Understanding Phenomenologically the Constitution of Knowledge When Working with Dynamic Geometry

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1 Introduction

The present chapter aims to examine the constitution of knowledge while working with dynamic geometry (DG). This view is grounded in the phenomenological perspective, with which we seek to understand how the presence and actions of a living-body[1] who is intentionally[2] with DG, open to what it can show while carrying out learning activities with it comes about. What we mean by this is: creating and occupying spaces and producing knowledge. We intend to create a teaching and learning situation where the subjects let themselves go so that they can live experiences with software, without premeditating what might come to pass.

[1] Understood as *Leib*, a body with intentional movement. It encompasses all lived experiences and is also the starting point for new experiences. It actualizes and is actualized in motion, assuming different perspectives and setting in motion in the life-world that is incessantly formed along with the incessant configurations and reconfigurations of this body (Merleau-Ponty, 2011).

[2] It is intentionality which characterizes consciousness in the pregnant sense of the term, and justifies us in describing the whole stream of experience as at once a stream of consciousness and unity of one consciousness (Husserl, 1972, p. 222).

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We intend to understand through this experience how the constitution of knowledge is exposed or can be exposed, based on our attentive analysis. The interrogation posed is: *How is knowledge constituted when working with dynamic geometry*? In order to tackle this question, we take insights from a study about the experiences lived by subjects who turn to DG, performing activities and taking advantage of the physical and logical possibilities presented by the software in which DG is projected. In this experience, the movement, perceptions, and knowledge constituted with the software take precedence in the analysis.

We understand and contend that every movement performed in DG configures changes that are correlated with such movement, whether they are manifested in the software and/or in the subject who performs the movement. The above-mentioned interrogation encompasses how these configuration changes are experienced, how they manifest themselves, and how they intertwine, constituting knowledge. This understanding, which emerged through broad debate about the proposed study, made us to realize that we were investigating the phenomenon: *the constitution of knowledge while working with dynamic geometry software.* We believe that it is possible to account for the interrogation and the phenomenon interrogated, by studying and bringing open insights resulting from the doctoral thesis entitled "Movement and the perception of movement in Dynamic Geometry environments" (Pinheiro, 2018). In this research, the actions of the living body were studied through the transcriptions of what was said by the subjects regarding the conduction of activities with DG.

We investigated *how movement and the perception of movement takes place while being with the computer and while students perform activities in a dynamic geometry environment*, aiming to understand the phenomenon of *movement-perception-knowledge* in the process of constituting geometric senses and meanings, by going to subjects who experienced ways of being with the computer, working with dynamic geometry. To this end, a group of undergraduate students in mathematics was invited to develop activities with other learning co-subjects, didactic resources, and dynamic geometry environment. The experiences lived by the subjects were described and analyzed through the movement of turning to the interrogation presented above and constituting meanings as they made sense to the researches. Such analysis constituted and exposed convergences, which enabled the researchers to visualize horizons for open synthesis, bringing reflections about the phenomenon interrogated, unveiling its structures, that is, what turned out to be constant among the different perspectives evidenced. In Pinheiro's research (2018), the process of analysis was comprised of four structures, which the author called Nuclear Ideas: *ways in which movement is evident, the evident perceptions that constitute grounds for new perceptions and argumentation, the unity that encompasses the moving-subject and that which is available to movement, and the constitution of knowledge when working with dynamic geometry*.

Although these four nuclear ideas speak of the theme of this chapter, we will focus on the fourth nuclear idea by deepening interpretations and broadening ways of understanding the phenomenon under investigation.

We focused on the actions, weaving understandings about the ways in which they materialize and advance in the totality of the subjects' living-experience with DG and with the activities developed with them, aiming to understand their implications for the constitution of knowledge. We understand that the core of DG is the *movement made by the living-body.*

2 The Movement of Living-Body and Its Expression in Dynamic Geometry Software Interfaces

Phenomenologically, we understand that the movement of the living-body is always intentional, "it is always a *movement towards*..." which, as it advances, instigates change while changing at the same time (Bicudo, 2010, p. 128, authors' translation).

Computing is a favorable area because it manifests a range of possibilities for movement and perception of change in a field where texts, icons, links, pictures, videos are constituted, which, when accessed through the computer interface,[3] expand and are expanded into a constant dialectic in which the computer opens to the perceptive subject a range of possible realizations. When materializing one of such possibilities, making movements, there is a modification/change in those who make the movement and, in turn, in the way of conducting the movement to the extent of the possibilities present in the logic of the software. Each one acts with the computer intentionally firing "commands that will result in specific tasks performed with the materiality available through the program with which they are operating" (Bicudo, 2014, p. 60, authors' translation).

The DG environment establishes a specific way of being with spatiality, opening possibilities for perceptions of changes that also occur in the interfaces. Therefore, understanding the movement and the perception of changes in DG software involves understanding such interfaces, the bifurcations of possibilities that manifest themselves in the logic that underlies such interfaces, as well as the movement that manifests different meanings, in those possibilities.

The interface of DG software "turns" to its programming face as it is able to generate, as a result of some commands/rules, figures and possibility of movement, and "turns" to the subject, who performs actions with it and is aware of their implications.

In one direction the machine, through its answers, encompasses the subject operating with it, showing possibilities of action and effects. In the opposite direction, the subject acts "on the machine, appropriating the senses, by which it gives itself, and manipulating them according to a broader sense of the world that the subject brings with him and are present when they use the machine" (Figueiredo, 2014, p. 134, authors' translation). The senses that open up to the subject evidence the

[3] Literally, *interface* refers to "something that links two faces, which touches both sides, and characterizes a boundary" (Figueiredo, 2014, p. 138, authors' translation).

programmers' intentionality when programming something that they hope those operating with the machine can intentionally do. DG interfaces, for example, are constructed by translating rules of geometry, such as Euclidean and Analytic, into commands to be triggered by the subject with the computer.

The movement of the subject makes it possible to *transpose* the software interface, not in the sense of having access to the programming codes involved, but to be able to foresee possibilities (or answers) for their actions. In this transposing movement, the subjects intentionally focus, perceive what is given to them through the interface, glimpse at horizons of possibilities that are opened to them by the software in response to what they aim to accomplish, and immediately put themselves into perceptive or reflexive movements with the interface. This *being-with-the-interface* occurs in the movement of transposition acts.

The interface of DG software presents operations and responses that trigger, on the subject who is working with it, actions that may provoke a desire to advance. So, it may arouse in this subject the curiosity, the desire to click, to move, to extend what the interface shows. "The subject acts with the interface, but we must note that he does it do it by issuing commands" (Bicudo, 2014, p. 63, authors' translation). But, we need to note that responding to commands is not a simple and mechanical task, because the person who acts is a complex carnal being who is always living concomitantly, with binary rationality and ways of being with others in the world.

The software manuals (guides) are commands that open the possibilities offered by the informational screen to the subject.

Merleau-Ponty (2011) understands the power that the subject has over things, when performing a task, as the force present in the motricity[4] of the living-body which defines, as being intentionally moving or being in movement, "the place where the living-body is, through the task of what he intends intend to accomplish, and the specific situations delineated by its harmonious unity with the horizon-world" (Bicudo & Kluth, 2010, p. 133, authors' translation).

Thus, we understand that the living-body is presence wherever there is something to be accomplished. Along with the subject's action with DG, a kinesthetic background[5] arises and brings about the software interface. In Husserl (2012), we understand that, there is always *fulfillment of senses* within the action and the subject who moves and is moving perceives the movement in this act and what it brings: knowledge, already explained and culturally materialized, configurations, deconfigurations, variants, invariants, or simply the expression of the movement performed.

A DG software interface is comprised of a set of geometrical and other information, which, when accessed, show possibilities of different movements and constructions, that can be expressed on the same screen, to which we are "plugged-in." In this act of accessing we perceive a set of things—shapes, points, icons, etc.—and

[4] Motricity is expressed by Merleau-Ponty (2011) as an intentional way of moving. It is the experience of the living-body willed to act as demanded by the life-world, which, in the present study is focused in the perspective of DG software. This is called kinesthetic movement.

[5] *Kinesthesis* contemplates all the "I move" and the "I do" the living-body performs freely.

precisely by perceiving them and by assuming an analytical attitude we can discern similarities or contiguities. "This does not mean simply that without any perception of the whole, we would not think of *noticing* the resemblance or the contiguity of elements, but literally that we would and would not be part of the same world and would not exist at all" (Merleau-Ponty, 2011, p. 16).

It is while being with the totality of a geometric construction that we realize its most intrinsic properties. In order to understand this construction, from a phenomenological perspective, one has to focus on it, leave it in suspension, so that its surroundings remain, but as a background, or field of past achievements, from which it can be differed. In cyberspace where DG is being worked on, the background mentioned contemplates the entire technological apparatus and its possibilities that boost the movement. This background is dynamic. The possible movement, with software, indicates new tones and perspectives along this background, which also vibrates the figure that renews itself *with* the subject that moves it while moving themselves.

Therefore, the interface is the window that shows us DG in its geometry and algebraic constitution. This interface is duly programmed to open possibilities of movement, whose performance occurs because the acting subject has recollections of motor, visual and kinesthetic experiences that lead the actions of the living-body. Remembering is bringing these understandings to present experiences through recollection (Merleau-Ponty, 2011). Thus, the subject knows which icon to click to fulfill the requirements of a task. The iconic character is present, alluding to a geometric world prior to the cybernetic environment of the software; a world that is close and familiar to the computer professional, who configured it, and the subject that inhabits it when performing actions with the software.

The work with the DG environment allows the subject new understandings. For example, during the task of studying the congruence of two triangles with DG, the subject can only move one triangle until it overlaps it with the other, in order to realize equality and/or inequality. The triangles and the overlapping motion are inert, and it is the presence of the moving-subject that enables the manifestation of diverse triangle and motion senses. In their achievements, the subject also experiences triangles, movements, and congruences, and "when working with geometry in software, these experiences are present again, but in their own way, with new configurations, and constitute the background for actions performed in it and for actions that can be performed" (Pinheiro, 2018, p. 65, authors' translation).

The possibilities of movement present in DG software, in cyber reality, enable the understanding of movement by the act of imagining movement and proceeding to vary movement. Husserl (2006, p. 153, authors' translation), referring to *geometer* says

> The geometer that thinks geometrically operates with imagery vastly more than when he does with percepts of figures or models; and this is true also of the 'pure' geometer, who dispenses with the method of algebra. In fancy it is true he must toil to secure clear intuitions, and from this labor the drawing and the model sets him free (Husserl, 1972, p. 182).

In the context of this freedom, drawings are shown as materializations that follow the constructions of the imagination and the pure eidetic thought that arises based on them, and mainly serve

> to fix the stages of the process already previously gone through, thereby making it easier to bring it back to consciousness once again. Even where the thinker "meditates" over the figure, the new processes of thought which link themselves on to it have fancy-processes as their sensory basis, and it is the results of this work of fancy which fix the new lines on the figures (Husserl, 1972, p. 183).

We understand that in the act of imagining movement in the computational interface, when we are working with software, it is not necessary to pursue its objectification with the aid of the mouse, but only to glimpse it and, subsequently, propose and test theories. The possibilities opened by the software and the intentions of the subject to perform actions, in view of specific tasks, help promote thinking about the geometric ideas present in the activities with DG. Such possibilities open a range of viable understandings about drawings, constructions, movements, and the realization of what was imagined as the accomplishment of a possible task or the solution to a problem or a response to a requirement. They enable the validation or invalidation of possibilities, suggested by the imagination. Imagining movement is already a way of realizing it and being immersed in it. In the imagination one can make a movement and experience its entire duration, even before making it and visualizing it materializing in the software interface.

Being in a DG environment, we can conjecture about possible movements as, at any given time, when we are sensibly with the software, we have the living-experience of making that movement, making it, and watching it happen. We move, we realize that we move, we are able to think about the movement performed or the movement yet to be performed.

The perceptive and reflective acts of the subject being-with the DG software interface expand it, transform, and transcend the intentions of the programmer.

Thus, the geometry presented within the software can be characterized as dynamic. This is so because it always demands the action of the subject who actualizes the program and, especially, who throw themselves in intentional attitudes expressed in movements of the living-body with the interfaces of the program and the tools available on the computer. This way, we understand that if there is no moving-subject, there is no DG, as there is no geometry, as understood by Detoni (2012). Even though the program establishes rules that support various possible movements, there only will be actualizations of those possibilities, if there is a subject who, in their carnality, acts intentionally.

Statements that the dynamics of DG occurs through the physical and logical possibilities of smartphones, tablets or desktop computers, and software can be found in studies in mathematics education, such as Richit (2005) and Silva and Penteado (2009).

However, we understand that this dynamism, besides being given by the potential of computer technology, also occurs with the presence of the moving-subject. This subject actualizes the action, enabling the potential to become reality in its way

of being dynamic. The statement supported by us that it is the subject that makes the action happen and, with that, triggers the potential making it a reality that sustains our claim that a phenomenological study in DG evidences the presence of the living-body, since it is in and with the living-body that the movement is actualized. Besides, the extension of this movement is materialized in the software interface.

The living-body moves while *being-with* and *in* this space and things gain new configurations as they act.

> The movements of the living-body are naturally vested with a certain perceptual significance, they form, with external phenomena, a system so well connected that the external perception "takes into account" the displacement of the perceptive organs, finds in them, if not the *express explication*, at least the reason for the changes that intervened in the spectacle, and thus can immediately understand them (Merleau-Ponty, 2011, p. 78, *emphasis added,* authors' translation).

The objects arranged in the software interface are presences that manifest themselves as *potency,* which "means characteristic of what is potent, has the strength to be, brings the potentialities to become" (Bicudo, 2010, p. 125, authors' translation). This potency comes to being (real) when the subject is with the software, which, when moving, intertwines the movement itself, producing change.

> It is the act that actualizes the potency, encompassing the movement in order to advance the happening. It includes a certain operation and is understood through change. This movement is important in Aristotelian philosophy, because it means "carrying out, effecting" what potentially exists while potentially existing. With this movement, the being goes from the potency to be to the act of being. Thus, the change of an object is the passage from a state of potency or potentiality (being potential) to a state of act or actuality (mode of being currently happening) (Bicudo, 2010, p. 125, authors' translation).

With DG software, the moving-subject performs movements. By responding to what is required in the activities to be performed, and actualizing these movements, the interface is configured with different fulfillments. "Existence occurs with the actualization of what is already potential. Thus, actuality is what presents itself as reality, although in dimensions of individualized actualizations, that is, in specific cases in relation to potency" (Bicudo, 2010, p. 125, authors' translation).

The act of moving objects in the software interface is the trigger for actualization. It transforms an intention to move into the movement itself. This act takes place with the materiality available. In the case of the computer world, the available materiality encompasses the reality opened by the informational screen.

Each movement before the DG software interface defines new experiences: focusing, blurring, moving, dragging. These experiences leave a kinesthetic trail in the living-body of the active subject(s). This is the kinesthesia that encompasses *man-computer-DG* and highlights the ways in which human motricity intertwines the nucleus of movements and perception and comprehension of movements, and may lead to the constitution of geometric knowledge, which occurs in the encounter between the moving-subject and the thing available in the DG software interface.

By being with the world unveiled in this interface, the subject goes towards objects available to perception; perceives their structure, also with the movement of their living-body and, by doing so, the object directly regulates their movements.

"This dialogue between the subject and the object, this recapturing by the subject of the sparse sense in the object, and by the object of the subject's intentions, which is the physiognomic perception, deposes around the subject a world that speaks to themselves and installs their own thoughts in the world" (Merleau-Ponty, 2011, p. 185, authors' translation).

The world we inhabit is not available to us as mere *world of things*, "but, in equal immediacy, as a *world of values*, as *world of goods*, as a *practical world.* "Without further effort on my part I found the things before me furnished not only with the qualities that befit their positive nature, but with value characters [….]" (Husserl, 1972, p. 93). Thus, we understand that the technological world, from which we highlight DG environments, is also available to us as a world of possible experiences that invites us to inhabit what is shown on the informational screen. What is shown and the screen itself are horizons of possibilities. Being with the DG interface is to organize, expand, study, and understand it, that is, what we already do in the life-world, spatializing, moving, being moved, undertaking projects that "polarize the world and make appear in it, as if by magic, a thousand signals that lead to action" (Merleau-Ponty, 2011, p. 161, authors' translation).

The understandings expressed herein so far enables us to think about how the constitution of knowledge occurs when working in environments of dynamic geometry, since they explain experiences in the sensory, perceptive, psychical, and spiritual spheres, from which different sensations and understandings emerge. We now intend to expose thoughts already articulated on how this constitution occurs, working on the *constitution of knowledge when working with dynamic geometry*.

In the articulations of the next topic, we will present part of the analysis described by Pinheiro (2018) about the Nuclear Idea highlighted here. The phrases shown in italics and between quotation marks are excerpts from the transcription of speech of research subjects, while performing the activities proposed, as well as interviews with them. Such clippings are brought in as a fulfillment of the understandings that were made possible about the referred Nuclear Idea, and which are presented in an articulated manner in the following text.

3 Constitution of Knowledge While Working with Dynamic Geometry

Phenomenologically, we do not think of space as simply physical, nor do we view it as an intellectual capacity; but more broadly as spatiality. As Heidegger claims in his *Being and Time* (Heidegger, 1999), it is an existential mode of being-in-the-world. Therefore, we believe that movement, understood here as generator of space, is not necessarily a materialization of motor (skills), but it is also known as a possibility, perceived even before it is carried out. Such possibility, once noticed, is actualized by means of intentionality of the moving-subject, who, in immediate control of their motor actions regulates the movement that he performs, assuming a

posture, as assumed by the subjects of the above-mentioned research, "*not being very eager, to make movements, but always mindful to everything happening on the screen*." In this mindfulness, other possibilities are bestowed upon the subject who glimpses at them, as can be seen in the following: "*The question of reflection [...], we knew where to click to do it, and, knowing it, we could do equal distances with it. But after thinking and moving a point, I was able to better understand its characteristics*." Through this *doing*, movement reveals itself as an actualizer, which animates the screen, producing changes that have characteristics of a mathematical property, allowing the subject to understand it better.

In the previous paragraph, production that starts with perception is exposed. First, there is movement especially as a perceptive act, which is performed and visualized as it generates intuitions for the subject who visualizes it, providing input for justifications and statements. When the subject turns to intuitions, trying to clarify them, there is a shift from perceptive acts to reflective acts. It is the realization of movement aiming at something, such as actualizing a desire to do something or thinking of doing something, fixing what is seen in the movement and validating conjectures and/or responses. This shift can be seen in the account: "*Visualization of movement has helped us communicate, test possibilities and also come up with solutions*," which puts perception (visualization) and acts of communication and testing as belonging to the same movement towards (re)solving a task.

When subjects claim they "have reached solutions," they often expose an argument based on movement and visualization of movement, understanding them as sufficient for their validation, which can be understood in the statements: "*how can we ensure the hose will be perpendicular? By moving and looking*"; "*You can see, right*? *It is apparent that they are equal.*" In another direction, one has the comprehension of the impossibility of validating only through the movement, which is expressed in: "*one cannot only move and think that it is right and ready, it may be wrong. So, we discussed how to validate. Reading the problem again, we found the keyword 'shortest distance'. If it is the shortest distance between two places, it has to be a straight line, right? We got this from mathematics.*"

We see the act of validating as relevant to the constitution of mathematical knowledge. In this text, validation is expressed, as shown in the previous paragraph, as being possible through movement and visualization of movement, as well as through revisiting and organizing mathematical properties. By analyzing the accounts of the subjects, we understand that in the successive movements and visualizations of their implications lies the constitution of certainty, and conviction, which we can highlight in the following: "*When we see the properties that do not vary, as much as we already know them, when we move, we are sure, it is the confirmation that it is the property, which has characteristics that we can study when we move.*"

The statement that is made regarding movements and visualizations, as well as that which is made through review and organization of mathematical concepts, are coherent to us and complement each other in a validation practice, which may contribute to demonstrative practice. When the subject says "*you can see, right? It is apparent that they are the same,*" they are not making empty statements, they have

an empirical foundation that allows them to assert themselves by drawing on a range of movements that show what they see and are sure of seeing. This understanding emerges from lines such as: "*GeoGebra helps a lot, the testing, the validation; we can do that by simply moving a point.*"

The succession of movements performed and visualized presents the subject with a "truth" which is perceived, and that can be an invariant that reveals itself in the variation. This invariant is visual, and with movement there is a way of showing it for what it is. Continuing the previous account, regarding an equality observed, another subject states: "*They are the same, because line CD divides the segment AE in half. So, it is the mediatrix of AE. So, any point on this line will be equidistant from A and E.*" Thus, the statement can be viewed another way, which is explanatory and explains mathematical properties that corroborate and fulfill with other meanings the statement of the first subject, thus projecting greater power of persuasion about the validity of the statement.

For validation, there is evidence of *previous knowledge supporting movements and perceptions*. In the previous account, the subject brings mathematical knowledge learned prior to the accomplishment of the proposed activity. Another subject states that in the classroom, during their undergraduate studies, they learned "*about translation, about reflection,*" movements they made in the development of field activities. However, we believe there are other previous knowledge brought by the subjects during the development of the activities and that guide the ways through which they were with DG. For instance, one subject understands that "*calmer movements gave me more answers. In the first activity, I learned this. In the others, it was easier, because I had learned it, so, that is why I say that it is not just about mathematical concepts,*" referring to how they learned by developing the activities. This account shows how prior knowledge of the way to perform movements unfolds into other activities. Another example is prior knowledge of the dynamic potential of DG software, which is revealed in lines such as: "*The challenge is to draw the point guard (of the basketball team in the activity), so I understood that in GeoGebra it could be moved, and that was it, I started moving and, moving ...*"; "*GeoGebra lets you move the ball. Then, we move it to the midpoint of the triangle.*"

Thinking in action enables us to acknowledge the possibilities of movement of an object in a given foreseen situation, as shown in utterances such as: "*After realizing that point G was moving, we might want to observe this movement in greater detail, and, before moving, we knew it could go left or right, so we moved left and right to see what would happen*". Equally important, is the previous knowledge of the functionality of the software tools, which is exposed in the utterance: "*we used the tools because we knew what each one did.*"

This prior knowledge comes from the subjects as a whole, who dispose of them in their perceptive openness, and not just as argumentative resources. They enable the subjects to think about the movement before actually performing it, as well as stop the movement to reflect on what emerges from it, and fix what is seen in the movement. These are investigative acts performed in the conduction of the activities. As previously stated, while investigating, the subject pays attention to the screen, which allows them to say that: "*When we were moving and paying attention*

to what was happening on the screen, we were able to observe some patterns, step by step, we were discovering things, organizing the information we got, and we were talking, too; discussing. This is all learning, as I was not used to working with activities like this in GeoGebra."

This was said by the subject when asked what they learned; others replied: "*we were in the middle of an investigation, which is what I liked the most about the activities; so in this respect I think we have learned to investigate; we were doing the steps and thinking, reflecting, seeing possibilities, so we tested to see if they worked.*" Thus, the investigation was not only a response to the requirements, but also something to be learned from the activities, working with the subject, the possibilities of computer technology available to make conjectures and create their own procedures.

It is worth noting that the background and/or starting point of almost all the investigation processes initiated by the subjects consisted of movement of geometric objects in the software interface. That shows the characteristics of the context that the subjects were experiencing, in which activities that required movement were presented, and software that opens possibilities for movement, thus exposing the subject to the relevance of movement to (re)solve mathematical problems projected in this context.

The perception of the dynamic context from which the investigations advanced guided the actions of the subjects, including studying a figure by setting it in motion. They understand that when "*working with construction and movement together, we had to move [...] with these movements we could better observe the buildings, understand what we did. So, studying how to construct is different from just constructing. This possibility that we had to better observe the construction in motion, was very interesting.*" Thus, movement is a way of studying the figure and its characteristics. In this case, the figure is given not as a static entity, whose properties have already been defined and described. Although the subject has intuitions about it, it is still a mystery, for, if the subject did not construct it, he would not know the settings before he moves it. When, within the possible configurations, the figure shows itself as a known figure, through movement, one can understand, as do the subjects, that it is possible to understand it "better,", because its properties stand out as invariant in the *figure-in-motion*. In this case, as pointed out by a research subject, one can "*set invariants as properties of a figure. For example, observe what does not vary in a square, then you can take what does not vary and define the square.*" In this utterance, the subject proposes an activity whose aim is to define a figure through perception and organization of the properties shown when it is moved.

In conducting investigations by setting the figures in motion, the subjects realized and understood that the preservation of the figure or its deconfiguration after a movement are consequences of the way it was projected on the screen, as can be understood in the utterance: "*did all the steps that the question required, and, in the end, we realized that these steps were linked by properties that we fixed, as perpendicularity, the parallel, reflection and other things. But, how could we see these relationships? It was through movement. That's where movement comes in, when we could drag the dots and see that everything fit together.*" This opens a discussion

about building and drawing/illustrating. Everything "fits" and shows the figure maintaining characteristics intuited before the movement, because care was taken regarding the requirements of the task; the figure was *constructed* through the projection of the properties that constitute it.

Contrary to that is the deformation of the figure without preservation of its characteristics, which occurs when it is not properly constructed, when the subject who constructs it does not rely on its properties, or even when they are aware of the properties, they are not intertwined with each other. This way of designing the figure in software resembles drawing/illustrating on paper or blackboard. This happened during the development of the activities, which led one of the subjects to state that: "*when I tried to make the final movements of the points, everything went wrong, everything got distorted, point G was kind of diagonally, the position of the capsule was not steady* (referring to time capsule activity). *But why? It's because I did not apply the properties correctly. The construction had to be done according to the properties. For example, instead of doing a perpendicular line, with the tool, I did it without measuring, roughly about 90°, then when I tried to move, everything changed; it all went wrong.*"

Thus, we have two accounts that deal with ways of designing a figure with the software interface. In one case, after the movements are conducted, the figure retains its characteristics, which were noticeable before the movement. In the other case, such characteristics are not preserved, causing deformation of the figure, generating figures which are different from those the subject intended before the movement. In the accounts of the subjects, we understand that this is evidence that "*in GeoGebra it is not enough to construct, one has to move them, because a construction can be maintained or not, depending on if the properties have been strictly followed, or not.*" That is, a figure projected on the DG software interface enables the subject to have intuitions, however, does not allow a statement that implies this figure. It is on an indeterminate horizon, whose vision is given by perceptual movements, including moving the mouse, moving the figure to validate the intuition.

Thinking about the constructed figure, a right triangle, for instance, when set in motion, we understand that there is a modification in the figure, because even while preserving its fundamental feature of having a straight internal angle, there can be variations in the dimensions of sides of the triangle.

The question that arises is: what does the movement present to the subject, who is aware of the software interface, seeing the configurations of this triangle? We understand that each stop in the movement yields a figure, which is similar to the first one (before the movement). However, this is a view that discretizes movement, thus presenting fragments and representation of fragments. Thus, we understand that by referring to motion as continuity, we cannot conceive that it gives us a succession of closely similar figures; we cannot claim that motion, in its duration, results in the same right triangle. By conjecture, we understand that motion enables the perception of the *right-triangle-in-motion*, with the understanding that it is being changed.

In that research project, the study of the figure, setting it in motion, is a way of constituting geometric knowledge that was worked on through activities assigned to

the subjects. They were developed so that movement, as a mode of resolution, would stand out. We believe that the subjects perceived movements already imbricated in the instructions of the activities, especially when they state that: "*they (activities) were created in a way that we knew we had to move to perceive (in the interface) some properties, some invariants, and then see the possibilities of solution.*" The activities and their requirements were the starting point for the subjects' actions with the software. Although we understood that the concepts worked on could be those that were shown at the moment of the subjects' accomplishment of the task, the activities contained previous conceptual demands, such as isometries.

Regarding geometric concepts, the subjects stated that: "*we worked with many concepts, it was interesting to see that in a single activity several concepts can be worked on. We started observing these concepts when we were constructing and then they became more visible when we moved the points that made the entire construction change. Except for the reflection, which was abundant during the activities, I think I know the other concepts, so I don't think I learned them, but I learned how to work with them, how I can think of connecting them within an activity. This we do by including the possibility of movement*." This statement shows that activities designed to be conducted with DG software are important tools for learning more than geometric concepts, and are important for the development of autonomous attitudes, since when presenting them as contrasted from a dynamic background, one can understand them by overcoming immediate potentialities, as the research subjects pointed out.

There is an understanding of the subjects who, while participating in the research, had an opportunity for professional formation as they were becoming mathematics teachers, to observe the relevance of working, basically with the same activities, but another public, as he manifested that "*if the activities were applied to elementary school students, they could learn many concepts*," opening possibilities for studying and understanding them as something new. About that, one subject stated: "*regarding the invariants, we saw the properties better when we moved the points, because they* were *preserved. We don't have to study properties, we just highlight them, because we already know them. But what if they were for school students (elementary school)? One could ask them to study what does not vary in that movement, so they would know more about those properties.*"

Thus, we understand that, for the subjects the most important in the constitution of knowledge which advanced in the context of that research project is the pedagogical and methodological knowledge, which enables them to see ways of *being teachers.* This knowledge is exposed when the subjects state "*what we learn here, I think, is more linked to the development, the exploration, the investigations that we were doing.* [...]. *So, the challenge was to apply these concepts*. *So, I think what we learned through this is how to apply such concepts in situations that require moving objects for them to appear*"—showing that they have learned about working with concepts in an activity—and "*how to conceive an activity* [...] *by putting in it factors that lead students to get their hands dirty and build and move objects*", they learned that "*you can have other types of exploration* [...]. *You can have different activities*

even while addressing the same themes"—evidencing a direction of learning that can contribute to the professional education of the subjects, who are future teachers.

Thus, during the work in the field, data which went beyond the research were constituted, presenting ways of working with school geometry. In the open context in this field, two *selves learn*, one is the self that is a student, who does the activities, and the (future) teacher who, during in the conduction of the activity, thinks of possibilities for their classes. Therefore, there are different views and actions running parallel to the activities, which can be understood in the utterance: "*while I was conducting the activities, I was also trying to think about how it was engendered there* (into the software), *because everything was neat, several ways of doing it were available, but, in the end, they all produced the same answers. Then, I was already thinking about my classes, because I like technologies a lot.*" In this account, there is a *self* which is the *student*, who is concerned with the task, trying to cope with the challenge it poses. There is also the other *self*, the (*future) teacher* who, while doing the task, is thinking about how to present it to their prospective students, or how to develop activities with which they will work. Therefore, these two *selves* are present in the same practice of a two-folded self who learns in order to teach.

Foreseeing professional and methodological practice and considering the requirements of the activities assigned to the students, we understand the subjects changed the focus of their view: from what was seen in their perception to understanding the way they were structured and the context in which they could be developed. Initially, subjects viewed that "*activities start from geometrical situations that are placed in a context where they can move. So, I learned that you can take a simple or self-contained activity and transform it[...]. Later, I will try to take geometry exercises and try to turn them into problems like these, which require construction and movement, it is much more inspiring; more challenging.*" Regarding the contextualization of an activity, we observe the subjects' understanding that it "*makes us learn about this context*" and that when it is directed to working with DG software, "*we have to create activities that make students move the points and see what happens when they move those points.*"

The learning resulting from conducting the activities (conceptual, methodological, and pedagogical knowledge) was expressed by the subjects during the conduction itself and during the interviews. They said that at different times they realized the continuity of the movement of constitution of knowledge that occurs when one is with DG and with learning co-subjects.

These are moments when the transition from subjectivity to intersubjectivity is apparent, when a subject perceives and recognizes the other as a subject of his learning, understanding that every act of learning occurs with the other. This transition is evident in utterances such as: "*There is also the involvement of the group. We were able to do the activities, sometimes on one computer, and everyone exchanged ideas about what we saw there, in the movement. Everyone could take the mouse and do something to show their ideas. So, we learned to respect the ideas of others, everyone had their turn, and could talk. Someone would say: move here, others, move there; they would pick up the mouse and move, then, in the end, everything worked out, everyone developed together, learned together as well.*" This account presents

what contributes to intersubjectivity in movement as a mode of expression, which occurred *in/with* the *living-body-mouse-figures-in-motion* which materialized the intentions of the subjects to give a dynamic and visual background to what they said.

In the movement of constitution of knowledge, the perceptions present in the subjectivity of each subject were expressed and shared in the intersubjectivity of the subjects who performed the activities together, and paid attention to what was being said by each subject who expressed themselves. Thus, dialogues about what was said took place, there were contributions, agreement, disagreement, articulations, organization, and improvement of what had been said. This occurred during the conduction of the activities, and later in the interview. What had been said was recalled, however, already presented in its final articulations, through the dialogued organization among the subjects. For us, this highlights a movement of constitution of knowledge.

We clarify and reaffirm that the subject who knows is not an inert subject, who is intellectually withdrawn while knowledge presents itself to them, whose role is solely to "keep their eyes open" following the rhythm of the presentation of knowledge. We understand that every movement of the subjects turning to other subjects, with the available informatics, opens spaces. We do not understand the *generation of space* as something that comes out of nowhere, which is totally new, but we want to emphasize creation presented through a view that does not presume an idea to be a given, but which views it as a mystery as the subjects involved in this movement problematize it. Thus, knowledge is constituted, and the moving-subject is at the center of this constitution, a creative center that also recreates itself with each new learning experience.

Nurtured by the openness given by the activities, to act and realize possibilities, the subjects stated that: "*we also learned to be more critical. Sometimes, we came up with solutions that seemed to be correct, because when we moved, we seemed to be coming up with the right solution, but then, when it came to testing, making a final construction, there was a slight difference.*" This criticality always puts the subject in a state of doubt, not surrendering or conforming to results before the validation process.

This criticality enables understandings such as*:* "*here, we work with properties; which we already know. But this is nonetheless learning. When working with properties, especially in software, which is a whole different world outside the notebook, we can see properties in motion. So, they (the properties), even though they are the same as in the books in which we learned, are also different, because in GeoGebra they are not the same thing, understand? In GeoGebra, there is a different approach, they can be moved.*" This account shows that geometric knowledge, even when already structured, is not contained in itself and does not make itself known in a single way. Thus, we understand that knowledge can be renewed in each space it is dealt with and discussed, and may be expanded by constituting itself with the possibilities opened in that space. More specifically, we understand, conjecturing, that geometry and its entities, taken tacitly by geometric thought, which does not show displacements, can be renewed when embraced by a geometric thought of spatialization in DG environments, which gives life to each entity, each property, each

geometric figure, setting and understanding them in motion. For example, through this manner of thought, a trapeze is no longer just a definition and its graphical representation. It is also a *trapeze-in-motion*, and everything that it entails also moves and can be known by being in motion. From this perspective, the view and the understanding of its own definition are renewed, as it is written within the movement, thus with the moving-subject

We see the constitution of knowledge which occurs with movement within DG environments as a particular case of the constitution of knowledge in general. In the life-world, where the technological world is present, we also learn through movement, for knowing is a constant flow that constitutes the knowledgeable-being at the same time as it constitutes the co-subject before themselves and the life-world. This implies that the subject, i.e., the person, is always a *being* in motion.

4 A Comprehensive Synthesis of the Ideas Articulated

In this chapter we sought to articulate an understanding of *how knowledge is constituted when working with dynamic geometry*. Therefore, bringing the analyzes carried out by Pinheiro (2018), we explain how the constitution of knowledge took place in a context in which the research subjects were conducting activities with the computer, in a DG environment. We expose perceptual acts that constitute the soil upon which the moving-subject weaves understandings. Such acts are apparent in the actualization and visualization of movement, manifesting ways in which the subject directs his actions towards the actualization of possibilities. Depending on the perceptive acts, the computer screen presents transformations that materialize as *figure-in-motion*. Along with this perceptive view a turnaround takes place, which leads the subject to reflect on the view and the intuitions generated, analyzing and expressing what is understood among the co-subjects, who, in turn, seek to understand what was said and contribute with what he understood through his individuality about what was discussed.

This exposes subjectivity and intersubjectivity, animating processes in which perceptions are shared, justified, and organized, seeking an articulation that is coherent and accepted by the group didactically involved in a discussion. The intentionality is geared towards the search to validate the thoughts through acts of verification that materialize, specifically, as movement performed with the mouse and expressed on the screen.

Thus, we have shown that, in the context presented, the starting point and also the background of the constitution of knowledge are the movements performed in an environment inhabited by groups of subjects. With the expression of what is perceived, the dialogue, the articulation of ideas, the organization, and explanation of the knowledge constituted in the subjectivity of each subject and in the intersubjectivity of the subjects is done through movements, producing knowledge that emerged from the performance of all these acts. Thus, the movement of the living-body reveals itself as the way of spacializing of a subject, who moves by *moving-*

perceiving-reflecting-understanding-expressing, and this action materializes as a unity in the *now* of the accomplishment of such acts by that subject.

By focusing on movement and perception, we understand that moving is an actualization that *actualizes-fixes-transforms-explores-tests-validates-shows-expresses,* indicating paths to knowledge and to the knowledge of oneself. We understand also that when movement is viewed as a phenomenon, it can be *actualized-perceived-understood.* Perception, as we have stated, is the primacy of knowledge from which what is perceived emerges, when interpreted and expressed through language by the perceptive subject, opening paths of the objectivation of knowledge.

Since moving, perceiving, and knowing are acts performed by the moving-subject; we understand that it is not possible to talk about movement in the software without talking about the software, just as it is not possible to talk about invariants without talking about the subject who engages their spatial-temporal presence with the occurrence of what remains and what varies, realizing permanence and variation.

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