Finite Difference Scheme for Special System of Partial Differential Equations

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Abstract The paper establishes conditions of existence and uniqueness of the bounded solution of a special system of linear partial differential equations of the first order. The system arises in the problem of a finite difference scheme of finding an approximate solution is elaborated.

Keywords First order linear partial differential equation · Numerical ethods · Finite difference scheme

1 Problem Statement

Further E^n is the Euclidean space of vectors x, (T denotes the transposition) with the norm $\|x\|$; Z is the set of integers; Z^n is the n-dimensional Cartesian product. We consider a problem of numerical solving of finding on $[0, T] \times E^n$ of a system of partial differential equations

$$\frac{\partial l^{(k)(t,x)}}{\partial t} + \sum_{i=1}^{n} f^{(i)}(t,x) \frac{\partial l^{(k)(t,x)}}{\partial x} + \sum_{i=1}^{n} g_k^{(i)}(t,x) l^{(i)}(t,x) = q^{(k)}(t,x), \quad k = \overline{1,n}. \tag{1}$$

With initial conditions

$$l^{(k)}(0,x) = r^{(k)}, \quad k = \overline{1,n}. \tag{2}$$

Further we assume that the following hypotheses be fulfilled.

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S. Pinelas et al. (eds.), *Mathematical Analysis With Applications*,
Springer Proceedings in Mathematics & Statistics 318,
https://doi.org/10.1007/978-3-030-42176-2_9

Assumption 1 *The problem (1)–(2) has continuous differentiable on* $[0, T] \times E^n$ *solution* $l(t, x)$ $\left(l^{(1)}(t, x), \ldots, l^{(n)}(t, x)\right)$ *such that partial derivatives* $\frac{\partial^2 l^{(k)}(t,x)}{\partial t^2}$ *and* $\frac{\partial^2 l^{(k)}(t,x)}{\partial {x_i}^2}$, $i = 1, \ldots, n, k = 1, \ldots, n$ *are continuous and bounded on* $[0, T] \times E^n$. *Also we assume existence of constants F, G such that*

$$\|f(t, x)\| \le F(t, x) \in [0, T]; \tag{3}$$

$$g_k(t, x) \le G(t, x) \in [0, T] \times E^n, \quad k = \overline{1, n}. \tag{4}$$

2 Finite Difference Scheme: Approximation, Stability, Convergence

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in E^n$; $\bar{e_k}$ be the unite vector of the axis $0x_k$ and $\tau = T/M$ (M is a natural). Denote $t_\nu = \nu\tau$; $\nu = 0, \ldots, M$; $x^\alpha = \alpha_1 h\bar{e_1}, \ldots, \alpha_n h\bar{e_n}$; $f_{\nu,\alpha} = f(t_\nu, x^\alpha)$.

In the region $[0, T] \times E^n$ we construct grids $\Omega_h^0 = (0, x^\alpha) : \alpha \in Z^n$, $\Omega_h^\nu = (t_\nu, x^\alpha) : \nu = 0, \ldots, M$; $\Omega_h^{'} = \{(t_\nu, x^\alpha) : \nu = 1, \ldots, M; \alpha \in Z^n\}$.

For grid functions $u_{\nu,\alpha} = \left(u_{\nu,\alpha}^{(1)}, \ldots, u_{\nu,\alpha}^{(n)}\right)$ defined on grids Ω_h^ν and $\Omega_h^{'}$ we use the corresponding norms

$$u_{\nu,\alpha} = \sup \Omega_h \|u_{\nu,\alpha}\|, u_{\nu,\alpha} = \sup \Omega_h^{'} \|u_{\nu,\alpha}\|.$$

Let $n_{\nu,\alpha}^+ = j \in 1, \ldots, n : f_{\nu,\alpha}^{(j)} > 0, n_{\nu,\alpha}^- = j \in 1, \ldots, n : f_{\nu,\alpha}^{(j)} \le 0.$

The difference numerical scheme corresponding to the problem (1)–(2) we construct in the following way.

On the grid $\Omega_h^{'}$:

$$\frac{u_{\nu,\alpha}^{(k)} - u_{(\nu-1),\alpha}^{(k)}}{\tau} + \sum_{i \in n_{\nu,\alpha}^+} f_{(\nu-1),\alpha}^{(i)} \frac{u_{(\nu-1),\alpha}^{(k)} - u_{(\nu-1),\alpha-\bar{e_l}}^{(k)}}{h} +$$
$$+ \sum_{i \in n_{\nu,\alpha}^-} f_{(\nu-1),\alpha}^{(i)} \frac{u_{(\nu-1),\alpha}^{(k)} - u_{(\nu-1),\alpha}^{(k)}}{h} + \sum_{i=1}^{n} g_{k,(\nu=1),\alpha}^{(i)} u_{(\nu-1),\alpha}^{(i)} = q_{(\nu-1),\alpha}^{(k)}, \quad k = \overline{1, n}. \tag{5}$$

On the grid Ω_h^0:

$$u_{0,\alpha}^{(k)} = r_\alpha^{(k)}, \quad k = \overline{1, n}. \tag{6}$$

From (5)

$$u_{\nu,\alpha}^{(k)} = \left(1 - \frac{\tau}{h}\sum_{i=1}^{n}\left|f_{\nu-1,\alpha}^{(i)}\right|\right)u_{\nu-1,\alpha}^{(k)} + \\ + \frac{\tau}{h}\sum_{i\in n_{\nu-1,\alpha}^{+}} f_{\nu-1,\alpha}^{(i)} \times u_{\nu-1,\alpha-\bar{e}_l}^{(k)} - \frac{\tau}{h}\sum_{i\in n_{\nu-1,\alpha}^{-}} f_{\nu-1,\alpha}^{(i)} \times u_{\nu-1,\alpha+\bar{e}_l}^{(k)}.$$

Solving the Eq. (5) with respect to $u_{\nu,\alpha}$ obtain

$$u_{\nu,\alpha} = \left(1 - \frac{\tau}{h}\sum_{i=1}^{n} f_{\nu-1,\alpha}^{(i)}\right) f_{\nu-1,\alpha}^{(k)} + \frac{\tau}{h}\sum_{i\in n} f_{\nu-1,\alpha}^{(i)} \times u_{\nu-1,\alpha-\bar{e}_l}^{(k)} - \frac{\tau}{h} + \\ + \frac{\tau}{h}\sum_{i\in n_{\nu-1,\alpha}} f_{\nu-1,\alpha}^{(i)} \times u_{\nu-1,\alpha+\bar{e}_l}^{(k)} - \tau\sum_{i=1}^{n} g_{k,\nu-1,\alpha}^{(i)} u_{\nu-1,\alpha}^{(i)} + \tau q_{\nu-1,\alpha}^{(k)}, k = \overline{1,n} \tag{7}$$

Because $u_{0,\alpha}^{(k)}$ are known from the initial condition (6) then by the formula (7) one can calculate layer by layer at first $u_{1,\alpha}$, $\alpha \in Z_n$, then $u_{2\alpha}$, $\alpha \in Z_n$, and so on.

Let us estimate the approximation order which the scheme (5)–(6) approximates the problem (1)–(2). Due to the Assumption 1 according to the Taylor series we have

$$\frac{l^{(k)}(t_\nu, x^\alpha) - l^{(k)}(t_{\nu-1}, x^\alpha)}{\tau} = \frac{\partial l^{(k)}(t_{\nu-1}, x^\alpha)}{\partial t} + \frac{\tau}{2}\frac{\partial^2 l^{(k)}(t_\nu, x^\alpha)}{\partial t^2} \tag{8}$$

$$\frac{l^{(k)}(t_{\nu-1}, x^\alpha) - l^{(k)}(t_{\nu-1}, x^\alpha - h\bar{e}_l)}{h} = \frac{\partial l^{(k)}(t_{\nu-1}, x^\alpha)}{\partial x_i} - \\ - \frac{h}{2}\frac{\partial^2 l^{(k)}(t_\nu - 1, \xi^{k,\nu,\alpha})}{\partial {x_i}^2}, \quad i = \overline{1,n}, \tag{9}$$

$$\frac{h}{2}\frac{\partial^2 l^{(k)}(t_\nu - 1, \eta_i^{k,\nu,\alpha})}{\partial {x_i}^2}, \quad i = \overline{1,n} \tag{10}$$

where

$$t_\nu \le \xi_{\nu,\alpha}^{k} \le t_{\nu,x^\alpha - h\bar{e}_l} \le \xi_i^{k,\nu,\alpha} \le x^\alpha, \quad x^\alpha \le \eta_i^{k,\nu,\alpha} \le x^\alpha + h\bar{e}_l. \tag{11}$$

From (6) follows that the initial condition (2) is approximated at Ω_h^0 exactly. Then due to (9)–(11) the residual between (1) and (5) on the solution $l(t,x)$ is equal to

$$\delta_{t,h}^{(k)} = \frac{\tau}{2}\frac{\partial^2 l^{(k)}\left(\xi_{\nu,\alpha}^{(k)}, x^\alpha\right)}{\partial t^2} - \frac{h}{2}\sum_{i\in n_{\nu-1,\alpha}}^{+} f_{\nu-1,\alpha}^{(i)}\frac{\partial^2 l^{(k)}\left(t_{\nu-1,\alpha}\xi_{\nu,\alpha}^{(k)}, x_\alpha\right)}{\partial t^2} \\ + \frac{h}{2}\sum_{i\in n_{\nu-1,\alpha}}^{-} f_{\nu-1,\alpha}^{(i)}\frac{\partial^2 l^{(k)}\left(t_{\nu-1,\alpha}\eta_i^{(k,\nu,\alpha)}\right)}{\partial x_i^2}, \quad k = \overline{1,n}$$

Due to the Assumption 1 the estimation $\|\delta\| \le c \times (\tau + h)$, $c = const$ is valid from which follows the following proposition.

Theorem 1 *If the Assumption 1 is valid then the difference scheme (5)–(6) approximates the problem (1)–(2) on its solution $l(t, x)$ with the first order with respect to τ and h.*

Let us show the stability of the difference scheme (5)–(6). It will be sufficiently for its convergence, because the initial condition (2) is approximated exactly on Ω_h^0.

Formula (7) shows the solvability of the difference problem (5)–(6). Let us obtain estimation of the solution of (5) corresponding to the zero initial conditions

$$u_{0,\alpha}^{(k)} = 0, \quad k = \overline{1, n}. \tag{12}$$

If

$$0 < \frac{\tau}{h} \le \frac{1}{nF}, \tag{13}$$

then from (3), (4), (7) follows

$$\sup_{\alpha} \|u_{\nu,\alpha}\| \le (1 + \tau G n) \sup_{\alpha} \|u_{\nu-1,\alpha}\| + \tau \|q_{\nu,\alpha}\|_h^{'}.$$

Then taking into account (12), obtain

$$\sup_{\nu,\alpha} \|u_{\nu,\alpha}\| \le \frac{T}{M} \|q_{\nu,\alpha}\|_h^{'} \left(1 + \frac{TGn}{M}\right)^M \times$$
$$\times \left[\frac{1}{(1 + \tau GM)^M} + \frac{1}{(1 + \tau GM)^{M-1}} + \cdots + \frac{1}{1 + \tau GM}\right]. \tag{14}$$

Taking into account that $(1 + \frac{TGn}{M})^M$ tends to e^{TGn} as $M \to \infty$ and therefore is bounded, then from (14) follows that the solution $u_{\nu,\alpha}$ of the problem (5), (12) satisfies the estimation $\left\|u_{\nu,\alpha}\right\|_h \le L \left\|q_{\nu,\alpha}\right\|_h$, $L = const$. This proves the following proposition.

Theorem 2 *If conditions (3), (4) and (8) are fulfilled then the scheme (5)–(6) is stable with respect to the right-hand side. From the stability and the approximation of the difference scheme follows its convergence.*

Theorem 3 *Let the Assumption 1 and conditions (3), (4) and (8) be fulfilled then the solution of the difference scheme (5)–(6) converges to the solution of the problem (1)–(2) with the first order by τ and h.*

Acknowledgements The research was supported by the Russian Foundation for Basic Research (project no. 17-01-00636).

Reference

1. Kim., A.V., Andryushechkina, N.A.: Real-time modeling of a system state during the process of more precise estimation of the initial position