Chapter 7
Multiplicative n-Hom-Lie Color Algebras

Ibrahima Bakayoko and Sergei Silvestrov

Abstract The purpose of this paper is to generalize some results on n-Lie algebras and n-Hom-Lie algebras to n-Hom-Lie color algebras. Then we introduce and give some constructions of n-Hom-Lie color algebras.

Keywords n-Hom-Lie color algebras · Color modules · Averaging · Semi-morphism · Morphism

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7.1 Introduction

The investigations of various q-deformations (quantum deformations) of Lie algebras began a period of rapid expansion in 1980s stimulated by introduction of quantum groups motivated by applications to the quantum Yang-Baxter equation, quantum inverse scattering methods and constructions of the quantum deformations of universal enveloping algebras of semi-simple Lie algebras. In [2, 22–25, 27–29, 35, 36, 50–52] various versions of q-deformed Lie algebras appeared in physical contexts such as string theory, vertex models in conformal field theory, quantum mechanics and quantum field theory, q-deformations of infinite-dimensional algebras, primarily the q-deformed Heisenberg algebras [34], q-deformed oscillator algebras and q-deformed Witt and q-deformed Virasoro algebras, and some interesting q-deformations of the Jacobi identity for Lie algebras in these q-deformed algebras were observed.

I. Bakayoko (✉)
Département de Mathématiques, Université de N'Zérékoré, BP 50, Nzerekore, Guinea
e-mail: ibrahimabakayoko27@gmail.com

S. Silvestrov
Division of Applied Mathematics, School of Education, Culture and Communication, Mälardalen University, Box 883, 72123 Västerås, Sweden
e-mail: sergei.silvestrov@mdh.se

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Hom-Lie algebras and more general quasi-Hom-Lie algebras were introduced first by Larsson, Hartwig and Silvestrov [33], where the general quasi-deformations and discretizations of Lie algebras of vector fields using more general σ-derivations (twisted derivations) and a general method for construction of deformations of Witt and Virasoro type algebras based on twisted derivations have been developed, initially motivated by the q-deformed Jacobi identities observed for the q-deformed algebras in physics, along with q-deformed versions of homological algebra and discrete modifications of differential calculi. The general abstract quasi-Lie algebras and the subclasses of quasi-Hom-Lie algebras and Hom-Lie algebras as well as their general colored (graded) counterparts have been introduced [33, 45–47, 59]. Subsequently, various classes of Hom-Lie admissible algebras have been considered in [53]. In particular, in [53], the Hom-associative algebras have been introduced and shown to be Hom-Lie admissible, that is leading to Hom-Lie algebras using commutator map as new product, and in this sense constituting a natural generalization of associative algebras, as Lie admissible algebras leading to Lie algebras via commutator map as new product. In [53], moreover several other interesting classes of Hom-Lie admissible algebras generalising some classes of non-associative algebras, as well as examples of finite-dimensional Hom-Lie algebras have been described. Since these pioneering works [33, 43, 45–48, 53, 55, 58], Hom-algebra structures have developed in a popular broad area with increasing number of publications in various directions. In Hom-algebra structures, defining algebra identities are twisted by linear maps. Hom-algebras structures are very useful since Hom-algebra structures of a given type include their classical counterparts and open more possibilities for deformations, extensions of cohomological structures and representations (see for example [3, 4, 20, 45, 56, 63, 64] and references therein).

Ternary algebras and more generally n-ary Lie algebras first appeared in Nambu's generalization of Hamiltonian mechanics, using a ternary generalization of Poisson algebras [54]. The mathematical algebraic foundations of Nambu mechanics have been developed by Takhtajan and Daletskii in [30, 60, 61]. Filippov, in [32] introduced n-Lie algebras. In [21], Leibnitz n-algebras have been studied. Properties and classification of n-ary algebras, including solvability and nilpotency, were studied in [10–17, 37]. The general cohomology theory for n-Lie algebras and Leibniz n-algebras was established in [57]. The structure and classification theory of finite-dimensional n-Lie algebras was considered in [49] and many other authors. For more details of the theory and applications of n-Lie algebras, see [31] and references therein.

Hom-type generalization of n-ary algebras, such as n-Hom-Lie algebras and other n-ary Hom algebras of Lie type and associative type, were introduced in [8], by twisting the identities defining them using a set of linear maps, together with the particular case where all these maps are equal and are algebra morphisms. A way to generate examples of such algebras from non Hom-algebras of the same type is introduced. Further properties, construction methods, examples, cohomology and central extensions of n-ary Hom-algebras have been considered in [5–7, 42, 43, 62, 65]. The construction of $(n+1)$-Lie algebras induced by n-Lie algebras using combination of bracket multiplication with a trace, motivated by the work of Awata et al. [9] on the quantization of the Nambu brackets, was generalized using the brackets of general Hom-Lie algebra or n-Hom-Lie and trace-like linear forms depending on the linear maps defining the Hom-Lie or n-Hom-Lie algebras [6, 7]. Generalized derivations of Lie color algebras and n-ary (color) algebras have been studied in [26, 38–41]. Derivations, L-modules, L-comodules and Hom-Lie quasi-bialgebras have been considered in [18, 19]. Super 3-Lie algebras induced by super Lie algebras in similar way have been considered in [1].

The purpose of this paper is to generalize some results on either n-Lie algebras or n-Hom-Lie algebras to the case of n-Hom-Lie color algebras. Then we introduce and give some constructions of n-Hom-Lie color algebras. Section 7.2 contains some necessary important basic notions and notations on graded spaces and algebras and n-ary algebras and used in other sections. Section 7.3 presents some useful methods for construction of n-Hom-Lie color algebras. In Sect. 7.4, Hom-modules over n-Hom-Lie color algebras are considered. Section 7.5 is devoted to generalized derivations of color Hom-algebras and their color Hom-subalgebras.

Throughout this paper, all graded linear spaces are assumed to be over a field $\mathbb{K}$ of characteristic different from 2.

7.2 Preliminaries

This section contains necessary important basic notions and notations on graded spaces and algebras and n-ary algebras used in other sections.

Definition 7.1 (1) Let G be an abelian group. A linear space V is said to be a G-graded if, there exists a family $(V_a)_{a\in G}$ of linear subspaces of V such that

$$V = \bigoplus_{a\in G} V_a.$$

(2) An element $x \in V$ is said to be homogeneous of degree $a \in G$ if $x \in V_a$. We denote $\mathcal{H}(V)$ the set of all homogeneous elements in V.

(3) Let $V = \oplus_{a\in G} V_a$ and $V' = \oplus_{a\in G} V'_a$ be two G-graded linear spaces. A linear mapping $f : V \to V'$ is said to be homogeneous of degree b if

$$f(V_a) \subseteq V'_{a+b}, \quad \text{for all} \quad a \in G.$$

If, f is homogeneous of degree zero i.e. $f(V_a) \subseteq V'_a$ holds for any $a \in G$, then f is said to be even.

Definition 7.2 (1) An algebra $(A, \cdot)$ is said to be G-graded if its underlying linear space is G-graded i.e. $A = \bigoplus_{a\in G} A_a$, and if furthermore

$$A_a \cdot A_b \subseteq A_{a+b}, \quad \text{for all} \quad a, b \in G.$$

(2) A morphism $f : A \to A'$ of G-graded algebras A and A' is by definition an algebra morphism from A to A' which is, in addition an even mapping.

Definition 7.3 Let G be an abelian group. A map $\varepsilon : G \times G \to \mathbb{K}^*$ is called a skew-symmetric bicharacter on G if the following identities hold for all $a, b, c \in G$:

(i) $\varepsilon(a, b)\varepsilon(b, a) = 1$,
(ii) $\varepsilon(a, b + c) = \varepsilon(a, b)\varepsilon(a, c)$,
(iii) $\varepsilon(a + b, c) = \varepsilon(a, c)\varepsilon(b, c)$,

If x and y are two homogeneous elements of degree a and b respectively and ε is a skew-symmetric bicharacter, then we shorten the notation by writing $\varepsilon(x, y)$ instead of $\varepsilon(a, b)$.

Example 7.1 Some standard examples of skew-symmetric bicharacters are:

(1) $G = \mathbb{Z}_2, \quad \varepsilon(i, j) = (-1)^{ij}$, or more generally

$$G = \mathbb{Z}_2^n = \{(\alpha_1, \dots, \alpha_n) | \alpha_i \in \mathbb{Z}_2\},$$
$$\varepsilon((\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)) := (-1)^{\alpha_1\beta_1+\cdots+\alpha_n\beta_n}.$$

(2) $G = \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \varepsilon((i_1, i_2), (j_1, j_2) = (-1)^{i_1 j_2 - i_2 j_1}$,
(3) $G = \mathbb{Z} \times \mathbb{Z}, \quad \varepsilon((i_1, i_2), (j_1, j_2)) = (-1)^{(i_1+i_2)(j_1+j_2)}$,
(4) $G = \{-1, +1\}, \quad \varepsilon(i, j) = (-1)^{(i-1)(j-1)/4}$.

Definition 7.4 An n-Lie algebra is a linear spaces V equipped with n-ary operation which is skew-symmetric for any pair of variables and satisfies the following identity:

$$[x_1, \dots, x_{n-1}, [y_1, \dots, y_n]] = \\ = \sum_{i=1}^{n} [y_1, \dots, y_{i-1}, [x_1, \dots, x_{n-1}, y_i], y_{i+1}, \dots y_n]. \tag{7.1}$$

Definition 7.5 An n-Hom-Lie color algebra is a graded linear space $L = \oplus L_a$, $a \in G$ with an n-linear map $[\cdot, \dots, \cdot] : L \times \cdots \times L \to L$, a bicharacter $\varepsilon : G \times G \to \mathbf{K}^*$ and an even linear map $\alpha : L \to L$ such that

$$[x_1, \dots, x_i, x_{i+1}, \dots, x_n] = -\varepsilon(x_i, x_{i+1})[x_1, \dots, x_{i+1}, x_i, \dots, x_n], \quad i = 1, 2, \dots n - 1. \tag{7.2}$$

$$[\alpha(x_1), \dots, \alpha(x_{n-1}), [y_1, y_2, \dots, y_n]] = \\ = \sum_{i=1}^{n} \varepsilon(X, Y_i)[\alpha(y_1), \dots, \alpha(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha(y_{i+1}), \dots, \alpha(y_n)] \tag{7.3}$$

where $x_i, y_j \in \mathcal{H}(L)$, $X = \sum_{i=1}^{n-1} x_i$, $Y_i = \sum_{j=1}^{i} y_{j-1}$ and $y_0 = e$.

Remark 7.1 Whenever $n = 2$ (resp. $n = 3$) we recover Hom-Lie color algebras (resp. ternairy Hom-Lie color algebras).

Remark 7.2 (1) When $\alpha = id$, we get n-Lie color algebra.
(2) When $G = \{e\}$ and $\alpha = id$, we get n-Lie algebra.
(3) When $G = \{e\}$ and $\alpha \neq id$, we get n-Hom-Lie algebra.

Definition 7.6 A morphism $f : (L, [\cdot, \dots, \cdot], \varepsilon, \alpha) \to (L', [\cdot, \dots, \cdot]', \varepsilon, \alpha')$ of an n-Hom-Lie color algebras is an even linear map $f : L \to L$ such that $f \circ \alpha = \alpha \circ f$ and for any $x_i \in \mathcal{H}(L)$,

$$f([x_1, \dots, x_n]) = [f(x_1), \dots, f(x_n)]'$$

Definition 7.7 (1) An n-Hom-Lie color algebra $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ is said to be multiplicative if α is an endomorphism, i.e. a linear map on L which is also a homomorphism with respect to multiplication $[\cdot, \dots, \cdot]$).
(2) An n-Hom-Lie color algebra $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ is said to be regular if α is an automorphism.
(3) An n-Hom-Lie color algebra $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ is said to be involutive if $\alpha^2 = id$.

Example 7.2 Let $G = \mathbb{Z}_2$, $\varepsilon(i, j) = (-1)^{ij}$, $L = L_0 \oplus L_1 =< e_2, e_4 > \oplus < e_1, e_3 >$,

$$[e_1, e_2, e_3] = e_2, \quad [e_1, e_2, e_4] = e_1, \quad [e_1, e_3, e_4] = [e_2, e_3, e_4] = 0,$$

and $\alpha(e_1) = e_3, \quad \alpha(e_2) = e_4, \quad \alpha(e_3) = \alpha(e_4) = 0$. Then $(L, [\cdot, \cdot, \cdot], \varepsilon, \alpha)$ is a 3-Hom-Lie color algebra.

Example 7.3 Let L be a graded linear space

$$L = L_{(0,0)} \oplus L_{(0,1)} \oplus L_{(1,0)} \oplus L_{(1,1)}$$

with $L_{(0,0)} =< e_1, e_2 >, \quad L_{(0,1)} =< e_3 >, \quad L_{(1,0)} =< e_4 >, \quad L_{(1,1)} =< e_5 >$.

The 4-ary even linear multiplication $[\cdot, \cdot, \cdot, \cdot] : L \times L \times L \times L \to L$ defined for basis $\{e_i\}, i = 1, \dots, 5$ by

$$[e_2, e_3, e_4, e_5] = e_1, [e_1, e_3, e_4, e_5] = e_2, [e_1, e_2, e_4, e_5] = e_3,$$
$$[e_1, e_2, e_3, e_4] = 0, [e_1, e_2, e_3, e_5] = 0$$

makes L into the five dimensional 4-Lie color algebra.

Now define on $(L, [\cdot, \cdot, \cdot, \cdot], \varepsilon)$ an even endomorphism $\alpha : L \to L$ by

$$\alpha(e_1) = e_2, \quad \alpha(e_2) = e_1, \quad \alpha(e_i) = e_i, i = 3, 4, 5.$$

Then $L_\alpha = (L, [\cdot, \cdot, \cdot, \cdot]_\alpha, \varepsilon, \alpha)$ is a 4-Hom-Lie color algebra. Observe that α is involutive (bijective).

Definition 7.8 A graded subspace H of an n-Hom-Lie color algebra L is a color Hom-subalgebra of L if

(i) $\alpha(H) \subseteq H$,
(ii) $[H, H, \dots, H] \subseteq H$.

Definition 7.9 Let $L_1, L_2, \dots, L_n$ be Hom-subalgebras of an n-Hom-Lie color algebra L. Denote by $[L_1, L_2, \dots, L_n]$ the Hom-subalgebra of L generated by all elements $[x_1, x_2, \dots, x_n]$, where $x_i \in L_i, i = 1, 2, \dots, n$.

(i) The sequence $L_1, L_2, \dots, L_n, \dots$ defined by

$$L_0 = L, \quad L_1 = [L_0, L_0, \dots, L_0], \quad L_2 = [L_1, L_1, \dots, L_1], \dots, \\ L_n = [L_{n-1}, L_{n-1}, \dots, L_{n-1}], \dots$$

is called the derived sequence.

(ii) The sequence $L^1, L^2, \dots, L^n, \dots$ defined by

$$L^0 = L, \quad L^1 = [L^0, L, \dots, L], \quad L^2 = [L^1, L, \dots, L], \dots, \\ L^n = [L^{n-1}, L, \dots, L], \dots$$

is called the descending central sequence.

(iii) The graded subspace $Z(L)$ defined by

$$Z(L) = \{x \in L | [x, L, L, \dots, L] = 0\} \tag{7.4}$$

is called the center of L.

Definition 7.10 A Hom-ideal I of an n-Hom-Lie color algebra L is a graded subspace of L such that $\alpha(I) \subseteq I$ and $[I, L, \dots, L] \subseteq I$.

Theorem 7.1 *Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ be an n-Hom-Lie color algebra with surjective twisting map $\alpha : L \to L$. Then, I_n, I^n and $Z(L)$ are Hom-ideals of L.*

Proof We only prove, by induction, that I_n is a Hom-ideal. For this, suppose, first, that I_{n-1} is a Hom-subalgebra of L and show that I_n is a Hom-subalgebra of L. For any $y \in \mathcal{H}(I_n)$, there exist $y_1, y_2, \dots, y_n \in \mathcal{H}(I_{n-1})$, such that

$$y = [y_1, y_2, \dots, y_n].$$

So, $\alpha(y) = \alpha([y_1, y_2, \dots, y_n]) = [\alpha(y_1), \alpha(y_2), \dots, \alpha(y_n)]$, which belong to I_n because I_{n-1} is a Hom-subalgebra. That is $\alpha(I_n) \subseteq I_n$.

For any $y_i \in \mathcal{H}(I_n)$, there exist $y_i^1, y_i^2, \dots, y_i^n \in I_{n-1}, i = 1, 2, \dots, n$ such that

$$[y_1, y_2, \dots, y_n] = [[y_1^1, y_1^2, \dots, y_1^n], [y_2^1, \dots, y_2^n], \dots, [y_n^1, \dots, y_n^n]].$$

I_{n-1} being a Hom-subalgebra, by hypotheses, $[y_i^1, y_i^2, \dots, y_i^n] \in I_{n-1}$ for $1 \leq i \leq n$, and so $[y_1, y_2, \dots, y_n] \in I_n$. Thus I_n is a Hom-subalgebra.

Now, suppose that I_{n-1} is a Hom-ideal. Let $x'_1, \dots, x'_{n-1} \in L$, $y \in I_n$, then there exist $x_1, \dots, x_{n-1} \in L$, $y_1, \dots, y_n \in I_{n-1}$ such that

$$\begin{aligned}[x_1', \ldots, x_{n-1}', y] &= [\alpha(x_1), \ldots, \alpha(x_{n-1}), [y_1, \ldots, y_n]] \\ &= \sum_{i=1}^{n} \varepsilon(X, Y_i)[\alpha(y_1), \ldots, \alpha(y_{i-1}), [x_1, \ldots, x_{n-1}, y_i], \alpha(y_{i+1}), \alpha(y_n)]\end{aligned}$$

As $[x_1, \ldots, x_{n-1}, y_i] \in I_{n-1}$, then $[x_1', \ldots, x_{n-1}', y] \in I_n$. So, I_n is a Hom-ideal of L. □

7.3 Constructions of n-Hom-Lie Color Algebras

In this section we present some useful methods for construction of n-Hom-Lie color algebras.

Proposition 7.1 *Let $(L, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ be an n-Hom-Lie color algebra and $\xi \in L_e$ such that $\alpha(\xi) = \xi$.*

Then $(L, \{\cdot, \ldots, \cdot\}, \varepsilon, \alpha)$ is an $(n-1)$-Hom-Lie color algebra with

$$\{x_1, \ldots, x_{n-1}\} = [\xi, x_1, \ldots, \ldots, x_{n-1}].$$

Proof With conditions in the statement,

$$\begin{aligned}&\{\alpha(x_1), \ldots, \alpha(x_{n-2}), \{y_1, \ldots, y_{n-1}\}\} = \\ &= [\xi, \alpha(x_1), \ldots, \alpha(x_{n-2}), [\xi, y_1, \ldots, y_{n-1}]] \\ &= [\alpha(\xi), \alpha(x_1), \ldots, \alpha(x_{n-2}), [\xi, y_1, \ldots, y_{n-1}]] \\ &= [[\xi, x_1, \ldots, x_{n-2}, \xi], \alpha(y_1), \ldots, \alpha(y_{n-1})] \\ &\quad + \sum_{i=i}^{n-1} \varepsilon(X, Y_i)[\xi, \alpha(y_1), \ldots, \alpha(y_{i-1}), [\xi, x_1, \ldots, x_{n-2}, y_i], \alpha(y_{i+1}), \ldots, \alpha(y_{n-1})] \\ &= \sum_{i=i}^{n-1} \varepsilon(X, Y_i)\{\alpha(y_1), \ldots, \alpha(y_{i-1}), \{x_1, \ldots, x_{n-2}, y_i\}, \alpha(y_{i+1}), \ldots, \alpha(y_{n-1})\}.\end{aligned}$$

which completes the proof. □

Corollary 7.1 *Let $(L, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ be an n-Hom-Lie color algebra and $\xi_i \in L_e$ such that $\alpha(\xi_i) = \xi_i$, $i = 1, 2, \ldots, k$.*

Then $L_k = (L, \{\cdot, \ldots, \cdot\}_k, \varepsilon, \alpha)$ is an $(n-k)$-Hom-Lie color algebra with

$$\{x_1, \ldots, x_{n-k}\}_k = [\xi_1, \ldots, \xi_k, x_1 \ldots, \ldots, x_{n-k}].$$

Corollary 7.2 *Let $(L, [\cdot, \ldots, \cdot], \varepsilon)$ be an n-Lie color algebra and $\xi \in L_e$.*

Then $(L, \{\cdot, \ldots, \cdot\}, \varepsilon)$ is an $(n-1)$-Lie color algebra with

$$\{x_1, \ldots, x_{n-1}\} = [\xi, x_1, \ldots, \ldots, x_{n-1}].$$

Theorem 7.2 *Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ be an n-Hom-Lie color algebra and β be an even endomorphism of L. Then*

$$L_\beta = (L, \{\cdot, \dots, \cdot\} = \beta[\cdot, \dots, \cdot], \varepsilon, \beta\alpha)$$

is an n-Hom-Lie color algebra.

Moreover suppose that $(L', [\cdot, \dots, \cdot]', \varepsilon, \alpha')$ is another n-Hom-Lie color algebra and β' be an even endomorphism of L'. If

$$f : (L, [\cdot, \dots, \cdot], \varepsilon, \alpha) \to (L', [\cdot, \dots, \cdot]', \varepsilon, \alpha')$$

is a morphism such that $f\beta = \beta' f$, then $f : L_\beta \to L_{\beta'}$ is also a morphism.

Proof First part is proved as follows:

$$\begin{aligned}
&\{\beta\alpha(x_1), \dots, \beta\alpha(x_{n-1}), \{y_1, \dots, y_n\}\} = \\
&= \beta([\beta\alpha(x_1), \dots, \beta\alpha(x_{n-1}), \beta[y_1, \dots, y_n]]) \\
&= \beta^2\Big([\alpha(x_1), \dots, \alpha(x_{n-1}), [y_1, y_2, \dots, y_n]]\Big) \\
&= \beta^2\left(\sum_{i=1}^{n} \varepsilon(X, Y_i)[\alpha(y_1), \dots, \alpha(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha(y_{i+1}), \dots, \alpha(y_n)]\right) \\
&= \sum_{i=1}^{n} \varepsilon(X, Y_i)\beta^2\Big([\alpha(y_1), \dots, \alpha(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha(y_{i+1}), \dots, \alpha(y_n)]\Big) \\
&= \sum_{i=1}^{n} \varepsilon(X, Y_i)\beta[\beta\alpha(y_1), \dots, \beta\alpha(y_{i-1}), \beta[x_1, \dots, x_{n-1}, y_i], \beta\alpha(y_{i+1}), \dots, \beta\alpha(y_n)] \\
&= \sum_{i=1}^{n} \varepsilon(X, Y_i)\{\beta\alpha(y_1), \dots, \beta\alpha(y_{i-1}), \{x_1, \dots, x_{n-1}, y_i\}, \beta\alpha(y_{i+1}), \dots, \beta\alpha(y_n)\}.
\end{aligned}$$

Second part is proved as follows:

$$\begin{aligned}
f(\{x_1, \dots, x_n\}) &= f([x_1, \dots, x_n]_\beta) = f\beta[x_1, \dots, x_n] = f[\beta(x_1), \dots, \beta(x_n)] \\
&= [f\beta(x_1), \dots, f\beta(x_n)]' = [\beta' f(x_1), \dots, \beta' f(x_n)]' \\
&= \beta'[f(x_1), \dots, f(x_n)]' = [f(x_1), \dots, f(x_n)]'_{\beta'} \\
&= \{f(x_1), \dots, f(x_n)\}'
\end{aligned}$$

This completes the proof. □

Taking $\beta = \alpha^n$ leads to the following statement.

Corollary 7.3 *Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ be a multiplicative n-Hom-Lie color algebra. Then, for any positive integer n,*

$$(L, \alpha^n[\cdot, \dots, \cdot], \varepsilon, \alpha^{n+1})$$

is also an n-Hom-Lie color algebra.

Taking $\beta = \alpha$ and $\alpha = id$ leads to the following statement.

Corollary 7.4 *Let $(L, [\cdot, \dots, \cdot], \varepsilon)$ be an n-Lie color algebra and α be an even endomorphism of L. Then*

$$L_\alpha = (L, \{\cdot, \dots, \cdot\} = \alpha[\cdot, \dots, \cdot], \varepsilon, \alpha)$$

is a multiplicative n-Hom-Lie color algebra.

Taking $\beta \in Aut(L)$, $\beta = \alpha^{-1}$ leads to the following statement.

Corollary 7.5 *Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ be a regular n-Hom-Lie color algebra. Then*

$$L_{\alpha^{-1}} = (L, \{\cdot, \dots, \cdot\} = \alpha^{-1}[\cdot, \dots, \cdot], \varepsilon)$$

is an n-Lie color algebra.

Taking $\beta = \alpha$ leads to the following statement.

Corollary 7.6 *Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ be an involutive n-Hom-Lie color algebra. Then*

$$L_\beta = (L, \{\cdot, \dots, \cdot\} = \alpha[\cdot, \dots, \cdot], \varepsilon)$$

is an n-Lie color algebra.

Theorem 7.3 *Let $(A, \cdot)$ be a commutative associative algebra and $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ be an n-Hom-Lie color algebra. The tensor product $A \otimes L = \sum_{g \in G} (A \otimes L)_g = \sum_{g \in G} A \otimes L_g$ with the bracket*

$$[a_1 \otimes x_1, \dots, a_n \otimes x_n]' = a_1 \dots a_n \otimes [x_1, \dots, x_n],$$

the even linear map

$$\alpha'(a \otimes x) := a \otimes \alpha(x)$$

and the bicharacter

$$\varepsilon(a + x, b + y) = \varepsilon(x, y), \forall a, b \in A, \forall x, y \in \mathcal{H}(L),$$

is an n-Hom-Lie color algebra.

Proof It follows from a straightforward computation. □

In the next definition, we introduce an element of the centroid (or semi-morphism) for n-Hom-Lie color algebra.

Definition 7.11 A semi-morphism of an n-Hom-Lie color algebra $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ is an even linear map $\beta : L \to L$ such that $\beta\alpha = \alpha\beta$ and

$$\beta[x_1, \dots, x_n] = [\beta(x_1), x_2, \dots, x_n].$$

Remark 7.3 Due to ε-skew-symmetry,

$$\beta[x_1, \dots, x_n] = [x_1, \dots \beta(x_i), \dots, x_n].$$

Theorem 7.4 *Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ be an n-Hom-Lie color algebra and $\beta : L \to L$ a semi-morphism of L. Define a new multiplication $\{\cdot, \dots, \cdot\} : L \times \dots \times L \to L$ by*

$$\{x_1, \dots, x_n\} = [x_1, \dots, \beta(x_i) \dots, x_n]$$

Then $(L, \{\cdot, \dots, \cdot\}, \varepsilon, \alpha)$ is also an n-Hom-Lie color algebra.

Proof The proof can be obtained as follows:

$$\begin{aligned}
&\{\alpha(x_1), \dots, \alpha(x_{n-1}), \{y_1, \dots, y_n\}\} = \\
&= [\alpha(x_1), \dots, \beta\alpha(x_i), \dots, \alpha(x_{n-1}), [y_1, \dots, \beta(y_j), \dots, y_n]] \\
&= [\alpha(x_1), \dots, \alpha\beta(x_i), \dots, \alpha(x_{n-1}), [y_1, \dots, \beta(y_j), \dots, y_n]] \\
&= \sum_{k<j} \varepsilon(X, Y_k)[\alpha(y_1), \dots, \alpha(y_{k-1}), [x_1, \dots, \beta(x_i), \dots, x_{n-1}, y_k], \\
&\qquad\qquad \alpha(y_{k+1}), \dots, \alpha\beta(y_j), \dots \alpha(y_n)] \\
&+ \varepsilon(X, Y_j)[\alpha(y_1), \dots, \alpha(y_{j-1}), [x_1, \dots, \beta(x_i), \dots, x_{n-1}, \beta(y_j)], \alpha(y_{j+1}), \dots, \alpha(y_n)] \\
&+ \sum_{k>j} \varepsilon(X, Y_k)[\alpha(y_1), \dots, \alpha\beta(y_j), \dots, \alpha(y_{k-1})[x_1, \dots, \beta(x_i), \dots, x_{n-1}, y_k], \\
&\qquad\qquad \alpha(y_{k+1}), \dots, \alpha(y_n)] \\
&= \sum_{k<j} \varepsilon(X, Y_k)[\alpha(y_1), \dots, \alpha(y_{k-1}), \{x_1, \dots, x_i, \dots, x_{n-1}, y_k\}, \\
&\qquad\qquad \alpha(y_{k+1}), \dots, \beta\alpha(y_j), \dots \alpha(y_n)] \\
&+ \varepsilon(X, Y_j)[\alpha(y_1), \dots, \alpha(y_{j-1}), \beta(\{x_1, \dots, x_i, \dots, x_{n-1}, y_j\}), \alpha(y_{j+1}), \dots, \alpha(y_n)] \\
&+ \sum_{k>j} \varepsilon(X, Y_k)[\alpha(y_1), \dots, \beta\alpha(y_j), \dots, \alpha(y_{k-1}), \{x_1, \dots, x_i, \dots, x_{n-1}, y_k\}, \\
&\qquad\qquad \alpha(y_{k+1}), \dots, \alpha(y_n)] \\
&= \sum_{k<j} \varepsilon(X, Y_k)\{\alpha(y_1), \dots, \alpha(y_{k-1}), \{x_1, \dots, x_i, \dots, x_{n-1}, y_k\}, \\
&\qquad\qquad \alpha(y_{k+1}), \dots, \alpha(y_j), \dots \alpha(y_n)\} \\
&+ \varepsilon(X, Y_j)\{\alpha(y_1), \dots, \alpha(y_{j-1}), \{x_1, \dots, x_i, \dots, x_{n-1}, y_j\}, \alpha(y_{j+1}), \dots, \alpha(y_n)\} \\
&+ \sum_{k>j} \varepsilon(X, Y_k)\{\alpha(y_1), \dots, \alpha(y_j), \dots, \alpha(y_{k-1})\{x_1, \dots, x_i, \dots, x_{n-1}, y_k\}, \\
&\qquad\qquad \alpha(y_{k+1}), \dots, \alpha(y_n)\}.
\end{aligned}$$

This completes the proof. □

Corollary 7.7 *Let* $(L, [\cdot, \ldots, \cdot], \varepsilon)$ *be an* n*-Lie color algebra and* $\alpha : L \to L$ *a semi-morphism of* L*. Then* $(L, \{\cdot, \ldots, \cdot\}, \varepsilon)$ *is another* n*-Lie color algebra, with*

$$\{x_1, \ldots, x_n\} = [x_1, \ldots, \alpha(x_i) \ldots, x_n]$$

Definition 7.12 Let $(L, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ be an n-Hom-Lie color algebra. An averaging operator of an n-Hom-Lie color algebra L is an even linear map $\beta : L \to L$ such that

(1) $\beta\alpha = \alpha\beta$
(2) $\beta[x_1, \ldots, \beta(x_i), \ldots, x_n] = [x_1, \ldots, \beta(x_i), \ldots, \beta(x_j), \ldots, x_n]$

Theorem 7.5 *Let* $(L, [\cdot, \ldots, \cdot], \varepsilon)$ *be an* n*-Lie color algebra and* $\alpha : L \to L$ *an averaging operator of* L*. Define a new multiplication* $\{\cdot, \ldots, \cdot\} : L \times \cdots \times L \to L$ *by*

$$\{x_1, \ldots, x_n\} = [x_1, \ldots, \alpha(x_i) \ldots, x_n]$$

Then $L_\alpha = (L, \{\cdot, \ldots, \cdot\}, \varepsilon, \alpha)$ *is an* n*-Hom-Lie color algebra.*

Proof The proof can be obtained as follows:

$$\begin{aligned}
&\{\alpha(x_1), \ldots, \alpha(x_{n-1}), \{y_1, \ldots, y_n\}\} = \\
&= [\alpha(x_1), \ldots, \alpha^2(x_i), \ldots, \alpha(x_{n-1}), [y_1, \ldots, \alpha(y_j), \ldots, y_n]] \\
&= \sum_{k<j} \varepsilon(X, Y_k)[\alpha(y_1), \ldots, \alpha(y_{k-1}), [x_1, \ldots, \alpha(x_i), \ldots, x_{n-1}, y_k], \\
&\qquad\qquad \alpha(y_{k+1}), \ldots, \alpha^2(y_j), \ldots \alpha(y_n)] \\
&+\varepsilon(X, Y_j)[\alpha(y_1), \ldots, \alpha(y_{j-1}), [x_1, \ldots, \alpha(x_i), \ldots, x_{n-1}, \alpha(y_j)], \\
&\qquad\qquad \alpha(y_{j+1}), \ldots, \alpha(y_n)] \\
&+ \sum_{k>j} \varepsilon(X, Y_k)[\alpha(y_1), \ldots, \alpha^2(y_j), \ldots, \alpha(y_{k-1})[x_1, \ldots, \alpha(x_i), \ldots, x_{n-1}, y_k], \\
&\qquad\qquad \alpha(y_{k+1}), \ldots, \alpha(y_n)] \\
&= \sum_{k<j} \varepsilon(X, Y_k)\{\alpha(y_1), \ldots, \alpha(y_{k-1}), \{x_1, \ldots, x_i, \ldots, x_{n-1}, y_k\}, \\
&\qquad\qquad \alpha(y_{k+1}), \ldots, \alpha(y_j), \ldots \alpha(y_n)\} \\
&+\varepsilon(X, Y_j)[\alpha(y_1), \ldots, \alpha(y_{j-1}), \alpha(\{x_1, \ldots, x_i, \ldots, x_{n-1}, y_j\}), \alpha(y_{j+1}), \ldots, \alpha(y_n)] \\
&\quad + \sum_{k>j} \varepsilon(X, Y_k)[\alpha(y_1), \ldots, \alpha(y_j), \ldots, \alpha(y_{k-1}), \{x_1, \ldots, x_i, \ldots, x_{n-1}, y_k\}, \\
&\qquad\qquad \alpha(y_{k+1}), \ldots, \alpha(y_n)] \\
&= \sum_{k<j} \varepsilon(X, Y_k)\{\alpha(y_1), \ldots, \alpha(y_{k-1}), \{x_1, \ldots, x_i, \ldots, x_{n-1}, y_k\}, \\
&\qquad\qquad \alpha(y_{k+1}), \ldots, \alpha(y_j), \ldots \alpha(y_n)\} \\
&+\varepsilon(X, Y_j)\{\alpha(y_1), \ldots, \alpha(y_{j-1}), \{x_1, \ldots, x_i, \ldots, x_{n-1}, y_j\}, \alpha(y_{j+1}), \ldots, \alpha(y_n)\}
\end{aligned}$$

$$+\sum_{k>j}\varepsilon(X,Y_k)\{\alpha(y_1),\dots,\alpha(y_j),\dots,\alpha(y_{k-1})\{x_1,\dots,x_i,\dots,x_{n-1},y_k\},$$
$$\alpha(y_{k+1}),\dots,\alpha(y_n)\}.$$

This ends the proof. □

Theorem 7.6 *Let $(L,[\cdot,\dots,\cdot],\varepsilon,\alpha)$ be an n-Hom-Lie color algebra and $\beta : L \to L$ an averaging operator of L. Define a new multiplication $\{\cdot,\dots,\cdot\} : L\times\cdots\times L \to L$ by*

$$\{x_1,\dots,x_n\} = [x_1,\dots,\beta(x_i)\dots,x_n]$$

Then $(L,\{\cdot,\dots,\cdot\},\varepsilon,\alpha)$ is also an n-Hom-Lie color algebra.

Proof It is similar to the one of Theorem 7.4. □

Taking $\alpha = id$, yields the following statement.

Corollary 7.8 *Let $(L,[\cdot,\dots,\cdot],\varepsilon)$ be an n-Lie color algebra and $\alpha : L \to L$ an averaging operator of L. Then $(L,\{\cdot,\dots,\cdot\},\varepsilon)$ is another n-Lie color algebra, with*

$$\{x_1,\dots,x_n\} = [x_1,\dots,\alpha(x_i)\dots,x_n]$$

Taking $\alpha = \beta$, yields the following statement.

Corollary 7.9 *Let $(L,[\cdot,\dots,\cdot],\varepsilon,\alpha)$ be an n-Hom-Lie color algebra and $\alpha : L \to L$ an averaging operator. Define a new multiplication $\{\cdot,\dots,\cdot\} : L\times\cdots\times L \to L$ by*

$$\{x_1,\dots,x_n\} = [x_1,\dots,\alpha(x_i)\dots,x_n]$$

Then $(L,\{\cdot,\dots,\cdot\},\varepsilon,\alpha)$ is also an n-Hom-Lie color algebra.

Theorem 7.7 *Let $(L,[\cdot,\dots,\cdot],\varepsilon,\alpha)$ be an n-Hom-Lie color algebra and $\beta : L \to L$ an averaging operator of L. Then the new bracket $\{\cdot,\dots,\cdot\} : L\times\cdots\times L \to L$ makes L into an n-Hom-Lie color algebra with*

$$\{x_1,\dots,x_n\} = [x_1,\dots,\beta(x_i)\dots,\beta(x_j),\dots,x_n].$$

Proof The proof is obtained as follows:

$$\begin{aligned}
&\{\alpha(x_1),\dots,\alpha(x_{n-1}),\{y_1,\dots,y_n\}\} =\\
&= [\alpha(x_1),\dots,\beta\alpha(x_i),\dots,\beta\alpha(x_j),\dots,\alpha(x_{n-1}),[y_1,\dots,\beta(y_k),\dots,\beta(y_l),\dots,y_n]]\\
&= \sum_{m<k}\varepsilon(X,Y_m)[\alpha(y_1),\dots,\alpha(y_{m-1}),[x_1,\dots,\beta(x_i),\dots,\beta(x_j),\dots,x_{n-1},y_m],\\
&\qquad\qquad \alpha(y_{m+1}),\dots,\alpha\beta(y_k),\dots,\alpha\beta(y_l),\dots,\alpha(y_n)]\\
&\quad +\varepsilon(X,Y_k)[\alpha(y_1),\dots,\alpha(y_{k-1}),[x_1,\dots,\beta(x_i),\dots,\beta(x_j),\dots,x_{n-1},\beta(y_k)],\\
&\qquad\qquad \alpha(y_{k+1}),\dots,\alpha\beta(y_l),\dots,\alpha(y_n)]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k<m<l}\varepsilon(X,Y_m)[\alpha(y_1),\ldots,\alpha\beta(y_k),\ldots,\alpha(y_{l-1}),\\
&\quad [x_1,\ldots,\beta(x_i),\ldots,\beta(x_j),\ldots,x_{n-1},y_m],\alpha(y_{m+1}),\ldots,\alpha\beta(y_l),\ldots,\alpha(y_n)]\\
&+\varepsilon(X,Y_l)[\alpha(y_1),\ldots,\alpha\beta(y_k),\ldots,\alpha(y_{l-1}),\\
&\qquad [x_1,\ldots,\beta(x_i),\ldots,\beta(x_j),\ldots,x_{n-1},\beta(y_l)],\alpha(y_{l+1}),\ldots,\alpha(y_n)]\\
&+\sum_{m>l}\varepsilon(X,Y_m)[\alpha(y_1),\ldots,\alpha\beta(y_k),\ldots,\alpha\beta(y_l),\ldots,\alpha(y_{m-1}),\\
&\qquad [x_1,\ldots,\beta(x_i),\ldots,\beta(x_j),\ldots,x_{n-1},y_m],\alpha(y_{m+1}),\ldots,\alpha(y_n)]\\
=&\sum_{m<k}\varepsilon(X,Y_m)\{\alpha(y_1),\ldots,\alpha(y_{m-1}),\{x_1,\ldots,x_i,\ldots,x_j,\ldots,x_{n-1},y_m\},\\
&\qquad \alpha(y_{m+1}),\ldots,\alpha(y_k),\ldots,\alpha(y_l),\ldots,\alpha(y_n)\}\\
&+\varepsilon(X,Y_k)[\alpha(y_1),\ldots,\alpha(y_{k-1}),\beta([x_1,\ldots,\beta(x_i),\ldots,\beta(x_j),\ldots,x_{n-1},y_k]),\\
&\qquad \alpha(y_{k+1}),\ldots,\beta\alpha(y_l),\ldots,\alpha(y_n)]\\
&+\sum_{k<m<l}\varepsilon(X,Y_m)\{\alpha(y_1),\ldots,\alpha(y_k),\ldots,\alpha(y_{m-1}),\\
&\qquad \{x_1,\ldots,x_i,\ldots,x_j,\ldots,x_{n-1},y_m\},\alpha(y_{m+1}),\ldots,\alpha(y_l),\ldots,\alpha(y_n)]\\
&+\varepsilon(X,Y_l)[\alpha(y_1),\ldots,\beta\alpha(y_k),\ldots,\alpha(y_{l-1}),\\
&\qquad \beta([x_1,\ldots,\beta(x_i),\ldots,\beta(x_j),\ldots,x_{n-1},y_l]),\alpha(y_{l+1}),\ldots,\alpha(y_n)]\\
&+\sum_{m>l}\varepsilon(X,Y_m)\{\alpha(y_1),\ldots,\alpha(y_k),\ldots,\alpha(y_l),\ldots,\alpha(y_{m-1}),\\
&\qquad \{x_1,\ldots,x_i,\ldots,x_j,\ldots,x_{n-1},y_m\},\alpha(y_{m+1}),\ldots,\alpha(y_n)\}
\end{aligned}
$$

$$
\begin{aligned}
=&\sum_{m<k}\varepsilon(X,Y_m)\{\alpha(y_1),\ldots,\alpha(y_{m-1}),\{x_1,\ldots,x_i,\ldots,x_j,\ldots,x_{n-1},y_m\},\\
&\qquad \alpha(y_{m+1}),\ldots,\alpha(y_k),\ldots,\alpha(y_l),\ldots,\alpha(y_n)\}\\
&+\varepsilon(X,Y_k)\{\alpha(y_1),\ldots,\alpha(y_{k-1}),\{x_1,\ldots,x_i,\ldots,x_j,\ldots,x_{n-1},y_k\},\\
&\qquad \alpha(y_{k+1}),\ldots,\alpha(y_l),\ldots,\alpha(y_n)\}\\
&+\sum_{k<m<l}\varepsilon(X,Y_m)\{\alpha(y_1),\ldots,\alpha(y_k),\ldots,\alpha(y_{m-1}),\\
&\qquad \{x_1,\ldots,x_i,\ldots,x_j,\ldots,x_{n-1},y_m\},\alpha(y_{m+1}),\ldots,\alpha(y_l),\ldots,\alpha(y_n)\}\\
&+\varepsilon(X,Y_l)\{\alpha(y_1),\ldots,\alpha(y_k),\ldots,\alpha(y_{l-1}),\{x_1,\ldots,x_i,\ldots,x_j,\ldots,x_{n-1},y_l\}),\\
&\qquad \alpha(y_{l+1}),\ldots,\alpha(y_n)\}\\
&+\sum_{m>l}\varepsilon(X,Y_m)\{\alpha(y_1),\ldots,\alpha(y_k),\ldots,\alpha(y_l),\ldots,\alpha(y_{m-1}),\\
&\qquad \{x_1,\ldots,x_i,\ldots,x_j,\ldots,x_{n-1},y_m\},\alpha(y_{m+1}),\ldots,\alpha(y_n)\}.
\end{aligned}
$$

This finishes the proof. □

7.4 Hom-Modules over n-Hom-Lie Color Algebras

In this section we consider Hom-modules over n-Hom-Lie color algebras.

Definition 7.13 Let G be an abelian group. A Hom-module is a pair (M, α_M) in which M is a G-graded linear space and $\alpha_M : M \longrightarrow M$ is an even linear map.

Definition 7.14 Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ be an n-Hom-Lie color algebra and (M, α_M) a Hom-module. The Hom-module (M, α_M) is called an n-Hom-Lie module over L if there are n polylinear maps:

$$\omega_i : L \otimes \dots L \otimes \underbrace{M}_{i} \otimes L \otimes \cdots \otimes L \to M, \quad i = 1, 2, \dots, n$$

such that, for any $x_i, y_i \in \mathcal{H}(L)$ and $m \in \mathcal{H}(M)$,

(a) $\omega_i(x_1, \dots, x_{i-1}, m, x_{i+1}, \dots, x_n)$ is a ε-skew-symmetric by all x-type arguments.
(b) $\omega_i(x_1, \dots, x_{i-1}, m, x_{i+1}, \dots, x_n) = -\varepsilon(m, x_{i+1})\omega_i(x_1, \dots, x_{i-1}, x_{i+1}, m \dots, x_n)$ for $i = 1, 2, \dots, n-1$.
(c) $\omega_n(\alpha(x_1), \dots, \alpha(x_{n-1}), \omega_n(y_1, \dots, y_{n-1}, m)) =$

$$= \sum_{i=1}^{n-1} \varepsilon(X, Y_i)\omega_n(\alpha(y_1), \dots, \alpha(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha(y_{i+1}), \dots, \alpha_M(m)) \\ +\varepsilon(X, Y_n)\omega_n(\alpha(y_1), \dots, \alpha(y_{n-1}), \omega_n(x_1, \dots, x_{n-1}, m)),$$

where $x_i, y_j \in \mathcal{H}(L)$, $X = \sum_{i=1}^{n-1} x_i$, $Y_i = \sum_{j=1}^{i} y_{j-1}$, $y_0 = e$ and $m \in \mathcal{H}(M)$.
(d) $\omega_{n-1}(\alpha(x_1), \dots, \alpha(x_{n-2}), \alpha_M(m), [y_1, \dots, y_n]) =$

$$= \sum_{i=1}^{n} \varepsilon(X, Y_i)\omega_i(\alpha(y_1), \dots, \alpha(y_{i-1}), \omega_{n-1}(x_1, \dots, x_{n-2}, m, y_i), \\ \alpha(y_{i+1}), \dots, \alpha(y_n)],$$

where $x_i, y_j \in \mathcal{H}(L)$, $X = \sum_{i=1}^{n-2} x_i + m$, $Y_i = \sum_{j=1}^{i} y_{j-1}$, $y_0 = e$ and $m \in \mathcal{H}(M)$.

Example 7.4 Any n-Hom-Lie color algebra $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ is an n-Hom-Lie module over itself by taking $M = L$, $\alpha_M = \alpha$ and $\omega_i(\cdot, \dots, \cdot) = [\cdot, \dots, \cdot]$.

Theorem 7.8 *Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ be an n-Hom-Lie color algebra, (M, α_M, ω_i) an n-Hom-Lie color module and $\beta : L \to L$ be an endomorphism. Define*

$$\tilde{\omega}_i = (\beta, \dots, \beta, \underbrace{id}_{i}, \beta, \dots, \beta), i = 1, 2, \dots, n.$$

Then $(M, \alpha_M, \tilde{\omega}_i)$ is an n-Hom-Lie color module.

Proof The items (a) and (b) are obvious. So we only prove (c), item (d) being proved similarly.

$$\begin{aligned}
&\tilde{\omega}_n(\alpha(x_1),\dots,\alpha(x_{n-1}),\tilde{\omega}_n(y_1,\dots,y_{n-1},m)) = \\
&= \omega_n(\beta\alpha(x_1),\dots,\beta\alpha(x_{n-1}),\omega_n(\beta(y_1),\dots,\beta(y_{n-1}),m)) \\
&= \omega_n(\alpha\beta(x_1),\dots,\alpha\beta(x_{n-1}),\omega_n(\beta(y_1),\dots,\beta(y_{n-1}),m)) \\
&= \sum_{i=1}^{n-1}\varepsilon(X,Y_i)\omega_n(\alpha\beta(y_1),\dots,\alpha\beta(y_{i-1}),[\beta(x_1),\dots,\beta(x_{n-1}),\beta(y_i)], \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \alpha\beta(y_{i+1}),\dots,\alpha_M(m)) \\
&\quad +\varepsilon(X,Y_n)\omega_n(\alpha\beta(y_1),\dots,\alpha\beta(y_{n-1}),\omega_n(\beta(x_1),\dots,\beta(x_{n-1}),m)) \\
&= \sum_{i=1}^{n-1}\varepsilon(X,Y_i)\omega_n(\beta\alpha(y_1),\dots,\beta\alpha(y_{i-1}),\beta([x_1,\dots,x_{n-1},y_i]), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \beta\alpha(y_{i+1}),\dots,\alpha_M(m)) \\
&\quad +\varepsilon(X,Y_n)\omega_n(\beta\alpha(y_1),\dots,\beta\alpha(y_{n-1}),\omega_n(\beta(x_1),\dots,\beta(x_{n-1}),m)), \\
&= \sum_{i=1}^{n-1}\varepsilon(X,Y_i)\omega_n(\beta\otimes\dots\otimes\beta\otimes id)(\alpha(y_1),\dots,\alpha(y_{i-1}),[x_1,\dots,x_{n-1},y_i], \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \alpha(y_{i+1}),\dots,\alpha_M(m)) \\
&\quad +\varepsilon(X,Y_n)\omega_n(\beta\otimes\dots\otimes\beta\otimes id)\Big(id\otimes\dots\otimes id\otimes\omega_n(\beta\otimes\dots\otimes\beta\otimes id)\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\alpha(y_1),\dots,\alpha(y_{n-1}),x_1,\dots,x_{n-1},m) \\
&= \sum_{i=1}^{n-1}\varepsilon(X,Y_i)\tilde{\omega}_n(\alpha(y_1),\dots,\alpha(y_{i-1}),[x_1,\dots,x_{n-1},y_i],\alpha(y_{i+1}),\dots,\alpha_M(m)) \\
&\quad +\varepsilon(X,Y_n)\tilde{\omega}_n(\alpha(y_1),\dots,\alpha(y_{n-1}),\tilde{\omega}_n(x_1,\dots,x_{n-1},m)).
\end{aligned}$$

This ends the proof. □

Corollary 7.10 *Let $(L,[\cdot,\dots,\cdot],\varepsilon,\alpha)$ be an n-Hom-Lie color algebra and $\beta: L\to L$ be an endomorphism. Then $(L,\{\cdot,\dots,\cdot\}_i,\alpha)$, with*

$$\{\cdot,\dots,\cdot\}_i = [\beta,\dots,\beta,\underbrace{id}_{i},\beta,\dots,\beta],\, i=1,2,\dots,n,$$

is an n-Hom-Lie color module.

Corollary 7.11 *Let $(L,[\cdot,\dots,\cdot],\varepsilon,\alpha)$ be a multiplicative n-Hom-Lie color algebra. Then, for nay $k\geq 1$, $(L,\{\cdot,\dots,\cdot\}_i^k,\alpha)$ is an n-Hom-Lie color module, with*

$$\{\cdot,\dots,\cdot\}_i^k = [\alpha^k,\dots,\alpha^k,\underbrace{id}_{i},\alpha^k,\dots,\alpha^k],\, i=1,2,\dots,n.$$

We end this section by giving some results for trivial graduation i.e. $G=\{e\}$.

Proposition 7.2 *Let* (M, α_M, ω_i) *be a module over the n-Hom-Lie algebra* $(L, [\cdot, \dots, \cdot], \alpha_L)$. *Consider the direct sum of linear spaces* $A = L \oplus M$. *Let us define on A the bracket*

$$\{x_1, \dots, x_n\} = [x_1, \dots, x_n],$$
$$\{x_1, \dots, x_{i-1}, \dots, m, x_{i+1}, \dots, x_n\} = \omega_i(x_1, \dots, x_{i-1}, \dots, m, x_{i+1}, \dots, x_n),$$
$$\{x_1, \dots, x_i, \dots, x_j, \dots, x_n\} = 0, \text{ whenever } x_i, x_j \in M.$$

Then $(A, \alpha_A = \alpha_L + \alpha_M)$ *is an n-Hom-Lie algebra.*

Proposition 7.3 *Let* $(M_1, \alpha^1_M, \omega^1_i)$ *and* $(M_2, \alpha^2_M, \omega^2_i)$ *be two modules over the n-Hom-Lie algebra* $(L, [\cdot, \dots, \cdot], \alpha)$. *Then* (M, α_M, ω_i) *is an n-Hom-Lie module with*

$$M = M_1 \oplus M_2, \quad \alpha_M = \alpha^1_M \oplus \alpha^2_M \quad \text{and} \quad \omega_i = \omega^1_i \oplus \omega^2_i.$$

7.5 Generalized Derivation of Color Hom-Algebras and Their Color Hom-Subalgebras

This section is devoted to generalized derivation of color Hom-algebras and their color Hom-subalgebras.

Here we give a more general definition of derivation, centroid and related objects.

Definition 7.15 For any $k \geq 1$, we call $D \in End(L)$ an α^k-derivation of degree d of the multiplicative n-Hom-Lie color algebra $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ if

$$\alpha \circ D = D \circ \alpha, \tag{7.5}$$

$$D([x_1, \dots, x_n]) = \sum_{i=1}^{n} \varepsilon(d, X_i)[\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)]. \tag{7.6}$$

Example 7.5 ([18]) The Laplacian of any Hom-Lie quasi-bialgebra $(\mathcal{G}, \mu, \gamma, \phi, \alpha)$ is an α^2-derivation of degree 0 of $(\Lambda\mathcal{G}, [\cdot, \cdot]^{\mu,\alpha})$, i.e.

$$L([X, Y]^{\mu,\alpha}) = [L(X), \alpha^2(Y)]^{\mu,\alpha} + [\alpha^2(X), L(Y)]^{\mu,\alpha}, \quad \forall X, Y \in \Lambda\mathcal{G}. \tag{7.7}$$

Example 7.6 Now, Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ be a multiplicative n-Hom-Lie color algebra. For any homogeneous elements $x_1, \dots, x_{n-1}$ of L and any integer $k \geq 1$, one defines the adjoint action of ΛL on L by

$$ad^{[\cdot,\dots,\cdot],\alpha^k}_{x_1,\dots,x_{n-1}}([y_1, y_2, \dots, y_n]) := \sum_{i=1}^{k} \varepsilon(X, Y_i)[\alpha^k(y_1), \dots \alpha^k(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha^k(y_{i+1}), \dots, \alpha(y_n)],$$

for any $y_1, \dots, y_n \in \mathcal{H}(L)$. Then $ad^{[\cdot,\dots,\cdot],\alpha^k}_{x_1,\dots,x_{n-1}}$ is an α^k-derivation of L of degree X. We call $ad^{[\cdot,\dots,\cdot],\alpha^k}_{x_1,\dots,x_{n-1}}$ an inner α^k-derivation. Denote by $Inn(L) = \oplus_{k\geq -1} Inn_{\alpha^k}(L)$ the space of all inner α^k-derivation.

The following proposition is proved by a straightforward computation.

Proposition 7.4 *Let D be an α^k-derivation of an n-Hom-Lie color algebra L and $\beta : L \to L$ an even endomorphism of L such that $D \circ \beta = \beta \circ D$. Then, for any non-negative integer s, $\Delta_s = D \circ \beta^s : L \to L$ is a $(\beta^s \alpha^k)$-derivation.*

Corollary 7.12 *If D is an α^k-derivation of an n-Hom-Lie color algebra L. Then Δ_s is an α^{k+s}-derivation of L.*

We denote the set of α^k-derivations of the multiplicative n-Hom-Lie color algebras L by $Der_{\alpha^k}(L)$. For any $D \in Der_{\alpha^k}(L)$ and $D' \in Der_{\alpha^k}(L)$, let us define their commutator $[D, D']$ as usual:

$$[D, D'] = D \circ D' - \varepsilon(d, d') D' \circ D.$$

Lemma 7.1 *For any $D \in Der_{\alpha^k}(L)$ and $D' \in Der_{\alpha^k}(L)$,*

$$[D, D'] \in Der_{\alpha^{k+s}}(L).$$

Denote by $Der(L) = \oplus_{k\geq -1} Der_{\alpha^k}(L)$.

Proposition 7.5 $(Der(L), [\cdot,\cdot], \omega)$ *is a Hom-Lie color algebra, with* $\omega(D) = D \circ \alpha$.

Definition 7.16 An endomorphism D of degree d of a multiplicative n-Hom-Lie color algebra $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ is called a generalized α^k-derivation if there exist linear mappings $D', D'', \dots, D^{(n-1)}, D^{(n)}$ of degree d such that for any $x_1, \dots, x_n \in \mathcal{H}(L)$:

$$D \circ \alpha = \alpha \circ D \quad \text{and} \quad D^{(i)} \circ \alpha = \alpha \circ D^{(i)}, \tag{7.8}$$

$$D^{(n)}([x_1, \dots, x_n]) = \sum_{i=1}^{n} \varepsilon(d, X_i)[\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D^{(i-1)}(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)]. \tag{7.9}$$

An $(n+1)$-tuple $(D, D', D'', \dots, D^{(n-1)}, D^{(n)})$ is called an $(n+1)$-ary α^k-derivation.

The set of generalized α^k-derivation is denoted by $GDer_{\alpha^k}$. Set

$$GDer(L) = \oplus_{k\geq -1} GDer_{\alpha^k}(L).$$

Definition 7.17 Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ be a multiplicative n-Hom-Lie color algebra. A linear mapping $D \in End(L)$ is said to be an α^k-quasiderivation of degree d if there exists a $D' \in End(L)$ of degree d such that

$$D^{'}([x_1, \dots, x_n]) = \sum_{i=1}^{n} \varepsilon(d, X_i)[\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)] \tag{7.10}$$

for all $x_1, \dots, x_n \in \mathcal{H}(L)$

We call D' the endomorphism associated to the α^k-quasiderivation D. The set of α^k-quasiderivations will be denoted $QDer(L)$. Set $QDer(L) = \oplus_{k\geq -1} QDer_{\alpha^k}(L)$.

Definition 7.18 Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ be a multiplicative n-Hom-Lie color algebra. The set $C_{\alpha^k}(L)$ consisting of linear mapping D of degree d with the property

$$D([x_1, \dots, x_n]) = \varepsilon(d, X_i)[\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)] \tag{7.11}$$

for all $x_1, \dots, x_n \in \mathcal{H}(L)$, is called the α^k-centroid of L.

We recover the definition of the centroid when $k = 0$.

Definition 7.19 Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ be a multiplicative n-Hom-Lie color algebra. The set $QC_{\alpha^k}(L)$ consisting of linear mapping D of degree d with the property

$$[D(x_1), \dots, x_n] = \varepsilon(d, X_i)[\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)], \tag{7.12}$$

for all $x_1, \dots, x_n \in \mathcal{H}(L)$, is called the α^k-quasicentroid of L.

Definition 7.20 Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ be a multiplicative n-Hom-Lie color algebra. The set $ZDer_{\alpha^k}(L)$ consisting of linear mappings D of degree d, such that for all $x_1, \dots, x_n \in \mathcal{H}(L)$:

$$D([x_1, \dots, x_n]) = \varepsilon(d, X_i)[\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)] = 0, \quad i = 1, 2, \dots, n, , \tag{7.13}$$

is called the set of central α^k-derivations of L.

It is easy to see that

$$ZDer(L) \subseteq Der(L) \subseteq QDer(L) \subseteq GDer(L) \subseteq End(L).$$

Proposition 7.6 *Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ be a multiplicative n-Hom-Lie color algebra.*

(1) *$GDer(L)$, $QDer(L)$, $C(L)$ are color Hom-subalgebras of $(End(L), [\cdot, \cdot], \omega)$:*

(1a) $\omega(GDer(L)) \subseteq GDer(L)$ *and* $[GDer(L), GDer(L)] \subseteq GDer(L)$.
(2b) $\omega(QDer(L)) \subseteq QDer(L)$ *and* $[QDer(L), QDer(L)] \subseteq QDer(L)$.
(3c) $\omega(C(L)) \subseteq C(L)$ *and* $[C(L), C(L)] \subseteq C(L)$.

(2) $ZDer(L)$ *is a color Hom-ideal of* $Der(L)$:
$\omega(ZDer(L)) \subseteq ZDer(L)$ *and* $[ZDer(L), Der(L)] \subseteq ZDer(L)$.

Proof (1a) Let us prove that if $D \in GDer(L)$, then $\omega(D) \in GDer(L)$. For any $x_1, \dots, x_n \in \mathcal{H}(L)$,

$$
\begin{aligned}
&(\omega(D^{(n)}))([x_1, \dots, x_n]) = (D^{(n)} \circ \alpha)([x_1, \dots, x_i, \dots, x_n])\\
&= D^{(n)}([\alpha(x_1), \dots, \alpha(x_i), \dots, \alpha(x_n)])\\
&= \sum_{i=1}^{n} \varepsilon(d, X_i)[\alpha^{k+1}(x_1), \dots, \alpha^{k+1}(x_{i-1}), D^{(i-1)}\alpha(x_i), \alpha^{k+1}(x_{i+1}), \dots, \alpha^{k+1}(x_n)]\\
&= \sum_{i=1}^{n} \varepsilon(d, X_i)[\alpha^{k+1}(x_1), \dots, \alpha^{k+1}(x_{i-1}), (D^{(i-1)} \circ \alpha)(x_i),\\
&\qquad\qquad \alpha^{k+1}(x_{i+1}), \dots, \alpha^{k+1}(x_n)]\\
&= \sum_{i=1}^{n} \varepsilon(d, X_i)[\alpha^{k+1}(x_1), \dots, \alpha^{k+1}(x_{i-1}), \omega(D^{(i-1)})(x_i), \alpha^{k+1}(x_{i+1}), \dots, \alpha^{k+1}(x_n)].
\end{aligned}
$$

This means that $\omega(D)$ is an α^{k+1}-derivation i.e. $\omega(D) \in GDer(L)$. Now let $D_1 \in GDer_{\alpha^k}(L)$ and $D_2 \in GDer_{\alpha^s}(L)$, we have

$$
\begin{aligned}
&(D_2^{(n)} D_1^{(n)})([x_1, \dots, x_n]) = D_2^{(n)}(D_1^{(n)}([x_1, \dots, x_n])) =\\
&= \sum_{i=1}^{n} \varepsilon(d_1, X_i) D_2^{(n)}([\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D_1^{(i-1)}(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)])\\
&= \sum_{i=1}^{n} \sum_{j<i}^{n} \varepsilon(d_1, X_i)\varepsilon(d_2, X_j)([\alpha^{k+s}(x_1), \dots, D_2^{(j-1)}(x_j), \dots, \alpha^{k+s}(x_{i-1}), D_1^{(i-1)}\alpha^s(x_i),\\
&\qquad\qquad \alpha^{k+s}(x_{i+1}), \dots, \alpha^{k+s}(x_n)])\\
&+ \sum_{i=1}^{n} \varepsilon(d_1 + d_2, X_i)([\alpha^{k+s}(x_1), \dots, \alpha^{k+s}(x_{i-1}), D_2^{(i-1)} D_1^{(i-1)}(x_i),\\
&\qquad\qquad \alpha^{k+s}(x_{i+1}), \dots, \alpha^{k}(x_n)])\\
&+ \sum_{i=1}^{n} \sum_{j>i}^{n} \varepsilon(d_1, X_i)\varepsilon(d_2, d_1 + X_j)([\alpha^{k+s}(x_1), , \dots, \alpha^{k+s}(x_{i-1}), D_1^{(i-1)}\alpha^s(x_i),\\
&\qquad\qquad \alpha^{k+s}(x_{i+1}), \dots, D_2^{(j-1)}(x_j), \dots, \alpha^{k+s}(x_n)]).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&([D_1^{(n)}, D_2^{(n)}])([x_1, \dots, x_n]) = (D_1^{(n)} D_2^{(n)} - \varepsilon(d_1, d_2) D_2^{(n)}(D_1^{(n)})([x_1, \dots, x_n])) =\\
&= \sum_{i=1}^{n} \varepsilon(d_1 + d_2, X_i)([\alpha^{k+s}(x_1), \dots, \alpha^{k+s}(x_{i-1}),
\end{aligned}
$$

$$(D_1^{(i-1)}D_2^{(i-1)} - \varepsilon(d_1, d_2)D_2^{(i-1)}D_1^{(i-1)}](x_i), \alpha^{k+s}(x_{i+1}), \dots, \alpha^{k+s}(x_n)])$$
$$= \sum_{i=1}^{n} \varepsilon(d_1 + d_2, X_i)([\alpha^{k+s}(x_1), \dots, \alpha^{k+s}(x_{i-1}),$$
$$[D_1^{(i-1)}, D_2^{(i-1)}](x_i), \alpha^{k+s}(x_{i+1}), \dots, \alpha^{k+s}(x_n)]).$$

Thus we obtain that $[D_1, D_2] \in GDer_{\alpha^{k+s}}(L)$.

(1b) That $QDer(L)$ is a color Hom-subalgebra of $(End(L), [\cdot, \cdot], \omega)$ is proved in the similar way.

(1c) Let $D_1 \in C_{\alpha^k}(L)$ and $D_2 \in C_{\alpha^s}(L)$. Then

$$\omega(D_1)([x_1, x_2, \dots, x_n]) = \alpha D_1([x_1, x_2, \dots, x_n])$$
$$= \varepsilon(d_1, X_i)\alpha([\alpha^k(x_1), \alpha^k(x_2), \dots, D_1(x_i), \dots, \alpha^k(x_n)])$$
$$= \varepsilon(d_1, X_i)[\alpha^{k+1}(x_1), \alpha^{k+1}(x_2), \dots, D_1(x_i), \dots, \alpha^{k+1}(x_n)].$$

Thus $\omega(D) \in C_{\alpha^{k+1}}(L)$. Moreover,

$$[D_1, D_2]([x_1, \dots, x_n]) = D_1D_2([x_1, \dots, x_n]) - \varepsilon(d_1, d_2)D_2D_1([x_1, \dots, x_n])$$
$$= \varepsilon(d_2, X_i)D_1[\alpha^k(x_1), \alpha^k(x_2), \dots, D_2(x_i), \dots, \alpha^k(x_n)]$$
$$-\varepsilon(d_1, d_2)\varepsilon(d_1, X_i)D_2[\alpha^s(x_1), \alpha^s(x_2), \dots, D_1(x_i), \dots, \alpha^s(x_n)]$$
$$= \varepsilon(d_1 + d_2, X_i)[\alpha^{k+s}(x_1), \alpha^{k+s}(x_2), \dots, D_1D_2(x_i), \dots, \alpha^{k+s}(x_n)]$$
$$-\varepsilon(d_1 + d_2, X_i)[\alpha^{k+s}(x_1), \alpha^{k+s}(x_2), \varepsilon(d_1, d_2) \dots, D_2D_1(x_i), \dots, \alpha^{k+s}(x_n)]$$
$$= \varepsilon(d_1 + d_2, X_i)[\alpha^{k+s}(x_1), \alpha^{k+s}(x_2), \dots, [D_1D_2](x_i), \dots, \alpha^{k+s}(x_n)].$$

So, $[D_1, D_2] \in C_{\alpha^{k+s}}(L)$ and finally $[D_1, D_2] \in C(L)$.

(2) By the same method as previously one can show that $\omega(D) \in ZDer_{\alpha^{k+1}}(L)$ and $[D_1, D_2] \in ZDer_{\alpha^{k+s}}(L)$, where $D_1 \in ZDer_{\alpha^k}(L)$ and $D_2 \in Der_{\alpha^s}(L)$. □

Proposition 7.7 *Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ be a multiplicative n-Hom-Lie color algebra.*

(1) *If $\varphi \in C(L)$ and $D \in Der(L)$, then φD is a derivation i.e.*

$$C(L) \cdot Der(L) \subseteq Der(L).$$

(2) *Any element of centroid is a quasiderivation i.e.*

$$C(L) \subseteq QDer(L).$$

Proof (1) For any $x_1, \dots, x_n \in \mathcal{H}(L)$,

$$
\begin{aligned}
\varphi D([x_1, \dots, x_n]) &= \sum_{i=1}^{n} \varepsilon(d, X_i)\varphi([\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_n)]) \\
&= \sum_{i=1}^{n} \varepsilon(d, X_i)\varepsilon(\varphi, X_i)[\alpha^{k+s}(x_1), \dots, \varphi D(x_i), \dots, \alpha^{k+s}(x_n)]) \\
&= \sum_{i=1}^{n} \varepsilon(d + \varphi, X_i)[\alpha^{k+s}(x_1), \dots, \varphi D(x_i), \dots, \alpha^{k+s}(x_n)]).
\end{aligned}
$$

Thus φD is an α^{k+s}-derivation of degree $d + \varphi$.

(2) Let D be an α^k-centroid, then for any $x_1, \dots, x_n \in \mathcal{H}(L)$,

$$
D([x_1, \dots, x_n]) = \varepsilon(d, X_i)[\alpha^k(x_1), \dots D(x_i), \dots, \alpha^k(x_n)], i = 1, 2, \dots, n. \quad (7.14)
$$

It follows that

$$
\sum_{i=1}^{n} \varepsilon(d, X_i)[\alpha^k(x_1), \dots D(x_i), \dots, \alpha^k(x_n)] = nD([x_1, \dots, x_n]). \quad (7.15)
$$

It suffises to take $D' = nD$. □

Lemma 7.2 *Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ be a multiplicative n-Hom-Lie color algebra. Then*

(1) *The ε-commutator of two elements of quasicentroid is a quasiderivation i.e.*

$$
[QC(L), QC(L)] \subseteq QDer(L).
$$

(2) $QDer(L) + QC(L) \subseteq GDer(L)$.

Proof For any $x_1, x_2, \dots, x_n \in \mathcal{H}(L)$,

(1) Let $D_1 \in QC_{\alpha^k}(L)$ and $D_2 \in QC_{\alpha^s}(L)$. We have, on the one hand

$$
\begin{aligned}
&[D_1 D_2(x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] \\
&= \varepsilon(D_1, D_2 + X_i)[D_2(\alpha^k(x_1)), \alpha^{k+s}(x_2), \dots, D_1(\alpha^s(x_i)), \dots, \alpha^{k+s}(x_n)] \\
&= \varepsilon(D_1, D_2 + X_i)\varepsilon(D_2, X_i)[\alpha^{k+s}(x_1), \dots, D_2 D_1(x_i), \dots, \alpha^{k+s}(x_n)] \\
&= \varepsilon(D_1, D_2)\varepsilon(D_1 + D_2, X_i)[\alpha^{k+s}(x_1), \dots, D_2 D_1(x_i), \dots, \alpha^{k+s}(x_n)].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&[D_1 D_2(x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] = \\
&= \varepsilon(D_1, D_2 + x_1)[D_2(\alpha^k(x_1)), D_1(\alpha^s(x_2)), \dots, \alpha^{k+s}(x_i), \dots, \alpha^{k+s}(x_n)] \\
&= \varepsilon(D_1, D_2 + x_1)\varepsilon(D_2, D_1 + X_i) \\
&\qquad [\alpha^{k+s}(x_1), D_1(\alpha^s(x_2)), \dots, D_2(\alpha^k(x_i)), \dots, \alpha^{k+s}(x_n)]
\end{aligned}
$$

$$= \varepsilon(D_1, x_1)\varepsilon(D_2, X_i)\varepsilon(x_1, D_1)$$
$$[D_1(\alpha^s(x_1)), \alpha^{k+s}(x_2), \ldots, D_2(\alpha^k(x_i)), \ldots, \alpha^{k+s}(x_n)]$$
$$= \varepsilon(D_2, X_i)\varepsilon(D_1, X_i)[\alpha^{k+s}(x_1), \ldots, D_1D_2(x_i), \ldots, \alpha^{k+s}(x_n)]$$
$$= \varepsilon(D_1 + D_2, X_i)[\alpha^{k+s}(x_1), \ldots, D_1D_2(x_i), \ldots, \alpha^{k+s}(x_n)],$$

and so

$$\varepsilon(D_1 + D_2, X_i)[\alpha^{k+s}(x_1), \ldots, [D_1, D_2](x_i), \ldots, \alpha^{k+s}(x_n)] =$$
$$= \varepsilon(D_1 + D_2, X_i)\Big([\alpha^{k+s}(x_1), \ldots, D_1D_2(x_i), \ldots, \alpha^{k+s}(x_n)]$$
$$-\varepsilon(D_1, D_2)[\alpha^{k+s}(x_1), \ldots, D_2D_1(x_i), \ldots, \alpha^{k+s}(x_n)]\Big) = 0.$$

It follows that

$$\sum_{i=1}^{n} \varepsilon(D_1 + D_2, X_i)[\alpha^{k+s}(x_1), \ldots, [D_1, D_2](x_i), \ldots, \alpha^{k+s}(x_n)] = 0.$$

Therefore $D' \equiv 0$, and $[D_1, D_2] \in QDer(L)$.

(2) Let $D_1 \in QDer_{\alpha^k}(L)$ and $D_2 \in QC_{\alpha^k}(L)$ with $|D_1| = |D_2|$. Then there exists $D_1' \in End(L)$ such that

$$D_1'([x_1, \ldots, x_n]) = \sum_{i=1}^{n} \varepsilon(D_1, X_i)[\alpha^k(x_1), \ldots, D_1(x_i), \ldots, \alpha^k(x_n)]$$
$$= [D_1(x_1), \alpha^k(x_2), \ldots, \alpha^k(x_n)] + \varepsilon(D_1, x_1)[\alpha^k(x_1), D_1(x_2), \ldots, \alpha^k(x_n)]$$
$$+ \sum_{i=3}^{n} \varepsilon(D_1, X_i)[\alpha^k(x_1), \ldots, D_1(x_i), \ldots, \alpha^k(x_n)]$$
$$= [(D_1 + D_2)(x_1), \alpha^k(x_2), \ldots, \alpha^k(x_n)] - [D_2(x_1), \alpha^k(x_2), \ldots, \alpha^k(x_n)]$$
$$+\varepsilon(D_1, x_1)[\alpha^k(x_1), D_1(x_2), \ldots, \alpha^k(x_n)]$$
$$+ \sum_{i=3}^{n} \varepsilon(D_1, X_i)[\alpha^k(x_1), \ldots, D_1(x_i), \ldots, \alpha^k(x_n)]$$
$$= [(D_1 + D_2)(x_1), \alpha^k(x_2), \ldots, \alpha^k(x_n)] - \varepsilon(D_2, x_1)[\alpha^k(x_1), D_2(x_2), \ldots, \alpha^k(x_n)]$$
$$+\varepsilon(D_1, x_1)[\alpha^k(x_1), D_1(x_2), \ldots, \alpha^k(x_n)]$$
$$+ \sum_{i=3}^{n} \varepsilon(D_1, X_i)[\alpha^k(x_1), \ldots, D_1(x_i), \ldots, \alpha^k(x_n)]$$
$$= [(D_1 + D_2)(x_1), \alpha^k(x_2), \ldots, \alpha^k(x_n)]$$
$$+\varepsilon(D_2, x_1)[\alpha^k(x_1), (D_1 - D_2)(x_2), \ldots, \alpha^k(x_n)]$$
$$+ \sum_{i=3}^{n} \varepsilon(D_1, X_i)[\alpha^k(x_1), \ldots, D_1(x_i), \ldots, \alpha^k(x_n)].$$

The conclusion follows by taking

$$D^{(n)} = D_1', \quad D = D_1 + D_2, \quad D' = D_1 - D_2, \quad D^{(i)} = D_1, \quad 2 \le i \le n-1.$$

This proved that $D_1 + D_2 \in GDe(L)$. □

Proposition 7.8 *If $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ is a multiplicative n-Hom-Lie color algebra, then*

$$QC(L) + [QC(L), QC(L)]$$

is a color Hom-subalgebra of $GDer(L)$.

Proof It follows from Lemma 7.2 by using the same arguments as in Proposition 2.4 in [41]. □

Proposition 7.9 *Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ is a multiplicative n-Hom-Lie color algebra such that α be a surjective mapping, then $[C(L), QC(L)] \subseteq Hom(L, Z(L))$. Moreover, if $Z(L) = \{0\}$, then $[C(L), QC(L)] = \{0\}$.*

Proof Let $D_1 \in C_{\alpha^k}(L)$, $D_2 \in QC_{\alpha^s}(L)$ and $x_1, \dots, x_n \in \mathcal{H}(L)$. Since α is surjective, for any $y_i' \in L$, there exists $y_i \in L$ such that $y_i' = \alpha^{k+s}(y_i)$, $i = 2, \dots, n$. Thus

$$\begin{aligned}
&[[D_1, D_2](x_1), y_2', \dots, y_n'] = \\
&= [[D_1, D_2](x_1), \alpha^{k+s}(y_2), \dots, \alpha^{k+s}(y_n)] \\
&= [D_1 D_2(x_1), \alpha^{k+s}(y_2), \dots, \alpha^{k+s}(y_n)] \\
&\quad -\varepsilon(d_1, d_2)[D_2 D_1(x_1), \alpha^{k+s}(y_2), \dots, \alpha^{k+s}(y_n)] \\
&= D_1([D_2(x_1), \alpha^s(y_2), \dots, \alpha^s(y_n)]) \\
&\quad -\varepsilon(d_1, d_2)\varepsilon(d_2, x_1 + d_1)[D_1\alpha^s(x_1), D_2\alpha^k(y_2), \dots, \alpha^{k+s}(y_n)] \\
&= D_1([D_2(x_1), \alpha^{k+s}(y_2), \dots, \alpha^{k+s}(y_n)]) \\
&\quad -\varepsilon(d_2, x_1) D_1[\alpha^s(x_1), \alpha^s D_2(y_2), \dots, \alpha^s(y_n)] \\
&= D_1\Big([D_2(x_1), \alpha^{k+s}(y_2), \dots, \alpha^{k+s}(y_n)]) \\
&\quad -\varepsilon(d_2, x_1)[\alpha^s(x_1), \alpha^s D_2(y_2), \dots, \alpha^s(y_n)]\Big) \\
&= D_1\Big([D_2(x_1), \alpha^{k+s}(y_2), \dots, \alpha^{k+s}(y_n)]) - [D_2(x_1), \alpha^s(y_2), \dots, \alpha^s(y_n)]\Big) = 0.
\end{aligned}$$

Hence, $[D_1, D_2](x_1) \in Z(L)$, and $[D_1, D_2] \in Hom(L, Z(L))$. Furthermore, if $Z(L) = \{0\}$, we know that $[C(L), QC(L)] = \{0\}$. □

Proposition 7.10 *Let $(L, [\cdot, \dots, \cdot], \varepsilon, \alpha)$ is a multiplicative n-Hom-Lie color algebra with surjective twisting α and H be a graded subset of L. Then*

(i) *$Z_L(H)$ is invariant under $C(L)$.*
(ii) *Every perfect color Hom-ideal of L is invariant under $C(L)$.*

Proof (i) For any $\varphi \in C(L)$ and $x \in Z_L(H)$, by (7.4), we have

$$0 = \varphi([x, H, L, \dots, L]) = [\varphi(x), \alpha^k(H), \alpha^k(L), \dots, \alpha^k(L)] = [\varphi(x), H, L, \dots, L].$$

Therefore $\varphi(x) \in Z_L(H)$, which implies that $Z_L(H)$ is invariant under $C(L)$.

(ii) Let H be a perfect color Hom-ideal of L. Then $H^1 = H$, and so for any $x \in H$ there exist $x_1^i, x_2^i, \ldots, x_n^i \in H$ with $0 < i < \infty$ such that $x = \sum_i [x_1^i, x_2^i, \ldots, x_n^i]$. If $\varphi \in C(L)$, then

$$\varphi(x) = \varphi\left(\sum_i [x_1^i, x_2^i, \ldots, x_n^i]\right) = \sum_i \varphi([x_1^i, x_2^i, \ldots, x_n^i])$$
$$= \sum_i [\varphi(x_1^i), \alpha^k(x_2^i), \ldots, \alpha^k(x_n^i)]) \in H.$$

This shows that H is invariant under $C(L)$. □

Proposition 7.11 *If the characteristic of* $\mathbb{K}$ *is* 0 *or not a factor of* $n - 1$. *Then*

$$ZDer(L) = C(L) \cap Der(L).$$

Proof If $\varphi \in C(L) \cap Der(L)$, then by (7.6) we have

$$\varphi([x_1, \ldots, x_n]) = \sum_{i=1}^{n} \varepsilon(d, X_i)[\alpha^k(x_1), \ldots, \varphi(x_i), \ldots, \alpha^k(x_n)],$$

and by (7.14), for $i = 1, 2, \ldots, n$,

$$\varepsilon(d, X_i)[\alpha^k(x_1), \ldots, \varphi(x_i), \ldots, \alpha^k(x_n)] = \varphi([x_1, \ldots, x_n]).$$

Thus

$$\varphi([x_1, \ldots, x_n]) = n\varphi([x_1, \ldots, x_n])$$

The characteristic of $\mathbb{K}$ being 0 or not a factor of $n - 1$, we have

$$0 = \varphi([x_1, \ldots, x_n]) = \varepsilon(d, X_i)[\alpha^k(x_1), \ldots, \varphi(x_i), \ldots, \alpha^k(x_n)], i = 1, 2, \ldots, n.$$

Which means that $\varphi \in ZDer(L)$.
Conversly, let $\varphi \in ZDer(L)$, Then

$$\varphi([x_1, \ldots, x_n]) = \varepsilon(d, X_i)[\alpha^k(x_1), \ldots, \varphi(x_i), \ldots, \alpha^k(x_n)] = 0, 1 \le i \le n$$

and thus $\varphi \in C(L) \cap Der(L)$. Therefore $ZDer(L) = C(L) \cap Der(L)$. □

Proposition 7.12 *Let* L *be an* n*-Hom-Lie color algebra. For any* $D \in Der(L)$ *and* $\varphi \in C(L)$

(1) $Der(L)$ *is contained in the normalizer of* $C(L)$ *in* $End(L)$ *i.e.*

$$[Der(L), C(L)] \subseteq C(L).$$

(2) *$QDer(L)$ is contained in the normalizer of $QC(L)$ in $End(L)$ i.e.*

$$[QDer(L), QC(L)] \subseteq QC(L).$$

Proof (1) For any $D \in Der(L)$, $\varphi \in C(L)$ and $x_1, x_2, \ldots, x_n \in \mathcal{H}(L)$,

$$\begin{aligned}
&D\varphi([x_1, \ldots, x_n]) = D([\varphi(x_1), \alpha^k(x_2), \ldots, \alpha^k(x_i), \ldots, \alpha^k(x_n)])\\
&= [D\varphi(x_1), \alpha^{k+s}(x_2), \ldots, \alpha^{k+s}(x_i), \ldots, \alpha^{k+s}(x_n)]\\
&\quad + \sum_{i=2}^{n} \varepsilon(d, \varphi + X_i)[\alpha^s\varphi(x_1), \alpha^{k+s}(x_2), \ldots, \alpha^k D(x_i), \ldots, \alpha^{k+s}(x_n)]\\
&= [D\varphi(x_1), \alpha^{k+s}(x_2), \ldots, \alpha^{k+s}(x_i), \ldots, \alpha^{k+s}(x_n)]\\
&\quad + \sum_{i=2}^{n} \varepsilon(d, \varphi + X_i)\varepsilon(\varphi, X_i)([\alpha^{k+s}(x_1), \alpha^{k+s}(x_2), \ldots, \varphi D(x_i), \ldots, \alpha^{k+s}(x_n)])\\
&= [D\varphi(x_1), \alpha^{k+s}(x_2), \ldots, \alpha^{k+s}(x_i), \ldots, \alpha^{k+s}(x_n)]\\
&\quad + \varepsilon(d, \varphi) \sum_{i=2}^{n} \varepsilon(d + \varphi, X_i)([\alpha^{k+s}(x_1), \alpha^{k+s}(x_2), \ldots, \varphi D(x_i), \ldots, \alpha^{k+s}(x_n)])\\
&= [D\varphi(x_1), \alpha^{k+s}(x_2), \ldots, \alpha^{k+s}(x_i), \ldots, \alpha^{k+s}(x_n)]\\
&\quad + \varepsilon(d, \varphi)\Big(\varphi D[x_1, x_2, \ldots, x_i, \ldots, x_n]\\
&\quad - [\varphi D(x_1), \alpha^{k+s}(x_2), \ldots, \alpha^{k+s}(x_i), \ldots, \alpha^{k+s}(x_n)]\Big).
\end{aligned}$$

Then we get

$$\begin{aligned}
&(D\varphi - \varepsilon(d, \varphi)\varphi D)([x_1, \ldots, x_n])\\
&\quad = [(D\varphi - \varepsilon(d, \varphi)\varphi D)(x_1), \ldots, \alpha^{k+s}(x_2), \ldots, \alpha^{k+s}(x_i), \ldots, \alpha^{k+s}(x_n)]),
\end{aligned}$$

that is $[D, \varphi] = D\varphi - \varepsilon(d, \varphi)\varphi D \in C(L)$.
(2) It is proved by using a similar method. □

Proposition 7.13 *Let L be an n-Hom-Lie color algebra. For any $D \in Der(L)$ and $\varphi \in C(L)$*

(1) *$D\varphi$ is contained in $C(L)$ if and only if φD is a central derivation of L.*
(2) *$D\varphi$ is a derivation of L if and only if $[D, \varphi]$ is a central derivation of L.*

Proof (1) From Proposition 7.12, $D\varphi$ is an element of $C(L)$ if and only if $\varphi D \in Der(L) \cap C(L)$. Thanks to Proposition 7.11, we get the result.
(2) The conclusion follows from (1), Propositions 7.11 and 7.12. □

If A is a commutative associative algebra and L is an n-Hom-Lie color algebra, the n-Hom-Lie algebra $A \otimes L$ (Theorem 7.3) is called the tensor product n-Hom-Lie color algebra of A and L. For $f \in End(A)$ and $\varphi \in End(L)$ let $f \otimes \varphi : A \otimes L \to A \otimes L$ be given by $f \otimes \varphi(a \otimes x) = f(a) \otimes \varphi(x)$, for $a \in A, x \in L$. Then $f \otimes \varphi \in End(A \otimes L)$.

Recall that if A is a commutative associative algebra, the centroid $C(A)$ of A is by definition

$$C(A) = \{f \in End(A) \mid f(ab) = f(a)b = af(b), \forall a, b \in A\}.$$

We now state the following

Proposition 7.14 *By the above notation, we have*

$$C(A) \otimes C(L) \subseteq C(A \otimes L).$$

Proof For any $a_i \in A, x_i \in \mathcal{H}(L)$, $1 \leq i \leq n$, and any $f \in C(A)$ and $\varphi \in C(L)$,

$$\begin{aligned}
&(f \otimes \varphi)[a_1 \otimes x_1, \dots, a_n \otimes x_n] = (f \otimes \varphi)(a_1 \dots a_n) \otimes [x_1, \dots, x_n] \\
&= f(a_1 \dots a_n) \otimes \varphi[x_1, \dots, x_n] \\
&= \varepsilon(\varphi, X_i)a_1 \dots f(a_i) \dots a_n \otimes [\alpha^k(x_1), \dots, \varphi(x_i), \dots, \alpha^k(x_n)] \\
&= \varepsilon(\varphi, X_i)[a_1 \otimes \alpha^k(x_1) \dots f(a_i) \otimes \varphi(x_i), \dots, a_n \otimes \alpha^k(x_n)] \\
&= \varepsilon(\varphi, X_i)[\alpha'^k(a_1 \otimes x_1) \dots (f \otimes \varphi)(a_i \otimes x_i), \dots, \alpha'^k(a_n \otimes x_n)].
\end{aligned}$$

Therefore, $f \otimes \varphi \in C(A \otimes L)$. □

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