83) are upper triangular.

Lemma 47 *The transition matrices between the bases* $\{u_i\}_{i=0}^d$, $\{u_i^{\Downarrow}\}_{i=0}^d$ *are given in the table below.*

From	To	Transition matrix
$\{u_i\}_{i=0}^d$	$\{u_i^{\Downarrow}\}_{i=0}^d$	$\exp_q\left(\frac{a}{q-q^{-1}}\widehat{\psi}\right)\exp_{q^{-1}}\left(-\frac{a^{-1}}{q-q^{-1}}\widehat{\psi}\right)$
$\{u_i^{\Downarrow}\}_{i=0}^d$	$\{u_i\}_{i=0}^d$	$\exp_q\left(\frac{a^{-1}}{q-q^{-1}}\widehat{\psi}\right)\exp_{q^{-1}}\left(-\frac{a}{q-q^{-1}}\widehat{\psi}\right)$

Proof Use Lemma 46. □

Lemma 48 *With respect to each of the bases* $\{u_i\}_{i=0}^d$, $\{u_i^{\Downarrow}\}_{i=0}^d$, $\{w_i\}_{i=0}^d$, *the matrices that represent* Δ *and* Δ^{-1} *are* $\exp_q\left(\frac{a}{q-q^{-1}}\widehat{\psi}\right)\exp_{q^{-1}}\left(-\frac{a^{-1}}{q-q^{-1}}\widehat{\psi}\right)$ *and* $\exp_q\left(\frac{a^{-1}}{q-q^{-1}}\widehat{\psi}\right)\exp_{q^{-1}}\left(-\frac{a}{q-q^{-1}}\widehat{\psi}\right)$ *respectively.*

Proof Use Theorem 3 and Proposition 3. □

We give the entries of the matrices representing Δ, Δ^{-1} in the following lemma.

Lemma 49 *The matrix in* (82) *is upper triangular. For* $0 \le i \le j \le d$, *the* (i, j)-*entry of* (82) *is*

$$\frac{\left(q-q^{-1}\right)^{j-i}[j]_q^![d-i]_q^!}{[i]_q^![j-i]_q^![d-j]_q^!}\prod_{n=1}^{j-i}\left(aq^{n-1}-a^{-1}q^{1-n}\right). \tag{84}$$

The matrix in (83) *is upper triangular. For* $0 \le i \le j \le d$, *the* (i, j)-*entry of* (83) *is*

$$\frac{\left(q-q^{-1}\right)^{j-i}[j]_q^![d-i]_q^!}{[i]_q^![j-i]_q^![d-j]_q^!}\prod_{n=1}^{j-i}\left(a^{-1}q^{n-1}-aq^{1-n}\right). \tag{85}$$

Proof Use Corollary 1 and Proposition 3. □

We finish the paper by giving the matrix that represents A with respect to $\{w_i\}_{i=0}^d$.

Lemma 50 *We give the matrix in* $\mathrm{Mat}_{d+1}(\mathbb{K})$ *that represents* A *with respect to* $\{w_i\}_{i=0}^d$. *For* $1 \le i \le d$, *the* $(i, i-1)$-*entry is 1. For* $0 \le i \le d$, *the* (i, i)-*entry is* $(a + a^{-1})q^{d-2i}$. *For* $1 \le i \le d$, *the* $(i-1, i)$-*entry is*

$$-q^{d-2i+1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}).$$

All other entries are zero.

Proof Let $\mathcal{A}$ denote the matrix that represents A with respect to $\{w_i\}_{i=0}^d$. By Lemma 29, $\mathcal{A}$ is tridiagonal with (i, i)-entry given by $(a+a^{-1})q^{d-2i}$ for $0 \le i \le d$.

We now show that the subdiagonal entries of $\mathcal{A}$ are all 1. Let $\mathcal{A}'$ denote the matrix that represents A with respect to $\{u_i\}_{i=0}^d$. Recall that this matrix is displayed on the left in (71). Observe that $\mathcal{A}$ is equal to $\exp_{q^{-1}}(-\frac{a^{-1}}{q-q^{-1}}\widehat{\psi})\mathcal{A}'\exp_q(\frac{a^{-1}}{q-q^{-1}}\widehat{\psi})$. It follows from this fact that the subdiagonal entries of $\mathcal{A}$ are all 1.

We next obtain the superdiagonal entries of $\mathcal{A}$. Let $0 \le i \le d$. Apply both sides of (37) to w_i. Evaluate the result using Proposition 3 and the fact that the w_i is an eigenvector for $\mathcal{M}$ with eigenvalue q^{2i-d}. Analyze the result in light of the above comments concerning the entries of $\mathcal{A}$ to obtain the desired result. □

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