Maria Aparecida Viggiani Bicudo *Editor*

Constitution and Production of Mathematics in the Cyberspace

A Phenomenological Approach

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Preface

In the chapters that compose this book, there are studies made by authors from the research group "Phenomenology in Mathematics Studies," coordinated by the first organizer of this book, Dr. Maria Aparecida Viaggiani Bicudo, and a chapter written by Dr. Angela Ales Bello, an Italian researcher from the Università Latarenense of Rome, who also contributed to the group. This group has existed and has been making studies and doing researches for the past 20 years (www.sep.org.br/fem). The theme that maintains it is the comprehension of mathematics in the ways it is constituted and produced as a practice in the everyday world and as an exact science, present in the civilization of the western world, thought about, in these manners, in environments of the constitution and production of the knowledge of this science and its applications, as well as of that practice. The phenomenological vision enlightens the conducting wire of our way of thinking, and therefore, of investigations of the group, in virtue of Edmund Husserl being a mathematician and having been dedicated to exposing, phenomenologically, the movement of constitution and production of this science, as well as exposing reflections about its history. Moreover, the studies brought by this philosophy have been clarifying our own inquiries and, with that, have also contributed to our comprehension of the world, science, and education. Other authors, such as Martin Heidegger and Maurice Merleau-Ponty, have been studied by the group, insofar their texts help to answer the posed questions.

Husserl makes a phenomenology of Lebenswelt, pointing out its structures (Husserl, 1970). However, if we look for ways of understanding concerning aspects of Life-World's "real," we come across aspects of reality that didn't appear in his studies, since they refer to something that's being produced ever since his passing, which occurred in 1938, and that has been imposed in our daily lives since 1960: the cyberspace.

This book describes ways of teaching and learning mathematics and also conducting investigations on mathematics, as well as being in cyberspace and with media. The reality of cyberspace is seen by the group as a way in which the "real" has of being actualized, that is, of becoming real as actions take place. The philosophy that underlies this way of thinking is the search for comprehension, exposing ways in which this comprehension is being articulated and exposed in text.

It is being stated, therefore, that the proposition of this book it is not explain the computing tools used and how to use them in the cyberspace, but to understand the way we are with media," aiming at teaching and learning of Mathematics in a way of thinking together with the students, the mathematics subject-matter, the computer, and the media being in cyberspace.

The differential of this book is to work phenomenologically with questions, activities and ways, as activities and ways that people who teach and learn are actualizing themselves in the process. Being highlighted the conception of people with which these authors work with: living-body, carnal totality constructed by physical, psychic and spiritual dimensions. The physical dimension is about the body, its structure and function, and its ways of sensing; the psychic dimension is about emotions and the aspects of cognition; and the spiritual dimension is about the action of judging. This approach requests that the constitution of knowledge is pointed, emphasizing the intentional movement made by the living-body that always puts itself in situations of attending to questions it seeks to comprehend, acting on the specialty in which it is and the temporality of its experiences. The spaciality and the temporality in which it moves concern Lebenswelt's own reality. The ways of experiencing them are specific, according to the aspect of the experienced reality. The ways of cyberspace of being "real" present its specificities as well. Taking on the phenomenological posture, the authors of the different chapters describe and analyze these manners, contributing to the comprehension about the complexity of the world in which we are researchers and teachers of mathematics.

We also present to you in this book words that may be odd to Anglo-Saxon readers since they may have a different interpretation in the Anglo-Saxon language. We are referring to words translated from the German and French languages. We highlight them below.

Readers will find, for example, the expression "living-experience" instead of "lived-experience," as was used in the English-translated versions of Maurice Merleau-Ponty's books. We use "living-experience" to manifest the idea of a *living-body* acting at the present moment, that is, moving at the present moment. Therefore, it is not about past actions. The expression "living-experiences" is also written that way instead of just "experience." Experience gives an empirical connotation, that is, performing/making experience. From a phenomenological point of view, we live the experience and we are aware of ourselves acting. In German, the word used by Husserl is *Erlebnis.*

Another expression usually seen in Anglo-Saxon versions is "lived-body" as a translation for "corpe-prope," which in Maurice Merleau-Ponty's books is translated to "proper-body" or "own-body." We use "living-body" to express more appropriately Husserl's idea of a living-body. We see it as a body that possesses life, a life that is dynamic and flowing.

Also often mentioned in the chapters is the expression "to actualize" instead of "to realize" or "to update," as what translators usually write. "To actualize" expresses the idea of "devenir," that is, of "becoming of a being" upon an action. To actualize expresses the idea the movement of being, *being.*

Anglo-Saxon readers are acquainted with the word “mind” and, in numerous occasions, use it to refer to consciousness. We do not use “mind” but express ourselves using the term “consciousness” or “to be aware of.” “Mind” brings with it the Cartesian conception of vision of duality of spirit and matter, even though by it we don’t mean that this vision is directly applied by those scientists that write “mind.” To Ales Bello (2016, p. 77, author’s translation), “The fundamental passage of Cartesian vision to psychology consists of not asking what is mind, but how it works. This is the non-philosophical attitude of cognitivism. The philosopher always inquires what is and how the mind works.”

“Mind” is a concept that expresses psychic acts. To cognitivism, cognition says about the acts that come from the mind. Citing Ales Bello (2016, p. 77, author’s translation), “We know that the word ‘cognition’ means knowing. The old Latin word is ‘conosco’; the origin is the same. If I say cognitivism, it means that I am interested in knowledge, I am interest myself in knowing.” In the sphere of classic cognitivism, that is, the decades of 1960 and 1970, “mind” refers to studies of the functioning of psychic acts, and in today’s sphere, we can encounter not only studies that rely on introspection but also studies that relate to the observation of behavior. However, with the advancement of computer sciences and neurosciences studies “mind” has become about the concept of seeing the brain’s way of operating the same way as a computer:

> While Descartes affirmed that only the body was like a machine, in the sphere of neuroscience and technology, it is also said that the spirit is like a machine. Nowadays, such vision forms itself with the development of technology in the period that covers the development of science up to this of technology. It is computers, cars, planes, etc., machines linked to humans (Ales Bello, 2016, p. 84, author’s translation).

Cognitivists, along with neuroscientists, have set out to understand what happens in the mind through a machine, for example, a robot, which consists of almost a double mind, and therefore, in its eyes, this can be understood from the outside, without needing introspection. “Mind” is a word from Anglo-Saxon culture, not from German culture, and it is not also present in the Latin language. We cannot find in phenomenology the word “mind.”

We do not take on cognitive and neuroscientific interpretations, even though we understand and accept that the living-body is a total concept, that is, consisting of neurons and their functions and psychic parts. But as we understand it, there is something else. There is afundamental question that says about the living-body iself as living, when we assume the pehnomenological posture, says about living-body knowing itself living:

> We don’t realize the concept of living, because we see living in the sense of the living-body. Saint Augustine had a beautiful saying “Intima scientia est qua nos vivere scimus”, which means that, deep down, us human beings know, are conscious, that we live. And what is that intimate science? It is conscience (Ales Bello, 2016, p. 88, author’s translation).

Therefore, we write consciousness that already bring the understanding that the theme of counsciousness is the underlying theme of the interpretation of human being. It talks about being aware that we are alive and “living the experience.”

Simply, Ales Bello (2016, p. 71, author's translation) stated that "conscience is perceiving yourself in so many activities."

Intropathy is a word we use to talk about the direct perception of another human being equal to me, as a living-body that feels pain and hunger, that senses, understands, and expresses itself. On many occasions, psychologists and other human science experts refer to the word *empathy*. This word is about putting oneself in someone else's shoes and understanding how this someone feels based on his or her own feelings. Intropathy is to see the other person intentionally look at another person, with whom it is in Life-World, and in a straightforward manner of capturing him in the perception as an equal. Empathy, otherwise, calls for a mediating act intermediate act to get together its own person with someone else and, in a loving act of empathy, understanding what this person is feeling.

We write "cyberspace's reality" to express a philosophical writing carries a philosophical vision that cyberspace is a way of being real as real is nature, society, and so on. It is not "virtual," as if it would not also be Life-World.

We highlight the way we write "with" or "with the computer" or "with media" and not, as commonly said: "I use the computer and other media." The difference lies in the concept, and it is strongly philosophical. The expression "use" implies that we see the computer and other media as utensils. In this case, we use them as a means to an end, for example, for the accomplishment of a task. Differently we understand "being-with-media" as Borba and Villarreal (2005) do however, with the philosophical conception of the totality of this word united by hyphens, different.[1] "Being together" to these authors is explicit in the connection of the human-being-with-the-media seen from a psychological and sociological perspective that comes from the Soviet school, as, for example, Vigotsky and his followers' works. From this perspective, they, like us, focus on the construction and production of knowledge, explaining it according to their views about the world and knowledge. On the phenomenological way, "with" and "being-with computer and other media in cyberspace says about the intentionality of conscience that is always aimed at a… "being aware about what is being lived." "Computer and other media" are means by which we accomplish things. The computer, given its logic and engineering capabilities and its operational and functioning possibilities, opens the opportunity for constituting and producing knowledge. I highlight the expression "constituting" knowledge and not "constructing" knowledge. The first is strongly phenomenological. It is about the interlacing of sensations felt by the living-body, perception, accomplished psychic acts, and spiritual acts, which, in their expression of sense and perception, constitute knowledge. The second is about cognitive construction.

This is a book organized into four parts.

Part I offers chapters that expose philosophical and didactic-scientific conceptions. At the request of Springer, we started the book by presenting ideas that are in line with the theme, which triggered the research presented herein, namely, mathematics education in/with cyberspace and the use of technologies. In this part, we

[1] About this theme, see studies published by Bicudo (2014).

also present chapters that show how phenomenology differs from other philosophies and the various ways of working with technology in contexts where mathematical knowledge is being constituted and produced. A nuclear Idea in the texts contained in this book is the understanding that the *constitution* of knowledge occurs in the carnality of the living-body and in the actions actualized by acts of conscience. Meantime, this constitution is not only circumscribed to the living-body, but transcends it through languagens expressed with the materiality available in the life-world. This is a movement of *knowledge production* that resides in mundane reality, in which the socio-cultural dimension imposes itself by force. However, as phenomenologists, without ignoring these dimensions in which science, customs, and morals play an important role in shaping a person, we turn to our "inner" selves, seeking to understand the vital aspects of the very carnality of our living- body: the hyletic dimension. Seeking to realize our perplexity in the existence of this dimension when one is with a computer and other media in cyberspace, constituting knowledge, we also offer our understanding of the ways in which the living-body moves in that reality.

Part II contains four chapters, which are focused on how mathematics education takes place in cyberspace being-with-media. These chapters describe the investigations carried out through computer activities with specific mathematical themes, such as dynamic geometry and sine function exploration. They detail activities performed with students: being in cyberspace, movements of the living-body, perception of movement, perception of the self and the other, and the possibility of being together with the other in this reality. In this part, there is also a chapter that deals with the development of websites and resources for teaching mathematics on the Internet.

Part III, "Teacher Education in Cyberspace Being-with-Media," contains three chapters whose investigations supporting the texts reveal the ways in which this education, which is also the formation of the person, takes place. This is not a linear formative movement that occurs without strangeness and perplexity. However, facing it opens the space for familiarities that, when experienced, in turn beget other perplexities.

Part IV, "Constitution and Production of Mathematical Knowledge Being with the Computer," is organized into four chapters. The theme of three of the chapters focuses on mathematics, understood as a science of the Western world, and computer science. The last chapter deals with the constitution of a mathematical object, based on studies of Husserl's logical investigations. In this chapter, it is stated that "The mathematical object is constituted as an ideality. It is an ideal object, which is supported chiefly by written language, as well as by other kinds of languages, such as pictorial; through pictures, figures, drawing, and so on. At the heart of mathematics conceived as a Western science, the reasoning present in pure logic is nuclear. Therefore, mathematical scientists are always conducting a thinking movement, of logical reflection, and after it is fulfilled with meaning, move forward to ideation. As stated above, *logical reflection* is effected as a result of theoretical connection and *theoretical reasoning*, constituting content in consummated acts of thought."

As organizer, I understand that it is important to impart our understanding of the constitution of the mathematical object within the scope of phenomenological philosophy. I believe that it is important to explain to the reader the reason for the inclusion in the present book of the chapter "The Constitution of the Mathematical Object in Acts of Evidence, From the Perspective of Edmund Husserl's First Logical Investigation," which is dedicated to uncovering the constitution of the mathematical object according to that author.

It is important to say that, as organizer of this book, I have been with the authors during the time the investigations were conducted and was able to gain insight into the logic behind the different chapters. This broader look allows me to state that the actions performed by the mathematicians who conducted the research with computers requires from them substantial knowledge of mathematical theories, which underlie their research, as well as about computers and the extent of their calculation and operation. For example, one of the investigations makes it clear that when we operate rational numbers with computers, we face the issue of limitation of computational memory. We need to be clear that a 32-bit computer stores a floating-point number mantissa with a maximum of 23 bits, causing a poor approximation of certain figures, such as number 0.1 and above. Thinking about these ideas, I understand that the primordial characteristic in mathematicians' knowledge movement is their intention of meaning regarding what they are investigating, about the logical reasoning that underlies the theory with which they are working, about the computer program and about the computer itself. They need to be aware of the reasoning they are making and how it is expressed, asking themselves continuously: what does this mean to my investigation and to the mathematical field in which I work? So, I understand that the mathematician can work out the ideality of the mathematics object being with computer, since they are aware of the question posed, the resources with which they operate, both from the standpoint of the mathematical tooling and of the computer itself.

An Afterword was also written, which provides a reflective analysis of what was treated in the different chapters, to conclude the present book.

Rio Claro, Brazil — Maria Aparecida Viggiani Bicudo

References

Ales Bello, A. (2016). Fenomenologia-psicopatologia-neurociências: e a consciência? In M. A. V. Bicudo & A. E. A. Antúnez (org.), *Fenomenologia, psicopatologia e neurociências: e a consciência?* (pp. 54–114). São Paulo: Universidade de São Paulo.

Bicudo, M. A. V. (org.) (2014). *Ciberespaço: possibilidades que abre ao mundo da educação*. São Paulo: Livraria da Física.

Borba, M. C., & Villarreal, M. E. (2005). *Humans-with-media and the reorganization of mathematical thinking: Information and communication technologies, modeling, experimentation and visualization*. New York: Springer.

Husserl, E. (1970). *The Crisis of Europeen sciences and transcendental phenomenology*. Evanston: Northwestern University Press.

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Contents

Part I
Philosophical Didactic and Scientific Conceptions

Mathematics Education in/with Cyberspace and Digital Technologies: What Has Been Scientifically Produced About It?

Maurício Rosa

1 Starting the Dialogue

The evolution of the digital domain, in the form of the computer network, has been appropriated by educators, and from this, the research about the possibilities of digital technologies (DT) brings to Education in Brazil and worldwide (de Oliveira, 2002; Kenski, 2003; Laurillard, 2008; Mansur, 2001; Underwood, 2009, among others) and has been conducted for decades. Specifically, in mathematics education, many studies point to the prominent potentialities of DT, inserting cyberspace in this context (Bairral, 2002, 2004; Bicudo & Rosa, 2015; Borba, 2004; Borba & Villarreal, 2005; Burton, 2009; Chronaki & Christiansen, 2005; Simmons, Jones Jr, & Silver, 2004; Zullato, 2007). Cyberspace can enhance the construction of online worlds and identities (Rosa, 2008; Rosa & Lerman, 2011; Rosa & Maltempi, 2006), as well as enabling the creation of a differentiated time/space for communication, interaction (Bicudo, 2018; Castells, 2003, 2005; Lévy, 2000) and, consequently, education (Hoyos, 2012; Tallent-Runnels et al., 2006).

With that in mind, this chapter presents important studies in the area of mathematics education that take the digital technologies as their core research, because this environment is what centralizes our efforts as researchers of this different time/space, which is a possible generator and/or articulator of the constitution of mathematical knowledge. Based on Husserlian phenomenology, taking Heidegger as prominent theorist, our research objective is to make philosophical analysis about what is presented to us with DT in terms of mathematics education. Assuming a meta-understanding of the educational practices carried out with DT, in particular, with the cyberspace, making a movement of reflection of the lived reality in this space, of critical comprehension of education courses, often offered at a distance, and in front of intentionality that keeps the subjects plugged

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into the cyber world, one dresses in a phenomenological attitude relying on a movement *to go back to the things themselves* and glimpse at answers on what this environment can offer to the mathematics education carried out with it.

Thus, this chapter aims to offer to the mathematics education community an analysis about studies on technology, in order to bring similarities and possible discrepancies in terms of "what" and "how" it can be investigated, by punctuating possible advances and other investigative opportunities when the subject is mathematics education in/with cyberspace (or cyber world) and with Digital Technologies interconnected with it. This analysis is based on the phenomenological perspective.

2 Research in Mathematics Education: "What" and "How" Do You Investigate When You Are in/with Cyberspace/Digital Technologies?

Winn and Bricken (1992) have already discussed the use of virtual reality (VR) to help students learn their school subjects, specifically, elementary algebra. The authors' research was concerned first and foremost with applying current studies on learning to improve students' classroom experiences. The authors affirm "Using a VR presentation system, the axioms of algebra can be, so to speak, built into behaviour of the world" (Winn & Bricken, 1992, p. 12). For them, the spatial algebra system, developed by Bricken, constitutes a totally new way of representing algebraic concepts and procedures. They described the design of system and give an example in which algebra is used for mathematics education purpose.

The virtual reality is conceived as a "[...] computer-generator, multidimensional, inclusive environment which can be accepted by a participant as cognitively valid" (Winn & Bricken, 1992). However, virtual reality (VR) is a term used with many meanings. Some researchers say VR is a specific collection of technologies, that is a head mounted display, glove input device and audio, for instance. Some other people are based on the idea of reality that happens with cyberspace. Isdale (1993) believes that "Virtual Reality is a way for humans to visualize, manipulate and interact with computers and extremely complex data". VR is understood as a space that has many immersive levels. Each level depends on the interaction with cybernetic world. Once one is connected in cyberspace, there are different ways for to feel themselves in the cybernetic world.

According to Lévy (2010, pp. 6–7)

> The cyberspace - which is the communication space opened by the global interconnection of computers - brings a new large scale 'many to many' communication configuration […] Cyberspace allows not only a 'one to one' and a 'one to many' communication but also a 'many to many' and the real time articulation between the three modes, which is very conducive to collective intelligence. These new possibilities are already used at a large scale for scientific, business, political, artistic (and so on) purposes.

Therefore, one way of connection with cyberspace is throw communication, one has an intentionality to launch themselves into communication. Another possible way is when one interacts hypertextually in/with this space. For this, "Cyberspace is not a medium, it is a metamedium" (Lévy, 2010, p. 6) because it goes beyond of understanding that cyberspace is just a digital resource.

In this sense, Hoyles (1995) has already mapped out and attempted to explain the pattern of exploratory computer-based learning environments. From the point of view of the student and student–software interaction, she has pointed five common categories of response out by referring to students' work with Cabri Géomètre and with Logo. Before this, Hoyles and Noss (1992) have analysed a series of episodes which occurred in a logo-based microworld constructed around the notions of ratio and proportion. They explored the patterns of pedagogy associated with on and off-computer activities. In both cases, the use of computer was considered like a way of understanding models/patterns of mathematical thinking. We consider 1990s the first step to provoke the mathematical thinking with cybernetic world. For example, Shaffer and Kaput (1998) suggest that computational media, from a cognitive-evolutionary perspective, are qualitatively different from many of the technologies in relation of educational change. They

> […] argue that mathematics education in a virtual culture should strive to give students generative fluency to learn varieties of representational systems, provide opportunities to create and modify representational forms, develop skill in making and exploring virtual environments, and emphasize mathematics as a fundamental way of making sense of the world, reserving most exact computation and formal proof for those who will need those specialized skills (Shaffer & Kaput, 1998, p. 97).

When those authors emphasize mathematics as a way of understanding the world, preserving accuracy and formal mathematical proof, they continue considering the work with virtual environments. In this sense, as early as 1998, they indicate that the transformation of digital technologies (especially cyberspace) already provides and will continue providing the mathematical thinking. Notwithstanding, other authors of mathematics education in the following years continued their research into the use of technologies, in particular the Internet as a promising base for cyberspace, on different levels of education. Thomas and Holton (2003) have intended to resignify the use of technology as a teaching tool, mostly, at university level. When they take technology into consideration, they mean with it not only graphic calculators and CD-ROM, but also all aspects of computers, including software and their use with the internet.

Although technological resources (software, dynamic geometry programs, CAS, LOGO Programming, etc.) were quite a fad in the 1990s and 2000s, the Internet opened different perspectives for mathematics education up. Then, working with cyberspace as a space for interaction and for many other possibilities was a challenge. Also, working in multiple contexts, which are hyperlinked and therefore open to so many online contexts, changed the way one teaches mathematics and also the way one may learn it. According to Chronaki (2003),

> An online context is a virtual space built on the internet (using hypertext and standalone software tools) where users can interact with tools and resources with the purpose to communicate, share and discuss or simply to search for information. These are all possible due to late developments of technological tools (e.g. web design, multimedia authoring, component oriented architecture) that enable amongst others a) the distribution, organisation and navigation of large amounts of information and tools, b) the asynchronous and real time communication between users, c) the construction of hypertexts that combine text, image and sound, d) the sharing of software tools addressing specific operations, and e) the visualisation of multiple relations amongst varied representations of information (Chronaki, 2003, p. 3).

Considering those aspects, Engelbrecht and Harding (2005) claim that Internet education in mathematics was being developed as a new mode of teaching with its own characteristics and possibilities, different from the traditional way of teaching. In their theory, it is shown that:

> Teaching practices are changing as d-learning (distance learning) is being replaced by e-learning (electronic learning), which in turn is making way for m-learning (mobile learning). The three modes are not exclusive and, in all likelihood, will blend even more in future (Engelbrecht & Harding, 2005, pp. 271–272).

The changes in teaching practices have been studied from different theoretical perspectives. As an example, Artigue (2007) considers the use of digital technologies as a window on theoretical issues in mathematics education. She highlights the development of the instrumental approach, which has its particularities. According to Artigue (2007),

> Making technology legitimate and mathematically useful from an educational point of view, whatever be the technology at stake, requires modes of integration that provide a reasonable balance between the pragmatic and the epistemic values of instrumented techniques. This, as shown by research, if one correctly reads its results, requires tasks and situations that are not simple adaptations of paper and pencil tasks. It requires situations that very often cannot be thought of in a paper and pencil environment. One can easily imagine that such tasks are not those that a teacher designs first when entering in the technological world with a paper and pencil culture, which has been and is still the general case. From this point of view, the research carried out in Grenoble with Cabri-géomètre studying the evolution over three years of the scenarios built by a group of teachers having differentiated relationships with technology is especially illustrative (Artigue, 2007, p. 73; Laborde, 2001).

Also, Laborde (2007, p.68) discusses

> […] the changes that technology brings about in classroom mathematical activity, the difficulties students encounter in solving problems in a technology-rich environment, the evaluation of potential assignments and the creation of conditions appropriate for learning, and the difficulties faced by teachers who use technology in deciding when and how to intervene.

A technology-rich environment is understood as a digital world, which changes the students and teachers' way of thinking.

From this perspective, Rosa and Lerman (2011) discuss the research about cyber mathematics, which rather has particular features for research in mathematics education. They suggest that cyberspace amplifies the natural environment in which the subjects show themselves, show who they want to be with cyberspace. It can change the construction of mathematical knowledge, because,

> The construction of mathematical knowledge potentially takes new turns when the student and the teacher show themselves through online identities. We can ask who is learning? Is it the student or his/her online identity? The important thing here is that we do not consider knowledge as something constructed inside the human body, individual, separate. Mathematical knowledge is a social construction; therefore, it is in social relations. If we have different identities (online identities), we will construct mathematical knowledge from our social relations with other online identities inside cyberspace. The knowledge that is constructed there is not just ours, students and teachers of mathematics (offline identities). This knowledge belongs to online identities as tractor drivers, doctors, teachers, or farmers in cyberspace. The online identities offer different emphases, motivations, goals, and rules for the student (or the teacher–researcher) and thus, although he or she is still the student, the new dimension extends the student; the online identities are not just part of the student, but her/his extended identity(ies) (Rosa & Lerman, 2011, p. 74).

In terms of research about teachers, Borba and Linnares (2012) showed us aspects about online environments in mathematics teachers' education. They identify some key topics that require further research: communities and networks of teachers in online environments; knowledge-building practices in technology-mediated work group interactions; online interactions among teachers; and sustainability of these communities and kinds of organizational structures. They call our attention for how online environments can support collaboration but it raises the issue of sustainability. They appoint that is not clear, what factors and how online communities keep being a "place" where learning and teaching takes place.

> The extensive literature in teacher education has pointed out the need of collaboration among teachers and among teachers and researchers in order to understand complex themes regarding mathematics education in general. Online communities facilitate finding a community that is connected to your interests, but this has not been sufficient to acknowledge its sustainability (Borba & Linnares, 2012, p. 703).

In my opinion, the online environments have many considerable particularities that guarantee their use in teachers' education, for instance. In the other hand, it is known that one should constantly ask themselves about the problems generated in online environments, as well as, what is missing in then. This kind of question is very important, because technology makes part of every one's life and reality, of our classes and the answers can help us constitute a specific discourse about DT. According to Clark-Wilson, Robutti, and Sinclair (2014):

> The role of digital technologies within this discourse has an increasing relevance as the society and government place demands on teachers to integrate technology into their classroom practices so that students can experience its potential as a powerful learning tool (Clark-Wilson et al., 2014, p. 1).

In addition, Bicudo (2018) advances about DT discourse and assumes a philosophical perspective to expose comprehensions about cyberspace, understood as an important aspect of the world reality. She presents ontological aspects of cyberspace, epistemological aspects in cyberspace and anthropological aspects. For her, the aspects are seen as intertwined and are to be considered an entanglement that unveils the complexity of the everyday reality in which one is living and in which cyberspace has been constituted. In the mathematic education area, Bicudo (2018) understands that the conception of thinking-with-technologies in Rosa (2008) found its base in the Heideggerian view.

Based on the perspective about the role of technologies in mathematical thinking, Rosa, Bairral, Gitirana, and Borba (2018) share the Brazilian conceptions of use and work with technologies in mathematics education. They discuss computer-supported collaborative learning and the exploration of collaboration as a principle for cognitive mathematical development. Also, they discuss: (a) factors involved in the design of digital resources for mathematics education, (b) the use of touchscreen devices in mathematical activity and the embodied nature of mathematical cognition, (c) the humans-with-media construct, which highlights the centrality of media in the productions of mathematical knowledge and (d) the initial and continuing development for teachers who work with these technologies, in classroom and/or in distance education, both towards cybereducation with mathematics teachers.

The discussion shows a diversity within the different theoretical ideas that supported the Brazilian research in mathematics education with digital technologies; however, there is one common point: all of them seem to look for, or state at front, that mathematics may be changing as different technology become present in the classroom. It would claim that mathematics is in change as touch technology, or different ways of using Internet are present in the production of knowledge by students or teachers, or by collectives that involve both of them. Mathematics in this perspective should not be considered as a result, but as process, mathematics in change. Its roots of the research in ethnomathematics strongly emphasized the notion that mathematics change as cultural groups differ. We may see in this chapter, with their different theories, mathematics process changing as technology differs.

Nevertheless, we will look to Heidegger for his phenomenological insight into technology in order to address this theme immersed in philosophy. We believe that in this way we can contribute in a particular view to the research in mathematics education with digital technologies, which for decades has been constituted and advanced through different theoretical perspectives.

3 Technology: A Heideggerian Perspective

Heidegger (1977) in his paper called "The Question Concerning Technology" discusses technology. In this same paper, he proposes a reflection way, which can open our human existence to the essence of technology. In his own words: "When we can respond to this essence, we shall be able to experience the technological within its own bounds. Technology is not equivalent to the essence of technology" (Heidegger, 1977, p. 3).

When the author highlights the essence of technology, he wants to say it does not correspond to a technological thing, which happens to be purely physical. One cannot experience the essence of technology considering only technological aspects. That is, each relationship with technology is not neutral, because, in current days, the people are together and chained to technology, whether they affirm or deny it.

If the current relationship with technology is considered neutral, the essence of technology is more likely not to be perceived. However, what must be the technology's essence then? Heidegger, 1977, p. 3) says:

> According to ancient doctrine, the essence of a thing is considered to be *what* the thing is. We ask the question concerning technology when we ask what it is. Everyone knows the two statements that answer our question. One says: Technology is a means to an end. The other says: Technology is a human activity.

In both cases, the definitions belong together, because to find ends and produce or utilize the means to them is a human activity. For example,

> The manufacture and utilization of equipment, tools, and machines, the manufactured and used things themselves, and the needs and ends that they serve, all belong to what technology is. The whole complex of these contrivances is technology. Technology itself is a contrivance, or, in Latin, an *instrumentum* (Heidegger, 1977, pp. 3–4).

According to the understanding of technology to which it is a means and a human activity, the instrumental and anthropological definition of technology is accepted and evidenced. Hardly anyone would disagree that is correct. Yet,

> The instrumental definition of technology is indeed so uncannily correct that it even holds for modern technology, of which, in other respects, we maintain with some justification that it is, in contrast to the older handwork technology, something completely different and therefore new (Heidegger, 1977, pp. 3-4).

Once all kinds of technology are a means to an end, so is modern technology. From power planes with its turbines and generators, to jet-aircrafts and the high-frequency apparatus, radar stations, computers, cellphones, stoves, pans and weather vanes are technologies. However, according to Heidegger (1977)

> Everything depends on our manipulating technology in the proper manner as a means. We will, as we say, "get" technology "spiritually in hand." We will master it. The will to mastery becomes all the more urgent the more technology threatens to slip from human control (Heidegger, 1977, p. 4).

The author assumes that technology is no longer a mere means and questions: what would be like to master it? Moreover, taking the misconception of instrumentality, how would we understand technology? So, the: "[…] instrumental definition of technology still does not show us technology's essence. In order that we may arrive at this, or at least come close to it, we must seek the true by way of the correct. We must ask: What is the instrumental itself?" Heidegger, 1977, p. 4).

When Heidegger questioned instrumentality, he broadens the idea that technology is a means to an end, since every end has a cause, and therefore discusses the four causes for the philosophy. In this discussion of the cause *materialis*, the cause *formalis*, the cause *finalis* and the cause called *efficiens*, he advances the understanding of some relevant themes such as *phisis* and *poiesis*, by highlighting the importance of revealing as the cause of these themes.

> What has the essence of technology to do with revealing? The answer: everything. For every bringing-forth/is grounded in revealing. Bringing-forth, indeed, gathers within itself the four modes of occasioning – causality – and rules them throughout. Within its domain belong end and means, belongs instrumentality. Instrumentality is considered to be the fundamental characteristic of technology. If we inquire, step by step, into what technology, represented as means, actually is, then we shall arrive at revealing. The possibility of all productive manufacturing lies in revealing. Technology is therefore no mere means. Technology is a way of revealing. If we give heed to this, then another whole realm for the essence of technology will open itself up to us. It is the realm of revealing, i.e., of truth (Heidegger, 1977, p. 7).

According to the author, when the origin of the word technology is taken into consideration, it is also possible to conclude so:

> The word stems from the Greek. Technikon means that which belongs to techne. We must observe two things with respect to the meaning of this word. One is that techne is the name not only for the activities and skills of the craftsman, but also for the arts of the mind and the fine arts. Techne belongs to bringing-forth, to poiesis; it is something poietic (Heidegger, 1977, p. 7).

Furthermore, the word *techne* is linked to the word *episteme*. Both words should be thought and analysed in a broader sense. It means that such idea provides an opening and this is a kind of revealing. The use of things themselves, the needs and ends that they serve, belong to a flow that runs between *episteme* and *techne*. When one builds something, they reveal themselves what should be produced, what it is and what is its finality. If one has the view of the final product to be produced, the manner of its construction is determined. Consequently, what is decisive in *techne* is not in the creation, manipulation or use of means, but in the revealing of what the action of building is proposing. "Technology is a mode of revealing" (Heidegger, 1977, p. 8). In addition, the author stated an intriguing question, in which he tried to comprise the concept of modern technology:

> What is modern technology? It too is a revealing. Only when we allow our attention to rest on this fundamental characteristic does that which is new in modern technology show itself to us. And yet the revealing that holds sway throughout modern technology does not unfold into a bringing-forth in the sense of *poiesis*. The revealing that rules in modern technology is a challenging [*Herausfordern*], which puts to nature the unreasonable demand that it supply energy that can be extracted and stored as such (Heidegger, 1977, p. 8).

To this extent, the current challenge concerning to technology is revealed by unlocking the energy that is hidden in nature. By doing so, what is unlocked is also transformed, and what is transformed is also stored, what is stored is in turn distributed and what is distributed is always reversed. Unlocking, transforming, storing, distributing and switching are ways to reveal. But the revealing never ends. The act of revealing shows its own intertwined paths by regulating its flow.

This regulation is based on the challenge of revealing the real, understood as what can be ordered, as a permanent reserve. This action, then, is called *Enframing*.

> Enframing means that way of revealing which holds sway in the essence of modern technology and which is itself nothing technological. On the other hand, all those things that are so familiar to us and are standard parts of an assembly, such as rods, pistons, and chassis, belong to the technological. The assembly itself, however, together with the

> aforementioned stockparts, falls within the sphere of technological activity; and this activity always merely responds to the challenge of Enframing, but it never comprises Enframing itself or brings it about (Heidegger, 1977, p. 12).

To Heidegger, *Enframing* is not just a human activity, nor a means within that activity. The merely instrumental and merely anthropological definition of technology is therefore unsustainable in principle. However, this can show itself quite intimidatingly. To claim what Heidegger calls the essence of technology, *Enframing* is not a means for any human activity to occur. It rather defies the rationality of what has always been presented to us as technology. So, one can consider this fact to be something monstrous, devilish, the very demonry of technology.

> There is no demonry of technology, but rather there is the mystery of its essence. The essence of technology, as a destining of revealing, is the danger. The transformed meaning of the word "Enframing" will perhaps become somewhat more familiar to us now if we think Enframing in the sense of destining and danger. The threat to man does not come in the first instance from the potentially lethal machines and apparatus of technology. The actual threat has already affected man in his essence. The rule of Enframing threatens man with the possibility that it could be denied to him to enter into a more original revealing and hence to experience the call of a more primal truth. (Heidegger, 1977, p. 16).

The *Enframing* is beyond the frames themselves. *Enframing* is beyond the action of placing pictures or frames, it is beyond the human activity of using technology and treating them as instruments or means to an end.

> Is then the essence of technology, Enframing, the common genus for everything technological? If that were the case then the steam turbine, the radio transmitter, and the cyclotron would each be an Enframing. But the word "Enframing" does not mean here a tool or any kind of apparatus. Still less does it mean the general concept of such resources. The machines and apparatus are no more cases and kinds of Enframing than are the man at the switchboard and the engineer in the drafting room (Heidegger, 1977, p. 17).

Thusly, tools and human beings are together in the world. They neither complement each other, nor supplement each other, they frame and exist together. If framing is considered to be the essence of technology, the presence of technology gives human input into what one can neither invent nor make in any way. There is no such thing as a human who, solely of itself, is only human, for it acts with the world in the world.

4 Traits and Notes: Phenomenological Perspectives for What Is Investigated in Mathematical Education with Digital Technologies

In almost 30 years of research on digital technologies and mathematical education revisited in this chapter, it was possible to identify in a significant amount of studies that technology is assumed as a means to an end, or as a human activity or even as an instrument. When technology is represented as an instrument, there is an implicit way a desire to master it. The idea of instrument materializes the human need for mastery and overcoming.

Nonetheless, when the essence of this instrument is questioned, the search for the essence of technology is similarly understandable, once the instrument itself does not account for the essence of technology. When one finally considers that the presence of the essence of technology happens in the concession of the human being, then the following becomes clear:

> The essence of technology is in a lofty sense ambiguous. Such ambiguity points to the mystery of all revealing, i.e., of truth. […] On the one hand, Enframing challenges forth into the frenziedness of ordering that blocks every view into the coming-to-pass of revealing and so radically endangers the relation to the essence of truth. […] On the other hand, Enframing comes to pass for its part in the granting that lets man endure – as yet unexperienced, but perhaps more experienced in the future – that he may be the one who is needed and used for the safekeeping of the coming to presence of truth. Thus, does the arising of the saving power appear (Heidegger, 1977, p. 19).

The power of the DT, under many interfaces that conceal it, contributes to a more humane view of these devices. So, digital technologies are always constituted by a skeleton structured by logic and, as this skeleton changes (that is, it is getting modified in) the way of being available, allowing interactivity between the user and the program. This interactivity is exposed through the instrumental relationship of some authors. What happens is that interfaces allow expanded approaches to the modes of the human subject, in their complexity, to the world. For example, distance education courses may take the form of a face-to-face class, in which distance education students perform activities that replicate those face-to-face classes or delineate new classroom designs, in which group communication and interaction take place. If technologies would be considered from their essence, there is no instrument or a means to an end, but the revealing of all possible things to be created and imagined.

By considering the *Enframing* as an action beyond the use of frames, beyond dividing tools and resources, it is possible to propose a reflection on the relation with DT and consider it like being-with-DT, thinking-with-DT and know-how-to-do-with-DT.

> Thus, the being-with-DT includes the creation of a new world or microworlds, where users are necessarily 'plugged' with the technological means; thinking-with-DT can allow the construction of mathematical knowledge in relations with the world and with others, including the (trans)formation of mathematical ideas with technological means (computer, software, video, etc.); and the know-how-to-do-with-DT shown by intentional actions taken with the world, with myself and others, actions that are performed in the activity in the construction of a product in practice (Rosa et al., 2018, p. 143).

In conclusion, the DT can be understood as a possible *Enframing* of cognitive changes.

> For this, it was considered a philosophical assumption that identifies the work with DT in the Heideggerian phenomenological perspective of being-there, and/or being in the world with (Heidegger, 1996), which in Rosa (2008) is presented in terms of connection with cyberspace as being-with, think-with, know-how-to-do-with-technologies. So, it is important to highlight that among the different theoretical bases on […] Digital Technologies, the changes that the media suffer constantly generate new questions about how to educate [students and] teachers to address these changes while they transform their own education and teaching practice. Thus, we have evolved a lot in terms of thinking about the

> Mathematics Teachers Education in line with the use of resources and technological environments in order to no longer characterize and invest in a reproductive practice, but in an expansion/transformation of existing mathematical thinking and that to be encouraged (Rosa et al., 2018, p. 143).

In the event, taking into consideration the studies on DT and mathematics education analysed in this chapter, it became clear that to those authors technology is understood as a resource. These studies claim the technological apparatuses to be simple instruments used to achieve a determinate goal of teaching, a means to end and a human activity.

In the other hand, the phenomenological perspective considers that, technology can be taken as an *Enframing,* once it participates in the act of being in the world *(Dasein)* effectively. In other words, the mathematics learning with technologies can take place when the students work in such an environment where *being-with, think-with, know-how-to-do-with-technologies* are actualized.

Although both perspectives aim to contribute to mathematics learning process, a discrepancy among them remains remarkable. The perspective, which considers technology as a resource, takes into account the DT as auxiliary tools to mathematics learning. The other, which postulates the concept of *Enframing,* ponders the DT as a participant of the learning process.

Thus, once the technological essence is not technological, any discussion or reflection on its essence, as well as, any confrontation to it must necessarily emerge in such a paradoxal prisma, which simultaneously confirms and denies it. As a professor and researcher on this area, I acknowledge the great difficulty in understand technology as something beyond its own technological aspects and characteristics. Even though, the advances of understanding on the essence of technology lay exactly on concepting of DT as the act of *Enframing*. DT is more than a mere object or tool, it is seen as a non-human actor in the mathematics education process.

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References

Artigue, M. (2007). *Digital Technologies: a window on theoretical issues in mathematics education*. Acesso em Novembro 20, 2019, em https://www.researchgate.net/profile/Michele_Artigue/publication/266577614_DIGITAL_TECHNOLOGIES_A_WINDOW_ON_THEORETICAL_ISSUES_IN_MATHEMATICS_EDUCATION/links/55f2cec108ae336d49887efd/DIGITAL-TECHNOLOGIES-A-WINDOW-ON-THEORETICAL-ISSUES-IN-MATHEMATICS-EDUCATION.pdf

Bairral, M. A. (2002). *Desarrollo profesional docente en Geometría: análisis de un proceso de Formación a Distancia.* Tese, Universidade de Barcelona, Barcelona, Espanha. Acesso em Agosto 20, 2019, em http://www.tdcat.cesca.es/TDCat-1008102-120710/

Bairral, M. A. (2004). Compartilhando e Construindo Conhecimento Matemático: análise do discurso nos chats. *Bolema – Boletim de Educação Matemática, 17*(22), 37–61.

Bicudo, M. A. V. (2018). Mathematics education actualized in the cyberspace: A philosophical essay. In P. Ernest (Ed.), *The philosophy of mathematics education today* (ICME-13 monographs). Cham: Springer.

Bicudo, M. A. V., & Rosa, M. (2015). Philosophical aspects present in questions regarding mathematics technology and education. *Studia UBB Philosophia, 60*, 47–54.
Borba, M. C. (2004). Dimensões da Educação Matemática a distância. In M. A. V. Bicudo & M. C. Borba (org.). *Educação Matemática: pesquisa em movimento* (pp. 296–317). São Paulo: Cortez.
Borba, M. C., & Linnares, S. (2012). Online mathematics teacher education: Overview of an emergent field of research. *ZDM Mathematics Education, 44*, 697. https://doi.org/10.1007/s11858-012-0457-3
Borba, M. C., & Villarreal, M. E. (2005). *Humans-with-media and the reorganization of mathematical thinking: Information and communication technologies, modeling, visualization, and experimentation*. New York: Springer Science.
Burton, D. (2009). Video in cyberspace: Does it increase engagement in an online course? In I. Gibson et al. (Eds.), *Proceedings of SITE 2009--Society for Information Technology & Teacher Education International Conference* (pp. 1783–1785). Charleston, SC: Association for the Advancement of Computing in Education (AACE). Retrieved November 8, 2019 from https://www.learntechlib.org/primary/p/30875/.
Castells, M. (2003). *A Galáxia da Internet: reflexões sobre a Internet, os negócios e a sociedade*. Tradução de Maria Luiza X. de A. Borges. Rio de Janeiro: Jorge Zahar
Castells, M. (2005). *A Sociedade em Rede* (6th ed.). Tradução de Roneide Venâncio Majer. São Paulo: Paz e Terra.
Chronaki, A. (2003). Mathematical learning as social practice: Basic dimensions for the design of an online context. In A. Gagatsis & S. Papastavridis (Eds.), *Proceedings of the 3rd Mediterranean Congress on Mathematics Education* (pp. 705–716). Athens, Springer, Cham.
Chronaki, A., & Christiansen, I. (Eds.). (2005). *Challenging perspectives on mathematics classroom communication*. Charlotte, NC: Information Age Publishing.
Clark-Wilson, A., Robutti, O., & Sinclair, N. (Eds.). (2014). *The mathematics teacher in the digital era* (Vol. 2). Dordrecht: Springer.
de Oliveira, L. M. P. (2002). Educação a distância: novas perspectivas a formação de Educadores. In M. C. Moraes, (org.). *Educação a Distância: fundamentos e práticas*. Campinas: UNICAMP/NIED.
Engelbrecht, J., & Harding, A. (2005). Teaching undergraduate mathematics on the internet. *Educational Studies in Mathematics, 58*, 253–276. https://doi.org/10.1007/s10649-005-6457-2
Heidegger, M. (1996). Being and time. Albany: State University of New York Press.
Heidegger, M. (1977). *The question concerning technology, and other essays*. New York: Garland Publishing, INC.
Hoyles, C. (1995). Thematic chapter: Exploratory software, exploratory cultures? In A. A. Di Sessa et al. (Eds.), *Computers and exploratory learning* (NATO ASI series (series F: Computer and systems sciences)) (Vol. 146). Berlin, Heidelberg: Springer.
Hoyles, C., & Noss, R. (1992). A pedagogy for mathematical microworlds. *Educational Studies in Mathematics, 23*(1), 31–57. https://doi.org/10.1007/BF00302313
Hoyos, V. (2012). Online education for in-service secondary teachers and the incorporation of mathematics technology in the classroom. *ZDM Mathematics Education, 44*, 775–786. https://doi.org/10.1007/s11858-012-0453-7
Isdale, J. (1993). *What is virtual reality?* Acesso em Novembro 12, 2019, from http://www.columbia.edu/~rk35/vr/vr.html
Kenski, V. M. (2003). *Tecnologias e ensino presencial e a distância*. Campinas: Papirus.
Laborde C. (2001). Integration of technology in the design of geometry tasks with Cabrygeometry. *International Journal of Computers for Mathematical Learning, 6*(3), 283–317.
Laborde, C. (2007). The role and uses of technologies in mathematics classrooms: Between challenge and modus vivendi. *Canadian Journal of Science, Mathematics and Technology Education, 7*(1), 68–92. https://doi.org/10.1080/14926150709556721
Laurillard, D. (2008). The teacher as action researcher: Using technology to capture pedagogic form. *Studies in Higher Education, 33*, 139–154.

Lévy, P. (2000). *Cibercultura* (2nd ed.). Tradução de Carlos Irineu da Costa. São Paulo: Editora 34.
Lévy, P. (2010). *Cyberspace as a meta evolutive step*. Acesso Outubro 10, 2019, em https://www.researchgate.net/profile/Pierre_Levy4/publication/267807775_Cyberspace_as_a_meta_evolutive_step/links/55e5b19308aede0b5736ef62.pdf
Mansur, A. (2001). A Gestão na Educação a Distância: novas propostas, novas questões. In E. Litwin (org.), *Educação a Distância: temas para o debate de uma nova agenda educativa*. Tradução de Fátima Murad. Porto Alegre: Art Med.
Rosa, M. (2008). *A Construção de Identidades Online por meio do Role Playing Game: relações com o ensino e aprendizagem de matemática em um curso à distância*. Tese, Instituto de Geociências e Ciências Exatas - Universidade Estadual Paulista, Rio Claro.
Rosa, M., Bairral, M., Gitirana, V., & Borba, M. (2018). Digital technologies and mathematics education: interlocutions and contributions based on research developed in Brazil. In A. J. Ribeiro et al. (Eds.), *Mathematics education in Brazil* (pp. 129–147). Cham: Springer. Acesso em Agosto 25, 2019, em https://doi.org/10.1007/978-3-319-93455-6_7
Rosa, M., & Lerman, S. (2011). Researching online mathematics education: Opening a space for virtual learner identities. *Educational Studies in Mathematics, 78*, 69–90.
Rosa, M. & Maltempi, M. V. (2006) Realização de Projetos a Distância: contribuições da colaboração à Educação Matemática. In *8° S*i*mpósio de educación matemática*, 8, Buenos Aires.
Shaffer, D. W., & Kaput, J. J. (1998). Mathematics and virtual culture: An evolutionary perspective on technology and mathematics education. *Educational Studies in Mathematics, 37*(2), 97–119. https://doi.org/10.1023/A:1003590914788
Simmons, S., Jones Jr., W., & Silver, S. (2004). Making the transition from face-to-face to cyberspace. *TechTrends: Linking Research and Practice to Improve Learning, 48*(5), 50–55. Retrieved November 8, 2019, from https://www.learntechlib.org/p/65619/.
Tallent-Runnels, M. K., Thomas, J. A., Lan, W. Y., Cooper, S., Ahern, T. C., Shaw, S. M., et al. (2006). Teaching courses online: A review of the research. *Review of Educational Research, 76*(1), 93–135. https://doi.org/10.3102/00346543076001093
Thomas, M. O. J., & Holton, D. (2003). Technology as a tool for teaching undergraduate mathematics. In A. J. Bisho et al. (Eds.), *Second International Handbook of Mathematics Education. Springer International Handbooks of Education* (Vol. 10). Dordrecht: Springer.
Underwood, J. (2009). *The impact of digital technology: A review of the evidence of the impact of digital technologies on formal education*. Coventry: Becta. https://dera.ioe.ac.uk/10491/3/A9RF934_Redacted.pdf
Winn, W., & Bricken, W. (1992). Designing virtual worlds for use in mathematics education: The example of experiential algebra. *Educational Technology, 32*(12), 12–19. Retrieved November 8, 2019, from https://www.learntechlib.org/p/170866/.
Zullato, R. A (2007). *Natureza da Aprendizagem Matemática em um Ambiente Online de Formação Continuada de Professores*. Tese, Instituto de Geociências e Ciências Exatas – Universidade Estadual Paulista, Rio Claro.

The View of the World and of Knowledge Made Explicit by Phenomenology: A *Turning Point* in the Way of Understanding Reality

Maria Aparecida Viggiani Bicudo

1 Understanding the Turning Point in the View of Knowledge and Reality

Here, I am referring to the change in the way reality and the object of knowledge is viewed in Western civilization. Traditionally, since the beginning of philosophical thought, with the ancient Greeks, and the first pre-Socratics, the question focused on *what real is*, with the fundamental question: *what is this, what is?* As widely known, the answers have since been different in terms of understanding the primordial substance of what is real, but the interrogation has remained, among the Greeks, and throughout the history of this civilization. It is important to point out that, even in Ancient Greece, little by little, the question about what is real, characterized as the great ontological question, has also come to be linked to an epistemological question: *how can we truly known what is*? As studies progressed, through the VIII, VII, VI, V, IV, and III centuries B.C., three questions that lie at the heart of Western philosophical thinking have arisen and remain to this day: what is it (what is real?); how do you know what is real? (which knowledge is true, correct, right?); what is value? (what is it worth?). Hence, the fields of ontology, epistemology, and axiology have since been constituted.

Through the course of history, up to the nineteenth and early twentieth centuries, what is real and reality, or the way of being real, was examined in a natural manner. Reality was just there, separated from man, who knows it, whether in this humanly lived-world, or in a world apart from this, as in the case described in Platonic philosophy. Until the fifteenth century, philosophical questions were approached from

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a cosmological point of view, including questions concerning the understanding of what is beyond possible logical links, which can be established through observable changes in daily life, for example.

Following the thread established by the title of this chapter, I highlight the worldview and logic of scientific thinking, as it has been established since the transition from the Middle Ages into the Modern Age, drawing on the teachings of Galileo, Newton, Descartes, pointing out, through these authors, a moment of construction of the ideas and methods of modern science. The search for valid knowledge of what real is, understood in the sense of what is correct, continues. However, it is now geared towards the objectivity of the world, taken in specific aspects to be studied, and not in its entirety and complexity, understood in the breadth of the cosmological vision.

The underlying logic of epistemological procedures aimed at knowing specific aspects of what is real privileges the hypothesis as a possible answer to an interrogation or problem. This hypothesis can be answered through objectively observed and measured experiments, which demonstrate the validity of the results obtained. The success of investigations conducted this way strengthened the confidence in this kind of procedure and helped establish the concept of real, understood as objectively given and taken in a natural way; that is, without questioning, just by asking "is this really so?" The investigator is understood as separate from the object.

I believe it is important to emphasize that in this historical and philosophical context what is real was taken in its physical aspects. It was interrogated in terms of the substances that constitute it, of the laws that govern it, of its origin, as posed by the pre-Socratic philosophers, determined by a substance or first principle. In prevailing modern philosophy, the knowledge of truth is in tune with the procedures dictated by a method, which has been proven successful in the pursuit of knowledge of what is real, delineated in terms of theoretical knowledge, the hypothesis raised, and its demonstration.

The success of this philosophy and its respective knowledge procedures has a direct impact on natural sciences and applications, which have also been proven fecund in the field of industry and, as a consequence, commerce. The first Industrial Revolution is an example of such changes, understood as important for the production of consumer goods and the generation of capital in England, France, later spreading throughout Europe and elsewhere. Science came to be seen as the truth that dictates what must be done to avoid mistakes and validate judgments made through different discourses.

Disciplines other than mathematics, physics, chemistry, biology began to aspire to *science status*. To achieve that, it was necessary to make the object of study explicit, accessible to being objectively investigated, and the research *method* understood as valid for knowing the object studied. History, sociology, and psychology are among the sciences which have pursued such status.

In the mid-nineteenth century many scholars, including philosophers, historians, and writers, began to question whether the humanities, or sciences of the spirit, could be studied according to the scientific method, as did natural sciences. The historical aspects, as much as the cultural, social and works of philosophy and literature, present in the debates mentioned, were understood as constitutive of worldly reality. Concomitantly, the subject of knowledge came to be understood not only from the point of view of its rationality, but as a more complex entity, as existential aspects, such as the anxiety, so well-described by Kierkegaard, the unconscious, dealt with by Freudian psychoanalysis, came to light. These questions, respective debates, and published works populate the understandings about what is real with *meanings* and *senses*. *Real* that comes to be understood in a more complex way, in comparison to the manner through which it had been understood within the realm of natural sciences.

I understand that the quest towards new research paradigms begins in the midst of these interrogations and arguments. I point out "Truth and Sense" (Piettre, 2005) and the ways both ideas are approached from different perspectives. This author directs his treatise in the above-mentioned article by interrogating and debating as follows:

> Does truth exist outside the realm of science, does it proceed only from logical deductions, does it thrive only on experimental ground, as positivists would like to believe? Now, the word truth also summons the interpretation of a literary or religious text, the interpretation of the work of art, that of human behavior or of utterances. Can truth be lying in what makes sense, in what has a meaning? However, the interpretation of meanings seems conjectural, whereas truth would necessarily be objective (Piettre, 2005, p. 28).

Focusing in the excerpt above, I ask: what does it mean to say "[...] whereas truth would necessarily be objective"?

Under the perspective of mathematical logic, and drawing on Alfred Tarski,[1] a statement is true if it designates an existing state of affairs. In order to explicate that, we have a state of affairs that is simply a situation, something that may or may not be real, indicated by an assertion. Objectivity refers to what is stated in the proposition that asserts the state of affairs. Propositional logic preserves truth to the extent that the assertion expresses an understanding of something. "Being true means there is a relationship between true statements and the states of things to which they refer" (Almeida, 2012). And yet, according to that author:

> It is important to realize that, whereas before being true and being false were predicates or properties solely of certain states of mind and the statements that expressed them, they are now predicates or properties primarily of the states of things referred to by beliefs and statements, and therefore it is necessary to differentiate between stating the truth (*alêtheuein*) and being true (*alêthe*), as well as between stating falsity (*pseudesthai*) and being false (*pseudes*) (Almeida, 2012, p. 268, author's translation).

Therefore, I understand that the truth, in the realm of logic, is expressed in the objectivity of the proposition, and it remains a statement of the truth, which differs

[1] Alfred Tarski (Warsaw, at the time the Russian Empire, nowadays Poland, on January 14, 1901—Berkeley, USA, October 26, 1983) was a Polish logician and mathematician. He immigrated to the USA in 1939 where he became a naturalized citizen in 1945. Tarski taught and carried out research in mathematics at the University of California, Berkeley, from 1942 until his death. He described himself as a "mathematician (and also a logician, and, perhaps, in a way, a philosopher). Accessed October 13, 2019 at https://pt.wikipedia.org/wiki/Alfred_Tarski.

from being true. That means, "being true" understood as a correspondence between what is there in the concrete world and what is being said about it.

From the standpoint of reality, understood as objectively existing, truth refers to the adequacy between what is known, through rigorous investigation, and that reality. The claim in this statement raises questions about the correspondence between seeing or observing reality and the statement about what has been seen or observed. In this view, it is believed that there is a *correspondence* or agreement between reality and the statement about reality.

This refers to Heidegger's (1970) view of truth, understood as adequacy or agreement, while examining it in the context of the traditional interpretation of truth. For that author (Heidegger, 1970), the current concept of truth is understood as the agreement between knowledge with its object. Considering *reality* as the object of knowledge, then truth equals the knowledge one has of it. However, knowledge is enunciated and such enunciation expresses the entity as it is seen/observed. According to that author, by keeping in agreement with the object of knowledge, the enunciation can hide the entity, keeping it veiled and disguised, since it is only linked to the way which the entity appears.

The conception of truth understood as agreement supports the possibility of explaining the statement. Piettre (2005, p. 29, author's translation) states: "Explaining the facts is to unfold them [...] linking them in a series of causal relations: to explain a disease is to search for the cause [...]." This understanding is in line with the procedures of natural sciences, whose purpose is to explain the facts and predict occurrences. Facts are understood as occurrences, happenings, episodes, circumstances, events, steps, acts, conjunctures, successes, achievements, adventures, sets of information about something. Thus, the goal is to explain the causes for certain occurrences to have happened and predict future events. Explanations, as accepted by natural sciences, which assume the scientific method, positivistic in character, are based on research about reality, viewing reality in a naturalistic way, i.e., without asking the question: what is reality? and conceive truth as a correspondence between what is real and what is stated about it.

Another possibility of understanding the truth, following the Heideggerian argumentation, is to take it as *Alethea* or unveiling of the entity itself, seeking the knowledge of the essence, that is, what is revealed under the scrutiny of the interrogation. This search has proven appropriate to the investigation of the sciences of the spirit.

In the wake of the discussions about whether the procedures used in natural sciences are appropriate or not for the investigations of the sciences of the spirit, in the late nineteenth century, Dilthey (*apud* Piettre, 2005, p. 30, author's translation) presents us with important work, opening horizons, so that we may tread more appropriate paths towards the knowledge of the matters dealt with by these sciences.

> Dilthey opposed sciences of the spirit (*Geisteswissenschaft)* to natural sciences. In natural sciences, such as physics or biology, certain objective facts can be guaranteed, while in sciences of the spirit, such as psychology or history, the facts are circumscribed by a series of gestures, behaviors and intentional acts, that is, spiritual or psychical phenomena that make sense (Dilthey apud Piettre, 2005, p. 30, author's translation).

Thus, the focus is on the meaning of gestures, the expression of intentional acts, that is, spiritual phenomena. However, meaning escapes accuracy, as it is not trapped in the agreement between the affirmation and the affirmed.

How to probe the meaning of the phenomena affecting humanities? How to know if the statements, even if they explain understandings, can be considered as knowledge?

An author who opened a path for work with the concept of *meaning* as early as the twentieth century is Dilthey.[2] He developed a whole philosophical framework focusing on the *Philosophy of Life*. It weaves a complex web of interconnected concepts that spell out the body of this philosophy, whose central themes are: life, totality of psychical life, experiences, history, objective expressions of experiences, hermeneutics. According to Amaral (1987)

> The expression 'life' reveals what is best known and most intimate in each one of us and, at the same time, most obscure, and indeed, something totally impenetrable. Life is an unfathomable mystery; starting point of all meditation, investigation, and thought lies in this unfathomable element. All knowledge is based on something never fully cognizable (Amaral, 1987, p. 12, author's translation).

This idea is known as the theory of impenetrability of life. Impenetrable, though not completely, as it is may be known through the description and analysis of historical-social facts. Such facts speak of the wholeness of the part-whole, in which the part refers to the psychophysical vital units, and the whole to the historical-social reality, as the unity of life. This whole is complex because it includes different projections of psychical structural connections linked to historically developed systems of psychophysical vital units. It is no longer possible to represent the complexity of this totality, much less deduce it according to logical laws, from metaphysical and ontological principles, but only to experience it. Thus, the investigation procedures of Diltheyan psychology are descriptive-analytical (Amaral, 1987). This kind of investigation is conducted by describing and analyzing experiences. Dilthey seeks analytical procedures that point towards the comprehension of the totality of historical-social facts and, by doing so, revives concepts of hermeneutics. According to Palmer (1969), the philosophy of contemporary thinkers understands hermeneutics as an interpretation of life, of experience lived and described, not exegetical.

Focusing on the ways through which life is revealed, the works of Dilthey opened the possibility of rigorously investigating human sciences. It evidences a logic

[2] Wilhelm Dilthey (19 November 1833–1 October 1911) was a German historian, psychologist, sociologist, and hermeneutic philosopher, who held G. W. F. Hegel's Chair in Philosophy at the University of Berlin. As a polymathic philosopher, working in a modern research university, Dilthey's research interests revolved around questions of scientific methodology, historical evidence, and history's status as a science. He could be considered an empiricist, in contrast to the idealism prevalent in Germany at the time, but his account of what constitutes the empirical and experiential differs from British empiricism and positivism in its central epistemological and ontological assumptions, which are drawn from German literary and philosophical traditions. Dilthey took some of his inspiration from the works of Friedrich Schleiermacher on hermeneutics. Accessed November, 6, 2019, at https://en.wikipedia.org/wiki/Wilhelm_Dilthey.

which is different from that of hard and natural sciences and, by doing so, the procedures of qualitative research are outlined.

Reviewing this thread of historicity, I understand that, since then, there has been a significant change in the manner we understand what is real, that is to say, the reality of the ways of being real, which is gradually being understood as a complexity that can be studied from different perspectives, even if one tries to understand its totality. The course described above highlights different ways of conceiving what is real and reality, as well as getting to know it, by pursuing truth, sense, and meaning. This understanding indicates a change in the possibilities for investigating reality from different perspectives. Therefore, it is not a matter of opposing scientifically produced knowledge about natural aspects of reality and knowledge of life as experienced in its myriad of facets. It is about understanding what the knowledge thus produced speaks of the world and the importance of developments that affect applied science, always approaching them in an inquiring way.

2 Understanding the Underlying Logic of the Sciences of the Spirit

The considerations above show that investigations carried out according to positivist procedures may indeed be appropriate to investigate physical and biological aspects of natural reality, for example. They also show that qualitative research is successful when focused on humans and their ways of being and being in the world, which develop into cultural and social dimensions. History is always intertwined in this reality, for it deals with the very "*évènement*" of human reality, which is in motion. Language, in its various modes of expression, is the medium that exposes events and sustains them in the materiality of the objects produced, such as texts, paintings, music, sculptures, and utensils. Human life itself occurs in the *here* and *now*, therefore, it happens in the spatiality and temporality of the intersubjective dimension of reality, which encompasses the constitution of communities and the production of social organizations.

Such aspects of human reality, understood in their complexity, call for an appropriate mode of investigation. As already pointed out throughout this text, the logic that underpins the sciences of the spirit is different from the logic that underpins the investigations of exact and natural sciences. The latter see the truth understood as the correspondence of a statement with the reality studied, the demonstration of certainty, according to the criteria laid down by the scientific methodologies followed. They aim at generalizations and explanations arising from generalized truths. Human sciences focus on meaning. In a qualitative way, they express understandings of the phenomenon studied; they do not aim at generalizations. They explain the meaning understood in the horizon of historical, cultural, and social grounds.

If *meaning* escapes objectivity and accuracy, then how can one penetrate phenomena regarding humanities through meaning? How can we know if the statements, even when explaining understandings, can be considered as knowledge generated in the dimension of science, in which knowledge differs from common sense or religious belief, for example? How to avoid being stuck in the subjective opinions, and opinions biased by emotions or preconceptions? If we move away from the objectivity of facts and their causal relationships, how can we know reality?

By posing such questions we move towards the understanding of the underlying logic of the sciences of the spirit, which, as stated previously, understand truth as *aletheia* or the unveiling of the entity itself, that is, of what one looks for and seeks to understand. Separate aspects of reality, understood as complex, such as the historical ones, those concerning man-world, man-man, and individual-society dialectic, demand appropriate manners so that what lies beneath the surface of the happening can be uncovered. These can be procedures based on content analysis, hermeneutic analysis, ethnography, studies that deal with symbolic representations, studies that assume the contradiction between man and nature, studies that focus on history, seen as the movement of content that "[…] engenders differences, polarities, conflicts, theoretical and practical problems […]" (Lefrevre *apud* Sanfelice, 2005, p. 145, author's translation).

Note that these procedures are qualitative. In order to cope with the qualitative, they deal with a description of what happened. They deal with interviews, reports, narratives; poetic texts, discourses and their intelligibility, with aspects of human subjectivity, and also intersubjectivity; with the movement of events, revealing their historical aspects; texts and contexts; words and their polysemy, language, both as grammar and expression, and as triggers of uses, customs, socio-cultural values of subjects; with social organizations and their presence in the determination of the subject. I mentioned a range of qualitative procedures, knowing that I did not exhaust them, but also that I gave a panorama so one can immediately understand, at first glance, differences in the way of conceiving humans, the world, and knowledge.

In the Western world, a manner of investigation has been articulated in an attempt to account for the qualitative character present in the complexity of human reality; it is called qualitative research. In this context, we work with the analysis of texts, but without taking them as truth; we act as interlocutors who think about the subject, object of the research; with reports, descriptions of what happened, etc. However, these data require interpretation that may be realized through the enlightening dialogue established between the researcher (including the research group); the interrogation; the content of the text, examined within its context; which is displayed, revealing the understanding of the theme under study. It is important that the research be carried out according to the underlying philosophical principles. For instance, in the case of dialectical materialism, the investigation must be dialectical. Sanfelice (2005, p. 144), while explaining Hegel's understanding of dialectics, states:

> […] Method should not disdain formal logic but run with it. So, what is this method? It is the consciousness of form, of the movement of content. And it is the 'content itself', the movement within itself, which drives it forward, including form. Dialectical logic adds to ancient logic, the empowerment of transitions, movements, developments, the 'necessary

and internal connection' of the parts within the whole […] […] (Sanfelice, 2005, p. 144, author's translation).

If the investigation deals with phenomenology, the movement must be that of phenomenology itself, that is, placing the phenomenon in suspension, leaving previous knowledge about it, around it; searching for the significant subjects who live the experience through which the phenomenon shows itself, describe it, analyze it, interpret it, through reductions. And, in this movement, move forward by articulating the meanings perceived in increasingly comprehensive ideas that indicate the structure of the phenomenon and, then, interpret these convergences, as well as the divergences underscored, in light of the interrogation formulated within the area of investigation.

What I sought to clarify in this part is that human sciences work within the logic of qualitative research, which seeks the meaning of what is investigated, examined in the entire movement of historicity and temporality. Such sciences deal with different conceptions of knowledge construction, constitution, and production, by embracing different philosophies that explain worldly reality and the ways through which the subject and society are constructed. In some cases, a separation between subject and reality prevails, even though both are seen within the historical-social movement as a whole, even if they are seen as inseparable. Other philosophical conceptions work with visions of the *constitutive inseparability* of subject-world. The latter being the phenomenological view.

Next, I will focus on phenomenology, a philosophy dedicated to understanding the pertinent matters of the sciences of the spirit. In an attempt to account for what was proposed in the title of this chapter, I present the *turning point* regarding the way to explain understandings about reality, the constitution of knowledge, which I understand to be present in the phenomenological philosophy of Husserl and his followers. I will highlight aspects that I view as important to understand teaching and learning in cyberspace, within this approach.

3 Understanding the *Turning Point* Within Phenomenology

The phenomenological philosophical thinking of Edmund Husserl[3] is directed towards the comprehension of reality, seeking to deepen, in a radical and original way, the knowledge of the very complexity of reality. It does not deny the objectivity of what is real, but understands that it

[3] Edmund Husserl (1858–1938) was a mathematician who left an impressive production about the interrogation that he pursued throughout his life "what is the nature of the objectivity of mathematics." The list of his publications, and the number of his class notes, are extensive. Husserl's greatest works were made public during his life. However, it is important to note that many of such works were organized by his students and followers. According to the Encyclopedia of Philosophy (2004), there are approximately 45,000 pages of stenographic notes based on his lectures and the research he conducted.

> […] It is always a reality for human beings, who must possess the instruments that enable him to know it. Thus, the *primacy of the question of knowledge is underscored*, not because everything is resolved in knowing, but because it is fundamental to reach the ultimate scope, to understand how things are done (Ales Bello, 2012, p. 104, author's translation).

Phenomenology does not deny the positive existence of the world and nature, as if they were an illusion, seeking validity achieved under the light of epistemology, as stated in Cartesian philosophy. Neither does this mean going to the other extreme, that of the implicit naive realism and the pragmatism, as if reality were objectively given in such a way that the propositions of science showed only a perfectly adequate correspondence between what is understood and what is stated. On the other hand, it does not deny science and all its production, historically and culturally present in the life-world (*Lebenswelt*).[4] However, it does not follow the prevailing practice in modern Western civilization, which views scientific knowledge as truth postulated about mundane reality. On the contrary, it radically critiques Western science, seeking to understand its roots, its constitution, and production.

Husserl believed that science is a second-order knowledge, in relation to that generated in the living-experience. As I understand it, the *turning point* comes about through the exposition of what is implied in this statement, which is the conception of knowledge, as explained by that author.

He sees himself charting *new territory* in the field of philosophical inquiry by talking about his work in the *Introduction,* which he wrote for the English translation of the first book of Ideas, dated 1931. This new landscape, metaphorically, refers to the horizon opened by phenomenology. However, this is a path he will not tread himself, given the complexity of this investigation, as well as all the meticulous investigative work that in different respects it demands, and the inevitable finitude of a man's life, in this case his own.

Thus, he expresses himself with a certain self-criticism:

> The author sees the infinite open country of the true philosophy, the "promised land" on which himself will never set foot. This confidence may wake a smile, but let each see for himself whether it has not some ground in the fragments laid before him as phenomenology in its beginnings. Gladly would he hope that those who come after will take up these first ventures, carry them steadily forward, yes, and improve also their great deficiencies, defects of incompleteness which cannot indeed be avoided in the beginnings of scientific work (Husserl, 1972, p. 22).

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[4] Life-world is a reality constructed in the historical and cultural moment that brings together present, past, and future. It is not a vessel in which we are placed or in which we drop knowledge, theories, etc., as if these were objects in their own empiricism. Rather, these are the spatiality and the temporality in whose dimensions we live with others, whether human or not, whose reality we in turn weave using articulated comprehensions, subjectively, and intersubjectively, which are materialized in available forms and contents. What is intersubjectively understood and kept, via repetition of successful actions, gradually forms itself, through the intertwining of senses and meanings into objectualities. Objectualities are objectivities built on the shift of subjectivity-intersubjectivity and, therefore, do not concern objectivity separately from this shift (Bicudo, 2018, p. 255).

The new ground, as I understand it, is the conception of reality of the life-world, not viewed as rationally created or referring to what is real as it presents itself perfectly in a non-mundane world. Nor is it about denying the world, doubting its existence, and then seeking to build it on the basis of clear and distinct truths, which would prevent fallacies, as brought by up Descartes. Husserl understands the world as being given to us immediately in the intentional view of the living-body, which apprehend the seen (noema) for *noema* for acts of consciousness, constituting and producing knowledge of the world. This new earth, which is unveiled in life-world (*Lebenswelt*), is what we see, where we move, live, produce science, put ourselves in a position of worship, and open ourselves to the divine. It is reality constructed in the historical and cultural moment that brings together present, past, and future.

As he argues, the objective world is there for you and me. As an investigator I, intentionally, put it on *epoché*; a movement made possible by reflection and, accepting myself, the investigator, as a human being in my own body and capable of realizing spiritual activities, I reflect on this reality and its characteristics, beyond what is shown in its manifolds.

In this line of thought, how is it possible to get to know worldly reality? From a phenomenological stand point, what reality are we talking about? The investigation of these two interrogations continues to intertwine them, as this reality is constituted in the movement of acts of knowledge, and processes of production of knowledge.

> The being or not being of the transcendent object towards the cognizing subject is a metaphysical problem, not theoretical-cognitive: the change in perspective of the lessons from 1906 to 1907 is propaedeutic, an essential step towards a strictly phenomenological analysis of the objectual correlate of acts of consciousness, which Husserl will realize a few months later with the lessons of *Die Idee der Phänomenologie* (1907) (Buongiorno, 2019, pp. 15–16, author's translation).

This change in perspective results from the critique of knowledge and the theory of knowledge that had been made by Husserl since *Richeche Logicke*, the fundamental thesis of his 1900/1 work. It focuses his interrogations on logic, taken at its fullest, understanding that it encompasses:

> A) all the formal disciplines of objective *a priori* (discipline of formal mathesis and real ontology); B) the *Rechetslehre* of knowledge as *noetic*[5] of the species of subjective evidence. C) the theory of knowledge, which examines the relations between cognoscitive acts, ideal meaning and the objective being (Buongiorno, 2019, p. 13, author's translation).

In order to conduct such critique, he realizes that it is necessary to proceed in a methodical manner, putting it in *epoché* and conducting a phenomenological *reduction* of all science and knowledge available at the time. It is necessary to clarify that "*epoché* does not mean *denial of all knowledge, but suspension of all unclear knowledge"* (Buongiorno, 2019, p. 14, author's translation) and that phenomenological reductions, which appeared in the lessons of 1906/7, bring about a decisive fact: "the inclusion, in the body of phenomenological investigation, of the *objectual*

[5] Logic specific term. Part of the discipline which studies the fundamental law of thought, known as the four principles: identity, contradiction, excluded middle and sufficient reason.

correlate of acts such as intentionally immanent givenness" (Buongiorne, 2019, p. 15, author's translation).

By conducting *epoché* and reduction, the data concerning the subjective phenomenon that can be investigated non-psychologically remain unquestionable, exploring the difference between empirical perception and phenomenological perception. While the former is anchored in the assumption of natural factual data, understood as occurring within the person, in their psychical experiences, the latter concentrates on phenomena. The investigation of empirical perception follows the model of natural sciences, which entails naturalizing consciousness and working in the dimension of nature considered as a unit of space-time, subjected to the exact laws of nature. This is how psychology, understood as health science, works with psychical phenomena immersed in the bias of the rationality of modern sciences. Conversely, phenomenological perception is understood as occurring in the *now*, the moment during which the act of perception itself occurs. Through radical criticism, phenomenology aims to describe and understand the acts of consciousness and their expression, also evidencing the movement of the psyche.

Phenomenology effects a shift in the way reality is seen. It is seen as an objectual correlate of acts of consciousness. It is not about consciousness *creating a* new reality; however, reality is intertwined by the invisible thread of intentionality of the interrogating gaze, which seeks to understand what it sees. Being aware does not mean reflecting on the actions we perform all the time, but being aware of the acts, while they are being actualized in/by the living-body. There is a stream of acts of consciousness, without us constantly evincing and describing the acts conducted. Consciousness is there, present, without being objectively seen, in the acts we perform. It is not a recipient or a thing, but an intentional movement. Intentionality is a core concept in phenomenological thinking. It is complex and difficult to describe through brief explanation. But it can be understood, at first glance, as an invisible thread that unites us to what we focus on with a watchful and inquisitive eye. This thread extends toward what we focus on in an interrogative way and brings the focus, sensed directly in the act of perception, into the acts of awareness, as perceived and not as reality itself.

"It is intentionality which characterizes consciousness, in the pregnant sense of the term, and justifies us in describing the whole stream of experience as at once a stream of consciousness and unity of one consciousness" (Husserl, 1972, p. 222). Intentionality is derived from the Latin *intendo*, *tendi, tendi, tentum*, which means tend towards one direction, extend, tend to open, become attentive, increase, sustain, become intense, affirm with force (Gaffiot, 1934). Such meanings enable my understanding of the movement of consciousness as an opening to the world, extending to everything, and becoming aware of the experiences in stream of its flow. Perceived experiences present themselves as *being awareness of*, therefore intentionally transformed into something understood. Intentionally aiming at something also[6] refers to a relationship with something which is external, transcendent

[6] "Also" because the experiences are all immanent to the stream of consciousness; however, they

and which is grasped in the movement of the act of perception of what is being perceived through the act itself.[7]

In this intentional view, consciousness encompasses everything, that is, the totality of the world which is now transformed as perceived,[8] no longer as objectively given in its reality. *This is the turning point.* We can only see from mundane reality what is immediately understood in perceptual acts, sensations, experiences intertwined in the articulating acts of consciousness and respective acts of expression of what is understood and articulated. One of the activities of intentionality is to be aware of the experiences that are flowing, happening. This modality exposes it as self-awareness. It is a movement made possible through reflection. This is the reflexive movement, through which it encompasses the living-experiences allowing lucidity. Interestingly the Greek word *phainomai* also means to shine, to sparkle. *"Being lucid" means seeing lucidly, shedding light.* (Bicudo, 1999, p. 19, author's translation).

> Reflection, as the foregoing analysis will have shown, is an expression for acts in which the stream of experience (Erlebnis), with all its manifold events (phases of experiences, intentionalities) can be grasped and analyzed in the light of its own evidence (Husserl, 1972, p. 200).

Thus, the movement of thinking about what is experienced advances the ways of understanding and talking about reality.

Having explained the *turning point,* I will now expose unfoldings regarding the ways understanding reality. Core ideas were emphasized throughout this Husserlian trajectory of constituting and producing knowledge, and understanding *truth,* some of which are explained below.

Phenomenon. It is understood as what is shown. But what is shown, to whom is it shown, how is it shown? These questions already indicate a radical change in attitude which keeps us from taking "real" as what is objectively given. They lead us to assume an attitude of critically and reflexively questioning what seems to be.

We live with two attitudes. The so-called natural attitude, in which researchers take reality as it is observed, i.e., in an uncritical or naturalized manner. They do not ask: what is nature? How can I know it? The other is called phenomenological. It is assumed by bracketing the world, suspending previous judgments about it, scientific or not, and, carefully, asking: what is the world? How can it be known to me? What is nature?

The natural attitude puts us in a position to observe and experience what we take as the object of investigation, developing ways of knowing how to do, to search for

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can be immanently directed when intentionality is geared towards other experiences or in a transcendent manner, when they aim at external objects in their surroundings.

[7] Husserl calls this relationship *noesis* and *noema* (Ales Bello, 2012, p. 122, author's translation).

[8] Perception is an act taken in its immediacy without intermediates explanations. It is taken in the very moment when it occurs when the essence of what is seen is clearly perceived as being true. Thus, it is understood as primacy of knowledge. What is clearly seen slips in the stream of consciousness and becomes obscure. Therefore, it demands acts of consciousness in order to proceed the constitution of knowledge.

the elements that the object of study entails, explain relationships and predict occurrences under specific circumstances. The phenomenological attitude always puts us in an attentive position, asking about what is seen and its way of being. The movement of the phenomenological reduction is evidenced in this attitude.

Therefore, I understand that the phenomenon shows itself to the person, who, as any human being, is always looking for the meaning of the world. How is it shown? In the living-experience[9] of the individual, in their corporeal reality, assumed as a living-body that actualizes acts demanded by the intention to respond to a need outlined by their inquiring gaze.

Thus, Husserlian phenomenology is seen as science (*o-logy*) of what is shown (phenomenon), advancing in a way rooted in the world, and which does not take the body of the individual as something made of two parts: spirit and matter, but as a complexity that exposes itself as a living-body that enables human beings to be seen, perceived by others and that also makes us view and touch the world, while making us perceive ourselves by touching what is in front of us, and being touched by what is there. It is a totality consisting of *flesh and bone*, that is, of physical/psychical and spiritual aspects. Thus, the human body is not reduced to a structure composed by the pair psyche/body, looked at in its materiality, which, in the works of Husserl (2002) and Merleau-Ponty (1962) is expressed by the word *Körper*. Husserl exposes his understanding of the living-body as an interweaving of different aspects; the physical—living-body sensations, psychical—feelings, emotions, cognitive acts and spiritual—acts of judgment. Thus, it shows actions nurtured by the intention and willingness to do something in a situation, that is, always moving toward something to be done. The concept of the living-body, a body that lives and feels which is living, is expressed by the word *Leib*[10].

Psychical and spiritual aspects do not allow us to speak of the living-body only from the perspective of corporeality. They underscore the complexity of such body. This complexity is intensified by highlighting the intentionality of the movement it makes. This intentional movement is kinesthetic.

This intertwining of sensations of psychical and spiritual acts actualized in the stream of consciousness delineates meanings about the world that are being created for the subject. The articulation of senses is configurated in expressions of what is being comprehended through language, which exposes what was understood by the subject, however through the polysemy of words themselves and the multiple forms of expression. What is expressed can bring out more and less of what is understood

[9] The living-experiences (*Erlebnisse*) speak, at first glance, about the life that flows, as it is lived. We live experienced acts being in motion for the duration of their temporality. At each moment we live the present moment of the act taking place. Psychical acts, such as perceiving, imagining, fantasizing, remembering, reflecting, which are inherent to human beings, even if they occur uniquely in each individual. Living-experiences flow, slips from now to what has been, making room for other living-experiences. We know we are living, but only by an act of consciousness do we realize what we are experiencing. This act is to perceive the experience as being lived and Husserl calls it "*Erlebniss*".

[10] Further explanations regarding the living-body can be found in the chapter entitled "Constituting mathematical knowledge being-with-media in cyberspace" of this book.

by the subject. Less because, it cannot make sense of everything that he has felt and understood. More as the forms of expression carry their historicity, bringing many other meanings. However, the expression presents to the world, through the materiality of the disposed resources, what happened in the subjectivity of an incarnated subject. Along with intropathy, this constitutes the sphere of intersubjectivity, in which experiences lived by different subjects are shared.

Leib always moves toward something one intends to accomplish. As it is a physical, psychical, and spiritual complexity, it is linked to the life-world by sensations; it immediately understands through the act of perceiving, moves intentionally, perceives themselves and the other in the carnality of the living-body that they see as similar and different. "I know" the other, as the other is there while "my own" living-body is here. However, through intropathy,[11] I feel the other as equal, because I know that the living-body of the other also feels sensations, emotions, and thinks, though not in the same way as I do. It is the genesis of the dual: same and different.

It is in this sphere that knowledge is being produced in the context of life among people, that is, the community, which is permeated by different understandings and possible agreements which, if successful and repeated, remain accepted.

This is a complex movement, as it encompasses the historicity of events, the values considered valid by the closest community, which that intertwine with those of other communities, that are not so close in spatial and temporal terms. The sphere of objectuality of knowledge is constituted by layers of sensations, feelings, understandings and the respective inter-subject expressions and understandings that intertwine and express the reality, which is lived, understood and ever-moving: life-world reality. Here, it is clear that there is a difference between the reality of the world empirically validated directly and naively, and eidetic reality, validated by transcendental phenomenology. The work of phenomenology is characteristically philosophical. It is intentional. It pursues paths of living-experience that lead to the radical origin that reveals the self-evidence of what is comprehended (Bicudo, 2011).

Radical origin—*Ürsprung*—is an important theme for understanding the issue of *truth* in the realm of phenomenology. In the *Origin of Geometry* (Husserl, 1970) the word refers to the intentional synthesis through which concrete objects themselves are constituted for the subject. Origin was treated in different ways throughout Husserl's work, but in that text, origin is viewed in the sense of constitution and not abstraction, as stated by Miller (1982). The issue of origin is intertwined with the analyzes of evidence, truth and being, which establish the distinction between *empty* and *fulfilled* intentions. In the former, the object is intended in a variety of ways, however it is not given in intuitive evidence. According to Miller (1982, p. 35), in this case, we might say, the object is intended in its absence. However, the object can also be given directly and intuitively, which characterizes the complete intentional act, meaning that the object is intended in its presence. This meaning is what

[11] Intropathy is knowledge of the other that occurs directly in the experiences in which the other is given (brought, exposed) to the self in its corporeality. It is a constituent perception of intersubjectivity. It is not, therefore, a theoretical concept or a predicatively constructed statement.

Husserl calls *original intuition*. This occurs in the *now*, in the moment lived, without mediation of the sign that can point and express what is intuited.

Derrida (1994) gives an enlightening explanation when talking about tip of the instant—*of now*—when original intuition occurs. It is the temporal present, indivisible unit, which cannot be known through sign. A perception or intuition of itself, by itself, in the presence of the act. It is not only the instance in which signification in general could occur,

> however it would guarantee the possibility of an original perception or intuition, in general, that is, non-signification as the principle of principles. And, after that, every time Husserl would assert the meaning of original intuition, he would remind us that it is the experience of absence, and the uselessness of the sign (Derrida, 1994, p. 70, author's translation)

This is the origin, which he sought in the *Origin of Geometry*. He wanted to show, through the exemplary constitution of that science, that the present brings

> "[…] the ideal and absolute certainty that the universal form of all experience (*Erlebnis*), and hence of life itself, has always been and will always be the present. There is only the present, there will always be the present. *Being* is presence or the change in presence" (Derrida, 1994, p. 63, author's translation).

This is the foundation of his proposal for the study of history: studying it from the present.

Truth resides in the *now* of the original intuition and slips into what has been, in the stream of consciousness within time (Husserl, 1994). It may be revisited in a movement of remembering, though it will have been already modified by acts of consciousness. Only then, in the expression of what is understood of original intuition, through articulation of acts of consciousness, signs, and meanings can be used for expression.

4 Understanding Cyberspace and Being-with-Media from a Phenomenological Perspective

I understand that cyberspace is in *Lebenswelt*. As mentioned in the chapter entitled "*Constituting mathematical knowledge being-with-media in cyberspace"* of this book; "cyberspace is defined as the space of communication, opened by the world-wide interconnection of computers and computer memories" (Lévy, 1999, p. 92, author's translation). This is a new structured means of communication. In that space, created with the support of the informational screen, the corporeal presence[12] of the communicating subjects and co-subjects[13] it is no longer necessary. However, space is created and kept open and moving (fast) by networks of interconnected

[12] We could also say: embodied presence.

[13] The term "co-subjects" refers to those who are with each other. In the research that supports this text (Barbariz, 2017) refers to the subjects who are with the researcher at the time of the research.

digital devices, including databases and the information available, thereby interconnecting people, documents, institutions, programs, and machines.

Its reality has specific characteristics, but it is not separate from this mundane reality in which we are immersed with others, intentionally plugged-in and moving. It is not a virtual reality, as seen by many authors, such as Lévy (1999), who, as I see it, understands it this way because it does not display the concreteness of objectively given objects, and the corporality of a subject present in the flesh. Conversely, I see it as a reality materialized by available science, techniques, and technology. Through the speed of its connections which bifurcate, expanding into networks which, in turn, divide even further, where people become present through the intentionality of what they set out to realize, in face of the requirements to which they open themselves. It is a movement of people, among people, who are with one another, intentionally intertwined, constituting and producing knowledge and reality with media.

The constitution and production of knowledge in cyberspace, as previously mentioned, demand a phenomenological posture, which is enabled by the concept of *being-with-media* (Borba & Villarreal, 2005), understood as a way of being and working together with available media.

Working in cyberspace within a phenomenological posture requires one to access the way through which spatiality expands through the actions performed by the subjects being with each other and with-media and actualizing actions. Actions which are affected in the present of the living-body who commands, making things happen together with the *other*. The *other* is understood as the interlocutor of intentional dialogue, which is being established, in a synchronous or asynchronous manner, as well as the mentor of the program with which they are working and the available equipment. The living-experience in time and space happens in the dimensional aspects of this reality: the present of the living-body and of time, which is actualized in dimensions of cyberspace.

As described, analyzed and reflected in the chapter "*Constituting mathematical knowledge being-with-media in cyberspace"*, in this book, the movement of constitution and production of knowledge is a dynamic movement that takes place in a temporal flow in which everyone intentionally involved with the subject-matter that is being dealt with is subject of their own actions and co-subjects of each other's actions, as they are present before one another and, at the same time, are aware of themselves. The questions posed by them can be presented and focused on everyone's thoughts, while articulating ideas and constituting knowledge. Beyond that, being-with-media they continue producing knowledge as they express their understanding and make it present in available materiality. Along with this materiality, the form is intertwined with the meanings understood by the subjects, their modes of expression and the design of the program with which they are working.

The constitution of intersubjectivity is also present in cyberspace. The subject, being-with-media and the others, in an intentionally focused way, that is, conscious, regarding the activity which is being conducted and the presence of co-subjects, expands the spatiality and temporality of the reality of the life-world, and, with that, the way of being-in-the-world and with others.

Intersubjective reality can be constituted in a teaching and learning situation, when there is an agent, a teacher, or a teaching aid, intentionally connected with the subjects with which they are, as well as the subject-matter they are dealing with and the media infrastructure. In the investigation conducted by Barbariz (2017) it was clear that she engaged with students' postings by reflecting on what she herself had studied and what students showed they understood. Realizing her actions and reflecting on them, she reports that she became aware of students' thoughts and her own, as she was committed that everyone could revisit what they had understood and advance their discussions by exposing examples and counterexamples. She reported that this led to more insights into the subject that was addressed.

Being-with in a conscious manner opens horizons for understanding oneself and the other. Intropathy; seeing the other as equal, with the possibility of thinking and expressing themselves intelligibly, through language, is a part of this movement which constitutes intersubjectivity and gives support so that a teaching and learning environment can predominate.

5 Comprehensive Summary

The phenomenological approach supports ways of working *in cyberspace, being-with-media* that, as I understand it, differ from other approaches that also deal with this dimension of reality. Indicative of the difference is how it examines actions triggered by the subjects who are with the media, aiming to understand the meaning that is being expressed in the intertwining of the complexity of the living-body with that of cyberspace, as well as the focus aimed at the intentionality of the actions and the matter and form of the informational screen. There is no separation among mind-body-machine. It is a way of being-with-it, using the resources available for the materialization of thought.

Focusing on the way through which the knowledge of the subject is constituted with/in this reality, seeking to understand sensations, intuitions, perceptions, evidence, judgments, openness to the other, possibilities of expression, revisited evaluations of what was accomplished and exposed through languages considered pertinent, by the subject, towards the activity that was developed, indicates a *change of the perspective* from which the subject intentionally directs their gaze, as soon as, they go beyond the work, under the light of the production made available within *Lebenswelt*.

In the different chapters of this book, among other themes, we sought to bring research that evince this movement. They focused on several themes. In the realm of mathematics; learning and teaching while being-with-media and the computer in cyberspace. These are also investigations which seek to promote understanding of how mathematicians themselves produce mathematics, while being-with-media; and how students see themselves in this movement of thinking about, and working with, dynamic geometry, as well as the other, in the distance, while maintaining a presence, being with the other.

References

Ales Bello, A. (2012). Coscienza Io Muondo – La fenomenologia di Edmund Husserl. In A. Ales Bello & P. Manganaro (orgs.), *...E la coscienza? Fenomenologia psico-patologia Neuroscienze* (pp. 101–240). Bari: Edizioni Giuseppe Laterza.

Almeida, N. E. (2012). Apontamentos gerais sobre os produtores e a produção de verdade em Aristóteles. *Dissertatio, 36*, 253–275.

Amaral, M. C. P. (1987). *Dilthey: um conceito de vida e uma pedagogia*. São Paulo: editora Perspectiva.

Barbariz, T. A. M. (2017*). A constituição do conhecimento matemático em um curso de matemática à distância*. Doutorado, Instituto de Geociências e Ciências Exatas - Universidade Estadual Paulista "Júlio de Mesquita Filho", Rio Claro.

Bicudo, M. A. V. (1999). *Fenomenologia: confrontos e avanços*. São Paulo: Editora Cortez.

Bicudo, M. A. V. (2011). The constitution of mathematical science from a phenomenological perspective. *RIPEM, 1*(1), 54–67.

Bicudo, M. A. V. P. (2018). Mathematics education actualized in the cyberspace: a philosophical essay. In P. Ernest (Ed.), *Philosophy of Mathematics Education Today* (pp. 253–270). Cham: Springer.

Borba, M. C., & Villarreal, M. E. (2005). *Humans-with-media and the reorganization of mathematical thinking: Information and communication technologies, modeling, experimentation and visualization*. New York: Springer.

Buongiorno, F. (2019). Uma introduzione generale alla dottrina della scienza: la rilevanza di HUA/XXIV nello sviluppo della fenomenologia husserliana. In E. Husserl (Ed.), *Introduzione Allá lógica e Allá teoria della conoscenza* (pp. 01–19). Brescia: Scholé.

Derrida, J. (1994). *A Voz e o Fenômeno*. Rio de Janeiro: Jorge Zahar Editor.

Gaffiot, F. (1934). *Dictionnaire Latin/Français*. Paris: Achete.

Heidegger, M. (1970). *Sobre a Essência da verdade. A tese de Kant sobre o ser*. São Paulo: Livraria Duas Cidades.

Husserl, E. (1970). *The crisis of European sciences and transcendental phenomenology*. Evanston: Northwestern University Press.

Husserl, E. (1972). Author's Preface to the English edition. In E. Husserl (Ed.), *Ideas – introduction to pure phenomenology* (pp. 5–22). New York: Collier Macmillan Ltd.

Husserl, E. (1994). *Lições para uma fenomenologia da consciência interna do tempo*. Lisboa: Imprensa Nacional da Casa da Moeda.

Husserl, E. (2002). *Idee per una fenomenologia pura e per una fenomenologia fenomenologica* (Vol. II). Torino: Einaudi.

Lévy, P. (1999). *Cibercultura* (2nd ed.). Tradução de Carlos Irineu da Costa. São Paulo: Editora 34.

Merleau-Ponty, M. (1962). *Phenomenology of perception*. London: Routledge & Kegan Paul Ltd.

Miller, J. P. (1982). *Numbers in presence and absence: A study of Husserl's philosophy of mathematics*. The Hague: Martinus Nihoff.

Palmer, R. E. (1969). *Hermeneutics – Interpretation theory in Schleiermacher, Dilthey, Heidegger and Gadamer*. Evanston: Northwestern University Press.

Piettre, B. (2005). Verité et Sens. *Revista Pesquisa Qualitativa, 1*(1), 27–72.

Sanfelice, J. L. (2005). Dialética e Pesquisa: seus embasamentos científico-filosóficos. *Revista Pesquisa Qualitativa, 1*(1), 135–154.

The Living Body and Cyberspace: The Hyletic Dimension in the Constitution of Knowledge

Angela Ales Bello

1 Introduction

When one hears terms like the "living" or "lived" body and hyletics, one immediately thinks of Edmund Husserl, especially since he identifies the hyletic dimension as the first layer of the constitution of the human being. This layer is closely linked to corporeity. He arrives at his conclusions through an analysis of the cognitive process, which yields surprising results. If one examines the history of western philosophy, one notes that Husserl's phenomenological investigations of human knowledge are original as he brings forward evidence of new aspects present in human beings that were previously unknown. While it is true that we find references to the necessity of the senses and intellect for grasping data that come from both the external world and human interiority, Husserl does not end his investigations with these aspects; rather, he mines human interiority to study our "lived experiences" [*Erlebnisse*], which are the privileged instruments deployed to understand how both internal and external reality are "constituted." Herein lies the novelty of the phenomenological method.

2 Body, Thing, and Real Space

How does one come to know corporeity at the level of lived experience? Husserl claims we do so through certain perceptions that presuppose sensations. The second volume of his *Ideas Pertaining to a Pure Phenomenology and Phenomenological*

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Philosophy, transcribed by his student Edith Stein, is quite revealing on this score. Perception, understood as an act lived by us, refers us back to sensations, especially important tactile and visual ones, that help constitute an act. These sensations are distinct because the former are localized in the body and the latter are not. Both, however, are fundamental for knowledge of physical things and for the reference back to the body itself. On the one hand, the body presents itself as a physical thing and, on the other hand, it is a sentient, feeling body [*Leib*]. As a living, sensing body, one finds diverse sensations though which we grasp inanimate, physical objects as well as our own bodies. When the body begins to relate physically with other materials (for example, blows, pressure, pushes, etc.), it presents localized sensations that are different than what happens when material things strike one another (Husserl, 1982).

The perception of a physical thing is connected to one's own living body, which is in contact with the physical thing. A series of extremely important perceptions that explain the reason why we feel our body and consider it our own arise from the foregoing connection. Sensations of touch as opposed to other sensations, for example, visual ones, best manifest this feeling of ownness because of their double reference. Husserl speaks of the touching-touched of the hand. The seen lived body is not a thing that sees and at the same time is seen by itself. But our own body, when it is touched by us (Husserl, 1982), both touches and is touched (Husserl, 1982, § 37). Hence, the role of visual sensation is different from that of touch. Vision alone does not give one's "own" body, nor does hearing. Vision and hearing simply yield the perception of a body, of the sound of a body. If touch did not exist, we would not be able to have the sensation of our bodies. We do not see our seeing, hear our hearing, but we can feel our touching touch.

Localized sensations, in addition to others derived from the senses, give us the possibility of grasping objects in space while carrying out functions foundational for the constitution of sensory things (Husserl, 1982, § 39). Here, we find ourselves in what Husserl calls the hyletic sphere. He uses the Greek term [*hylé*] to delineate an unexplored, higher realm of possibility that is vital for the constitution of physical objects. This sphere does not only cover the relation of one's own body with the external world but also includes other groups of sensations that are called "sensorial feelings" like pleasure, pain, tension and relaxation, well-being and dis-ease, all of which lie at the base of affectivity and valuing. They constitute the material [*hylé*] with which feelings and values are formed.

Both external and internal sensations are understood by Husserl as having a common hyletic base formed by egological and non-egological aspects that constitute the substrate of the entire life of consciousness. Intentional functions of a higher order that move toward the spiritual dimension are connected to this hyletic dimension. The conscious registering or awareness of all the aforementioned aspects resides in lived experiences [*Erlebnisse*]—the acts that testify to the flow of the whole of life that are lived by us. Perceptual acts refer to corporeity, whereas feelings of pleasure, pain, and forms of tension and release, all refer to the psyche. They also refer to the spirit when we undertake acts of intellect and/or will.

To understand the meaning of lived experiences, we must continue to pursue our discussion of sense experience. Whereas localized sensation may remain in the finger, the actual perception is not localized. Perception, understood as an intentional lived experience that tends toward the perceived thing, is distinct from all localizations: It is the means by which we can enter into the complexity of interior life. It is a principal way that opens a horizon of inquiry focused on the sensory sphere and sensorial feelings, on the one hand, and spiritual feelings and valuing, on the other hand.

The hyletic sphere characterizes the body proper as a lived body [*Leib*] insofar as it is sentient. The lived body also allows us to grasp its being a crossroads between interiority and exteriority due to its being "different" than other material things because its sensitivity to stimuli is a real quality that has a different source than the extensional qualities of the thing. The body is "matter" for the psyche, which manifests itself through psychic states successively developed in feelings. Psyche not only express through the body, but it could be said that psyche has its own body.

Additionally, the living body possesses other qualities that allow us to affirm further that it is different than other physical things: The lived body is a center of orientation. This means that things appear and disappear according to what appears closer or further away from the body. All this happens in relation to my body, from which I am unable to exit except by virtue of the "imagination" and, hence, not really, that is, as Husserl notes, vis-à-vis the relation of possibility and impossibility. In other words, such an exit remains "impossible" because it is unrealizable. Impossibility is identified with what is imaginary.[1] Hence, we can affirm that the body is a zero point of orientation.

The body has "possibilities" and "limits"; it can spontaneously move itself, modify its position, and it executes mechanical movements from the inside and in response to external events. It also may be moved by what we undergo in psyche. The limit is defined by what presents itself as impossible, that is, what is incompatible with the capacities that the body possesses. From the viewpoint of conscious lived experiences, the possibility of completing a movement corresponds, even implies, a series of lived bodily experiences that run the gamut from the very simple perception of space to the management of one's own body to the perception of space in relation to other objects or to what constitutes a perceived obstacle. Whence does movement arise? The term "arise" can signify the base that determines movement or what a base signifies. Here, we are dealing with a spontaneous, reactive movement determined by a received blow or push or desire to move toward something. It can also refer to a willed intention to reach a certain end. The latter meaning is possible only if the fulfillment of the intentions present in diverse lived experiences is real, otherwise it would remain only imagined or desired, but not realizable. Strictly speaking, the imaginary is that which is incompatible with real possibility. This

[1] I treat the question of the relation between possibility and impossibility in my article, Ales Bello (2010), pp. 217–228.

being said, one must not underestimate the power of the imagination, for it bespeaks the great human capacity to construct alternative worlds and spheres.

It is through the imagination that we enter into the world of cyberspace, which we can only access by understanding what corporeity and space are, in particular, how they come to be constituted as real in relation to the bodily and kinesthetic movements in which we find ourselves fully immersed. Husserl analyzes the constitution of space in tandem with that of the things of the physical world. In his work *Thing and Space: Lectures of 1907* (Husserl, 1997), he begins with perception, underscoring the fact that the things of the physical world are only grasped in a perspectival manner, through adumbrations: We seize only a part of the whole and we grasp the whole though our understanding. If we did not possess such a capacity, we would not be able to grasp reality in three dimensions.

All what Husserl describes occurs in a temporality that is identified with flow of the givens of the perception or of an apprehension, that is an apperception: This is the flow of the consciousness of these givens. The temporal framing is necessary, according to Husserl, for grasping the sense of spatiality, the thing always fills a time and, therefore, has a duration. The thing and space are intimately connected and, hence, it is necessary to turn to the constitution of the thing to understand the genesis of space. Husserl highlights the role of the sense of touch and sight that are necessary for the constitution of the thing. For example, if we wish to describe how we come to know a sheet of white paper, we must individuate the sensations of touch from the vision of the hand. They may have different developments, but together they contribute to the formation of space and thing. Also, while we perceive, we experience certain psychic reactions. Husserl notes that, for example, if I hold a metal sphere in my hand, the spatial tactile determination remains unmodified even though the temperature may drop, moving from a hotter to a cooler state. Though the cold and the heat are felt in the hand, they are also felt as reactions in psyche. Here, we find ourselves in the hyletic sphere that contains sensations and reactions to sensations of cold, heat, pain, etc.—Phenomena of primary experience whose sense we recognize are given in the aforementioned manner. The sense or essence of perception is to make manifest the object, and this is different than what is brought forward in acts of phantasy or fictive imagination, though we also need phantasy.

3 Phantasy and Virtual Space

Husserl makes important contributions to the study of phantasy, the formation of images, and memory. His analyses of these phenomena date from 1898 to 1925, and are collected in volume 23 of *Husserliana* titled, *Phantasie, Bildbewusstsein, Erinnerung* [*Phantasy, Picture-Consciousness and Memory*] (Husserl, 1980). Perception is key to these texts insofar as Husserl shows the difference between perceptual knowledge and that given by phantasy and memory, both of which distance themselves from sense impression: An apprehension through perception is different than that of phantasy and memory. If the perceptual moments are indispensable as it is the source of contact with the external world, the content that stems from it is elaborated in a manner important for the constitution of our world. Here, the world is not only the world of physical things but also a world that is constituted from within, thanks to the operations executed by the faculties we possess as well as our capacities, potentialities, including memory and phantasy. They allow us to represent to ourselves objects known through perception. Perceptions make presents things in their actuality [*Gegenwärtigung*], whereas memory and phantasy make things present again, that is, they presentify content [*Vergegenwärtigung*].

Physical objects are present in "flesh and blood," whereas in a painting or picture, the following are given: (1) a physical image (*physisches Bild*) in which is present (2) a representing object (*repräsentirendes Objekt*), and (3) a represented object (*Bildsubjekt*) (Husserl, 1980). According to Husserl, there is a large difference between the representing and represented object. He gives the example of a black and white photo. The boy pictured in the photo really has blond hair and rosy cheeks (represented object), but in the photo the colors are gray and violet (representing object). Husserl also notes that if the photo were to depict the child as he were in real life, which our contemporary photos do, it would be impossible to speak of an "image of which we are conscious" or "an awareness of an image" [*Bildbewusstsein*]; rather, we would speak of the thing itself. Husserl first affirms the foregoing claim, but he subsequently corrects himself, remarking that even if the image were to represent perfectly what is represented, we are not dealing with the real child, but with something that represents something else, albeit it is identical or similar to it (Husserl, 1980). Today, we could say that we here encounter the virtual, non-real object.

Husserl distinguishes three types of images: physical objects (painting or picture); symbolic images or those of phantasy; and what he calls "spiritual" images. Whereas the first two types are linked to perception, the third one Husserl calls "spiritual." In the case of the latter, we are not dealing with a "thing-image" contained in spirit, but, from a phenomenological perspective, one here has consciousness of something though not the concrete existence of the thing. In the painting, one finds the canvas and color, as physical objects, but not the lion, though it may be depicted in the painting. Here, we are dealing with an interweaving of sensations that the observer lives in him- or herself, and the apprehension and the sense, both of which help construct the image with the sensations, together generate the consciousness of an image (Husserl, 1980). In images of phantasy, we find an interweaving of "phantasms," illusory images (*Phantasmata*) that present something, which has sense or meaning based on a series of apprehensions (Husserl, 1980). Phantasy operates on objects that are grasped in a new apprehension. Husserl gives the following important example: If we see the sign for an integral (*ƒ*) and we do not know what it is, the sign is grasped, but not its sense. This is why it is necessary to attribute a significance to that sign, thereby entering into the realm of symbolization.

The symbolization could be a reference, as in the case of the picture of the Madonna of Raphael, as cited by Husserl. In fact the picture that represents the

Madonna does not represent the Madonna, for when I say that it is a Madonna, I do not base my statement on a perception; rather, the perception changes its quality. It takes on the quality of the representation through its likeness with the original—this is a quality of seeing in an image something that transcends the image itself (Husserl, 1980). It is at this point that the processes of symbolization and memory come into play. The representation of Raphael's Madonna in wood reminds us of the original that lies in Vienna's Kunsthistorisches Museum. "Images can function as analogical signs for memory" (Husserl, 1980, p. 35, author's translation). The object represents itself in an associative manner.

"Symbolic presentation" of writing may be classified in two ways. Originally, writing presented itself though images, for example, hieroglyphs. Later, the signs distanced themselves from the images and became artifacts, for example, the alphabet and algebraic signs. This transition also applies to art, especially forms in which phantasy comes to play a major role. It is not sufficient that one be conscious of an image of a "subject," understood as the subject of representation and, therefore, as a represented object. Also important is the manner or mode in which the object is represented, that is, the choice the artist makes in terms of this or that possibility of representation. The artist seeks the most ideal representation, the most beautiful representation from an aesthetic viewpoint. This does not happen, of course, in the everyday products of the imagination.

Relevant for our discussion of cyberspace is Husserl's discussion of the dynamics of theater, which involve both the actor and spectator. Together, in them, we find a doubling: The actor is a human being who has a story, but in acting s/he becomes "other" to him- or herself. Likewise, spectators, who are not children, know that they are viewing a fiction. Husserl notes that they nonetheless actively immerse themselves in the drama. As they watch the scenes unfold before them, they hope and fear, love and hate, but "[a]ll of this 'as' phantasy, in the mode of "as if"" (Husserl, 1980, p. 517, author's translation). We create then an alternate world and we live at two levels, namely, the everyday and that of fiction.

The body participates in the aforementioned processes. First, it consents to see and hear, it follows as immobile or as moving, with a sense of well-being or disease, understood as psychic reactions to what is occurring. In the world of phantasy, however, the hyletic dimension is active as are parts of the spiritual dimension, including intellect and will. All persons can evaluate the quality of the acting and can decide whether to continue or not watching the show. The space of the theater seems to become a concrete, physical space, and in a certain sense it is, which is not the case with movies. But the space of the theater is also virtual in that the scenery refers to an alternate reality, a fiction. For example, buildings are represented on flat surfaces that are designed to simulate the tri-dimensionality present in reality. They are nonetheless physical images. Similar to the foregoing case of the theater, we live on two levels, namely, the perceptual and phantasmatic (with all of its images). Though the space of the theater is limited, it still remains an artifact in that it is a real space. Actors move with their bodies in this space.

Different from the space of a painting, statue, or theater scenery is the photograph, which lies at the base of film and the digital hardware that creates cyberspace. In fact, what we call cyberspace requires highly developed technologies in order to operate, for example, advanced computers and monitors/screens, in which "presentation occurs though signs" (*Vorstellung durch Zeichen*), to borrow Husserl's language. Of course, Husserl would never have known this technology. These signs, however, appear on the screen through a very complex digital or analogical process. The technology is so advanced that the distinction between reality and phantasy can easily be blurred.

I cannot treat here all cybernetic technologies, for example, those in which one can "pilot" or direct vast artificially constructed systems, that is, cybernetic beings that are different than living organisms. These cybernetic entities are in advanced stages of development and will soon be more present in our lives. They are different than machines, which are formed by parts assembled according to human design, which gives defined and limited functions to them. Here, one can think of the distinction between artificial intelligence and machine learning, the former being cybernetic and the latter belonging to the realm of machinery run by statistical probability and memory. We are always searching for more refined algorithms. In cybernetic reality, we always remain within an "artificial universe" that, like artificial intelligence, seeks to imitate the reality produced and used by the human brain. This is a Promethean project in that human beings show the desire to become a Creator-God. I maintain that all that human constructs will always remain artificial: It is impossible for humans to create completely new organisms. The concept of entelechy proposed by Aristotle and then reworked by Hedwig Conrad-Martius is useful to understand the difference between the organic and the artificial. Entelechy is a principal that designates the inner development of an organism; a machine analogously has a motor. The motor is constructed of assembled pieces, and, left on its own, it would have no life, for it has no life like the plant or animal.

The instruments deployed to create cyberspace are artificial and they are "things" made of the physical world: We see and touch them. We can have bodily contact with them. But what they display at the visual level, though here the discussion of the interaction of the blind with such machines becomes different as alternate technologies are needed in order to be able to interact with them, is neither visible nor touchable through an immediate everyday perception of a thing. If we look at a representation of a human being, though similar to a real human being, we clearly see that it is not the original. Furthermore, we seek to understand the particular angle the artistic representation develops. For example, a portrait may seek to show the person or use her as symbol for something else, as happens in film and advertising.

If the image is accompanied by a written text, we operate as with books or a newspaper, both of which deploy writing, understood as a symbolic form, to communicate a message. An interesting case is that of Skype, which allows us to communicate through a virtual space in which we seem really present to one another, a case of the abovementioned "as if" of the imagination. It is as if we are in a double, ideal world, namely, that of geometry and the virtual world of dynamic geometry.

The foregoing discussion opens the possibility of a third space,[2] which is not reducible to a simple interaction between the real and the virtual; rather, here we are speaking of a third entity that lives and is in contact with virtual space. This third space would create a sort of unique world that is objective in the sense that one could move in it as in the real world. But, here, there is also the risk of real disembodiment, which we will discuss later.

4 The Virtual Space of Geometry

To develop the foregoing discussion of virtual space, I would like to focus my attention on dynamic geometry, which challenges our understanding of the static sense of geometry. Husserl, as a trained mathematician, was deeply attentive to the insights and power of geometry: He tried to grasp phenomenologically the sense of the "knowledge" offered by geometry, especially as it was the fruit of human "creation." Does phantasy enter into such a form of knowledge?

Husserl devoted a lot of thought to the birth of geometry, as Appendix III of *The Crisis of the European Sciences and Transcendental Phenomenology* (Husserl, 1970) testifies. Here, he claims that he does not wish to offer another history of geometry; rather, he claims to be searching for the ultimate sense of this discipline, which has its origins in Ancient Greece, by examining the spiritual operations that gave rise to it. He explores the inner subjectivity of the first geometers to grasp the constitutive operations of the field. We are dealing here with spiritual/cultural products: Culture is a product of the human spirit. Such products ground an ideal "objectivity" that is equal for all and to which everyone has access. Husserl reminds his readers that the Pythagoreans claimed that geometry exists only once, as universal spatio-temporal individuations, though it can be formulated in many different and personal ways. For example, this happens in the case of a straight line consisting of a series of points. Geometry is passed down through oral and written language, and it refers to ideal objectivities.

According to Husserl, these ideal objectivities that historically have been at the base of geometry are connected to praxis. He affirms that in technical practice one tends to find privileged forms that mathematicians are constantly seeking to improve. Surfaces, planes, angles, and lines, all of these objects have at their base of production a technology of measurement, but the move to geometric surfaces, planes, lines, and I would add, points, requires a qualitative leap from the level of sense experience to spiritual ideality—a leap that human beings are capable of making. Here, we come into contact with the extraordinary aspect of theoretical fields of

[2] Federica Buongiorno discusses the possibility of a third space from a phenomenological perspective in her work, *Cyberspazio e ambienti digitali: Localizzare il soggetto negli spazi tecnologico-mediali* [*Cyberspace and Digital Environments: Localizing the Subject in Technological-Mediatized Spaces*], forthcoming in *Humana-Mente*. She refers to important theorists' work like Kosari and Amoori (2018), pp. 163–165.

study. They are born from praxis, gradually distancing themselves from it to construct an ideal world that is not a world of pure phantasy, but the world of pure intellect.

Even the intellectual world of ideal objectivities needs phantasy, for the "as if" is required for this world to show or prove itself. The ideal is not in and of itself representable, yet the possibility of representation lies in the capacity to construct with phantasy the geometrical image. Furthermore, the ideal surface is not linked to perception, but in order to be grasped perceptually, it must be expressed in forms that do not exist as such in perception; rather, they constitute world of forms translated into images when we seek to reproduce and offer to visual perception material through a drawing. This transmission certainly was the case for geometry up until the advent of cybernetics.

The means of transmission in various past epochs was the clay tablet, parchment, or paper. What was written on them remained static. Human beings have now constructed something upon which we can not only manually write but which also has the capacity for further expression through software, for example, the computer or digital hardware. Static and dynamic representations are now possible, which will assist us to understand future transformations both in real and virtual time. Here, the medium of transmission is complex: What is transmitted has as its object an ideal world, expressed through phantasy. Virtual representation has produced forms that have taken on a variable objectivity. The triangle can be expressed in a variety of forms, though its essential structure remains the same and is not subject to change by variegations of phantasy, for it would lose its essence and properties. The triangle is not a refiguration of a physical object; rather, it is the refiguration of an ideal object that requires the medium of an image elaborated in phantasy and that is physically expressed. This process is the same for both the triangle drawn on paper and the one constructed on a computer. The latter has the advantage of being able to create objects in motion that the drawing does only with great effort. We know, for example, that Walt Disney drew the countless micro movements of his characters that were subsequently photographed. Today, software can produce more quickly and precisely the same effect.

We now turn to the question of hyletics. We spoke earlier about the kinesthetic movements of the body. Knowledge of movement is primary for humans and is first seized through the experience of one's own physicalness. Knowledge of external movement comes about through an encounter with bodily movements and psychic reactions. This continues to be the case no matter the part deployed. While following the visual perception of a geometric change while trying to grasp its ideal significance, a feeling of satisfaction or frustration may arise, that is, a psychic reaction. The human being is always and completely present with his or her bodily, psychic, and spiritual layers. Sight dominates the bodily level, but hands are also necessary, for the arms move the hands that eventually make machines work. If these two physical instruments were not present, one would not be able to launch the cognitive process.

The foregoing process must pass through the psychic level, ultimately culminating in intellectual activity. Yet, we can also invert this structure in the sense that knowing classical geometry at the intellectual level and being pushed by the desire to make it dynamic, we complexify it through our phantasy, by using symbols. We can also trace it on the machine through our managing hands and our guiding sight as well as hearing, all of which work together. It is interesting to note that, at the level of intellectual research, the instrument we use the most can also be deployed virtually, thereby allowing the body to come into contact with the virtual programmer. According to certain researchers, true advances in virtual technology are made when software can recreate and even perfect human capacities and characteristics that can facilitate real-time human interaction with virtual technologies. The machine is only a medium, that is, an instrument, though it can execute complex tasks in a very short time that human beings are unable to perform. Here, we come to the question of artificial intelligence, behind which there is always human intelligence that organizes and projects it[3]. We also find here a sufficient example that confirms the interconnection between the bodily, the psychic, and the intellectual layers of our humanity.

5 Questions Connected to Cyberspace

We have discussed the human being with its living body, who uses an instrument to place itself in contact with a virtual reality. Another theme we investigated was the presentation of the living body in cybernetic space in which one finds the attempt to reproduce bodily movements similar to the manner in which geometric figures are made to move on a computer screen. The living, real body observes the virtual and may even identify with it; it can also find itself in the third space represented by an autonomous world, which is a product of the human mind. Here, one finds oneself as if in a dreamlike state in which the body has been annihilated. This state certainly can arise in infants, adolescents, or in adults suffering from various forms of mental illness. Those thinkers who support the existence of a third space deploy its potentialities to expand our cognitive horizons. Even if this third space were to exist, what remains important is the awareness of the need to privilege the intersubjective dimension. This means, then, that physical sensation alone is insufficient to grasp reality. Yet, a detachment from the physical runs the risk of producing the absolutization of a false world: This world may be used, but it must not be substituted for concrete, existential reality. One need not penalize the world of phantasy or the possibility of imagining a different space while working within them; rather, we must become aware of all aspects that constitute human life.

As with all human activity, excessive identification creates dependency and, hence, we can see why important and timely questions about education arise when we use virtual technologies, whose design and programming tend to favor

[3] See, … *e la coscienza? Fenomenologia Psico-patologia e Neuroscienze [*…And what about consciousness? Phenomenology Psychopathology, and Neuroscience], eds. Angela Ales Bello and Patrizia Manganaro (Bari: Giuseppe Laterza, 2012).

identification and the loss of the self. Advantages and disadvantages are always present. The intellectual and moral formation of the human being through the development of critical capacities is the best antidote against the possible dispersion and loss of the self. Through an analysis of the hyletic we move to the moral dimension of human beings, for the whole complex structure of human being becomes equally involved in the experience of both the real and the virtual.

References

Ales Bello, A. (2010). L'impossibile reductio ad unum dell'essere umano. *Archivio di Filosofia, LXXVIII*(1), 217–228.

Husserl, E. (1970). *The Crisis of European Sciences and Transcendental Phenomenology* (D. Carr, Trans.). Evanston, IL: Northwestern University Press.

Husserl, E. (1980). *Phantasie, Bildbewusstsein, Erinnerung 1898–192*. Dordrecht: Kluwer Academic Publishers.

Husserl, E. (1982). *Ideas pertaining to a Pure Phenomenology and to a Phenomenological Philosophy: First Book* (F. Kersten, Trans.). Dordrecht: Kluwer.

Husserl, E. (1997). *Thing and Space: Lectures of 1907* (R. Rojcewicz, Trans.). Dordrecht: Springer.

Kosari, M., & Amoori, A. (2018). Third Space: The Trialectics of the Real, Virtual and blended Spaces. *Journal of Cyberspace Studies, 2*(2), 163–185.

Part II
Mathematics Education in Cyberspace
Being-with-Media

Understanding Phenomenologically the Constitution of Knowledge When Working with Dynamic Geometry

José Milton Lopes Pinheiro, Maria Aparecida Viggiani Bicudo, and Adlai Ralph Detoni

1 Introduction

The present chapter aims to examine the constitution of knowledge while working with dynamic geometry (DG). This view is grounded in the phenomenological perspective, with which we seek to understand how the presence and actions of a living-body[1] who is intentionally[2] with DG, open to what it can show while carrying out learning activities with it comes about. What we mean by this is: creating and occupying spaces and producing knowledge. We intend to create a teaching and learning situation where the subjects let themselves go so that they can live experiences with software, without premeditating what might come to pass.

[1] Understood as *Leib*, a body with intentional movement. It encompasses all lived experiences and is also the starting point for new experiences. It actualizes and is actualized in motion, assuming different perspectives and setting in motion in the life-world that is incessantly formed along with the incessant configurations and reconfigurations of this body (Merleau-Ponty, 2011).

[2] It is intentionality which characterizes consciousness in the pregnant sense of the term, and justifies us in describing the whole stream of experience as at once a stream of consciousness and unity of one consciousness (Husserl, 1972, p. 222).

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We intend to understand through this experience how the constitution of knowledge is exposed or can be exposed, based on our attentive analysis. The interrogation posed is: *How is knowledge constituted when working with dynamic geometry*? In order to tackle this question, we take insights from a study about the experiences lived by subjects who turn to DG, performing activities and taking advantage of the physical and logical possibilities presented by the software in which DG is projected. In this experience, the movement, perceptions, and knowledge constituted with the software take precedence in the analysis.

We understand and contend that every movement performed in DG configures changes that are correlated with such movement, whether they are manifested in the software and/or in the subject who performs the movement. The above-mentioned interrogation encompasses how these configuration changes are experienced, how they manifest themselves, and how they intertwine, constituting knowledge. This understanding, which emerged through broad debate about the proposed study, made us to realize that we were investigating the phenomenon: *the constitution of knowledge while working with dynamic geometry software*. We believe that it is possible to account for the interrogation and the phenomenon interrogated, by studying and bringing open insights resulting from the doctoral thesis entitled "Movement and the perception of movement in Dynamic Geometry environments" (Pinheiro, 2018). In this research, the actions of the living body were studied through the transcriptions of what was said by the subjects regarding the conduction of activities with DG.

We investigated *how movement and the perception of movement takes place while being with the computer and while students perform activities in a dynamic geometry environment*, aiming to understand the phenomenon of *movement-perception-knowledge* in the process of constituting geometric senses and meanings, by going to subjects who experienced ways of being with the computer, working with dynamic geometry. To this end, a group of undergraduate students in mathematics was invited to develop activities with other learning co-subjects, didactic resources, and dynamic geometry environment. The experiences lived by the subjects were described and analyzed through the movement of turning to the interrogation presented above and constituting meanings as they made sense to the researches. Such analysis constituted and exposed convergences, which enabled the researchers to visualize horizons for open synthesis, bringing reflections about the phenomenon interrogated, unveiling its structures, that is, what turned out to be constant among the different perspectives evidenced. In Pinheiro's research (2018), the process of analysis was comprised of four structures, which the author called Nuclear Ideas: *ways in which movement is evident, the evident perceptions that constitute grounds for new perceptions and argumentation, the unity that encompasses the moving-subject and that which is available to movement, and the constitution of knowledge when working with dynamic geometry*.

Although these four nuclear ideas speak of the theme of this chapter, we will focus on the fourth nuclear idea by deepening interpretations and broadening ways of understanding the phenomenon under investigation.

We focused on the actions, weaving understandings about the ways in which they materialize and advance in the totality of the subjects' living-experience with DG and with the activities developed with them, aiming to understand their implications for the constitution of knowledge. We understand that the core of DG is the *movement made by the living-body.*

2 The Movement of Living-Body and Its Expression in Dynamic Geometry Software Interfaces

Phenomenologically, we understand that the movement of the living-body is always intentional, "it is always a *movement towards*..." which, as it advances, instigates change while changing at the same time (Bicudo, 2010, p. 128, authors' translation).

Computing is a favorable area because it manifests a range of possibilities for movement and perception of change in a field where texts, icons, links, pictures, videos are constituted, which, when accessed through the computer interface,[3] expand and are expanded into a constant dialectic in which the computer opens to the perceptive subject a range of possible realizations. When materializing one of such possibilities, making movements, there is a modification/change in those who make the movement and, in turn, in the way of conducting the movement to the extent of the possibilities present in the logic of the software. Each one acts with the computer intentionally firing "commands that will result in specific tasks performed with the materiality available through the program with which they are operating" (Bicudo, 2014, p. 60, authors' translation).

The DG environment establishes a specific way of being with spatiality, opening possibilities for perceptions of changes that also occur in the interfaces. Therefore, understanding the movement and the perception of changes in DG software involves understanding such interfaces, the bifurcations of possibilities that manifest themselves in the logic that underlies such interfaces, as well as the movement that manifests different meanings, in those possibilities.

The interface of DG software "turns" to its programming face as it is able to generate, as a result of some commands/rules, figures and possibility of movement, and "turns" to the subject, who performs actions with it and is aware of their implications.

In one direction the machine, through its answers, encompasses the subject operating with it, showing possibilities of action and effects. In the opposite direction, the subject acts "on the machine, appropriating the senses, by which it gives itself, and manipulating them according to a broader sense of the world that the subject brings with him and are present when they use the machine" (Figueiredo, 2014, p. 134, authors' translation). The senses that open up to the subject evidence the

[3] Literally, *interface* refers to "something that links two faces, which touches both sides, and characterizes a boundary" (Figueiredo, 2014, p. 138, authors' translation).

programmers' intentionality when programming something that they hope those operating with the machine can intentionally do. DG interfaces, for example, are constructed by translating rules of geometry, such as Euclidean and Analytic, into commands to be triggered by the subject with the computer.

The movement of the subject makes it possible to *transpose* the software interface, not in the sense of having access to the programming codes involved, but to be able to foresee possibilities (or answers) for their actions. In this transposing movement, the subjects intentionally focus, perceive what is given to them through the interface, glimpse at horizons of possibilities that are opened to them by the software in response to what they aim to accomplish, and immediately put themselves into perceptive or reflexive movements with the interface. This *being-with-the-interface* occurs in the movement of transposition acts.

The interface of DG software presents operations and responses that trigger, on the subject who is working with it, actions that may provoke a desire to advance. So, it may arouse in this subject the curiosity, the desire to click, to move, to extend what the interface shows. "The subject acts with the interface, but we must note that he does it do it by issuing commands" (Bicudo, 2014, p. 63, authors' translation). But, we need to note that responding to commands is not a simple and mechanical task, because the person who acts is a complex carnal being who is always living concomitantly, with binary rationality and ways of being with others in the world.

The software manuals (guides) are commands that open the possibilities offered by the informational screen to the subject.

Merleau-Ponty (2011) understands the power that the subject has over things, when performing a task, as the force present in the motricity[4] of the living-body which defines, as being intentionally moving or being in movement, "the place where the living-body is, through the task of what he intends intend to accomplish, and the specific situations delineated by its harmonious unity with the horizon-world" (Bicudo & Kluth, 2010, p. 133, authors' translation).

Thus, we understand that the living-body is presence wherever there is something to be accomplished. Along with the subject's action with DG, a kinesthetic background[5] arises and brings about the software interface. In Husserl (2012), we understand that, there is always *fulfillment of senses* within the action and the subject who moves and is moving perceives the movement in this act and what it brings: knowledge, already explained and culturally materialized, configurations, deconfigurations, variants, invariants, or simply the expression of the movement performed.

A DG software interface is comprised of a set of geometrical and other information, which, when accessed, show possibilities of different movements and constructions, that can be expressed on the same screen, to which we are "plugged-in." In this act of accessing we perceive a set of things—shapes, points, icons, etc.—and

[4] Motricity is expressed by Merleau-Ponty (2011) as an intentional way of moving. It is the experience of the living-body willed to act as demanded by the life-world, which, in the present study is focused in the perspective of DG software. This is called kinesthetic movement.

[5] *Kinesthesis* contemplates all the "I move" and the "I do" the living-body performs freely.

precisely by perceiving them and by assuming an analytical attitude we can discern similarities or contiguities. "This does not mean simply that without any perception of the whole, we would not think of *noticing* the resemblance or the contiguity of elements, but literally that we would and would not be part of the same world and would not exist at all" (Merleau-Ponty, 2011, p. 16).

It is while being with the totality of a geometric construction that we realize its most intrinsic properties. In order to understand this construction, from a phenomenological perspective, one has to focus on it, leave it in suspension, so that its surroundings remain, but as a background, or field of past achievements, from which it can be differed. In cyberspace where DG is being worked on, the background mentioned contemplates the entire technological apparatus and its possibilities that boost the movement. This background is dynamic. The possible movement, with software, indicates new tones and perspectives along this background, which also vibrates the figure that renews itself *with* the subject that moves it while moving themselves.

Therefore, the interface is the window that shows us DG in its geometry and algebraic constitution. This interface is duly programmed to open possibilities of movement, whose performance occurs because the acting subject has recollections of motor, visual and kinesthetic experiences that lead the actions of the living-body. Remembering is bringing these understandings to present experiences through recollection (Merleau-Ponty, 2011). Thus, the subject knows which icon to click to fulfill the requirements of a task. The iconic character is present, alluding to a geometric world prior to the cybernetic environment of the software; a world that is close and familiar to the computer professional, who configured it, and the subject that inhabits it when performing actions with the software.

The work with the DG environment allows the subject new understandings. For example, during the task of studying the congruence of two triangles with DG, the subject can only move one triangle until it overlaps it with the other, in order to realize equality and/or inequality. The triangles and the overlapping motion are inert, and it is the presence of the moving-subject that enables the manifestation of diverse triangle and motion senses. In their achievements, the subject also experiences triangles, movements, and congruences, and "when working with geometry in software, these experiences are present again, but in their own way, with new configurations, and constitute the background for actions performed in it and for actions that can be performed" (Pinheiro, 2018, p. 65, authors' translation).

The possibilities of movement present in DG software, in cyber reality, enable the understanding of movement by the act of imagining movement and proceeding to vary movement. Husserl (2006, p. 153, authors' translation), referring to *geometer* says

> The geometer that thinks geometrically operates with imagery vastly more than when he does with percepts of figures or models; and this is true also of the 'pure' geometer, who dispenses with the method of algebra. In fancy it is true he must toil to secure clear intuitions, and from this labor the drawing and the model sets him free (Husserl, 1972, p. 182).

In the context of this freedom, drawings are shown as materializations that follow the constructions of the imagination and the pure eidetic thought that arises based on them, and mainly serve

> to fix the stages of the process already previously gone through, thereby making it easier to bring it back to consciousness once again. Even where the thinker "meditates" over the figure, the new processes of thought which link themselves on to it have fancy-processes as their sensory basis, and it is the results of this work of fancy which fix the new lines on the figures (Husserl, 1972, p. 183).

We understand that in the act of imagining movement in the computational interface, when we are working with software, it is not necessary to pursue its objectification with the aid of the mouse, but only to glimpse it and, subsequently, propose and test theories. The possibilities opened by the software and the intentions of the subject to perform actions, in view of specific tasks, help promote thinking about the geometric ideas present in the activities with DG. Such possibilities open a range of viable understandings about drawings, constructions, movements, and the realization of what was imagined as the accomplishment of a possible task or the solution to a problem or a response to a requirement. They enable the validation or invalidation of possibilities, suggested by the imagination. Imagining movement is already a way of realizing it and being immersed in it. In the imagination one can make a movement and experience its entire duration, even before making it and visualizing it materializing in the software interface.

Being in a DG environment, we can conjecture about possible movements as, at any given time, when we are sensibly with the software, we have the living-experience of making that movement, making it, and watching it happen. We move, we realize that we move, we are able to think about the movement performed or the movement yet to be performed.

The perceptive and reflective acts of the subject being-with the DG software interface expand it, transform, and transcend the intentions of the programmer.

Thus, the geometry presented within the software can be characterized as dynamic. This is so because it always demands the action of the subject who actualizes the program and, especially, who throw themselves in intentional attitudes expressed in movements of the living-body with the interfaces of the program and the tools available on the computer. This way, we understand that if there is no moving-subject, there is no DG, as there is no geometry, as understood by Detoni (2012). Even though the program establishes rules that support various possible movements, there only will be actualizations of those possibilities, if there is a subject who, in their carnality, acts intentionally.

Statements that the dynamics of DG occurs through the physical and logical possibilities of smartphones, tablets or desktop computers, and software can be found in studies in mathematics education, such as Richit (2005) and Silva and Penteado (2009).

However, we understand that this dynamism, besides being given by the potential of computer technology, also occurs with the presence of the moving-subject. This subject actualizes the action, enabling the potential to become reality in its way of being dynamic. The statement supported by us that it is the subject that makes the action happen and, with that, triggers the potential making it a reality that sustains our claim that a phenomenological study in DG evidences the presence of the living-body, since it is in and with the living-body that the movement is actualized. Besides, the extension of this movement is materialized in the software interface.

The living-body moves while *being-with* and *in* this space and things gain new configurations as they act.

> The movements of the living-body are naturally vested with a certain perceptual significance, they form, with external phenomena, a system so well connected that the external perception "takes into account" the displacement of the perceptive organs, finds in them, if not the *express explication*, at least the reason for the changes that intervened in the spectacle, and thus can immediately understand them (Merleau-Ponty, 2011, p. 78, *emphasis added,* authors' translation).

The objects arranged in the software interface are presences that manifest themselves as *potency,* which "means characteristic of what is potent, has the strength to be, brings the potentialities to become" (Bicudo, 2010, p. 125, authors' translation). This potency comes to being (real) when the subject is with the software, which, when moving, intertwines the movement itself, producing change.

> It is the act that actualizes the potency, encompassing the movement in order to advance the happening. It includes a certain operation and is understood through change. This movement is important in Aristotelian philosophy, because it means "carrying out, effecting" what potentially exists while potentially existing. With this movement, the being goes from the potency to be to the act of being. Thus, the change of an object is the passage from a state of potency or potentiality (being potential) to a state of act or actuality (mode of being currently happening) (Bicudo, 2010, p. 125, authors' translation).

With DG software, the moving-subject performs movements. By responding to what is required in the activities to be performed, and actualizing these movements, the interface is configured with different fulfillments. "Existence occurs with the actualization of what is already potential. Thus, actuality is what presents itself as reality, although in dimensions of individualized actualizations, that is, in specific cases in relation to potency" (Bicudo, 2010, p. 125, authors' translation).

The act of moving objects in the software interface is the trigger for actualization. It transforms an intention to move into the movement itself. This act takes place with the materiality available. In the case of the computer world, the available materiality encompasses the reality opened by the informational screen.

Each movement before the DG software interface defines new experiences: focusing, blurring, moving, dragging. These experiences leave a kinesthetic trail in the living-body of the active subject(s). This is the kinesthesia that encompasses *man-computer-DG* and highlights the ways in which human motricity intertwines the nucleus of movements and perception and comprehension of movements, and may lead to the constitution of geometric knowledge, which occurs in the encounter between the moving-subject and the thing available in the DG software interface.

By being with the world unveiled in this interface, the subject goes towards objects available to perception; perceives their structure, also with the movement of their living-body and, by doing so, the object directly regulates their movements.

"This dialogue between the subject and the object, this recapturing by the subject of the sparse sense in the object, and by the object of the subject's intentions, which is the physiognomic perception, deposes around the subject a world that speaks to themselves and installs their own thoughts in the world" (Merleau-Ponty, 2011, p. 185, authors' translation).

The world we inhabit is not available to us as mere *world of things*, "but, in equal immediacy, as a *world of values*, as *world of goods*, as a *practical world.* "Without further effort on my part I found the things before me furnished not only with the qualities that befit their positive nature, but with value characters [….]" (Husserl, 1972, p. 93). Thus, we understand that the technological world, from which we highlight DG environments, is also available to us as a world of possible experiences that invites us to inhabit what is shown on the informational screen. What is shown and the screen itself are horizons of possibilities. Being with the DG interface is to organize, expand, study, and understand it, that is, what we already do in the life-world, spatializing, moving, being moved, undertaking projects that "polarize the world and make appear in it, as if by magic, a thousand signals that lead to action" (Merleau-Ponty, 2011, p. 161, authors' translation).

The understandings expressed herein so far enables us to think about how the constitution of knowledge occurs when working in environments of dynamic geometry, since they explain experiences in the sensory, perceptive, psychical, and spiritual spheres, from which different sensations and understandings emerge. We now intend to expose thoughts already articulated on how this constitution occurs, working on the *constitution of knowledge when working with dynamic geometry*.

In the articulations of the next topic, we will present part of the analysis described by Pinheiro (2018) about the Nuclear Idea highlighted here. The phrases shown in italics and between quotation marks are excerpts from the transcription of speech of research subjects, while performing the activities proposed, as well as interviews with them. Such clippings are brought in as a fulfillment of the understandings that were made possible about the referred Nuclear Idea, and which are presented in an articulated manner in the following text.

3 Constitution of Knowledge While Working with Dynamic Geometry

Phenomenologically, we do not think of space as simply physical, nor do we view it as an intellectual capacity; but more broadly as spatiality. As Heidegger claims in his *Being and Time* (Heidegger, 1999), it is an existential mode of being-in-the-world. Therefore, we believe that movement, understood here as generator of space, is not necessarily a materialization of motor (skills), but it is also known as a possibility, perceived even before it is carried out. Such possibility, once noticed, is actualized by means of intentionality of the moving-subject, who, in immediate control of their motor actions regulates the movement that he performs, assuming a posture, as assumed by the subjects of the above-mentioned research, "*not being very eager, to make movements, but always mindful to everything happening on the screen*." In this mindfulness, other possibilities are bestowed upon the subject who glimpses at them, as can be seen in the following: "*The question of reflection [...], we knew where to click to do it, and, knowing it, we could do equal distances with it. But after thinking and moving a point, I was able to better understand its characteristics*." Through this *doing*, movement reveals itself as an actualizer, which animates the screen, producing changes that have characteristics of a mathematical property, allowing the subject to understand it better.

In the previous paragraph, production that starts with perception is exposed. First, there is movement especially as a perceptive act, which is performed and visualized as it generates intuitions for the subject who visualizes it, providing input for justifications and statements. When the subject turns to intuitions, trying to clarify them, there is a shift from perceptive acts to reflective acts. It is the realization of movement aiming at something, such as actualizing a desire to do something or thinking of doing something, fixing what is seen in the movement and validating conjectures and/or responses. This shift can be seen in the account: "*Visualization of movement has helped us communicate, test possibilities and also come up with solutions*," which puts perception (visualization) and acts of communication and testing as belonging to the same movement towards (re)solving a task.

When subjects claim they "have reached solutions," they often expose an argument based on movement and visualization of movement, understanding them as sufficient for their validation, which can be understood in the statements: "*how can we ensure the hose will be perpendicular? By moving and looking*"; "*You can see, right*? *It is apparent that they are equal.*" In another direction, one has the comprehension of the impossibility of validating only through the movement, which is expressed in: "*one cannot only move and think that it is right and ready, it may be wrong. So, we discussed how to validate. Reading the problem again, we found the keyword 'shortest distance'. If it is the shortest distance between two places, it has to be a straight line, right? We got this from mathematics.*"

We see the act of validating as relevant to the constitution of mathematical knowledge. In this text, validation is expressed, as shown in the previous paragraph, as being possible through movement and visualization of movement, as well as through revisiting and organizing mathematical properties. By analyzing the accounts of the subjects, we understand that in the successive movements and visualizations of their implications lies the constitution of certainty, and conviction, which we can highlight in the following: "*When we see the properties that do not vary, as much as we already know them, when we move, we are sure, it is the confirmation that it is the property, which has characteristics that we can study when we move.*"

The statement that is made regarding movements and visualizations, as well as that which is made through review and organization of mathematical concepts, are coherent to us and complement each other in a validation practice, which may contribute to demonstrative practice. When the subject says "*you can see, right? It is apparent that they are the same,*" they are not making empty statements, they have an empirical foundation that allows them to assert themselves by drawing on a range of movements that show what they see and are sure of seeing. This understanding emerges from lines such as: "*GeoGebra helps a lot, the testing, the validation; we can do that by simply moving a point.*"

The succession of movements performed and visualized presents the subject with a "truth" which is perceived, and that can be an invariant that reveals itself in the variation. This invariant is visual, and with movement there is a way of showing it for what it is. Continuing the previous account, regarding an equality observed, another subject states: "*They are the same, because line CD divides the segment AE in half. So, it is the mediatrix of AE. So, any point on this line will be equidistant from A and E.*" Thus, the statement can be viewed another way, which is explanatory and explains mathematical properties that corroborate and fulfill with other meanings the statement of the first subject, thus projecting greater power of persuasion about the validity of the statement.

For validation, there is evidence of *previous knowledge supporting movements and perceptions*. In the previous account, the subject brings mathematical knowledge learned prior to the accomplishment of the proposed activity. Another subject states that in the classroom, during their undergraduate studies, they learned "*about translation, about reflection,*" movements they made in the development of field activities. However, we believe there are other previous knowledge brought by the subjects during the development of the activities and that guide the ways through which they were with DG. For instance, one subject understands that "*calmer movements gave me more answers. In the first activity, I learned this. In the others, it was easier, because I had learned it, so, that is why I say that it is not just about mathematical concepts,*" referring to how they learned by developing the activities. This account shows how prior knowledge of the way to perform movements unfolds into other activities. Another example is prior knowledge of the dynamic potential of DG software, which is revealed in lines such as: "*The challenge is to draw the point guard (of the basketball team in the activity), so I understood that in GeoGebra it could be moved, and that was it, I started moving and, moving ...*"; "*GeoGebra lets you move the ball. Then, we move it to the midpoint of the triangle.*"

Thinking in action enables us to acknowledge the possibilities of movement of an object in a given foreseen situation, as shown in utterances such as: "*After realizing that point G was moving, we might want to observe this movement in greater detail, and, before moving, we knew it could go left or right, so we moved left and right to see what would happen*". Equally important, is the previous knowledge of the functionality of the software tools, which is exposed in the utterance: "*we used the tools because we knew what each one did.*"

This prior knowledge comes from the subjects as a whole, who dispose of them in their perceptive openness, and not just as argumentative resources. They enable the subjects to think about the movement before actually performing it, as well as stop the movement to reflect on what emerges from it, and fix what is seen in the movement. These are investigative acts performed in the conduction of the activities. As previously stated, while investigating, the subject pays attention to the screen, which allows them to say that: "*When we were moving and paying attention* *to what was happening on the screen, we were able to observe some patterns, step by step, we were discovering things, organizing the information we got, and we were talking, too; discussing. This is all learning, as I was not used to working with activities like this in GeoGebra.*"

This was said by the subject when asked what they learned; others replied: "*we were in the middle of an investigation, which is what I liked the most about the activities; so in this respect I think we have learned to investigate; we were doing the steps and thinking, reflecting, seeing possibilities, so we tested to see if they worked.*" Thus, the investigation was not only a response to the requirements, but also something to be learned from the activities, working with the subject, the possibilities of computer technology available to make conjectures and create their own procedures.

It is worth noting that the background and/or starting point of almost all the investigation processes initiated by the subjects consisted of movement of geometric objects in the software interface. That shows the characteristics of the context that the subjects were experiencing, in which activities that required movement were presented, and software that opens possibilities for movement, thus exposing the subject to the relevance of movement to (re)solve mathematical problems projected in this context.

The perception of the dynamic context from which the investigations advanced guided the actions of the subjects, including studying a figure by setting it in motion. They understand that when "*working with construction and movement together, we had to move [...] with these movements we could better observe the buildings, understand what we did. So, studying how to construct is different from just constructing. This possibility that we had to better observe the construction in motion, was very interesting.*" Thus, movement is a way of studying the figure and its characteristics. In this case, the figure is given not as a static entity, whose properties have already been defined and described. Although the subject has intuitions about it, it is still a mystery, for, if the subject did not construct it, he would not know the settings before he moves it. When, within the possible configurations, the figure shows itself as a known figure, through movement, one can understand, as do the subjects, that it is possible to understand it "better,", because its properties stand out as invariant in the *figure-in-motion*. In this case, as pointed out by a research subject, one can "*set invariants as properties of a figure. For example, observe what does not vary in a square, then you can take what does not vary and define the square.*" In this utterance, the subject proposes an activity whose aim is to define a figure through perception and organization of the properties shown when it is moved.

In conducting investigations by setting the figures in motion, the subjects realized and understood that the preservation of the figure or its deconfiguration after a movement are consequences of the way it was projected on the screen, as can be understood in the utterance: "*did all the steps that the question required, and, in the end, we realized that these steps were linked by properties that we fixed, as perpendicularity, the parallel, reflection and other things. But, how could we see these relationships? It was through movement. That's where movement comes in, when we could drag the dots and see that everything fit together.*" This opens a discussion about building and drawing/illustrating. Everything "fits" and shows the figure maintaining characteristics intuited before the movement, because care was taken regarding the requirements of the task; the figure was *constructed* through the projection of the properties that constitute it.

Contrary to that is the deformation of the figure without preservation of its characteristics, which occurs when it is not properly constructed, when the subject who constructs it does not rely on its properties, or even when they are aware of the properties, they are not intertwined with each other. This way of designing the figure in software resembles drawing/illustrating on paper or blackboard. This happened during the development of the activities, which led one of the subjects to state that: "*when I tried to make the final movements of the points, everything went wrong, everything got distorted, point G was kind of diagonally, the position of the capsule was not steady* (referring to time capsule activity). *But why? It's because I did not apply the properties correctly. The construction had to be done according to the properties. For example, instead of doing a perpendicular line, with the tool, I did it without measuring, roughly about 90°, then when I tried to move, everything changed; it all went wrong.*"

Thus, we have two accounts that deal with ways of designing a figure with the software interface. In one case, after the movements are conducted, the figure retains its characteristics, which were noticeable before the movement. In the other case, such characteristics are not preserved, causing deformation of the figure, generating figures which are different from those the subject intended before the movement. In the accounts of the subjects, we understand that this is evidence that "*in GeoGebra it is not enough to construct, one has to move them, because a construction can be maintained or not, depending on if the properties have been strictly followed, or not.*" That is, a figure projected on the DG software interface enables the subject to have intuitions, however, does not allow a statement that implies this figure. It is on an indeterminate horizon, whose vision is given by perceptual movements, including moving the mouse, moving the figure to validate the intuition.

Thinking about the constructed figure, a right triangle, for instance, when set in motion, we understand that there is a modification in the figure, because even while preserving its fundamental feature of having a straight internal angle, there can be variations in the dimensions of sides of the triangle.

The question that arises is: what does the movement present to the subject, who is aware of the software interface, seeing the configurations of this triangle? We understand that each stop in the movement yields a figure, which is similar to the first one (before the movement). However, this is a view that discretizes movement, thus presenting fragments and representation of fragments. Thus, we understand that by referring to motion as continuity, we cannot conceive that it gives us a succession of closely similar figures; we cannot claim that motion, in its duration, results in the same right triangle. By conjecture, we understand that motion enables the perception of the *right-triangle-in-motion*, with the understanding that it is being changed.

In that research project, the study of the figure, setting it in motion, is a way of constituting geometric knowledge that was worked on through activities assigned to the subjects. They were developed so that movement, as a mode of resolution, would stand out. We believe that the subjects perceived movements already imbricated in the instructions of the activities, especially when they state that: "*they (activities) were created in a way that we knew we had to move to perceive (in the interface) some properties, some invariants, and then see the possibilities of solution.*" The activities and their requirements were the starting point for the subjects' actions with the software. Although we understood that the concepts worked on could be those that were shown at the moment of the subjects' accomplishment of the task, the activities contained previous conceptual demands, such as isometries.

Regarding geometric concepts, the subjects stated that: "*we worked with many concepts, it was interesting to see that in a single activity several concepts can be worked on. We started observing these concepts when we were constructing and then they became more visible when we moved the points that made the entire construction change. Except for the reflection, which was abundant during the activities, I think I know the other concepts, so I don't think I learned them, but I learned how to work with them, how I can think of connecting them within an activity. This we do by including the possibility of movement*." This statement shows that activities designed to be conducted with DG software are important tools for learning more than geometric concepts, and are important for the development of autonomous attitudes, since when presenting them as contrasted from a dynamic background, one can understand them by overcoming immediate potentialities, as the research subjects pointed out.

There is an understanding of the subjects who, while participating in the research, had an opportunity for professional formation as they were becoming mathematics teachers, to observe the relevance of working, basically with the same activities, but another public, as he manifested that "*if the activities were applied to elementary school students, they could learn many concepts*," opening possibilities for studying and understanding them as something new. About that, one subject stated: "*regarding the invariants, we saw the properties better when we moved the points, because they* were *preserved. We don't have to study properties, we just highlight them, because we already know them. But what if they were for school students (elementary school)? One could ask them to study what does not vary in that movement, so they would know more about those properties.*"

Thus, we understand that, for the subjects the most important in the constitution of knowledge which advanced in the context of that research project is the pedagogical and methodological knowledge, which enables them to see ways of *being teachers*. This knowledge is exposed when the subjects state "*what we learn here, I think, is more linked to the development, the exploration, the investigations that we were doing.* [...]. *So, the challenge was to apply these concepts. So, I think what we learned through this is how to apply such concepts in situations that require moving objects for them to appear*"—showing that they have learned about working with concepts in an activity—and "*how to conceive an activity* [...] *by putting in it factors that lead students to get their hands dirty and build and move objects*", they learned that "*you can have other types of exploration* [...]. *You can have different activities* *even while addressing the same themes*"—evidencing a direction of learning that can contribute to the professional education of the subjects, who are future teachers.

Thus, during the work in the field, data which went beyond the research were constituted, presenting ways of working with school geometry. In the open context in this field, two *selves learn*, one is the self that is a student, who does the activities, and the (future) teacher who, during in the conduction of the activity, thinks of possibilities for their classes. Therefore, there are different views and actions running parallel to the activities, which can be understood in the utterance: "*while I was conducting the activities, I was also trying to think about how it was engendered there* (into the software), *because everything was neat, several ways of doing it were available, but, in the end, they all produced the same answers. Then, I was already thinking about my classes, because I like technologies a lot.*" In this account, there is a *self* which is the *student*, who is concerned with the task, trying to cope with the challenge it poses. There is also the other *self*, the (*future) teacher* who, while doing the task, is thinking about how to present it to their prospective students, or how to develop activities with which they will work. Therefore, these two *selves* are present in the same practice of a two-folded self who learns in order to teach.

Foreseeing professional and methodological practice and considering the requirements of the activities assigned to the students, we understand the subjects changed the focus of their view: from what was seen in their perception to understanding the way they were structured and the context in which they could be developed. Initially, subjects viewed that "*activities start from geometrical situations that are placed in a context where they can move. So, I learned that you can take a simple or self-contained activity and transform it[...]. Later, I will try to take geometry exercises and try to turn them into problems like these, which require construction and movement, it is much more inspiring; more challenging.*" Regarding the contextualization of an activity, we observe the subjects' understanding that it "*makes us learn about this context*" and that when it is directed to working with DG software, "*we have to create activities that make students move the points and see what happens when they move those points.*"

The learning resulting from conducting the activities (conceptual, methodological, and pedagogical knowledge) was expressed by the subjects during the conduction itself and during the interviews. They said that at different times they realized the continuity of the movement of constitution of knowledge that occurs when one is with DG and with learning co-subjects.

These are moments when the transition from subjectivity to intersubjectivity is apparent, when a subject perceives and recognizes the other as a subject of his learning, understanding that every act of learning occurs with the other. This transition is evident in utterances such as: "*There is also the involvement of the group. We were able to do the activities, sometimes on one computer, and everyone exchanged ideas about what we saw there, in the movement. Everyone could take the mouse and do something to show their ideas. So, we learned to respect the ideas of others, everyone had their turn, and could talk. Someone would say: move here, others, move there; they would pick up the mouse and move, then, in the end, everything worked out, everyone developed together, learned together as well.*" This account presents what contributes to intersubjectivity in movement as a mode of expression, which occurred *in/with* the *living-body-mouse-figures-in-motion* which materialized the intentions of the subjects to give a dynamic and visual background to what they said.

In the movement of constitution of knowledge, the perceptions present in the subjectivity of each subject were expressed and shared in the intersubjectivity of the subjects who performed the activities together, and paid attention to what was being said by each subject who expressed themselves. Thus, dialogues about what was said took place, there were contributions, agreement, disagreement, articulations, organization, and improvement of what had been said. This occurred during the conduction of the activities, and later in the interview. What had been said was recalled, however, already presented in its final articulations, through the dialogued organization among the subjects. For us, this highlights a movement of constitution of knowledge.

We clarify and reaffirm that the subject who knows is not an inert subject, who is intellectually withdrawn while knowledge presents itself to them, whose role is solely to "keep their eyes open" following the rhythm of the presentation of knowledge. We understand that every movement of the subjects turning to other subjects, with the available informatics, opens spaces. We do not understand the *generation of space* as something that comes out of nowhere, which is totally new, but we want to emphasize creation presented through a view that does not presume an idea to be a given, but which views it as a mystery as the subjects involved in this movement problematize it. Thus, knowledge is constituted, and the moving-subject is at the center of this constitution, a creative center that also recreates itself with each new learning experience.

Nurtured by the openness given by the activities, to act and realize possibilities, the subjects stated that: "*we also learned to be more critical. Sometimes, we came up with solutions that seemed to be correct, because when we moved, we seemed to be coming up with the right solution, but then, when it came to testing, making a final construction, there was a slight difference.*" This criticality always puts the subject in a state of doubt, not surrendering or conforming to results before the validation process.

This criticality enables understandings such as*:* "*here, we work with properties; which we already know. But this is nonetheless learning. When working with properties, especially in software, which is a whole different world outside the notebook, we can see properties in motion. So, they (the properties), even though they are the same as in the books in which we learned, are also different, because in GeoGebra they are not the same thing, understand? In GeoGebra, there is a different approach, they can be moved.*" This account shows that geometric knowledge, even when already structured, is not contained in itself and does not make itself known in a single way. Thus, we understand that knowledge can be renewed in each space it is dealt with and discussed, and may be expanded by constituting itself with the possibilities opened in that space. More specifically, we understand, conjecturing, that geometry and its entities, taken tacitly by geometric thought, which does not show displacements, can be renewed when embraced by a geometric thought of spatialization in DG environments, which gives life to each entity, each property, each geometric figure, setting and understanding them in motion. For example, through this manner of thought, a trapeze is no longer just a definition and its graphical representation. It is also a *trapeze-in-motion*, and everything that it entails also moves and can be known by being in motion. From this perspective, the view and the understanding of its own definition are renewed, as it is written within the movement, thus with the moving-subject

We see the constitution of knowledge which occurs with movement within DG environments as a particular case of the constitution of knowledge in general. In the life-world, where the technological world is present, we also learn through movement, for knowing is a constant flow that constitutes the knowledgeable-being at the same time as it constitutes the co-subject before themselves and the life-world. This implies that the subject, i.e., the person, is always a *being* in motion.

4 A Comprehensive Synthesis of the Ideas Articulated

In this chapter we sought to articulate an understanding of *how knowledge is constituted when working with dynamic geometry*. Therefore, bringing the analyzes carried out by Pinheiro (2018), we explain how the constitution of knowledge took place in a context in which the research subjects were conducting activities with the computer, in a DG environment. We expose perceptual acts that constitute the soil upon which the moving-subject weaves understandings. Such acts are apparent in the actualization and visualization of movement, manifesting ways in which the subject directs his actions towards the actualization of possibilities. Depending on the perceptive acts, the computer screen presents transformations that materialize as *figure-in-motion*. Along with this perceptive view a turnaround takes place, which leads the subject to reflect on the view and the intuitions generated, analyzing and expressing what is understood among the co-subjects, who, in turn, seek to understand what was said and contribute with what he understood through his individuality about what was discussed.

This exposes subjectivity and intersubjectivity, animating processes in which perceptions are shared, justified, and organized, seeking an articulation that is coherent and accepted by the group didactically involved in a discussion. The intentionality is geared towards the search to validate the thoughts through acts of verification that materialize, specifically, as movement performed with the mouse and expressed on the screen.

Thus, we have shown that, in the context presented, the starting point and also the background of the constitution of knowledge are the movements performed in an environment inhabited by groups of subjects. With the expression of what is perceived, the dialogue, the articulation of ideas, the organization, and explanation of the knowledge constituted in the subjectivity of each subject and in the intersubjectivity of the subjects is done through movements, producing knowledge that emerged from the performance of all these acts. Thus, the movement of the living-body reveals itself as the way of spacializing of a subject, who moves by *moving-*

perceiving-reflecting-understanding-expressing, and this action materializes as a unity in the *now* of the accomplishment of such acts by that subject.

By focusing on movement and perception, we understand that moving is an actualization that *actualizes-fixes-transforms-explores-tests-validates-shows-expresses,* indicating paths to knowledge and to the knowledge of oneself. We understand also that when movement is viewed as a phenomenon, it can be *actualized-perceived-understood.* Perception, as we have stated, is the primacy of knowledge from which what is perceived emerges, when interpreted and expressed through language by the perceptive subject, opening paths of the objectivation of knowledge.

Since moving, perceiving, and knowing are acts performed by the moving-subject; we understand that it is not possible to talk about movement in the software without talking about the software, just as it is not possible to talk about invariants without talking about the subject who engages their spatial-temporal presence with the occurrence of what remains and what varies, realizing permanence and variation.

References

Bicudo, M. A. V. (2010). Realidade virtual: Uma abordagem filosófica. *Ciências Humanas e Sociais em Revista, 32*(1), 121–134.

Bicudo, M. A. V. (2014). A perplexidade: Ser-com-o-computador e outras mídias. In M. A. V. Bicudo (Ed.), *Ciberespaço: Possibilidades que se abrem ao mundo da educação* (pp. 37–66). São Paulo: Editora Livraria da Física.

Bicudo, M. A. V., & Kluth, V. S. (2010). Geometria e fenomenologia. In M. A. V. Bicudo (Ed.), *Filosofia da educação matemática: Fenomenologia, concepções, possibilidades didático-pedagógicas* (pp. 131–147). São Paulo: Editora UNESP.

Detoni, A. R. (2012). A geometria se constituindo pré-reflexivamente: propostas. *Revista Eletrônica de Educação, 6*(2), 187–202.

Figueiredo, O. A. (2014). A questão do sentido em computação. In M. A. V. Bicudo (Ed.), *Ciberespaço: Possibilidades que se abrem ao mundo da educação* (pp. 313–342). São Paulo: Editora Livraria da Física.

Heidegger, M.(1999). Being and Time. New York: Harper & Row, Publishers.

Husserl, E. (1972). *Ideas general introductions to pure phenomenology*. New York: Collier Books.

Husserl, E. (2006). *Ideias para uma Fenomenologia pura e para uma filosofia fenomenológica: Introdução geral à fenomenologia pura/Edmund Husserl* (5th ed.). São Paulo: Ideias & Letras.

Husserl, E. (2012). *A Crise das Ciências Europeias e a Fenomenologia Transcendental: Uma introdução à filosofia fenomenológica*. Rio de Janeiro: Forense Universitária.

Merleau-Ponty, M. (2011). *Fenomenologia da percepção* (4th ed.). São Paulo: Martins Fontes.

Pinheiro, J. M. L. (2018). O movimento e a percepção do movimento em ambientes de geometria dinâmica. In *Tese de Doutorado*. Rio Claro: Instituto de Geociências e Ciências Exatas – Universidade Estadual Paulista.

Richit, A. (2005). Projetos em geometria analítica usando o software de geometria dinâmica: Repensando a formação inicial docente em matemática. In *Dissertação de Mestrado*. Rio Claro: Instituto de Geociência e Ciências Exatas - Universidade Estadual Paulista.

Silva, G. H. G., & Penteado, M. G. (2009). *O trabalho com GD em uma perspectiva investigativa*. In Anais do 1° Simpósio Nacional de Ensino de Ciência e Tecnologia, Curitiba, 4–6 June 2009.

Constituting Mathematical Knowledge Being-with-Media in Cyberspace

Maria Aparecida Viggiani Bicudo

1 Introducing the Subject of the Chapter

As already pointed out, all chapters in this book are articulated by the theme "the understanding and production of Mathematics while in cyberspace and with computers and other media" taken phenomenologically.

Since we, as the authors, always focus on a potential reader, we will bring the understanding which we articulated, throughout our studies, about the constitution of knowledge in Mathematics, as being in *Lebenswelt* with others. This is a complex subject. Its complexity is intensified because our investigations are focused on understanding a way of being from the mundane reality experienced by us, which is where we are intentionally with others at a distance, supported by the informational[1] screen that technology provides us. So, we immersed ourselves in the complexity of *Lebenswelt*, today.[2]

During our readings about Informatics, more specifically in Mathematics Education, we came to understand that Informatics is linked to Cybernetics in its origin. Cybernetics was a word coined by Wiener (n.d.), deriving from the Greek word *kubernetes* or pilot, from which the word *governor* may also have originated, as he wrote. He defends the thesis that the understanding of a society only occurs

[1] Informational screen (Bicudo & Rosa, 2010).

[2] In Crisis of European Science (1970), Husserl makes a very relevant and enlightening phenomenology of *Lebenswelt*. However, it is a work written in the 1930s, when cyberspace was not yet a reality.

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through the study of its messages and the communication facilities (means) available to such society.

The studies and their materialization in computer products and their realization in users' practice opened the constitution of a communication space that is interconnected by systems, called *cyberspace*. This is defined as "the space of communication, opened by the worldwide interconnection of computers and computer memories" (Lévy, 1999, p. 92, authors' translation). This is a new structured means of communication. This space created with the support of the informational screen does not require the presence of the communicating subjects and cosubjects. However space it is not necessary, however, space is created and kept open and moving (fast) by networks of interconnected digital devices, including databases and the information which is available, thereby interconnecting people, documents, institutions, programs, machines.

We started to live our daily lives also in this space. Thus, nowadays, we live in this mundane web of reality that is being made at every moment, also in the speed of the diffusion of messages and the dynamics of the bifurcations that widen the network and new arrangements of what they say, as well as of the sensory materiality that it produces. Reality, understood as a "way of being real" (Bicudo & Rosa, 2010) experienced in life-world, necessarily encompasses the reality sustained by cybernetics and cyberspace. It is in this reality that we are constituting from the mathematical knowledge with others and exposing it in productions formatted in the symbiosis of spoken language,[3] in the grammar of structured language through which we communicate in the culture in which we live and in the form of the computational language itself that sustains the computer programs we work with.

The world of education has changed, both through accessibility to information, and by the very structure of the system and the informational screen that supports it. The manner through which education, teaching, and learning take effect has been accelerated and modified. On-site courses are common in schools, semi-presence-based courses, that combine face-to-face instruction, and distance learning called Distance Education. The latter has been widely criticized by educators, philosophers, and psychologists, for example, for not allowing, according to them, the physical encounters between people, when the much proclaimed and prestigious "face-to-face" situation, understood as vital in educational environments and the situation in which teaching and learning take place, ceases to occur. Questions such as: is distance education even possible? In this environment, is the constitution of knowledge possible or is there only linear learning made possible by a deterministic programming of problems/questions and answers? Is the constitution of intersubjectivity possible in this space?

By understanding the complexity of this subject and based on previous studies carried out by us and our research group "Phenomenology in Mathematical Education" (Bicudo, 2010, 2014; Bicudo & Silva, 2018; Bicudo & Rosa, 2010), we

[3] Spoken language refers to the expression of understandings made by the subject expressed in sound, structured according to the spoken language present in the socio-cultural context in which they live.

conducted research focused on the phenomenology of the constitution of mathematical knowledge in a Distance Education environment. The encounters between teacher, researcher, and students were described phenomenologically, producing data that were also analyzed according to phenomenological reductions.[4] For the construction of data, a Distance Education course entitled "Geometry: what is it about?"[5] was designed, programmed, and actualized.[6] The course, in its entirety, is found in Barbariz (2017).[7]

2 The Research Carried Out

The guiding interrogation of the research which supports this text is the *constitution of knowledge by being with Mathematics, at the computer and the co-subjects*. By assuming Husserlian phenomenology, we seek to go to the thing, as it shows itself to those who experience it and, in order to do so, promoted a distance course in

[4] The phenomenological reduction is characterized as a thinking movement, exposing, through the articulations of ideas, the complexity of senses, and meanings that intertwine in broader ideas. The movement of exposing the understandings of the phenomenon encompasses the process of highlighting the phenomenon of the background from which it arises, expressing understandings open to the researcher about its senses and meanings. Through Husserl we understand that the research phenomenon is put in suspension or evidence or *epoché*. This means that the researcher's preconceptions about the object investigated are put in suspension so that the researcher is on alert and their previous experiences do not deterministically lead the research. The reduction movement, still considering what is said by the afore mentioned author, is taken as intentionality, determined by the acts of consciousness, in motion. As intentional subjects, we are open to the life-world that presents itself to our perception, so, the reduction movement can help us understand essential aspects of this world. The reduction movement presents itself as important for the investigation, because, through reduction, acts of conscience are exposed, that is, it is possible that one becomes aware of them so that, by reflection, the cognitive roots of the affirmations themselves are explicit (Bicudo, 2011). It should be clarified that reduction, within the scope of phenomenology, does not refer to a simplification or a summary of what is presented in the text, but to an articulating thought movement in which the meanings are intertwined with more meanings and by the assigned meanings, put into language, they are shaped into ideas that encompass them in a comprehensive whole.

[5] Actualization as Baumann (2013, authors' translation) says of a creative action that, when performed, makes present possibilities present [...]. Such, updating is understood here as a movement to make current, to make the ideas present [in the project] through actions that may or may not innovate. By dismembering the word, we can reaffirm that updating is the action of making current.

[6] In the case of the research mentioned here, despite recognizing that the expression "distance learning" can refer to any kind of teaching and learning process that considers the teacher and the learner in this condition, such as correspondence teaching or by using media such as radio and television, we are focusing on those processes that take place through computer intermediation and with the Internet.

[7] This investigation, which led Taís Barbariz to the title of "Doctor" was supervised by me, Maria Aparecida Viggiani Bicudo, very closely. Taís defended her doctorate thesis on 3/16/2017 and passed away on 04/03/2018.

Mathematics and, more specifically, Geometry. We experienced the course[8] throughout its conduction, including: discussions about the focus of the philosophy of the course and the way of looking at Geometry; the outline of the proposal and design; the selection of texts to be used; the ways in which the meetings would be described; data analysis procedures; the ways of conducting the phenomenological reductions; the designation of Nuclear Ideas[9] so that their names reflected the meanings experienced and articulated; the critical-reflexive discussion of the Nuclear Ideas.

The course "Geometry: what is it about?" was guided by a philosophical view about Geometry with the students, seeking to understand its underlying ideas. This way of working has moved us away from the teaching of "how to do" geometry. As the target audience—which would later become research co-subjects—consisted of teachers who taught geometry and who are exercising their profession, we assumed that they would be familiar with this science and its teaching. The texts selected for reading and discussion should be in line with the goal: thinking about the ideas that are essential to geometry. An important author who was studied in the course was Hans Freudenthal (1986)[10] whose works contributed to the selection of other texts.

The Instructional Design of the course was conceived as a project designed to meet the specific characteristics of thinking about Geometry. The course was structured as follows: two modules, with four classes each. Each class offers texts, videos, and links and streamlines some forums through questions or provocations, inviting participants to contribute their insights on the material provided and their professional experiences by recording and analyzing them. For every two classes available on the platform, a synchronous meeting is scheduled. To prepare for this meeting, the material used in each class includes an audiovisual summary of all content studied, also showing the provocations in the form of questions or proposals for reflection on the subject. Additionally, a space (forum) is offered for students to respond to the provocations, presented by the teacher–researcher regarding the summary delivered, and also to express doubts and contributions to the meeting; live broadcasting, when the teacher comments on students' posts, answers to questions already posed in their messages, and responds to requests submitted synchronously through comments from participants online at the time; synchronous encounters: four, in addition to the opening class, which was also online. The total period of the course is twelve weeks:

[8] This tells us about the professor–researcher, Taís Barbariz, who conducted the course and me, Maria Bicudo, who was with her throughout all stages of the research.

[9] Nuclear Ideas—NI—in the research cited in this article, refers to the broader convergences articulated through successive movements of phenomenological reduction.

[10] Hans Freudenthal, (Luckenwalde, Germany, September 11, 1905– October 13, 1990), the creator of Realistic Mathematics, was a Dutch-born mathematician. He made substantial contributions to algebraic topology and was also interested in literature, philosophy, history, and Mathematical Education.

eight directed to the availability of classes and synchronous meetings and four for dissemination, call for registration, closing and publication of results.[11]

Aware of the movement of the project, the researchers analyzed the data according to two distinct moments of the course: the designing of the project itself and the moment of the update when they were with the co-subjects, with the computers and other interconnected digital devices and geometry, working with the model brought in the *being-with-media* construct (Borba and Villarreal, 2005), as it is understood as a way of being and working together with the available media. Thus, insights, doubts, difficulties, and the openness to understandings that occurred during the discussions for outlining course design were described. These descriptions were analyzed, highlighting teacher–researcher's perceptions about herself: her anxieties, joys, and be aware that knowledge is being constituted.

The living-experiences[12] (Erlebnisse) of both researchers—the teacher–researcher and the supervisor—occurred in streams in which the experiences lived by one (teacher–researcher) were brought in by the recollection of memories, when they realized what had happened, already advancing into analytical-reflective thinking and the experiences of the other (the supervisor) took place in while listening carefully to what was reported, realizing what was being shown, while engaging in analytical-reflexive thinking. Together they lived moments of understanding and they reflected on what was understood, so that what was said by one of them was intertwined with the thinking of the other. Bringing these lived experiences to mind, I understand[13] the constitution of knowledge taking place in intersubjectivity in which the subjectivities involved are intentionally linked to thinking *while thinking*.

[11] The course uses the Moodle platform; its conduction for the constitution of research data was made possible by CECEMCA—Center for Continuing Education in Mathematical, Scientific and Environmental Education; made available by UNESP Rio Claro campus, and is certified by the Dean of Institute of GeoSciences and Exact Sciences of UNESP. It is described in its entirety in Barbariz (2017).

[12] The living-experience (Erlebnis) tells, at first glance, about the life that flows, as it is lived. We live experienced acts being in motion for the duration of their temporality. At each moment we live the present moment of the act taking place. Psychical acts, such as perceiving, imagining, fantasizing, remembering, reflecting, which are inherent to human beings, even if they occur uniquely in each individual. Living-experiences flow, slide from now to what has been, making room for other living-experiences. We know we are living, but only by an act of consciousness do we realize what we are experiencing. This act is to perceive the experience as being lived and Husserl calls it "Erlebnisse."

[13] I speak in the first person singular, for Taís is now present in the flow of my memory. In the text there is a variation between *me* and *us*. When I speak in the first person singular, it is me, Maria, revisiting the memory of what was experienced with Taís and also the understandings that were opened to me; when I speak in the first person plural, I bring the thought and what was accomplished by the both of us.

The data analyzed concerning the *first moment*, which consisted in designing of the course, refer to the lived experiences of Taís, the teacher–researcher. She wrote the notes on the description of what was perceived and felt as she was studying the texts, articulating them according to the outlined philosophy, and organizing them in a sequence that, with comings and goings, ultimately proved to be didactically sound. However, writing always occurs in the flow of recollection, of the act of perceiving and feeling, as, even though actions last, that is, have a duration in time, when occurring, they are in the present, in the very now; whose living experience, once passed, can only be recaptured in recollections of those acts experienced, when one realizes what happened (Husserl, 1977, 2013, n.d.). It is this recollection that is systematically taped on by the teacher–researcher, describing it as directly and faithfully as possible. This means: no interpretations and no judgments of value intersperse the description. This becomes the data taken for the analysis and interpretation carried out after the account, when she looks at the descriptions of the experiences, with the supervisor.

The *second moment*, which is related to the construction of data, is characterized by the updating of the course, when it takes place in the meetings of the teacher–researcher with the co-subjects, students who make up the group, with the texts studied and the informational screen. The course activities were recorded on Moodle. The dialogues of each forum were analyzed and also recorded on the platform. As we looked at the descriptions of such dialogues, we realized that if we focused on what was said by one of the participants, who stood out in the different forums, we would bring the manifestations of the other co-subjects. This is because the dialogues take place between co-subjects and do not flow linearly, but intertwine ideas of the co-subjects that present themselves in the language expressed. Thus, when we bring the clippings of what was said by one of the co-subjects, we find speeches of others who are with them and, more importantly, the expression of collective thinking. With this understanding, we elected one of the dialogues as exemplary.[14]

The descriptions concerning the activities of the first and second moments of the course update were analyzed by highlighting Units of Sense (US[15]), understood here as those that express a first meaning that, for researchers, is shown by the intentional reading of the text contained in the description. In this movement the formulated question remains as a counterpoint and contributes to weave the plot of the text. This way, the US are not taken in isolation, but as a whole, present in the description.

We analyzed these US seeking for the meaning they have for us who have formulated the interrogation and, in order to express such meaning through language, we established a dialogue between the meaning contained in what is said for the researchers and the words that are historically and culturally present in the region of inquiry in which the investigation is taking place, as well as the historical-social context. In this movement, with the aid of hermeneutic work, the US are transformed

[14] Detailed data is found in Barbariz (2017).

[15] T.N.: all abbreviations, initialisms, and acronyms correspond to the original words in Portuguese.

into Units of Meaning—USg. All the US and USg, by enumeration, are listed so that even as we proceed to phenomenological reductions, advancing toward the articulation of broader ideas that speak of perceived senses and worked meanings, we can make a retroactive movement, so that those first senses may be retrieved when the movement of interpretation demands it. The articulations of our thinking were from US to USg, performing the so-called Idegtaphic Analysis is an analysis of the individuals (Bicudo, 2011) and then to broader ideas, proceeding to Nomothetic Analysis (Bicudo, 2011), Nomothetic Analysis is a more comprehensive ideas, also expressed in words, which we call Convergences of Meaning and significate (CSs). We then continue to carryt out the reduction movement, articulating the CSs from the analysis of both moments of the course, preparation and updating, constituting the Nuclear Ideas (IN). Searching for what these INs say and, by addressing the research question, we understood that we could further realize a convergence of senses and meanings. We call these convergences Comprehensive Nuclear Ideas (INA). INA were open to the interpretation of the analysis performed, as we sought to explain our understanding of the phenomenon examined under the light of a dialogue with the authors read and the data constituted during the course.

Addressing a potential reader of this text and being concerned with explaining the way we carried out (Tais and I) this research to help its understanding, I included an excerpt from APPENDIX 1 published in Tais' thesis, as an example, which shows the US numbered from 1 to 60 and the respective USg also numbered from 1 to 60 (Barbariz, 2017, p. 92, authors' translation).

> **Hermeneutic Excerpt**: Virtual space–the space made possible by the computer.
>
> **Understanding what is said**: The realization of the course, through its effective integration onto the platform, causes enthusiasm in the teacher.
>
> **What is said**: feeling of satisfaction of the researcher.
>
> **USg**: 013—feeling of satisfaction of the researcher.

Excerpt from Appendix 2 (Barbariz, 2017, p. 92, authors' translation) in which, for each unit of highlighted speech, it is shown what the teacher understands from what is said by the subject and, then, what is understood by *what is said*, followed by, the *Units of Meaning* and their respective number subsequent to the last one (60) of the first moment of the research, concerning course planning, up to 158.

> **What is said**: The student is with a co-subject, text and internet; they search greater technical knowledge.
>
> **USg 68**: The student is with a co-subject, text, and internet
>
> **UGg 69**: The student seeks more technical knowledge (Barbariz, 2017, p. 92, authors' translation).

Through the phenomenological reduction movement, when we ask ourselves what US and USg say about the interrogation, the research guideline, we articulated

43 Convergences of Sense and Meaning.[16] By looking closely at the SCs, we understand that it is possible to articulate yet another reduction movement, and, as we sought to gather senses and meanings expressed in their utterances, while moving towards broader articulations and returning to the US and USg, we had articulated 10 Nuclear Ideas[17] and, by the same procedure, we articulated 05 INAs, namely:

INA 1: Self-perception teacher/researcher (articulates IN 01, 03, 05);
INA 2: Self-perception while constituting knowledge (articulates IN 02);
INA 3: Ways of being with one another (articulates IN 06);
INA 4: Perception of the co-subject as constituting knowledge (articulates IN 07, 08);
INA 5: Perception of the course actualized (articulates IN 04, 09, 10).

The interpretation of INAs occurs through a dialogue between the research interrogation, the careful reading of the descriptions that review what is said by the co-subjects, the study of significant authors, the area of research in which the research is inserted, the INAs which, as previously explained, articulate the senses and meanings of USs, USgs, CSs, and IN. Thus, it is a whole that unravels the constitution of knowledge while in cyberspace. However, by being willing to provide a comprehensive synthesis on the subject that is expressed in the title of this chapter, I will explain my understanding of the research movement that is materialized in the doctoral thesis of Barbariz (2017), by focusing on the constitution, and in doing so, highlighting the mathematical knowledge while in cyberspace with the other co-subjects with whom we intentionally dialogue about a specific subject which we seek to understand.

[16] Perception of the self while questioning the execution of what was proposed. Opening horizons of understanding of geometry; Perception of the institutional means for the actualization of the course; Perception of the complexity of course planning; Perception of experiences lived as a student, a teacher, and a researcher; Willingness to understanding pedagogical practice; Self-perception of the researcher when conducting her own research; Psychical ways of being a teacher/researcher; Perception of the possibility of course actually happening; Self-perception as a teacher–researcher; Perception of glitches in the technological apparatus; Perception of the actual beginning course; Focus on the technological apparatus supporting the course; Perception of the discrepancy between the activity developed as a teacher and that expected as a researcher; Perception of tensions and compatibilities between the course which was planned and what was effected; Perception of change in the way of being a teacher; Perception of concern regarding the research. Perception of oneself producing knowledge; Ways to be with each other at a distance; Perception of mathematics teaching practice according to the available materiality; Perception of dynamics of the course; Perception that others and the computer are present in the very act of designing the course; How to make search records; Perception of the assessment of the way of being, of others and themselves; Concern about student attendance and participation (Barbariz, 2017, p. 97, authors' translation).

[17] Self-perception designing the course; Self-perception producing knowledge; Self-perception as a teacher/researcher; Perception of the movement of actualizing the course; Perception of the psychical ways of being a teacher/researcher; Perception of ways of being with others at a distance; Perception of others articulating ideas; Perception of others constituting knowledge; Perception of the importance of language in actualizing the course; Perception difficulties of articulation by the subjects (Barbariz, 2017, p. 98, authors' translation).

3 Understanding the Constitution of Mathematical Knowledge While in Cyberspace

Constitution of knowledge is a word that makes sense at the core of phenomenological philosophy. This is because it refers to a movement when subject's experiences happen, understood as an embodied living-body[18] which, in a totality that cannot be separated into parts, brings the characteristics of its organism, which is sensorially plugged to the life-world by the sensory organs, however, not casuistically, but always intentionally directed toward something that awakens its quest for understanding. An organism that, during the flow of living, functionally encompasses all its articulated characteristics (genetic; neural; organic, sensory, etc.). The physicality of the living-body can be studied if the researcher keeps a natural posture,[19] by observing how certain organs function, what their characteristics are when looking from a health perspective, for example. When we look at a living-body, what immediately strikes us is its somatic aspect. But a longer look shows that it always moves toward something it seeks to understand or do while expressing its motricity. However, along with the physical-sensory aspects there are those psychical aspects, which are evidenced by liking, disliking, feeling pleasure, and pain, which can also be studied by evidence of behaviors that may even be related to certain occurrences, such as stimuli, for example, which "cause" certain responses. But also, with these aspects there are psychical aspects, which also encompass our cognitive acts, and the spiritual ones, which encompass acts of judgment, such judgments may include *bigger than*, *smaller than*, *prettier than*, etc.

Psychical comes from the word *psyche*. According to Ales Bello (2015) this word originates from the Greek language and indicates a divine and immaterial part of the human being, complex and referring to the spiritual aspect par excellence; yet it is a word derived from epistemologically divine acts, that is, the psyche is traditionally understood as being from a territory that is not part of our body, as it is immaterial, that is, we cannot touch it. The psyche is also the territory of acts of reaction and impulse, that cannot be controlled, since they come spontaneously. For example, when we say to another person*: I am thirsty right now*, the other person, who is with us may, may not be thirsty at that time, but that does not prevent such person from understanding what we are feeling, because feeling the need to drink

[18] The living-body makes us appear and touch the world, while making us perceive ourselves by touching what is in front of us and being touched by what is there. It is a totality consisting of "flesh and bone," that is, of physical/psychic and spiritual aspects. Thus, the human body is not reduced to a structure composed by the pair psyche and body, looked at in its materiality, which, in the works of Husserl (2002) and Merleau-Ponty (1994) is explicit by the word Körper. Husserl exposes his understanding of the living-body as an interweaving of the physical, psychic, and spiritual aspects. Thus, it shows actions nurtured by the intention and willingness to do something in a situation, that is, as always moving toward something to do. The conception of the living-body—a body that lives and feels that it is living is expressed by the word *Leib*.

[19] The natural attitude is to take the world as existing in its objectivity.

water is common among people. The other person cannot measure the thirst I am feeling, but they understand what *being thirsty* means.

Psychical and spiritual aspects do not allow us to speak of the living-body only from the perspective of corporeality. They highlight the complexity of such body. This complexity is intensified by highlighting the intentionality of the movement it makes. Phenomenologically, intentionality is considered to be the characteristic of consciousness. Being consciouss does not mean reflecting on the actions we perform all the time, but that we have the perception that we perform them. There is a stream of acts of awareness, without us constantly evincing and describing the acts conducted. Consciousness is there, effected in the acts we perform. Ales Bello (2015) exemplifies this mode of consciousness through a metaphor of the transparency of a glass. In "Person and Community" (2015, p. 28, authors' translation), she tells us about the experience of looking out of the house through a closed glass window. She states, "We see the people, the garden, and things. We can describe them. However, we do not speak about the transparent glass. It goes unnoticed."

Intentionality is a core concept in phenomenological thinking. It is complex and difficult to describe through brief explanation. But it can be understood, at first glance, as an invisible thread that unites us to what we focus on with a watchful and inquisitive eye. This thread extends toward what we focus on in an interrogative way and brings the focus, perceived directly in the act of perception, into the acts of awareness as perceived and not as reality itself.

Perceiving is an act that highlights to the individual the reality of what is perceived. It takes place in the "now" and, in terms of the constitution of knowledge, it "impregnates" the spirit of intuitions still empty, as it demands other experiences that occur in the totality of the living-body, which is a complex organism, made of flesh and blood, and which combines in a single whole the sensory, the psychical, and the spirit, as said above. By taking place in an instant, the act of perceiving already speaks of the temporality that runs through experiences in their movement of coming about.

The living-experiences are fluid and encompass the performance of the acts performed by the living-body that knows himself while is performing them, because he is alive. By being fluid and always being, that is, happening, they slip from the "now," the instant of enactment, to the what "has been," has already occurred and, in this dynamic, it brings to the "now" what is in a situation of about to happen, the "not yet." The stream of consciousness (Husserl, n.d.) then reveals itself in its continuity, where the present presupposes the past and brings the future to itself. By sliding from the *now* to what *was*, the conduction of the act of remembering making it possible to relive what was accomplished in the present moment (in the now) that has already passed. According to Ales Bello (2015), *apud* Barbariz (2017, p. 145, authors' translation) "in remembering, what is perceived, although not noticeable in flesh and blood, is present in its own way."

My understanding is that, for the people who took the distance course "Geometry: what is it about?," what was brought by remembering is present by intertwining what is remembered with the question that is posed and requested by the teacher, by articulating it with the speech of others that also took the course with them. Metaphorically, it is like a symphony in which the chords follow each other in a harmonic way, whose harmony is materialized in the totality of the piece. The theme of music: the course proposal; the vitality: the willingness of all to take it; the execution: the joint work intentionally directed towards a goal.

Phenomenologically, I will make an effort to clarify the ways in which I understand the constitution of knowledge reveals itself in a movement, while the students and the teacher–researcher are in cyberspace, understood here as one of the modes of spatiality and temporality of *life-world*.

I came to realize that in the conception of *life-world* described and analyzed by Husserl, as, for example, in Crisis (Husserl, 1970), cyberspace should also be considered as a way of being and, in it, we experience our actions. With technologies, the ways of being in the world show themselves in specificities that were not present before cybernetics came to be. I understand that, with the reality of cyberspace, our ways of being space-temporally have changed, as well as putting ourselves together with others, therefore, constituting the community and knowledge. In order to highlight my understandings, I take the research carried out by Barbariz (2017), in which I participated, as a researcher-advisor, as already mentioned in this text.

In the research carried out (Barbariz, 2017), the people who attend the course are in the *life-world* whose reality is also present in the materialities available on the Moodle platform. This means that this world experienced by them encompasses the ground in which nature, the animal world, the human community and organizations, culture, history, sciences and their applications, technology and the possibilities open for the dynamic visualizations of shapes, colors, and movements are present, therefore, expanding and accelerating possibilities of communication between individuals.

Let us focus on the course "Geometry: what is it about?" and zoom into the ways people experience their spatiality.

There is a person, a living-body, in their entirety, present in the course, moving in front of the computer screen and they can feel different emotions, impulses when they are driven by different forces. They may be feeling that it is easy or difficult to understand what is being said in the indicated texts; articulating in their thinking as they seek to write the contributions to be read by those in the course, and they may feel tired, unwilling, happy, sad, frightened, indecisive, willing to move forward; working hard to provide explanations about something already clear to them and which will pave the way for completion of the tasks required; heeding others, their co-subjects. Their actions, which are intentionally directed to what "must be done," this command, dictated by the prevailing force in this living-body, unfolds in searches: on the Internet, seeking to find and bring texts other than those indicated in the reading list of the course or posted on the Moodle platform by the teacher, seeking to mine examples and situations that help clarify what they want to convey through their texts posted on the platform, or merely to mark their presence, even though not fully coping with what is required in terms of the logical and argumentative core of the question. The spatiality in which they live is constituted of places where, physically and sensorially, they are, in the flesh, in front of a computer and other devices they can operate; they are, concomitantly, before media they run and which bring the presence of other people who are taking the course with them. Then, spatiality has already changed, becoming more complex. Other "here and now" become present, and they welcome those psychically; experiencing emotions, by being included in the movement of understanding and, through cognitive and spiritual actions, detaching themselves from that specific space in which is their physicality and experiencing actions by entering other "here and now." They experience this reality intentionally focused on what is said, by speaking; through sound and enunciation, or writing; they enter the dialogical movement by addressing others.

Both sound and writing of the subjects utterances are conveyed with the support of computer programs. These programs, while making the broadcasting possible, also logically format ways of saying and expressing themselves. At the same time these activities occur, their records are perfomed. The range of possibilities that computer programs opens up to the memory of the event is extensive by enabling the recording of sounds, writing, scenes in which the body language of the people can be frozen for analysis and understanding. The recollections of past "*nows*" brought to the present in the stream of consciousness by intentionally seeking to resume actions that have already taken is sharpened and streamlined by the documented record of what is spoken as well as the activities performed, storing them for recall. However, it is the intentionality of the individual in action, who enquires about what happened that sets the recollection in motion and weaves the narrative that tells about his understandings.

The programmed software remains a resource of texts to be worked on. It is made available, and can be displayed on the computer screen, provided that the person, the subject who triggers the intended search, initiates the movement of finding them. Moreover, its structure is logical and remains present in the activities performed by the person, without the person being aware of each action. This does not mean such structure is not important. Its presence is seen as Heidegger states in Being and Time (1962) as "simply given", that is, as being there in the world without us being aware of it and which goes unnoticed in everyday and ordinary dealings in front of a computer. Metaphorically, this presence could be compared to the air one breathes. One breathes. But, in terms of prominence in daily life affairs neither air nor respiration stands out; they just exist and are vital.

However, although one is "there" before this openness to the world, the knowledge of each program materialized into software demands a psychological investment from those who intend to work with it. In the case of course project in point, the tools offered by Moodle were important and required detailed learning on the part of the teacher–researcher, in order to use them when the situation required. At the same time, the Virtual Instructional Designer provided by the tool cannot be ignored or be littled, as it was by being with it that the course took shape and could be viewed in its mediatic materiality.

This tool which gave support to the reality experienced in the course provides fast and dynamic bifurcations. The speed of the communications triggered, which populate cyberspace, allows participants to visit other sites, while driven by the energy that comes from their desire to seek knowledge and cope with the requests posted on the platform. They search for themes and authors concurrently, but focused on the course and its proposal. They find themselves linked to many people—enrolled in the course—who inhabit different geographic regions, and are also devoted to the subject under study as much as they are.

Let us continue to take a closer look at the course "Geometry: what is it about?" and highlight the possibilities for participants to experience time.

Time is experienced in the duration of acts performed by persons themselves, realizing what is posted by the teacher or fellow students, the desire to know what was said about themselves and about others, as well as the subject of the course, for instance. By moving in cyberspace, it becomes possible for the activities of the members of the course to take place concomitantly, in the "here and now" if they occur synchronously, or in a "here and now" intended as concomitant, but that does not take place in the same "now." This characteristic of happening in the same "now" is brought about in the sense that there is an intentional connection between the person who speaks and the person they speak to, which occurs in asynchronous activities. The presence of these people, even if they are in different places and times, is felt. This presence is revealed in their speech, even though they may be silent or confusing and caused by different situations: a student who sends a message on the platform to answer any questions regarding the movement of the course itself; a student who gets in touch with the teacher, using the available channels, to ask a particular question about any points addressed; a student who participates in a forum, commenting a provocation or posting from a colleague or teacher; the teacher responding to the posting from one or more students. The speaker's presence is felt in all such situations, and possibly others. In the excerpt below, we see a student expressing their thoughts about a text, an object of specific activity, in which Freudenthal states that the "deductive structure of traditional Geometry was not exactly a didactic success" and explains that this failure is probably due to the fact that it was not introduced to students the way Socrates did, as we can see in the dialogue between Socrates and the slave [...]

> Forum 111. Student 8 states in their text: For Freudenthal (1986), Geometry is not only an irresistible part of a deductive science, it is also the oldest and most explicit example of didactics. According to another author, Fainguelernt (1999, p. 20), "Geometry is considered a tool for understanding, for description and interrelation with the space in which we live" (Barbariz, 2017, p. 172, authors' translation)

I understand that the speech, the dialogues, the communications made via messages are ways through which these subjects *are with one another*, *being with the other and the media.* It is important to highlight that, although the people enrolled in the course do not know each other in person, that is, they have never been together in the physicality of the world, the dialogue has taken place. This was evidenced by their being intentionally linked. The presence of these people is strongly felt, even in the silence of those who do not speak. They are there, because they sought the course, enrolling, showing their willingness to the opening created by those activities. They are present in the concern of the teacher–researcher in carrying out the course *project* and trigger its actualizing movement.

The constitution and production of mathematical knowledge made possible by focusing on what Geometry is about has taken place in different ways, highlighting nuances of the movement performed by the co-subject individually and all together, in the harmonic totality of the course.

In line with this statement, I highlight that the participants of the course, when posting their ideas regarding the provocations of the teacher or colleagues, referred to each other, therefore establishing an intentional dialogue, which lays in the field of possibilities that can be actualized; they stand together, exposing themselves and seeking what the other, with whom they are, thinks, without blocking the movement triggered by the eagerness to know about Geometry, beyond its know-how. In the ways they behave there is evidence of solicitude, shown in the attempt to help one another, by indicating readings, with examples. The constant resumption by the professor-researcher of what has been addressed in the course, seeking to keep everyone's reasoning alive is shown as a driving motto.

> Forum 1.2.4—The teacher comments: Student 18, The purpose of the Forum is to relate the text of this class to the comment made by the author in the previous class. What do you think Freudenthal meant by "more solid foundations"? Is "living space" a foundation? What did you understand about space being "suspect to a genuine mathematician" if taken as the object of physical research? Comment! (Barbariz, 2017, p. 112, authors' translation).

The teacher–researcher saw herself understanding and creating ways for the possible students of the course to constitute knowledge of Geometry, as they gained understanding of the proposed activities. As reported by Barbariz (2017), during the conduction of the movement that was established, the experiences she had lived came to her mind when she sought to learn and teach Geometry, gearing her thoughts about what she had already experienced and leading her to move toward the opening provided by the texts by Freudenthal (1986). These can be expressed by some excerpts from her statements:

> US 007—The breadth of what is described in the texts opened possibilities for activities with students (Barbariz, 2017, p. 102, authors' translation).

> US 012—Seeing the material that I had been preparing for all these months taking shape is undoubtedly emotional (Barbariz, 2017, p. 102, authors' translation).

Thus, the flow of memory interweaves experiences which are enlaced when thinking about the motion, intentionally directed to the knowledge of Geometry.

In cyberspace there is also the constitution of intersubjectivity, which calls for acts of intropathy[20] and language. In the case of the course "Geometry: what is it

[20] Intropathy is knowledge of the other that occurs directly in the experiences in which the other is given (brought, exposed) to the self in its corporeality. It is a constituent perception of intersubjec-

about?" the teacher and students experienced the condition of "being together" among themselves, with Geometry and the informational screen, thereby constituting and producing knowledge. In the description of the activities performed and the recollections of the moment of their accomplishment, Barbariz (2017) explains that when she experienced the course update and got involved with the activities of the forums, she, by assuming the position of a teacher, returned to the course content, as she had understood Freudenthal's point. She reported that, concurrently, she engaged with students' postings by reflecting on what she herself had studied and what students showed they understood. Realizing her action, she reports that she became alert to provoke students' thoughts and her own, as she was committed that everyone could revisit what they understood and advance their discussions by exposing examples and counterexamples. She reported that this action led her to more insights into the subject that was addressed. She stated "US 008—Many new ideas for the initial approach to this field of Mathematics emerge while in contact with the way the author addresses it" (Barbariz, 2017, p. 152, authors' translation).

In this movement of constitution of mathematical knowledge, the teacher and students were intentionally focusing on the objective of the course, seeking to perform the activities required, exposing their thoughts about what was requested, and the way through which their thoughts were articulated.

I understand that the "being together" experienced by the members who took the distance learning course, object of analysis, and reflection herein is a dynamic movement that takes place in a temporal flow in which everyone is subject of their own actions and co-subjects of each other's actions, as they are present to one another, and at the same time take notice of themselves. The questions presented are focused on everyone's thoughts, while articulating ideas and constituting knowledge.

In the cyberspace where the subjects and co-subjects are present, aiming to complete a course in Geometry, the constitution of knowledge becomes evident in different activities. They are interconnected by the intentionality that puts them in action to complete the course and meet its demands. Each living-body is set in motion in spatiality and temporality, where one discerns what needs to be done. In this case, it means to focus on Geometry, seeking deeper understanding. The thread that supports this interconnection is strengthened by the very intentionality of the teacher–researcher who is attentive throughout the action of updating the course "Geometry: what is it about?" She remains aware during all steps: carrying out the pedagogical project, selection of study topics, searching for texts from authors who meet the goals of her proposal, i.e., thinking about Geometry, the consonance of the activities and the tooling on the platform with which she was working, details of the students' institutional accreditation, the enrollment of students, their attendance during the course. Their co-subjects are also alert. They connect to the platform at the scheduled times, read the texts, express their understanding about the activities they are performing. This way, they remain together.

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tivity. It is not, therefore, a theoretical concept or a predicatively constructed statement.

According to Barbariz (2017), the act of being alert, formulating provocations relevant to what was requested and written by the students, in order to boost their comprehension, bringing examples, and counterexamples, led to memories of experiences in which she was, with other students and in other types of courses, while teaching Geometry. The interconnectedness of lived and moving experiences in the "now" came together with evidence, shedding light onto their understandings. She stated that her practice as a mathematics teacher was also modified by what she learned from the course content and the experiences reported by the students. Her initial approach was not to accept the texts of students who were not directly aligned with the question posed. Often, students would not focus on the text made available in the platform and would visit other sites, bringing in other authors who somehow addressed the theme. She states in Barbariz (2017) that at first, she wanted to reject what was brought by the co-subjects. But, in discussions about what they said in the underlying messages, she understood that this was a way for them to be present. She stated that she concluded with the thesis supervisor, that it was up to her, the teacher–researcher, to revisit what was posted and with her co-subjects' dialogue through questions, and provocations. The excerpt "Forum 1.2.4." on page 80 of this article exemplifies this statement.

4 Comprehensive Summary

Bringing a synthesis of understanding about the constitution of knowledge while being in cyberspace-with-media, I realize that Barbariz and I have clarified many aspects.

The intentionality of the teacher–researcher and the researcher-supervisor of the research kept the question already posed in the title of the course "Geometry: what is it about?" alive. The intentionality was filled by the desire to carry out the research and the will to look at Geometry from perspectives that are different from those that point to merely grasping some topics of this field in order to be able to teach its concepts, ways of demonstrating and solve problems, and apply it pragmatically to everyday requests. We wanted to work with fundamental ideas underlying Geometry. This desire led us to seek a significant and recognized author in this area who pursued these essential ideas. We found Freudenthal. Studying his work took a lot of effort. Starting with the translation from English to Portuguese of the text to be read in the course, we faced issues pertinent to science itself and the way of addressing them. At the same time, it was necessary to obtain informational tools support which would aid in the materialization of the course and that had institutional endorsement. In finding them, we need to devote ourselves to the study of their functions.

The experiences of this temporality were seen and reported as of a moment when the project was being designed, bringing into the present some past experiences; teaching Geometry through distance learning courses, and the future, foreseeing the students of the course in action. Even though the works of the author chosen as reference for the activities opened horizons and brought insights and evidence about the field, the steps proved to be important because they filled what was being studied with meaning, and advanced the movement of knowledge onto new paths. In terms of feelings, we noticed joy and satisfaction, while always attentive to "there is more." From the standpoint of *what is coming*, the students were already present there, as we placed ourselves in a position of teaching provocatively, although we did not know who the students or their expectations would be. The feelings aroused were anxiety and, sometimes, anguish due to the emptiness of what is “not yet materialized.” But we became aware of the knowledge that was already being constituted in us. The presence of the other is remarkable in this movement. Whether in the search for the author with whom to dialogue, or the anticipation of the participation of potential students, the presence of students taking the course proved to be one of the invariants of the phenomenon studied. The dialogues, through speech, when performing synchronous activities, or written, when posting activities, evidenced the presence of all, intentionally focused on the topics that were studied. These speeches linked answers from the teacher–researcher, leading her to find alternatives to challenge the students, as well as bringing more understanding about aspects of Geometry and its teaching.

In this movement of constitution of knowledge of Geometry, while in cyberspace and being-with-media, I highlight the intentionality that interconnects all actions; the experiences of people who participated in the course, who, even at a distance, were present and expressed that they were aware of what they experienced; self-awareness in action; the search for understanding; the presence of the other, stirring up uncertainties, challenges, evidence; the recollection of past living experiences brought to the present by virtue of the willingness to know the focus, in this case, essential ideas of Geometry, charging the action with meaning; the invisibility of the informational screen, without which people could not be together knowing each other, plotting their statements, that is, expressing themselves. This screen supports the reality of cyberspace. But this would not be a reality without the actions of people intentionally driven to do something. It is a way of being with computational tools, because they capture our attentive eye, require reasoning and ways of expressing it within their logic, but which open themselves to possibilities through existing interfaces between logic and people’s actions; they involve our willingness to do, our emotions, that is, we have experiences by being with it.

By specifying what I understand about the constitution of knowledge in Geometry, I point out that in the research carried out, this understanding refers to the way through which the examination of essential ideas of Geometry took place in face of the texts of Freudenthal available to students on the informational screen. These thoughts were triggered by the questions and the activities proposed by the teacher–researcher. Such reasoning did not seem to be progressing in a straight line of chronologically marked sequences. However, it was evidenced many times by the act of the co-subject wandering away from the Geometry text and related activities, seeking to *go around* it, bringing other geometry texts. I understand that this *going around* encompasses recollections of teaching and learning situations that had been experienced by the co-subjects in which activities including aspects of Geometry were worked on, i.e. attempts to solve the questions visualized with the mathematical tools that they had mastered, the contributions of colleagues regarding what was being discussed, the researcher's remarks, resulting from her way of being intentionally alert to what was said by a fellow co-subject to others.

The production of knowledge was also revealed, as what was experienced perceived, analyzed, and reflected upon was demonstrated through language and materialized by the available means in Moodle, as we were writing texts and presenting them to the community of mathematical educators, thus making ourselves available for dialogue. The knowledge produced brings our understanding of what was investigated, that is given to others, remaining in temporality and materiality through the form and material available.

What I see in regards to new studies; the ideas brought up in this text open pedagogical philosophical horizons for the investigation of the way the living-body feels with cyberspace.

References

Ales Bello, A. (2015). Pessoa e comunidade. In *Comentário: Psicologia e ciência do espírito de Edith Stein*. Belo Horizonte: Ed. Artesã.

Barbariz, T. A. M. (2017). *A constituição do conhecimento matemático em um curso de matemática à distância*. Doutorado, Instituto de Geociências e Ciências Exatas - Universidade Estadual Paulista "Júlio de Mesquita Filho", Rio Claro.

Baumann, A. P. P. (2013). *A atualização do projeto pedagógico nos cursos de formação de professores de matemática dos anos iniciais do ensino fundamental: Licenciatura em Pedagogia e licenciatura em Matemática*. Doutorado, Instituto de Geociências e Ciências Exatas - Universidade Estadual Paulista "Júlio de Mesquita Filho", Rio Claro.

Bicudo, M. A. V. (Ed.). (2010). *Filosofia da educação matemática: Fenomenologia, concepções, possibilidades didático-pedagógicas*. São Paulo: Editora UNESP.

Bicudo, M. A. V. (Ed.). (2011). *Pesquisa qualitativa segundo a visão fenomenológica*. São Paulo: Cortez.

Bicudo, M. A. V. (Ed.). (2014). *Ciberespaço: Possibilidades que abre ao mundo da educação*. São Paulo: Livraria da Física.

Bicudo, M. A. V., & Rosa, M. (2010). *Realidade e Cibermundo: Horizontes filosóficos e educacionais antevistos*. Canoas: Ed. ULBRA.

Bicudo, M. A. V., & Silva, A. A. (2018). Análise de descrições de vivências em situações de constituição de conhecimento. In C. Brandão et al. (Eds.), *A prática na investigação qualitativa: exemplos de estudos* (pp. 153–178). Aveiro: Ludomedia.

Borba, M. C., & Villarreal, M. E. (2005). *Humans-with-media and the reorganization of mathematical thinking: information and communication technologies, modeling, experimentation and visualization*. New York: Springer.

Fainguelernt, E.K. (1999). Educação Matemática: representação e construção em geometria. Porto Alegre: Artes Médicas Sul.

Freudenthal, H. (1986). *Didactical phenomenology of mathematical structures*. Boston: Mathematics Education Library.

Husserl, E. (1970). *The crisis of European sciences and transcendental phenomenology*. Evanston: Northwesternon University Press.

Husserl, E. (1977). *Cartesian meditations: An introduction to phenomenology*. The Hague: Martinus Nijhoff.

Husserl, E. (2002). *Idee per una Fenomenologia pura e per uma filosofia fenomenológica. Tradução de Enrico Filippini* (Vol. II). Torino: Einaudi.
Husserl, E. (2013). *Meditações cartesianas e conferências de Paris*. Rio de Janeiro: Forense Universitária.
Husserl, E. (n.d.). *Lições para uma fenomenologia da consciência interna do tempo*. Lisboa: Imprensa Nacional-Casa da Moeda.
Heidegger, M. (1962). *Being and time*. New York/Evanston: Harper/Row.
Lévy, P. (1999). *Cibercultura. Tradução de Carlos Irineu da Costa* (2nd ed.p. 34). Editora: São Paulo.
Merleau-Ponty, M. (1994). *Fenomenologia da Percepção. Tradução de Carlos Alberto Ribeiro de Moura*. São Paulo: Martins Fontes.
Wiener, N. (n.d.). *Norbert Wiener*. Retreived Setembro 13, 2014, from https://pt.wikipedia.org/w/index.php?title=Norbert_Wiener&oldid=43379334

Phenomenological Study of a Digital Simulator of the *Sine Function* Contextualized in a Teaching and Learning Activity

Verilda Speridião Kluth

1 Introduction

The research group, *Fenomenologia em Educação Matemática*,[1] has systematically pursued, through several research projects, the answer to the question: How does mathematical thinking occur in the presence of educational software? In Kluth and Moura (2014), in the article *Preâmbulos investigativos sobre o uso de softwares como ato propulsor ao pensar matemático* (Investigative preambles on the use of software as a driving force for mathematical thinking), we describe research which investigated the theme with children interacting with a software called *Supermercado Virtual* (Virtual Supermarket), which, while proposing purchasing activities leads to mathematical calculations. In this research, we take as reference the *tacit cogito* described by Merleau-Ponty (1994), as that which refers to the act of perception, whose living-experience was not yet expressed in words, but in which underlies all explicit thought. "From this research it is understood that the mathematical thinking that takes place as a result of the encounter between student and software is thinking whose soil is constituted by the intrinsic rules of the chosen educational software" (Kluth & Moura, 2014, p. 245, author's translation).

Thus, when one is in cyberspace, a ground of mathematical thinking is inaugurated that is constituted by the possibilities arising from the way the educational software makes the mathematical object present.

The research presented herein is no exception to this rule, it also investigates how mathematical thinking occurs in the presence of educational software. Now, however, from another perspective, as we focus on the fact that the mathematical content

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required by the education system, at its different levels, always proposes previously known mathematical objects, which encompass a long historical trajectory and construction, embedded in concepts, definitions, methods, and demonstrations, whose underlying ideas are not always apparent to teachers and students.

At the beginning of higher education, it is expected that the framework of mathematical knowledge will be worked on so that mathematical ideas and tools for mathematical work are exposed, understood, and can support the advancement of the understanding and possible applications of mathematical knowledge by students and teachers.

Thus, educational software, which are made available on platforms, make their way into the educational scenario as a possibility, not only to work with teaching activities, but also as a reference to tackle possible difficulties during students' autonomous study.

We believe that the teaching and learning possibilities of such resources need to be known, which include the mathematical thinking they foster. Among the questions that arise in our dialogue with ourselves, a broader interrogation arises: Would the products generated by the objects of cyberspace have the potential to "present" the mathematical object in various forms and understandings that underlie its historicity as object as a human construction?

This question leads us on a path that we believe sets us on the road to investigate it, without getting lost in a wide universe of investigative possibilities. Since trigonometry is a very complex subject for students in high school[2] and higher education, we set out to study the *sine* as mathematical object. Such complexity is enhanced by addressing the *sine function* from the perspective of differential calculus. For the realization of our research objective, we found a digital learning object,[3] more precisely a simulator of the *sine function*, to be worked on by high school teachers, which included aspects of the mathematical historicity of the *sine*. Thus, the guiding interrogation of the research presented in this chapter is: How does the simulator, in activities performed with students, makes the *sine* present in its different possible forms of technical work and understandings of its historicity?

This guiding interrogation calls for a study; it places us in search of a theoretical perspective that underpins, and can give subsidies to encompass it as a cohesive and coherent whole, which does not favor dichotomies or carries the intention of relating conceptual perspectives between man, world, knowledge without a unifying view of these themes. The Husserlian phenomenological work brings ideas in this regard that make sense to us and that are in line with our way of looking at the interrogation.

Thus, the objective of the present article is describing the research resulting from the guiding interrogation: How does the simulator, in activities performed with students, makes the *sine* present in its different forms of possible technical work and

[2] High school in the Brazilian educational system comprises basic education with a duration of 9 years. In most cases it caters to students aged 15–18.

[3] http://curriculomais.educacao.sp.gov.br/.

understandings of its historicity? The research is based on studies on how phenomenology understands cyberspace in its technological and philosophical aspects, and how the constitution of mathematical thought takes place under a Husserlian perspective, which inspires the articulations included in the structure of the research being conducted.

2 Explanation of the Guiding Interrogation

We will now explain why the guiding interrogation makes sense, when we examine it under a phenomenological light. We sought to understand cyberspace with a focus on its nature, following the question: what is it, virtual reality?

Bicudo and Rosa (2010) examined the work of several authors and philosophers pointing to the characteristics of what is virtual, from the perspective of what is real and not-real, as well as the current and non-current perspective. Such authors concluded as follows:

> However, opposing real to virtual is an oversimplification of complex concepts, and assuming this aspect of being opposites with respect to a physically material and palpable reality, according to conceptions of modern physics, masks important meanings regarding conceptions of reality (Bicudo & Rosa, 2010, p. 25, author's translation).

The authors approach Granger's (1995) ideas about the actual and non-actual, where they find arguments about the "virtual and actual" relationship through the framework described by Bicudo and Rosa (2010) "as the scientific and technological apparatus itself which supports cyberspace actualization". This understanding takes them to state that "the reality of cyberspace is virtual because it already has its basis in science, notably mathematics" (p. 28, author's translation). However, the modes of cyberspace actualization are quite unique and not determinable, since there are innumerable possibilities offered by its logic and informational screen, actualized by analysts, operators, and users of such systems.

Following this approximation, Bicudo and Rosa (2010) visited the work of many other authors who discuss the reality of cyberspace and articulated the phenomenological thoughts of Merleau-Ponty and Husserl as an anchor for assertions that show cyberspace blossoming into cyberworld, where reality is understood in the indissolubility of time/space in which the action of the subject, enhanced by the computer, creates spatialities and temporalities. It is therefore a modality of that which is real.

> Although considered by authors specialized in this subject as differentiated from the world lived in the dimension of mundane reality, the virtual world, characteristic of cyberspace, is one of the dimensions of the life-world, as much as science. Its complexity, however, is ingrained in its own constitution - as it is grounded in science and technology - and on the multiplicity of ways it is shown (Bicudo & Rosa, 2010, p. 74, author's translation).

Therefore, the virtual world, originated from cyberspace, is only (another) facet of mundane reality. So, it is a driver of knowledge of this reality resulting from possible articulations empowered by technologies, which are formed around a basic element of the whole computational system—the code. The code is rooted in the presence or absence of an electric current transmission, which later translates into electrical circuits, producing affirmative or negative responses which, when associated with number *one*, when there is current, and *zero*, when there is no current, make powerful instruments available, understood as interfaces. They are, in this case, generators of numeric codes that translate a fact dealt with by physics, whose existence the average does not even need to know. de Figueiredo (2014, p. 139, author's translation) states: "But this is the core principle of interfaces: they hide the code. This might seem like a paradox, on one hand the computer is strongly based on code, but on the other, it avoids revealing such codes as long as possible. This makes computation a hidden reality."

The hidden feature of the interface makes it a machine governed by a set of operational rules whose reasons for existence are submerged in its processes.

> Interfaces are automatons with a myriad of arms, joints, levers and movement. In this variety, the sense of machine is often lost. The program window I move on the screen, with its shadows, transparencies, and movement, so true to the movement I make with the mouse, is an unquestionable reality for me (Figueiredo, 2014, p. 145, author's translation)

The importance of being attentive to the experiences promoted by cyberspace interfaces and the unveiling of the unknown underlying educational software or computational teaching objects available on platforms emerges from the fact that cyberspace interfaces may be considered indisputable by those who use them. Such aspects enable us to show their potential for education in general and, in the case of the research presented herein, the mathematical thinking constituted while being with the simulator concerning sine function and its historicity (Fig. 1).

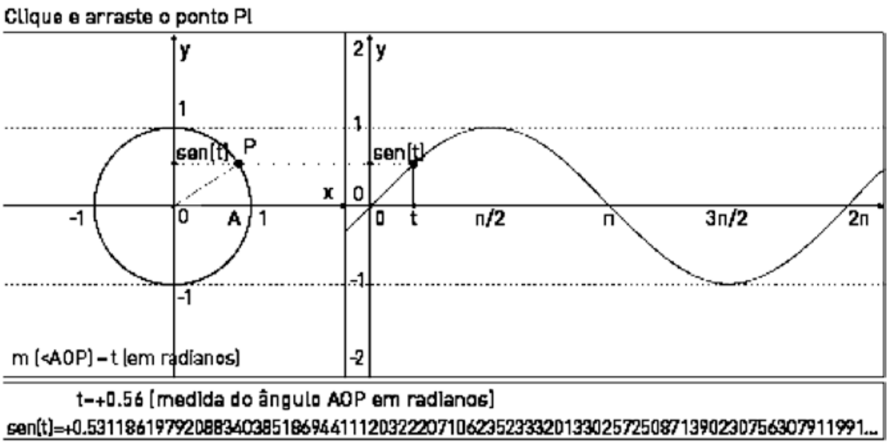

Fig. 1 Simulator screen for the sine function graph using the radian as a measure of angles (Figure authored by Cardeira and the first author of this chapter; also displayed in Cardeira and Kluth (2019))

Geogebra[4] was the simulator chosen for the research in point, as shown below. It synchronizes the movement of a point P in the trigonometric circle, aligned with the sine value of angle AOP; with the movement of another point of the *sine function* graph represented in the Cartesian system. Thus, it presents two interconnected graphs.

By relating the *sine* in the trigonometric circle to the sine function via the Cartesian system graph, the simulator of the *sine function* becomes an example of the mathematical maturation that occurs around the mathematical object: *sine*. In itself thought of as a *formal ontology*.[5] Thus, the simulator concentrates its historicity—according to Bicudo (2013) who states that: "computing is a fitting juxtaposition of formalisms" (p. 7, author's translation) and with formal mathematical procedures, in the sense that they are formally organized (procedures involving inductive thinking and deductive axiomatic mathematical thinking) and expressed in a specific language, whether through geometry or analytical geometry.

For the purpose of this research, the historicity of the object *sine* presents as its end, its way of being in the relation between the set of all *sine* values in real numbers and their respective angles measured in radians, which culminates in a function, which coincides with the goal of the simulator we investigated. Opportunely, in this article, we will describe the construction of the *sine function* immersed in Western culture intertwined with its constitution in a phenomenological perspective.

In order to elucidate the interrogation, we must clarify some points regarding the meaning of the word *living-experience* which, according to Gadamer (1997), Dartigues (1992), Merleau-Ponty (1994), Bicudo (1999), and Husserl (2001), encompasses all acts of consciousness, whose essential structure is intentionality as the conducting thread of the accomplishment of the *ego cogito*, and which shows the movement of mathematical thinking in phenomenological reduction.[6]

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[4] https://www.geogebra.org/m/KGWhcAqc.

[5] *Formal ontology* is a term used by Husserl, in his doctoral thesis *Philosophie der Arithmetik*. Then, the term designated natural numbers, in their symbolic characteristics, as drivers of the numerical system, which comprises integers, rational, irrational, negative, and complex numbers, the latter composed of elements later named by Husserl *imaginary entities*. Later, the concept of formal ontology was expanded and gained the status of being: "the study of a system and its internal structure in terms of derivable relations, in domains where there is a language in which the noema is presented as an assertion system" (Kluth, 2005, p. 126, author's translation) Noema is here is defined as the intended mathematical object.

[6] Phenomenological reduction is a mode of reflection; an attitude that aims to neutralize any and all ingenuity. Moura (1989) points to the naivety of the natural attitude by stating: "If the natural attitude is that through which the investigation is directed towards the objects, the phenomenological attitude is distinguished from it first of all by the "anti-natural habitus of reflection" (LUI)./111). The phenomenological attitude is determined only by the conversion of the acts themselves into objects, acts that remain anonymous in the "natural attitude." (p. 116, author's translation).

3 Mathematical Thought Under the Husserlian Phenomenological Perspective of *ego-cogito-cogitatum*

Pugliesi (2001) states that Husserl, as well as Descartes, seeks a unified basis for knowledge. As stated by Cantista (1996), Descartes was particularly interested on what refers to the cogito; Husserl was interested on what it encompasses. Corroborating the above-mentioned scholars, Husserl (2001) while intertwining his thoughts to those of Descartes, categorically stated:

> Nowadays, the nostalgia of a living philosophy has led to many rebirths. We ask: the only truly fruitful rebirth would not consist in resurrecting Cartesian meditations, clearly not to embrace them in their entirety, but to unveil at once the deep meaning of a radical return to the pure *ego cogito,* and then to revive eternal values that stem from it? At least this is the path that led to transcendental phenomenology (Husserl, 2001, pp. 23–24, author's translation).

This chapter is not intended to sing praise to Husserlian transcendental phenomenology and its developments, but we understand that the above-mentioned excerpts give an idea of the cultural and philosophical ground that inspired it. Thus, in transcendental phenomenology, we seek support to understand the educational object of cyberspace in its potentialities for presenting mathematical content to humans being with it, as it takes the *ego cogito* as the ultimate foundation of knowledge and proposes a series of investigations into the nature of knowledge and the cognoscenti subject. The investigations take place in the sense that the one who realizes the first experience of the object and the transcendental experience of the self, that regarding to the experience of others, is the subject themselves who, together with other cognoscenti subjects, not necessarily contemporary, builds the apodictic of the object.

Phenomenological reduction of the *cogito*, described as

> Everything that is the "world", every spatial and temporal being exists to me, that is, it is true for me; the very fact that I have experience of it; perceive it, remember it, think of it in any way, make judgments related to its existence or value, desire it, and so on. All this, as it is known, Descartes designated with the word *cogito* (Husserl, 2001, p. 38, author's translation).

led Husserl to declare, in the wake of such thoughts, that the world is what exists for me, and that my conscience has value in a similar *cogito*. Thus, all existential validity of the world comes from *cogitationes*. Consequently, we cannot live, experiment, think, build a scientific life if not founded in the world that is "in me and takes from me its meaning and its validity" (Husserl, 2001, p. 38, author's translation). While making considerations about the thoughts of Descartes, the philosopher also concludes, that: "The domain of natural existence, therefore, has only a second-level authority and always presupposes a transcendent domain." (Husserl, 2001, p. 39, author's translation). Which means that natural existence is defined by *cogitationes*, that is, by what transcends it. Hence the expressions I think, therefore I exist, or I think, so I am. This transcendent domain of the "I am," pointed out by Descartes, becomes the focus of Husserlian investigation based on the idea of apodictic evidence of the "I am"—*ego cogito*—as the reduction of my natural self and my psychical life to my transcendental self.

Transcendence as referred to herein

> […] is inherent, in the specific sense of everything that is part of the world, even though we cannot give this "world" and its determinations any meaning other than that extracted from our experiences, representations, thoughts, value judgments, and actions. This way we cannot justify the attribution to this world of overt existence, except from our own evidence and deeds. (Husserl, 2001, p. 44 author's translation).

That being said, Husserl ventures into the exploration of the endless field of transcendent experience, considering that the *ego* can systematically explain itself through that. His analysis is initially focused, no longer on the *ego cogito*, but on the multiple *cogitationes,* understood as the object of perception, recollection, judgment, etc. That is, he turns his gaze to the stream of consciousness that forms the life of that self (myself, the self of the subject who reflects). "At any moment, the identical self may direct its reflective gaze upon this life, whether it be perception or representation, judgment of existence, value or volition. The self can observe it at any moment, explain it and define its contents" (Husserl, 2001, p. 50, author's translation).

Then, Husserl explains that in transcendental reduction (*epoché)* it is necessary to broaden the contents of the *ego cogito*, as all *cogito*, that is, all states of consciousness, retains something that it already carries in itself, which is the respective *cogitatum*. Each *cogito* broads its contents in its own way. For instance: the perception of the triangle assumes it in a perceptual way; the memory of the triangle takes it as a memory; the predicative judgment of the triangle assumes it in the particular way of the predicative. And so on. These states of consciousness are intentional states.

The word intentionality means this fundamental and general particularity of consciousness to be the consciousness of something, in its capacity as a *cogito* containing *its cogitatum*.

Husserl still provides a very important analysis in distinguishing spontaneous acts of consciousness from reflexive acts, although reflexive acts are perceptive acts of a new category. In spontaneous perception, we apprehend the house, not its perception. In transcendental reflection, we only "turn towards" this act itself and the perceptive orientation "about" the house. This way of reflecting upon the world or the things of the world presents them not as existence but as phenomena to be understood.

This translates into a modified experience, the transcendental experience proposed by phenomenology, which transcends the natural reflection that places us on the realm of the world, of the world presented as existing. It can, therefore, be asserted that transcendental reflection alters the primitive state, that of spontaneity. For Husserl (2001):

> The task of reflection is not to reproduce the primitive state a second time, but to observe it and make its content explicit. The passage to this reflective attitude naturally gives rise to a new intentional state which, in its uniqueness of "relating to the previous state", makes conscious, even evident, this very state, not any other state. (Husserl, 2001, p. 52, author's translation)

On the other hand, refection, through this new attitude, leads to the understanding that the world, as well as everything that exists, is nothing more than something “of value,” that is, which only exists for me as *cogitatum* of my *cogitationes*, as objects of my universal descriptive observations.

Husserl (2001) states that, it is “through the real perception process that the most precise determination takes place, confirming or invalidating the anticipations, but always indicating new horizons and opening new perspectives” (p. 63, author’s translation). The awareness of something can transform into constantly renewed modes of consciousness, while remaining as consciousness of an identical object, intentionally inherent, with an identical objective sense.

> The object can be said to be an identity pole, always presented with a “pre-conceived” “sense” “to be” realized. It is, in every moment of consciousness, the indicator of a noetic[7] intentionality that belongs to it through its meaning. Such intentionality can be researched and made explicit. (p. 63, author’s translation).

In transcendental reduction, the mathematical object is a pole of identity from which stems “preconceived meaning” to be actualized. Thus, clarifying the mathematical thought that actualized and actualizes it consists in a phenomenological investigation that pursues every moment of consciousness, which indicates the noetic intentionality that belongs to such mathematical thought through its meaning. From this understanding we can say that mathematical thinking is a very particular pairing of *cogito-cogitatum*—as an act of consciousness of something is one or more appropriation of the world in the form of *cogitationes*, as forms of consciousness. These forms carry and surpass the current meaning of the mathematical object by being able to: specify the intention understood as the guiding thread of the constitution of the mathematical object; and to fulfill it intuitively through later perceptions or recollections that “the self” can carry out.

Under this perspective, when we ask about the possibility of a cyberspace object “to present” the mathematical object *sine* in various forms and understandings that underlie the historicity of the mathematical object, we are asking about the presentation of the multiplicity of aspects of the corresponding modes of presentation, which foster the constitution of the synthetic unity as an objective unity, which must be perceived, remembered, and reflected upon.

Each aspect considered presents itself as a “unit” of “multiplicities” that succeed each other. An example is the sine as a ratio between sides of the right triangle. The very fact that it is the defining angle of an internal geometric region of the triangle that preserves the equality of the ratios, regardless of the length of its sides, and whose horizon is the possibility of pairing all the results and synthesizing them in a hypotenuse triangle of unitary length; and finally, envision the sine of the right triangle, in the trigonometric circle, which culminates in its presentation as a function

[7] Noetic and noematic have meanings in phenomenology when seen in the relation of man to the world, in the noesis–noema relationship. This relation tells of the human supposition which is the intentional act of consciousness—concerning noesis and the object of supposition—the noema, which is a correlate object of the world, which is understood as the natural, constituted, and being constituted world.

in relation to all possible sines. Each revealing aspect of this historicity is also "representation" of a "pre-conceived" "sense" yet "to be" realized. Each revealing aspect of this historicity is also a "representation" of "preconceived meaning" "to be realized." "Thus, the *cogito* is aware of its *cogitatum* not in an undifferentiated act, but in a structure of multiplicities of well-defined noetic and noematic character, a structure which is essentially coordinated with the identity of that particular *cogitatum*" (Husserl, 2001, p. 58, author's translation)

We understand that to pursue the interrogation *how does the simulator, in activities performed with students, makes the sine present in its different possible forms of technical work and understandings of its* historicity? is to analyse the driving potential of the object—simulator—in providing the learner, in the presence of the object, with a broadened understanding of *sine* in the flow of intentional syntheses, possible to any consciousness, which creates unity and constitutes the unity of objective meaning. According to Husserl (2001), this is "the realm of essential structures, of an essential nature, capable of being encompassed by rigorous concepts" (p. 67, author's translation).

4 In Search of Comprehensions Regarding the Simulator

Under the guidance of the ethical and methodological principles of phenomenological qualitative research, an architecture for the collection of testimonies was built, which was implemented in the computer lab of a school in Diadema, a city in the metropolitan area of São Paulo, Brazil, after school hours. Nine third-year high school-students participated in the activities. Assuming a phenomenological posture of understanding the constitution of mathematical objects, as explained before in this article, we developed an activity in the form of questions and writing orientations organized in four steps, to be performed by the students participating in the research while they were in the presence of the simulator.

The first step, with spontaneous exploration characteristics, consisted of the following tasks: (1) Describe what you see on the computer screen; (2) What do you understand from what you see? (has to do with math?); (3) What high school grade are you in? (4) Have you ever studied the *sine* of an angle? (5) Do you know the right triangle? Write about it. (6) Do you know what the hypotenuse and opposite cathetus are? What are they?

The second step, is called "Let us now look at the circle," with the purpose of directing the gaze to the circle. The following situations were suggested: (1) Knowing that the *sine* is equal to the ratio of the opposite cathetus and the hypotenuse, find the *sine* value in the circle and describe where it is represented in the circle. (2) Does the *sine* value of an angle depend on the length values of the sides of the right triangle or does it always have the same value? (3) Now, look at the radius of the circle. It is written on the screen that it is 1. What does this 1 mean to you: (a) That the radius is equal to 1? or (b) that the value of the radius length is being taken as a unit? Can you see a reason for the radius value to be one?

The third step of the activity is *Let us now look to the other side of the drawing (of the curve)*, whose purpose is to steer the students' attention towards the function graph. Then, the following tasks are proposed (1) is the circle in the screen related to the other side of the drawing? (the curve). How do they relate? (2) Have you studied the *sine function*? (3) What is being represented on the *y* axis that refers to the *sine* of an angle? (4) What do you think the curve represents?

In the fourth step, the student is invited to: *Go back to the screen and make comments about what you see.* This is a very open moment, whose aim is to search for possible articulations seen, which were not asked about or focused on.

Part one of steps one and two is intended to examine the mathematical syntheses already elaborated by the declarants, which underlie their descriptions of what they see on the screen. The purpose of this procedure is to place in *epoché*[8] the spaces of mathematical comprehension that supported their statements or statements that may be filled by the intrinsic *cogitationes* to the construction of the body of mathematical knowledge.

The aim of steps three and four is to understand the possible articulations that are constructed between what is mathematically placed on the screen, considering the screen as an interface that interprets the relations via movement of graphical elements. But they also seek the underlying mathematical syntheses that project through the articulations observed.

The students worked without the interference of the researcher conducting the activity at all times, except for the initial explanations required for accessing the platform and indicating that they should explore the simulator for 15 min before doing the activity. The students worked in pairs, except for one, and responded individually, in writing, to the tasks proposed in the activity. Completion of the proposed activity took 60 min. Shortly after, with the accounts in hand, a lecture was given seeking to explain some of the improprieties exposed about the theme by the students.

Although aware that there was some guidance in the procedure for the activity which predefines it, and that, in a certain way, the software is not speaking for itself, we believe that this predefinition is necessary, as what we are looking for regards mathematics. We understand that the orientation is mathematically coherent with the proposal of the learning object. Moreover, this previous direction is aligned with the direction indicated by the guiding interrogation of the research, regarding the potential of presenting the built-in historicity of the *sine* within the body of mathematical knowledge, which heralds an intrinsic objective convergence between the simulator and math. Added to this is the fact that the phenomenological analysis of the statements looks for the meaning of the *sine* and its function as revealed by the simulator, that is, the essentiality perceived by a thinking subject.

[8] The word *epoché* means highlighting something we want to know. This evidence builds when we initially disconnect from what is already known about it. It can be said that to make epochpe is to make this something an intentional object of reflection.

5 Analytical Procedures and Results

The analysis performed in the research presented herein was conducted assuming a phenomenological posture and respective world view and knowledge. In the wake of this understanding, we heard the accounts of research subjects, who brought up their experiences when performing the activity. This analysis was also inspired by the phenomenological hermeneutics of Gadamer (1997),[9] who investigates the structure of the question and answer as posed in human deeds.

Thus, the analysis of the accounts is reflexive, when they are read and questioned, about the answers given to the guiding interrogation: how does the simulator, in activities performed with students, makes the *sine* present in their possible different forms of technical work and understandings of its historicity?

Hence, the analysis takes place in perspectives that intertwine, in the proposed research activity; the simulator and its objective and the subject who experiences the activity in the dimension of the *ego-cogito-cogitatum,* a procedure that seeks answers to the guiding interrogation in the respondents' statements, organizing them according to the questions they answer. This way, nuclei of meaning of the answers to the guiding interrogation are articulated.

In the analysis, the answers from the participants' statements tackle two questions: What is the simulator's potential to *makes the sine present*? What mathematical or non-mathematical understandings underlie the description given of what respondents see on the computer screen showing the simulation? The information expressed in the answers was articulated in five meaning cores that reveal important aspects regarding the investigated theme and the subjects' living experiences.[10] We will now briefly present one of the five cores.

(1) *Casting a spontaneous look onto the simulator as a whole—The potentiality of the simulator to present the sine*

The answers considered in the analysis are related to the following task:
(1) Describe what you see on the computer screen

The simulator shows the *sine function* graphically and offers explanations about *what the sine is* through examples of *sine*, represented in graphs. It shows the *sine* in two Cartesian planes, in the trigonometric circle as corresponding to the central angle AÔP and, as the *sine* increases or decreases, the second plane represents the

[9] Reference on this matter: Kluth (2005, 2007), Kluth and Moura (2014), Kluth (2017).

[10] For Gadamer (1997) there are only experiences to the extent that something is experienced or intended in them. This intentionality, because it is operative, gives the experience a lasting character. Thus, it is not restricted to the immediacy of the fact, but reveals itself as a phenomenon when learned in corporeality. It is the concrete element upon which the meaning of the world is constituted for us.

angle in the graph. An inversion can be observed here: the change in the *sine* value determines a change in the angle.

(2) *Casting a spontaneous look onto the simulator as a whole—Mathematical comprehensions underlying spontaneous exploration*

The answers considered in the analysis are related to the following task:
(1) What do you understand from what you see? (regarding mathematics);
(2) Do you know the right triangle? write about it.
(3) Do you know the hypotenuse and opposite cathetus? What are they?
All respondents had already studied the *sine* of angles.

There are accounts that, when referring to the Cartesian plane that represents the *sine function*, made use of the terms "*sine function*" and "*sine function* graph," which, in the students' view, can express the same thing, although mathematically those are distinct. There was one respondent that stated that the *sine* is a wave, which we believe to refer to a function graph.

Even though some of the students claimed to know the right triangle, five of the respondents did not actually spell it out, nor did they clearly describe the hypotenuse and the opposite cathetus. Interestingly enough, one of them, when asked how they understood what they saw, resorted to the memory of the definition of *sine* as the ratio of opposite cathetus and the hypotenuse.

Two of the respondents knew what a triangle and hypotenuse were, however, they fumbled as they tried to explain their understanding of the opposite cathetus. Only two of the respondents showed mathematical understanding of what the right triangle, hypotenuse, and opposite cathetus were. They realized the relationship between the angle amplitude and *sine* value through the movement enabled by the simulator.

(3) *Casting a look onto the trigonometric circle—the potentiality of the simulator to present the sine*

The answers considered in the analysis are related to the following task:
(1) Knowing that the *sine* is equal to the ratio between the opposite cathetus and the hypotenuse, locate the *sine* value in the circle and describe where the circle is represented;
(2) What is the relationship that determines the sine of an angle in the circle?

Among nine respondents, five did not answer the question. For deponent 7, the simulator represents the trigonometric circle in the Cartesian plane and brings the possibility of *sine* interpretation in this medium. They saw the interpretation of *sine* in the right triangle within the circle. Interestingly, this deponent was unable to speak precisely and clearly about what they knew about the right triangle, hypotenuse, and opposite cathetus.

For respondent 1, the simulator brings an interpretation of *sine* in the Cartesian plane using the trigonometric circle. When asked about the transition from the flat geometric interpretation to that brought by the simulator, the student focused on the geometric interpretation of the right triangle rooted in analytical geometry, and found no symbolic elements on the screen to support this transition.

However, for two of the respondents, the simulator represents the trigonometric circle in a Cartesian plane and introduces de possibility of interpretation of the *sine* through this medium. They saw the *sine* on the *y* axis and determined the relationship of dependency between the values of the *sine* and the angle.

(4) *Casting a look onto the trigonometric circle—mathematical understanding which underlies spontaneous exploration.*

The answers considered in the analysis are related to the following task:
(1) Does the value of the *sine* of an angle depend on the length of the sides of the right triangle or does it always have the same value?
(2) Now, notice the radius of the circle. On the screen, it is written that it is equal to 1;
(3) What does this 1 mean to you: (a) That the radius is equal to one? or (b) the radius length value is being taken as a unit?
(4) Do you see any reason for the radius to have a value of 1?

When asked about changing the *sine* value of an angle, if the length of the sides of the triangle defining it was increased or decreased, two respondents did not answer, two showed a lack of understanding of the ratio, and five respondents showed understanding of the ratio, answering that the *sine* value does not change.

Only two respondents demonstrated having understood that the radius is taken as a unit of measurement. None of the respondents knew the reason for taking the radius as a unit of measure.

(5) *Casting a look onto the sine function graph—the potentiality of the simulator to present the sine*

The answers considered in the analysis are related to the following task:
(1) Is there a relation between the circle on the screen and the other side of the drawing (the curve)?
(2) What is the relation?
(3) What is represented on the *x* axis regarding the *sine* of an angle?
(4) What is represented on the *y* axis regarding the *sine* of an angle?
(5) What do you think the curve represents?
Respondents 4 and 5 had never studied the *sine function*.

Of the nine respondents, only one did not answer "yes" to the question about the relationship between the trigonometric circle and the function graph. Only three students answered the question about the localization of *sine* values on the *y* axis correctly. We will highlight here three analyses resulting from the respondent's accounts which synthetize responses to the five questions asked.

For respondent number 1, the simulator presents the relationship between the graphs (of the circle and of the *sine function*) because they deal with the same elements, *sine*, angle because they are graphically represented, however, under different perspectives. Questions number 3 and 4 were not comprehensibly answered. The respondent got confused with the nomenclature used in the simulator, and did not understand that x refers to the angle corresponding to the *sine* and that x also designates the axis of the angles. Moreover, the simulator allows the respondent to invert the dependent variable—*y* and the independent variable—*x*, as if the angle value were given by the *sine* value, as the simulator moves the function graph, indicating the connection between the circle and function graphs, via *y* axis, which leads the student to invert the variables.

For respondent 9, the simulator has great potential to present the *sine* historicity; however, in this version it does not present the *sine function* to the students, showing its relationship with the trigonometric circle, since the simulator allows an inverted understanding of the variables, as it uses a straight line, as described by the respondent, that leaves a point in the circle, which determines the amplitude of the center angle in the circle and ends at a point on the *y* axis, which represents the value of the *sine*.

For respondent 2, the simulator presents the relationship between the circle and the function graph, because as point P moves in the circle, the degree of the angle changes. The respondent was confused about the nomenclature used, and did not understand that x refers to the angle corresponding to the *sine*, and that x designates the axis of the angles. The responded offered a response which would have been correct if they were analyzing the circle, which is quite understandable, since the simulator uses the same line to represent the *x* axis of both graphs. In the circle, the *x* axis represents the cosine value and, in the function graph, represents the angle value. Although the simulator has two origins, one for each graph, this went unnoticed by the respondent.

Only one respondent provided an answer to item IV during the activity. He declared that the simulator shows an understanding through graphs.

6 Reflection About the Simulator Researched

Let us return to the guiding interrogation: *How does the simulator, in activities carried out with students, makes the sine present in its different forms of possible technical work and understandings of its historicity*? against the background of the analysis of the accounts and the results.

Although the simulator conveys the *sine* through various modes of presentation, either by what is on the screen or by the students' recollection of their experiences with the *sine*, when articulating the transition of the *sine* from an angle in the trigonometric circle with the representation of the *sine function*, it opens up possibilities of interpretation which lead students away from the mathematical understanding of the *sine function*, by implying that this transition results from the value of the *sine,* rather than the value of the angle; an essentiality of *sine*, which is to be dependent on the value of the angle.

Using the same line to define the *x*-axes of the two systems, the one that presents the *sine* in the trigonometric circle, and the one that has the *sine function* is misleading, and can result in misinterpretation, as the systems involve different variables: cosine x and angle opening.

We believe these details to be of outmost importance, as they are questionable, from a mathematical standpoint, since they offer two origins for the same line, one for each Cartesian system, thus opening the possibility of misconceptions which are incompatible with mathematical assertions, though these could be easily corrected by the software used. A suggestion would be for the software to present two separate Cartesian systems, and present the interconnection of the graphs via independent variable.

All this, when observed in the field of computer programming, are details that can be technically solved. They do not minimize the potential of simulators for presenting formal objects, such as the *sine*, which may be seen as formal ontologies in the historicity of their constitutions. However, as we have seen, for this presentation to be appropriate to the constitution of the object, its development must take into account not only the intrinsic coherence of mathematics, but also how this digital object is experienced.

Even when taking the precautions described above, we must not forget that each student has their own understanding of what is being presented, which forms a unity of resemblance of consciousness, which determines a state of consciousness in relation to what is presented and, in addition, the understanding of the *coming-to-be* of the mathematical object might not interconnect with the current understanding. We were able to understand this clearly in one of the accounts, which stated that the articulation of the two graphs was visible because they were common elements, and not of the objective sense of such elements. In the approach for this research we refer to a pairing association, through which two contents presented by the simulator are

> [...] expressed and intuitively shown in the unit of a consciousness and, this way, purely passively, that is, whether they are "noticed" or not, while they appear distinct, phenomenologically, they create a unit of resemblance; therefore, they always appear as pairs (Husserl, 2001, p. 126, author's translation).

Thus, the *sine*, placed in the ratio of the opposite cathetus with the hypotenuse in any right triangle, is paired up with the *sine* of the trigonometric circle, two "distinct" objects that are at the same time "together." In a mutually retrieving connection that gives rise to objective overlap, mutually transmitted through its elements. In the case in point, the mutual transmission can be expressed by the permanence of the relationship in the ratio between projected lengths in the *x* and *y* axes of the Cartesian system when the right triangle is realized in the trigonometric circle.

Pairing transforms the state of consciousness, which may not be properly expressed by the learner into a rigorous concept, but remains as a ground of understanding as was shown in the account of subject 7 when asked about the constancy of the *sine* value of the same angle, which may be revitalized in the presence of the simulator.

Therefore, the potentiality of the simulator to present the *sine* is related to the *cogitationes* already elaborated in the *ego-cogito-cogitatum* relationship, so the pairing is consummated. In other words, the potentiality of the simulator to present the *sine* is not totally assured in its mathematical coherence, but in the relation of the states of consciousness of those who are with the simulator, those who perceive, think about what is seen and perceived, and the very way the simulator presents its content.

Through the analysis conducted in this research, we understand that the way the simulator analyzed is constructed, it is not capable of presenting mathematical function to students. What is seen, learned, of the system, when interpreted and reflected upon, lead to a misconception of the characteristics of the variables, of being dependent on one another, but when inverted, no longer define a mathematical function. That is, the students recognize the graph, know that its name is *sine function* graph; however, they cannot go any further, considering the objective mathematical sense of the content.

We cannot mistake the potential of computing to present formal elements with the potential of a simulator, and assume a naive attitude as mathematical educators. We must be aware, as the simulator, in all its potential to open horizons, can lead to the awareness of things that are not necessarily what was proposed to be understood of mathematics.

The simulator is seen by students as something existing and unquestionable and as a safe haven for understanding something. Therefore, a means for learning. Thus, the student seeks in the simulator an answer to the question posed by the activity or their doubts, whatever they may be, because the simulator becomes the background of possible understandings in the learners' field of view, especially when they somehow recognize the object.

What we mean is that simulators are not necessarily sources of meaning for mathematical objects. Interfaces, however elaborate, may miss something of the intrinsic *cogitationes* of mathematical thinking that constitute its syntheses. Therefore, in such cases, their use should be accompanied by other resources that complement them.

References

Bicudo, M. A. V. (1999). A contribuição dafenomenologia à educação. In M. A. V. Bicudo & I. F. Cappelletti. (org.), Fenomenologia uma visão abrangente da educação (11-51). São Paulo: Olho d'água.

Bicudo, M. A. V. (2013). *Relatório final do projeto apresentado ao CNPQ sobre a pesquisa O ciberespaço: a sua realidade e possibilidades que abre ao mundo da educação*. [S.I.:s.n.].

Bicudo, M. A. V., & Rosa, M. (2010). *Realidade e Ciberespaço – horizontes filosóficos e educacionais antevistos*. Canoas: Editora da ULBRA.

Cantista, M. J. P. (1996, November). O cogito de Descartes na Génese do pensarTranscendental. Evidência Cartesiana e Vivência Fenomenológica. In *Actas doColóquio internacional* (pp. 18–20). Porto.
Cardeira, F. A., & Kluth, V. S. (2019). Compreendendo a função seno por meio do uso de TIC. In V. S. Kluth, J. L. B. Silva, & E. I. Santos (orgs) *Práticas Pibidianas nos subprojetos de Matemática e Pedagogia do PIBID Unifesp – reflexões sobre ser e fazer-se professor*. (s/p) Jundiaí: Paco.
Dartigues, A. (1992). *O que é fenomenologia?* São Paulo: Moraes.
de Figueiredo, O. A. (2014). A questão do sentido em computação. In M. A. V. Bicudo. *Ciberespaço – possibilidades que abre ao mundo da educação* (pp. 109–149). São Paulo: Editora Livraria da Física.
de Moura, A. R. (1989). *Crítica da razão na fenomenologia*. São Paulo: Nova Stela Editora.
Gadamer, H. G. (1997). *Verdade e método – Traços fundamentais de uma hermenêutica filosófica*. Tradução de Flávio Paulo Meurer. Petrópolis: Vozes.
Granger, G. G. (1995). *Le probable, lepossibleetlevirtuel*. Paris: Odile Jacob.
Husserl, E. (2001). *Meditações Cartesianas – Introdução à Fenomenologia*. Tradução de Frank de Oliveira. São Paulo: Madras.
Kluth, V. S. (2005). *Estruturas da álgebra –Investigação fenomenológica sobre a construção de seu conhecimento*. Rio Claro: Tese de Doutorado, Instituto Geociências e Ciências Exatas - Universidade Estadual Paulista.
Kluth, V. S. (2007). O Movimento da construção das estruturas da álgebra: uma visão fenomenológica. *Bolema, 20*(28), 95–113.
Kluth, V. S. (2017). *As estruturas da álgebra – uma investigação fenomenológica*. Saarbrücken: Novas edições acadêmicas.
Kluth, V. S., & Moura, S. P. R. (2014). Preâmbulos investigativos sobre o uso de softwares como ato propulsor do pensar matemático. In M. A. V. Bicudo (Ed.), *Ciberespaço – possibilidades que abre ao mundo da educação* (pp. 221–247). São Paulo: Editora Livraria da Física.
Merleau-Ponty, M. (1994). *Fenomenologia da percepção*. São Paulo: Martins Fontes.
Pugliesi, M. (2001). Prefácio. In E. Husserl. *Meditações cartesianas – Introdução à fenomenologia* (pp. 9–18). Tradução de Frank de Oliveira. São Paulo: Madras.

Resource and Website Developers for Teaching Mathematics on the Internet

Marli Regina dos Santos

1 Introduction

The Internet and access to this network have increased the dissemination of content and information as well as teaching and learning possibilities of various subjects. On the one hand, there are users who seek support and available resources on the Internet to inform, know, learn, or enhance knowledge. On the other, subjects who, individually or collectively, turn to the ways of teaching in this space, focusing their attention on possibilities, specificities, and demands.

This text addresses the second group above mentioned and their experiences, which occur in the Internet collective, together with potential users; colleagues with whom they share ideas; technologies, in their possibilities and limits; mathematics and conceptions about teaching it. It reports on the living experiences of three developers of spaces and/or resources for the teaching of mathematics on the Internet (free websites, channels, groups, and/or blogs), highlighting motivations, interactions, and demands that emerge when reflecting on their experiencing with technology, Internet, mathematics, and the co-subjects.

In creating resources and websites for teaching mathematics on the Internet, the cyberspace is understood as the *locus* (Bicudo, 2014) where people live experiences of teaching and learning mathematics, re-elaborate understandings, share situations, and can intropathically perceive themselves as equals, that is, able to communicate and share understandings mediated by the tools and resources available in this space that connects them, even if their presence is not always physical or synchronous.

To focus on the resources and spaces elaboration for the teaching of mathematics on the Internet, from the standpoint of those who elaborate them, points to several perspectives through which we can investigate this phenomenon. Such elaboration indicates several aspects that subjects can highlight when turning to their work, such

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as experiences with technologies and the Internet, professional, personal, or financial motivations, as well as technical characteristics involved. In all these aspects, something stands out: they are experiences that involve the dimension of a human experience, individual and collective, lived together with others, colleagues, authors, technologies, articulating understandings, and meanings that intertwine within the Internet and beyond it.

The human relationships established guide analyzes that go beyond the individual accounts of those who create spaces and resources in the internet, covering the mutual relationships in which the co-subjects are linked one with other in the cyberspace by at least one common interest: the mathematical knowledge.

Therefore, seeking an understanding of the aspects involved in the elaboration of resources and spaces for teaching mathematics when we are-with-media, emphasizing the Internet leads us to take a step back and look at the phenomenon and its environment. At first, such elaboration brings us to the motivations and inspirations of the creator (or creators). Giving a step back, we can see that it also involves the ground where it occurs and the processes of knowledge construction, highlighting actions and interactions with other subjects. Therefore, the elaboration of resources and spaces is evidencing to be a phenomenon that appears in the experience shared with others that *inhabit* the common space of the Internet, being-with-media (Borba & Villarreal, 2006), who wish to make content available and enable access to it.

In order to explain, analyze, and understand aspects involved in this elaboration from the perspective of its creators, we focused our attention to the way which developers of website and resources for teaching mathematics look at their living experiences when they are elaborating, (re)assigning to meaning them, attentive to the subject that triggers actions, and the background that supports them.

2 Delineating the Study

Directing our inquiries, the discussion that we present was constituted by focusing on the living experiences of subjects with the elaboration of resources for the teaching of mathematics on the Internet, with emphasis on Basic Education.[1] Initially, we looked at spaces on the Internet created for teaching and dissemination, such as websites, blogs, channels, and profiles, focused on mathematics, either those resources and/or materials available and shared whose creation was their own as those that disseminate several resources, from different sources. Here, the word *resource* indicates a variety of teaching materials and tools, such as videos, audios, didactic games, applets, animations, experiments, educational software, software activities, handouts, books, texts, exercise lists, teaching sequences, teaching tutorials, guides.

[1] In Brazil, this level of education corresponds to the age group of 7–15 years. It is equivalent to Elementary School and Middle School (USA).

In order to identify spaces for teaching mathematics on the Internet, we consulted, on the Internet itself, pages for dissemination and evaluation of specific websites, which present rankings based on specific criteria, such as number of monthly visits, frequency and updating of posts, quality and originality of content, user interaction, and navigability. Based on these rankings and the experiences as a teacher and user of Internet spaces, we sought information and data that could assist in the selection of possible subjects of investigation.

Given the range of possibilities, some selection criteria were included in order to establish a direction for data collection. We focused on free spaces and websites and excluded those that focus on the early grades because of the specificity of mathematics teaching at this level. Institutional websites, such as IMPA,[2] were also excluded as they are usually created by a dissemination team for disseminating the institution. We also only looked into Brazilian websites, even though they use and disclose materials and resources from external sources, such as the videos from the Khan Academy.

In this search directed by websites, blogs, and channels, it was apparent that many of them are created by mathematics teachers with the initial objective of helping or motivating their students, integrating technologies into learning activities directed to their own classes. Therefore, there is an initial motivation related to teaching practice and interaction with students outside the formal teaching spaces, but there is also a desire to extrapolate the boundaries of the classroom and the target audience, interacting and disseminating mathematics, reaching students and educators from the most diverse regions and realities, through the Internet. A fact that caught our attention was that, with the exception of those websites aimed at early childhood education, the great majority of spaces for teaching mathematics on the Internet is created by male teachers.

Once the criteria had been established and several teaching spaces available on the Internet consulted, five subjects were selected and contacted for the interview. Of those, only three sent their answers to the open questions sent to them, enabling the elaboration of the analysis that will be presented.

Considering the profile of the participants, they will be called *educators* throughout this text.

Educator M3 participated in the creation of a proposal funded by the Ministry of Education (MEC) implemented within the framework of a renowned higher education institution in Brazil, the State University of Campinas—UNICAMP, by an interdisciplinary team coordinated by a Professor of the mathematics department of that institution, focusing on the elaboration of activities and resources for high school and teacher practice, as explained on the website:

> The educational resources in this collection address almost all high school mathematics content in Brazil in a varied way, and it is up to the teacher, according to pedagogical coordination staff and school management, to choose the items that best fit their program, respecting the characteristics of the teacher and the reality of their students.[3]

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[2] Institute of Pure and Applied Mathematics (Brazil).

[3] Available at https://m3.ime.unicamp.br/principal/justificativa.

The other two educators participating in the research worked privately providing and disseminating mathematics teaching content, resources, and actions, and both have more than one address or web page. Unlike the previous one, they do not focus specifically on the elaboration of teaching resources, but on the dissemination and promotion of activities, materials and discussions about mathematics and teaching mathematics.

Educator Edi congregates in his blog (as indicated by the page logo), the most varied types of resources for teaching mathematics. His interest in technology in teaching and the promotion of discussions on various topics specific to mathematics, education, technologies, and mathematics education stand out. As explained:

> Its initial aim (that of the blog) was to share mathematical knowledge, increase and improve my communication with my students and fellow teachers.
>
> The goal was to boost my school activities with my students, making downloadable files available, exercise lists, assessment schedules, simulations, etc. Over the years, the blog has gained greater proportions, and another format of publication has emerged, as I started writing articles of my own.[4]

Educator Leo, in addition to the spaces for aggregating and disseminating resources for teaching mathematics, created a teacher discussion group through the exchange of e-mails and messages in *Google Groups*. In the spaces he created, he also publishes specific works and studies related to education, such as childhood education and inclusion in Mathematical Education. In addition to the website, he created groups and profiles on Facebook focusing on Mathematics Teaching. In *Portal dos Professores de Matemática* (*Mathematics Teachers Portal)* Facebook group,[5] one can find the following description:

> Grupo Nacional de Professores de Matemática (National Group of Mathematics Teachers,), concerned and interested in improving the teaching and learning of this subject. The main objective is to SHARE knowledge and information.
>
> Everything that is shared in this group is on our group's website:
>
> http://www.professoresdematematica.com.br

Going on, we sought an understanding of the proposals of the subjects when examining their actions while developing spaces and websites for teaching mathematics on the Internet, systematically organizing their responses, looking carefully at all of them, looking for their meaning and articulating, phenomenologically.[6]

In order to do so, we conducted an open interview with each educator, aiming to understand the motivations for the creation of the teaching space(s) and resource(s)

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[4] Available at: https://www.prof-edigleyalexandre.com/p/sobre.html.

[5] Previous address: http://www.professoresdematematica.com.br/inicio.html. Current address: http://professoresdematematica.com.br/. Facebook group address: https://www.facebook.com/groups/profsmat/about/.

[6] In phenomenology, these understandings derive from successive articulations that occur in the reduction movement and lead the researcher's thinking to articulate a broader convergence of the ideas expressed in the descriptions (in the case of this research, in the answers of the subjects), which indicate the invariants, that characterize the structure of the phenomenon investigated.

on the Internet, the interactions that took place during the creation and maintenance of these spaces, their expectations and their understandings of mathematics and technologies. In addition to recording, organizing, and analyzing the responses of educators, we also explored their webpages, focusing our attention on their characteristics and the material resources, texts and information available, seeking to understand the associated conceptions and attitudes, even if implicitly.

We reviewed the responses of the educators, guiding our understandings and analysis through the guiding question: *What is shown when educators who create spaces and resources for teaching mathematics on the Internet turn their gaze to their lived experiences*[7] *in the elaboration process?*

This question directed us to the analysis and interpretation of the subjects' answers by focusing on senses and meanings constituted in the temporality of their experiences. The present work is not in any way intended to uncover and present linear steps for the elaboration of resources and websites in order to create a guide on how to produce them, nor analyze the educational potential of resources created, but rather to understand motivations, actions, and modes of being with the other that guide the process of their creation.

As evidenced by the research on the websites, in general, we highlight the fact that many spaces for teaching mathematics on the Internet are created by teachers and educators with affinity for the subject whose initial objective is promoting it and stimulating practices among students in their classes as well as other teachers' practice, integrating learning technologies into classroom activities. Therefore, there is a motivation related to the practice of teaching mathematics and the desire to extrapolate the physical and spatial limits in the promotion of this knowledge.

The following analyzes focus on the constructions that take place on the realm of the Internet, focusing on mathematics and its teaching, in the context of the intersubjectivity that is configured when the subjects are with each other being-with-media (Borba & Villarreal, 2006). In this sense, the analysis presented here encompasses the expression of: motivations, intuitions of creators, interactions with the co-subjects in the collective space that underpins the elaborate, understandings about mathematics, conceptions about its teaching and working with technologies, expectations about what is available, and the production that emerges from it.

3 Aspects of Creating Resources and Spaces for Teaching Mathematics on the Internet: Motivation, will, Action, and Expectations

The possibility of disseminating mathematics and approaches to its teaching on the Internet is strengthened through its potential to expand the reach of users and access to interactive and dynamic materials, enabling a different look at this science. The open horizons with and through the Internet are broad and, in general, different

[7] The lived experiences are taken as reported by the educators, then interpreted by the researcher aiming to understand of what is being interrogated.

from those of traditional teaching spaces. Such potentiality associated with motivations and expectations indicates a fertile ground for the elaboration of spaces and resources.

M3 explains that the quality and authenticity, the open source, and possibility of adaptation by teachers were promising for implementing the institutional project for the creation of resources for teaching mathematics available on the Internet. For Edi, the blog is a favorable space for presenting ideas and getting comments and seems to encourage new themes and content to be discussed and (re)examined collectively. Leo sees in the open space of the Internet the possibility of creating a rapidly expanding community of teachers and this has led him to a constant work towards improvement, seeking to reach a wide audience of mathematical educators.

> The main motivation is the dissemination of everything related to the universe of teaching and Mathematical Education. Everything directed to teachers and future teachers who teach mathematics at all levels. (Leo)

> We seek to provide resources that teachers choose according to their preferences and could become a standard source of resources for mathematics teachers. (M3)

> Share knowledge. Everything I learned in college (not just mathematics), I am pleased to share. I cannot keep everything just for myself. (Edi)

> It has always been clear for us that the material does not cover the whole curriculum (…) (It) offers elements that can be incorporated here and there (M3)

> Social media networks also reveal efficiency and, in a way, more agility in the dissemination and exchanges among teachers. (Leo)

> The initial idea was to make my used files available to my students (…) teaching mathematics differently using computer programs (Edi)

> The need arose to create a specific group to solve student doubts. (Leo)

Exchanges of ideas and doubts, sharing materials, information, and links, and the size of the project are fascinating. The wide reach and access to a massive range of content and information, which are actualized as the technology demands or changes over time, highlight the dynamism and immensity that the Internet allows to embrace. Among the main motivations in the interviews are improving the interaction between teachers and students; taking advantage of the spaces and resources available on the Internet inside or outside the classroom; student promotion; share experiences; inspire users; show mathematics from another perspective; motivate for the subject; aggregate information; meet demands; propose content and approaches; promote the teacher's professional growth.

It is prominent that the possibility of giving and hearing opinions, interacting with others (who are often unknown) presenting conceptions and approaches that can intervene with those interested in mathematics, or even those who are apathetic to the subject, by showing its "beauty" through available technologies, is what fuels the desire to implement teaching spaces or resources on the Internet.

Future expectations also appear as motivations for action. The desire to share available resources with the other, to become a source for teaching mathematics, to promote the training of teachers and students, nurture the creation, maintenance, and modification of the projects created by the educators.

User feedback and surprise with the effects of what they propose also fuel and promote continuity and improvement. Edi explains that *a well-structured, stabilized work may even yield unexpected results* (Edi). He presents the positive report he received from a teaching colleague, and points out:

> This is one of the cases that makes me very happy, thrills me and makes it worth keeping this blog. It is an indescribable satisfaction to know that the blog has reached teachers, students and schools in other Brazilian states, and with such beautiful results. (Edi)

The personal expectations, the possibilities open with the Internet, and the results obtained intertwine with the volition in triggering the action of elaborating websites and resources. The Internet and the potential of teaching technologies fascinate many educators, attracting more people interested in not only consulting, but also creating spaces, and disseminating content and information on the Internet. On the other hand, implementing such a proposal often does not go beyond this desire, whether due to the lack of specific knowledge, time, or other reasons. We may ask ourselves: Why does not every educator, faced with the desire to present a proposal for teaching on the Internet, venture into this achievement?

By studying the person in their individuality and the coming-to-be that is actualized in the temporality and spatiality of the experiences that occur alongside the co-subjects, Edith Stein[8] invites us to reflect on these interrogations. Stein analyzes the human being in his bodily, psychical and spiritual dimensions (Ales Bello, 2014, 2015; Stein, 2000) and explains that each person, being a living-body, connects with their environment and expresses themselves to the world, embracing it as they turn to it. The psychical aspect is shown in the manifestations as emotion and desire. The spiritual dimension, on the other hand, is revealed in acts of determination, such as decision-making, choices, judgments, and values. We are structurally the same, but the structures are uniquely activated in face of each one's individual characteristics, personality, dispositions, moods, and even factualities to which we are subjected (Ales Bello, 2014; Stein, 2000).

In temporal and spatially located experiences, from the perspective of our living-body,[9] we are driven by our feelings, analysis, and actions, from which a relationship between "free acts" and decision-making emerges. Quotation marks indicate that such acts are not an unregulated action that occurs only through the desire for subjectivity. Motivation is an assumption for the free act, but it is not a determinant of action, which is linked to decision-making, and that can be started, completed, or

[8] Edith Theresa Hedwing Stein (Wroclaw, October 12, 1891—Auschwitz-Birkenau, August 9, 1942) was a German philosopher and theologian, disciple and assistant to Edmund Husserl, the founder of phenomenology.

[9] Living-body "understood as *Leib,* body with intentional movement" (Bicudo, 2010, p. 135, author's translation).

even blocked before even starting (Ales Bello, 2014). Thus, the free act is different from an impulse, because by becoming conscious of their will, the person can reflect on actions and postures, bringing out values, judgments, and considerations, actualizing possibilities for action.

Therefore, motivation is present in the movement of acts of being (of the person), but not in a cause and effect association, as a predictable sequence of acts and motivations (Ales Bello, 2014). Free acts presuppose a desire, a motivation, but do not determine a predictable action, explains Ales Bello (2000). Therefore, it is not a connection of simultaneous or successive phases of the flow of experiences, nor of an associative connection, but of the fact that one experience is completed by another. This author (2014) presents an example given by Stein: certainly, our moods tomorrow will be linked to our past moods, our chores, our physical conditions, among other factors, but there is no security for future predictability about what our mood will be like a deterministic action (Ales Bello, 2014).

The motivational field and personality of each subject guide their action with the values, predictions, analyzes, which is not detached from the background where it occurs, i.e., from the life-world[10] (Lebenswelt) with others, subjects, and things around. Therefore, even if a potentially feasible action or project does not materialize at any given time, this does not mean that it will not be possible to come to fruition later, along with the desires, feelings, and interactions with the partners, intertwined with the actualizations that the experiences promote, culminating in the motivational nexus for the realization of a *project* a posteriori, but whose seed was already sown in the midst of the initial motivations and may or may not be materialized. Thus, even if an action does not occur simultaneously to the motivation and decision-making of an individual or group of subjects, we cannot exclude the importance of the various experiences undergone for triggering it into a future project.

Back to the interrogation, although at some point the individual does not transcend the way they use the Internet to teach mathematics, even feeling the desire to venture into the possibilities that it opens for creation, they may come forward towards the *pro-ject* that "lunges forward," actualizing itself in actions and programs (Bicudo, 1999, p. 11, author's translation). By lunging forward, the *pro-ject* embraces the future and the past: it demands a turning to past living experiences and a forward-looking projection, where new doors open and others close (Bicudo, 1999).

The *pro-ject,* even though it is an individual demand, presupposes desire and involves life stories and co-subjects, bringing the feeling of belonging to the world with others. In the words of Joel Martins (1992): being situated, I am never locked in a world like an object in a box. My freedom, the fundamental power that I enjoy, being the subject of all my living experiences, does not differ from my insertion in the world (Martins, 1992, p. 53, author's translation).

Therefore, even if motivations, desires, and challenges are presented to everyone, each one faces the Internet in a unique way, and the materialization or not of an effective action to elaborate resources or teaching spaces is related to the individual

[10] Life-world "is not a recipient, a thing, but a space that extends as actions are carried out and whose horizon of understanding expands as meaning is being made for each of us and the community in which we are inserted" (Bicudo, 2010, p. 23, author's translation)

characteristics of each person, in their horizon of possibilities and their historicity that are not detached from the influences that the intersubjective world exerts on subjectivity. Everyone has free will to choose what to do or not, but they are also motivated and motivate longing and actions in a kind of attractive force that people exert on each other.

The most diverse experiences with the Internet and its pedagogical possibilities carry the possibility of the *coming-to-be* in which people are able to conceive, analyze, value, and move, not deterministically, but by the arising of factualities and actualizations that occur in this space, where the expectations met can, together with the possibilities and updates of Information and Communication Technologies (ICT), foster new motivations, leading to a (re)organization.

Today, creating a teaching resource or space on the Internet is an easier task than not so long ago, and predictably harder that it will be in the near future. Motivation and will also re-signify in the face of these changes and facilities that arise, and may make the link between motive and action for the conduction of a teaching project on the Internet more reasonable.

So, even though, initially, some aspects prove to be important for undertaking a project on the Internet, such as mathematical knowledge, affinity with technologies and teaching experiences, beyond these demands, such project highlights expectations, motivations, interactions, and longings, which connect developers and users, intertwined in the *care* and *pre-occupation* which occur in the elaboration of spaces for teaching mathematics on the Internet, their actualization and maintenance.

4 Being with the Other on the Internet: Intropathy in the Elaboration of Resources and Teaching Spaces

In cyberspace actions and interactions emerge, in the ways of being with each other in the collective that the Internet makes possible, focusing on didactic-pedagogical activities aimed at teaching and learning mathematics. This organization nurtures a number of relationships, such as those individuals who participate directly in the elaboration of resources and websites, as well as those who indirectly, and sometimes without realizing it, are part of this elaboration or its maintenance.

In interviews with educators, the *other* (helping colleague, team, authors, students with individual characteristics, teachers, etc.) arises from their answers, in the most diverse contexts, as a support that motivates the action, private, or collective, expanding it as interactions, discussions, demands, and impacts take place. The other participates proposing, requesting, suggesting:

> Some members request their demands and others try to help from their own experience. (Leo)

> The creation of groups has always been influenced by users and members' demands. Someone always asked for help in some respect. So, I would search for an answer or consider the replies of other members, and those replies would feed the professoresdematematica.com.br website, because I knew that request would come up several times. (Leo)

> In elaborating the participation of basic education teachers, it focused on the beginning of the project (…) a few sporadic reports of resources that were tested by them. (M3)

> The more user interaction, the more inspiration I have for writing more articles (Edi)

Edi explains that the creation of his blog was, itself, lonely work, but the others encourage its actualization, since it is a favorable space to present ideas and get comments, which leads to new themes and content:

> The more user interaction (subscribers or readers) the more inspiration I get to write more articles, by the way, most articles resulted from this kind of inspiration. (Edi)

In the elaboration of resources for teaching mathematics with the UNICAMP project, M3 emphasizes the importance of the interdisciplinary team, in which personal experiences guide joint actions, motivate the creation, point to paths that can lead to success. The classroom experiences of some members of the team as well as the specific training of others (such as in arts) drive discussion, verification, and the project itself. M3 highlights the importance and appreciation of each individual, with their education and experiences, for the success of the action:

Decisions were based on team members' experience (…) These experiences were highly valued within the project (M3). M3 also highlights the project coordinator as someone capable of "*heading a large team (…) establishing the roles of each member"* (M3).

Apart from those who participate, directly or indirectly, in the elaboration of resources and websites by educators, giving their opinion, suggesting or giving feedback, there are those on the other side of the Internet teaching–learning interface, the prospective users, who, even without realizing, are present in this elaboration, as it presupposes a *pre-occupation* and *care* (Heidegger, 2002)[11] with the other. This proves to be a foundation for actualizations and (re)elaborations.

> I have always worried about early-childhood mathematics. I lecture in in-service training for teachers at this level of education and it is my passion! (Leo)

> *to structure and maintain a blog that is always up- to-date and with content that catches the eye of Internet users* (Edi)

> *we always thought that the audience who would browse our material would be the teachers, not students. This is reflected in the way we organize the official portal.* (M3)

> *Many students with doubts accessed the group PROFESSORES DE MATEMÁTICA (MATHEMATICS TEACHERS) seeking help. But this was not the purpose of the group, but exchanges between teachers. Therefore, the need arose to create a specific group to solve students' doubts.* (Leo)

[11] In his work, Being & Time, Heidegger says that "Dasein's totality of being as care means anticipating oneself-being-already-in-existence (in the world) and on the verge of (those who come to meet within the world)."

Reflecting on the “fear” of people of mathematics, Edi questions: *Why do people feel this way? Did not they learn mathematics in school? Did they have a lousy teacher? Or did they just dislike mathematics? But what if they had numeracy skills, would they like mathematics?* (Edi). He adds: *My role as a mathematics teacher goes far beyond making a student understand mathematical calculations* (Edi).

When elaborating a resource for teaching mathematics through the Internet, the questions and expectations that arise, already during planning, regarding the possibilities and the results, bring the other, unveiling its presence:

> Questions like “do you think this is feasible for the classroom?”, “do you think any teacher would use this?”, “do you think this is suitable for high school students?” were usual (M3)

When elaborating a teaching activity (or disseminating an existing one), in a certain way, possibilities are offered for the fulfillment of meanings for those who perform it, so that, in this interaction, cognitive operations are established that allow the user to advance, aiming at success in the activity. Although the paths traveled are not those devised by the developer, the action takes place in the materiality made possible by the activity or resource and expands into new potentially possible actions—perhaps not imagined by the creator.

In the action of turning to the subjects of learning, in an imaginative act, there is a direction for the possible users, which allows reflections on the developer’s own experiences as to the content or theme addressed. Along with the specific language in cyberspace, the ways of being connected and the underlying interests of each one, concerns and attention towards the other are shown in the choices and paths taken, aiming at success, even if it is not a conventional class, in which one can look for signs of learning by being with the learner, noticing his reactions. This brings us to some thoughts on *intropathic act* and its developments (Ales Bello, 2000; Stein, 2000).

Ales Bello (2003, 2015) explains that intropathy is an act of openness to the understanding of the other, in which the identity and the distinction between subjects are emphasized: in the intropathic act I realize the other, who is not me and who is a stranger to me, recognizing common senses and feelings linked to my own experiences.

When physically addressing another person in teaching, for example, we can look into their eyes and perceive feelings, such as difficulty, doubt, ease, in the intropathic act. On the Internet, it is not always possible to look and feel directly the subject of learning, but one can, imaginatively, intuit and reflect on senses and meanings expressed in an available resource. Thus, even if they do not always happen directly (in the presence of the other), intropathic living experiences occur in the teaching actions on the Internet, in the elaborations that take place in cyberspace, leading to reflections on the difficulties, successes, expectations, *giving shape* to the other; different and at the same time similar to me, placing individualities with each other, without confounding subjectivities.

At the individual level, intropathy can trigger actions, as it supports ways of being with the other along with the desire to interact, transform, create objects and techniques, giving rise to new acquisitions and elaborations. At the collective level, it opens understandings with the subjects who look at each other and understand each other, dialogue, create, and transform their environment.

We are inserted in interpersonal and collective environments, in various forms of organization. This brings us to the understandings of the individualities within each specific associative context and leads us to the different possibilities of meaning, sense, decision-making, and actions in relation to the other—who is analogous to me—that can also understand among objects, symbols, creations, and recreations that provide the objective conditions in which subjects relate, such as language, ethical values, customs, directing behaviors of individuals.

5 Being-with-Media: (Re)configurations and Reach in Mathematics Teaching

Beyond esthetic and interactive aspects supported by the media, the language and particularities of mathematical knowledge drive understandings about its teaching on the Internet. The aspects evidenced in the actions of teaching mathematics via the Internet, such as accessibility to what was made available, attention to the levels of intuition, formal deduction, the contexts in which the disposition can be worked raise concerns of the elaborators.

The educators explicate their experiences with teaching and learning mathematics by focusing on their experiences of elaborating resources and spaces on the Internet,

> The (coordinator) was contacted for offering some courses for mathematics teachers that were well evaluated internally. (M3)

> several team members had experience in basic education. (M3)

> there was a certain accumulated energy (due to the courses offered) that could have great effect on this project. (M3)

> I was invited by several institutions and entities to give lectures and short courses. (Leo)

> The blog is my nook of writing and sharing everything I think and reflect about mathematics in various contexts. (Edi)

> I believe it is impossible to conduct satisfactory classes just through the blog. (Edi)

Thus, the elaboration of spaces and resources refers to the teaching experiences of the subject and indicate conceptions, postures, and understandings that are reflected in what is elaborated. The conceptions about mathematical knowledge and its teaching stand out in the characteristics of what they propose to create or disseminate, such as highlighting the mathematical applications in everyday life, the exploration of facts about mathematics, the application of technological resources for teaching, to the work with history.

> I think there are teachers, with various approaches: traditional, constructivist, critical mathematics, etc. ... In fact, every teacher has their own. Debates are always respectfully of different points of view. (Leo)

> A mathematics class is too discursive to fit a text (…) Student feedback is even more discursive. (Edi)

The conceptions of the educators about mathematics and its teaching expand through the Internet along with the resources and spaces elaborated by them. Thus, beliefs, conceptions, and attitudes also spread throughout the Internet, reaching other spaces, educational or not.

M3 explains that in the teaching resources created in the project, a certain configuration was planned for each type of media. In experiments, for example, *a question is proposed, attempts are made, data is generated (…) when appropriate, some mathematical formalization is done.* For software we adopted the *question script format (…) Something like a computer guided discovery.* Some resources reveal concern about relating mathematics to literature, for example, approaching it *in a much lighter way* (M3).

Thus, attention must be paid to the desired formalism and degree of difficulty with the choices as to what to teach and the choice of tools and formats in relation to the available media.

The importance of technologies in projects also permeates the possibilities that it opens to the various spaces for teaching mathematics, beyond the cyberspace itself:

> There were two uses for digital technologies capabilities: (1) correcting a wide range of objective responses and (2) creating interactive environments, similar to dynamic geometry environments, where students can interact with mathematical elements that would not be possible without the computer. (M3)

> the great relevance of technology is the dissemination of good practices, communication speed. It would be worse without the Internet. (Leo)

> Facebook proved to be more efficient regarding the dissemination of events, exchanges of experiences, relevant information on mathematics education and teaching (Leo)

Noteworthy are the help to the teacher; the creation of interactive environments; possibilities of dissemination; speed; reach; in-service training; the didactic exploration of content through ICT.

The limitations, difficulties, and clashes that are present while being-with-media are also shown in the elaboration of resources and websites by Educators:

> *I had built the Laboratório Sustentável de Matemática (Sustainable mathematics Lab) blog.* The project was already spectacular, but the blog environment at the time was not correctly set up. (Edi)

> *we could not find good examples that would inspire our audios (M3).*

> *At that time, I was a new to the blogosphere and did not understand the technical part of a blog (Edi)*

> *MEC did not allocate any more funds for the project to be disseminated or resources to be maintained (M3)*

> *(as the infrastructure that would be taken to schools did not happen) I believe we have a much smaller reach and impact than we could have.* (M3)

We understand that some limitations imply difficulties for the action and reach that resources and websites could have.

In reflecting on their own teaching experiences with classroom media, M3 highlights the infrastructure difficulties found, which eventually transformed a *movie showing* in *an event and not something natural (M3):*

> whenever I wanted to use something like a video, I had to go through an epic hassle at school (pick up and install the projector, bring notebook, change room layout, struggle with the curtains to get good lighting, etc.). (M3)

The meager feedback from users on the way they reach or use resources and spaces shows a gap to be filled.

> we do not know how resources are being used via mobile phones or computer, individually or in the classroom. (M3)

> Video lessons are more appropriate (not yet produced), but still need the feedback from those watching. (Edi)

The creation of resources and teaching spaces for mathematics on the Internet allows access to content in a *ground* of experiences that is contextualized in the infrastructure made possible by the technologies to which one has access, even if it is updated, explaining ways of experiencing teaching situations, establishing a promising partnership between mathematics and technologies, in a new way of teaching and learning which allows ideas and proposals to be adapted, implemented, and promoted.

The concerns of the educators about mathematical knowledge and its particularities highlight the pedagogical contours that occur with ICT, seeking to adapt and explore potentialities. But, reflections on its actual uses and reach in mathematical learning reveal that there is still a long way to go towards ample inclusion in the most varied spaces.

In the movement of thinking on what to teach, to whom we teach, with what objective and with which focus we teach, what types of resources available to implement, which media to work with, how to present something on the Internet, it occurs a (re)construction of latent knowledge in the elaboration, planning, and execution of a teaching resource or space itself.

In the collective of educators and students, the other who seeks is seen as an individual similar to the developer, that is, an individual potentially capable of interacting with the resource elaborated by them. In this movement of identification, ways of being with mathematics and its teaching are re-elaborated in line with the possibility that being-with-media opens.

6 Reflections that Emerge from an Understanding of the Elaboration of Resources and Spaces for Teaching Mathematics on the Internet

Throughout the study, it was possible to understand the articulation between mathematics teaching practice and the elaboration of resources and spaces for teaching of mathematics on the Internet. The latter arises in the midst of the demands and the practice of the former, aiming to explore open possibilities with technologies, and contribute to the practice, directly or by the continuous education that the Internet can promote. In a movement sometimes of reciprocity, the Internet allows one to glimpse and venture through paths to be explored along with the ideas that emerge from the practice, at the same time, as it is a subsidy that renews the on-site spaces, amidst the diverse experiences of mathematics teaching and learning that take place in the ground of the life-world.

Viable paths emerge for the practice of the subject (in the various spaces), promoting new understandings about mathematics and its teaching, requiring adaptations, new approaches, curriculum changes, revealing the dynamic aspect of knowledge production.

The Internet will not bring ready and definitive solutions to the teaching of mathematics; however, it can promote not only the research of resources that the teacher can implement, but also the exchange of ideas between educators, and of those with their students. It can assist the teacher in various tasks by expanding ways of teaching, assessing, and communication itself, as well as interactions with learners.

The advent of the Internet has raised some concern about the extinction of the teaching profession or the lack of importance of the role of the teacher before a generation of *digital citizens*. But, on the contrary, insertion of the Internet in the field of teaching/learning increasingly indicates the paramount importance of educational practice in promoting teaching that is focused in a creative and open practice, helping students manipulate the giant and chaotic encyclopedia that emerges in the midst of actions on the Internet, whose lore can hardly be lonely leafed.

Just as the various social spaces are (re)signified with new cultural acquisitions, the teaching–learning spaces for mathematics, along with the available technologies, reveal the emergence of thinking and reflecting on the new challenges of the teaching profession facing the cyber reality in which we are immersed. In it the teacher no longer holds the monopoly of their specific knowledge, but can offer students paths that open to knowledge, mediating learning, strengthening the relationship between what is human and technological, the tools and understandings, the presence and being present with the Internet.

Although the dimension of a collective production is difficult to observe without the temporal or historical detachment it requires, this study reveals traces of ways of interaction and human relations that are connected to the collective making of this construction which emerges with the media and mathematics teaching. One can see a configuration of motivations, interactions, and actions that expand into new interactions and actions, where individualities are sometimes distinguished, either in their unitary or general characteristics, sometimes diluted among others.

Reflecting on mathematical knowledge and the elaboration of resources or websites for its teaching made possible by the Internet, as a researcher and educator who, so far, has not ventured into the possibility of creating websites and programs, put us in the middle this construction. It also leads us to glimpse the current possibilities and the importance of actions for the integration of different spaces for teaching mathematics (cyberspace, the classroom, the street). Managing and integrating them in a harmonious way requires the planning of moments that are diversified and integrated into the school curriculum, which needs to be studied, analyzed, and understood within the scope of mathematics education.

We weave and are wrapped in the network that is created with the Internet. We are taken into a collective that constitutes and influences our experiences, integrating experiences, making us feel parts of a whole. In the educational field, this whole interweaves people, expectations, possibilities, openings, and predictions, wishes that may come true or not, search, care, dissemination, authenticity, solution, improvements, news, feedback, recognition, intropathy, collaboration, ease, difficulty, suspicion. Therefore, whatever the space, educating is a complex process and, beyond the focus on concepts, teaching also helps individuals cope with a range of sensations, feelings, reasoning, which even help to work on aspects of self-confidence. As educators, the others invite us to take responsibility and, even if the scenario takes on different configurations, the ontological basis of the actions of teaching and learning aimed at education remains: caring about the *coming-to-be ourselves, the other, the world.*

References

Ales Bello, A. (2000). *A fenomenologia do ser humano*. Tradução de A. Angonese (Ed.), Bauru: Edusc.

Ales Bello, A. (2003). *L'universo nella coscienza: introduzione Allá fenomenologia di Edmund Husserl, Edith Stein, Hedwig Conrad-Martius*. Pisa: Edizioni ETS.

Ales Bello, A. (2014). *Edith Stein: paixao pela verdade*. Curitiba: Juruá.

Ales Bello, A. (2015). *Pessoa e Comunidade: Comentários - Psicologia e Ciências do Espírito de Edith Stein*. São Paulo: Ed Artesa.

Bicudo, M. A. V. (1999). *Fenomenologia: uma visão abrangente da educação*. São Paulo: Olho d'Água.

Bicudo, M. A. V. (2010). *Filosofia da Educação Matemática: fenomenologia, concepções, possibilidades didático-pedagógicas*. São Paulo: Editora UNESP.

Bicudo, M. A. V. (org.). (2014). *Ciberespaço: possibilidades que se abrem ao mundo da educação*. São Paulo: editora Livraria da Física.

Borba, M. C., & Villarreal, M. E. (2006). *Humans-with-media and the reorganization of mathematical thinking: information and communication technologies, modeling, experimentation and visualization*. New York: Springer.

Heidegger, M. (2002). *Ser e tempo*. Vozes: Petrópolis.

Martins, J. (1992). *Um enfoque fenomenológico do currículo: educação como poíesis*. São Paulo: Cortez.

Stein, E. (2000). *La struttura della persona umana*. Roma: ed. Città Nuova.

Part III
Teacher Education in Cyberspace
Being-with-Media

Cybereducation with Mathematics Teachers: Working with Virtual Reality in Mathematics Activities

Maurício Rosa and Rosana Piovesan Pinheiro

1 Introduction

In 2008, when we began studying and constructing the concept of Cybereducation, it became apparent that research involving the education of mathematics teachers and the use of digital technologies (DT) was necessary, so that the use of DT would not be considered as mere support/aid to mathematics classes, but DT would be mostly viewed as participants in the process of producing mathematical knowledge. We state that because using DT in the education of mathematics teachers was seldom considered in the descriptions of courses for education of mathematics teachers in Brazil (Gatti & Nunes, 2009). Under this perspective, researching how to consolidate the use of DT for the production of mathematical knowledge, making the classroom an even more encompassing space, regarding the number and diversity of technologies available for the education of mathematics teachers show themselves as being of great relevance. Nowadays, as a result of our view, we have modified the research about "use" to research regarding work with DT in mathematics education, because we no longer deal with the mere use of DT, but with the reflexive and articulated work with mathematics, especially state-of-the-art technologies, which is still an relevant issue to be investigated.

The theme of teacher education and the kind of research underlying it revolve around several issues concerning pre-service (initial) and in-service education of mathematics teachers (Perez, 2004). Hence, such issues must also be examined regarding cybernetic environments and, by extension, the work with DT as well as VR[1] DT, once the answers to such interrogations need to point towards relevant analyses and reflections about the teacher education process within these environments.

[1] Although in this text we use the phrase virtual reality, already configured in the area of computer science, we will assume the perspective of virtual as highlighted in Bicudo and Rosa (2010), which will be mentioned in a later session.

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Ponte (2004), even though not mentioning work with DT, converges on the idea that the practice of research regarding the education of mathematics teachers would evolve, if not only the aspects of education and didactics were examined, but also the aspects of their mathematical formation, contributing to the improvement of the education process and also to the credibility of the mathematics teacher, and of mathematics education in general.

We agree with da Ponte (2004), and consider the mathematical dimension of teachers' education to be of paramount relevance to their professional education, as well as the pedagogical and, nowadays, more prominently, technological formation, as a result of our current knowledge-based society. Meanwhile, on-site, distance, semi-presential, and mobile teacher education in the mathematical, pedagogical, and technological dimensions have been investigated for some years (Caldeira, 2016; Dantas, 2016; Pazuch, 2014; Rosa & Caldeira, 2018; Seidel, 2013; Vanini, 2015). However, education which includes high-immersion VR has seldom been examined and therefore it became the subject of this study, which seeks to reveal, howbeit partially, how the work with mathematics activities-with-VR is conducted, in the sense of "being-thinking-knowing-doing-mathematics-with-DT" within a Cybereducation process. That is, the concept of Cybereducation with mathematics teachers, which will be examined here, is considered as being the education developed for mathematics teachers to work with DT. Thus, it reaches aspects, dimensions, and possibilities of online mathematics education, and also seeks to yield results that show teachers who will work with immersive VR technologies, discussing mathematics.

We will present Cybereducation with mathematics teachers based on theoretical aspects concerning our research interrogation, namely: "*how is the work with mathematical-activities-with-VR effected under the perspective of the mathematical dimension of Cybereducation of mathematics teachers?*" as, when preparing mathematics teachers to work with DT, we aimed to analyze *a pedagogical proposal for working with mathematical-VR-activities aiming to conduct Cybereducation with mathematics teachers.*

Thus, the present research conceives of the work with VR DT as a way to potentialize the production of mathematical knowledge, that is, such technological resources are part of the process of thinking mathematically with DT. Therefore, it should be recognized that education of mathematics teachers must go beyond a notion of use of DT in mathematics education, which views this use as a way to accelerate the educational process (the calculator makes calculations, for example), or only to support mathematics teaching (the computer channels students' attention when performing a graphical construction task, for example), but investigate the mathematical educational process which happens *with* DT, in particular the DT in VR, in terms of the inseparability and/or potentiality of mathematical thinking. It is, think about the work with DT and not just its use.

2 Theoretical Aspects of Cybereducation Mathematics Teachers with VR

The sharing of ideas and technological resources, as well as the discussion of such resources, is part of the mathematics teachers' education in the technological, specific (mathematical), and pedagogical dimensions (Richit & Maltempi, 2009). Such dimensions, when inserted in Cybereducation, are enriched by the possibility of articulating ideas which are probably from distinct cultural perspectives. This fact can contribute significantly to the elaboration of commonly explored mathematical conjectures once it shares points of view conditioned by adjacent cultures. This, then, highlights both the specific and pedagogical dimensions, which are not detached from the technological dimension present in the immersion of teachers in the cyberworld.

Thus, initially,

> *Formation* (education) refers to the process of becoming, in which the contour of the image that pursues the model is realized. However, it is more than that. This process is not carried out in such a way as to serve a technical purpose external to it, but rather springs from the internal process of constitution and formation, remaining in constant evolution and improvement (Bicudo, 2003a, p. 28, emphasis added, authors' translation).

The formation/education of a mathematics teacher is a constant movement, based on the idea present in the culture of a people regarding the ideal teacher; it encompasses technical aspects, but involves personal (psychological), social (sociological), cognitive (philosophical), and cultural evolution. Regarding the education of teachers who will work with DT, understood under the focus of the three presented dimensions (mathematical, pedagogical, and technological), we assume that the technical-methodological aspects are directly intertwined with those of the "being" who inhabits the cyberworld. Therefore, this formation/education requires a particular/specific *form-a(c)tion* process.

Regarding to the focus of this investigation, which highlights what we initially termed "Cybereducation of Mathematics Teachers" (Rosa, 2010), therefore, we manifest its advance and transformation in understanding which it is no longer a specific *form-a(c)tion* (Bicudo, 2003a) of teachers or for teachers, but "*with*" teachers (Nacarato, 2005). Thus, perceiving the background of this landscape that is shown in the formation/education with mathematics teachers, figures emerge from the intentionality of our perspective of it. Under this perspective the figures unveil themselves by the specific *form-a(c)tion*, which is the action of forming understood as

> […] artistic and plastic configuration, forming the image. Realizes the plasticity, the movement, the fluidity which act on the form. However, the direction of this movement is not chaotic, but outlined in the ground of the culture of a people, from which a desired image of man and society emerges, and which reflects the conceptions of world and knowledge; ground in which the worldview of such people puts down its roots; where the materiality necessary for the form to be realized is found (Bicudo, 2003a, p. 29, authors' translation).

This phenomenological vision is all-encompassing. The formation with teachers, under this perspective, considers the *form-a(c)tion* movements. That is, the composition of the desired form of the image of man, society and also the teacher, arises and is actualized with the movements triggered and guided by aspects imbricated in "[…] the way of life of a people, their yearnings, traditions and customs […]" (Bicudo, 2003a, p. 31, authors' translation). The actions are being configured by the plastic means available on this ground-horizon to act in the artistic movement of realizing this dynamic image. This means that the desired form of the mathematics teacher is never completed as a ready and finished object, after artistic and plastic actions in a crystallized form (Rosa, 2015), but rather taken in this movement of happening, realizing what happens, self-evaluating in relation to the aspects that should be observed (maintained, transformed, discarded), by considering the desired outcome: education of mathematics teachers in the very movement of this *form-a*(*c*)*tion*.

In Rosa (2011), relevant concepts for "Cybereducation" were taken into consideration, which we believe are important in the construction of the word "Cybereducation." *Cyber* expresses the notion of Internet or communication between computer networks, therefore, concerning aspects of working with technologies, in particular, DT (implicitly considering the idea of universality). Education understood as *form-a*(*c*)*tion* (Bicudo, 2003a) surrounded by specific (in this case, mathematical), pedagogical, and technological flows, which permeate the process of *forming-with-technology*, including the use of cybernetic environments and any technological apparatus, which happens to be connected and/or produced as enhancers of mathematical cognition, rather than agility, motivation, and/or fad. This means that it is not the mechanical or technical use of technologies dissociated from the teaching and learning processes, but as means that effectively participate in the production of mathematical knowledge and/or form-action of teachers (Rosa, Bairral, Gitirana, & Borba, 2018).

Moreover, according to Rosa (2011), this term was not created as many others which exist in the perspective in the cyberculture jargon (such as the "cyber thing"), but it is an elaborated concept from the interweaved aspects as specific (in this case, mathematical), technological, and pedagogical aspects (Richit & Maltempi, 2009). So, this study on technologies is underpinned by an understanding of Heidegger's philosophy of *being-in-the-world-with* (Heiddegger, 2012). Although we do not assume each of the dimensions, specific, pedagogical, and technological, as parts of a whole, we seek to characterize them in the dynamic totality of Cybereducation, highlighting them as figures in this background (Cybereducation itself).

In Rosa (2015), specific (mathematical) aspects, that is, ideas, definitions, concepts and other relations are studied so that learning teachers and teachers understand the multiple relations with their spatiality and temporality, with their lived time (Bicudo, 2003b), that is, with their reality, whether the reality is assumed in its dimensions of the physicality of things in the physical world, or in its "virtual" dimensions, or the aspects related to the teaching and learning of these relationships, or of the implicit relations to mathematics itself as language, as a tool and/or field of study. Under this perspective, it is important that specific mathematical formation (understood as the production of mathematical knowledge, that is, mathematical practice) fosters the perception that (Bicudo, 2003b),

> The practice of mathematics teachers […] develops within an *educational* context, which requires a fundamentally different view. In this context, more descriptive definitions, alternative forms (more accessible to students at each of the educational levels) for demonstrations, debate or presentation of concepts and results, in-depth reflection regarding the origins of students' mistakes, etc., are core values associated with school mathematical knowledge (Moreira & David, 2007, p. 21, emphasis added, authors' translation).

Thus, the *specific dimension* of Cybereducation with mathematics teachers reflects the effort for creating bridges between theory and practice, transcending this dichotomy and advancing towards evidencing/engendering a presence of the *praxis* of future teachers, or mathematics teacher (Vanini & Rosa, 2011) in the context of cyberspace and/or work with DT. This presence can emerge in a different way (from that of mundane reality), for there is a phenomenologically subject's *presence*[2] in the cybernetic world when the subject intentionally addresses something they want to do and understand. "The [mathematical] cognition […] [takes place in] embodiment from the dynamics of bodily processes" (da Nóbrega, 2010, p. 31, authors' translation); "[…] [mathematical] knowledge is incorporated, that is, it refers to the fact that we are bodies, with an infinite number of sensorimotor possibilities, and we are immersed in multiple contexts" (da Nóbrega, 2010, p. 79, authors' translation). Thus, we must understand in our VR-connected body-proper[3], thinking cyberspace as a differentiated space/time, as well as, a means of articulating mathematical aspects.

The DT of VR in the present study, in particular, provide a technological teaching and learning environment, which can be configured as ground for the culture of a contemporary people, encompassing the ideas of connected society, network society, knowledge-based society with the current "generation @." Thus, the image of man and society that reflects the conceptions of world and knowledge interconnected via computer network emerges from cyberworld, whose specific aspects generate new demands for mathematics teachers who will, for example, act in this space. Similarly, immersive VR can create possibilities for thinking mathematically, which would not be possible, without the connection to these VR capabilities. Immersive VR arises through interfaces, which use references to mundane reality as a way to break the computer screen barrier and enable interactions, which abstract laws of classical physics regarding to the user, such as visualizing three-dimensional environments, moving themselves within them and manipulating virtual objects (Kirner & Siscoutto, 2007).

According to Kirner and Siscoutto (2007, p. 7, authors' translation) VR "[…] is an "advanced user interface" for accessing computer-based applications, providing real-time visualization, movement, and user interaction within three-dimensional

[2] "[…] understood as a human being who is always in the world" (Bicudo, 2003b, p. 75, authors' translation), carrying "[…] with them possibilities of being and becoming in face of the choices made in the actualization of the way of being in daily life" (Bicudo, 2009, p. 148, authors' translation).

[3] Bicudo (Bicudo, 2009, p. 152, authors' translation): "[…] it is the body seen as whole, that is without separation between spirit and matter, which is exposed as intentional carnality moving in the world spatially/temporally, so as to act in relation to what it perceives as requiring action"

computer-generated environments." We agree with those researchers, because when immersed in VR, one can visualize, move, and interact with the cyber environment, however, when considering immersive VR, there are other interfaces, which do not depend on the computer; smartphones and VR glasses bring other possibilities in terms of the degree of immersion and cognition.

> […] emphasizes features such as multisensory device utilization, three-dimensional space navigation, application context immersion, environment simulation and real-time interaction. Summarizing other definitions of VR, it can be said that it is an advanced interface technique, through which users can perform immersion, navigation and interaction, using multisensory channels, in a three-dimensional computer-generated synthetic environment (Pasqualotti & Freitas, 2001, p. 81, authors' translation).

Just as Pasqualotti and Freitas (2001) state that, with VR the user can perform immersion, navigation, and interaction, it is possible to postulate that VR can also be considered as the junction of three basic ideas, which are not exclusive to VR, but are intertwined in this context: immersion, interaction, and involvement (Morie, 1994). The idea of immersion is linked to the feeling of being immersed in the environment, which is possible through visual impressions, sound, movements of the body and controls, that is, sensory and kinesthetic senses. In interaction, the cyber environment is instantly modified through user input, simulating scenes in response to given commands. Involvement, on the other hand, is linked to intentionality when performing a certain task; VR allows for passive (exploration of a virtual environment) and active (user interaction with a virtual world) involvement (Brandão et al., 1998).

There are several educational actions that can be created in/with immersive VR, which enable a "happening" in potency and not act, but that can cognitively favor teaching and learning by being actualized cybernetically. This can be observed in actions that allow the laws of nature to be surpassed, studied, and understood without the direct interference of the physical environment of mundane reality. Thus, mathematical models can be studied in order to naturally disregard certain variables, such as friction, in a model related to mechanical physics, for example, and can generate important studies of different mathematical aspects related to the topic in question. Thus, simulation and mathematical experimentation can be explored with applications that generate images and even movements, in order to reproduce physical phenomena, which are qualitatively different in relation to visualization and comprehension.

In this sense, it is possible to observe that VR augments

> […] the representation of [mundane] reality, opening possibilities for a new educational approach based on games, allowing the exploration of several multimedia resources. Its use changes teaching dynamics, strategies and behavior of both students and teachers (de Carvalho, Haguenauer, & Victorino, 2005, p. 5, emphasis added, authors' translation).

By understanding and assuming these ideas, a work plan is proposed in order to analyze comprised of mathematics-with-VR activities aiming at education of mathematics teachers. Such work plan sheds light onto the ways through which mathematics is shown in VR and also served as a means of investigating the processes of education of mathematics teachers with VR.

3 Research Methodology

In this study, qualitative research's paradigm was adopted to support the investigation of the relationship between the view about knowledge, the world, and methodological investigative procedures. Actions, reactions, and manifestations may be unlimited when in a VR environment. That is, the strength to lift giant blocks, zero friction, multi-sense gravity, geometric figures and movements mixed with actions in immersive digital games are possibilities, which are in the world and can enhance the production of knowledge. Thus, it is assumed that they need to be particularly considered when the research environment is characterized by VR. Therefore, it was sought to show *"How"* the work with mathematical-activities-with-VR is effected, under the perspective of Cybereducation in mathematics, with mathematics teachers, which, in this case, took place within a specific temporality/spatiality. Thus, the research proposal was to analyze a 60-hour university extension course called "*CyberImersion*: *Building-Mathematical-Activities -with-Virtual-Reality*."

The course aimed to guide mathematics teachers in the development of mathematical-activities-with-VR, by using VR goggles and applications using this kind of reality (games, for instance). The use of technological resources regarding mathematics as a result of the volition and need of the participants of the course itself as well as the theoretical foundation with VR, are derived from the perspectives of mathematical education with DT.

In order to discuss the conceptual construction of mathematical ideas, as well as the relationship among them and VR, the course was developed as Cybereducation considering the different dimensions, which this *form-a*(*c*)*tion* requires. The course consisted of face-to-face meetings; distance learning, through guided study and reading of articles; stages for working with the proposed activities; construction of new mathematical-activities-with-VR by the participants, and the conduction of such activities at the respective schools, where the participants taught. The 22 participants of the specialization course were organized in groups in order that they could work better with the proposed activities and develop the design of their activities. In the group, each member had a task in the development of the activities. They took turns; while one participant played with VR goggles, the others analyzed the strategies of the game, keeping in mind the proposed activities, through mirroring of the screen on smartphones or notebooks. Such mirroring was achieved through applications. *AirMore* is an application which allows the connection of mobile devices to a computer or notebook via wireless networks. This application must be installed on the smartphone and paired up with the notebook by means of a QRCode.

The extension course was conducted as follows: seven three-hour long face-to-face sections, which took place on Saturday mornings at *Faculdade de Educação* (Institute of Education), at the central campus of *Universidade Federal do Rio Grande do Sul* (UFRGS), Brazil, which totaled 21 on-site hours. Distance learning, through readings, exchange of materials and discussions forums conducted through Google Drive, totaling 14 h. Ten hours were dedicated to guided studies, in which the groups remotely met, via Skype or WhatsApp, one of the course advisors to inform on the conduction of the activities that were being created. And, finally, 15 h were devoted to the execution and development of activities in the schools where the teachers/participants themselves worked, under the supervision of the advisors (authors of the present chapter).

By doing this, it was assumed that it was feasible to create the pedagogical proposal of the course aimed at Cybereducation of mathematics teachers with VR, and analyze it with the goal of listing indicators for the *form-a*(*c*)*tion* of teachers in a VR environment, through mathematics education. Thus, it is appropriate to identify and present excerpts of the data produced and analyze them by examining the understandings unveiled in the articulation of the ideas presented by the participants. Some of such ideas, present in the excerpts, are in line with those presented in the theoretical framework.

4 Data Immersed and Revealed in the Realization of Mathematical Possibilities of VR

The data were produced during the extension course. Throughout the process, these data were examined through the transcripts of the videos recorded during the meetings. Under this perspective, this section will focus on the activities conducted with VR, which allowed participants to experience the potentialities of VR, as they were engendered under the perspective of working with DT as participants in the production of mathematical knowledge.

Furthermore, it is relevant to clarify that this study is part of a larger investigation and that several other excerpts were produced, establishing these and other categories, which were not reproduced here, due to space constraints. From the descriptions analyzed, we believe that the participating teachers augmented/enhanced their mathematical conjectures in terms of spatial recognition, through VR. We then proceed with the emerging category of a previous verification of the data, by examining and later grouping units of meaning, which resulted in the category called: "augmenting/enhancing spatial conjectures with VR."

The mathematics shown in this study is unveiled in the movement of production of mathematical knowledge and is presented by the reflections/discussions highlighted in the *form-a*(*c*)*tion* performed. The following excerpt presents discussions on a topic that emerged in a moment of reflection/discussion throughout the investigation, triggered by an activity developed with the game Infinity[4] VR.

The first episode presented happened on October 27, 2018, during the meetings of the CyberImmersion course. The excerpt presents the discussion around the activity proposed with the VR game, which had the following installation instructions (Table 1):

[4] An immersive first-person exploration game, in which, with a console, via Bluetooth, players manipulate their dislocation through a sinuous path in order to find a door.

There were also instructions for the activity itself (Table 2):

The interactions relayed herein represent the dialogue which reveals an important aspect identified in the work with VR in terms of the mathematical dimension, when displaying a spatial discussion and enhancing/potentializing mathematical conjectures in relation to this "new" space. VR characterizes non-traditional educational

Table 1 VR applications installation instructions

• Install the Infinity VR app on the smartphone (link: https://play.google.com/store/apps/details?id=com.onceuponafox.infinityvr&hl=en_US)
• Turn on the console (made available along with virtual reality headset) and establish the connection with the smartphone via Bluetooth
• After that, push the configuration button and change the name of the Bluetooth
• Open Infinity VR, push the buttons "@" and "D" on the console, in this order, and position the smartphone on the VR headset 
• Put the VR headset on, set the zoom so that image is crisp and clear and access the game using the console, which has already been connected via Bluetooth
• Using the VR headset, point to "Play" and select it using the "command button" on the upper part of the console  ⁎ "Tecla de comando" means *command button*
• After that, select "discovery" using the same process
• Finally, point your head towards the desired game level, which is referred to as "Area" and, again, select it by using the command button on the console
READY! You are inside Infinity universe. Here are additional instructions to play and achieve the objective of the game
Note 1: the same command button on the console, which selects the commands previously, will enable the player to move forward
Note 2: The path is controlled by head movements
Note 3: The command button must be pushed slowly

Table 2 Instructions for the mathematics-with-VR activity

• Each participant should play level 1 and 2 of Infinity VR in order to familiarize themselves and understand the objective of the game
Note: To go back to level 1, the participant must "look down" until the player finds the link "Menu", and use the command button to select the symbol that resembles a "little house", in order to return to the main menu, where the other player must select again "Play", then "Discovery", and, after that, the "area 1" level
• Each member of the group plays again level 1 while the other member writes down the number of moves each player made to get to the door
Note 1: the steps may be identified through a beep which the game sounds or can be counted by the number of clicks on the command button
Note 2: lunges are counted as another step
• Enter the number of the steps on the table presented in the annex. Discursively record the path each of the participants took. Based on the record made, compare the results and answer the questions herein
After that, the following questions were presented
– From the paths taken by the group, players can advance to another level with a different number of steps, even when no one made a mistake during the trajectory? Why?
– Describe the trajectories in a way that other people can reproduce them
– How can we describe the paths taken with a graphic representation?
Note 1: If the number of steps of the participants is the same, just describe the path taken. If different, consider why the number of steps was different
Note 2: The trajectory must be described in such way, so that anyone would be able to cover it by taking the same number of steps

streams regarding classroom geometry, which indicates a mathematical-thinking-with-VR. In this episode, VR can be seen as a participant in Cybereducation, as it enables the reflection/discussion regarding space and geometric localization in the realm of VR.

Day 27/10—Video 7—Group: Andreia, Anuar, Marcos, Naira[5]

[00:08:10.00] Marcos: I would have already jumped, already. One step forward, a leap into the future, isn't it? Because, that leap that you take there is your future, if you don't, you cannot go on, right?

[00:08:32.05] Anuar: You will have to try to manage to fall on that platform, that is in the center, so you will have to walk around and fall there.

[00:09:00.00] Naira: I can go up through those stairs, can't I?

[00:09:11.07] Marcos: Those stairs are fake, if I am not mistaken.

[00:09:14.00] Anuar: It [the stairs] is blocked there, look.

[00:09:17.06] Andreia: Humm, there…

[00:09:19.09] Anuar: Go upstairs until the top, then you will see.

[5] Use of participants' names authorized through free and informed consent. In order to obtain a more understandable comprehension of the participants' colloquial language, some adaptations were needed in matters of translation.

[00:09:21.17] Naira: There is a ramp on it. Oh, no! it [the stairs] is blocked over there.
[00:09:24.19] Anuar: It is blocked over there!
[00:09:25.16] Marcos: That shaded area over there is a straight wall.
[00:09:28.03] Anuar: Right behind you, if you turn 180 degrees. Look, you will have to fall on that platform there, so that you can find the door, somehow.
[00:09:36.10] Naira: Ahh, I will have to go either to the right or to the left?
[00:09:38.11] Anuar: Yes, either right or left to get to the center and then, fall there again
[00:09:45.27] Marcos: Yeah, when you fall there, you change your dimension. Do you understand?
[00:09:49.01] Naira: Ahh
[00:09:50.15] Marcos: You come into a parallel dimension to the vertical one, but the vertical one becomes parallel to you.
[00:09:54.25] Naira: Am I supposed to fall there?
[00:09:59.08] Marcos: Exactly!
[00:10:08.00] Naira: But this does not make much sense…
[00:10:17.00] Marcos: Ok, keep on walking forward, walk, keep walking, there is a flight of stairs at the end.
[00:10:42.18] Marcos: Turn around a little bit over there, for us to see which side you have to fall.
[00:10:46.25] Naira: The door is there (pointing with her finger)
[00:10:55.08] Marcos: There, you can fall down there. Look, there... You just have to go a little bit to the left.
[00:11:02.22] Anuar: Don't worry! You won't hurt yourself.
[00:11:17.50] Marcos: There is a delay, right?
[00:11:19.03] Andréia: Yes, sometimes.
[00:11:21.09] Marcos: It seems that the delay is getting longer.
[00:11:25.01] Naira: These stairs?
[00:11:31.15] Marcos: That is the door, you can go down. There is the door. You may go straight ahead. There, at the end, is the door.
[00:11:44.17] Naira: Now, it is the last one.
[00:11:53.00] Marcos: Go there, to the green spot, to change, to make your bridge. Look at the green (spot). Look at the green (spot) and click with the button to rotate the bridge.
[00:12:16.13] Naira: How about the door, where is it?
[00:12:20.26] Marcos: You have to rotate the bridge anyway, to close it and then you can return.
[00:12:27.24] Andreia: It is stuck.
[00:12:29.00] Marcos: It is stuck here; I cannot help you now.
[00:12:35.00] Marcos: It is unlocked. That's it; now the bridge is closed.
[00:13:02.07] Anuar: Can you predict where you have to fall?
[00:13:05.24] Naira: I cannot. Not very well.

[00:13:14.04] Marcos: This looks like that Rubik cubic game. Do you know that cube that you have to find the corrected position? Don't you think this has the same "vibe"?
[00:13:44.08] Naira: This is the last one.
[00:13:46.08] Marcos: Yes, this is the last one.
[00:13:46.14] Naira: Now I have to come back to the menu, right? Isn't it?
[00:14:08.00] Andreia: I will keep recording.
[00:14:09.00] Marcos: Yes!
[00:14:11.00] Naira: My perception was like falling. I kept looking for a straight line all the time (laughs). But I got a little dizzy, I think it is because I shook my head a lot. [Andreia takes the VR headset to play herself].
[00:14:29.00] Marcos: When you change the angle of your head, this way (showing how, moving his head to the side), that level of your ear also changes, doesn't it?
[00:14:39.01] Naira: I think so.
[00:14:39.23] Marcos: Yeah.
[00:14:40.28] Andreia: This one up here? [points to the control button]
[00:14:42.15] Naira: Yeah!
[00:14:43.26] Marcos: Be careful… with labyrinthitis (laughs)
[Andreia adjusts the focus of the VR headset]
[00:15:27.00] Marcos: I will get a little closer.
[00:15:34.00] Naira: I did not realize it, but I kept holding the VR headset with my arm and it even hurt…I realized that my arm hurt…
[00:16:02.02] Marcos: She watched three times [peers playing]. So, she must fly by now. You are an expert now, huh?
[00:16:44.13] Andreia: Do I have to fall here?
[00:16:46.27] Marcos: Yes.
[00:16:47.00] Anuar: Yep!
[…]
[00:18:21.04] Naira: You go along the stairs and, sometimes, you arrive at the same place.
[00:18:29.00] Anuar: In the physical world this is impossible, right?
[00:18:38.22] Marcos: But when you start thinking, it makes sense, right? The whole question is the change… Which I find interesting… The atmospheric change, do you understand? Because you change quadrant. The professor told me it is not a quadrant, actually. It is a dimension. The dimension changes.
[00:18:58.27] Anuar: The dimension? I don't think so, because [we are in] one dimension, two dimensions and three dimensions. We are in three dimensions in this world.
[00:19:05.11] Marcos: Yes.
[00:19:06.05] Naira: Doesn't it change the plane?
[00:19:07.13] Anuar: That's it.

[00:19:07.17] Andreia: Where should I go now? [Andreia speaks together with the colleagues, Anuar and Marcos continue the discussion and Naira tries to help Andreia]

[00:19:08.13] Marcos: No, it is not the plane. It is with dimensions, otherwise wouldn't be able to change, you wouldn't need to fall. It wouldn't change your atmospheric perspective.

[00:19:15.02] Naira: So, you will have to fall now.

[00:19:16.07] Anuar: You change the plane where you are, xy or xz. This way.

[00:19:20.09] Naira: Here, we are stuck.

[00:19:22.22] Marcos: Yes, then you change the dimension. In that sense, you do. You shouldn't see a plane. You have to see a square and the whole center of the square turning right, left, up and down.

[00:19:35.12] Anuar: Then, we are talking about only three dimensions. The graph is all in three dimensions. It is the planes we are talking about in three dimensions.

[00:19:47.05] Marcos: But you change dimensions. Each dimension has a plane, see? When you change dimensions, it is a different plane. That is why sometimes we are under, sometimes we are over. What seems cool to me is this anchorlike role of our feet, isn't it? And that is the atmospheric issue I am talking about.

[00:20:07.22] Anuar: Ah, is that the atmospheric issue?

[00:20:09.06] Marcos: Yes, which is the pressure always downwards, always towards the feet.

[00:20:15.19] Anuar: But there is no gravity there. Ahh, there is gravity, because one falls downward.

[00:20:19.21] Marcos: If not, you wouldn't fall.

[00:20:20.14] Anuar: Surely, it is true.

[00:20:21.03] Marcos: If not, you wouldn't fall, you'd float.

[00:20:24.11] Anuar: And, this is kind of a cliché.

[00:20:25.18] Naira: Oh, yes.

[00:20:26.23] Anuar: To the ground, for a fact!

[00:20:30.26] Marcos: And, your ground changes. According to the dimension where you are in, your ground changes: sometimes your ground is the opposite of the other, of the other dimension. Just check it! Everything coverts towards the zero point. Notice how interesting this situation is.

[00:20:44.11] Anuar: Yes, if you think about it, your feet are always in two dimensions and there is a third one, the height, in which you will not be. So to speak, your feet will not be.

[00:20:55.23] Marcos: Sometimes you are, because when you go up, you manage to go up. So, in certain moments, you are going up, you are changing the…

[00:21:01.26] Anuar: You change the axis.

[00:21:03.30] Marcos: Yeah… (agreeing), you change the axis. At many moments, you change the axis.
[00:21:08.04] Anuar: But we cannot see the planes, right? Xy, if we consider the axis x, y and z, right? At one moment, you are in the xy, then you go to yz, and another time, you go to zx.
[00:21:32.12] No, don't think like that! Think this way: think about a square, or rather, sorry, think about a sphere, but not a circle. Because a circle is flat. A sphere has dimensions.

This excerpt shows the group, *Andreia, Anuar, Marcos*, *and Naira,* conducting the activities of the game Infinity with VR. One of the group participants played while the others watched and discussed their actions by visualizing the mirroring of what happened in the VR environment on a laptop screen. When analyzing the chosen excerpt, we were able to show that in [00:09:28.03] *"Right behind you, if you turn 180 degrees. Look, you will have to fall on that platform there, so that you can find the door, somehow*.*"* Anuar is thinking-mathematically-with-the-VR-environment. This thinking-with-VR articulates the mathematical dimension of Cybereducation, once Naira is stimulated to interact by turning 180°, that is, in [00:09:36.10] *"Ahh, I will have to go either to the right or to the left*,*"* to think-mathematically-with-the-VR-environment. In this sense, Anuar in [00:09:38.11] reiterates Naira's conjecture, which is spatially located, when mentioning *"Yes, either right or left to get to the center and then, fall there again*.*"* Advancing the production of knowledge regarding spatial localization, Marcos in [00:09:45.27] states *"Yeah, when you fall there, you change your dimension. Do you understand?"* referring to the change in direction in corporeal terms, without suffering the action of gravity in the initial sense. That is, Marcos infers that, if Naira kept walking and climbed vertically, she would end up upside down without falling and, from her perspective, she would have continued in a straight line. Then, he continues [00:09:50.15] to state *"You come into a parallel dimension to the vertical one, but the vertical one becomes parallel to you."*

For the authors, Naira, Anuar, and Marcos developed their mathematical practices in a VR educational context, which requires a different view (Moreira & David, 2007) from that, which they are used to resorting to in their classrooms. The mathematical dimension is then evidenced/generated when the practice of mathematics teachers is made present (Vanini & Rosa, 2011) in the context of VR; therefore, it gives rise to teacher knowledge articulated to VR. This presence is effected differently (in relation to mundane reality), because there is phenomenologically a *presence* of the subject in terms of the cybernetic world, that is, the VR environment. This can be identified when Marcos in [00:18:38.22], however incorrectly interpreting of what the professor had previously said, states that *"But when you start thinking, it makes sense, right? The whole question is the change… Which I find interesting… The atmospheric change, do you understand? Because you change* *quadrant. The professor told me it is not a quadrant, actually. It is a dimension. The dimension changes."* Trying to configure the change in reality itself. By being in VR, he refers to a change of atmosphere, when referring to the perception of this reality through spatial localization. In this sense, Anuar disagrees with Marcos stating in [00:18:58.27] that *"The dimension? I don't think so, because [we are in] one dimension, two dimensions and three dimensions, we are in three dimensions in this world."* After that, Marcos agrees and Naira in [00:19:06.05] asks: *"Doesn't it change the plane?"* then, Marcos thinking-mathematically-with-a-VR-environment claims [00:19:08.13] *"No, it is not the plane. It is with dimensions, otherwise wouldn't be able to change, you wouldn't need to fall. It wouldn't change your atmospheric perspective."* Anuar, then, goes further in terms of spatial perception: [00:19:16.07] *"You change the plane where you are, xy or xz. This way."*

Both Marcos and Anuar demonstrate pedagogical actions between themselves in order to explain mathematically what happens in the cyberworld of VR, and this contributed to the expansion/potentialization of mathematical conjectures that were being explored. In the case of Marcos, in [00:19:22.22], he insists on the change of dimension, ascribing a particular meaning to what he believes to be a dimension in VR, when stating *"Yes, then you change the dimension. In that sense, you do. You shouldn't see a plane. You have to see a square and the whole center of the square turning right, left, up and down."* Anuar, on the other hand, articulating the mathematical dimension with VR, in the same sense, makes conjectures, which seem to be more plausible, if approached from the point of view of academic mathematics. In [00:19:35.12], he reveals *"Then, we are talking about only three dimensions. The graph is all in three dimensions. It is the planes we are talking about in three dimensions,"* once, even when VR gives a different perception, through its multisensory devices, still navigates in three-dimensional spaces. Immersion in the context of application, simulation of environments, and interaction in real time leads us to synthesize other perceptions of the VR environment. Thus, it is pertinent to state that there is an advanced interface technique, whereby the user can perform immersion, navigation, and interaction in a computer-generated three-dimensional synthetic environment using multisensory channels. (Pasqualotti & Freitas, 2001, p. 81, authors' translation) that enables the expansion of ideas and conjectures, altering one's worldview. We state that after analyzing Marcos' fertile discourse when in [00:19:47.05] he states: *"But you change dimensions. Each dimension has a plane, see? When you change dimensions, it is a different plane. That is why sometimes we are under, sometimes we are over. What seems cool to me is this anchorlike role of our feet, isn't it? And that is the atmospheric issue I am talking* about." He addresses the same aspects that Anuar does, but uses another jargon while seeking to differentiate the world in which he is immersed, as gravity continues to act, even when the "being" depends on the "up-side-down" perspective. Meanwhile, Anuar in [00:21:08.04] says: *"But we cannot see the planes, right? Xy, if we consider the axis x, y and z, right? At one moment, you are in the xy, then you go to yz, and another time, you go to zx"* mathematically attributing to it the idea of inversion.

Thus, these *actions-with* (*being-with*, *thinking-with*, and *knowing-how to do-with*) coined by Rosa (2008) and materialized in the process *becoming-with, thinking-with, and* knowing-how to do-with-*DT* that are expressed by the corporeity of the embodied-self. We state that, as "[mathematical] cognition refers to corporeality, of the dynamics of corporal processes" (da Nóbrega, 2010, p. 31, authors' translation), that is, "[...] refers to the fact that we are bodies, with a multitude of sensory-motor possibilities, and are immersed in multiple contexts"(da Nóbrega, 2010, p. 79, authors' translation), including VR, and this also makes the participants know-how-to-do-with-VR, as, for example, this was evidenced when they were moving in VR. Meanwhile, articulating the reality through the space-temporality where we are, in this case, as Bicudo and Rosa (2010) stated, in terms of the transformation present in the same reality, but that, in mundane or virtual terms, localizes where in spatiality/temporality one is-together-with-the-cybernetic-world, enabling one to mathematically perceive the space/time inhabited and broadens/potentializes conjectures which, nowadays, would not be possible to effect without VR DT.

5 Conclusion

The results obtained show that the participants broadened/enhanced mathematical conjectures in terms of spatial recognition with virtual reality, as VR characterizes different aspects, which are not common in terms of classroom geometry, and such environment is relevant in the dynamics of mathematical-thinking-with-VR. We state this because Naira, Anuar, and Marcos experienced mathematical practice in VR differently from their practice in the classroom, once they moved in a scenario where they stayed upside down for a long period of time, defying the laws of gravity, or rather experiencing a gravity not yet tested by them, as it is a different perception from that realized in mundane reality. Moreover, they moved objects of supposed great size and mass, which they could not do, considering their human reality. Thus, the mathematical dimension is generated with VR, because the knowledge produced emerges from the articulation of social mathematics built in an immersive cyberworld and the issues situated in this technological context of VR.

The pedagogical actions for mathematically explaining what happens in the VR cyberworld contribute to the elaboration of mathematical conjectures, which were being explored with VR. There are new understandings for dimension, giving particular meaning to what can be understood as dimension in VR. This is because, being in VR, they called dimension "atmospheric change," dealing with the perception of this reality through spatial location. In this perspective, even while bringing up different perceptions, VR emphasizes navigation in three-dimensional spaces. Immersion in cyber environments can lead us to consider perceptions of this environment. There is an underlying technique to this perceptual movement, as the projection, articulation, and programming of scenarios and interfaces promote possible perceptions of spaces and spatial conjectures with different aspects from those commonly known. This allows us to expand ideas and conjectures, changing the way we view and understand the world.

Thus, we believe that working with mathematical-activities-with-VR in a sense of being-with-and-thinking-with-VR, since technology, in this case, acts in accordance with the teacher and, consequently, with the student, in terms of constructing the environment, conditioning the constitution of knowledge, allows us to go beyond the memorization of mathematical techniques, requiring us to inquire "what if?". It is worth mentioning the effective role of technology as a participant in the process, as, without it, the production of mathematical knowledge would not be the same.

In VR, one moves educationally in continuous streams that blend, intersect but never recede. Like a river, one moves, articulates, educates, projects, leaps, and never goes back, in the sense of having to do mechanical activities, mathematically repetitive, of rote memorization, for instance. Thus, it is up to us to increasingly involve ourselves in the study of cyberculture with emerging technologies in mundane reality; technologies in different fields, advanced technologies, such as: games, applications, software, graphic interfaces, as, there is an increasing need to worked with them, not only with teachers, but students, in order to investigate their potentialities and limitations in educational terms. Then, in this sense, we agree with Marcos, when he said: "*I would have jumped, already. One step forward, a leap into the future.*"

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References

Bicudo, M. A. V. (2003a). A formação do professor: um olhar fenomenológico. In M. A. V. Bicudo. (org.), *Formação de Professores?* (pp. 7–46). Bauru: EDUSC.

Bicudo, M. V. A. (2003b). *Tempo, tempo vivido e história*. Bauru: EDUSC.

Bicudo, M. V. A. (2009). *O estar-com o outro no ciberespaço* (Vol. 10, pp. 140–156). Campinas: Educação Temática Digital.

Bicudo, M. A. V., & Rosa, M. (2010). *Realidade e Cibermundo: horizontes filosóficos e educacionais antevistos*. Canoas: Editora da ULBRA.

Brandão, E. J. R., Trentin, M. A. S., Lebedeff, T. B., Mortari, M. I. M., Oro, N. T., & Pasqualotti, A. (1998). A realidade virtual como proposta de ensino-aprendizagem de matemática para deficientes auditivos. In *Seminário de Pesquisa em Educação da Região Sul–ANPED*. Florianópolis: UFSC.

Caldeira, J. P. S. (2016). *Conexões entre professores de matemática em Cyberformação mobile*. Dissertação de Mestrado, Programa de Pós-Graduação em Ensino de Ciências e Matemática, Universidade Luterana do Brasil, Canoas.

da Nóbrega, T. P. (2010). *Uma Fenomenologia do Corpo*. São Paulo: Editora Livraria da Física.

da Ponte, J. P. (2004). A formação matemática do professor: uma agenda com questões para reflexão e investigação. Intervenção no painel "A Matemática e diferentes modelos de formação". In A. Borralho, C. Monteiro, & R. Espadeiro (Eds.), *A Matemática na formação do professor* (pp. 71–74). Lisboa: Secção de Educação e Matemática da SPCE. Acesso Maio 24, 2018, em http://www.educ.fc.ul.pt/docentes/jponte/docs-pt/03-Ponte(Evora%20SPCE).pdf.

Dantas, D. M. (2016). *A Criatividade Tecnológica na Construção de Atividades com Professores de Matemática em Cyberformação*. Dissertação de Mestrado, Programa de Pós-Graduação em Ensino de Ciências e Matemática, Universidade Luterana do Brasil, Canoas.

de Carvalho, F. S., Haguenauer, C. J., & Victorino, A. L. Q. (2005). Utilização de Jogos Interativos no Ensino a Distância Via Internet. In *12 Congresso Internacional de Educação a Distância*, Florianópolis. 12 à 22, Setembro 2005. Acesso Agosto 20, 2006, em http://www.abed.org.br/congresso2005/por/pdf/040tcc5.pdf

Gatti, B. A., & Nunes, M. M. R. (2009). *Formação de professores para o ensino fundamental: estudo de currículos das licenciaturas em Pedagogia, Língua Portuguesa, Matemática e Ciências Biológicas*. São Paulo: Textos FCC.

Heiddegger, M. (2012). *Ser e Tempo* (6th ed.). Petrópolis: Vozes.

Kirner, T. G., & Siscoutto, R. (2007). Realidade virtual e aumentada: conceitos, projeto e aplicações. Livro pré-simpósio. In *Anais do 9° Symposium on Virtual and Augmented Reality*, Petrópolis, 28, Maio, 2007.

Moreira, P. C., & David, M. M. M. S. (2007). *A formação matemática do professor: licenciatura e prática docente escolar*. Belo Horizonte: Autêntica.

Morie, J. F. (1994). Inspiring the future: Merging mass communication, art, entertainment and virtual environments. *Computer Graphics, 28*(2), 135–138.

Nacarato, A. M. A. (2005). Escola como *locus* de formação e de aprendizagem: possibilidades e riscos de colaboração. In D. Fiorentini & A. M. Nacarato (Eds.), *Cultura, formação e desenvolvimento profissional de professores que ensinam matemática* (pp. 175–195). São Paulo: Musa Editora e GEPFPM-Prapem-FE/Unicamp.

Pasqualotti, A., & Freitas, C. M. D. S. (2001). Experimentação de ambiente virtual para melhoria do ensino-aprendizagem de Matemática. *Bolema, 14*(16), 79–101.

Pazuch, V. (2014). *Cyberformação semipresencial: a relação com o saber de professores que ensinam matemática*. Tese de Doutorado, Programa de Pós-Graduação em Ensino de Ciências e Matemática–Universidade Luterana do Brasil, Canoas.

Perez, G. (2004). Prática reflexiva do professor de Matemática. In M. V. Bicudo & M. C. Borba (Eds.), *Educação Matemática: pesquisa em movimento* (pp. 250–263). São Paulo: Cortez.

Richit, A., & Maltempi, M. V. (2009). Educação a Distância e Formação Continuada de Professores de Matemática: um olhar sob a perspectiva da teoria dialética. In 6 *Congresso Iberoamericano de Educação Matemática*, Puerto Montt, Chile, 4 à 9, Janeiro, 2009.

Rosa, M. (2008). *A Construção de Identidades Online por meio do Role Playing Game: relações com o ensino e aprendizagem de matemática em um curso à distância.* Tese de Doutorado, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Rio Claro.

Rosa, M. (2010). Cyberformação: a formação de professores de Matemática na Cibercultura. In *Anais do 10° Encontro Nacional de Educação Matemática*, Salvador, 7 à 9, Julho, 2010.

Rosa, M. (2011). Cultura Digital, Práticas Educativas e Experiências Estéticas: interconexões com a Cyberformação de Professores de Matemática. In: *34° Reunião Anual da ANPED*, Natal, 2 à 5, OUTUBRO, 2011.

Rosa, M. (2015). Cyberformação com Professores de Matemática: interconexões com experiências estéticas na cultura digital. In M. Rosa. et al. Educação Matemática, Tecnologias Digitais e Educação a Distância: pesquisas contemporâneas (57–93). São Paulo: Editora Livraria da Física.

Rosa, M., Bairral, M., Gitirana, V., & Borba, M. (2018). Digital technologies and mathematics education: interlocutions and contributions based on research developed in Brazil. In A. J. Ribeiro et al. (Eds.), *Mathematics Education in Brazil*. Cham: Springer.

Rosa, M., & Caldeira, J. P. S. (2018). Conexões Matemáticas entre Professores em Cyberformação Mobile: como se mostram? *Bolema, 32*(62), 1068–1091.

Seidel, D.J. (2013). *O Professor de Matemática online: percebendo-se em Cyberformação*. Tese de Doutorado, Programa de Pós-Graduação em Ensino de Ciências e Matemática, Universidade Luterana do Brasil Canoas.

Vanini, L. (2015). *A Construção da Concepção da Cyberformação por Professores e Tutores de Matemática Online na Formação Continuada e na sua Prática: uma análise bourdieana*. Tese de Doutorado, Programa de Pós-Graduação em Ensino de Ciências e Matemática, Universidade Luterana do Brasil Canoas.

Vanini, L., & Rosa, M. (2011). O papel da educação matemática na vida de um professor de matemática. In *Atas da 13 Conferência Interamericana de Educação Matemática*, Recife, 26 à 30, Junho, 2011.

Mathematics in African Cultural Creations: Open Horizons to the World of Mathematics Education and the Formation of the Person

Tânia Baier and Maria Aparecida Viggiani Bicudo

1 Introducing the Theme of This Chapter

Throughout the pedagogical experience of the authors of this chapter, dialogues with undergraduate students in mathematics and with fellow teachers of mathematics at various levels of education, questions were raised about the fact that mathematics textbooks distributed in Brazil present rare references to fractals and make no reference to *fractal* objects, elaborated through iterative processes, created in a self-similar way by traditional African peoples. It has been argued that the themes presented in such books prioritize the creations of the ancient Greek civilization, which underlies the development of modern European science, devaluing mathematical creations of other cultures. Interactions with mathematics teachers revealed their difficulty in working with their fellow teachers of other disciplines to develop activities that cover topics of African culture, according to Brazilian educational legislation, which determines that Afro-Brazilian and African history and culture are to be taught in basic education. Instigated by the controversy raised by dialogues with the students of the course for training mathematics teachers of a University of the State of Santa Catarina, in the south of Brazil, we proposed to conduct a distance course for teachers working in elementary schools[1] located in different regions

[1] In Brazil, elementary school comprises 9 years; children start elementary school between the ages of six and seven.

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throughout that state, focusing on the theme: "Mathematics in African cultural creations," focusing on "fractals." It was a short course through which it was possible to provide teachers with examples of objects of African culture, point out aspects that encouraged thinking about fractals, and carry out different educational and pedagogical activities with children and adolescents. However, our experience with objects of African culture, the theory of fractals, the vision of the world and education that we assume, which encompasses concepts of the formation (*Bildung*) of the person, of constitution and production of knowledge, more prominently linked to mathematics, has led us to expose our thoughts about the horizons that such activities open to the world of mathematical education and the formation of the person. The aim of this chapter is to describe the distance course conducted with African cultural objects, which were presented and analyzed, both from a mathematical standpoint, and the openness made possible to the culture of the other, highlighting ethnomathematics approaches, mathematical patterns, values of beauty and community, as well as highlighting the anticipated possibilities for us to work phenomenologically with such students.

2 The Course Offered and Themes Examined

First of all, it is important to explain the significance of approaching African culture in Brazil, from the standpoint of the ancestry of many of its inhabitants. Although in the southern region of the country there is a prevalence of people of European descent, primarily German and Italian, there are also African descendants. And, when considering the population of the entire country, the number of people of African descent is estimated at over 50%.

According to https://agenciadenoticias.ibge.gov.br the results of the household surveys carried out by *Instituto Brasileiro de Geografia e Estatística* (IBGE/Census Bureau), at the time of the Population Census of 2010, Caucasians made up over half of the Brazilian population. Since then, black and brown people have exceeded the white population. In 2012, IBGE started asking the residents interviewed to self-declare their skin color in a questionnaire with five options: white, brown, black, Asian, or indigenous. IBGE research shows that, over time, the reduction of white people and the increase of black and brown people in the Brazilian population has been a trend. The data collected revealed that between 2012 and 2016, the Brazilian population grew 3.4%, reaching 205.5 million; the number of inhabitants who declared themselves as white fell by 1.8%; the number of self-declared browns grew 6.6% and blacks 14.9%. In May 2019, the results of the 2018 census were released showing that 19.2 million people self-declared as black, totaling 4.7 million more than in 2012, which corresponds to a 32.2% increase in the period.

The distance learning course described in this chapter was focused significantly on the work of Ubiratan D'Ambrosio, a very important scholar and internationally renowned as the "father" of ethnomathematics, whose work corroborates the views set forth regarding teaching fractals and African objects. He states:

> Ethnomathematics is the recognition that, in every culture, there are ideas and practices that have similarities of objectives, practices, methods and theories, with what, in the Western tradition, we call *mathematics*, is a recent construct (D'Ambrosio & Rosa, 2008, p. 95).

> According to D'Ambrosio's point of view, preparing for citizenship and stimulating creativity are the fundamental objectives of education [… .] Even when practiced in the limited ethnographic style, ethnomathematics helps to build respect of other cultures, hence other individuals, avoiding inequality, bigotry and arrogance (D'Ambrosio & Rosa, 2008, p. 94).

Miarka (2011) presents the explanation of Ubiratan D'Ambrosio regarding the etymology of the word "ethnomathematics" and highlights the basis of the ethnomathematics program:

> In order to explain ethnomathematics, (D'Ambrosio) separates the word into 3 parts: *mathema*, *tics* and *ethno*. Regarding ethnicity, it is considered as culture, involving the communication of a group, the social context, the cosmology of a group, etc. In this scenario, ways are developed to explain phenomena that occur in a group. These modes are called *matema*. Technique, is in turn a way of continuing these ways of understanding. These techniques, which enable the continuity of *mathema*, are called tactics. […] He (D'Ambrosio) considers that the basis of the ethnomathematics program is the tripod composed of respect for each other's wishes and needs, solidarity and cooperation (Miarka, 2011, p. 82).

We understand that working with African culture creations opens up a possibility of dealing with the mathematics present in that culture, and its own expression; with mathematics, understood as science of the Western world, and also work with students aiming to understand the difference in cultures, and ways of knowing and doing mathematics.

So, when contacted, elementary school teachers were willing to study this theme, since the didactic-pedagogical orientations coming from governmental instances require the work with African culture themes in school, and they had no knowledge of activities that could be developed with children and adolescents.

Considering this disposition, we invited mathematics teachers working in the final years of elementary school, through electronic mail, to express their understanding of the development of pedagogical actions in mathematics classes, incorporating African cultural creations. They stated that they did not know how to approach African culture objects in mathematics classes. When asked how they saw the possibility of working with this theme in their classes, their answers were quick, thus showing a lack of reflective thinking about the cognitive and articulation aspects of mathematical knowledge, expressed in different ways, in every culture. We believe that they were seduced by the possible didactic resources which working with African themes could bring to arouse students' interest. The teachers' answers also brought up fragments of political speech which is often given at ideological level, such as the need for appreciation of African creations and respect for black people. Here is what they said:

- The appreciation of African cultural creations is important for the general culture of the students.

- Showing images of African artifacts which are similar to Euclidean geometric objects, shown in the course book, is a strategy to peak students' interest towards mathematics class.
- Colorful forms of African cultural creations can motivate students in art classes; however, in mathematics class such forms have no relation to curriculum content.
- Exotic images printed in African masks and costumes are beautiful curiosities that can be shown in art classes, but do not relate to mathematics content.
- Getting to know African folklore is an interesting theme for geography class.
- The history of mathematics can be explored in mathematics classes, showing the evolution of primitive forms in African crafts to the geometry elaborated in Greece.
- During the school year there is not enough time to teach the whole mathematics curriculum and sometimes it is not possible to examine geometry content in depth.
- The time available in mathematics classes should be spent teaching content which will be assessed by national examinations.
- Themes related to African culture are not mandatory for mathematics classes and should be studied in history, geography, and art classes.
- It is interesting to talk about African cultural creations on *Dia Nacional da Consciência Negra* (Black Pride Day).
- Experiencing a little bit of the life and culture of black people, and all the positive things they have to offer, helps foster respect and arouses students' curiosity about African culture.

Aiming to break with this view expressed by the teachers who focus on a more utilitarian understanding, we proposed to offer at least possibilities of understanding beyond the folkloric and ethical aspects. We decided to initiate a movement to foster understandings about the work with fractal geometry and African objects. To this end, we offered the abovementioned course "Mathematics in African cultural creations."

The course designed and set in motion by us, sought to contribute to the improvement of mathematics teaching following the principles of ethnomathematics, as understood by D'Ambrosio. It was conducted through teaching activities related to contemporary mathematical concepts, developed since the middle of the twentieth century, and creations of traditional African peoples. We believe that it is important to focus on fractal objects and cyber ideas. The former because their thinking is present in systemic thought. The latter, for being present in contemporary works of art, science fiction film, and computer games, for example. Articulating them proved to be encouraging, in different perspectives, which we intend to show throughout this text.

Given the territorial dispersion of the 11 mathematics teachers working at the time, we decided to offer a distance learning course, through electronic correspondence, organized in ten modules. For each module, texts were prepared with explanations of possible articulations of fractal geometry with objects of African culture as well as images of such objects.

Thus, the topics covered in the course were chosen based on the knowledge of aspects of African cultural heritage, examined concurrently with concepts of fractal geometry. This is a current subject almost never addressed in mathematics classes in elementary school. The modules were designed by articulating these ideas. The initial modules focused on fractal geometry concepts, iterative processes, recursiveness, and self-similarity present in artistic artifacts, in traditional African villages and contemporary architectural projects.

We consider it important to point out that fractal geometry is an area of mathematics created in the second half of the twentieth century, and African peoples developed fractal forms in their traditional cultural productions, however, without using this name. In the images of African cultural creations selected to compose the modules of the course "Mathematics in African cultural creations" it is possible to recognize the presence of recursive processes, in the generation of fractal forms, as well as the self-similarity property. The schemes of geometric patterns that clarify the amount of iterations generated by the recursive process were also exposed. The didactic activities presented in the other modules were developed, aiming to incorporate playful activities in the mathematics classes. The traditional African *owari* game and its simplified version were chosen because they involve cyber concepts related to the presence of self-organizing patterns generated by recursive processes. To enable the replication of didactic games, the modules described their construction process with low-cost materials. This way, mathematics laboratories in elementary schools could be created or, if they already existed, they could be updated with didactic activities related to African cultural creations.

The ten modules of the course "Mathematics in African cultural creations" were as follows: (i) Understanding of participants on how to approach African cultural creations in mathematics classes; (ii) The ethnomathematics perspective of Ubiratan D'Ambrosio; (iii) Fractals in the architecture of traditional African peoples; (iv) Exploring fractal structures of African villages through software; (v) Fractals in handcrafted fabrics; (vi) Fractals in African artistic objects; (vii) Fractals in contemporary African architecture; (viii) Exploring fractal structures through software; (ix) Concepts of cybernetics in the *owari* game; (x) African games.

We should clarify that modules (iv) and (viii) were designed to be developed with software. However, such activities could not be carried out because the teachers had neither informatics resources nor functioning computers in their schools.

3 The Conduction Movement of the Course "Mathematics in African Cultural Creations"

In this movement, we saw ourselves transcending the organization and writing of texts, as well as the dynamics of the course in action, and entering into an analytical and reflective thinking about the issues that unfolded, according to our understanding. In this part, we seek to describe this movement, aiming to expose the intertwining of ideas of fractal geometry and objects of African culture, as well as the understanding of education and educational attitudes, which lead to the knowledge of the other, of the different, seeing ourselves also as different before the other, whom we see as different and equal at the same time.

The following describes the modules chosen by the teachers, the issues addressed, and our understanding of what was proposed and what was achievable.

First Module Dealt with the teachers' own understandings, which were presented by the teachers themselves, by answering a request to explain such understandings, taking them as a trigger of what would be treated in the course. Focusing on their accounts, it was possible to have a dialogue with the participants, even though mediated by e-mail, emphasizing African cultural objects, both from the perspectives the teachers themselves had already explained, as well as opening possibilities for advancing towards mathematical ideas and concepts of education that, albeit silently, arise in mathematics classes.

We talked about pedagogical practice, as being the characteristic of teacher action, of any discipline and any area of knowledge. This is because, when teaching, one assumes values, ways of being with the other and with knowledge, that is the focus of teaching, which encompass a vision of education that goes beyond teaching.

Revisiting this understanding expressed by these teacher-students, we focused on the way we understand education and pedagogy.

We believe that pedagogical practice engenders in the triggered action itself the learning of what is being achieved. It brings with it both cognitive actions and the orders perceived in the attitudes assumed by those who are educating/teaching. When pedagogical practice is linked to the action of educating, it is directed towards a goal, defined by the desire to promote the way of being and becoming of the other, of ourselves, of a people. This desire runs through a range of close and distant personal and social purposes. It carries visions of the world and well-being of the person, the community, and society, that speak of the historical-cultural ideals of a culture and, more broadly, of humanity. Thus, it gives a direction to the event that happens alongside teaching and education.

The act of educating requires looking at the other and at ourselves, taking into account the conduction of an action—the act in movement of actualizing oneself, aiming at the becoming of oneself and the other. This coming-to-be is understood as a temporal becoming, whose movement lingers in the now, while the action is taking place, and which shifts to the "has been" and drags to the present moment the "not yet," of what is happening to the person who is the target of educative actions. It is action impregnated with the commitment to "take care" of oneself or the other so that it is somehow already visualized by the subject who triggers the action of educating.

Care is understood by Heidegger as an ontological characteristic of human beings; it is previous to all attitude and situation or a way *of being in the world* (*Dasein*). In the ontic dimension, that of everyday occupations, *Dasein* tends to assume the care as concern or as indifference. This author tells us that each human being always sees themselves as care, in a pre-ontological interpretation. That is, even before conducting all the philosophical interrogation work about the being, triggered by the inevitable questions which we all ask ourselves: Who am I? What am I? Why am I or Why do I exist? They perceive themselves as being care, as concerned care.

This tell us that while we live we are care and that we are *concerned* occupation with what *must be*, understood as much as the weight of the maxim *you must be*, as of *becoming*, of *devenir*, of what *will be*. However, this concern admits various ways of showing itself in actions and reactions. It can be, for example, selfless, impersonal. Why do we say *concerned*? Because, whatever manner the action may be effected and, as it is being effected, the existence of the human being is being actualized, for it is projected into the world, by just being, in some way. The way through which concern is expressed sets a tendency, temporalizes and historicizes.

Now, caring, in the action of promoting education, is caring *how* and *about what*? At first, unknowingly, one might think of a division: we care about teaching, contents, and pedagogical procedures, and have concern about education, anticipating desirable outcomes.

Such separation is misleading and dangerous, as while teaching, one educates. While educating, there is always implicit content, there is an action conducted with such content, which impresses form onto it. There is always a *reason* for teaching *something* in *any given way*. This means that the concern with its consequences is implicit in teaching, whether clearly or not. We are concerned and care about it at the same time. Engaging in teaching is to understand what to teach and how to teach it; is to be aware of the questions "*what is this teaching for*? Why *teach this way*? *With whom will I perform the teaching actions*? *Where will the other and I conduct such teaching actions*?"

These are concerns regarding what will happen in relation to the possibilities foreseen in view of the consequences of interventive action, conducted *with* the other or with oneself[2]. C*are* is ontologically present in this act of taking care, as we will *care* as long as we live. Bicudo (2011) views education

> […] as caring, in the sense of looking after, being together with the other, being solicit, so that this presence is freed to become care, that is, so that it is also in the ontological dimension. It is an attentive being-with, which prevents us from banalizing daily-life in its monotony and in the dealings of public demands, when one is everybody and nobody at the same time. Being with the student entails seeing them, heeding them, thinking and living in a world where we are with the other. It is to live the openness of possibilities of being-in-the-world-with, both taking care and caring. However, never enshrouded by the uniformity and mediocracy of that which is with everyone (Bicudo, 2011, p. 85).

Second Module "The ethnomathematical perspective of Ubiratan D'Ambrosio" brought about the view of that scholar regarding ethnomathematics. We favored this view given the wide range of topics addressed by this author, encompassing issues regarding culture, science, and mathematics education. He presents an important program about "ethnomathematics" (D'Ambrosio & Rosa, 2008) through which he

[2] Here we refer to the Care of the Self as posed by Foucault (1985), for example.

criticizes the teaching of mathematical content separate from mathematical creations of different cultures.

His proposal, as we understand it, reveals a vision that science is historical-cultural thinking that is actualized through the contextualized actions of different peoples, triggered by social and cultural demands, and materialized with materials available in the life-world which we inhabit. The movement of thinking, according to our understanding, is carried out by every living human being, who is in a challenging situation, and who feels compelled to move forward, realizing what is presented to them.

Focusing on the science of western civilization, seen for centuries as advanced and superior to the knowledge produced in other cultures, and examining it under the light of what has been said in the previous paragraph, we move towards the understanding that there is an intertwining of different productions of knowledge, which sustain each other and are expressed in practice. Some of them already theorizing and formalized as a result of idealizations[3], such as those present in ancient Greek culture, for instance. Others directly applied to the conduction of practical activities. Some, as the former, are expressed primarily by written and propositional language, others, as the latter, expressed by artistic language or by the materiality of instruments or other ways of exposing reasoning realized. This perspective of knowledge nullifies a hierarchy of values, put in terms of "the knowledge of this culture is better than of that one," and points to the recognition of differences. However, in terms of school education, we understand that it is not just about showing the difference and beauty of colorful objects and shapes, which sometimes are seen as exotic—a posture that often brings with it the overvaluation of knowledge of other cultures to the detriment of ours—but of entering the movement of contextualized thinking, which is made present, in its own way, in those objects.

In educational terms, this vision opens horizons for understanding the other, seen as a different person from each of us, taken in their uniqueness, or that of the other culture, understood as the historical-cultural making of a people, looking at them as equals and as different. From the point of view of philosophical and cultural anthropology, understanding the same/different complexity is important to transcend the tendencies to look at each other, as in the case of African descendants in Brazil, as deserving compassion and help from the descendants of Europeans, because they are always indebted to them. We are equal and different and, it is only the dimension of this understanding, and of the respect it generates, that it is possible to build a

[3] The constitution of *idealities*, understood in the realm of Husserls' phenomenological philosophy is complex. It entails: the capacity for unlimited repetition of self-evidence, by the act of remembering, constituting an identity of the structure throughout the chain of repetitions, when the idealizing operation, which is consummated with the objectifying act, intervenes. However, even with this act, it is not yet actualized as presented in the Lebenswelt. In order to be so, Husserl points to the act of intropathy and language, which make it possible for the objectifying act to become ideally objective, and, as such, capable of being transmitted and resumed passively by consciousness or in the mode of active production when it is possible for consciousness to intentionally reactivate the original spiritual act.

society of solidarity and mutual respect. Edith Stein worked on this idea of the complexity of the same/different in "*A Dual Anthropology*," mentioned by Ales Bello in "*Tutta Colpa di Eva*" (2017). This dual anthropology defends the idea that the difference between masculinity and femininity is sustained alongside the specific unity of human beings. We bring this idea of the specific unity of the human being, looking at it from the perspective of the understanding of phenomenology, which exposes the complexity of the human being, describing it as constituted by the functional and dynamic intertwining between physical, psychical, and spiritual bodies (Husserl, 2002). This complexity speaks of equality, as everyone: feels sensations and perceives what their gaze inquires; feels emotions, pleasure and pain, joy and sadness, reason; performs spiritual acts that speak of judgments about sensations, emotions, and reasoning, which may be linked and explained in logical sequences and forms. It also speaks of difference. Each individual actualizes feelings, psychical and spiritual acts in a specific way, following their organic tendencies and responding to the demands of the world around them, where they are with others. Likewise, from a cultural perspective, there is a practice common to the constitution of communities and culture which, at its core, comes to be in the historicity of the events that happen to the people who are part of it, being with each other, in different social organizations that co-exist.

Silva (2017) explains that the organization of community life is a complexity that requires the presence of individuality and collectiveness, understood as two aspects of the person's way of life. Ales Bello (2015) *explains* that each individual lives as a person and as a community, but we are not absorbed by the community, because we always remain a personal self. The sphere of intersubjectivity that is established between subjects is the constitutive nucleus of the community; intropathy and language underpin the realization and constitution of this sphere. The acts performed by people with one another, for their duration and the productions materialized, set in motion the historicity of the events and produce a culture which is specific to that community. This is the "how" of the process. The "way it happens" differs in temporalities and spatialities, hence the existence of different cultures and cultural products. Again, the same and the different, that is, the duality.

Ethnomathematics, as conveyed by D'Ambrosio, and accepted by us as the guiding proposition of the course "Mathematics in African cultural creations," is structured with values that converge with the abovementioned view regarding respect for the other, understood as equal and different from me. Aware of the dominant ideologies that emphasize differences, but whose meaning of difference is understood as "different, thus inferior," and also aware of the inequalities and violence that this stance entails, this author presents "Peace" as the first value of his "Program from ethnomathematics."

D'Ambrosio considers that the main stream of ethnomathematics "[…] is restore the dignity of individuals, recognizing and respecting their roots. Recognizing and respecting one's own roots does not mean ignoring and rejecting the roots of the other" (D'Ambrosio, 2001, p. 42). However, as we understand it, it is a movement of knowing oneself and the other, understanding the reality of the life-world, and therefore of the object studied.

The course we designed and set in motion sought to contribute to the improvement of mathematics teaching following the principles of ethnomathematics, as understood by D'Ambrosio, and also by following our understanding of the person and culture as dual; by conducting teaching activities related to contemporary mathematical concepts, developed from the mid-twentieth century, as well as creations of traditional African peoples. We understand that focusing on fractal objects and cyber ideas is important because they are present in contemporary works of art, science fiction films, and computer games, for example. Articulating them with objects of African culture has proven to be a way to understand scientific and cultural roots.

Third Module "Fractals in the architecture of traditional African peoples." The word fractal was coined by Benoit Mandelbrot to describe the irregular shapes of geometric objects related to natural phenomena. He chose the term "fractal" from the Latin adjective "*fractus*," which means irregular or broken. Fractal objects are constructed by adding or removing many parts indefinitely, they display a property known as self-similarity: at all scales each part has a similar appearance to the initial form (Mandelbrot, 1977). Iteration is the recursive mathematical process used in the generation of fractal objects, which can be iterated by geometric constructions or algebraic expressions. The iteration rule can be repeated indefinitely, and when the iterative process of fractal generation is used in architectural design projects, a finite number of iterations are defined and the construction is fixed onto a previously chosen iteration.

Recursive mathematical processes and the property of *fractal* objects, called self-similarity, can be identified in the architectural composition of buildings in Ba-ila, in southern Zambia. "Figure 1" shows the format, created by the process which today is called recursive; in a down-scale application the construction pattern can be described as a ring of rings, also constituted by smaller rings. In these buildings, some "rings" are cattle pens and others are houses where there is the sacred altar. "Figure 2" shows the computationally constructed with geometric scheme (iterative process of fractal generation) highlighting the first three iterations, which reveal the same fractal structure on all scales. The chief's relationship with their people is expressed by the word "*kulela*," which can be translated as govern, but also means breastfeed and nurture. The same word, "*kulela*," applies to a mother who cares for her child and also to the village chief, seen by the people as the "caring father." The spiritual significance of family ties is embodied in the descending scale format of the residences and the architectural layout of the village.

In current mathematical language, one can say that the self-similarity property is present: the shape of the chief's residence is similar to the shape of the residences and floor plan of the whole village (Eglash, 1999).

Studying the geometric shape of the Ba-ila village, one of the course participants reported that, when he sought more information on fractal geometry, he realized the similarity of the African village's geometric layout with the self-similar shapes of the border of the fractal found in the "Mandelbrot set." All participants agreed and some expressed a change of heart. They believed that the recommendations of ethnomathematics researchers regarding approaching other cultures in mathematics

Fig. 1 Fractal design on the facade of the library of the University of Dakar. *Font*: https://structurae.net/en/structures/cheikh-anta-diop-university-university-library

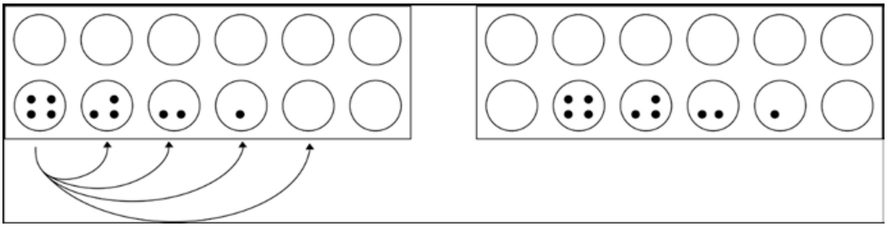

Fig. 2 The "marching group" pattern in the owari game. *Fonte*: designed by the authors

classes would be difficult. However, they came to realize that comparative image analysis makes this approach significant.

The buildings of Mokoulek, a village of the Mofou people located in the Mandara Mountains of Cameroon, are created by stacking stones that cover the region where they reside. Fractal self-similarity is present in the construction of their buildings with small circular constructions and other large circular constructions spiraling within three large stone enclosures that spiral from a central point.

By examining the layout of this building, it is possible to understand that there is a type of "prescription" or algorithm which determines how the system can expand in order to accommodate the growth brought about by the success of agricultural activities (Eglash, 1999). Visualizing the algorithm is an important didactic-pedagogical activity to be conducted with students.

Nankani buildings that follow a self-similar fractal pattern consist of cylinders within cylinders that become smaller as the architectural layout of the buildings seems to rotate from the central courtyard. According to Eglash, this geometric arrangement materializes the movement of life from the womb to the delivery room, then crawling across the courtyard, from the courtyard to the village and, finally, from the village to the world (Eglash, 1999).

By focusing our attention on the Mokoulek and Nankani's figures of the buildings of Ba-ila, we consider that working with their forms, and exploring the operations carried out for their construction stimulates reasoning towards understanding the process of production of the buildings. If the didactic-pedagogical activity focuses on these processes, moving towards understanding the presence of rings, interactivity, self-similarity; exploring their meaning and anticipating possibilities of expressing them through the mathematical language they master, then the attitude of looking at creations of African cultures as folkloric, exotic colors and shapes that wash over us, embellishing mathematics classes, is overcome.

To investigate the living-experiences of these students and teachers, *how* they present themselves, how they pose questions, their attempts and ways of expressing what was understood could lead to the comprehension of constitutive cognitive processes of mathematical knowledge, examining the mathematical practice performed by African peoples, and their production, focusing on the buildings. We understand that by moving forward with the activities of the course, it is important to address fractal themes so that students can move toward the knowledge of the mathematical operations and languages present in contemporary mathematical theories, as well as operations present in computing. Such studies open possibilities to articulate understandings of the mathematical ideas present in the mathematical production of both cultures, highlighting the "equal and different." At the same time, these are important activities for teaching and learning mathematics.

We understand that teaching and educating, while assuming the conception and attitude exposed when working with African creations (or from other cultures) opens possibilities for students to see themselves as unique and different, as being individual subjects and, at the same time, belong to a community and its culture.

Fifth Module "*Fractals* in handmade fabrics." A fractal pattern appears in traditional Fulani weaving: a camel-wool blanket is woven at the occasion of wedding celebrations. Weaving expresses the belief in spiritual energy being intertwined with the pattern employed and with each successive iteration this energy increases. In successively smaller scales, the geometric shapes are arranged in a similar way to the diamonds created in ancient Greece.

Sixth Module "Fractals in artistic African objects." A fractal structure is in a sculpture created by a Mangbetu folk artist, which is a traditional Congolese culture. Four heads overlap, each carved in a self-similar manner on a successively smaller scale. In the geometric scheme drawn on the image of the sculpture, the sequence of squares is constructed through an iterative process dividing the side of each square in half in order to establish the length of the side of the next square (Eglash, 1999).

Seventh Module "*Fractals* in contemporary African architecture." "Fractals have also appeared in large-scale public artworks, such as on the façade of the University of Dakar library" (Eglash, 1999, p. 216). The design of this façade, shown in Fig. 1, was created through the self-similarity property of fractal geometry, in which the alternation of small-scale painted rectangles coincides with the alternation of the volume of the larger building. The sequence of successively smaller rectangles is inspired by fractal patterns of traditional clothing woven by traditional African peoples.

The design of the façade of Lideta Mercato Shopping Mall, designed by Xavier Vilalta, derives from a traditional Ethiopian fractal pattern used in local fabrics. The same architect designed the Melaku Center, a space for teaching and working for the inhabitants of Mek'ele, capital of Tigray, a region in the North of Ethiopia.

Course participants accessed information at the website of the architect, Xavier Vilalta, and enjoyed the images of these two projects. The appreciation of the aesthetic aspect of the projects was commented, without detriment to the spiritual meanings of traditional African peoples. When comparing the geometric forms present in the textbooks adopted in schools, where the teachers-students work, with the geometric forms present in African creations; there was a striking absence of meaning in the former, which originated in ancient Greece, and the appreciation of cultural aspects in the geometric features present in the architectural works which were studied. The approximation of the volume of the buildings with the shapes of objects created by traditional African peoples can be seen in Vilalta's projects.

Ninth Module **"**Cybernetic concepts of the *owari* game.**"** Mankala-type games are found in most African countries and were brought to the Americas during the tragic slave trade period. The game consists of a board with two, three, or four rows of pits carved in stone, wood, or in the ground, where stones or seeds are transferred from one cavity to another. This game has specific names and rules in the various regions where it is popular. The double-row board is used to play in the North, West, and parts of East Africa, receiving various names such as Wari, Oware, Ayo, and Giuthi. In East and South Africa, the most widely known is a game with a four-row board called Bao, Nchuba, and Mweso, and in Ethiopia there are three-row boards (Zaslavsky, 2000).

One of the course participants compared owari to chess, whose board is painted on the tables in public schools all over the region. The rules of chess aim at eliminating the opponent's pieces, whereas in the original owari the player cannot let his opponent "starve" and donates seeds according to the rule that, in case all the opponent's pits are left empty; the player must place seeds inside them when it is their turn to play.

The first rule to move the seeds in the owari game consists in taking the four seeds from one pit, as shown on Fig. 2 placing one seed in each of the contiguous pits.

Eglash (1999, p. 106) described the pattern formed by the disposition of seeds using the cybernetic concept of "marching group," as shown in Fig. 2, and defined "owari as a one-dimensional cellular automaton […] The patterns noted by traditional owari players offer a great deal of insight into self-organizing behavior. The marching group is an example of a constant pattern. In nonlinear dynamics, the constant pattern is called a point attractor."

Owari involves ideas similar to those developed during WWII at the beginning of the *Cybernetics* movement, which investigates the neural mechanisms underlying mental phenomena, trying to express them in precise mathematical language. Analogies between computers and the brain are investigated, and the functioning of the brain examined through mathematical logic. Neuroscientists, mathematicians, social scientists, and engineers research "communication patterns, especially in close ties and networks. Their investigations led to concepts of feedback and self-regulation and, later, to self-organization" (Capra, 1998, p. 56).

The early ideas of the creators of cybernetics originated in comparisons between organisms and machines, but the fundamental difference between cyber machines and Descartes' mechanistic conception was established by Norbert Wiener's conception of feedback. In the last decades of the twentieth century, mathematical tools were created to deal with systemic complexity and, in current mathematical language, a feedback loop corresponds to a special kind of nonlinear process known as iteration. Iterating generally means repeating. Iterative processes happen in several situations. For instance, the process of cell division is repetitive; interest-bearing banking transactions are iterative; the Greeks used iteration to get better approximations for the value of π, geometric iterations produce fractals. In other words, the concept of iteration is related to the topic "compound function," which is dealt with in Brazilian elementary education with little emphasis, because the iterative process is seldom used for the production of modern science, although it assumes a fundamental role in the construction of contemporary science. In many areas of science, the spontaneous emergence of order has been found in complex, seemingly random systems, and the concept of self-organization is now widely used among systemic thinkers (Baier, 2005).

The themes of "iteration" and "patterns," focused on the course, object of this section, are fundamental mathematical concepts for the construction of science aligned with systemic thinking. Since the middle of the twentieth century "a new language has emerged, aimed at understanding the complex and highly integrative life system. Chaotic attractors, fractals, dissipative structures, self-organization and autopoietic networks are some of its key concepts" (Capra, 1998, p. 19). The study of patterns, not significant in a mechanistic worldview, is not valued in school curricula; however, it becomes important in systemic thinking. Today, it has become essential for understanding living systems.

Tenth Module "African games." For this module two books, Zaslavsky (2000) and Cunha (2016), with detailed descriptions of African games, were recommended to the teachers, who were invited to study them and play the games with their students. Only one teacher actually conducted the activities, Geovana Zappelline[4]. After reading

[4]This teacher was also a student of the undergraduate mathematics teacher-training course at

Fig. 3 Children building the game. *Fonte*: Baier (2018)

Fig. 4 Children playing. *Fonte*: Baier (2018)

Zaslavsky's book (2000), the teacher chose eight games from several African countries. She divided the 32 students of her sixth-grade class into eight teams and assigned one game to each team, through a draw. Each team was encouraged to research the most relevant characteristics of country where the game originated; for instance, population, demographic density, culture, traditional dance, and food. The photos in Figs. 3 and 4 show 10-year-old children building the games and studying the respective rules. After that, each team presented the game to their classmates.

In her account, Zappelline explains that working with African themes in classes enabled students to get to know "[…] a little more about Africa, its cultures and some ways to enjoy games." In this teacher's view, "We live in a world where prejudice is widespread; experiencing a little of the lives of black people and all they have to offer, helps promote respect and provoke curiosity about African cultures." We understand from her account that, in addition to children developing activities in which they learned to seek information and interpret it according to the context required and realize the culture of black people, assuming an attitude of respect towards the different, they also developed specific mathematical content, by building and understanding the rules of the games. These are activities that require logical reasoning.

FURB, a public university located in Blumenau (SC), and chose African games as the theme of Practicum V, in 2018, whose advisor was Dr. Tânia Baier, one of the authors of this chapter.

4 Going Back to the Title

We understand pedagogical practice as a characteristic of teachers' actions in any discipline and any field of knowledge. We believe it because, when teaching, one assumes values, ways of being with the other and with knowledge, the focus of teaching, which include a view of education that goes beyond teaching. While teaching, we educate. While educating, there is always some implicit content, there is always an action effected with such content and, which impresses form onto it, and there is always a direction which steers the conduct of the teacher who teaches. This means that the concern with its consequences is implicit in teaching, whether clearly or not. We care and are concerned with both education and teaching, simultaneously.

We understand that to engage in teaching is to realize what to teach and how to teach it; and the question "*why is this teaching important*"?; and the complexity of the way the person (the student *with* whom we are) is—understood in the movement of becoming, thus encompassing the way of becoming; of becoming a person; the historical and cultural context where teaching occurs; of the differences in knowledge produced by different cultures, entering the movement of contextualized thought, which is made present in teaching objects and activities; triggering attitudes of respect towards the other, understood in their dual complexity: equal and different from me.

This view enables us to focus on the absence of a hierarchy of values put in terms of "the knowledge of this culture is better than of that culture" and points to the recognition of cultural and personal differences. In educational terms, this opens horizons for understanding the other, seen as a different person from each of us. Underlying this philosophy is the understanding of the importance of teaching activities being driven by the pursuit of understanding and respect for the same and the different, laying the foundations of a society of solidarity and mutual respect.

The spotlight which lit the path of the course "Mathematics in African cultural creations," designed and set in motion by us, stemmed from the principles of ethnomathematics, as explained herein, and our understanding of the person and the culture as dual. Thus, we understand that focusing on fractal objects and cybernetic ideas is important because these ideas are present in systemic thinking, contemporary artwork, science fiction films, and computer games, for instance. Articulating them with objects of African cultures opens possibilities for understanding different cultures, examining their beliefs, artistic and scientific creations.

Teaching mathematics with fractal geometry, analyzing it from the point of view of mathematics within Western culture, as well as that of the architectural and artistic objects and games, can conduce to learning mathematics itself and foster, in the people involved, respect for each other, seeing them as different but also, and more importantly, as equal.

References

Ales Bello, A. (2015). *Pessoa e Comunidade. Comentário: Psicologia e Ciência do Espírito de Edith Stein*. In Tradução de M. Mahfoud e Ir. J. T. Garcia. Belo Horizonte (Ed.) Artesã.

Ales Bello, A. (2017). *Tutta colpa di Eva. Antroplogia e religione dal femminismo alla gender theory*. Roma: Lit. Edizioni Srl.

Baier, T. (2005). *O nexo "Geometria Fractal—produção da ciência contemporânea" tomado como núcleo do currículo de matemática do ensino básico*. Tese de Doutorado, Instituto de Geociências e Ciências Exatas-Universidade Estadual Paulista, Rio Claro.

Bicudo, M. A. V. (2011). A Fenomenologia do Cuidar na Educação. In A. Peixoto & A.F. Holanda. (coord.), *Fenomenologia do cuidado e do cuidar—Perspectivas multidisciplinares* (pp. 85–91). Curitiba: Juruá.

Capra, F. (1998). *A Teia da Vida: Uma Nova Compreensão Científica dos Sistemas Vivos* (3nd ed.). São Paulo: Cultrix.

da Cunha, D. A. (2016). *Brincadeiras Africanas para a Educação Cultural*. Castanhal: Edição do autor.

D'Ambrosio, U. (2001). *Etnomatemática. Elo entre as tradições e a modernidade*. Belo Horizonte: Autêntica Editora.

D'Ambrosio, U., & Rosa, M. (2008). Um diálogo com Ubiratan D'Ambrosio: uma conversa brasileira sobre etnomatemática. *Revista Latinoamericana de Etnomatemática, 2*, 88–110. Retrieved August 26, 2018, from http://www.redalyc.org/articulo.oa?id=274020253007

Eglash, R. (1999). *African fractals: Modern computing and indigenous design*. New Brunswick, NJ: Rutgers University Press.

Foucault, M. (1985). *A história da sexualidade: o cuidado de si*. Rio de Janeiro: Edições Graal.

Husserl, E. (2002). Idee per una Fenomenologia pura e per uma filosofia fenomenológica. (Vol. II, Enrico Filippini, Trans.). Torino: Einaudi.

Mandelbrot, B. B. (1977). *The fractal geometry of nature*. New York: W. H. Freeman and Company.

Miarka, R. (2011). *Etnomatemática: do ôntico ao ontológico*. Tese de Doutorado, Instituto de Geociências e Ciências Exatas-Universidade Estadual Paulista, Rio Claro.

Silva, A. A. (2017). *A produção do conhecimento em educação matemática em grupos de pesquisa*. Tese de Doutorado, Instituto de Geociências e Ciências Exatas-Universidade Estadual Paulista "Júlio de Mesquita Filho," Rio Claro.

Zaslavsky, C. (2000). *Jogos e atividades matemáticas do mundo inteiro—diversão multicultural para idades de 8 a 12 anos*. Porto Alegre: Artes Médicas Sul.

Familiarity-Strangeness: Movements on Actualization of a Project of Mathematics Teachers Technological Formation

Luciane Ferreira Mocrosky, Fabiane Mondini, and Nelem Orlowski

1 Introduction

The challenge of working with mathematics teachers aiming at their formation (Bildung), under a contemporary perspective in which digital technologies (DT) are present, has been the ground of our professional activities and our analysis and reflections on teaching and learning, in a context which highlights a certain way of thinking, creating, understanding, and producing science.

From the investigations we have been developing in the Research Group on Phenomenology on Mathematics Education (FEM),[1] we started to develop a research project, in 2015, aimed at understanding how the production of mathematics is seen while one is in cyberspace, being-with-computer-and-other media.

[1] http://fem.sepq.org.br/.

Actualization in chapter title denotes "*Verwirklichen*" in German cf. the Index of English Expressions in Heidegger (1951)

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Among all those studies conducted within FEM, the dwelling of teachers stood out due to the strangeness that familiarity with the technological world raises, when we are intentionally focused on understanding teaching and ways of teaching, aiming at students' learning.

In this investigative path, in which reflective thinking is emphasized, technological education was highlighted and remains to be understood. The reason for this is that, in Brazil, such educational programs are rooted on the demands of industrial education, requiring professionals who are capable of knowing what to do with the technological apparatus available in the most diverse productive areas. This tells us that familiarly in technological education is reflected in the architecture of that whose epitome is the machine. This conception is manifested in the current school context, which puts more emphasis on computers, the applications and programs embedded in it, than on the possibilities of thinking while being with digital technologies (Obata et al., 2018).

We understand that technological education has been supported by the pillars of urgency that determine needs and precision, often anchored in the exposure of a product which supports having the devices and, therefore, knowing how to use them. The foregoing conception is strongly evident in the pursuit of (mathematics) teachers for educational programs which enable them to teach using activities and models that are applicable and transferable to classrooms, due to certain teaching contents. In the studies, we further this understanding by allowing openings so that technological education, within the scope of mathematics education, can be shown as something less utilitarian and more as an environment where we are with one another and with-media. However, it still requires attention so that we can transcend the idea of the availability of equipment present in schools.

When DTs are understood as a product, discourses converge on *having* and *using* them: institutions which form teachers *must have* digital whiteboards, laboratories, and other state-of-the-art technological devices, so that teachers can learn how to *use* them and, as a result, address the possibilities of approaching teaching contents using technological devices in the classroom (Mocrosky et al., 2016). Along this path, there is a linear cadence that highlights DT as a product, supporting the formulation of discourse that emphasizes how much its scarcity is disruptive, therefore making it difficult for teacher formation programs to result in pedagogical practice: lack of equipment; and when they exist, lack of maintenance thereof; lack of teacher-formation *(Bildung)* to deal with the technology present in their contemporaneity; lack of pedagogical proposals or lack of understanding of approaches in daily school routine, among others (Mocrosky et al., 2018).

When we look at the technological education through the lens of the formation *(Bildung)* of the person, we move away from defining it as something available to education, just as any educational technology that emphasizes available resources, which may or may not be used.

This journey, which started in 2015, has had developments. In the technological education scenario, we have invested in studies that aim to understand the formative movement *of* and *for* teachers who teach mathematics, in diverse school environments. In this text, we are interested in highlighting the formation and coming to be a mathematics teacher who, among many other things, lives, learns, and teaches in a substantially technological world, considering that the movement undertaken to learn requires the *involvement* of those who are willing to learn, before one can display a possible end product as a development in a given process.

It is true that not having DT in school and not knowing how to use them is a gap for teaching, however, this does not translate the complexity of the theme, since having it and knowing how to use it do not always envisage the horizon of teaching mathematics. In this sense, this text aims to establish a dialogue regarding the construction of paths for teaching mathematics with technologies, assuming the Heideggerian ideas of *dwelling*[2] with DT.

2 Construction Through Letting-Dwell

The construction[3] of knowledge is discussed in the educational academic environment from several theoretical perspectives. Regardless of the approach, there is a consensus that learning requires a path of personal constitution[4] because in the school curriculum the disciplinary ways in which science is organized and its teaching contents only make sense in the learner's commitment. Therefore, constituting and producing knowledge has been at the forefront of the discourse of schools. But what has been built in school and the way it was built requires, an attitude of lingering, just remaining, feeling alive and in harmony with the environment in order to dwell in such construction.

From the beginning of this text, we can argue that we only dwell in what we build and, conversely, those who build foresee the dwelling as its purpose. But we could pose de question: would all constructions be habitable? Heidegger (2002), in *On Being and Time*, discusses the ideas of living and constructing, exemplifying them with everyday situations. Although the text does not address education, such writings provide openings for understanding in the educational context. This philosopher provokes thinking about building and living by saying:

> Bridges and hangars, stadiums and power stations are buildings but not dwellings; railway stations and highways, dams and market halls are built, but they are not dwelling places. Even so, these buildings are in the domain of our dwelling. That domain extends over these buildings and yet is not limited to the dwelling place. The truck driver is at home on the

[2] In Heidegger (1951) dwelling is bound with the idea, from the contributions, of being as appropriation.

[3] "Delving into texts by philosophers and psychologists dedicated to the study of knowledge, these terms indicate various positions, which lead to conceptions about different realities and their knowledge, however, not completely deviating from the meanings pointed out. In constructivism, for example, the underlying idea is that of a construction resulting from the joining of elements or interactions: historical, social, linguistic, etc." (Bicudo & Silva, 2018, p. 156, authors' translation).

[4] According to Bicudo & Silva (2018, p. 157, authors' translation) the constitution of knowledge is actualized in the intertwining of senses subjectively experimented, which merge and enable the perception of an object and its form, made present in the flow of consciousness, which, through its acts, flows in the movement of articulating senses experimented and expresses what is understood, intersubjectively, through language.

> highway, but he does not have his shelter there; the working woman is at home in the spinning mill, but does not have her dwelling place there; the chief engineer is at home in the power station, but he does not dwell there. These buildings house man. He inhabits them and yet does not dwell in them, when dwelling means merely that we take shelter in them. (Heidegger, 1951, p. 1).

With Heidegger one can understand that we dwell in a building, even if it was not built by us, if we understand by dwelling feeling good where we are, as when we are in the coziness of our home. Therefore, the highway was not built by the truck driver, but they somehow dwell in it when their truck makes them feel at home.

This philosopher also tells us that building already introduces the dwelling: "the way in which you and I are, the manner we humans are on the Earth, is dwelling" (Heidegger, 1951, p. 3). To build is to dwell, and to dwell is the way that we, mortal beings, are on Earth from birth to death (Heidegger, 1951, p. 280). By being on Earth with other mortals, we co-belong to Earth, and as such, we aim to build. But we build only when we linger *in* and *with* things; when they remain present, because they have been carefully designed in the possibilities of building housing; this means that only when we are able to dwell, we can build. Building is at the core of dwelling, which, in turn, is the feeling of peace given from a shelter, remaining peaceful "in the freedom to belong, to be in the world" (Heidegger, 1951, p. 5).

> The bridge hangs "lightly and forcefully" over the river. The bridge not only links preexisting margins. It is only at the bridge crossing that the margins appear as margins. The bridge lets them rest in their own way, facing each other. Through the bridge, one side separates from the other. [...] The bridge is ready for the inclemency of the sky and its ever-changing essence, both for the calm and joyful flow of the waters, and for the churning of the sky with its harsh storms, for the melting of the snow in torrential waves, crashing down over the gap among the pillars. Even there, where the bridge covers the river, it keeps the current facing skyward by receiving it at the opening of the arch and therefore releasing it again (Heidegger, 1951, p. 5).

We dwell when our body and thoughts intertwine with a place, which we build *during* and *for* the becoming of our existence; when *we feel at home*. Therefore, the bridge is a construction, in a certain space and time, that allows us to inhabit. We dwell when there is harmony between us and what is built.

When we bring the verb *dwell*, based on Heideggerian works, to speak of DTs in forming and being formed as persons—from a perspective of mathematics education—we intend to present them, as already said, as a *being there* and feeling alive and in harmony with life and the environment. In this sense, we are assuming an attitude different from that of utilitarianism.

Dwelling can be seen in the light of what has been built, as the possible result of being exposed to a community, or, moreover, the way of cultivating the growth of what is being built, of what is always on way, on the move: development and involvement are faces of a way of dwelling.

Dwelling primarily identifies the habitual, of what is frequently present and is in the framework of the tradition of human dealings. Immersed in the habitual, humans go on paths, often without questioning things or interrogating themselves about them. The familiarity we have with things leads to a human way of dwelling. However, the familiar, the habitual, bring openings for strangeness, which happens when, in the experience with the familiar, something bothers us, provokes us, and calls us to look beyond appearances. This perplexity puts us in a state of alertness, awareness, attentiveness, and therefore strangeness *with* and *in* things; it is laid out as a condition for openness to knowledge, to let-dwell.

When we are provoked, in the strangeness, we can perceive the world from multiple perspectives, giving rise to different forms and actions, or even possible ways of continuing formation. With Bicudo (2003), we understand that such formation consists in the uniqueness of teaching, shaping a way of being a teacher, so as to always be in search of the condition of being in ideal form, revealed by the real teacher. A challenge that is imposed at every moment during our professional lives.

In the daily life of teachers, strangeness comes with questions, many of which emerge from teaching and reflecting on what was taught and as well as from the academic discourse that points to possibilities. In this context, the technological formation of a teacher moves between what is familiar and what is strange. However, it is in the *being there* thinking about the work done and about what was planned and actualized[5] that the possible meanings of strangeness gains prominence and is projected into understanding.

Therefore, in our research about technological education with mathematics teachers, we highlighted paths for the teacher to dwell in their own professional practice.

3 Digital Technologies and Teacher Formation: Constructions on the Way

By considering dwelling as cultivation for the growth of a work, so that the construction happens and maintains its presence, we highlighted the movement of DT in the formation and being formed of mathematics teachers, focusing on the historicity of teaching of mathematics in Brazil.

Posing the question "What is being built when technology and mathematics teaching are at the backbone of teacher formation?" we investigate this subject and we found out that it has permeated teacher mathematics education.

For Borba, Scucuglia, and Gadanidis (2014), the theme technology in mathematics education can be synthesized in four phases.[6] According to the authors, the

[5] Actualization as Baumann (2013) says of a creative action that, when performed, makes present possibilities present [...]. Such, updating is understood here as a movement to make current, to make the ideas present [in the project] through actions that may or may not innovate. By dismembering the word, we can reaffirm that updating is the action of making current.

[6] The first phase included the use of LOGO software, which enables the construction of geometric objects. The second phase, in the mid-1990s, fostered the creation of educational software with the spread of personal computers, thereby stimulating the offer of teacher education for the use of these technologies. In the beginning of 1999s, with the advent of the internet, a new phase was outlined, becoming mainly used as a mode of communication between teachers and students. Finally, the fourth phase is characterized by authors as that of the fast internet, the presence of various modes of communication in cyberspace (Borba et al., 2014).

concern regarding the promotion of technological education for teachers and research on such theme emerged in the 1990s. Since then, more and more researchers have been interested in investigating the use of technologies in *teacher-formation (Bildung)* and it is in this scenario that research in mathematics education, which takes technologies into account, became relevant.

In pre-service formation programs DT have been included in research in the field of mathematics education, pointing in the direction of an intertwining between pre-service formation and DT. So, future teachers have the possibility to experience DT during undergraduate studies. By working in this way, it is understood that DT are not only resources in disciplines, but are present throughout the curricular organization of teacher certification courses.

In this regard, Richit & Maltempi (2010) point out that DTs favor the specific, technological, pedagogical formation, and other aspects of pre-service formation, warning that the practice of thematic projects with software cannot be restricted to a single experience, or to the context of one discipline. There is a need for students and teachers to take ownership of DT and the different ways they can be used in mathematics teaching and learning, so that they can promote pedagogical actions with technology. The prevalent idea in the authors' discussions reveals technology as an opportunity to seek new ways to learn mathematics.

In the same direction, Motta (2017) states that DTs, in pre-service formation of teachers, promote significant classroom impacts and create an environment that provides students with new ways of thinking and acting, therefore stimulating the development of their own learning in an active process. The author also considers DTs as tools for meaningful learning, which is understood as the opportunity to provide new ways of thinking and acting.

The abovementioned authors corroborate the discussions of Bairral (2007), who, in his research regarding the use of DTs, shows two key moments: the emphasis on how students learn in virtual environments and the studies in which technology is made available as a learning tool, independent of the teacher's action. The author highlights this second moment by describing studies he has developed with students, in which there was *autonomy to learn* by positioning themselves as authors, in the production of mathematical knowledge.

Bairral (2015, p. 487, authors' translation) considers it fundamental that research clarifies how they conceive the "individual-technology-learning relationship and how the cognitive-discursive aspects are considered in a formative setting." According to the author, the perspective of humans-with-media (Borba & Villareal, 2005) allows us to overcome the dichotomized understanding of technology and humans, as it emphasizes these two aspects in the construction of knowledge in an articulated way. For Bairral (2015), the production of knowledge entails changes of a cognitive nature, showing that research needs to problematize epistemological aspects. What is explained by the researcher draws attention and illustrates the importance of thinking about the meaning of epistemology in the context of teacher formation in a technological perspective.

Therefore, researchers tell us about pre-service formation of mathematics teachers, from a technological perspective, that technologies have appeared as a possibility for learning and building mathematical knowledge. Similarly, to these authors, we think that technological formation is already present in the way a future teacher will think about mathematics and, consequently, how they will deal with it. Such researchers tell us of dwelling as a possibility in which the disciplinary modes of mathematics present themselves in teacher certification courses, bearing in them the technological possibility, revealing themselves as a formative aspect that is fundamental to the future teacher.

We focus on in-service formation and we conducted a survey about studies that deal with this theme. We found out that they revealed other aspects.

Bittar (2011) analyzes the change in teachers' practice when they start working with DTs. She investigates possibilities of integrating technology with teacher practice and asks *what it means* for a teacher to be prepared for the insertion of technology in their classroom practice. She argues that both initially and in the context of in-service formation, the teachers need more than knowledge and mastery of technological resources (computers, tablets, etc.). She states that "technology must be used for the purpose of giving students access to properties or aspects of a concept; or even to mathematical activities that are different from those usually addressed in a paper and pencil environment" (Bittar, 2011, p. 159, authors' translation).

Her research proposes the *incorporation* of technology into the pedagogical practice of mathematics teachers and, based on psychology theories seeks to investigate the possibilities of integration, that is, when teachers create ways to work with technology in the classroom, aiming to foster the learning of students.

In the same direction, Penteado (2000) had already shown that technologies require significant changes in pedagogical practice, both in operational questions (such as space, time organization) and epistemological questions "(such as the production of new meanings for the content to be taught)" (Penteado, 2000, p. 23, authors' translation).

In this scenario of constant change, we understand teacher formation in a technological perspective, in which we highlight the *insertion* of technology in teacher practice and, with it, the need to create ways of teaching and learning mathematics.

This reveals an epistemological dimension of change, as if there were an orientation towards the ways through which teachers come to dwell in the technology in their teaching practice. This move to start dwelling shows a teacher who, like the truck driver, did not build the road, but because they travel on it, moves in search of places, and things. The road reveals pathways, however, treading the road creates senses. The journey can tell us if we are just travelling on the road built or taking the time to understand its layout, bifurcations, the surroundings of places that we see in the landscape visited and mind the direction of the traffic, letting ourselves dwell in the to-and-fro that leads to certain places, between departures and arrivals.

This understanding is a reminder that even on sign posted roads, with mindful drivers, there are limitations that make it difficult to travel. About the complexities of travelling towards the technological formation of the mathematics teachers, Borba and Penteado (2010) state that it is necessary for the teacher to allow themselves to leave their comfort zone, guided by what is familiar in the pedagogical routine, to enter a risk zone, given that DTs bring unpredictability, uncertainty, eventuality of technical problems, the loss of control that generates indiscipline, in short, quite a lot of strangeness. The authors provoke, exposing difficulties, including those that challenge teaching knowledge. They return to these aspects and advance them by proposing that the unknown and the risks are accepted as horizons for teaching, lessening the limitations that have been denouncing gaps and shortcomings in a computerized school environment. This change of view of technology requires that teachers open up to the new, "possibilities for [...] development of students, development of teachers, development of teaching and learning situations" (Borba & Penteado, 2010, p. 66, author' translation).

In the wake of the reflections exposed by Borba & Penteado (2010), Rosa et al. (2012, p. 99, authors' translation) announced Cyberformation, emphasizing the transformative formation of teachers with DTs. The purpose of the researchers is that teachers think with technologies, "in order to reflect/discuss problem situations related to the mathematical concepts in question, in order to transform/foster the processes of in-service formation." From this perspective, they point out that teachers are invited to analyze the potentialities of materials (tasks or resources made with DTs) for the production of mathematical knowledge and, in the same formation movement, they were encouraged to develop their own materials in line with the reflective process of thinking with DTs.

The perspective shown is that of formation with technology, through perspective of collaboration, production of knowledge and creation, surpassing the idea of use, which, although still present in the scope of the research, has been amplified by the needs which the use itself demands, such as collaboration that boosts the production of knowledge that, in turn, enables the production of ways for working with technology in classrooms.

With these authors, we understand the perspective of production through which production and product occur concurrently. The production and the result are constituted together, and it is not always possible to distinguish the point in which one has a finished product and at which point it is used. The use itself correlates with the Heideggerian mode that is made explicit as *pro-ducing*.

> Pro-ducing is used to differentiate production in its Greek meaning from the one that occurs within modern rationality. This means the production that has at its core the "seeing, contemplating what manifests," leading the becoming, from which it stands out by the representation given by the materialized deed, which uncovers things, in order to put it into a dominant position of use, forcing us to go beyond the possibilities, actualizing the "make being." Therefore, "pro-duction" means to bring forth, to make appear, having in itself the initial movement of what is effected in order to dwell in what results from that movement (Mocrosky & Bicudo, 2013, p. 411, authors' translation).

As stated by the researchers, pro-ducing describes a construction that is sustained in the dwelling of human possibilities and not only of the realization that is exposed. This means that evidences of the need to build are part of the productive movement, "that is, of what has already proved to be emerging to be done and what has the possibility of becoming" (Mocrosky, 2010, p. 294, authors' translation), so that "letting-dwell" is the goal.

The expression *letting-dwell* is borrowed from Heidegger, and it brings us the understanding that building encompasses dwelling, although it is possible to understand both as needed in a relationship that binds them, where building and dwelling come together as relation of means and end. Heidegger warns that such an understanding does not reveal the essential relations between these two dimensions, so that the way of thinking that is hinted here must show, on the other hand, that thinking, just as building, belongs to dwelling.

> Building and thinking are, each according to their nature, essential for dwelling. But both are also inadequate for dwelling as long as each concerns itself with its own affairs in separation, instead of listening to one another. Building and thinking are able to listen if both, belonging to dwelling, remain within their limits and know that one, as much as the other, comes from the workshop of a long experience and continual practice (Heidegger, 1951, p. 15).

Indeed, for the philosopher, inhabiting tells us about the possibilities of men to understand themselves as being-in-the-world, that is, in their own condition that they are in the world and such condition includes cultivation, that means, the protection. Therefore, inhabiting is about letting us be together with things, in the sense of being at home where being together with the things themselves reveals what is proper to existence. Following this argumentation, we ask: to us what is proper to the existence in the context of technological formation?

This foray in the realm of mathematics education research and teacher formation with technology leads us to initially think that classroom technologies, as well as the research on the subject, are developing harmoniously. However, the question of lack of structure is still a relevant aspect in the discussion of DTs in teacher formation and, consequently, in school. This argument is supported by the research of Javaroni & Zampieri (2015) when their research group inventoried the studies that had been carried out by GPIMEM—Research group on informatics, other media and mathematics education (*Grupo de pesquisa em informática outras mídias e educação matemática*), seeking to map the structural conditions of schools in São Paulo, based on Silva, Medeiros, and Morelatti (2014), Chinellato (2014), in addition to the research carried out by Paulo & Firme (2014).

Javarone's study revealed systematic carelessness and lack of investment, given that pedagogical guidelines are formulated but, most of the time, there is no continuity of the proposals or public policies in Brazil. These are punctual projects and actions that, over the short span of their lives, lose support and physical and human resources which would require continuous financial investments. This is still one of the perspectives in which the phenomenon of technology is shown when questioned in the context of teacher formation and in schools. This perspective is closely connected to the idea that teachers still resist the presence of technologies in the classroom. In a way, this ends up configuring the teacher as the one who denies, who refuses to dwell in this technological formation context.

> At this moment, dwelling takes a primordially temporal meaning of coming together or becoming used to something, or yet in the sense of cultivating; it is a form of closeness that is defined, above all, by the permanence with loved ones in general, a permanence that gives to the being-in-the-world a kind of protection, a certain familiarity based on the comfort of

> what is always the same. Habit, habitual, and inhabiting, however, are not separate from their other being: the unfamiliar, the foreign, the uninhabitable, the "nowhere" that distresses, and erupts in the core of something we imagined knowing (Saramago, 2000, p. 74, authors' translation).

In a meeting with the researchers whose work was studied, we proposed thinking of technological formation as a search for what is resisted, that is, thinking about the *feeling of not being at home*. Thus, through the study of those authors, resulted the idea that technological education is primarily structured in the production of knowledge *in* and *for* teacher formation, based on letting-dwell, as strangeness-familiarity.

4 Dwelling: Strangeness and Common-Unity

Although technology often proves to be an instrument for producing something, we understand that more than knowing technology, one must understand oneself with it in order to go beyond having it and knowing how to use the equipment. Therefore, we move to another instance of knowledge, namely, to use what we have or what we are about to have for pedagogical purposes. This knowledge has geared schools and teacher formation, establishing a cadence on having, knowing, and using.

While *having* and *learning* are verbs that often indicate the acquisition of physical and human resources, when humans use the physical domain, a durability is shown as letting us be in what is achieved by having and knowing. As one of the characteristics of a technological society is the speed with which things happen and change, the *time of change* is naively revealed as the fundamental feature of society.

With Ubiratan D'Ambrosio, we thought: "technology, understood as the convergence of knowing (science) and doing (technique), are intrinsic to the solidarity search to survive and transcend" (D'Ambrosio, 1999, p. 159, authors' translation). Therefore, we are directed to understanding that formation with technology (technological formation) could come from thought that aims at unifying the formation of mathematics teachers and technology in order to understand ourselves as human and, in this sense, reclaim the aspect of being formed, as opposed to *having formation for*.

We believe that the sense of the technological world is hidden, as well as technological education. While the formation of mathematics teachers has been shown by *having* (to be trained for), the meaning of being (being formed) is veiled. We understand the meaning of being, as well as Detoni (2014), who, based on the Heideggerian work, explains.

> Methodologically, Heidegger questions the meaning of being, and questioning itself becomes an investigative task. All questioning is a search. [...] the conscious search can turn into "inquiry" if what is questioned is liberatingly determined [...]. I interpret this characteristic of freedom as an exercise of the researcher occupying possible openings in the horizons that are being made in what they investigate, but also in themselves, since they are the being who is questioning. It is inalienable that those who ask are shown as beings. In

> Heidegger's words, we do not know what being is. But, even when we ask what "being" is, we keep ourselves in an understanding of the "is" (Detoni, 2014, p. 54, authors' translation).

Understanding technology requires an understanding of temporality. According to Bicudo (2003), the question of time has always been implicit in man's way of being: "it is an ontological characteristic of the human way of being," so, time as temporality (ways of being in time), indicates the consubstantiality of time and space; in other words, it indicates that our ways of dwelling in the world, our spatialization, can be understood as "living time."

To explain what is understood by living time, the researcher introduces different concepts that guide the understanding of space and time. And, in this research, highlights that to understand living time is relevant to the presence of the "observer, time and space in the constitution of the universe," in which the existence of a homogeneous and unique time would not make sense. Based on these explanations, the researcher clarifies how such way of viewing reality, "constituted by the web weaved by the observer/time/space," is in accordance with the conception of reality and phenomenological knowledge. That is to say, in phenomenology, the primacy of knowledge is the perception that is always given through profiles, perception of the living-body in the life-world, and that is expressed by language. Language which is articulated in discourse, and at the same time explains and communicates, that is, contributes, along with entropathy,[7] so that "the intersubjective knowledge is constituted, which as it is historically and culturally communalized and accepted, becomes objective" (Bicudo, 2003, p. 35, authors' translation).

In face of these considerations, we envision the possibility of examining the dynamics of the movement of the constitution of knowledge among teachers, aiming at formation under a technological perspective, or technological education, bringing more elements to our discussion, such as collaboration, common-unity, and openness to what is new. As explained by Paulo and Ferreira (2016, p. 318, authors' translation), we understand "the potentially communicative cyberspace as a locus in which communication is expanded, common disposition must be shared, revealing the being-with," i.e., as the possibility of experiencing the expression of what is understood in the movement of constitution of knowledge about mathematics, its teaching and learning with teachers in *forma(c)tio*.

For Heidegger, building and inhabiting are part of the same movement. Then, we delve into understanding the constructions on the way to make the ideas our home,

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[7] Silva (2017), when investigating the question of the constitution of the *community* based on Ales Bello, defines intropathy as an "experience in which the other is perceived by the self," that is, the knowledge of the other occurs in the experience of intropathy. This experience occurs immediately when we perceive the other. The researcher also describes that, according to Husserl, intropathy is one of the most important acts that are experienced and that differently from the act of perception, requires a perception, but is not limited to it; it links experiences that have already occurred (memories, recollections) and brings the "becoming-present"—the presenting. Intropathy entails the act of distinguishing between animate and inanimate beings, bringing the possibility of understanding the other as living (with a psychical life). Therefore, "Intropathic action is the origin of the possibility of communication that together with language constitute the core of intersubjectivity" (Silva, 2017, p. 28, authors' translation).

contributing with other faces of the buildings started, now we stand in the way of building to expose the product. In any case, they are perspectives in which the produced is a construction that is sustained in the inhabiting of human possibilities and not just of the exposed realization. This means that the productive movement includes evidences about the need to build, "that is, what has already emerged to be done and what it has the possibility of becoming" (Mocrosky, 2010, p. 294, authors' translation) aiming to "let-dwell."

Then, following our arguments, we ask: What may be imminent in terms of technological formation? How to let-dwell?

Heidegger (1987) interprets the meanings of teaching and learning, in the context of what it means to think, that is, to accept the calling to the task of teaching to what is proper of teaching: learning.

> Learning is harder than teaching; therefore, only those who can truly learn—and only to the extent that they can—can truly teach. The true teacher differs from the student only because they can learn better and want to learn more authentically. In all teaching it is the teacher who learns the most. The hardest learning consists of embracing effectively and deeply what is there to be known and what we have always known. Such learning, the only one to which we indulge here, requires us "to let us being in" permanently on what is apparently closest, for example, in the question "what is a thing?" We constantly ask the same evident inutility, from a utilitarian perspective: what is a thing, what is a utensil, what is a man, what is a work of art, what is the state, what is the world (Heidegger, 1987, p. 80, authors' translation).

Therefore, we think of *Bildung* (formation) as a potency to move teachers to create spaces in which technology opens itself to understanding, beyond utility, bringing temporality, guided by the meaning it has in the life-world. One needs to dwell in the technological world, and therefore digital technologies no longer are, but become possible to be. One has to be in the technological world, which means to move so that the movement constitutes it, so that it can be understood in it as presence (as opposed to the western way of being of representation, of calculated and technical deeds), to be open to let-dwell. The letting-dwell that calls for cultivation, whose original meaning is related to culture.

The word culture, in turn, comes from Latin, it is a verb and means the *act of planting and cultivating plants or carrying out agricultural activities*. Over time it began to mean *cultivating the mind*. This term originates from another Latin word *colere*, which means *taking care of plants*. We propose to think about cultivating, harvesting, the potential what one plants. The seed carries potential. A seed carries the potential to spawn what has been planted. Human beings are beyond potential, they are possibility. Being one thing not just by chance; it must be planted. When we plant, we cultivate actions, thoughts, there is an opening to possibilities.

By understanding formation as caring for oneself to care for the other, we consider that a core trait of formation is care. Care that the educator should have to offer something to others, but primarily the care that the person needs to take of themselves. If we are not open to care for our own formation, however cautiously the other prepares something, there will be no openness. It is about receptivity, openness, longing. It is cultivation. To cultivate the possibilities of allowing oneself to dwell in the technological formative context, in which the guiding sense is *being* technological and not *having* technology.

The imminence of technological formation for mathematics teachers has been announced. However, other questions are posed which require further investigation: What do we intend to cultivate technologically in mathematics education? What kind of culture of technology formation is under way?

References

Bairral, M. A. (2007). *Discurso, interação e aprendizagem matemática em ambientes virtuais a distância*. Seropédica: edur.

Bairral, M. A. (2015). Pesquisas em Educação Matemática com Tecnologias Digitais: algumas faces da interação. *Perspectivas da Educação Matemática, 8*, 1–21.

Baumann, A. P. P. (2013). *A atualização do projeto pedagógico nos cursos de formação de professores de matemática dos anos iniciais do ensino fundamental: licenciatura em pedagogia e licenciatura em matemática.* Tese de Doutorado. Instituto de Geociências e Ciências Exatas—Universidade Estadual Paulista, Rio Claro.

Bicudo, M. A. V. (2003). A Formação do Professor: Um Olhar Fenomenológico. In M. A. V. Bi-cudo (Ed.), *Formação de Professores? Da incerteza à compreensão* (pp. 7–46). Bauru: EDUSC.

Bicudo, M. A. V., & Silva, A. A. (2018). Análise de vivências em situação de constituição de conhecimento. In B. Catarina et al. (Eds.), *A prática na Investigação Qualitativa: exemplos de estudos* (pp. 158–178). Aveiro: Edicão Ludomedia.

Bittar, M. (2011). A abordagem instrumental para o estudo da integração da tecnologia na prá-tica pedagógica do professor de matemática. *Educar em Revista, 1*(1), 157–171.

Borba, M. C., & Villarreal, M. E. (2005). *Humans-with-media and the reorganization of Mathematical thinking: Information and communication technologies, modeling, experimentation and visualization*. New York: Springer.

Borba, M. C., & Penteado, M. G. (2010). *Informática e Educação Matemática* (4nd ed.). Belo Horizonte: Autêntica.

Borba, M. C., Scucuglia, R., Silva, R., & Gadanidis, G. (2014). *Fases das Tecnologias Digitais em Educação Matemática: Sala de aula e internet em movimento*. Belo Horizonte: Autêntica.

Chinellato, T. G. (2014). *O uso do computador em escolas públicas estaduais da cidade de Limeira/SP*. Dissertação de Mestrado, Instituto de Geociências e Ciências Exatas—Universidade Estadual Paulista “Júlio de Mesquita Filho”, Rio Claro.

D’Ambrosio, U. (1999). A História da Matemática: questões historiográficas e políticas e reflexos na Educação Matemática. In M. A. V. Bicudo (Ed.), *Pesquisa em Educação Matemática: concepções e perspectivas* (pp. 97–115). São Paulo: UNESP.

Detoni, A. R. (2014). O estar-presente à distância: possíveis contribuições de Martin Heidegger. In M. A. V. Bicudo (Ed.), *Ciberespaço: possibilidades que se abrem ao mundo da educação* (pp. 93–108). São Paulo: Editora Livraria da Física.

Heidegger, M. (1951). *Building dwelling thinking* (Adam Bobeck, trans.). Retrieved December 10, 2019, from https://www.academia.edu/34279818/Building_Dwelling_Thinking_by_Martin_Heidegger_Translation_and_Commentary_by_Adam_Bobeck_

Heidegger, M. (1987). Que é uma coisa? In *Tradução de Carlos Morujão* (2nd ed.). Lisboa, Portugal: Edições 70.

Heidegger, M. (2002). *Ser e Tempo* (10th ed.). Petrópolis: Vozes.

Javaroni, S. L., & Zampieri, M. T. (2015). O Uso das TIC nas Práticas dos Professores de Matemática da Rede Básica de Ensino: o projeto Mapeamento e seus desdobramentos. *Bolema, 29*(53), 998–1022.

Mocrosky, L. F. (2010). *A presença da ciência, da técnica, da tecnologia e da produção no curso superior de Tecnologia em Fabricação Mecânica*. Tese de Doutorado, Instituto de Geociências e Ciências Exatas—Universidade Estadual Paulista, Rio Claro.

Mocrosky, L. F., & Bicudo, M. A. V. (2013). Um estudo filosófico-histórico da ciência e da tecnologia sustentando a compreensão de educação científico-tecnológica. *Acta Scientiae, 15*(3), 406–419.

Mocrosky, L. F., Paulo, R. M., Mondini, F., & Orlovski, N. (2016). Sobre precisão e necessidade: um pensar acerca da tecnologia e educação matemática. In M. A. Kalinke & L. F. Mocrosky (Eds.), *A Lousa digital e outras tecnologias na Educação Matemática* (pp. 171–185). Curitiba: CRV.

Mocrosky, L. F., Mondini, F., & Orlovski, N. (2018). A quem interessar possa! In R. M. Paulo et al. (Eds.), *Ser professor com tecnologias: sentidos e significados* (pp. 43–58). São Paulo: Cultura Acadêmica.

Motta, M. S. (2017). Formação inicial do professor de matemática no contexto das tecnologias digitais. *Contexto & Educação, 32*(102), 170–204.

Obata, J. Y., Mocrosky, L. F., & Kalinke, M. A. (2018). Tecnologia, Educação e Educação Tecnológica: herança e endereçamentos. *Revista de Educação Ciência e Tecnologia, 7*(1), 1–22.

Paulo, R. M., & Firme, I. C. (2014). O Programa Acessa Escola: um Espaço para Atuação com as TIC. In *Aprendizagem online: anais do 3° Congresso Internacional das Tic na Educação Lisboa*, 14–16 de novembro de 2014. Retrieved January 28, 2019, from http://ticeduca2014.ie.ul.pt/downloads/AtasDigitais/Atas_Digitais_ticEDUCA2014.pd.

Paulo, R. M., & Ferreira, M. J. A. (2016). Comunicação no Ciberespaço: diálogos acerca da Matemática. *REVEMAT, 11*(Edição especial), 256–267.

Penteado, M. (2000). Possibilidades para a formação de professores de Matemática. In M. C. Borba & M. Penteado (Eds.), *Informática em ação: formação de professores, pesquisa e extensão* (pp. 23–35). São Paulo: Editora Olho d'Água São Paulo.

Richit, A., & Maltempi, M. V. (2010). Desafios e Possibilidades do Trabalho com Projetos e com Tecnologias na Licenciatura em Matemática. *Zetetiké, 18*(2), 12–23.

Rosa, M., Pazuch, V., & Vanini, L. (2012). Tecnologias no ensino de matemática: a concepção de Cyberformação como norteadora do processo educacional. In *anais do 13° encontro Gaúcho de Educação Matemática*, Lajeado, 2–5 de junho de 2012.

Saramago, J. (2000). *A caverna*. São Paulo: Companhia das Letras, São Paulo.

Silva, E. C., Medeiros, D. O., & Morelatti, M. R. M. (2014). Avaliação dos laboratórios de informática das escolas estaduais de presidente prudente no contexto do programa acessa escola. In *anais do 1° Seminário Internacional de observatórios de educação e formação*, Porto, 15 de março de 2014.

Silva, A. A. (2017). *A produção do conhecimento em educação matemática em grupos de pesquisa*. Tese de Doutorado, Instituto de Geociências e Ciências Exatas—Universidade Estadual Paulista, Rio Claro.

Part IV
Constitution and Production of Mathematical Knowledge Being with the Computer

Interactivity as a Way to Express and Learn Mathematical Ideas About Change, Dependency, and Restriction

Orlando de Andrade Figueiredo

1 Introduction

There is something fascinating and appealing about interactive devices, and this is the starting point of this exploration. We could go beyond mere curiosity and acknowledge that there is a serious and profound epistemological principle underlying interactive experiences. The interactive media must be acknowledged as way to express a distinct modality of knowledge, with its own specificities. Dynamic geometry has been well studied and discussed in mathematical education classes (see chapter "Understanding Phenomenologically the Constitution of Knowledge When Working with Dynamic Geometry" in this book), so, here we highlight other applications that, while having their own characteristics, have some points in common with dynamic geometry. We expect that the analysis herein presented will also cover, in that it clarifies and explains, dynamic geometry. These applications are Graspable Math (Weitnauer et al., 2016) and DynaLin (Figueiredo, 2010). The first application provides a playful and interactive context for learning the transformations of algebraic equations in order to resolve them (elementary school). The second application is a kind of DynaGraph (Goldenberg et al., 1992) designed to assist the teaching and learning of linear transformations in Linear Algebra (higher education).

To support these propositions, we will refer to a first person, epistemologically. In other words, the reader finds here a description of a perspective on the phenomenon, and such perspective highlights it. Therefore, it is not, in principle, a thesis or theory about the experience.

The flow of ideas begins with a discussion of the Graspable Math application and the pedagogical principle that can be inferred from its use (Sect. 2), and which is in line with the proposition defended that interaction is a qualitatively unique

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experience. Next, the DynaLin application is presented as another example to reinforce the previously highlighted points (Sect. 3), especially the Rank–nullity Theorem of Linear Algebra. Triggering the movement supporting the asserted points, an explanation of the framework of the first-person knowledge is given, especially the perception of movement, whose motto is the experience with kinetic sculptures (Sect. 4). Dependency and interactivity deserve an exclusive section (Sect. 5). Finally, open questions which the research yielded are summarized (Sect. 6).

2 Dynamic Algebra Notation Systems and the Teaching of Equation Solving

Graspable Math is a software application used for teaching to solve algebraic equations. On Graspable Math, the user transforms the equation with mouse actions, and the application responds by authorizing the transformations that are compatible with algebra definitions. It is hoped that, by observing what is or is not allowed by the application, equation transformation procedures are recognized and learned. Embedding the procedures for transforming equations into an interactive device is a special way of highlighting such procedures, therefore evidencing them for students. This pedagogical approach takes advantage of the features of interactive devices.

The technology itself is full of procedures. Every hardware and software have their procedures for use. Initially, there were manuals that came with these devices to show users how to operate them. A manual provides a static discourse about the operation of a device. Over time, some classes of technological devices, such as games, for example, have abandoned the idea of a manual in favor of learning what is embedded in the use of the technology itself. Game designers use the strategy of creating early stages of the game for pure exploration of their rules. There is an effectiveness in the learning that takes place interactively and through online observation of responses of the device to the actions of users. This approach can also be seen in mobile and web applications. User interfaces are designed to be as self-explanatory and as intuitive as possible. A feature in this regard is the use of balloon help, which give tips on functionality. Even if none of this worked, there is always a natural inclination for people to learn to use devices by trial and error, and by observing other people using them. This can be found in everyday situations, such as the handling different remote controls, increasingly present in our everyday lives, or even in the operation of mobile applications, also abundant and varied.

If technology uses its own interactive feature to make itself understood as a set of procedures, then why not apply the same principle to mathematical procedures? This is a way to understand the Graspable Math application: Its designers want to *express* their knowledge about handling equations in the form of constraints that are perceived in the answers that Graspable Math gives to actions on the equations handled that are represented there. This fact is the common thread for a greater understanding, which is the objective of the present essay: that a unique form of knowing is manifested in interactivity. Moreover, as a consequence, interactivity is a special way of expressing formal ideas. These are observations that deserve more recognition from the scientific community in the areas of epistemology, cognition, and education.

The authors of Graspable Math classify it as a dynamic algebra notation system. The term is very appropriate, as its main functionality is the movement of mathematical symbols. Figure 1 illustrates the operation to solve an equation in a hypothetical dynamic algebra notation system. In the framework of human–computer interaction (a sub-area of computer science that addresses the study of interfaces between humans and computers, among other things), an application such as this uses the mechanism of *direct manipulation*. While direct manipulation interfaces are desirable on the one hand, they contain some design glitches. Graspable Math designers came up with good interface solutions for direct handling of the terms of an equation and coefficients, including the operations associated with the distributive property, such as the "highlighting" operation. There is also a feature to review the process as history, which is similar to the process with paper and pencil. However, not all operations are easily solved by direct manipulation, such as applying the formula to quadratic equations. In this case, as well as many other cases, the designers chose to provide the user with a menu of operations, which can be opened to select a complex procedure to be used for the equation.

Starting from experiments with dynamic geometry (implicitly assumed here) and dynamic algebra notation systems, I gradually take this direction for grounding the interaction as a cognitive horizon on its own terms. Inevitably, this discussion arises imbricated in an educational perspective. In part, the problems of education can be formulated in terms of *intersubjectivity*. I will return to intersubjectivity later on, but, for now, it is enough that we paraphrase as follows: the problems of education can be formulated in terms of communication and expression between teacher and student. In other words, it is possible to obtain excerpts of educational situations in

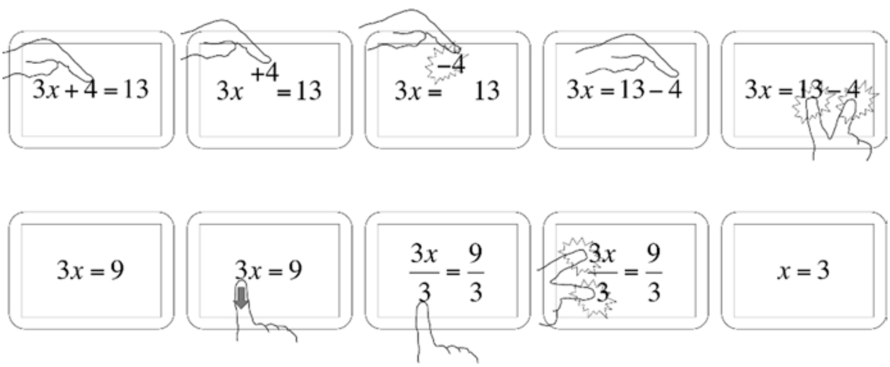

Fig. 1 Example for solving an equation on a hypothetical dynamic algebra notation system

order to find the teacher busy trying to make themselves understood, in reaching a point where they are convinced that they have had an encounter with the student. This meeting of subjectivities is never clear, because there is no way to compare subjectivities, at most, there is evidence of points of contact. When someone thinks about something, there is much more going on with them regarding such thought than they themselves realize. If they try to express themselves, the first resort is language in its various forms, from vernacular, through mathematical symbols, body language, voice intonation, and other modes of expression. There are also other elements that the subject of knowledge themselves may not consciously recognize. Knowledge is a network of subjective experiences. We find, in mathematical education literature, similar positions, such as the notion of *conceptual image* proposed by Vinner (1992). The author seeks to understand the cognition of mathematical concepts, such as mathematical functions, and seeks to criticize some widespread understanding among educators that precise mathematical definitions are sufficient to account for the learning of concepts. Vinner (1992) shows that a student articulates several experiences beyond the formal definition of concepts. The author wants to introduce educators to the reality that the formal definition is just a visible facade of something much broader that happens with a student, which is the conceptual image. And my purpose here aligns with this idea. What I propose is a form of becoming aware of the tacit and pre-predicative cognitive landscape, and the recognition of a modality, or quality of knowledge, which is the knowledge pertinent to interactive experiences. This is the main movement that carries with it another movement: that we can use this modality of knowledge (moments of interactive experiences) as a resource for expression (intersubjectivity).

In the experience with interactive devices, we can highlight elements such as change and response. An interactive device, due to its very nature, changes in some way. And, it often changes in response to the subject's action. There is a subtlety about the change and the response in interactive devices that deserves to be stressed right now, and refers to the notion of representation. A sign (for example, a word or a symbol) acts in place of that which it represents. The sign is the representation of the thing that is represented. The word "sky" alludes (points us) to the experience of the actual sky. The word "sky" is an index. Likewise, the symbol "4" is a representation of an abstract idea, the quantity of four. The sign has this property of representing something, often something that is absent; I speak about the night although it is day; the night is not here yet, however, that does not prevent me from evoking it through the sign. When an interactive device, in turn, changes something about itself, it does not represent the change, because its change is never an absence. The change that occurs within the device is a *presence*. If there are 4 rectangles on the screen, and there is a change to 5 rectangles, the experience of the change itself was not represented, it was experienced. Representing would be to create a language of symbols for a change, such as a musical score, or the drawing of the steps in a choreography, or even use language, as I am doing now.

There is a third element in the experience with interactive devices: the experience of *constraint*. Constraint manifests itself, in the response of the device, as an impediment to some degree. That happens when I try to drag a geometric point in the dynamic geometry application, but it is not free for any movement; when I try to handle an equation variable in the dynamic algebra notation system according to a rule that does not match what I had in my mind, and nothing happens. These are not representations of constraints, but constraints themselves. And the math teacher who teaches algebraic equations often aims to help the student avoid certain mismatching operations, that is, they want to communicate constraints in the process of solving the equations. In traditional teaching, this is done through lectures and by correcting exercises. With dynamic algebra notation systems, all this abundance of information is embedded in the interaction constraints of the application. It is reformulated in terms of another form of knowledge.

3 DynaGraphs

DynaGraphs were proposed by Goldenberg et al. (1992) as a dynamic representation of mathematical functions to be applied in mathematical education. Figure 2 illustrates the basic shape of a DynaGraph, made up of two horizontal bars, each with a pin that can be moved (or moves) along its length. The user drags the piece, representing x, left or right and sees the coherent response of the piece resting on the numerical line above, which represents the value $f(x) = x(x - 2)$. The idea of mathematical function manifests itself here by observing the linked movement between moving parts. There is an original and naive experience of *dependency and causality* in the most trivial events in life, and, with DynaGraphs it would be possible to bring mathematical ideas into this space. Goldenberg and Cuoco (1998) see dynamic representations as a resource to support the development of complementary images of mathematical function.

DynaLin is a special case of DynaGraph, applicable to the teaching of Linear Algebra in higher education (Figueiredo, 2010). It is a representation of linear transformations from plane to plane. In the computational interface (Fig. 3), one sees two flat areas, side by side. The hollow circles indicate the origins of the planes. The black dots are the pins. You can move the left pin and see the effect on the position of the right pin, which moves concomitantly. On the toolbar, at the top of the interface, there is a selector of linear transformations. The program is preloaded with some linear transformations. One can see the formula that defines each one.

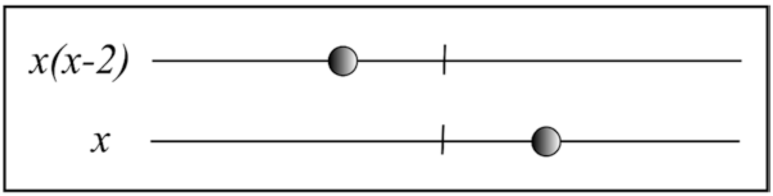

Fig. 2 A DynaGraph. As the part on the x axis is moved, $x(x - 2)$ it moves accordingly (Adapted from Goldenberg & Cuoco, 1998)

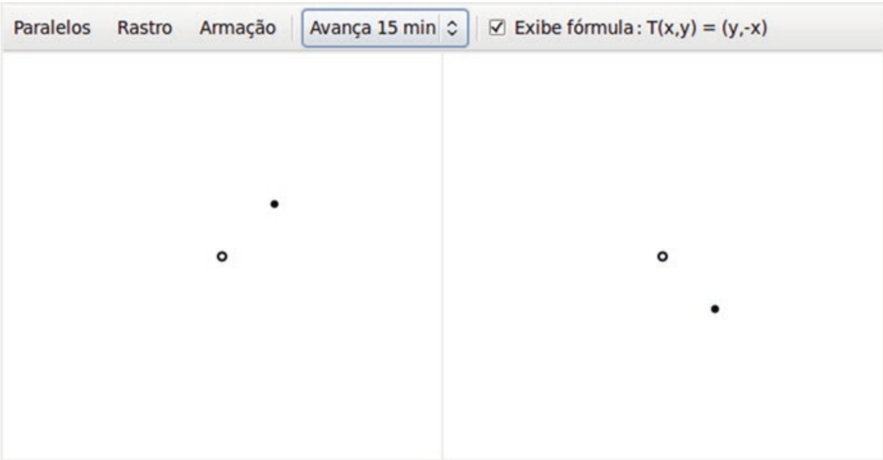

Fig. 3 DinaLin in normal mode. On the left side is the representation of the domain, on the right side, the contra domain (Figueiredo, 2010)

Additionally, they are given suggestive names such as Advance 15 min, Mirror and Hill. The objective is that, through experimentation, the student will be able to make comparisons between these linear transformations. The characteristics of linear transformations should be especially prominent, as the student explores the movement of the pins for each of them.

There are some supporting tools that can be activated in the interface in order to assist the observation of certain properties.

- The "Frame" tool is used for observing the property of linearity.
- The "Parallels" tool is used for observing the eigenvectors.
- The "Trace" tool is used for observing the kernel of linear transformations.

Linearity is best understood in the presence of nonlinear transformations. This is why we included the "Take Off" transformation (which is nonlinear) in the set of preloaded transformations. The activation of the "Frame" tool brings new elements to the interface (Fig. 4): the movement pin receives the label u and a line segment appears, which connects it to the origin; a second movement pin (v) with a line segment; and a third (non-movable) pin representing the vector sum of the previous ones; in the codomain, besides the images of the previous three, there is a fourth pin, which represents the vector sum of the images $T(u)$ and $T(v)$. A transformation T is linear if, and only if, for any u and v vectors with any k scalar, both properties are valid: $T(u + v) = T(u) + T(v)$ and $T(k \times u) = k \times T(u)$. The focus of the tool is on the first property. In cases which the transformation loaded in the system it is linear, the pins corresponding to $T(u) + T(v)$ and $T(u + v)$ coincide along the entire path of movement of pins u and v.

The intuition associated with the notion of a linear transformation eigenvector is that the original vector and the transformed vector have the same direction, that is,

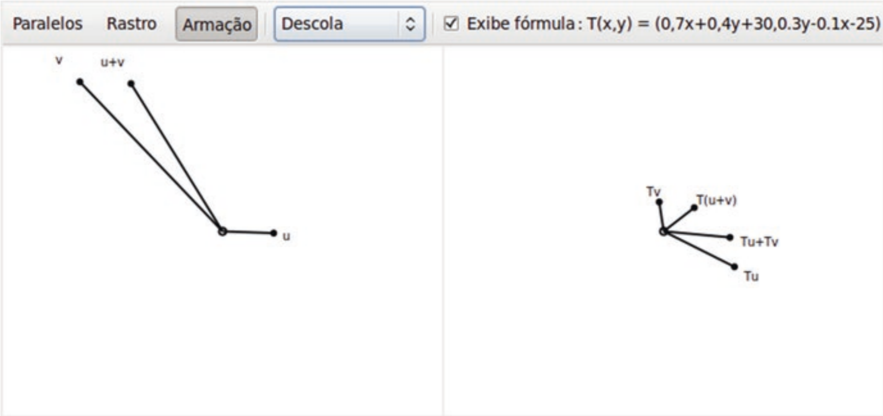

Fig. 4 The transformation “Take Off” when the “Frame” tool is on (Figueiredo, 2010)

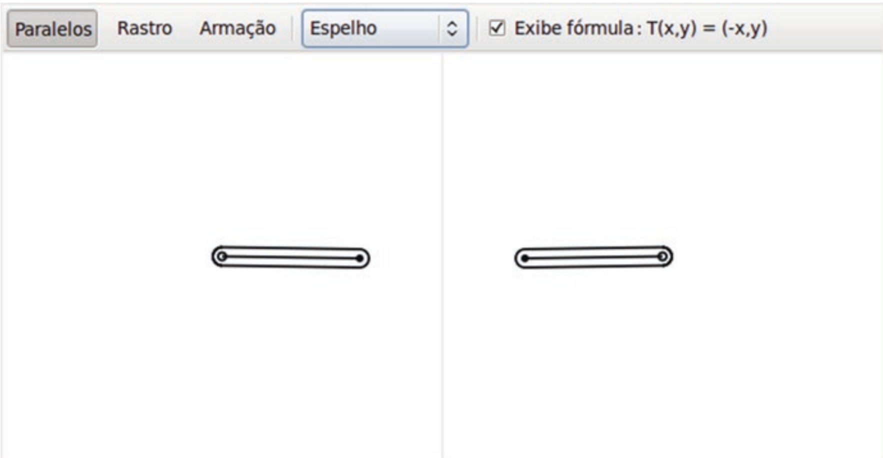

Fig. 5 The “Mirror” linear transformation when the “Parallels” tool is actuated (Figueiredo, 2010)

they are parallel. To explore this property through experimentation, students must press the “Parallels” button, thereby enabling a device that displays straight line segments connecting the two pins to their respective origins (Fig. 5). Additionally, there is a special indication when the line segments are parallel (the halos surrounding them in the figure). In some cases, such as “Advances 15 min,” the line segments are never parallel. In others, such as the “Mirror,” the parallelism of the line segments only takes place in some situations.

The kernel of a linear transformation is the set of all the vectors in the transformation domain, whose images result in the null vector. All transformations have a kernel, but in some of them it is only the origin of the domain (in this case we say the kernel is trivial). With the help of the “Trace” tool, the purpose is to explore

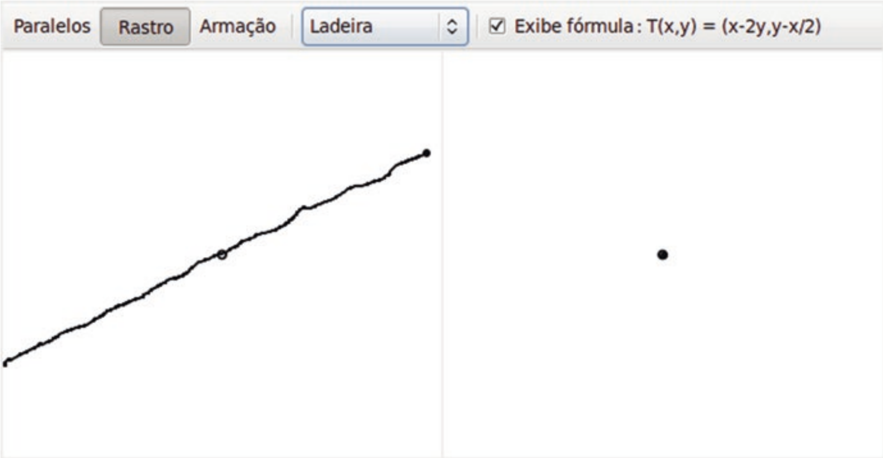

Fig. 6 Result obtained with the "Trace" tool and the Hill transformation (Figueiredo, 2010)

linear transformations that have a nontrivial kernel. In order to do so, one must scan the domain, seeking to ensure the image remains at the origin. The "Trace" tool (Fig. 6) enables us to record the traversed region (the kernel).

Exploring linear transformations with DynaLin has the potential to foster learning in two respects. First, by the empirical observation that nontrivial kernels are straight lines. This is a fact that emerges and should cause surprise, opening the possibility for the teacher to make the connection with the theory, which explains and predicts such fact. Second, the Rank–nullity Theorem, which also emerges naturally. According to that theorem, the kernel dimension of the linear transformation and the image dimension are linked, since together they always add up to the dimension of the domain. In the case of DynaLin transformations, whose domain and codomain always have dimension 2, the only interesting case is when the linear transformation kernel has dimension 1 and the image, according to the theorem, also has dimension 1. The finding that the image is also a straight line, that is, that the movement of the pin in the right plane is restricted and confined to a straight line that goes through the origin, is also a fact that should emerge to students' perception and arise their curiosity, creating an opportunity for discussion about the theorem on the part of the teacher.

The emergence of the Rank–nullity Theorem in the design and use of DynaLin is the relevant fact for the main discussion proposed here, as it answers the question: Is there a qualitative difference in the assimilation of mathematical knowledge when it is approached with the support of a DynaGraph? This points out that the way of learning functions through interactive devices is unique in some regards, and is it realized in the application of DynaLin? The following discussion leads to a positive response. And it is anchored in my own testimonial as the tool designer, and who have undergone a change in the way I understand the Rank–nullity Theorem.

Exploiting nontrivial kernels was an expected feature in the process of creating DynaLin. However, the observation that the image of linear transformation in these cases is also confined to a straight line was not expected. I became a subject of the research I was developing. And I went through a qualitative change in the way I understand the Rank–nullity Theorem, to the point that its justification became evident ever since. This new perspective, which clarified the theorem, is not within the scope of its formal demonstration, it took place in another cognitive modality. I can summarize such justification as follows: during the experiment I had a strong *sense of projection*, that is, while I moved the pin freely in the domain and observed the response restricted to a straight line in codomain, I recalled my previous experiences with function graphs and their axes. One aspect of these experiments is the loss of dimensionality, that is, projecting a whole plane on an axis implies mapping entire straight lines into points. Particularly, there must be a straight line to be projected at the origin, and this is precisely the kernel. In other words, the existence of a nontrivial kernel is a consequence of the projection on a single point (the origin) for the whole set of points (the kernel), and this, in turn, is the consequence of the effect of an overall projection of the whole plane (domain) over a straight line (the image), in a loss of dimension, which is precisely quantified in the theorem. Prior to performing the activity with DynaLin, when the theorem presented only algebraically to me, this loss of overall dimensionality and, as a result, of the kernel, was not evident; only within the cognitive space of things that move interactively, and as a result of this quality of knowledge, that this came to the foreground and became apparent.

4 First-Person Knowledge

In the previous sections, we came into contact with the basis of this research, that is, didactic situations of mathematical education in which interactive tools favor different modalities of enunciating and learning mathematical issues. We also began devising the aim of the present essay: understanding how interactive experience is a form of knowledge in itself. And, if we can convince ourselves of this, then we will have laid the epistemological basis for the use of the dynamic geometry, of dynamic algebra notations systems, and DynaGraphs in mathematics education. Then, the methodological issue becomes paramount. How can one find this special knowledge shown in the interaction? In order to do this, we need to step back and ask ourselves: How do we understand knowledge in general?

The first hints of a methodology for approaching knowledge have already appeared in previous sections. We have seen a broadening of the *description* of the panorama of experience in a comprehensive perspective, from a particular way of being aware of that experience. This differs fundamentally from theorizing, conceptualizing, proposing theses, for example. It is, therefore, about opening ourselves to the possibilities of getting in touch with our own experience. It is less a finding of new facts and information, than a finding within ourselves of different attitudes and skills about what we have experienced in our daily lives. For example, we want to be able to turn to the whole of the experience and, in an analogy, as in a symphony, focus and bring to the foreground of our hearing a particular instrument, that is, a particular quality of knowledge, in this case, knowledge that is particular to the interaction.

In the context of philosophy, the use of the expression “first person” is commonly associated with phenomenology. It is emblematic that phenomenological texts are partly concerned with representing and refounding phenomenology itself. I have set off to do something similar in a modest way. What I call here “first-person knowledge” is a personal formulation of some ideas that are inspired by phenomenology. This “first-person knowledge” is the epistemological and methodological foundation of the present essay. However, I want to point out that it is a work in progress, a personal philosophical program in its early stages, and thus, and also due to space limitations, in this chapter, I will focus more on merely introducing the framework of “first-person knowledge,” rather than justifying it and relating it to phenomenology texts. I hope, however, that this mere introduction will fulfill the role of drawing attention to the great methodological and epistemological differences between the schools of thought existing in philosophy.

The terms “first person” and “third person” alone say a lot. The dominant orientation in common sense, and even in science, is that of the third person. A contrast between the third-person orientation and the first-person orientation can be pointed out through the contrast between the phrases: “The world” and “The world for myself,” or “This is the way the world is” and “This is how my being unveils the world in me now.” In first-person orientation, the subject mandatorily seeks to be aware of their own cognitive movements, which make the panorama of cognition happen. In other words, the subject seeks to be aware of their own cognitive processes and includes them in the landscape. In an analogy, it is as if the photographer could step back and make the camera itself appear in the photograph; it is as if, in a movie, the cameras were shown, along with the director, the counter, and everyone involved in the shooting. It is not a trivial maneuver. Perception is special among cognitive processes. Addressing first-person perception sounds like a paradox, something like “perceiving perception.” This is a great source of difficulty in epistemology. To address perception as an objective thing is to fall into a basic prejudice. When Merleau-Ponty (2014) says that the world is what we perceive, people may even stop for a moment to admire the rhetorical strength of this proposition, and then to go on with their lives without operating the profound transformation to which it invites. If indeed they had paid attention to the philosopher, they would have operated disruptive epistemological transformations. The first-person perception is misunderstood by almost everyone, almost an arcane. Following Merleau-Ponty’s hint that we do not live the world taken objectively, but the world perceived, then we can understand the reason for the difficulties. Perception paints pictures of this world we live in, the perceived world, and hides in it. Perception does not *want* us to be aware of its presence. It creates the show and withdraws as much as possible. There is, therefore, a technique of first-person perception that consists of seeking out experiences in which this disguise fails, even if just a little.

The elements of first-person knowledge are:

- primacy of subjectivity;
- recognition of intersubjectivity;
- introspection or attention to one's own experiences;
- experiences of exceptionalities;
- broadening of the notion of *qualia*.

The subjective nature of the experience is underestimated. And in order to start recognizing this, one must question the opposite pole, which is, objectivity. At the root of this picture is the fact that we tacitly extend to the *other* the world we live in. The "world in which I live is the same where you live" is not a theoretical, ethical, even sociological, cultural, or anthropological position—it is something even deeper and more essential to our human nature. It is taken from this most original recondite of being, which, in the end, is the target of first-person knowledge, the pre-predicative aspect of existence. In other words, the vague notion of objectivity that manifests itself in everyday life is a loss, a "myth." Almost everything that appears to us in our experience, contrary to what an initial and superficial intuition suggests, is only for us. Tacitly, we expect that it will be the same way to the other, but we do not realize that it is not. Discussing subjectivity has long been critical to philosophy. The novelty proposed here is to place the issue in the movement of extending one's subjectivity to the other.

A recurrent contradiction to the idea of subjectivity is to identify it with solipsism, that is, a form of isolation of the cognizant subject. First of all, there is a precariousness at the formal (logical) level, such as when you deny the sentence "all balls are blue" with the sentence "no balls are blue" (you go from one extreme to the other, concealing other possibilities). Second, and most importantly, we *know* that we are not isolated, that humans are in contact with each other is a reality; the issue is to shed light onto this realm of experience and deepen the understanding of how this happens. Certainly, as the myth of objectivity points out, it is not in the most transparent way. Phenomenology brings notions such as *intersubjectivity* and *empathy* to designate the cognitive mechanisms shown in the analysis of the phenomenon of being together. Opposing the notion of transparency (in the sense that the other is completely open to me) is that of *opacity*. The world is opaque, the other is opaque. This also involves the notion of perspectival knowledge. There are always several perspectives regarding the phenomenon, and one should not consider the simultaneous experience of perspectives, on the contrary, an experience is marked by the predominance of one perspective. Additionally, perspectives are always revealing themselves; it does not even make sense to speak of "all" perspectives, because our being cannot cope with exhausting the possibilities of the phenomenon being seen. This is particularly noticeable regarding the other. The following account illustrates the opacity of the other. It is a hypothetical but credible formulation. Someone has had an accident in the past with a paperweight. In the presence of a paperweight, this person still experiences, even if in the background, some kind of bad memory and discomfort. It is possible that the person does not thematize this. If this person and I are in the presence of a paperweight, such memories are completely opaque to me. Although I project onto the other the same paperweight experience which I have, there is a difference that I cannot even suspect. Perhaps, over a long period, our relationship might afford opportunities for such memories to be shared in some way, perhaps in a conversation. We are naturally inclined to extend our subjectivity to the other, forgetting about the differences, but on the other hand, in everyday situations, we may intuit that there are differences. If we observe with focused and intentional attention, we will find that, in everyday experience, some instances when different ways of being are felt. For instance, when we have difficulty making ourselves understood in a discussion, or when we operate more or less easily something that the other has mastered in a different way. The root of the difficulties, as mentioned earlier, lies in the fact that we naturally forget these differences. Education can be understood as an effort of intersubjectivity by the agents of the educational process. However, the positions are asymmetrical, in the sense that the teacher is expected to take the initiative to bring new knowledge to students. This purpose is within the scope of intersubjectivity.

We seek to understand interactivity as a form of knowledge in itself. And to do so, we enter this field of first-person knowledge. Getting in touch with this *landscape of cognition* presents its own difficulties and challenges. First, because one may have the impression that one speaks of something foreign and alien, when, in fact, it is the same experience here and now, the place where we have always been and from which we never departed. But, above all, it is a matter of completely transforming this experience, as appropriate ways of observing it are discovered. Specifically addressing the difficulties in this, first of all is the fact that the cognitive schemes with which we deal in everyday life are of no use in this stance, they can no longer organize this landscape, on the contrary, they themselves are the source of difficulties. In fact, *they are part of the landscape*. We intend to operate a cognitive maneuver that corresponds to taking a step back even from our most basic cognitive schemes, with which we organize our everyday experiences. Our cognition did not evolve to favor cognitive schemes that would allow us to handle cognition itself. What we have are cognitive schemes for everyday life, and now we need to use them to–indirectly, by similarity, by analogy, by projection, by extension—address cognition. Examples of basic cognitive schemes are: the notion of inside/outside, the notion of overlapping layers, the notion of cause/effect, etc. Analytical philosophy is the approach where reflection allows itself to be subjected to these schemes without thematizing them. It is our cognition trying to use schemes as rulers to frame the experience. However, not thematizing the one's own schemes is an absolute lack of rigor. This adverse effect is found not only in common sense, but also in the sciences and third-person philosophy.

We have already addressed subjectivity and its improper association with solipsism. Subjectivity is also wrongly associated with a vague notion of *interiority*. Here, we have the inside/outside (interior/exterior) scheme trying to prevail over cognition itself. The key point here is to properly address the notion of *exteriority*, and, therefore, of the world. It is possible to find a propitious perspective in which the interior/exterior dichotomy is overcome, and the loose end for us to begin with lies in exteriority. Part of the confusion lies in the fact that we are not able to discern between the *thing* and the *perception of the thing*. (To be precise, we really only have the perception of the thing, that is, the thing itself does not exist for us.) An excellent exercise for experimenting these possibilities is to thematize the experience of color. Why is it important to bring the discussion about interiority? Speaking of *introspection*, a word that unfortunately, by construction, became strongly associated with the notion of interior. First-person knowledge is an exercise of introspection, but not *any* introspection. It is not a *withdrawal* from experience, retiring from it, but rather a *plunge* into it. It is a skill that one exercises to modulate their own experience. And that comes from learning and discovering new perspectives. Such discovery of a skill is analogous to the moment when a skater or skier learns a new trick for the first time; before that moment they try and fail, until an insight occurs and they enter a new condition. In the first-person knowledge framework, these skills are a form of knowledge, a form of knowing. The specific introspection referred to here is, in itself, knowledge, and it is the access to the root of all knowledge. It is methodology and epistemology. And to put it bluntly, this section, from beginning to end, is about this method of introspection, an action of seeking to realize what is going on within us.

The landscape of experience changes as we thematize it. We have a superposition (i.e., simultaneity, like a *symphony*) of: perceptions of things, sensory qualities, cognitive schemas, space-time intuitions, mental speech utterances, ongoing interactions with other people, and other actions, including even the pre-predicative experience. And all of this takes place in temporal flow. Gradually, we have to dive and immerse ourselves in this perspective. We no longer rely on the cognitive scheme that is able to isolate things, to separate them clearly. There is no sharpness. At most, there is a *putting in evidence*, bringing to the foreground, as the listener who seeks to discern a musical instrument in a symphony. We have, above all, the glance, the peek, the apparition. We have no contours, we have no edges, we have the overlap of these things without a contour and without edges. This strange world here and now is the same experience as ever, though in the first person.

Perception is a special character in the landscape of the first-person experience. It is the loose end to start pulling the thread. In particular, optical illusions and other perceptual exceptions deserve attention. It is regrettable that these moments are addressed as mere curiosities and nothing else. A serious researcher should not trivialize the exceptionalities of perception, quite the contrary. During the experience of an optical illusion, a certain tone of precariousness is present. It is as if something that is not well determined is defective, it is wrong. By a strange mechanism, we naturally put all this aside, we sweep it under the rug, and go on with life. An indispensable exercise is to bring normal and exceptional experiences to the same level. That is, doing the exact opposite of what this terminology (normal/exceptional) invites us to do. In the presence of an optical illusion, we know that what we perceive is not reality. In ordinary life we identify perception with reality. We need to extend the precariousness that is established by the optical illusion, and thereby learn to step back, during a normal experience, to have our own perception in front of us. In the optical illusion, the misperception stands out. We need to achieve the same effect in the ordinary experience.

The difficulty with having a foreground perception in the landscape of cognition and the flow of experiences is that perception is the horizon from which everything springs. This seems to be an impossible cognitive maneuver, an unreachable perspective. A popular analogy for this is to say that fish in water do not realize that they are in water. This is why Merleau-Ponty sought the misalignment so much; as a kind of exceptionality. As we have seen, the elements of the landscape of cognition are overlapping, or in other words, aligned. We have a perfectly matched whole, and we want to highlight perception. The philosopher proposes the experience of observing a human face upside-down. He reports strangeness. This is an experience of the exceptional kind. Some way perception tries to fit this experience into the frame of a right-side up face. The eyes, which are now below, precariously come to resemble a mouth. And the mouth, which is above, the eyes.

> If someone is lying on a bed, and I look at him from the head of the bed, the face is for a moment normal. It is true that the features are in a way disarranged, and I have some difficulty realizing that the smile is a smile, but I feel that I could, if I wanted, walk round the bed, and I seem to see through the eyes of a spectator standing at the foot of the bed. If the spectacle is protracted, it suddenly changes its appearance: the face takes on an utterly unnatural aspect, its expressions become terrifying, and the eyelashes and eyebrows assume an air of materiality such as I have never seen in them. For the first time I really see the inverted face as if this were its 'natural' position: in front of me, I have a pointed, hairless head with a red, teeth-filled orifice in the forehead and, where the mouth ought to be, two moving orbs edged with glistening hairs and underlined with stiff brushes (Merleau-Ponty, 2014, p. 263).

The reader is required to experiment and live this strangeness. No description, no expression in language can replace this experience as a modality of knowledge. This experience is knowledge, and it is irreducible to forms of self-expression, just as a discourse on color does not replace the experience of color (given the difficulties of talking about colors with the visually impaired). Moreover: like any knowledge process, this form of knowing is subjective, but by becoming aware of the philosopher's experience of bewilderment through his account, and discovering something that inexplicably matches our own experience, we establish a bridge between subjectivities, an encounter. I recognize myself in the philosopher's anecdote. Introspection placed us within the perspective of subjectivity, showing that the distance from the other is greater than what is commonly acknowledged, and we now experience an approximation, which is consistent under the light of this epistemology. But let us go back to the effort to make the perception of the face clear in the foreground. In our everyday experience of a face, various cognitive elements operate in alignment. There is the perception of the face itself, the rational knowledge of what a face is, among others. In the upside-down experience of a face, a dissonance from the normal experience arises. On the one hand, we have the information and also the rational deduction that the face is upside-down. On the other hand, we have the perception of the face trying to establish something else. The perception of the face, thereby, dawns on us, even if at a glance, in snapshots. This is an opportunity to discover a new repertoire of perceptions: the perception of the perception of a face. If this new cognitive posture is learned, like a skater who, by touching, trying, and missing, learned the spinning jump, now, we can find it again in our everyday experience. We can use this new ability to discern the face perception into an exception and apply it to our everyday experience. This is fundamental, and because we aim at interactive devices, I illustrate what it would be like to apply this procedure to the perception of movement. Dutch artist Theo Jansen is the author of a series of kinetic sculptures called *Strandbeest* (Jansen, 2016), a Dutch neologism to denote “beach creatures.” These are huge articulated assemblies that, driven by the wind, move as if of their own accord. Jansen has developed his own mechanical joint design technique. Figure 7 shows a kinetic sculpture composed of many legs. As it moves, the sculpture brings up the image of a row of marching soldiers. Or a giant legged bug. The experience as a whole is one of surprise and fascination. For a moment, the sculpture seems alive. Videos with these sculptures are popular. The observation that these works of art arouse my interest, as well as that of other people is, again, an encounter, an indication of an inter-subjective bridge between me and the artist and other admirers of his work. Above all, the observation of the *Strandbeest* kinetic sculptures places our perception in a situation of exceptionality. The perception of being alive and the perception of orderly movement (marching of soldiers), which are usually the beginning of the experience with animals or military parades, are seen here in an unusual situation. On the one hand, one knows, for various reasons, that it is a contraption, an artifice; on the other hand, due to the unusual movement, the realization of a living being begins to emerge. Again, this “impression” is the perception placed outside its normalcy range, and on rare occasions, such as this, we can learn to recognize the perception as originating experience.

Fig. 7 A kinetic sculpture by Theo Jansen

Back in daily life, with creatures or military marches, we must recognize that same element imbedded in the whole of the experience. With directed and intentional attention, one can pinpoint that specific modulation in the flow of experiences. The effectiveness of introspection lies in an *ability to tune into* a certain quality of experience. It is as if the thought had its own *qualia*.

Qualia is a specific term in philosophy that refers to basic sensory experiences, and its qualities, such as redness, brightness, the pitch of a musical note, volume, timbre, softness, roughness, sweetness, saltiness, the scent of rose. More precisely, for example, it is the *redness of red*. An important property of *qualia* is that they are tangible elements in the landscape of consciousness. One can circumscribe them, delimit them, locate them, point them out. One can speak of them in language, even though their actualization in consciousness is absolutely subjective (we may speak of red, but the experience of red is one's own). The tune one seeks in the introspection of the framework of first-person knowledge is for something with inverse properties. That is why an extension of the notion of *qualia* is proposed. Strictly speaking, *qualia* refers to experiences of the most basic sensory content, which requires no maneuvering to be noticed. We can extend this notion of "experience that is being perceived" to also encompass those experiences that require deliberate effort and developed skill. It is in this realm that we can find change, dependency, constraint, and interactivity. But, before that, let us make the experiment with just a pen. Proposing that there are *qualia* of thought (or perception) is like proposing that in consciousness there is a quality of experience that is especially developed for pens. This is counterintuitive. For us, it is less questionable to admit that nature has provided us with cognitive resources for dealing with colors, sounds, voices, smells, and the like, than with artificial objects, abstract concepts, and so on. But the fact is that we seem to be *open and adaptable beings*, that is, ready to realize unique experiences for everything that makes up our world. The pen is not the image in front of me. *Gestalt* has shown that the same image can unfold into totally different experiences, perceptions. The pen is a unique and specific experience, without contours, that is triggered by the image of a pen, and it is present here and now, in this pen, in front of me, supporting the whole experience. If the images, textures, sounds and the like which I experience with the pen make sense, it is because the perception of the pen—this experience of its own qualities—prevails and predominates in consciousness, therefore unifying everything. Similarly, in the case of kinetic sculptures, there is a perception of animality, which goes beyond moving images. Moving images are the trigger that brings out this quality of experience, that we seek to learn to tune. Merleau-Ponty (2008) discusses how the perception of animality springs from the perception of movement, when analyzing the experiences of Albert Michotte. In one of the experiments, the projection of two moving traces on a wall, when presenting certain patterns, establishes a chasing experience.

In short, epistemology understood as first-person knowledge consists of a form of introspection that one rediscovers for themselves in their subjectivity and practice. According to this framework, knowledge of the experience takes place in terms of the experience itself. It is there, during the moving experience, that the elements show themselves as experiences. All art consists, by directed and intentional attention, of learning to recognize the elements and to pinpoint them into the flow of experiences that overlap and align. This is the background without which no language, no theorization, no rationalization, no science can stand. This is where we seek to find change, dependency, constraint, and interactivity.

5 Dependency and Interactivity in a First-Person Perspective

Bergson (2007) studies the perception of change. That research makes perfect sense within the work of this philosopher, who has devoted himself to the difficult interrogations concerning time, duration, memory, and movement. By the way, Bergson is also a philosopher who, not by chance, recognizes intuition, beyond reason, as a legitimate space for philosophical reflection. The difficulties associated with movement, such as the Zenon's paradoxes, would need to be addressed, according to Bergson, using our intuition. Our proposal of introspection is in line with this. In the present section, I will deal only with the perception of dependency that manifests from change, but is related to constraint and interactivity as well. Dependency plays a major role in the use of DynaLin.

The unsuspecting reader may find the first-person approach strange. Here, they find a set of descriptions of the obvious. Trite facts parade before them, and they may even wonder what philosophy is all about, after all. The justification is as follows: the long descriptions of banal facts are invitations to experience these same facts from another perspective, now trying to be in tune with this background knowledge.

I propose a fictional experiment to show the perception of dependency (Fig. 8). There are several circles arranged randomly over an area. The circles can be handled: when touched, they can light up or not; and also, can be moved throughout the area. Sometimes, when a circle is lit or moved, another circle shows some behavior, such as lighting or moving. The mere fact that a circle can respond to an event that occurred with another circle is enough to establish a sense that there is a *bond* between them. And this takes place under the loosest conditions possible: behavior

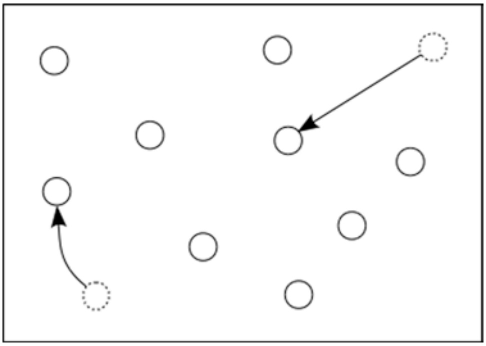

Fig. 8 Experience to show the perception of dependency (Figueiredo, 2010)

does not necessarily need to be repeated; it does not need to follow a deterministic pattern (for example, when doing the reverse path with the handled circle, the other circle does not necessarily need to travel in its inverse path); no movement is required (this is the case where the circles just light up). It all comes down to isolated and possibly non-deterministic events. In each such event, the momentary impression of bond is realized.

In addition to the experience described above, the perception of dependency can be observed in descriptions of everyday situations, as in the following accounts:

- A child runs and jumps. They want to reach further. They repeat the jump by using additional *pull* into their body and get the result. This child *somehow knows* that there is a relationship between the body strength of the jump and the resulting reach. It is a relationship that holds some proportionality: the greater the pull, the greater the distance travelled after the jump. In other words, the child knows that the range of the jump *depends* on the physical strength they exert. Characterization can be a little more complex. In addition to the physical strength, there are other parameters that influence the result achieved: the direction, the posture of the body as a whole, especially the feet. As the child jumps farther and farther, they learn to acknowledge these relationships tacitly.
- A similar situation: A child throws a stone to hit a fruit. Again, they operate an intricate network of dependencies that now includes not only knowledge of one's own body, but of the environment as well. In this case, the wind is an additional parameter to be evaluated and compensated for in order to achieve the desired result. If the wind is blowing against the child, a greater force must be applied in the opposite direction. The child needs to gauge *how much* more force must be used based on their assessment of the wind speed.
- A faucet has a ratio of dependency between the turning angle and the water flow. Again, this relationship is more or less proportional. Of course, the tap has some limits. At one extreme it is shut off for good, at the other it is wide open. Many faucets are defective, they leak, show unexpected behaviors. People learn to know the idiosyncrasies of faucets and they offset those to get the desired result. An electric shower is even more interesting as it shows a conjunction of dependencies, as, in addition to the flow rate, the water temperature is also considered. As people regulate flow rate, it is clear that they *understand* these dependencies.
- Tuning a radio receiver (the dial) resembles a faucet because of the knob, a round part that also needs to be turned, and because this operation is also limited on one and the other sides. In older receivers, the dial had a graduated frequency scale with an indicating arrow that moved as a function of the spin of the knob. Electric and electronic devices are full of controls that appeal to this sense of perception of dependency: the stereo volume control, fan speed control, etc.
- Before those, there were purely mechanical devices. A living room blind can have its palettes rotated by sliding a cord. The measure of turn can be controlled by the distance the cord is pulled. And there is much more: car windows, the throttle, the clutch and the brake, the perception of the gear ratio, etc.

In short, human beings are constantly recognizing and operating relations of dependency. And we do this with absolute familiarity, as if they were part of us, as if they were a natural and deeply present characteristic within us. Only dependency has been analyzed here because of the limited amount of space allowed for this text. The same procedure could have been applied to change and constraint. In the previous section, we had an appropriation of animality through movement (a variety of change).

An experience with interactive devices is, of course, a good place to find first-person interactivity. With focused and intentional attention, we should strive to tune into the elements of the interactive experience, such as, for instance, the immediate response, the moving forward (proceeding), the cancelling, the accepting (agreeing, approving, clicking on *OK*), deflecting, pausing, returning, repeating. I stand in front of a computer with a text editor on the screen. My actions on the keys reverberate as letters that sprout on an interface that mimics a sheet of paper. In another situation, I drag a circle in a dynamic geometry application. Here and now, at the core of the experience are these elemental experiences which make up an *interacting cognition*, that is, a way of being and knowing the interaction, whether with the world, people, animals, electronic and mechanical devices, and so on.

6 A Research in Movement

The issue here is whether interactive devices are new media that, like none before, allow us to carve out experiences of change and response. Just as sculptures are not a representation of volume as an elementary experience, but the presence of volume itself, interactive devices *would not be* a representation of change, answer, and interactivity, but a means of realizing them. And within an experience of change and answer, dependency and constraint may arise.

These findings should be evident when we modulate our experience from a broad perspective, in which we turn to this background that sustains it, the flow of elementary experiences, the secret pre-predicative foretaste of experience.

If we apply the same cognitive procedure to the practice of mathematics, can we recognize there, mixed up within the whole, moments of change, dependency, and constraint, but this time semiotically represented in mathematical language, symbology, and knowledge? The question that arises is how far we could transpose them into interactive experiences, and how favorable this media is to them. Dynamic geometry, dynamic algebra notation systems, and DynaGraphs could therefore be understood as the first steps in this direction. However, the pedagogical developments are a subject of research in mathematical education, whose path has been indicated by dynamic geometry, for example. One can conjecture that we are underway towards a conjuncture in which static discourses and interactive experiences coexist and complement each other.

References

Bergson, H. (2007). *Creative mind: An introduction to metaphysics*. Mineola, NY: Dover.
Figueiredo, O. A. (2010). *Sentidos de percepção e educação matemática*. Tese de Doutorado, Instituto de Geociências e Ciências Exatas, UNESP, Rio Claro.
Goldenberg, E. P., Lewis, P., & O'Keefe, J. (1992). Dynamic representation and the development of a process understanding function. In G. Harel & E. Dubinsky (Eds.), *The concept of function* (pp. 235–260). Washington, DC: Mathematical Association of America.
Goldenberg, E. P., & Cuoco, A. A. (1998). What is dynamic geometry? In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 351–367). Mahwah, NJ: Lawrence Erlbaum Associates.
Jansen, T. (2016). *Theo Jansen: the great pretender* (3rd ed.). Rotterdam: nai010.
Merleau-Ponty, M. (2008). *The world of perception*. New York: Routledge.
Merleau-Ponty, M. (2014). *Phenomenology of perception*. New York: Routledge.
Vinner, S. (1992). The function concept as a prototype for problems in mathematics learning. In G. Harel & E. Dubinsky (Eds.), *The concept of function* (pp. 195–213). Washington, DC: Mathematical Association of America.
Weitnauer, E., Landy, D., & Ottmar, E. (2016). Graspable math: Towards dynamic algebra notations that support learners better than paper. In *Future technologies conference (FTC), proceedings* (pp. 406–414).

Real Numbers and Computational Challenges Under a Phenomenological Perspective

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1 Looking into the Production of Knowledge

When assuming a phenomenological perspective, the production of mathematical knowledge is understood as a movement that encompasses different aspects of human life. In this perspective, knowledge is produced by worldly beings, who are always in the world with other beings, in a production movement that occurs through the intersubjectivity of the members of a community. Wiltsche (2018, p. 5) states that "every perceptual encounter with things necessarily involves an implicit relation to an open community of subjectivities, without which the thing could not be constituted as objectively existing." This way of understanding the production of knowledge is strongly linked to Husserlian philosophy, which is assumed in this chapter as a way of focusing on the production of mathematical knowledge about real numbers.

The phenomenological gaze with which our research was produced led us to understand that the conception of knowledge that grounds Western sciences is produced by a process of understanding living-experiences, which allows intelligible articulations for a community, of what was previously intuitive for formal and logical terms. We will examine real numbers and also the developments arising from the advent of digital technology, from this perspective.

Attentively examining mathematics and the historical process of its development, it is possible to observe a transition between a pragmatic status, as a counting and measuring tool, and a level of science which is reliable for other sciences, and

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to produce abstract theoretical knowledge. However, when viewed from the perspective of phenomenology such development is understood in a different way.

In assuming this attitude, we understand that the structure of the production of mathematical knowledge is strongly linked to human productions. We need to be aware that this production results from the possibilities available at a given time. In this sense, we understand that Western mathematical sciences, which we are now familiar with, were developed in a community and materialized in a culture, not as a result of a priori apodictic facts, taken as truths, but through a gradual process of constitution and production that, through an articulating and theorizing movement also came to produce such apodictic facts.

As mathematics became the science we know today, its productions were oriented towards a generalizing scope, grounded in logic and algebra. This movement has brought many advances to the academic scene, but also paradoxes. It is not difficult to list famous cases of contradiction to newly formulated theories. Notably, the nineteenth century was marked by paradoxes and the effort to overcome a crisis in the foundations of mathematics.

The joint efforts of many mathematicians to establish the foundation of mathematics led to a process known as the arithmetization of analysis which, among other objectives, sought to give real numbers a solid enough structure so that they could be elaborated to write mathematics in a way which was free of paradoxes. This view, however, has given real numbers greater relevance, since their structure ensures the consistency of other areas of mathematics.

Although it is possible to criticize[1] the model that emerged from this production movement, it is the most accepted both historically and culturally. We must remember that, at the time the foundation of analysis was established, the research was already firmly based on algebraic structures and on *Cantor Set Theory*. However, change was dawning in the horizon, as the twentieth century was met by the advent of digital technology, with computers. Computers, however, do not display much affinity with the algebraic abstraction elaborated in the nineteenth century, considering the physicality of electrical circuits. Such circuits, in turn, are oriented by Boolean logic and the virtuality of the data, written in binary language.

Then, a new challenge arises for those who intend to understand the intricacies of mathematics: making sense of real numbers by producing mathematics by being with the computer. We can see a favorable scenario for these challenges. However, overcoming the crisis of fundaments has also created a structure that is difficult to refute.

The efforts to ground mathematics, especially those made in the nineteenth century, and the scientific authority granted by the logic applied in this process, established a culture of infallibility. So, disputing new productions implies questioning the very structure of mathematics. For this reason, it might become even more difficult for the current movement to update a mathematical object with the

[1] It is worth pointing out Hermann Weyl's stern *critique,* in his book *Das Kontinuum* (1994), to the whole process of arithmetization of mathematical analysis, which, he contends, is full of paradoxes and cyclical definitions.

computational perspective. This is why we look closely at the production of mathematical knowledge and its developments in light of digital technology.

Within the scope of this chapter, we seek to shed light onto significant historical events, by presenting our understandings and philosophical reflections about them. We will approach the problem that involves real numbers from their mathematical perspective, bringing their production movement through the work of Dedekind and Cauchy. Then, we will address real numbers from the perspective of computers, bringing the concept of floating point along with its possibilities and limitations.

We will present how formalization of the concept of real numbers, in the nineteenth century, encompasses the physicality of the world and the geometrical understandings of mathematicians at that time, as well as the reasons why the possibilities of cyberspace and its digital features make it necessary to address this kind of number again.

By working with this theme, we put ourselves in the place of mathematical educators who, when assuming the phenomenological vision, cope with confrontations facing those who endeavor to produce mathematics while being with computers.

2 Historical Context

The nineteenth century was marked by disputes regarding the truth and the foundations of mathematics. Such criticisms have spread to various areas of knowledge, as a result this period was dubbed the *Foundational Crisis*. In 1901, Bertrand Russell presented a paradox regarding the theory underlying the mathematics production of the time, Cantor's Set Theory, which marked the beginning of the movement that sought to lay the foundations for the sciences.

Cantor was an important nineteenth century mathematician and the first to theorize the existence of different types of infinity. His work made it possible to calculate the cardinality of sets with infinite elements and create the concept of set density, in addition to the distinction between infinity types. According to Silva (2007, p. 13, author's translation), Cantor's theory arises "From the need for proper treatment of the arithmetic continuum, but it soon became a theory of infinite totalities considered absent". In his production, Cantor, along with Dedekind, while studying infinite sequence problems, realized that the structure of real numbers was inconsistent and required rigorous foundation.

In line with Eves (1990), we agree that the structure of mathematics today depends largely on the consistency of real numbers. Following this reasoning, by assuming the phenomenological perspective to understand and conceive mathematics, we understand that consistency implies that the studies of those who produced the foundations we have today converged to the fact that numbers are constitutive of mathematics, notably real numbers. So, we understand that conception would enable the production of more meanings and more knowledge.

The perspective that considered real numbers as constitutive of mathematics has fostered research in the area of this science known as *Real Analysis*. The production obtained in this area provided a foundation for mathematics. However, it was a process that required a lot of effort from great mathematicians. The search for this rich basis for mathematics was called *arithmetization of analysis*, and it can be said that this movement began in the late eighteenth century with Lagrange. However, only in the nineteenth century, with the work of Gauss, an efficient move to break away from intuitive ideas and towards a formalization of concepts such as limits, continuity, and differentiability was achieved.

A detailed development of this movement can be found in Eves (1990, pp. 563–576), whose driving force was the newly developed Differential and Integral Calculus, which therefore was developed intuitively by mathematicians of that time.

According to Eves (1990), Jean-le-Rond D'Alambert pointed to the need for a theory of limits in 1754. However, it was not until 1797 that a rigorous attempt was made for the formalization of calculus by mathematician Joseph Louis Lagrange "Based upon representing a function by a Taylor's expansion" (Eves, 1990, p. 564). Although this attempt ignored important aspects about series convergence and divergence, it influenced later research, which allowed us to reach the current formalization.

Throughout the nineteenth century, research to consolidate mathematical analysis was developed. Among which is the work of Gauss, who, according to Eves (1990), in 1812 presented the first really adequate consideration regarding the convergence of an infinite series. However, according to that author, the first major advancement in this process can be attributed to Cauchy. By drawing on D'Alambert's ideas, Cauchy was able to develop a theory of boundaries and thus satisfactorily define the concepts of continuity, differentiability, and definite integration.

However, towards the end of the nineteenth century, examples of functions that contradicted Cauchy's proposal began to emerge. Riemann, for example, defined a continuous curve for irrational values but discontinuous for rational values. For Eves (1990, p. 564), "Examples such as these seemed to contradict human intuition and made it increasingly apparent that Cauchy had not struck the true bottom of the difficulties in the way of a sound foundation of analysis."

The grounding proposals, presented in the nineteenth century, were the target of criticism, based, for example, on constructing mathematical structures that contradicted the clear view of intuition, that is, the intuitive evidence. This scenario is recurrent in the history of mathematics. According to our understanding, it results from the fact that the production of knowledge was based on meanings that needed to be revisited. That is, the concepts that were necessary for the formalization of the analysis required understanding of the roots of certain structures, such as real numbers, which still rested in an intuitive sphere.

By focusing Cauchy's proposal, we can understand that the concepts with which he worked were based on the real number system, and that the system was not yet formalized. To that end, Weierstrass advocated a program of formalization of analysis, which demanded that the real number system itself be formalized. This would be followed by a coherently logical-deductive dynamic, understanding that the concepts necessary for the foundation of analysis depend largely on the immanent properties of numbers.

We understand, with Eves (1990), that the arithmetization process of analysis, proposed by Weierstrass, consisted of "a program wherein the real number system itself should first be rigorized, then all the basic concepts of analysis should be derived from this number system" (Eves, 1990, p. 564).

Nowadays, we know that the success of the *Weierstrassinian program* not only supported the analysis of real numbers, but also established a relationship between Euclidian geometry and real numbers, and between algebra and real numbers. Therefore, "In fact, today it can be stated that essentially all of existing mathematics is consistent if the real number system is consistent. Herein lies the tremendous importance of the real number system for the foundations of mathematics" (Eves, 1990, p. 565).

The discussion presented by Lopes (2006) on the *arithmetization* process of analysis points us to three distinct paths to be followed, which are listed according to their most notorious advocates:

> Hankel (1839 - 1873) and Frege (1848 - 1925) defended the traditional idea that analysis should be founded on the notion of continuous quantity.
>
> Dedekind, Weierstrass (1815 - 1897) and Cantor (1845 – 1918) defended the notion that quantity should be substituted by a rigorous arithmetic construction of real numbers, that is, a construction based in the notion of natural or rational, which proved to be less problematic than the notion of continuous quantity.
>
> Heine (1821 - 1881), Thomae (1840 - 1921) and Hilbert (1862-1943) defended that the fundamental concepts of analysis could and should be constituted simply in a formal manner, disregarding, as much as possible, philosophical matters (Lopes, 2006, p. 3, author's translation).

One of the above-mentioned paths presents Hankel's and Frege's understanding that work with numerical systems and operations, which are built as a combination of basic symbols, produces new symbols. However, Hankel assumed that a formal numerical system "Should be generated from a finite set of basic symbols by a countable sequence of definite operations applications" (Lopes, 2006, p. 5, author's translation), and thus the irrational numbers could not be formally constructed, since it would not be possible to define all the operations that could be admitted regarding real numbers.

The view of Weierstrass regarding analysis is based on the belief that numbers are *aggregates* of certain elements. This way, he can characterize the numbers as *basic units* and *parts of units*. Weierstrass' arithmetic construction associated *quantity* with numbers and, for this reason, his demonstrations are based on the concept of transformations between quantities. Such demonstrations only later were attributed to numbers.

The ideas of Weierstrass influenced Cantor and Heine, who produced constructions of numerical systems with the same understanding, but in different ways. Cantor worked with Weierstrass' conceptions regarding quantities and Cauchy's concept of convergence,[2] to characterize a number derived from the convergence of a series of *first-species numerical quantity* rationals. This quantity was associated with a point on a line.

Thus, Cantor was able to construct real numbers in a way which is similar to the method followed by Weierstrass. However, in carrying out this process, for the first time in the history of mathematics, Cantor saw that real numbers constitute an unnumbered set. It was evident to him that there were different types of infinity, and this led his research in another direction. Notably the *Theory of Transfinite Sets*. According to Burton (2007).

> Not satisfied with merely defining infinite sets, Cantor proposed something even more shocking and impious—endowing each set with a number representing its plurality. This would allow him to distinguish infinite sets by "size," and to show, for example, that there are "more" real numbers than there are integers (Burton, 2007, p. 672).

Dedekind's works follow the same path taken by Cantor, as he also assumes Weierstrass' understanding to present his arithmetic construction of real numbers, based on the geometrical idea of points. That is, an intuitive notion of continuous quantity of line segments was associated with rational numbers.

Dedekind observes that both a numeric set and a string extended can be separated into two sets by one element. He takes the concept of continuity of a rope that is broken by a single cut and extends this conception to numbers, stating that there is an element that can also separate a set into two parts. This element in his theory is unique, and was denominated *Cut*.

The third path mentioned above, followed by Heine, draws on Weierstrass' concepts of infinite sets and quantities, but including Cauchy's convergence criterion for the construction of number systems. However, what was most noteworthy about his approach was the fact that "He sought to avoid philosophical problems in a surprisingly naive way, viewing numbers as *tangible* symbols without realizing how vague his idea was" (Lopes, 2006, p. 9, *author's emphasis, author's translation*).

The effort to separate the construction of real numbers from intuition, by means of equivalence class criteria, Cauchy convergences, and tangible quantities was criticized by Frege. Such procedure was not accepted by the mathematical community, also proving to be a very controversial construction for mathematical sciences.

In light of the above, we observe that convergences and divergences of ideas have guided many mathematicians in their search for formal foundations that characterize real numbers. This led to different approaches for constructing these numbers, namely, the construction by *cuts*, by *equivalence classes*, or even by *intervals*.

However, we believe that there are two processes that are more widespread within the mathematical community. Both of them propose that real numbers be obtained by extending the concept of rational number: the Dedekind Cuts method

[2] A sequence $\{x_n\}$ in a metric space (S, d) is called a Cauchy sequence if it satisfies the following condition (called the Cauchy condition): for any $\varepsilon > 0$ there is an integer N such that $d(x_n - x_m) < \epsilon$, whenever $n \geq N$ and $m \geq N$ (Apostol, 1981, p. 73).

and the Cauchy Sequences method. So, we will focus on those and talk about them further.

3 Dedekind Cuts

The construction of real numbers via Dedekind cuts was first published in 1872. Aiming to overcome the absence of nonrational numbers, his argument takes into account three properties of numbers and their corresponding properties for points in a line. To this end, Dedekind takes two rational numbers and establishes the relations of order between them. Thus, by considering *a, b* two rational numbers, and the relationships of being larger (>) or smaller (<), well defined, it is possible to establish the following properties:

First Property **(on numbers)**

1. If $a > b$, and $b > c$, then $a > c$. Whenever a, c are two different (or unequal) numbers, and b is greater than the one and less than the other, we shall, without hesitation because of the suggestion of geometric ideas, express this briefly by saying: b lies between the two numbers a, c.
2. If a, c are two different numbers, there are infinitely many different numbers lying between a, c.
3. If a is any definite number, then all numbers of the system R fall into two classes, A_1 and A_2, each of which contains infinitely many individuals; the first class A_1 comprises all numbers a_1 that are $<a$, the second class A_2 comprises all numbers a_2 that are $>a$; the number a itself may be assigned at pleasure to the first or second class, being respectively the greatest number of the first class or the least of the second. In every case the separation of the system R into the two classes A_1, A_2 is such that every number of the first class A_1 is less than every number of the second class A_2 (Dedekind, 2007, p. 3).

Similarly, taking two points p, q on a straight line L, and the relationships of right and left, well defined, it is possible to establish the following properties:

Second Property **(on points)**

1. If p lies to the right of q, and q to the right of r, then p lies to the right of r; and we say that q lies between the points p and r.
2. If p, r are two different points, then there always exist infinitely many points that lie between p and r.
3. If p is a definite point in L, then all points in L fall into two classes, P_1, P_2, each of which contains infinitely many individuals; the first class P_1 contains all the points p_1, that lie to the left of p, and the second class P_2 contains all the points p_2 that lie to the right of p; the point p itself may be assigned at pleasure to the first or second class. In every case the separation of the straight line L into the two classes or portions P_1, P_2 is of such a character that every point of the first

class P_1 lies to the left of every point of the second class P_2 (Dedekind, 2007, pp. 3–4).

The correspondence seen by Dedekind between numbers and points, and between the set of real numbers and the line, brings with it the incompleteness of rational numbers. At this point, we understand that, for the correspondence to be considered valid, that is, for all the numbers to rest on the line, it would take more than rational numbers. It was then necessary to expand the concept of number, so, besides rational numbers, the irrational numbers were also well defined.

Dedekind's construction consists in showing that not all cuts are produced by rational numbers and, therefore, it is necessary to broaden the concept of rational number and the subsequent construction of real numbers as the union of all rational and irrational cuts. Thus, for Dedekind, real numbers are constructed as sets of rational numbers, with the property of continuity, by overlapping a geometric element, the line, and respecting the principle of good ordering and arithmetic operations. Therefore, Dedekind extends the notion of number by using the idea of continuity of the line.

4 Cauchy Sequences

In his theory for the construction of real numbers, Cantor used a procedure which was different from that of Dedekind. He considered the use of *rational numbers* and the concept of *fundamental succession,* nowadays called *Cauchy sequence*, to establish a relation between the numbers and the line.

A fundamental sequence or succession can be defined as:

Definition A sequence $\{r_n\}$ is said to be a Cauchy sequence, or a normal sequence, or a fundamental sequence, which is symbolically represented as $\mathcal{C}\{r_n\}$, when this sequence satisfies the so-called Cauchy condition*:*

$$\mathcal{C}\{r_n\} \Leftrightarrow \forall \delta > 0 \exists N \in \mathbb{N} \mid \forall n > N, \forall p > 0 \Rightarrow r_n - \delta < r_{n+p} < r_n + \delta$$

This means that, for each positive rational number δ, there is an order $N \in N$, such that all the elements r_{n+p} of higher order to a given element r_n, of order $n > N$, are in the rational neighborhood $(r_n - \delta, r_n + \delta)$ of r_n.

While for Cauchy the whole sequence that enjoyed this property converged to a single real number, Cantor claimed that to assume the existence of a real number was a logical mistake on the part of mathematicians. If we consider that real numbers were not yet defined, then the convergence of all Cauchy series is an intuitive fact, which Cantor was trying to formalize by algebraic means, thus avoiding paradoxes.

Although Cauchy's assumption could be taken as intuitive, we understand that in the process of production of knowledge, meanings are produced in the movement of understanding living-experiences. This allows us to talk about the intuitive elements that are present in the movement of formalization of knowledge. Specifically, by focusing on the production of mathematical knowledge in real numbers, we see that the possibility of the existence of a number to which sequences can converge has made it possible for Cantor to formally construct real numbers.

By proving that every fundamental sequence was convergent, according to Cauchy's principle of convergence, and considering that there are sequences that do not converge to rational numbers, Cantor understands that it is necessary to define real numbers, and does so by using the artifice of equivalence classes.

Cantor's construction takes *real numbers* as an equivalence class of fundamental sequence convergences. Unlike Dedekind's theory that the number is a *cut*, which separates a continuum into two parts, in Cantor's conception, the number is the representative of a set that has a neighborhood of points. Both conceptions give *real numbers* a close connection to a geometric line, which gives this set a property of being continuous and respecting basic arithmetic rules.

Although these two conceptions differ epistemologically, regarding number structure, modern mathematicians have worked closely with logic to characterize different constructions of real numbers as isomorphisms. That is, despite the fact that their elements are conceptually different, it is understood that their behavior is the same.

The considerations made in this text allow us to see that the mathematical knowledge about real numbers has been produced through understandings of intuitive elements. They were subject to the temporal limitations of each researcher, and mathematicians were motivated by an interrogative demand. There was a need to seek answers to the challenges. This movement led to a process of generalization of concepts and their formalization according to the rules of logic. For Longo (2001), this process is characteristic of the production of mathematical knowledge and is strongly linked to the phenomenological perspective of investigation. According to him, our intuition regarding the mathematical object is about common elements. He understands that meanings emerge through a plurality of acts of experience. Such meanings are laid down to the mathematical community and, in an intersubjective movement, they are constituted as a formalized mathematical object.

5 Real Numbers and Computing

Considering that computers are electronic devices designed to perform programmable operations and process different types of data, we understand that their physicality is based on electrical circuits and components that allow them to perform these operations, at high speed, in a systemic and orderly manner, provided they have been previously programmed.

Detailing the operations of computers, as well as the technical knowledge of electronics, logic, and mechanics, specific to that kind of device is beyond the scope of this paper. However, we assume that computers operate within the Boolean logic and their physical structure is grounded on binary arithmetic. That said, it is possible to understand that what is done computationally is executable in a bitwise electrical circuit.

As technology advances, the physical distance between the user and the computer processor is shrinking. Nonetheless, the number of interface layers is growing. The purpose of the interfaces that act in the computer-user relationship is to facilitate processing access and expand the possibilities for the user to operate with the computer. However, they also distract the user. If, on the one hand, the use of computers becomes more intuitive with the insertion of interface layers, on the other, it is more difficult to access the mechanical and electrical dimensions, which are part of the necessary tools in order that a click of a button can command the desired operation, for example.

By focusing on the dimension of interfaces of computational numbers, we might be led to view them under an intuitive perception of numbers. They are considered so in order to facilitate access to information. For example, mass media publishes numbers, indexes, and prices without proper disclosure of the structure that enabled their calculation.

We understand that, historically, many concepts have been discussed based on this mode of logical-intuitive knowledge. However, this practice has proven to be a source of paradoxes and misunderstandings. In order to avoid misunderstandings, we have inquisitively focused on the structuring dimension of computers and also examined how real numbers are updated by it. The computer performs calculations at the most different levels. As complex as the operations may be, they all need a number structure, which is compatible with electrical circuit processing.

According to Knuth (1979), the history of mathematics presents several numbering systems, and each one contributes differently to what is the culturally produced by man. Following this reasoning, the choice of a computational numerical system also brings consequences for the understanding of computation we have assumed. According to this author:

> The first American high-speed computers, built in the early 1940s, used decimal arithmetic. But in 1946, an important memorandum by A. W. Burks, H. H. Goldstine, and J. von Neumann, in connection with the design of the first stored-program computers, gave detailed reasons for the decision to make a radical departure from tradition and to use base-two notation (Knuth, 1979, p. 186).

Corroborating this understanding we find that since the above-mentioned memorandum, the numerical base adopted for all computers is binary. Although the numerical system with which Western civilization operates is based on decimal arithmetic, it is known that since the studies of Leibniz regarding the binary basis, it is possible to operate with fewer symbols for the representation of numbers. This economy of symbols may have been decisive in the process of consolidation of computational language.

In Knuth (1979), we find some ways to virtually represent a rational number on a binary basis. These include the *fixed-point* technique, the *2-complement* technique, and the *floating-point* technique. *Fixed-point* and *2-complement* techniques have already been used. But, currently, the implementation of algorithms is based on floating-point arithmetic.

6 Floating Point

The floating-point structure was developed to increase the speed and capacity of computers and, currently, is the most widely used in processors worldwide. Its presence in computing is taken in a naturalized way, so the user does not need to think about how to use it, and what its limitations are, but we can understand them as an intuitive element, with which we can produce knowledge.

The idea of floating point can be explained as follows: Every rational number consists of two parts, an integer and a decimal, which are separated by the radix point. When we apply the fixed-point technique, our separator is always placed at the same bit of a computational variable. So, the number of digits, decimals, or integers remains fixed. However, the floating-point technique is characterized by the dynamism of the position of this point, that is, the point that separates the integers from the decimals can change position by changing the number of digits of one of the parts.

This technique is used, for example, in scientific notation, which is characterized by a number belonging to the open range (0, 10) multiplied by the power of ten. This way, the power exponent will define where the separator will be placed. Similarly, we define the number x in floating-point notation as:

$$x = \pm d \times \beta^e$$

where d is the mantissa represented by algorisms of the base chosen; β is the numerical base; e is the exponent which will indicate where the separator is located.

Below, we present an example of floating-point notation number with a decimal base and another with a binary base.

$x = 5.132$ decimal notation number
$d = 0.5132$ mantissa
$\beta = 10$ decimal Base
$e = 1$ exponent
$x = 0.5132 \times 10^1$ decimal-base normalized floating point
$y = 0.10111$ binary notation number
$d = 0.1011100000$ mantissa
$\beta = 2$ binary base
$e = 5$ ou $(101)_2$ exponent
$y = 0.1011100000 \times 2^5$ binary-base normalized floating point.

When this number is computationally processed, each element is stored in one *bit* of a variable. It is also necessary to reserve a specific space for the mantissa signal.

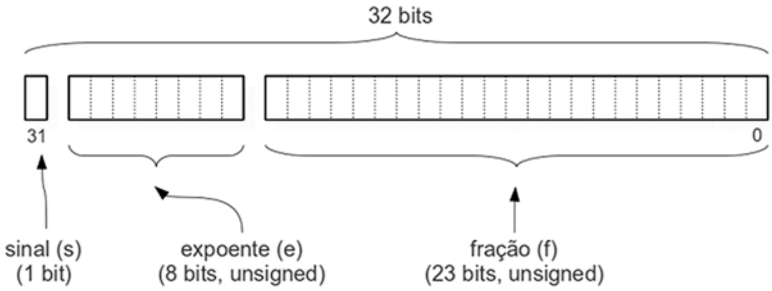

Fig. 12.1 Representation of a 32-bit float structure. Fonte: https://bitismyth.wordpress.com/2011/05/27/o-problema-do-ponto-flutuante-parte-3/

Figure 12.1 illustrates the 32-bit representation of a number. It is also possible to have 64- and 128-bit representations, but the organization is preserved. We move forward in our argument by presenting the interrogations that arise when working with this kind of representation. It is important to clarify that, even though this chapter aims to examine understandings of the computational structure of *real numbers*, we are not concerned with specific arithmetic methods, as we believe that they are beyond the scope of our proposal.

This being our understanding, we first question whether it is possible to obtain the completeness of real numbers in this kind of representation. The ordered field of real numbers ensures that there will be no "gaps," since it is always possible to display *infimum and supremum* in any subset of the body. That means, between two numbers there is always another number. However, to be able to display some decimal numbers, the computational representation adopted would require an infinite number of bits.

A 32-bit structure, as shown above, can store different numbers. However, its accuracy is six or seven decimal places. By stating this, we are not saying that it is not possible to store numbers smaller than 10^{-7} on a computer. But rather that if a number has more than seven decimal places, there will be a rounding process which will result in loss of precision of the number.

Large numbers are also restricted in some computational structures. Domestic devices that are set to 32 bits, for example, are capable of storing numbers smaller than 4×10^{10}, and the accuracy will be restricted to seven numeric orders, or, at most 15, when a 64-bit memory structure is adopted.

We can see that there is an infinity of real numbers that surpasses computational capacities. Therefore, the completeness of numbers is not maintained. So, the evidence that Dedekind and Cauchy used to ground the theory of real numbers, namely, the existence of a number that always limits a sequence of rational numbers cannot be shown computationally.

Considering that our conception of completeness is consonant with cultural and temporal conceptions; and that this conception is largely grounded in a geometric concept of the line, which was established by the nineteenth century mathematicians to construct the real numbers; we believe, as we move into the field of computer science today, this geometric understanding influences the way we conceive numbers.

The following reasoning can help us understand this influence: thinking that the geometric representations we make of different mathematical concepts are convenient, because they can be associated with our worldliness, which is physical and geometrical. We can accept this statement in a naïve way, that is, without posing philosophical questions. However, reality actualized by computers is permeated by digitalities, with elements that are not physical.

Another point that we can question is related to the basis chosen for computational development. As mentioned above, computers operate with binary base. When Leibniz developed his research on this numbering system, his perplexity stemmed from the simplicity of the symbols with which it was possible to write any number. His production had no elements that involved concepts such as those of different infinity types, or set enumerations; neither is it a formalization of numbers that required completeness, that is, that required the need to make any number present in a geometric physicality.

However, even in the field of rational numbers, with which Leibniz could have worked, there are problems with binary-based representation. In order to understand what happens, let us think that decimal numbers are written as a sum of ratios of type $\frac{1}{2^n}$. Unlike the integer part of a number, the combination of these fractions cannot represent certain decimal numbers, which result in periodic tithes.

Numbers that have simple decimal-base notation, such as the number 0.1, can have infinite representations in the binary base. In this case the number 0.1 is written in the base binary as (0.0001100110011001100 ...). Taking into account the maximum decimal digits available on a high-performance computer, it would still only be an approximation of the number 0.1.

Thus, we understand that when we operate rational numbers with the computer, we face the limitations of computational memory. We need to be clear on the fact that a 32-bit computer stores a floating-point number mantissa of up to 23 bits, causing a bad approximation for certain numbers, as in the example given 0.1, which has infinite digits in its mantissa and will be truncated in 23 digits.

The two points presented herein are challenging for mathematicians to articulate, and produce a formalization of real numbers with computers as rigorous as that conducted in the nineteenth century. However, in assuming the phenomenological perspective we must remember that it is our living-experiences that allow us to talk about what we live, and it is the acts of consciousness that allow us to produce the meanings necessary for the process of formalization of mathematical knowledge.

We believe that, for the production of scientific knowledge these first meanings can be called intuitive, and require subsequent formalization based on valid logical structures. This is the horizon that is being opened for discussion. And, as said before, we must be careful with the intuitions that may generate paradoxes, or problems such as the ones mentioned. But, being human, we keep moving striving for the production of knowledge.

7 Articulating Comprehensions

The movement of production of mathematical knowledge from a phenomenological perspective embraces dimensions other than the scientific ones. All knowledge is produced by worldly human beings who are in the world with other beings and are conscious of their historicity. In this sense, the knowledge that underlies sciences is produced by a movement of subjectivity, intersubjectivity, and objectivity. This movement brings what is culturally and historically available, providing material for the consummation of form.

It is possible to observe that the path that led us to the current state of formalism in relation to real numbers was treaded taking into account the living-experiences regarding the most intuitive elements, and through logical processes, obtained scientific structure. Although this path is not unique, academic culture prompts us to adopt one standard as more acceptable.

However, with the advent of digital technology and cyberspace, other horizons open up and new elements become part of our life-world. Thus, we need to revisit the formalized concepts in order to understand our actions and ourselves in this scenario. In the beginning of this text we stated that such a task would not be simple, since the work done for the foundation of mathematics, in the nineteenth century, rendered its structure rigid and safe. The foundation laid out for this science, based on algebra and non-Boolean logic, does not apply directly to cyberspace, which brings us to the challenges of an age of knowledge produced with digital media.

Our investigation led us to even question what can be done computationally. By bringing the foundation of mathematics to computer science, new challenges and questions arose. In this direction, we adopted the position of trying to understand what the process of production of mathematical knowledge about real numbers was, and how computer science has been dealing with this concept.

We further believe that there are still situations which are analogous to the historical period prior to the arithmetization of analysis. That is, situations that are based on intuitive understandings of objects, in this case, the mathematical object called real numbers, which, although plausible and, to some extent pragmatic, is still a source of paradoxes.

This understanding is linked to the fact that the movement of making geometric and physical representations of mathematical beings is not innocuous. The concept of completeness, for example, rooted in physicality, was written algebraically, but the nuances of what is virtual and digital were not contemplated in the historical process for the development of analysis and construction of numbers. Therefore, we have numbers that are not computationally representable, at least not in the way Dedekind constructed them or mathematics, as a Western science, disseminates.

History has shown that there is more than one way to construct numbers and that, although they are isomorphic among themselves, the epistemologies of their constitution are distinct. This allows us to glimpse at alternatives for the formalization necessary to work in cyberspace.

We have posed questions to ourselves that point to a paradigm-shifting perspective, just as Russell's paradox moved the entire mathematical community to rethink the foundations of mathematics. We asked: Could the physical materiality that limits computational memory be overcome by computational speed and time? Could we allude to our generalizing capacity to the inexhaustible iterative processes of computers? Is there another structure for real numbers that does not require the presence of the entire geometric line?

We believe that interrogating the fundaments of mathematics, in face of the production of knowledge with computers, opens horizons for research. And we understand that the historical knowledge regarding the movement of production of knowledge; the phenomenological attitude which guides us in the quest for understanding our life-world; and a philosophical way of interrogating what is around us, are ways of tackling the challenges presented in this text.

References

Apostol, T. M. (1981). *Mathematical analysis* (2nd ed.). Massachusetts: Addison-Wesley Publishing.

Burton, D. M. (2007). *The history of mathematics: An introduction* (6th ed.). New York: McGraw–Hill Companies.

Dedekind, R. (2007). *Essays on the theory of numbers*. [S. I.]: Project Gutenberg. Acesso Outubro 14, 2018, em https://www.gutenberg.org/files/21016/21016-pdf.pdf

Eves, H. (1990). *An introduction to the history of mathematics with cultural connections* (6th ed.). Orlando, FL: Saunders College Publishing.

Knuth, D. E. (1979). *The art of computer programming* (Vol. 2, 2nd ed.). Massachusetts: Addison-Wesley Publishing.

Longo, G. (2001). The mathematical continuum: From intuition to logic. In J. Petitot (Ed.), *Naturalizing phenomenology: Issues in contemporary phenomenology and cognitive science* (pp. 401–428). Stanford, CA: Stanford University Press.

Lopes, P. C. R. (2006). *Construções dos Número Reais*. Dissertação de Mestrado, Departamento de Matemática e Engenharias, Universidade da Madeira, Funchal.

Silva, J. J. (2007). *Filosofia da Matemática*. São Paulo: Editora UNESP.

Weyl, H. (1994). *The continuum: A critical examination of the foundation of analysis*. Mineola, NY: Dover Publications.

Wiltche, H. A. (2018). Models, science, and intersubjectivity. In F. Kjosavik et al. (Eds.), *Husserl's phenomenology of intersubjectivity: Historical interpretations and contemporary applications* (pp. 1–14). Abingdon, UK: Routledge.

The Mathematician Producing Mathematics (Being) with Computers

Rosa Monteiro Paulo and Miliam Juliana Alves Ferreira

1 Introduction

The production of knowledge has been the focus of discussions and research within the scope of mathematics education, in different perspectives and philosophical conceptions. For a quantitative idea of the academic work on this topic developed in Brazil, a search for "*production of knowledge*" in the catalog of thesis and dissertations of CAPES,[1] already applying some filters, we found 14,958 results. Limiting the search to "*Digital Technologies and production of knowledge*" returned 163 results, among thesis and dissertations presented and approved in Brazil. Of those, we investigated some related to mathematics education and we understand that discussions take place in different approaches and with different objectives. Some expose the conception of assumed knowledge and others expose research procedures which open in various directions.

[1] CAPES: Coordenação de Aperfeiçoamento de Pessoal de Nível Superior is a foundation of the Ministry of Education (MEC) in Brazil. This entity is of fundamental importance in the expansion and consolidation of stricto sensu postgraduate (masters and doctorate) in all states of the Federation. At <https://catalogodeteses.capes.gov.br/catalogo-teses/#!/> there is a catalog of theses and dissertations presented in Brazilian Universities.

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Our research group, Phenomenology in Mathematics Education (FEM),[2] has been examining and developing significant work in order to comprehend the constitution and production of knowledge, including with digital technologies. The group project seeks to clarify the conceptions of the constitution and production of knowledge we assume and, in this particular text, focuses on the production of mathematicians when they are with the computer.

It is worth mentioning that the Research Group in Informatics, other Media and Mathematics Education (GPIMEM),[3] has sought to understand the production of knowledge with technologies, although assuming a conception different from the phenomenological. This is also a research group of the Post-Graduation Program in mathematics education (PPGEM) of São Paulo State University (UNESP), Rio Claro. The work of this group assumes the theoretical construct *humans-with-media* of Borba and Villareal (2005), which subsidizes the discussion of production with the computer. Most of their work investigates classroom technologies, the use of software, distance learning, and other topics related to knowledge production with technologies in the scope of research in mathematics education. One of their objectives is to overcome the dichotomy between man and technology. To achieve that, they work with the reorganization of thought proposed by Tikhomirov (1981), exposing the possibilities of current interfaces and the ideas of Lévy (1993) about collective thinking, advancing discussions that allow them to expose what is desired.

The work we will bring in this text stems from the ideas by Borba and Villareal (2005), which lead us to look closely at *how* mathematicians find themselves producing mathematics with the computer and explain the sense and meaning that this technology has in their production.

The research developed by Ferreira (2019), supervised by the first author of this text, is based around the interrogation: "*How does the mathematician who produces mathematics with the computer express the ways by which they understand the presence of this technology in their production?*" The *how* written in the beginning of the phrase expresses the investigative intent. It aims at the ways in which the mathematician is at the computer doing, producing, and revisiting their deed to realize what has been done. So, to understand what we are interrogating, in a phenomenological stance, we established a dialogue with mathematicians in the field of pure and applied mathematics who, to produce what they produce, are with the computer. The analysis and interpretation of what was constituted in the research, made possible to bring the strength of *being-with*, which, under the Heideggerian perspective, is what makes man-production-computer inseparable.[4]

[2] FEM: Fenomenologia em Educação Matemática.

[3] Grupo de Pesquisa em Informática, outras mídias e Educação Matemática.

[4] Ferreira's PhD research (2019) presents her research about this theme. She made it possible to clarify for us the process of producing mathematical knowledge with the computer.

2 A Path to Understand What Is Interrogated

Understanding the constitution and production of mathematical knowledge was a desire awakened at the end of Ferreira's masters research (2014). In that research she sought to understand *how the dialogue on mathematical content happened and was possible in social media groups*. That research allowed us to understand how the co-subjects, participants of the chosen groups that discuss mathematics, understood the contents of mathematics and organized ways of exposing themselves to others and sought alternatives of language to communicate their comprehensions. However, at the end of the research it was not clear to us how to foster the constitution of mathematical knowledge being-with-media. This uneasiness was intensified and gave rise to a new project which led us to investigate *how does the mathematician, who produces mathematics with the computer, express the ways in which they understand the presence of this technology in their production?* However, the interrogation required to delineate the investigative procedures, such as *choosing the mathematician who produces mathematics with the computer*. This identification would bring possibilities. The mathematician we wanted to talk to was the professional who researches (produces) in the scope of pure mathematics or applied mathematics and who is with the computer in order to produce what they produce. Those would be the mathematicians chosen for dialogue in the research.

So, we turned to mathematicians who produce with technologies, more specifically with the computer. We intended to understand how they perceive themselves producing mathematics and how they understand the presence of this technology in their production.

The authors dedicated to understanding mathematics in different contexts, such as Borba (2002), Kalinke and Almouloud (2013), emphasize that the media, and the context in which knowledge production occurs, interferes with the way mathematics is produced. Consequently, with mathematics production itself. For the first author, thought is reorganized by the presence of technologies, such as working with the computer. For the two other authors, the computer potentializes transformations. They argue that a text constructed in a multilinear form, such as hypertext, when read on a computer screen, gains power and dynamism. This enables the reader to assign it as much dimension as they want, since "while the page is a structural unit, the screen is a temporal unit" (Kalinke & Almouloud, 2013, p. 203, authors' translation) that enhances ways of reading and writing, inaugurating a new state or condition for those who read or write on the screen.

Arguments and statements such as these led us to question whether mathematics, understood as science in the Western world, would be modified when worked on with a computer. And if so, it was important for us to understand the meaning of this for the producer. We raise the questions to ourselves, such as: What would be the modification of mathematics, from the perspective of its producer? These concerns and provocations lead us to seek to understand the production experience that is experienced and understood by the mathematician when producing with the computer. We understand that beyond the questions we addressed ourselves, a broader and deeper interrogation arises: How does the mathematician who produces mathematics with the computer perceive themselves producing and perceives the characteristics of mathematics which is produced with the computer?

3 Being-with and Being-with-the Computer

The *being*, as Heidegger (2005) describes it, is always in the world *with* others. This "*other*" can be a person or not, which allows us to say that *being-with* refers to an ontological way of being of human beings or, as Heidegger (1962, p. 119) states, the *with* identifies and marks "the world is always the one that I share with others. The world of Dasein is a world-with [Mitwelt]. Being-in is being-with others."

The being-with points to an ontological way of the Dasein[5] being. It is always with. This means that it is not abstract, but is being-with-in-the-world, with persons or any kind of entity. Ontologically, it is together, so there is no separation between it and the world. Those beings with whom it is, may have the characteristics of being-there or not. They can be there as instruments, for instance. The *Daseins'* way of being-there interrogates itself and the proper being. This characteristic reveals it as an active subject in the world, who differs from instruments and objects, animals and from nature, in general. The Dasein is there, that means, always in the openness with, interrogating and willing to understand. So, given its ontological mode of being, we can understand that it is always together in its surroundings, open to the there. This mode of being permits the encounter. This being-there is the only being who has the possibility of worrying about its way of being. This being can understand and relate to the world, as well as recognize the possible existence of other beings endowed with the same capacities/skills/faculties. Therefore, it can recognize the other as one who similarly has possibilities to be open to the there. And to the other beings-there and mundane entities.

Assuming the phenomenological attitude, we mean the *being-with* as *being-in-the-world* without subject-object separation or, to put it another way, to assume that the being is *in-the-world-with*, means understanding that the world is always a shared world and that the *being-there* is the one who has the possibility to make it become, to make it present. Therefore being-with-the-computer and other media takes sense and meaning and can be understood in the context of the production of mathematical knowledge, since, according to Barbariz (2017, p. 39, authors' translation), in "being with the computer, there is a way of being of humans that already projects this way of being together."

Understanding that "there is no human being without a world and humans are never separated from it, makes the dichotomous subject-object conception unfeasible" (Bicudo, 2014, p. 45, authors' translation); *being-with* is shown as a way for the

[5] *Dasein* is a German word translated into Portuguese as presence or *being-there*. "This being, which we ourselves always are and which, among the other possibilities of being, has the one of questioning, we designate with the term *Dasein*" (Heidegger, 2005, p. 33, authors' translation).

human being to be with others, including the computer. In this way of *being-with*, the computer is not just a physical object or a machine, it is a whole made up of units (hardware, software, cyberspace, Internet, etc.) that leverages actions of the *Dasein* in its way of intentionally being-with.

However, saying that the computer is an intentional object means to say that it is not a tool or utensil, because if action is leveraged, it opens the possibility for the production of mathematical knowledge. As a tool, utensil, or instrument of use, the computer would be associated with doing, with usefulness: an object that serves for doing. As an intentional object it is imbued with open possibility, with the ways of that being to be in the world. It carries potential. It is in this sense that we turn to the mathematician who produces mathematics with the computer, seeking to see how they perceive their production.

4 Constitution and Production of Mathematical Knowledge

Knowledge production can be understood through different philosophical currents and in different aspects. However, as we have already clarified, in our research we assume the phenomenological stance, following the ideas of Edmund Husserl and his successors, such as Martin Heidegger and Merleau-Ponty. It is in this ground that the constitution of knowledge, as well as its production, are being understood.

The constitution of knowledge occurs in the dimension of living experience, in the life-world, having as primacy perception; it encompasses intentional acts of consciousness, which means that there are "ways of senses that make the living-subject intertwined and gradually constitute a form that becomes present to consciousness, so that the subject can *realize* what they understand of the life-world" (Rosa & Bicudo, 2018, pp. 12–13. emphasis added, authors' translation).

In Barbariz (2017, p. 143, authors' translation), it is understood that the constitution of knowledge is treated as a complex movement that embraces "this presence of the act of the subject (of doing) that occurs in the now, which can be intentionally revived in remembering, [this doing is] intertwined by perceived challenges of being-with-the-other, in a spatiality materialized in specific ways." Thus, it can be understood that the constitution of knowledge is an act of the subject who knows. However, it is not an isolated act, and neither are acts of isolated subjects, since the being is always *with* and, therefore, the acts of constitution are bound by the presence of the other, co-subjects with whom one is in the life-world; binding the subjectivity of the other, opening up to intersubjectivity and

> [...] is consummated in materialized actions and that, by the agreement between intentional subjects (people) living in a community, and the repetition of what has been agreed upon, becomes objective, capable of being resumed, repeated, understood, vivified in sensory, psychical and spiritual acts (Rosa & Bicudo, 2018, p. 17, authors' translation).

The production of knowledge is what, in this link, occurs in being-with-others, in the dialectic subjectivity-intersubjectivity-objectivity and, in research, when we turn to the production of mathematical knowledge, we understand it as a movement, in which the knowledge constituted by the subject—the mathematicians with whom we dialogue—is expressed through a language, is shared with others, co-subjects with which one dialogues with the intention to expose thought, but it is also openness to dialogue, to discussion. This openness takes place in a specified community,[6] for example, mathematicians, whose intention is to validate what is exposed. What is constituted is validated in the subjectivity of the subject, who originally thought it, and that was expressed through language, is capable of remaining as objective knowledge, whose meaning can always be reactivated. The object of knowledge, in this context of production of mathematical knowledge, is the mathematical object.

5 The Research in Dialogue with the Mathematician

Seeking to understand *how the mathematician, who produces mathematics with the computer, expresses the way in which they understand the presence of this technology in their production*, a first question needed to be clarified: who is this mathematician? Through D'Ambrosio (2013) we could understand the profile of this mathematician and seek a dialogue with them. According to this author,

> in the academic world, especially since the seventeenth century, there has been a 'professionalization' of mathematicians. The production of those professional mathematicians must be recognized for meeting criteria of rigor, formalism, and even methods. Thus, it is very well-established who the mathematicians are and what the production of those professionals is. Gradually, the recognition criteria have been better defined and journals and specialized academies were created. Thus, today it is easy to identify individuals recognized as mathematicians. In general, we can consider those identified by the International Mathematical Union, according to criterion for elaborating the International Directory of Mathematicians. The criterion is to have articles indexed in *Mathematical Reviews/Zentralblatt fûr Mathematik*, which are the international references in the area (D'Ambrosio, 2013, pp. 18–19, emphasis added, authors' translation).

We interviewed mathematicians who fit this profile and who are working with pure or applied mathematics using computers in their production. We contacted them via e-mail. Seven of them agreed to talk to us about the subject focused. We conducted five face-to-face interviews and two via Skype, according to the availability of the interviewees. These were semi-structured interviews, which took place in the form of dialogue, aimed at opening a spontaneous conversation. The face-to-face interviews were recorded in audio using a regular recorder or a smartphone. For the interviews conducted via Skype, we worked with the aTube Catcher software, which captures the computer screen and records audio. All interviews were transcribed and resulted in a text open to interpretation.[7]

[6] Community, according to Ales Bello (2015), can be understood as a way for people to organize themselves to be together, with a certain purpose (or common goal).

[7] Following an ethical protocol all interviewees signed a letter of transfer of rights to the researchers (Ferreira, 2019) authorizing the disclosure of their names. So, it was possible to identify them in the text.

For the analysis of the interviews, we followed the procedures of phenomenological research. We start with the ideographic analysis whose intention is to highlight relevant aspects of the speech of each interviewee; those are called Meaning Units (US). Then, we begin the nomothetic analysis through an articulated thinking movement, which seeks for convergences and divergences of what was said to expose general aspects and to show how the interrogation is understood.

In order to make explicit in this text the way which we proceeded during each phase of the analysis—ideographic and nomothetic—we exemplify with a clipping of the interview of Professor Dr. Nicolau Saldanha, noting that the full analysis is in the PhD thesis (Ferreira, 2019). In this clipping we bring the interpretation made, in the form of a table.

In the first column of the table we identify the mathematician interviewed using the initials of their name and a number, which refers to the excerpt of their account, shown in the second column. In the third column there is a rewriting of the interviewee's speech by the researcher, just to put it in formal language and make the text easier to read. In the fourth column we number the meanings attributed to the interviewee's speech (U.S.), also identifying them. For example, in the following Table *NS.1.1–1: test examples* indicates that we understood in Prof. Nicholas Saldanha speech that, for him, the computer is relevant to test examples. His account was the first we analyzed, so NS.1, the first US excerpt NS.1.1, in addition to being the first of all US, NS.1.1–1. The last column of the table was added later, when we organized the U.S. into a new table for convergence and found that numbering them neatly would make it easier for the reader to locate them (Table 1).

Once the ideographic analysis was finished, we had obtained 301 U.S. However, some of them were repeated and we opted for the construction of a summary table, in which we highlighted 178 *Ideas* that remain in the range of those U.S. Continuing the interpretation of what mathematicians said, we saw that 178 Ideas converged into 11 invariants. As we turned to invariants,[8] interrogating the sense and meaning

Table 1 Clipping of the interview with Prof. Nicolau Saldanha

<table>
<tr><th>Code</th><th>Subject account excerpt</th><th>Articulated assertion—researcher rewriting</th><th>Unit of meaning—U.S.</th><th>Organization of U.S.</th></tr>
<tr><td rowspan="2">NS.1</td><td rowspan="2">I have combinatorial work, so, in this combinatory area it is often interesting to have examples, test examples, you often want to do many cases to see what happens and then a computer experience is very useful</td><td rowspan="2">He claims to have combinatorial work and it is often interesting to have examples and test them for various cases, so a computer experiment is very useful</td><td>NS.1.1–1: Test examples</td><td>2</td></tr>
<tr><td>NS.1.2–2: Verify cases</td><td>6</td></tr>
</table>

[8] Husserl, in Formal Logic and Transcendental Logic (1965, p. 33, authors' translation) speaks of an "invariant core" that defines the "essence" of what is shown in and to consciousness as what persists, what remains the same.

Table 2 Invariants and open categories: second reduction

Invariants	Open categories
Computer or software features Communication/visual expression or through visualization	The computer as a potency for mathematical production and expression
The thinking supported/subsidized by mathematics The producing with Validation and the demonstration *x* verification	Mathematical thinking when being-with the computer
Investigation possibility Impossibility or difficulty to do without computer Diversity manifests Openness to the new	Ways of investigating within production with the computer

of what they said, they constituted "open categories." The invariants and the open categories are shown in Table 2.

In the interpretive movement, it became clear to us that the mathematician, with whom we dialogued, considers *the computer as a potency for mathematical production and its expression*, also considers that there is *mathematical thinking when being-with the computer* and that *ways of investigating within computer production* are enhanced. These are the three categories of analysis that allow us to talk about the interrogation.

In Ferreira (2019), each of these categories is discussed, making it possible to understand the meaning of what was expressed by the interviewee focusing on the researchers' interrogation. For the purpose of this text we will focus on one of such categories: *mathematical thinking when being-with the computer*.

6 Mathematical Thinking When Being-with the Computer

The mathematicians interviewed expose the importance of mathematical thinking for the production of knowledge with the computer. This assertion shows the strength of *thinking which is supported or subsidized by mathematics which* reveals *ways of producing with the computer*. In this way of producing, the *validation and demonstration* are in *contrast to verification*, bringing *concern and care* regarding *accuracy*.

To understand the ideas implied in these statements, we believe it is important to focus on the question "what is this mathematics?" In the authors such as Silva (2002), one can understand that one of the fundamental characteristics of mathematics is its universality. In others, such as Davis and Hersh (1986), one can comprehend that *what mathematics is* does not allow itself to be apprehended in a temporal and spatial now, because it is fluid and is shown in a historical construction that embraces multiple conceptions. To these authors, over time, these conceptions are constituted by a certain community of mathematicians who formulates a definition, aiming to expose the meaning that mathematics has for them. This formulation involves the perspective of its producer, the mathematician who constitutes knowledge and who belongs to this community at a certain point in time and space.

Thus, understanding what mathematics is requires clarification of mathematical thinking. This thinking requires knowledge of content, a validation that follows certain rules accepted by the community, and ways of expressing it through mathematical language. This mathematical language has "a symbolic system with its own symbols, which are related according to certain rules whose appropriation/.../ is inseparable from the process of knowledge construction" (Lorenzatti, 2009, p. 9, authors' translation). Therefore, it is a rigorous language whose possibility of expression opens up in the flow of what is produced. However, it is also a language that must express to the community of mathematicians the meaning of what has been understood, interpreted, and produced.

In the accounts of the mathematicians with whom we dialogued, rigor is a constant concern and is characterized both by the language that communicates the production and the manner through which what is produced is validated. The requirement of rigor leads the mathematician to be aware of what is shown to them, while exploring something they investigate, including, and especially with-computers. Thus, the analysis of what is obtained in a computer simulation requires caution, taking into account the accuracy of what is obtained. This accuracy, as our research co-subjects pointed out, requires awareness of the possibility of *errors* that the computer can make when analyzing an answer. Errors that can be detected as a result of the clarity given by the rigor that the mathematical community demands for the validation of what is obtained. The computer is *a partner* that, as in any partnership, must have its hypotheses and results verified, analyzed and validated to legitimate what is obtained.

Professor Ralph Teixeira emphasizes that the mathematician who works "seriously with the computer has to be aware that errors may exist" (Teixeira, 2017. In: Ferreira, 2019, p. 190, authors' translation). Therefore, one should be concerned and ensure that what is investigated does not lead to the disclosure of unreliable results. He points out that the computer may conduct rounding that compromises the validity of the result obtained, due to its characteristics and to the numerical capacity of a particular software. Besides, there is a possibility of a computer glitch that might compromise its functioning. The way mathematicians talk about their concerns and the care they should take in their research with the computer, articulates and expresses the relevance of validating what is produced in mathematics. According to them, there is always a need for a demonstration of what is obtained through the explorations. They argue that what one can investigate with the computer "makes you see" or "gives conviction," but does not prove it (at least, not yet).

They make it clear that the computer, in this partnership, shows them something that they wanted to understand but had not visualized. This "showing" elucidates some idea that they had already sensed or even suggests paths for investigation that had not occurred to them. There are situations where, with the computer, it is possible to validate a conjecture. However, this "validation" is something that helps and contributes to building a path for the demonstration.

What they say can be understood by the statement of Irineu Bicudo (2002) regarding the mathematical demonstration that must satisfy the community of experts. That community is responsible for mathematical production.[9] It defines the type of speech that conveys what is produced, as well as stipulates what should be accepted as valid. Strict proof is the guiding thread of the truth regime of what is produced in mathematics characterizing its style (Garnica, 2002). Thus, proof is a demonstration.

The demonstration that mathematicians understand and defend as necessary is what allows the veracity of a statement to be established (the demonstration thesis) and, according to them, must be done through a language that follows certain standards required by the community. Through demonstration it is possible to argue for the acceptance of what is obtained, making it valid or refuting it. This manner of thinking and doing is maintained even when one works with a computer.

Thus, as Irineu Bicudo (2002, p. 80, authors' translation) states, "to speak in mathematics is to speak of demonstration." With the computer, ideas are tested, conjectures are "discovered," but, as members of a certain community, to publish what was "discovered," the mathematician needs to validate. This statement is corroborated by Professor Márcio Silva who states:

> It [Technology] becomes a very potent tool for some areas [...], but that really only becomes results in mathematics, not because of technology, but because of mathematics itself, because of the mathematical system (Silva, 2016. In: Ferreira, 2019, p. 126, authors' translation).

What MS says reveals the issue of relevance, generality, and veracity of what can be identified through technologies. However, computer demonstration is a point of disagreement within the mathematical community and the answers to the questions *does the computer prove?* or *does the computer demonstrate?* are conflicting. Some say yes, and some say no.

The Four Color Theorem was much quoted in the accounts of the mathematicians. It involves determining the minimum number of colors to color a map so that countries with a common border have different colors, regardless of whether existing or created countries are registered on it (Sousa, 2001). In 1852, Francis Guthrie conjectured that the minimum number of colors was four. More than a hundred years later, in 1976, it was possible to prove that the conjecture was correct by obtaining the famous Four Color Theorem. Kenneth Appel and Wolfgang Haken, with the help of an IBM 360 computer, presented the demonstration.

According to Sousa (2001), the news excited many teachers throughout mathematics departments. But the euphoria faded away when they learned that the demonstration included more than a thousand hours of high-speed computer use. The proof was too long to be done manually. The reason for this "cooling down" of mathematicians was that computer might have made a mistake that was hard to

[9] We understand here mathematics as a science of the Western world.

detect. That is, there was more enthusiasm for the possibility of seeking a demonstration than for what had been proven.

The computer-enabled demonstration of the Four Color Theorem mobilized mathematicians to discuss its acceptance. According to Prof. João Sampaio, a dilemma has been introduced in the community. Although the community is unanimous in considering that the computer showed or made possible to see the map colored with only four colors, so that countries with a common border have different colors, the same does not happen with the acceptance of what was obtained. The reason being that the result could not yet be validated as they deemed necessary, via formal or rigorous language.

On the other hand, João Sampaio says that the investigation with the computer gave rise to new research problems and new theories, such as the graph theory.

> There was that controversy about the Four-Color Theorem, which was computer-demonstrated in the 1970s, and for which a formalized demonstration was never achieved. This can also happen, of course. But the team had to spend hours programming the verifications. (Sampaio, 2016. In: Ferreira, 2019, p. 67, authors' translation).

> A new paradigm was created. The team had to rethink the issue of mathematical truths not being constructed by human thinking alone, they may have the use of technology /…/ this is a problem that arouse in a somewhat playful way and even raised a mathematics research problem, originating graph theory /.../ and ended up having a positive answer /.../ this statement is true, although no one has been able to formally demonstrate this as mathematicians do, but with the use of computers, verifying a large sequence of particular cases. [...] today it seems that this has been overcome, there have even been simplifications of the algorithm, the program, but nothing beyond that. The computer is still there as a demonstration element, the program. (Sampaio, 2016. In: Ferreira, 2019, p. 68, authors' translation).

The dilemma is understood in accepting what becomes clear in computer investigation. With the computer, it is possible to analyze several cases, but still it cannot be said that there is validity for each and every case, as required in a mathematical demonstration. To mathematicians, mathematical demonstration requires that "step-by- step" which can be checked, verifying that there are no mistakes, "gaps" or errors, and if it is valid for each and every case, and can be accepted by what they call "purist" mathematicians.

There is no consensual answer to the question *does the computer demonstrate?* For Prof. Márcio Silva, the process of mathematical validation is via demonstration and, with the computer, one cannot validate what is investigated, there are only conditions to verify.

> The process of mathematical validation is different, it comes via demonstration, via formal language, formal demonstration, so I would not say validation, but verification. (Silva, 2016. In: Ferreira, 2019, p. 196, authors' translation).

This author claims that the computer helps in the investigation and enables the verification of results. However,

> Is not validation. Because validation is not useful, it may even become a conjecture, but it is nonetheless a conjecture. If you do not present a formal demonstration of it, it still remains a conjecture. (Silva, 2016. In: Ferreira, 2019, p. 196, authors' translation).

> The technology is great for launching conjectures, but it is just that, to launch conjectures. To demonstrate you need to turn to the formal axiomatic system of mathematics and, from that, write a demonstration. There is no escaping that. (Silva, 2016. In: Ferreira, 2019, p. 196, authors' translation).

On the other hand, Prof. Ralph Teixeira states that an investigation with the computer could be as valid as what is formally demonstrated, if care is taken to clearly analyze and expose what is obtained.

> Just as the person who wrote a classic article has to explain all the things they have done for us to read, criticize, and try to find errors, people who produce computer-aided demonstrations, algorithms, they must publish the algorithm for us to read, verify, contest, see if something can go wrong. So, are they susceptible to error? Yes. As much as classics, for example. (Teixeira, 2017. In: Ferreira, 2019, p. 197, authors' translation).

We asked him if he had any of his technology production exposed this way. He answered no, as his research is not produced with technologies. That is to say, his investigation is based on the GeoGebra software,[10] which allows one to visualize certain aspects of what is interrogated, but does not use the software to perform simulations. He then uses the formal axiomatic system to validate what is shown in the software when conducting investigations.

> As much as we see, what motivates us, as much as we see the theorem in GeoGebra, we still feel the need to demonstrate formally.
>
> /.../ if you take one of the last articles we have published, they are articles about discrete mathematics for which we use GeoGebra to make figures /.../ we mention that we used GeoGebra /.../ but in fact, actually, the theory that is written in the article /.../ does not need GeoGebra to be constructed. We used GeoGebra to discover the theorems, but then we will prove them in the classic way, like everyone does. (Teixeira, 2017. In: Ferreira, 2019, p. 197, authors' translation).

We understood from their accounts that, although through the computer there is openness to mathematical investigation, in expressing what they produce, they resort to formal demonstration. Prof. Nicolau Saldanha, for example, seeks to justify this by stating that "there is this official line of mathematics which says: either the thing is proven or it is not. If it is proven, there is no doubt about it, it is absolutely right." (Saldanha, 2017. In: Ferreira, 2019, pp. 92–93, authors' translation).

So, we questioned ourselves: What then is produced with the computer? From what the mathematicians interviewed claim, the computer contributes to "corroborating a distrust, a suspicion, a conjecture, but the construction of mathematical knowledge, its consolidation, really resides in the realm of ideas," according to Prof. João Sampaio (Sampaio, 2016. In: Ferreira, 2019, p. 198, authors' translation).

[10] GeoGebra is a free dynamic mathematics software. It has tools for constructing mathematical objects allowing visualization of function graphs, 2- and 3-dimensional geometric figures, and other mathematical content. It is available for download at: https://www.geogebra.org/download

From this argument we understood that the characteristic of mathematical production is demonstration. For the community of mathematicians, mathematical knowledge becomes firm and can be publicized when it has been formally proven.

However, their argument also made clear that the computer has a supporting role in investigation. As Professor Nicolau Saldanha states, the computer is a partner.

> That coworker who thinks very differently from you and who knows the things you know badly; knows badly the things you know well. That is cool /.../ because everyone does something well and they go together. So, the computer is very much that, it is a partner, let us say, that does something very well, such as looking at a billion examples. However, other things, the human mind does better, so it is a partnership between two... different skills. So, it works in the most varied ways. (Saldanha, 2017. In: Ferreira, 2019, p. 262, authors' translation).

This partnership between the mathematician and the computer enhances the work, allows a logical dialogue, enables seeing and makes possible to advance in terms of understanding what is shown and making conjectures. We highlight words such "partner," "makes it clear," "brings clarity," "suggests new conjectures," "gives conviction," which are recurrent in their accounts. We interpret them as characterizing the way the interviewees view the computer in their production. Even if the rigor of mathematical research requires that they demonstrate their findings with the computer through mathematical language, they claim that the way of being-with the computer is what enables them to investigate.

If we consider that investigation is characteristic of mathematical production, the way the mathematician is—with the computer shows that it gives them the possibility to advance in their production. As Ponte (2003, p. 21) claims, the investigation with the computer,

> requires a rationality very different from mere opinion. It presupposes an effort for clarity in concepts, reasoning and procedures from those who perform it. It presupposes reflection, debate, and in-depth criticism by the peer community. This, of course, requires that ideas be presented in a sufficiently detailed and rigorous way to be understood and debated. It requires a stronger argumentative rationality than simple justification *ad hoc* and it requires knowing the general paradigm or theoretical framework by which this rationality can be measured. (Ponte, 2003, p. 21, authors' translation).

In the analysis of what is said by the mathematicians interviewed, we understand that there is a way to produce knowledge with the computer, because investigation is leading their work and opening them up to research. The investigation becomes the ground for the constitution of knowledge, being present in the process of the mathematician testing their hypotheses, verifying, choosing mathematical models, or electing the most appropriate software for that case, to analyze what has been seen, to weigh the results obtained, to compare them in the light of what they know, to express their understandings, to open themselves to dialogue with the co-subjects, with other mathematicians in their community. In the investigation process, the way the mathematician *is–with* the computer is what allows them to broaden their understanding of concepts and see other possibilities they had not thought of. So, the mathematical knowledge produced with-computers is expressed in a language, validated, and available for analyses. It remains in the historical and social context of a certain community; remaining as a possibility of reactivating its original evidence (Husserl, 1970).

7 Comprehensive Syntheses: Resuming What Was Investigated

In this research we understand that being-with the computer opens possibilities for understanding and expression of the investigated theme. In this case, the computer is not only taken as a tool or utensil, but as a partner that enhances research. Being-with shows a posture of the subject who works with-computers. It requires that one opens oneself to what is shown in the action triggered.

The movement of mathematical thinking being-with computers highlights the importance of knowledge of mathematical content, as well as the chosen software itself. This means that mathematicians face mathematics with an attitude of understanding what they seek to know, and not as theoretical knowledge to be applied without analyses and reflection. At the same time, they seek to understand what appears on the screen in response to the commands they activated. The choice of software also requires this knowledge, as well as of its operational characteristics and numerical capacity. In addition, the mathematicians who work with-computers investigating mathematical issues need to be aware of bugs that can lead to errors.

Mathematicians claim that the computer does not know everything and does not do everything. It is as a coworker. We trust it, but we remain attentive to what it does. We do not want to criticize or invalidate what it shows. But we want to make what it (the computer) shows clear. This way, we keep arguing, questioning, debating.

The production with-computers requires this attentive partnership. Those who work with-computers need to assume a responsible attitude. It is like a team whose members work together. This work, the production of mathematical knowledge, requires demonstration, since what is investigated must be questioned, validated, and presented to the community through a certain language that brings rigor.

We agree with Gadamer (2002) that language is what makes it possible to express the articulated acts in the movement of consciousness. Through this research, we understood that the thinking movement of mathematicians was done with the computer, and the language expresses an intertwining between those who question, propose problems, think; the programmer of the software; and the functioning of the computer.

References

Ales Bello, A. (2015). Pessoa e comunidade: comentários. *Psicologia e ciências do espírito de Edith Stein*. Tradução Miguel Mahfoud e Ir. Jacinta Turolo Garcia. Belo Horizonte: Editora Artesã.

Barbariz, T. A. M. (2017). *A constituição do conhecimento matemático em um curso de matemática à distância*. Tese de Doutorado, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Rio Claro.

Bicudo, I. (2002). Demonstração em Matemática. *Bolema, 15*(18), 79–90.

Bicudo, M. A. V. (2014). A perplexidade: ser-com-o-computador e outras mídias. In M. A. V. Bicudo (Ed.), *Ciberespaço: Possibilidades que abre ao mundo da educação* (pp. 33–62). São Paulo: Editora Livraria da Física.

Borba, M. C., & Villareal, M. V. (2005). *Humans-with-media and the reorganization of mathematical thinking: Information and communication technologies, modeling, experimentation and visualization*. New York: Springer.

Borba, M. C. (2002). Coletivos seres-humanos-com-mídias e a produção de Matemática. In Anais do I SBPEM - *Simpósio Brasileiro de Psicologia da Educação Matemática*, Curitiba, Paraná (vol. 1, pp. 135–146).

D'Ambrosio, U. (2013). Por que e como ensinar história da Matemática. *Rematec, 8*(12), 7–21.

Davis, P. J., & Hersh, R. (1986). *A experiência matemática* (3rd ed.). Rio de Janeiro: Editora Francisco Alves.

Ferreira, M. J. A. (2014). *A expressão no ciberespaço: um voltar-se fenomenologicamente para o diálogo acerca de conteúdos matemáticos*. Dissertação de Mestrado, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Rio Claro.

Ferreira, M. J. A. (2019). *A constituição e a produção do conhecimento matemático ao ser-com o computador*. Tese de Doutorado, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Rio Claro.

Gadamer, H.-G. (2002). *Verdade e método II: complementos e índice* (4th ed.). Tradução de Ênio Paulo Giachini. Petrópolis: Vozes.

Garnica, A. V. M. (2002). As demonstrações em Educação Matemática: um ensaio. *Bolema, 15*(18), 91–99.

Heidegger, M. (1962). *Being and time. Translated by John Macquarrie & Edward Robinson*. Oxford: Blackwell Publishers Ltd..

Heidegger, M. (1971). *Sobre a essência do fundamento*. Tradução de Ernildo Stein. São Paulo: Livraria das Cidades.

Heidegger, M. (2001). *Ensaios e conferências* (8th ed.). Petrópolis: Vozes.

Heidegger, M. (2005). *Ser e Tempo* (15th ed.). Petrópolis: Vozes.

Husserl, E. (1965). *Logique Formelle et logique transcendantale*. Paris: Presses Universitaires de France.

Husserl, E. (1970). The origin of geometry. In E. Husserl (Ed.), *The of European sciences and transcendental phenomenology*. Evanston: Northwestern University Press.

Kalinke, M. A., & Almouloud, S. A. (2013). A mudança da linguagem matemática para a linguagem web e as suas implicações na interpretação de problemas matemáticos. *ETD – Educação Temática Digital, 15*(1), 201–219.

Lévy, P. (1993). *As tecnologias da inteligência: o futuro do pensamento na era da informática* (2nd ed). Rio de Janeiro: Editora 34.

Lorenzatti, E. J. C. (2009). Linguagem Matemática e Língua Portuguesa: diálogo necessário na resolução de problemas matemáticos. *Revista Conjectura, 2*, 89–99. Acesso Janeiro 24, 2019, em https://www.ucs.br/site/midia/arquivos/linguagem.pdf

Ponte, J. P. M. (2003). *Investigar, ensinar e aprender*. Acesso Janeiro, 24, 2019, em http://www.educ.fc.ul.pt/docentes/jponte/docs-pt/03-Ponte(Profmat).pdf

Rosa, M., & Bicudo, M. A. V. (2018). Focando a constituição do conhecimento matemático que se dá no trabalho pedagógico que desenvolve atividades com tecnologias digitais. In R. M. Paulo et al. (Eds.), *Ser professor com Tecnologias: sentidos e significados* (pp. 13–44). São Paulo: Cultura Acadêmica. Acesso em Janeiro, 24, 2019, em http://www.culturaacademica.com.br/catalogo/ser-professor-comtecnologias/?fbclid=IwAR26zpLdx0wJ1e3uJCSGQcyElGlRPyGvsyFQvXyiM1Y7b_iOf_Ztwf-ZJc

Silva, J. J. (2002). A demonstração matemática da perspectiva da lógica matemática. Bolema, 15(18), 68–78.

Sousa, L. (2001). O Teorema das Quatro Cores. *Millenium, 24*, 125–151. Acesso em Janeiro, 24, 2019, em http://www.ipv.pt/millenium/Millenium24/12.pdf

Tikhomirov, O. K. (1981). The psychological consequences of the computerization. In J. Werstch (Ed.), *The concept of activity in soviet psychology*. New York: Sharp.

The Constitution of the Mathematical Object in Acts of Evidence, from the Perspective of Edmund Husserl's First Logical Investigation

Jamur Andre Venturin and Flávio de Souza Coelho

1 Initial Considerations

As the quest to understand Edmund Husserl's *First Logical Investigation* continued, we noticed that the aim of the *Investigation* was to explain the foundations of pure logic. On the other hand, insofar as Husserl speaks of pure logic, and understanding that this is the beginning of his phenomenological philosophical project, we realize that the elaboration of the initial concepts of phenomenology are always actualized through Husserl's life as he advances as he furthered and deepened the analysis of his own work, criticizing it. Thus, the phenomenological concepts are realized, always advancing in terms of opening understandings of their meanings.

Another noteworthy point is the way Husserl conducts his analyzes, namely from a philosopher's point of view. The movement is performed in zigzag, explaining different sides of what he investigates. His style brings about the difficult task of reading his production, which requires patience and care in order not to fall into the trap of conceptual determinism. The object of the investigation is spread throughout the text as a phenomenon surveyed. Therefore, it is always possible to make several points and understandings when following his movement of analysis. In this case, we understand the reader is required to have an encounter with the entire text in order to be able to evaluate the theme in question. As an example, we can cite the question of "essence," thematized by Husserl in different nuances, but always seeking to open meanings to understand ways through which essentialities are shown. This is a warning.

With this investigative movement, assuming this work by Husserl as ground for understanding, we seek to clarify how mathematical objects are constituted. We are

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aware that posing an interrogation, as a guide for our reading, could condition the theme presented to a path to be followed. However, at the same time, we understand that this movement would open a possibility to examine the target without presupposing any assumptions. The delimitation and openness afforded by the interrogation can be evaluated, from Gadamer's point of view, when stating that: "Asking, is putting out in the open" (2012, p. 474, authors' translation), as there is no immediate answer to the question. However, according to Gadamer (2012), even when posing an open question there is a demarcation of the investigative horizon. For that author, the importance of defining the investigative horizon is justified by the need for the inquiry to direct the investigation, including establishing a perspective of what is interrogated.

All that has been said shows the need and importance of conducting a hermeneutic reading of Husserl's text, guided by our interrogation, namely how is the movement of constitution of the mathematical object, from a phenomenological perspective? We emphasize that, in the work analyzed, Husserl does not specifically deal with the mathematical object, but this object is approached in a non-thematized way throughout his paragraphs. In light of this scope, we interrogated the text, seeking to understand the ways by which he approaches to this object that, within the focus of this investigation, we thematize as mathematical object. In order to show how far we realized the question; we provided the understanding in written form. Doing so, we agree with Gadamer when he points out that, "All understanding is interpretation and all interpretation develops through the *medium* of language which intends to let the object speak, and, at the same time being the interpreter's own language" (2012, p. 503, author's emphasis, authors' translation).

For Gadamer, language is understood as "the *medium* of hermeneutic experience," as it is with language that the agreement among interlocutors, or between the text and the interpreter, is affected. So, it is the language that makes it possible to intersubjectively understand an interrogation. Gadamer ponders that when an understanding movement takes place, the meaning, which was expressed and materialized through language, shows the interpretation of the interpreter through the *common language* generated in that movement. For this reason, "all understanding is interpretation," that is, there would be no understanding without interpretation. Besides, without the possibility of expression through a *common language*, we could not have a hermeneutic experience.

This *common language* is produced, when interlocutors agree on a subject matter as well. It is the *medium* through which what is taken as common meaning during a conversation is explained. That is, the *common language* expresses what is understood through the investigative process (Gadamer, 2012).

In the present research, the *common language* was agreed upon by us with the texts. We reached this agreement through analyses, interpretations, and reflections regarding what stood out during the studying process. Such movement led us to produce a discourse, presented herein, in written language.

Under this perspective, experiencing agreements is to perceive oneself as being in agreement. This way of understanding such experience is contrary to a view which assumes a prior agreement, as an adjustment of ideas that confirm what was initially questioned. According to Gadamer "As the conversation proceeds, both [interlocutors] submit to the truth of the matter at hand, which unites them in a new community" (2012, p. 493, authors' translation). In short, it is an agreement in the spiritual dimension understood in an epistemological perspective, which intertwines with the ontological perspective. In other words, it is *mutual understanding* (Husserl, 2012).

In order to be rigorous in our investigative procedure, it is important to indicate the work, volume, and edition analyzed in the production of the present work. We studied the first two editions, volumes 1 and 2, or parts thereof, and what they deal with in the *Logical Investigations*. Volume 1 of the first edition, published in 1900, entitled *Logical Investigations*: *Prolegomena to Pure Logic,* and volume 2, also, of the first edition, *Logical Investigations: Investigations for Phenomenology and Theory of Knowledge*, 1901. We must clarify that we decided to study these works in order to answer the question: Why did Husserl write these works? To answer this question, let us make a brief historical retrospective of the Husserlian phenomenological project between 1890 and 1913.

According to Martinelli (2018), in *Philosophy of Arithmetic* (1890) Husserl asserts that the laws of logic are supported by psychology. However, Frege criticizes the work of Husserl, stating Husserl had confused number with its representation: number is ideal (objective) and its representation is psychical (subjective). In this direction, the above-mentioned author asseverates that "For Frege, when trying to explain the psychological origin of numbers, Husserl confused number with the representation of number, falling into a psychologism" (Martinelli, 2018, p. 48, authors' translation). As a result of such discussion, Husserl wrote the first edition *of Logical Investigations*: *Prolegomena to Pure Logic,* in 1900, stating that it would be a mistake to fundament the laws of logic in psychologism,[1] as, at that time, psychology sought to prove its truth by proceeding with the inductive method, considering empirical procedures and thus, through observations that led to generalizations, as Martinelli states. The author asserts that for Husserl this practice of establishing truths was contingent, that is, the laws of logic based in the laws of nature are error-prone and, for the philosopher, the laws of logic are "conducive to thought" (Martinelli, 2018, p. 50, authors' translation).

For Martinelli, the mistake regarding psychologism, seen by Husserl, consisted in saying that the act of judging what is real, founded in empirical science, would be the ground of logical laws. In this sense, she exposed the criticism of psychologism made by Husserl and by logicists who defended logic: "As it is a science of facts and an experimental science, psychology clarifies *how thought occurs*, which leads to empirical generalizations, leading to contingent generalizations, as Husserl puts it. However, logic deals with the *necessary rules*, of *how we should think*; interested in analyzing the ideal connections between propositions" (Martinelli, 2018, p. 51, emphasis added, authors' translation). This means that, for psychologism, "how

[1] Psychologism is understood as the process of psychologically analyzing what is not in the field of psychology; a reduction of the object in question to psychology (Peres, 2017).

thought comes to be" is based on a physical-mathematical model. There are laws, and therefore truths being generated through experiences, which can be indeterminate rather than absolute. On the other hand, for logic, there are *necessary rules* that guide the way we should think. As we understand with Martinelli, the rules which support the constitution of new propositions must be general, true, and therefore absolute.

This way of referring to the laws of pure logic remains the path followed by Husserl in *Investigações Lógicas: segundo volume, parte I: Investigações para a Fenomenologia e a Teoria do Conhecimento*, segundo volume, 1901. However, to describe the theory of knowledge, he characterizes phenomenology as *descriptive psychology*. It seemed to us that he retrogressed to the psychologist ideas disputed in the previous work (Husserl, 2012 (1901).[2] Hence, he presented the second edition of Volume 2, in 1913. According to Peres (2017), the second edition of *Logical Investigations*, Volume 2, 1913, modify the 1901 text. Husserl now presents the theme of *transcendental reduction* that was not presented in a prominent way in 1901, but was fragmented, as Husserl himself pointed out in the 1907 texts.[3] Peres states that details regarding transcendental reduction appeared in *Ideas. General Introduction to Pure Phenomenology,* in 1913. Peres asseverates that with the revision to Logical Investigations, Husserl decharacterizes phenomenology viewed *purely* from a psychological standpoint, as "psychology (of essence or fact) as far as it presupposes a certain concept of reality cannot fundament epistemology with the necessary radicalism" (Peres, 2017, p. 118, authors' translation), thus turning to transcendental phenomenology.

That being said, we state that our formulations of judgment are derived from the analysis of the *First Logical Investigation*, entitled *Expression and Meaning*, in *Investigações Lógicas: segundo volume, parte I: Investigações para a fenomenologia e a teoria do conhecimento*, segunda edição, 1913.[4]

After presenting these considerations, we will present the organization of this work.

In the first part, a description of the themes/concepts necessary for understanding the constitution of the object is presented; in the second part, the way through which Husserl links pure logic and meaning; and, finally, considerations regarding the constitution of the mathematical object.

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[2] We indicate 1901, because given the structure of the second edition of Logical Investigations, Volume 2, it is possible to access, in volume 2, the text of Logical Investigations, Volume 1, first edition. For further information, see the Portuguese translation of Logical Investigations, volume 2, Part I, Investigations for Phenomenology and Theory of Knowledge, Second Edition, 1913.

[3] Cited in the Introduction of the German editor Walter Biemel of "The Idea of Phenomenology", while mentioning Edmund Husserl's manuscripts of September, 1907, BII 1 (Husserl, 2008a).

[4] The Sixth Logical Investigation was published in 1921, separately from part I of the second volume.

2 Nuclear Concepts for Understanding *Meaning*, *Expression, and Object*

In order to understand how the object is engendered, we will follow the path laid out by Husserl (2012), analyzing and showing the distinction between *sign* and *expression,* as they cannot be understood as synonymous. Husserl states that "Every sign is a sign for something, but not every sign has 'meaning', a 'sense' that the sign 'expresses'" (Husserl, 2008b, Chap. 1, paragraph 1).[5] And "To mean is not a particular way of being a sign in the sense of indicating something" (Husserl, 2008b, Chap. 1, paragraph 1). In order to highlight the difference between sign and expression, we give the example present in the First Investigation: "The chalk-mark of the robber is a mere indication, while a proper name is an expression" (Husserl, 2008b, Chap. 1, paragraph 16). According to Husserl, a *proper name* is an expression and even sometimes functions as an index (characterizing sign), as it shows, points to, and manifests a desire to explain the *object* represented*,* which is being seen and named. Hence the reason to understand the proper name functioning as an *index* (characterizing sign), which is manifested. The proper name is understood as manifesting the object, representing it, but the function of the manifestation is secondary to the *meaning*, as the author attests, because its objective is to express and present an object that is signified here and now. That is, the primacy of the proper name is not the representation of the object (manifestation), but the signified object. The name speaks of the object, while the "chalk outline of the thief" is understood only as an index. Being an index, it just points to whatever one wants the other person to see. It serves as a distinction. It is neither intended as an object or an ideality.

Husserl treats the themes of the First Investigation, *expression* and *meaning*, as phenomenon under investigation. As his work always moves forward, opening interpretations, it does not cause strangeness when he puts under examination the word *expression*, seeking to point out the difference between *sign* and *expression*. The latter understood as the "sound of word animated with signification" in *communicative discourse*. The philosopher exposes that in communicative speech *what is said* by the subject (an articulated sound-complex) is the *spoken word.* But this will only be so if there is a psychical opening among interlocutors and that which is being spoken is heard by the other subject. "Such sharing becomes a possibility if the auditor also understands the speaker's intention" (Husserl, 2008b, Chap. 1, paragraph 7). What is pronounced could only be understood in mutual agreement, from the point of view that there are corresponding physical and psychical experiences of discourse, that is, it is understood that there are living-experiences between the subjects who speak to each other, which converge. So, they lead interlocutors to establish spiritual "bonds" and, therefore, "submit to the truth of the matter" (Gadamer, 2012, p. 493, authors' translation). In this direction, the expressions besides fulfilling the function of indicating assume the jurisdiction of the index, characterizing a

[5] In order to preserve academic rigor, direct quotations were taken from the work published in English.

sign. As they point in one direction, they are understood as drivers of thought, whose purpose is meaning. That is, the comprehension of what is presented in one articulated sound, constituting the spoken word, in order to perceive the subject who manifests themselves as a person. This is not conceptual knowledge: "When I listen to someone, I perceive him as a speaker, I hear him recounting, demonstrating, doubting, wishing etc. The hearer perceives the intimation in the same sense in which he perceives the intimating person [...]" (Husserl, 2008b, Chap. 1, paragraph 7). Here, we recognize that mutual understanding takes place in the dimension of intersubjectivity through the spoken word.

The breakdown of what we have written regarding the spoken word (sound-complex) is detailed in paragraph number nine of the above-mentioned work. It states that the *physical appearance of expression,* understood as a moment of presence through a physical state[6] with *intention of meaning, that is,* which confers meaning, understood as an essential act, with *intuitive completeness*, that is to say, fulfillment of meaning, constitutes "[...] its relation to an expressed object (Husserl, 2008b, Chap. 1, paragraph 9). With these three acts, we believe that the movement of constitution of the object makes it public, thus experiential, as we shall see later. Husserl (2012) states that:

> A *name*, e.g., names its object whatever the circumstances, in so far as it *means* that object. But if the object is not intuitively before one, and so not before one as a named or meant object, mere meaning is all there is to it. If the originally *empty* meaning-intention is now fulfilled, the relation to an object is realized, the naming becomes an actual, conscious relation between name and object named (Husserl, 2008b, Chap. 1, paragraph 9, *author's emphasis*).

Let us explain what was announced in the citation above, in particular, the *intention of the act of meaning*. It is understood as the primordial act for mutual understanding in communicative discourse. In Husserl (2012), *essential acts* have the purpose of conferring meaning, or are understood as intentions of meaning. With these acts the expression is animated; it is "[...] a verbal sound infused with sense" (Husserl, 2008b, Chap. 1, paragraph 9), and the extra essential acts are those that fulfill the intention of meaning that "[...] (confirming, illustrating) it more or less adequately, and so actualizing its relation to its object" (Husserl, 2008b, Chap. 1, paragraph 9). These last acts constituted in the living-experience are understood as units of knowledge and are called acts *of fulfillment of meaning.* Thus, meaning, depicted in the expression, is constituted by appearing, with meaning intent, and the act that fulfills meaning. Finally, Husserl points out that the intention of meaning act is primordial to what is manifested in communicative discourse.

We understand "the intention of meaning of the fundamental act in a co-communicative discourse" with Derrida (1994) who asseverates that only through voice it is possible to overcome worldliness. That is, any other form of communication between interlocutors requires an empirical, hence a sensorial representation.

[6] In agreement with Onate, with understand physical state as "[...] is a concrete sign phenomenon" (2016, p. 91, author's translation).

This is not the case with the sound of a voice. It surpasses the empirical and aims for ideality. Derrida states that

> *The voice is heard.* Phonic signs [...] are "heard" by the subject who utters them in the absolute proximity of their present. The subject does not have to go outside themselves to be immediately affected by their expression activity. My words are "alive," because they do not seem to leave me: they do not scape me, out of my breath, into a visible distance; they do not cease to belong to me, to be at my disposal, "without accessories" (1994, p. 86, *author's emphasis*, authors' translation).

When we communicate with significant languages,[7] such as writing and illustrative figure, what is intended is represented. In turn, representation has a "form" that characterizes it, making it dependent on a sensory act, therefore, subject to contingency. That is, it is not subject to mutual understanding, "[…] so, the ideality of the object consists only in its being-for a non-empirical consciousness" (Derrida, 1994, p. 86, authors' translation).

Derrida observes that, for Husserl, communicative discourse is the ideal mode of communication, because "when I speak, it encompasses the phenomenological essence of this operation that *I hear myself at the time* I speak. The signifier animated by my breath and the intention of meaning [...] is absolutely close to me" (1994, p. 88, *author's emphasis,* authors' translation). Therefore, it is with the voice, or as Husserl states, with the "sound of words animated with meaning" that the ideal unity is understood.

Thus, we believe that it is with the sound of a voice that meaning is grasped by the interlocutor. It is *to be aware of...* It is the clear and obvious intuition, animated by the listener here and now. It is a sense-giving act. Hence, as a result, we understand that illustrative figures are not necessary during interlocution. So, we turn our attention to the speaker who speaks of the object grasped. Regarding writing and voice, Gadamer (2012, pp. 229–230, authors' translation) states

> There is nothing as strange and demanding to perception as writing. Not even an encounter with people who speak a foreign language can be compared to this strangeness and estrangement, as the language of gesture and tones of voice encompass a moment of immediate comprehensibility.

For him, the language of gestures and the tone of voice, in addition to announcing the presence of the speaker, enable the immediate understanding of what is being said. Thus, we understand that the phrase "the sound of a word animated with meaning" by the listener constitutes the act of meaning of those who are united by a communicative purpose.[8] This event generates mutual comprehension and, as long as it signifies, an objective common reference.

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[7] Derrida talks about the possibility of surpassing the absence of a voice: "Mutism and deafness go together. A deaf individual can only participate in a colloquy by shaping their deeds in the form of words whose *telos* allows them to be heard by the speaker" (1994, p. 89, author's translation).

[8] Still, the complexity of communicating to other individuals what is learned can be overcome if the interlocutors are focused on what is expressed in a *pure* way. "Pure," in the sense Ales Bello (2004) points to, means that there are no external affects in the phenomenological analysis; which is performed in the phenomenological field without interference from other philosophical thoughts.

In short, in the dimension of intersubjectivity, the spoken word, or as Derrida puts it, the voice, is animated with sense by the listener, originating meaning. Thus, the spoken word can be understood as a way of describing intentionality, since "every domain of being has its own way of being targeted by intention" (Levinas, 1997, p. 40, authors' translation). In this case, the listener, by intending the sound of spoken word, animates it with sense. This movement gives origin to meaning.

Let us now examine how the acts that originate meaning are thematized with the emergence of expression. We are seeking comprehensions regarding: "the expression itself, its sense and its objective correlate [that is, the physical appearance of what the expression conveys]" (Husserl, 2008b, Chap. 1, paragraph 11), exemplified with the declarative phrase: *the three heights of a triangle intersect at point*.

Husserl states that whoever says it, and whenever it is pronounced the objective validity of this expression will have identical meaning. This is so because the content conveyed in the expression itself is not subjective, but rather expressed in objective language. It is also possible to trace such expression geometrically, which displays the corresponding (physical) objectivity. Husserl points out that the real psychical judgment, which is made about what is stated regarding the intersection of triangle heights may differ, but the value of the assertion does not change. This is a *geometric truth.*

Thus, the statement *the three heights of a triangle intersect at point* is an *ideal unit.* It is *properly* understood as content objectified by this written expression. Husserl points out that the understanding of *the ideal unity as intellectually grasped and the judgment as actualized by a real act* could cause confusion. It could be understood as "My act of judging is a transient experience: it arises and passes. But what my assertion asserts, the content that the three perpendiculars of a triangle intersect in a point, neither arises nor passes away" (Husserl, 2008b, Chap. 1, paragraph 11). Therefore, the judgment promulgated in a real act cannot be confounded as being true, since it is capable of contradicting itself with the ideal unity grasped intellectually.

From the point of view of Levinas (1997), Husserl's *intellection* is understood as the evidence. In the words of Lévinas "The relationship between object and subject is not a simple presence of one with the other, but the understanding of each other, the intellection; and this intellection is evidence" (1997, p. 32, authors' translation). This leads us to understand the *intellectually grasped* expression as the evidence apprehended.[9] It is the *apprehension* of what is inquired in the investigative search. However, the clear evidence does not occur in a passive situation of the inquiring subject. On the contrary, to cope with what is under investigation, it is required acuity, effort and subtlety in the mobilization of acts triggered in the movement of investigating, in order not to be surprised by *pseudo evidence*, in which empirical judgment dwells. In this case, there is a facticity that is therefore subject to indeterminacy. In order to overcome it, the demonstration of the propositional is conducted, for example.

[9] Within the text, to refer to "apprehended evidence," we wrote "intellectually grasped."

For Husserl, expression becomes objective reference insofar as it signifies. In other words, the expression names the object in light of meaning. The philosopher points out that in objective reference, which points to the object, two situations may occur: the first, when expression expresses the object itself; the second, when the ideal correlate is being filled with meaning, thus constituting the fulfilling sense. In the latter, the intention of signification is animated with meaning by the sound of the word. It is filled with acts of fulfillment of meaning, with the connections of thoughts evidenced by indices. These acts confirm, reinforce, and illustrate them and thus evidence the correlated object, which is designated in the same way as the intention of meaning, and thus, pointing to it. Husserl states that with this process, the clarity of the intended correlate is affected and the fulfilling sense is triggered. It should be noted that, in the second situation, the *content* and the *object* present in a perceptual assertion are apparently different. The clarity is not perceived immediately. Despite that, they can become identical in the acts of fulfillment of meaning, as "In the unity of fulfilment, the fulfilling content coincides with the intended content, so that, in our experience of this unity of coincidence, the object, once intended and "given," stands before us, not as two objects, but as *a single one*" (Husserl, 2008b, Chap. 1, paragraph 14, *author's emphasis*). Husserl also states that the intended object emerges as an idea and is linked with its correlate, in the act of fulfillment of meaning, also as an idea, guaranteeing identical meaning (content), or ideality (essence).

Husserl observes that, in certain expressions, meaning could sound strange to the listener. He also states that, in his time, there would be misconceptions about meaning and the total absence thereof. So, for him, it was necessary to clarify the required difference. He asseverates that: "It is part of the notion of an expression to have a meaning:" (Husserl, 2008b, Chap. 1, paragraph 15). Consequently, he points out that expressions without meaning have no corresponding object and cannot be considered as expressions. He exemplifies that with the word *abracadabra*, stating that it does not have a unitary meaning, and does not aim at a correlated object. That is why he understands that the core of the expression is presenting meaning. *Abracadabra* does not objectively point to an object, but rather at a surprise to be fantastically revealed. On the other hand, in analyzing the name *mountain of gold*, even though there is no such corresponding object, and therefore no meaning, in so far as meaning is associated with the object expressed in the expression, one has the understanding of the nonexistence of the correlate.

As a result of what we have described, Husserl felt the need to clarify "[...] Here men generally distinguish objectlessness from meaninglessness" (Husserl, 2008b, Chap. 1, paragraph 15). In order to do that, the presented "expressions" such as round *square, squared circle,* and *regular decahedron* understood as contradictory because there is no respective corresponding object. For the analysis of such cases, we bring Marty's objection, brought by Husserl:

> "If the words are senseless, how could we understand the question as to whether such things exist, so as to answer it negatively? Even to reject such an existence, we must, it is plain, somehow form a presentation of such contradictory material" … "If such absurdities are called senseless, this can only mean that they have no rational sense" (apud Husserl, 2008b, Chap. 1, paragraph 15).

Husserl agrees with Marty, and understands that total meaninglessness occurs, for example, when we mention the word *abracadabra*. Now, in the cases mentioned *mountain of gold*, *round square, quadrangular circle*, and *regular decahedron* there is no corresponding object, but, on the other hand, as Marty pointed out, there would be a rational understanding that such object does not exist. In these cases, it is not possible to affect fulfillment of meaning that confirms and actualizes the objective reference.

Therefore, an expression will have meaning when it is possible to establish a correspondence of what was intended with its fulfillment. That is, an expression has meaning if there is a corresponding unitary illustration, as Husserl states. He also claims that we must understand that the ideal content occurs with the relation of the ideal unity, the intended meaning, with the fulfillment of meaning, that is, the corresponding illustration. That content is not subordinate to the contingent. Finally, in the philosopher's words: "We apprehend this ideal relation by ideative abstraction based on an act of unified fulfillment" (Husserl, 2008b, Chap. 1, paragraph 15).

There is indissolubility of meaning content expressed in the expression with the corresponding objective reference which is represented objectivity. Husserl points out that *illustrative images of fantasy* must be avoided and should not be understood as possible meanings. This is because the search for fantasy images would be the task of descriptive psychology, with the intention of presenting only the apparent. In making this observation, this images which illustrate fantasy would gain the status of being part of the expression. However, the possibility of understanding the meaning of an expression with *fantasy images* is not denied. He questions whether the representations of fantasy should constitute what is being expressed, taking into account, that the representation of fantasy may not be what should be intellectually grasped. That is, an apprehension that clearly highlights what it *is*.

On this issue, he warns us that we represent nonsense, for example, when we say that the sum of the internal angles of a triangle would be either greater or smaller than the sum of two right angles, or what would the representation of a *chiliagon* (polygon with thousand sides) be. He observes that, in these two examples, the representations of fantasy are "crude images" that would not represent the totality of what was thought. He states that *image* would only be a recourse to "show" the *intellection,* that is, what is understood through acts of consciousness. It would be also a support materialized whose purpose would be geometric idealization and, also, a process to help understanding what is being focused on. In this sense, he claims that sensible images do not carry meaning within themselves, but rather fulfill the phenomenological function of enabling the visualization of a concept, while functioning, aiding comprehension.

When the symbol which is perceived is comprehensively embraced, the materiality of the symbol is manifested in written language and in sensitive images. This does not alter its already constituted meaning. Therefore, what the materiality of the symbol alters is the living-experience of tracing or representing the symbol, but the comprehended sign does not change. The experience of tracing the symbol shows what is shown. On the other hand, "It is in this sense-giving act-character" (Husserl, 2008b, Chap. 1, paragraph 18) and not when the symbol is represented sensuously by images and parallel representations. This is the reason why Husserl states that: illustrative intuition is not required in the process of apprehending understanding. He states that both scientific thought and certain everyday actions do not require illustrative figuration. That is, they do not need to be exemplified.

> […] we rather live entirely in the consciousness of meaning, of understanding, which does not lapse when accompanying imagery does so. One must bear in mind that symbolic thinking is only thinking in virtue of a new, intentional act-character: this distinguishes the meaningful sign from the mere sign, i.e. the sounded word set up as a physical object in our mere presentations of sense (Husserl, 2008b, Chap. 2, paragraph 20).

That is, signification is realized in the act of imparting meaning without being conditioned to illustrating intuition. Here, the jurisdiction of the sign is to make idealized meaning explicit without taking its place. The idea that the function of the sign is to view/point to something is exemplified by the game of chess. In this game there is it does not matter the material from which the pieces are made; what is relevant and makes them phenomenal objects are the rules which regulate how each piece should move. Thus, the rules of the game give meaning to the game.[10]

Following these arguments, we understand with Sokolowski (2010), that the concreteness of the chess pieces becomes secondary. The focus of intellect is oriented at the piece perceived as transcendental Again here, the idea of the transcendency associated with intellection becomes clear. The physical appearance of the object expressed by expression and illustrative figures, which fulfills the understanding with meaning, is not required. So, Husserl solves the problem of the actual judgments being dependent on experiential acts, as psychologists asserted, thus turning to ideal judgments.

Finally, Husserl observes that meaning and its possible circumstantial fluctuation is caused by the signification of words, which is circumstantial, given that meaning is a "truth in itself"[11] of what is expressed. Therefore, the author states that the meanings of occasional, subjective expressions are fluctuating and, therefore, meaning fluctuates, and can be replaced by another objective expression aimed at the exact, ideal meaning.

[10] For Husserl the need to go back to "corresponding intuition," for the clarification of the meaning represented symbolically, it occurs in cases in which the meaning of the content of a concept are fluid. In this case, the jurisdiction of illustrative intuition aims at identifying the topological limits of the differences shown with which the expressions express, and guarantee an effective judgment of the corresponding meaning.

[11] "The use of quotation marks in the expressions *in-themselves* and *truths themselves* is intended to mark the Husserlian detachment from meaning they enjoy in Kantian philosophy, but also, and above all, to indicate that they must be understood in a strictly phenomenological semantic context, to be gradually clarified in the sequence of Logical Investigations and other works published, as well as the other private texts currently available" (Onate, 2016, p. 101, author's emphasis, author's translation).

3 Meaning and Pure Logic

We turn our attention to what Husserl says about *pure logic* and psychologism aiming at understanding the intimate interconnection between pure logic and meaning. Pure logic is concerned with ideal units, understood as meanings, and also with making the *ideal essence of meanings*. These acts are independent of psychological ones and even of the possible grammar that involves them. When this occurs, the field in which scientific researchers operate is definitely that of *pure logic*.

Thus, logic is understood as the underpinning of sciences, as it establishes the theoretical connection of laws dealing with the object of study formulated by science. Moreover, while theorizing, scientific researchers present an assertion whose objective validity is based on the ideal laws of logic. These laws, as stated before, present the ideal unity. Husserl points out that meaning entails the ideal unity that presents itself in the *multiplicity* of the lived-perceptual field. Therefore, it must be distinguished from both contingent expression and contingent experiences. So, from that which is not guaranteed by pure logic, and causes confusion. That said, he states:

> If all given theoretic unity is in essence a unity of meaning, and if logic is the science of theoretic unity in general, then logic evidently is the science of meanings as such, of their essential sorts and differences, as also of the ideal laws which rest purely on the latter (Husserl, 2008b, Chap. 3, paragraph 29).

Hence, we find that logic *is both* a science of meanings and a field in which its pure laws, that is, the logical laws, are founded on meanings. Or, rather, as Martinelli (2018) observes, "[...] Logic is about the *necessary rules* of *how we should think*, and it is occupied in analyzing the ideal connections between propositions" (p. 51, *emphasis added,* authors' translation). It is clear that, in order to express mathematical reasoning, we search for a mathematical truth and, to do that, we work with logical laws which sustain and formulate such reasoning.

In this horizon of comprehension, Lyotard (1954) pondering about Husserlian reasoning, states that, when the mathematician is performing movements of "thinking about," of reasoning, they experience a flow of reasoning that leads them to "ideal reasoning." In short, this is free of subjective reasoning,"[...] the mathematician reasons correctly when, through this subjective flow, attains the objectivity of true reasoning" (Lyotard, 1954, p. 17, authors' translation).

To understand *pure logic* as a science of meanings is contrary to the way traditional logic deals with the relationships established between the objects announced in the propositions. That is, traditional logic mobilized psychological elements, so that the interpretation of judgment, of associations, of inferences, of consequences, and so on, could be guaranteed within the framework of empirical psychology. However, Husserl does not present the critique of psychologism in the Investigations, as he had done previously in the prolegomena. He argues that it is no longer possible to follow the path of psychologism, because it would drive traditional logic. He understands that the traditional psychology cannot encompass the object of logic. In order to highlight "how" the logic of meanings is operated in the scientific field, Husserl states that when theorizing, within the investigative process, the scientist presents the objective unity. However, without conducting judgment, understood here as real act. That is, what the scientist intends is the ideal, objective meaning announced by the expressions. They aim at the theoretical concept, understood as "unity of ideal meaning," aiming to derive other states of affairs, without performing real psychical acts. They even intend to establish new objectified propositions, which present the ideal unity or ideal meaning. Hence, the emphasis on propositions being understood as key elements for the elaboration of reasoning.

As previously explained, Husserl observes that, in reasoning, there is no judgmental act based on the empirical-psychological relationship. The act of reasoning is simply to establish *the ideal relationship of possible assertions*, always explaining the result in objective expressions.

In this direction, Husserl states that intellectually grasped thought contains the ideal meaning, that is, the essence. Thus, scientific researchers would know that they do "[…] not *make* the objective validity of thoughts and thought-connections, of concepts and truths, as if he were concerned with contingencies of his own or of the general human mind, but that he *sees* them, *discovers* them" (Husserl, 2008b, Chap. 3, paragraph 29, *author's emphasis*). He adds that, taking into account the establishment of theories, all science is "[…] in its objective content, of *one* homogeneous stuff: it is an ideal fabric of *meanings*" (Husserl, 2008b, Chap. 3, paragraph 29, *author's emphasis*).

Thus, Husserl emphasizes that the demonstration of a proposition is closely associated with the relations of *principle* and *consequence*, understood as ideal units free from psychological acts of judgment, as ideal contents, or as propositions. Therefore, the ideal legality of the demonstrative process would not be subject to psychological judgment, but rather to ideational reflection. On the other hand, he attests that indices are devoid of this ideal legality, but fulfill the function of "pointing." He exemplifies such a statement with the *state of thing A* indicates *the state of thing B*, without any rational connection to the logical foundations necessary for this realization. In this case, an empirical relational connection occurs and credulous learning takes place. In another direction, he warns that the rational connection of the *state of thing A* to *the state of thing B* is realized and guaranteed with the demonstration of a proposition. This is understood, because the demonstration is taken as an *intellectual* act. Here, we have ideal legality that does not depend on who realized it. That is, ideal legality is not subordinated to empirical judgments; it is a purely logical act.

We emphasize that, for Husserl, what is identical and also what is variable belong to the expressions, from the standpoint of psychological content. Whereas, from the point of view of logical content, what is uniquely identical, that is, ideal, is what belongs to the expressions. Let us look at an example of logical expression expressed in the work analyzed: π is a t*ranscendent number*. Meaning in this proposition is ideal; it does not vary. As we interpret what is expressed, symbolized by "π," considering its historicity, fulfills the function of representing something that is systematically repeated. This leads to the understanding that regarding an expression with "unitary significance," in logical terms, the essence does not vary. It is identical, regardless of who announces it or when. The content is identical on a case by case basis. Still, the proposition "π is a transcendent number," is apprehended, as the author states, and brings an object represented by the sign "π," as said intellectually:

> I see that, wherever there is talk of the proposition or truth that *π is a transcendental number*, there is nothing I have less in mind than an individual experience, or a feature of an individual experience of any person. I see that such reflective talk really has as its object what serves as a meaning in straightforward talk. I see lastly that what I mean by the sentence in question or (when I hear it) grasp as its meaning, is the same thing, whether I think and exist or not, and whether or not there are *any* thinking persons and acts. (Husserl, 2008b, Chap. 4, paragraph 31, *author's emphasis*).

Husserl justifies the existence of the *identity of species*, understanding that it encompasses the various multiplicities of the species of meaning. With the meanings, we have what Husserl calls *general objects*. He observes that the existence of such general objects is in a metaphysical dimension, restricted to worldliness. This means that general objects understood as an idea, a concept, etc., are generated and understood as statements for legitimating judgments. The philosopher observes that *meaning* and the class of *general objects* do not coincide. Thus, each species generates another meaning. He exemplifies with the non-coincidence of the number 4 seen as representation and as meaning. If you take the number 4, and its possible meanings, where the number 4 appears as an object such as a species 4, which represents an object with operational properties, plus a species 4, associated with counting, you will have the number 4, with its individual representations. Such number would be associated with different meanings. According to Husserl: meaning, with its object, is understood in the field of what is individual; while meanings that form general objects, with their different possibilities of representation, are understood in the field of what is general.

4 Constitution of the Mathematical Object

Understanding that the *expression* aims at meaning, and also that it is associated with the *proper name*, results from the fact that the proper name acts as the meaning, since the *nomos* identifies a name that was announced and theorized into a proposition. The *nomos* refers to an intended object, that appears in the proposition. Husserl (2012) while talking about the object, in this study, refers to an idea, or concept, totaled in sense and meaning, without the appeal of sensory impressions. Chaui (1988) clarified these words: "The idea (sic) or concept is the overcoming of infinite transcendent perspectives of the thing by the immanent unity of meaning" (p. XIII, authors' translation). In agreement with the author, we understand that Husserl considers that things, "[...] are characterized by their perspectivism, their incompleteness, the possibility of always being targeted by new *noesis* that enrich and modify them. Ideas (sic) or concepts, on the contrary, are characterized by their globalizing, whole and finished aspect" (Chaui, p. XII, authors' translation).

To exemplify these understandings, we bring an analysis of the Pythagorean Theorem. In the theorem, the object in point is the idea presented by the proposition, which once *rationally articulated*, makes the triangle appear, showing the relationship between the catheti (sides) and the hypotenuse. In this case, the *name* and *object* would be intimately experienced if we said: Rectangle Triangle Theorem. Thus, we would have the *name* meaning the object. Moreover, when the Pythagorean Theorem is mentioned, the proper name clarifies the relationship between the catheti (sides) and the hypotenuse of the right triangle. Therefore, the name Pythagorean Theorem, aims at an *object*, namely: *the object is* the *idea* which is made present in the proposition about the right triangle and its metric relations. The object is not the figure of a right triangle. On the other hand, the figure of the right triangle is an *index* that points to the *object*, which the proper name is Pythagorean Theorem. However, *pointing to* is a secondary action.

The idea present in Pythagorean Theorem is the ideal unity. It is clearly understood by others when the proposition is communicated. That is, the *object* declared and "made present" in an *expression* can manifest itself through written, spoken, pictorial languages, for instance. However, we have seen that *ideality* is preserved when people communicate with the *spoken word*. It is understood, from our study, that other kinds of language require physical intermediate support in order to be enunciated. So, these types of languages lead communication to assume a physical and, therefore empirical configuration. In this context, when the interlocutors are open to intersubjective dialogue, they recognize each other as a similar who have equivalent concerns and *what is said*, that is, the spoken word, is intended by the listener who animates it. Thus, it generates the possibility of *mutual understanding*. A horizon is open to other subjects' living-experiences of evidence, of what they apprehend. Thus, the *object* announced and theorized in a proposition is *intellectually grasped* by the scientific researcher and, therefore becomes public, experiential knowledge in the theoretical field when it is understood within communicative discourse.

It could be argued that the *sound* that carries the word pronounced is also physical. That is true. However, it emanates directly from the senses of the living-body that *says* the word. This word, when written, is already distanced from the senses of the living-body, because it requires a physical medium to be exposed: writing on a sheet of paper, for example. In turn, the written word can be taken up by the subject who said it or wrote it or by other subjects, reactivating the meaning it brings.

Given the reciprocity in the connection of pure logic and meaning, understanding them as coexistent in the *object-mathematical world*, has eliminated all possibility of the laws of logic being understood as sustained in psychologism. So, the *object-mathematical world* is not subjugated by experimental method, capable of generating what is contingent. In this way, there is no empirical-psychological link in the establishment of general rules, or in the demonstrative process. In this sense, what the scientific researcher does while *demonstrating* a proposition is to intellectually deduce a conclusion by articulating other ideal contents, so it is clear that the demonstration tends to signification.

Claims regarding *pure logic and meaning* make us wonder "why is demonstration necessary, to talk about meaning and, consequently, about the mathematical object?" The title of paragraph 34 of the First Investigation is" §34 In the act of meaning we are not conscious of meaning as an object." That means that when a *first thought* which articulates theoretical *entities* arises, it must be put to the test. In this sense, Husserl attests that the object assumes the dimension of object, strictly speaking, as a result of the act of *logical reflection* of thought, which enables abstraction, understood as ideation. That is, the *mathematical object* is constituted by *logical reflection*; which, in turn, clarifies the understanding that a *first thought tends*[12] *towards meaning*. *Logical reflection* is affected as a result of theoretical connection and *theoretical reasoning*, constituting "[...] the *contents* of the thought-acts just performed [...]" (Husserl, 2008b, Chap. 4, paragraph 34). With theoretical reasoning, for example, we seek conjectures that require their validity to be established and sustained, through demonstration guaranteed by other results already shown to be true, with theoretical connection, thus making it necessary to establish logical connections of what had been expressed in the form of a proposition. Thus, deductions are made *with* theoretical connections, which lead the researcher to conclusive propositions. According to Husserl, we will say *therefore mathematically* only when we are sure of the meanings of the premises that are considered in the logical inferences for the constitution of the object.

This explanation shows that for a proposition to express a mathematical object and constitute itself as an *ideality*, which engenders a *mathematical object*, must necessarily be rationally connected to the theoretical body through which it is being examined. Otherwise it has no meaning; it is pure speculation; it is pseudo evidence.

References

Ales Bello, A. (2004). *Fenomenologia e Ciências humanas*. Trad. e Org. Miguel Mahfoud e Marina Massimi. Bauru: Edusc.

Chaui, M. (1988). Husserl: Vida e Obra. In E. Husserl. *Investigações Lógicas: sexta investigação: elementos de uma elucidação fenomenológica do conhecimento* (pp. V–XV). Seleção e tradução de Zeljko Loraric e Andréa Maria Altino de Campos Lorapric. São Paulo: Nova Cultural.

Derrida, J. (1994). *A voz e o fenômeno: introdução ao problema do signo na fenomenologia de Husserl*. Tradução de Lucy Magalhães. Rio de Janeiro: Jorge Zahar.

Gadamer, H. G. (2012). *Verdade e Método I: traços fundamentais de uma hermenêutica filosófica* (12th ed.). Tradução de Enio Paulo Giachini. Petrópolis: Vozes.

Husserl, E. (2008a). *A Ideia da Fenomenologia* (p. 70). Tradução de Artur Morão. Lisboa: Edições.

Husserl, E. (2008b). *Logical investigations volume 1* (International Library of Philosophy) [Kindle edition]. Retrieved from https://www.amazon.com

[12] "tend towards" is used here, in the sense of fostering, projecting, furthering; as we understand that, according to Husserl, intellectual thought cannot directly bring meaning.

Husserl, E. (2012). *Investigações Lógicas: segundo volume, parte I: Investigações para a fenomenologia e a teoria do conhecimento*. Tradução de Pedro M. S. Ales e Carlos Aurélio Morujão. Rio de Janeiro: Forense.

Levinas, E. (1997). *Descobrindo a existência com Husserl e Heidegger*. Lisbon: Instituto Piaget.

Lyotard, J. F. (1954). *A fenomenologia* (p. 70). Lisbon: Edições.

Martinelli, Á. V. (2018). As origens do pensamento de Edmund Husserl: do psicologismo à fenomenologia. *PERI Revista de Filosofia, 10*, 36–57. Acesso Dezembro 7, 2018, em http://www.nexos.ufsc.br/index.php/peri/article/view/2021/2307

Onate, A. M. (2016). *Ficção e tempo na filosofia de Edmund Husserl*. Porto Alegre: Editora FI: EDIPUCRS. Acesso Janeiro 31, 2019, em http://ebooks.pucrs.br/edipucrs/Ebooks/Pdf/978-85-397-0861-1.pdf

Peres, S. P. (2017). Psicologia eidética e teoria do conhecimento nas Investigações lógicas de Husserl. *Psicologia USP, 1*, 118–124. Acesso Março 8, 2019, em http://www.scielo.br/scielo.php?script=sci_arttext&pid=S0103-65642017000100118&lng=en&nrm=iso

Sokolowski, R. (2010). Introdução à Fenomenologia (2nd ed). Tradução de Alfedro de Oliveira Moraes. São Paulo: Loyola.

Constitution and Production of Mathematical Knowledge in Cyberspace Being-With-Media

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1 Introduction

Throughout history, ways of understanding science and the production of scientific knowledge have been closely linked to the socio-scientific-cultural context of each era, considering different conceptions of the world, reality, and knowledge, which support different meanings and senses about what science is, how science is understood, and how its truths are established.

Phenomenology, as a philosophy, critiques positivism, or the natural attitude, anchored in the conception of the world—understood as factual reality—in which knowledge is taken for granted and seen as unquestionable. Husserl (1907/1997, p. 41) states that "For natural thought, the possibility of knowledge is obvious... there is no opportunity to raise the question of the possibility of knowledge in general," i.e., one fails to question what enables the correspondence between cognitive

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experiences and the "*things*" to be known. Thus, positive sciences end up "immersed in the natural attitude, whose exercise expresses the relationship between a spontaneous (empirical or psychological) consciousness and the natural world, empirically revealed to this consciousness, in its facticity" (Tourinho, 2010, p. 383, authors' translation).

Husserl, in his project for constituting philosophy as a science of rigor, although he knows that apodictic evidence is fundamental to philosophy, considers that it cannot be based on the empirical evidence of the natural attitude, since

> As perfect as empirical perception may be, it will always be the perception from a point of view and, as such, it can only reveal "aspects" or "perspectives" of the perceived thing (*perceptum*) which, in turn, will not be revealed in its fullness, but only partially. Still, the belief about what we perceive empirically goes far beyond what empirical perception actually reveals. In this sense, it can be said that the thing, seen empirically, will always be a "mixture of what is seen and not seen" (Tourinho, 2010, p. 383, authors' translation)

Husserl opted for the exercise of *epoché* or "suspension of judgments" in relation to the existence of things. "By suspending judgment on the facticity of the world, I do not fail to experience the 'thesis of the world,' however, I no longer use it. I try to keep it out of the loop" (Tourinho, 2010, p. 384, authors' translation).

Phenomenological *epoché* causes a dislocation of attention, from the facts of the world, as considered by the natural attitude,

> into the realm of a transcendental subjectivity, within which and from which 'phenomena' - as pure idealities - will reveal themselves as 'absolute evidences' for a transcendental consciousness, endowed with the ability to truly see such phenomena as they stand in full evidence. It is, as Husserl himself states, 'pure sight' (Tourinho, 2010, p. 384, authors' translation).

Thus, by assuming the phenomenological attitude "the world[1] reveals itself in and to pure (or transcendental) consciousness as a 'horizon of senses' /…/ where pure means /…/ what cannot be thought of in terms of empirical data" (Tourinho, 2010, p. 385, authors' translation).

It may be stated, as conceived by Husserl (1907/1997, p. 45, authors' translation), that phenomenology is both a method and an attitude; "a strictly philosophic thinking attitude and a strictly philosophic method." Therefore, to assume a phenomenological attitude means to assume an analytical and reflexive attitude, which seeks to behold what is shown to consciousness in its "donation."

What is shown, is manifested in perception, in the *now*. It brings what is focused through intentional gaze in its originality. Thus, *a priori* truths are not established by assuming a phenomenological attitude; they are constituted for the subject who seeks meaning in what is perceived. This is also the position we take when researching phenomenologically, constituting knowledge in the realm of mathematics education. In the research

[1] World, as it is conceived from a phenomenological perspective, is treated in a more encompassing way in the texts of Bicudo (first and second part) and perception in the text of Ales Bello (first part) and Pinheiro, Bicudo, and Detoni (second part).

> generalities are explored [...], which are supported by articulations successively made with the meanings of what is being expressed. This research allows us to understand characteristics of the investigated phenomenon and, in doing so, give rise to possibilities of attainable comprehension, when the interrogation of the phenomenon is directed to different contexts from those in which the investigation was carried out. They support important articulating reasoning for decision-making regarding policies, education, and research, and slowly set the stage for surveys with rigorous analysis and interpretation (Bicudo, 2012, p. 19, authors' translation).

So, what has this manner of doing research produced in the field of mathematics education? This question can be partially answered by turning our attention to the research conducted by the members of FEM,[2] particularly those regarding the project entitled "*A compreensão e a produção da matemática ao se estar no ciberespaço e junto ao computador e outras mídias* (Comprehension and Production of Mathematics in Cyberspace, with the Computer and Other Media)." Some results of this project, developed in the period between 2015 and 2019, have been shared in this book. In the present text, we try to pull the threads that weave the network of knowledge that has been constituted by the researchers that participate in this project, focusing on the question "*How does the constitution and production of mathematical knowledge come about when being-with-media in cyberspace*?"

Considering the work of the members of the study group FEM, the intent is to present a comprehensive synthesis of the theme *constitution and production of mathematical knowledge while being-with digital technologies*.

The question "*how does it come about*?" displays the desire to turn to the original sense. By reading and analyzing the texts that compose the book, namely those that focus on the constitution and production of mathematical knowledge by mathematicians, mathematical educators who are concerned with teaching and learning of mathematics, teacher education, and contemporary educational practices (methodological and pedagogical), always *with-the-media* in cyberspace, we proceed hermeneutically, aiming to clarify the question above and show what is understood from the production of the group.

Through this journey, that aims to show understandings, we are interested in the ways of being, thinking, doing, teaching, and learning mathematics, with digital technologies, understood as technological apparatuses that are available, in our daily lives, and enhance ways of understanding and producing mathematical knowledge. They are used to study the behavior of certain mathematical content, with the aid of software; to test hypotheses; watch video lessons or read texts; study a given content; disseminate material produced, which suggests and potentializes actions; promote theoretical advances, which are only possible with the aid of computers, and educate teachers *for* and *with* digital technologies.

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[2] Detailed information regarding the research conducted by FEM can be found at http://fem.sepq.org.br/

2 Constituting and Producing Mathematical Knowledge with Digital Technologies

The purpose of the above-mentioned FEM project is to understand the mathematical knowledge which has been produced in this world augmented by technology, in contemporaneity. Therefore, we clarify that mathematical knowledge may be that which is engendered by a specific community of mathematicians, and for the purposes of this group it can also address teaching, whose guiding thread is the education of individuals, so that people learn mathematics. This means that these deeds navigate between the scientific realms of exact, social and human sciences, as well as technology. In order to focus on any of these, one assumes the Heideggerian perspective of technology conceiving it

> Not [as] a thing, which would be machinism; not even another simple knowledge: it is about the construction of the world in given conditions. Technology is conforming an era, expressing a way of being that opens a world, as it embodies the correspondence between a process of positioning reality and a way of thinking (Rüdger, 2006, p. 19, authors translation).

Therefore, the understanding of teaching and learning environments, which encompasses the physicality of space, in chronological times for school tasks, is open. However, it expands with the internet. We are together here and there, overcoming geographically established barriers of near and far, questioning the 24-hour-day, through the permanent dialogue "of" and "in" the *now*. The research of the group exposes the understanding of cyberspace as belonging to mundane reality, open to living-experience (*Erlebinis*) in ways of *being-with* others. Time-space escape the chronological-geometric dimensions, expanding physical and temporal limitations, intertwining intersubjectivity through the *presentification* of co-subjects, based on intentionality and the experience of what is real, unveiling truths, and realizing projects.

> In ancient times, the Greeks themselves referred to exact time as *Cronos* and to subjective time as *Kairos*. /.../ time is shown in the confluence of meanings and the organization of societies, which are structured according to certain times (Sabóia, 2007, p. 178, authors' translation).

The chronological time and physical space in which things happen, in person or at a distance, are important and drive the organization of school education. However, providing education does not always mean promoting openness to learning. In this sense, ways of *being-with-the-other* are emphasized in the research conducted by FEM, pointing to face-to-face, as well as distance education, as facets of living-reality. This is a fertile ground for research and for transformative pedagogical practices that aim to escape commonplace. Regarding technologies, having equipment and knowing how to use it must be placed in a linear continuum. Only then should we seek, in chronological academic time, the space for a possible articulation of meanings for teaching, in order to achieve openness to learning.

Such meanings are routinely revealed so that mathematical learning can occur from a perspective through which DT are pedagogically explored, as evidenced in the texts contained in the second part of this book. While transposing the gaps revealed in the absence of conditions to guarantee the availability of equipment in schools, such texts look towards the relevance of teacher education, to foster the comprehension of time, which enables us to consolidate what has been beneficial to educational practices, through technology. This movement points to the original meaning of technology, which is not instrumental, but refers to production, i.e., *techné*. *Techné* is understood as (concrete) action of knowledge. It is *poiesis*, a way of becoming present through perception and thought (Heidegger, 1936/1977).

This way of understanding technologies is underlined in the studies of FEM. It aims to overcome the discourse of a persisting vicious opposition of the need for technical knowledge to "deploy," fix, insert TD in schools, and technological knowledge to make use of them. In the studies presented in the present book, the teaching–learning movement is the trigger for research. In such movement, teacher and student, though assuming different roles, intermingle in the school environment where, while teaching, individuals learn, and while learning, they also teach, thus placing learning at the core of teaching.

According to Heidegger (1987) the meaning of teaching and learning can be interpreted by focusing on thought.

> Learning is harder than teaching; thus, only those who can truly learn - and only to the extent that they can - can truly teach. The true teacher differs from the student merely because they can learn better and want to learn more authentically. In all teaching it is the teacher who learns the most.
>
> The most difficult learning consists in embracing what there is to know and that we have effectively always known to the core. Such learning, the only kind to which we give ourselves, requires us to permanently dwell on what is apparently closest, for example, on the question «*what is a thing*? » (Heidegger, 1987, p. 80, author's translation).

The investigations of FEM, which focus on teaching and learning, as can be seen in the second part of the book, turn to ways of *thinking mathematically* in spaces and at times created/made possible *with* technology. This thinking regarding the constitution of mathematical knowledge, by the individual who learns and, and being formal or in a school environment, shows the path of who and what is being taught.

The movement of learning, of constituting knowledge, encompasses "seeing; contemplating what is manifested, which leads to *becoming*, uncovering what is intuitive, putting it in a dominant position of use, impelling it beyond possibilities, […]" (Mocrosky, 2010, p. 60, authors' translation).

In the specificities of mathematics, or the ways of constituting mathematical knowledge, as Heidegger (1983) states, we interpret the *mathematical* meaning of things as "learning to know what is already known; the presupposition of all academic work. The search for *letting learn* or *understand* aims at the original meaning, which is the foundation of any and all know-how." Therefore, *knowledge* does not lie in clarifying or justifying procedures. It reveals the understanding of what is done, the meaning that is being constructed and the possibilities which are being opened, so that something can be done in one way or another, which *requires learning to know and to meditate about what is being known*. The texts of the first part of the book bring philosophical and didactic-scientific views, which enable the understanding of the way to produce knowledge with technologies. However, the text by Venturini and Coelho, in the fourth part of the book, specifically discusses the constitution of the mathematical object.

The search for a method, or a proper way of doing, takes on the sense of unveiling, considering the technique as a way of manifesting, which opens the way for *learning to know and to meditate about what is being known* (Heidegger, 2012). Technique is shown by the attentive, inquiring, and curious eyes of those who are out there, producing mathematics, or treading the road to knowledge, through its intricacies. In the group, this mode of production can be found in research that contemplates mathematics itself, and the way of conceiving its production, with technologies, by the producers themselves, and seeks to uncover the meaning of such work for those who produce it, as explained in texts in the fourth part of this book.

For teachers, who do not produce mathematics as such, or not in the way that such production is viewed by mathematicians, the constitution of knowledge is brought up by the possibility that, while being-with mathematics, which gives meaning to what they do, they can explain the mathematical way of being of school content, in the most diverse formative paths. The constitution of knowledge of teachers occurs in the movement of the living-experiences with mathematical content, present in teaching situations, in which they perceive themselves teaching and learning. This perception unveils/produces meaning for *mathematical practice*, for the knowledge conveyed in the transmission of academic content, and organized in their common daily teaching activities. The constitution of knowledge of the teacher requires an action that is not based on a previous process, but which *lets itself be apprehended*. It seeks the way of being of what is presented in acting, in the creation of a method, a technique, understood as unveiling, which aims to explain mathematically the meanings hidden within school content. In their practice, teachers seek to *dis-close* the meaning of content, advancing towards the understanding of the original meaning, which requires turning to the ways of doing, and interrogating what has been done and why. This view is dealt with specifically in the third part of the book.

The research of FEM, materialized in this book, shows that the questions of time and temporality are fundamental for the planning of formative actions, for the constitution of mathematical knowledge, by subjects who are with digital technologies. It is evident that temporality needs to be supported by permanence and care for others. According to Bicudo (2011, p. 91), this care is inherent in education itself.

> [ ] Thus, education is viewed as caring, in the sense of helping, being with the other, of being solicitous, so that *Dasein* is released in the direction of becoming its cure, that is, also in the ontological dimension. It is being-with in an attentive manner, not letting ourselves be trivialized by everyday sameness, or the pursuit of the demands of public dealings, when one is, at the same time, everyone and no one. This care for the student means seeing, feeling, thinking and living-with the world where we are with others. It is living in the openness of possibilities of being-in-the-world-with, concerned and busy. But, never just through the dullness and mediocrity of being like everyone else (Bicudo, 2011, p. 91, authors' translation).

It is a time to be, to know, to understand. Time that is understood as a condition of man to get to know the world, the material and social conditions of the constitution and production of knowledge. In the case of school environments, it is a time when the reality experienced by teachers needs to be considered. A time of *being thought about*. It is temporality.

Temporality. A movement that expresses continuity, permanence, without becoming paralyzed, fixed in a punctual situation in the *now*. Temporality that projects, transcends the living-moment, and reaches the space of the classroom, in which knowledge produced collectively is again set in motion for resignification, so that the meaning is available to students, clarifying what is being focused on.

However, beware! We live in an instrumentalized society and this cannot be synonymous with technological education. As explained by Skovsmose (2001), the instrumentalization of the use of tools does not necessarily bring technological knowledge. Technological education can be built to the extent that technological thought can serve as the basis for the reasoning required to solve a problem, while reflective thinking evaluates and questions technological solutions for the benefit of the majority. From this perspective, reflective knowledge will be epistemologically supported by technological problems, "but deal with them in a technological way" (Skovsmose 2001, p. 84, authors' translation).

The "technological manner of dealing with" requires understanding of technology itself, which, as stated by Heidegger,

> it is nothing more than the inheritance we received from the tradition of Western thought. This legacy needs to be gained every day. However, in this conquering, it both imprisons and frees us. It imprisons us when we simply appropriate what is imposed onto us through culture, customs, values, without meditation. It sets us free when we think of its essence. Therefore, in order not to lose their roots, man must be able to think about the essence of technology. Thinking of this essence means surpassing technology, not in the sense of depreciating or annihilating it, but rather to step inside it, to understand it more radically. Thinking the essence of technology is thinking about our own essence (Rafael, 2007, p. 6).

3 Final Considerations

When turning to what was produced by FEM, displayed in the texts that make up this book, we seek to understand the theoretical advances that are shown regarding the constitution and production of mathematical knowledge while being-with digital technologies. When inquiring about what each text says and from the careful reading of what was produced and expressed, aiming at a hermeneutic interpretation, we highlight the group's advances regarding the meanings and senses of the constitution and production of mathematical knowledge by being-with digital technologies, as well as the technology itself.

Since *Being and Time* (1927), Heidegger encouraged us to think[3] about the veiled senses of *being-in the world*. When we consider digital technologies in contemporary times, we can do so in two ways: via traditional metaphysics,

[3] "'sense' designates the projective realm. And, indeed, with its own intent, in tune with the single question regarding the 'meaning of being,' the clearing of the being who opens themselves, who comes into existence in this projection. Such projection, however, occurs in a project played. It is the one which, in the project played, happens appropriately as that which is essentialized in the truth" (Heidegger, 2012, Vol. II, p. 12, authors' translation).

inherited from modernity that has become an ontology of the thing, which “takes the ‘thing’ as a paradigm of representation for all that is” (Barbosa, 1998, p. 2, authors’ translation), causing the being to be measured or considered by what they “have.” In this attitude, which plagues contemporaneity, the being falls into oblivion and the thing is overvalued, that is, the apparatus highlighted in detriment of people.

The members of FEM, in their willingness to investigate “Understanding and producing mathematics while in cyberspace and with-computers and other media” chose the second path, which favors thinking about being-in the world through fundamental (phenomenological) ontology. Under this perspective, the beings give themselves as an actualizing presence; always being. By taking this attitude as an investigative procedure, we draw lines, however faint, about technological education in mathematics education, in which, by *being-with* digital technologies in cyberspace, computerized learning environments can contribute with ways of thinking about mathematics and forming people.

The theoretical advances presented in this book, in the light of philosophy, express the closing of a cycle that enables FEM members to understand the constitution and production of mathematical knowledge by being-with media in cyberspace. However, we know that this is a dialectical cycle, where new questions arise at the same time as understandings are reached, because we cannot imprison the being in the finitude of the now. The b*eing* is always *being*, therefore *becoming* (*devir)*, open to possibilities.

References

Barbosa, M. F. (1998). A noção de ser no mundo em Heidegger e sua aplicação na psicopatologia. *Psicologia: ciência e profissão, 18*(3), 2–13. Acesso 21 dezembro 2019, em http://www.scielo.br/pdf/pcp/v18n3/02.pdf

Bicudo, M. A. V. (2012). A pesquisa em educação matemática: a prevalência da abordagem qualitativa. *Revista Brasileira de Ensino e Ciência e Tecnologia, 2*(5), 15–26.

Bicudo, M. A. V. (2011). *Pesquisa qualitativa segundo a visão fenomenológica*. São Paulo: Cortez.

Heidegger, M. (1936/1977). *A Origem da Obra de Arte*. Tradução de Maria da Conceição Costa. Rev. Artur Morão. Lisboa: Edições 70.

Heidegger, M. (1983). *Que é metafísica*. Coleção Pensadores. São Paulo: Abril Cultural.

Heidegger, M. (1987). *Que é uma coisa*? Tradução de Carlos Morujão. Lisboa: Edições 70.

Heidegger, M. (2012). *Ser e tempo. Tradução de F. Castilho*. Campinas: Editora da Unicamp.

Husserl, E. (1907/1997). *L’idée de la phénoménologie*. Cinq leçons. Paris: PUF.

Mocrosky, L. F. (2010). *A presença da ciência, da técnica, da tecnologia e da produção no curso superior de Tecnologia em Fabricação Mecânica*. Tese de Doutorado, Instituto de Geociências e Ciências Exatas—Universidade Estadual Paulista, Rio Claro.

Rafael, M. A. (2007). A questão da tecnologia no pensamento de Martim Heidegger ou Uma possível leitura da conferência “serenidade” (1959). *Revista Eletrônica do Grupo PET*—Ciências Humanas, Estética e Artes da Universidade Federal de São João Del-Rei, *3*(3), 1–10.

Rüdger, F. (2006). *Martin Heidegger e a questão da técnica: prospectos acerca do futuro do homem*. Porto Alegre: Sulina.

Sabóia, I. B. de. (2007). Cronos e Kairós: reflexões sobre temporalidade laboral e solvência social. Fortaleza. Acesso Abril 26, 2018, em www.repositorio.ufc.br/bitstream/riufc/2227/1/2007_dis_IBDSaboia.PDF.

Skovsmose, O. (2001). *Educação Matemática crítica: a questão da democracia*. Campinas: Papirus.

Tourinho, C. D. C. (2010). Fenomenologia e ciência humanas: a crítica de Husserl ao positivismo. *Revista Filosofia, 31*(22), 379–389.

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