Chapter 40
On Pade Approximants Series Solutions of MHD Flow Equations with Heat and Mass Transfer Due to a Point Sink

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Abstract The objective is to revisit the problem pertaining to the similarity boundary layer equations governing magneto-fluid dynamic steady incompressible laminar boundary layer flow for a point sink with an applied magnetic field, heat and mass transfer. The series solution method has been effectively implemented to a related integro-differential equation. The condition at infinity is applied to a related Pade approximants of the obtained series solution. The features of the flow characteristic, heat and mass transfer have been analyzed and discussed with respect to the pertinent parameters viz magnetic and suction/injection parameters. It has been found that the magnetic filed increases the skin friction but it reduces the heat transfer. Comparison of the obtained results for some particular cases of the present study has been done with the earlier results and they have been found in good agreement.

Keywords MHD flow equations · Heat and mass transfer · Pade approximants · Incompressible laminar boundary layer

40.1 Introduction

The prophecy of the flow field and heat transfer in MHD boundary-layer flows plays a basic role in various branches of technology such as in vortex chambers, MHD power generators, nuclear reactors, geophysical fluid dynamics, etc. The exploration of the boundary layer flow of an electrically conducting fluid on a cone due to a point sink with an applied magnetic field is relevant in the study of conical nozzle

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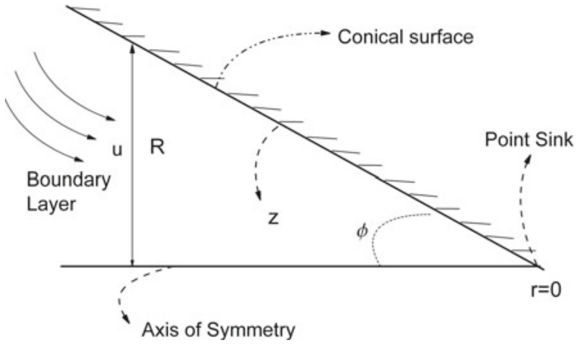

Fig. 40.1 Axi-symmetric boundary layer flow of an electrically conducting fluid in a circular cone at the vertex

or diffuser-flow problems and hence has been undertaken by Choi and Wilhelm [3]. The preceding problem in the absence of magnetic field, mass flux diffusion and heat transfer has been studied in [4]. The same problem was studied by Takhar [9] to include the effects of the magnetic field, mass flux diffusion and heat transfer. The problem was solved by shooting method on the lines of Soundalgekar et al. [6].

The objective of the present research paper is to obtain the solution of the third order non-linear differential equation pertaining to the similarity boundary layer equations governing magneto-fluid dynamic steady incompressible laminar boundary layer flow for a point sink with an applied magnetic field, heat and mass transfer by using series solution method as suggested by Wazwaz [10]. This problem has already been tackled by Takhar et al. [9]. In this research paper, the condition at infinity has been applied to a related Pade approximants of the obtained series solution. The features of the flow characteristic have been analyzed. The results of this research work have been compared graphically with those of [9] who obtained the solutions of the same problem by applying numerical treatment for the pertinent parameters. Our results have been found in excellent agreement with those given in [9].

40.2 Governing Equations

We have considered the steady laminar incompressible axisymmetric boundary layer flow of an electrically conducting fluid in a circular cone at the vertex (Fig. 40.1). The hole can be regarded as a three-dimensional sink.

A magnetic field B_0, fixed relative to the fluid, is applied in z-direction. The magnetic Reynolds number is assumed to be small so that the induced magnetic field can be neglected in comparison with the applied magnetic field. The wall and the free-stream are maintained at a constant temperature and concentration. The Hall effect and the dissipation terms are neglected. The effect of mass transfer (suction and injection) has been included in the analysis. It is assumed that the injected gas possesses the same physical properties as the boundary layer gas and has a static temperature equal to the wall temperature. Both gases are assumed to be perfect gases. The boundary layer equations under the foregoing assumptions are:

$$(ru)_r + (rw)_z = 0 \tag{40.1}$$

$$uu_r + wu_z = -\rho^{-1} P_r + \nu u_{zz} - \rho^{-1}\sigma B_0^2 u \tag{40.2}$$

$$uT_r + wT_z = \alpha T_{zz} \tag{40.3}$$

$$uC_r + wC_z = DC_{zz} \tag{40.4}$$

where

$$-\rho^{-1} P_r = UU_r + \rho^{-1}\sigma B_0^2 U, \quad U = -\frac{m_1}{r^2}, \quad m_1 > 0 \tag{40.5}$$

The boundary conditions are given by

$$\begin{cases} u(r,0) = 0, \quad w(r,0) = w_w, \quad T(r,0) = T_w, \quad C(r,0) = C_w \\ u(r,\infty) = U, \quad T(r,\infty) = T_\infty, \quad C(r,\infty) = C_\infty. \end{cases} \tag{40.6}$$

Applying the following transformations

$$\begin{cases} \eta = \frac{{m_1}^{1/2} z}{(2\nu r^2)^{1/2}}, \quad ru = \psi_z, \quad rw = -\psi_r, \quad \psi = -(2m_1 \nu r)^{1/2} f \\ u = Uf'(\eta), \quad w = \left(\frac{m_1 \nu}{2r^2}\right)^{1/2} (f - 3\eta f') \\ \frac{T - T_\infty}{T_w - T_\infty} = g(\eta), \quad \frac{C - C_\infty}{C_w - C_\infty} = G(\eta) \\ M = \frac{2\sigma B_0^2 r^3}{m_1 \rho}, \quad P_r = \frac{\nu}{\alpha}, \quad S_c = \frac{\nu}{D}, \quad K_w = w_w \left(\frac{2r^3}{m_1 \nu}\right)^{1/2} \end{cases} \tag{40.7}$$

to Eqs. (40.1)–(40.4), we find that the Eq. (40.1) is satisfied identically and Eqs. (40.2)–(40.4) reduce to self-similar equations given by

$$f''' - ff'' + 4(1 - f'^2) + M(1 - f') = 0 \tag{40.8}$$

$$g'' - P_r f g' = 0 \tag{40.9}$$

$$G'' - S_c f G' = 0 \tag{40.10}$$

The boundary conditions (40.6) reduce to

$$f(0) = K_w, \quad f'(0) = 0, \quad f'(\infty) = 1 \tag{40.11}$$

$$g(0) = 1, \quad g(\infty) = 0 \tag{40.12}$$

$$G(0) = 1, \quad G(\infty) = 0 \tag{40.13}$$

where r and z are the distance along and perpendicular to the cone; R is the radius of the cone ($R = rsin\phi$), ϕ is the semi-vertical angle of the cone; u and w are velocity components along r and z directions; ψ and f are the dimensional and dimensionless stream functions; C and T are concentration and temperature; g and G are dimensionless temperature and concentration; η is similarity variables; σ, ν and ρ are the density, kinematic viscosity and electrical conductivity; B_0 is the magnetic field; α and D are thermal diffusivity and binary diffusion coefficient; U is the inviscid flow velocity; m_1 is the strength of the point sink; P_r and S_c are Prandtl number and Schmidt number; M is the magnetic parameter; K_w is the mass transfer parameter of the sink flow. The subscripts r and z denote derivative w.r.t. r and z; the subscripts w and ∞ denote conditions at the wall and in the free stream; and prime denotes derivative with respect to η.

It is here remarked that the mass transfer parameter K_w will be treated as constant if the velocity normal to the wall w_w varies as $r^{-3/2}$ as $m_1\nu$ is constant for our mathematical analysis (Ref. Takhar et al. [9]). Also, the magnetic parameter M can be treated locally as constant for a fixed r as first considered by Takhar [7] and then by Takhar and Nath [8]. Also in a sink flow, $K_w < 0$ for suction and $K_w > 0$ for injection (cf. Takhar et al. [9] and Schlichting and Gersten [5], pp. 294–298).

The Pade Approximants constitute the best approximation of a function by a rational function of a given order. Developed by Henri Pade, Pade approximants often provide better approximation of a function than Taylor Series truncating does and they may still work in cases in which the Taylor Series does not converge. For these reasons, Pade approximants are used extensively in computer calculations and it is now well known that these approximants have the advantage of being able to manipulate polynomial approximation into the rational functions of polynomials. Pade approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function (Refs. [2, 10]). The $[\mathcal{L}/\mathcal{M}]$ Pade approximant to a formal power series is given by $[\mathcal{L}/\mathcal{M}] = P_{\mathcal{L}}(x)/Q_{\mathcal{M}}(x)$, where $P_{\mathcal{L}}(x)$ is a polynomial of degree at most $\mathcal{L}$ and $Q_{\mathcal{M}}(x)$ is a polynomial of degree at most $\mathcal{M}$. Without loss of generality, we can assume $Q_{\mathcal{M}}(0)$ to be 1. Furthermore, $P_{\mathcal{L}}(x)$ and $Q_{\mathcal{M}}(x)$ have no common factors. This means that the formal power series $A(x)$ equals the $[\mathcal{L}/\mathcal{M}]$ approximant through $\mathcal{L} + \mathcal{M} + 1$ terms. It is a well known fact that Pade approximants will converge on the entire real axis if $f(\eta)$ is free of singularities on the entire real axis. More importantly, the diagonal approximants are the most accurate approximants, therefore we will construct on diagonal approximants. Using the boundary condition $f'(\infty) = 1$, the diagonal approximants $[\mathcal{M}/\mathcal{M}]$ vanish if the coefficients of numerator vanish with the highest power in the η. Choosing the coefficients of the highest power of η as equal to zero, we get a polynomial equation in η which can be solved very easily by using the built-in utilities in the most manipulation languages such as Scilab, Matlab and Mathematica.

40.3 Mathematical Analysis

Here we consider the initial value problem (40.8) with the boundary conditions

$$f(0) = K_w, \qquad f'(0) = 0, \qquad f''(0) = \beta_0(say), \tag{40.14}$$

$$f'(\infty) = 1 \tag{40.15}$$

The Eq. (40.8) can be written as

$$f''' = ff'' - 4(1 - f'^2) - M(1 - f') \tag{40.16}$$

Integrating both sides of (40.16) from 0 to η and using the condition $f''(0) = \beta_0$, yields

$$\begin{aligned} f''(\eta) &= \beta_0 - (4+M)\eta - MK_w + \int_0^\eta f(t)f''(t)dt \\ &+ 4\int_0^\eta f'(t)^2 \mathrm{dt} + Mf(\eta) \end{aligned}$$

Integrating both sides of (40.3) from 0 to η and noting the condition $f'(0) = 0$, yields

$$\begin{aligned} f'(\eta) &= (\beta_0 - MK_w)\eta - (4+M)\frac{\eta^2}{2} + \int_0^\eta \int_0^\eta f(t)f''(t)dtdt \\ &+ 4\int_0^\eta \int_0^\eta f'(t)^2 dtdt + M\int_0^\eta f(t)dt \end{aligned} \tag{40.17}$$

so that by integrating again, we obtain

$$\begin{aligned} f(\eta) &= K_w + (\beta_0 - MK_w)\frac{\eta^2}{2} - (4+M)\frac{\eta^3}{6} \\ &+ \int_0^\eta \int_0^\eta \int_0^\eta f(t)f''(t)dtdtdt + 4\int_0^\eta \int_0^\eta \int_0^\eta f'(t)^2 dtdtdt \\ &+ M\int_0^\eta \int_0^\eta f(t)dtdt \end{aligned}$$

The following integro-differential equation

$$\begin{aligned} f(\eta) &= K_w + (\beta_0 - MK_w)\frac{\eta^2}{2} - (4+M)\frac{\eta^3}{6} + \int_0^\eta \frac{(\eta - t)^2}{2} f(t)f''(t)dt \\ &+ 2\int_0^\eta (\eta - t)^2 f'(t)^2 dt + M\int_0^\eta (\eta - t)f(t)dt \end{aligned} \tag{40.18}$$

is obtained from (40.17) upon converting the triple integral to a single integral. To determine a solution of (40.18), we use the series solution method used by Wazwaz [10]. Therefore, we express $f(\eta)$ as a power series of the form

$$f(\eta) = \sum_{n=0}^{\infty} a_n \eta^n \tag{40.19}$$

Substituting (40.19) into (40.18), and using only a few terms for simplicity reasons, we get

$$\begin{aligned}
&a_0 + \eta a_1 + \eta^2 a_2 + \eta^3 a_3 + \eta^4 a_4 + \eta^5 a_5 + \eta^6 a_6 + \eta^7 a_7 + \cdots \\
&= K_w + (\beta_0 - M K_w)\frac{\eta^2}{2} - (4 + M)\frac{\eta^3}{6} + \int_0^{\eta} \frac{(\eta - t)^2}{2} (a_0 \\
&+ t a_1 + t^2 a_2 + t^3 a_3 + t^4 a_4 + t^5 a_5 + t^6 a_6 + t^7 a_7 + t^8 a_8 \\
&+ t^9 a_9 + \cdots \big) \big(2a_2 + 6t a_3 + 12t^2 a_4 + 20t^3 a_5 + 30t^4 a_6 \\
&+ 42t^5 a_7 + 56t^6 a_8 + 72t^7 a_9 + \cdots \big)\, dt + 2\int_0^{\eta} (\eta - t)^2 \\
&\big(a_1 + 2t a_2 + 3t^2 a_3 + 4t^3 a_4 + 5t^4 a_5 + 6t^5 a_6 \\
&+ 7t^6 a_7 + 8t^7 a_8 + 9t^8 a_9 + \cdots \big)^2 dt \\
&+ M \int_0^{\eta} (\eta - t) \big(a_0 + t a_1 + t^2 a_2 \\
&+ t^3 a_3 + t^4 a_4 + t^5 a_5 + t^6 a_6 \\
&+ t^7 a_7 + t^8 a_8 + t^9 a_9 + \cdots \big)\, dt
\end{aligned} \tag{40.20}$$

which gives after integration and simplification

$$\begin{aligned}
&a_0 + \eta a_1 + \eta^2 a_2 + \eta^3 a_3 + \eta^4 a_4 + \eta^5 a_5 + \eta^6 a_6 + \eta^7 a_7 + \cdots \\
&= K_w + \frac{1}{2}(-K_w M + \beta_0)\eta^2 - \frac{1}{6}(4 + M)\eta^3 + \frac{1}{3}\eta^3 a_0 a_2 \\
&+ \frac{1}{12}\eta^4 a_1 a_2 + \frac{1}{30}\eta^5 a_2^2 + \frac{1}{4}\eta^4 a_0 a_3 + \frac{1}{10}\eta^5 a_1 a_3 \\
&+ \frac{1}{15}\eta^6 a_2 a_3 + \frac{1}{35}\eta^7 a_3^2 + \frac{1}{5}\eta^5 a_0 a_4 + \frac{1}{10}\eta^6 a_1 a_4 \\
&+ \frac{1}{15}\eta^7 a_2 a_4 + \frac{3}{56}\eta^8 a_3 a_4 + \frac{1}{42}\eta^9 a_4^2 + \frac{1}{6}\eta^6 a_0 a_5 \\
&+ \frac{2}{21}\eta^7 a_1 a_5 + \frac{11}{168}\eta^8 a_2 a_5 + \frac{13}{252}\eta^9 a_3 a_5 + \frac{2}{45}\eta^{10} a_4 a_5 \\
&+ \frac{2}{99}\eta^{11} a_5^2 + \frac{1}{7}\eta^7 a_0 a_6 + \frac{5}{56}\eta^8 a_1 a_6 + \frac{4}{63}\eta^9 a_2 a_6 \\
&+ \frac{1}{20}\eta^{10} a_3 a_6 + \frac{7}{165}\eta^{11} a_4 a_6 + \frac{5}{132}\eta^{12} a_5 a_6 + \frac{5}{286}\eta^{13} a_6^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{8}\eta^8 a_0 a_7+\frac{1}{12}\eta^9 a_1 a_7+\frac{11}{180}\eta^{10} a_2 a_7+\frac{8}{165}\eta^{11} a_3 a_7\\
&+\frac{9}{220}\eta^{12} a_4 a_7+\frac{31}{858}\eta^{13} a_5 a_7+\frac{3}{91}\eta^{14} a_6 a_7+\frac{1}{65}\eta^{15} a_7^2\\
&+\frac{1}{9}\eta^9 a_0 a_8+\frac{7}{90}\eta^{10} a_1 a_8+\frac{29}{495}\eta^{11} a_2 a_8+\frac{31}{660}\eta^{12} a_3 a_8\\
&+\frac{17}{429}\eta^{13} a_4 a_8+\frac{19}{546}\eta^{14} a_5 a_8+\frac{43\eta^{15} a_6 a_8}{1365}+\frac{7}{240}\eta^{16} a_7 a_8\\
&+\frac{7}{510}\eta^{17} a_8^2+\frac{1}{10}\eta^{10} a_0 a_9+\frac{4}{55}\eta^{11} a_1 a_9+\frac{37}{660}\eta^{12} a_2 a_9\\
&+\frac{1}{22}\eta^{13} a_3 a_9+\frac{1}{26}\eta^{14} a_4 a_9+\frac{46\eta^{15} a_5 a_9}{1365}+\frac{17}{560}\eta^{16} a_6 a_9\\
&+\frac{19}{680}\eta^{17} a_7 a_9+\frac{4}{153}\eta^{18} a_8 a_9+\frac{4}{323}\eta^{19} a_9^2+\cdots
\end{aligned}
\tag{40.21}
$$

Equating the coefficients of like powers of η in both sides leads to

$$
\begin{aligned}
&a_0 = K_w, a_1 = 0, a_2 = \frac{\beta_0}{2}, a_3 = \frac{1}{6}(-4 - M + K_w\beta_0),\\
&a_4 = \frac{1}{24}\left(-4K_w - K_w M + {K_w}^2\beta_0 + M\beta_0\right),\\
&a_5 = \frac{1}{120}\left(-4{K_w}^2 - 4M - {K_w}^2 M - M^2 + {K_w}^3\beta_0\right.\\
&\left.+\,2K_w M\beta_0 + 9\beta_0^2\right),\\
&a_6 = \frac{1}{720}\left(-4{K_w}^3 - 8K_w M - {K_w}^3 M - 2K_w M^2 - 112\beta_0\right.\\
&\left.+\,{K_w}^4\beta_0 - 28M\beta_0 + 3{K_w}^2 M\beta_0 + M^2\beta_0 + 37K_w\beta_0^2\right),\\
&a_7 = \frac{1}{5040}\left(448 - 4{K_w}^4 + 224M - 12{K_w}^2 M - {K_w}^4 M + 24M^2\right.\\
&-\,3{K_w}^2 M^2 - M^3 - 492K_w\beta_0 + {K_w}^5\beta_0 - 123K_w M\beta_0 + 4{K_w}^3 M\beta_0\\
&\left.+\,3K_w M^2\beta_0 + 104{K_w}^2\beta_0^2 + 48M\beta_0^2\right),\\
&a_8 = \frac{1}{40320}\left(1968K_w - 4{K_w}^5 + 984K_w M - 16{K_w}^3 M\right.\\
&-\,{K_w}^5 M + 111K_w M^2 - 4{K_w}^3 M^2 - 3K_w M^3 - 1456{K_w}^2\beta_0\\
&+\,{K_w}^6\beta_0 - 696M\beta_0 - 364{K_w}^2 M\beta_0 + 5{K_w}^4 M\beta_0 - 174M^2\beta_0\\
&\left.+\,6{K_w}^2 M^2\beta_0 + M^3\beta_0 + 250{K_w}^3\beta_0^2 + 282K_w M\beta_0^2 + 459\beta_0^3\right),\\
&a_9 = \frac{1}{362880}\left(5824{K_w}^2 - 4{K_w}^6 + 2784M + 2912{K_w}^2 M\right.\\
&-\,20{K_w}^4 M - {K_w}^6 M + 1392M^2 + 340{K_w}^2 M^2 - 5{K_w}^4 M^2\\
&+\,170M^3 - 6{K_w}^2 M^3 - M^4 - 3640{K_w}^3\beta_0 + {K_w}^7\beta_0 - 4212K_w M\beta_0\\
&-\,910{K_w}^3 M\beta_0 + 6{K_w}^5 M\beta_0 - 1053K_w M^2\beta_0 + 10{K_w}^3 M^2\beta_0
\end{aligned}
$$

$$+ 4K_w M^3 \beta_0 - 12424\beta_0^2 + 555{K_w}^4\beta_0^2 - 3106M\beta_0^2 + 1060{K_w}^2 M\beta_0^2 + 207M^2\beta_0^2 + 4141K_w\beta_0^3\big), \ldots \tag{40.22}$$

Accordingly, the solution of (40.8) in a series form is given by

$$\begin{aligned} f(\eta) &= K_w + \eta^2\frac{\beta_0}{2} + \frac{\eta^3}{6}(-4 - M + K_w\beta_0) \\ &+ \frac{\eta^4}{24}\left(-4K_w - K_w M + {K_w}^2\beta_0 + M\beta_0\right) \\ &+ \frac{\eta^5}{120}\left(-4{K_w}^2 - 4M - {K_w}^2 M - M^2 + {K_w}^3\beta_0 + 2K_w M\beta_0 + 9\beta_0^2\right) \\ &+ \frac{\eta^6}{720}\big(-4{K_w}^3 - 8K_w M - {K_w}^3 M - 2K_w M^2 - 112\beta_0 + {K_w}^4\beta_0 \\ &- 28M\beta_0 + 3{K_w}^2 M\beta_0 + M^2\beta_0 + 37K_w\beta_0^2\big) \\ &+ \frac{\eta^7}{5040}\big(448 - 4{K_w}^4 + 224M - 12{K_w}^2 M - {K_w}^4 M + 24M^2 \\ &- 3{K_w}^2 M^2 - M^3 - 492K_w\beta_0 + {K_w}^5\beta_0 - 123K_w M\beta_0 + 4{K_w}^3 M\beta_0 \\ &+ 3K_w M^2\beta_0 + 104{K_w}^2\beta_0^2 + 48M\beta_0^2\big) \\ &+ \frac{\eta^8}{40320}\big(1968K_w - 4{K_w}^5 + 984K_w M - 16{K_w}^3 M - {K_w}^5 M \\ &+ 111K_w M^2 - 4{K_w}^3 M^2 - 3K_w M^3 - 1456{K_w}^2\beta_0 + {K_w}^6\beta_0 \\ &- 696M\beta_0 - 364{K_w}^2 M\beta_0 + 5{K_w}^4 M\beta_0 - 174M^2\beta_0 + 6{K_w}^2 M^2\beta_0 \\ &+ M^3\beta_0 + 250{K_w}^3\beta_0^2 + 282K_w M\beta_0^2 + 459\beta_0^3\big) + \cdots \end{aligned} \tag{40.23}$$

$$\begin{aligned} f'(\eta) &= \beta_0\eta + \frac{\eta^2}{2}(-4 - M + K_w\beta_0) \\ &+ \frac{\eta^3}{6}(-4K_w - K_w M + {K_w}^2\beta_0 + M\beta_0) \\ &+ \frac{\eta^4}{24}(-4{K_w}^2 - 4M - {K_w}^2 M - M^2 + {K_w}^3\beta_0 + 2K_w M\beta_0 + 9\beta_0^2) \\ &+ \frac{\eta^5}{120}\big(-4{K_w}^3 - 8K_w M - {K_w}^3 M - 2K_w M^2 - 112\beta_0 + {K_w}^4\beta_0 \\ &- 28M\beta_0 + 3{K_w}^2 M\beta_0 + M^2\beta_0 + 37K_w\beta_0^2\big) \\ &+ \frac{\eta^6}{720}\big(448 - 4{K_w}^4 + 224M - 12{K_w}^2 M - {K_w}^4 M + 24M^2 \\ &- 3{K_w}^2 M^2 - M^3 - 492K_w\beta_0 + {K_w}^5\beta_0 - 123K_w M\beta_0 + 4{K_w}^3 M\beta_0 \\ &+ 3K_w M^2\beta_0 + 104{K_w}^2\beta_0^2 + 48M\beta_0^2\big) \\ &+ \frac{\eta^7}{5040}\big(1968K_w - 4{K_w}^5 + 984K_w M - 16{K_w}^3 M - {K_w}^5 M \end{aligned}$$

$$+111K_wM^2 - 4{K_w}^3M^2 - 3K_wM^3 - 1456{K_w}^2\beta_0 + {K_w}^6\beta_0$$
$$-696M\beta_0 - 364{K_w}^2M\beta_0 + 5{K_w}^4M\beta_0 - 174M^2\beta_0 + 6{K_w}^2M^2\beta_0$$
$$+M^3\beta_0 + 250{K_w}^3\beta_0^2 + 282K_wM\beta_0^2 + 459\beta_0^3\big) + \cdots \tag{40.24}$$

The heat and concentration equations can easily be solved by assuming that we already posses suitable accurate solutions to the velocity equation. It is very much precise to obtain solution of Eq. (40.9) and consequently (40.10) by replacing g by G and P_r by S_c as

$$g(\eta) = 1 - \frac{\int_0^\eta e^{P_r \int_0^x f(y)dy} dx}{\int_0^\infty e^{P_r \int_0^x f(y)dy} dx} \tag{40.25}$$

$$g'(\eta) = -\frac{e^{P_r \int_0^\eta f(y)dy}}{\int_0^\infty e^{P_r \int_0^x f(y)dy} dx} \tag{40.26}$$

and heat transfer parameter is given by

$$g'(0) = -\frac{1}{\int_0^\infty e^{P_r \int_0^x f(y)dy} dx}. \tag{40.27}$$

40.4 Results and Discussion

We here note that the series solution (40.23) can also be obtained by using the series solution near the ordinary point $\eta = 0$ directly into the nonlinear equation (40.8). Further, the result (40.23) may also be determined by using the Adomian's decomposition method (Adomian [1]). Our goal now is to determine the constant β_0 by using the condition $f'(\infty) = 1$. It is easily seen that this condition cannot be applied directly to (40.24). We can achieve our goal by representing the series (40.24) by a rational function $h(\eta)$ by using the powerful Pade approximants $[\mathcal{L}/\mathcal{M}]$ of this series. In Baker [2], a Pade approximant to the power series (40.24) is defined as

$$[\mathcal{L}/\mathcal{M}] = \frac{P_{\mathcal{L}}(\eta)}{Q_{\mathcal{M}}(\eta)} \tag{40.28}$$

where $P_{\mathcal{L}}(\eta)$ and $Q_{\mathcal{M}}(\eta)$ are polynomials of degrees at most $\mathcal{L}$ and $\mathcal{M}$ respectively. Besides, we may consider $Q_{\mathcal{M}}(0) = 1$, and $P_{\mathcal{L}}(\eta)$ and $Q_{\mathcal{M}}(\eta)$ have no common factors.

In the following, we have determined the Pade approximants [2/2] of (40.24). The Pade approximants [3/3] and [4/4] can also be determined in a parallel manner. To determine the Pade approximants [2/2] to $f'(\eta)$ of degree 4, it requires choosing of A_0, A_1, A_2, B_1 and B_2 so that the coefficients of η^k for $k = 0, 1, 2, 3, 4$ are zero in the expression

$$\left(\beta_0\eta + \frac{\eta^2}{2}(-4 - M + K_w\beta_0) + \frac{\eta^3}{6}(-4K_w - K_wM + {K_w}^2\beta_0 + M\beta_0)\right.$$
$$\left. + \frac{\eta^4}{24}(-4{K_w}^2 - 4M - {K_w}^2M - M^2 + {K_w}^3\beta_0 + 2K_wM\beta_0 + 9\beta_0^2)\ldots\right)$$
$$\times\left(1 + B_1\eta + B_2\eta^2\right) = \left(A_0 + A_1\eta + A_2\eta^2\right) \tag{40.29}$$

Expanding (40.29) by putting $M = K_w = 0$ (Rosenhead [4]) and equating the coefficients of η^k for $k = 0, 1, 2, 3, 4$ to '1' yields Pade approximant [2/2].

Consequently, the Pade approximant [2/2] is given by

$$[2/2] = \frac{\beta_0\eta + \frac{1}{32}\left(-64 + 3\beta_0^4\right)\eta^2}{1 + \frac{3\beta_0^3\eta}{32} + \frac{3\beta_0^2\eta^2}{16}} \tag{40.30}$$

Applying the condition $f'(\infty) = 1$ to (40.30) gives

$$\beta_0 = f''(0) = 2.3928. \tag{40.31}$$

In a manner parallel to our above discussion, the approximants [3/3] and [4/4] are obtained and the values of the constant $\beta_0 = f''(0)$ are found to be very close to that of (40.31). Again, the value of $\beta_0 = f''(0)$ is calculated for different values of the pertinent parameters K_w (for both suction and injection) and $M = 0, 1, 2$ which are tabulated below in the Tables 40.1 and 40.2.

From the Table 40.1, it is obvious that the skin friction i.e. $f''(0)$ increases along with increasing values of the mass suction parameter ($K_w < 0$) and magnetic parameter (M) for the different Pade approximants. From the Table 40.2, it is obvious that the skin friction i.e. $f''(0)$ increases along with increasing values of magnetic parameter (M) but decreasing values of the mass injection parameter ($K_w > 0$) for the different Pade approximants.

Thus these numerical results given in Tables 40.1 and 40.2 are highly in conformation with the graphical results given in Figs. 40.2 and 40.3 respectively. The effects of the mass suction/injection parameter K_w and the magnetic parameter M have been depicted in the Fig. 40.2. The present results have been compared with Takhar et al. [9]. From the figure, it is evident that the present results are in good agreement with those obtained by [9].

The heat transfer and mass flux diffusion parameters have been found with the help of the Eq. (40.27) and using the values in Tables 40.1 and 40.2. Hence for the sake of brevity, it has not been described here. The heat transfer and mass flux diffusion parameters increase with an increase in suction ($K_w < 0$) and they decrease with an increase in the injection parameter ($K_w > 0$).

From the Figs. 40.2 and 40.3, it is clear that the magnetic parameter increases the skin friction but decreases the heat and mass flux diffusion parameters. However, the effect of magnetic field on $-g'(0)$ and $-G'(0)$ is too little as compared to its effect on skin friction. In the Fig. 40.4, the velocity profiles have been found to be very

Table 40.1 Values of $\beta_0 = f''(0)$ for different values of M and $K_w < 0$

M	K_w	$[\mathcal{L}/\mathcal{M}]$	β_0	Present case	Takhar et al. [9]
0	–2	[2, 2]	4	3.51069	3.5182
		[3, 3]	3.29017		
		[4, 4]	3.51069		
	–1	[2, 2]	3.10165	2.87481	2.8772
		[3, 3]	2.67469		
		[4, 4]	2.87481		
0.5	–2	[2, 2]	4.09212	3.60755	3.6162
		[3, 3]	3.42054		
		[4, 4]	3.60755		
	–1	[2, 2]	3.20699	2.95113	3.0231
		[3, 3]	2.82088		
		[4, 4]	2.95113		
1	–2	[2, 2]	4.18121	3.70103	3.7124
		[3, 3]	3.53775		
		[4, 4]	3.70103		
	–1	[2, 2]	3.30797	3.05035	3.1121
		[3, 3]	2.94427		
		[4, 4]	3.05035		

much steep due to suction ($K_w < 0$) and an opposite tendency is seen for injection ($K_w > 0$). Here the temperature and concentration parameters have similar nature for $P_r = S_c$. That is why, only temperature profiles have been plotted. From the Fig. 40.5, it is clear that the Prandtl number P_r and Schmidt number S_c respectively have significant effects on temperature and concentration profiles respectively. It is evident that the effects of mass transfer on $f''(0)$, $-g'(0)$, $-G'(0)$ are well pronounced as compared to those of the magnetic field. From the Fig. 40.5,

40.5 Concluding Remarks

1. Our results are in excellent agreement with those obtained by Takhar et al. [9].
2. The skin-friction increases with increasing magnetic field. The skin-friction is greater for suction parameter ($K_w < 0$) as compared to injection parameter ($K_w > 0$).
3. The heat transfer and mass diffusion parameters $-g'(0)$ and $-G'(0)$ decrease with increasing magnetic parameter, whereas they increase with increasing mass transfer parameters; i.e., suction and injection.
4. The velocity profiles increase with increasing magnetic parameter and decrease with increasing mass transfer parameters.

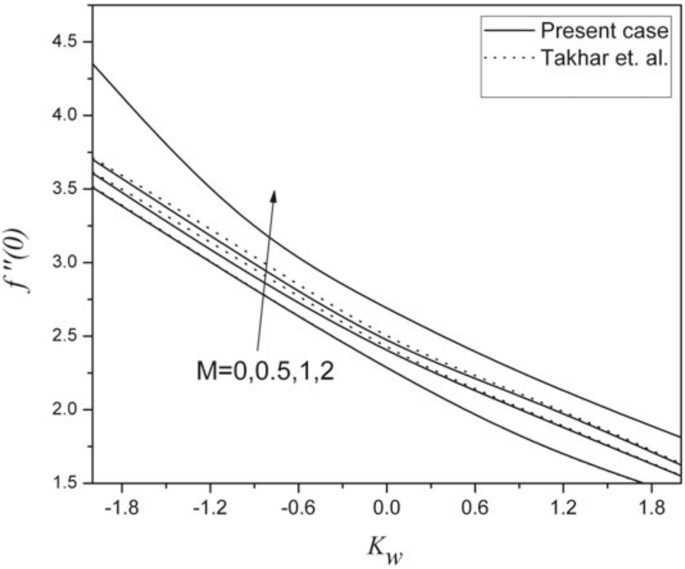

Fig. 40.2 Variation of skin friction with K_w for M = 0, 0.5, 1, 2 (Comparison with [9].)

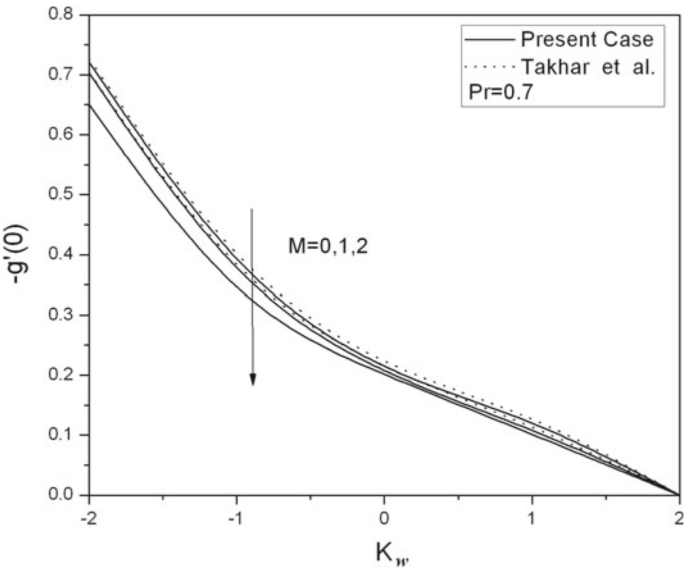

Fig. 40.3 Variation of heat transfer and mass diffusion parameters with K_w for M = 0, 0.5, 1, 2 and $P_r = S_c = 0.7$ (Comparison with [9].)

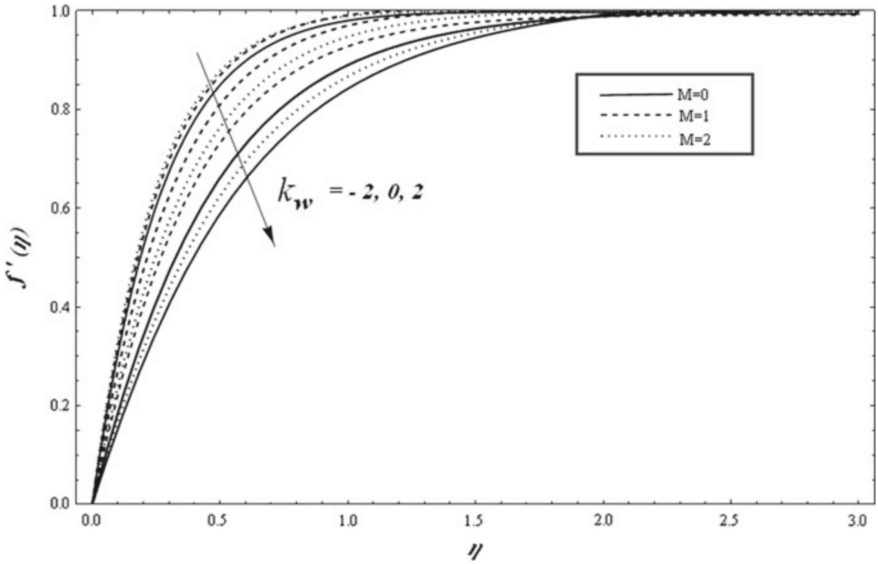

Fig. 40.4 Variation of velocity profiles with K_w for M $= 0, 1, 2$ ($K_w = -2, 0, 2$)

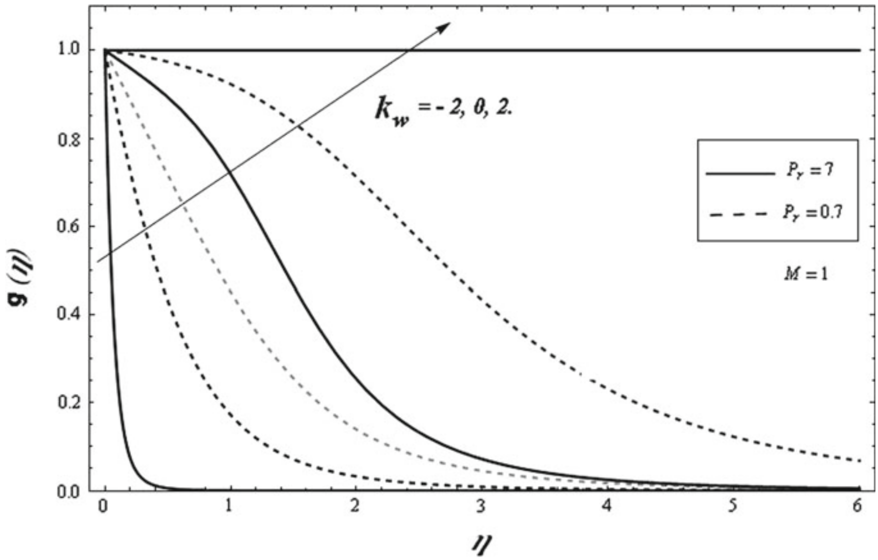

Fig. 40.5 Variation of temperature profiles for M $= 1$, $P_r = 0.7$ and $P_r = 7$ ($K_w = -2, 0, 2$)

Table 40.2 Values of $\beta_0 = f''(0)$ for different values of M and $K_w > 0$

M	K_w	$[\mathcal{L}/\mathcal{M}]$	β_0	Present case	Takhar et al. [9]	Rosenhead [4]
0	0	[2, 2]	2.3928	2.27229	2.2728	2.273
		[3, 3]	2.22496			
		[4, 4]	2.27229			
	1	[2, 2]	1.84939	1.75217	1.7505	–
		[3, 3]	1.75217			
		[4, 4]	1.79594			
	2	[2, 2]	1.44298	1.41415	1.4121	–
		[3, 3]	1.41415			
		[4, 4]	1.42072			
0.5	0	[2, 2]	2.50437	2.3797	2.392	–
		[3, 3]	2.29964			
		[4, 4]	2.3797			
	1	[2, 2]	1.96066	1.96066	1.973	–
		[3, 3]	2.10025			
		[4, 4]	1.90371			
	2	[2, 2]	1.54922	1.54922	1.5529	–
		[3, 3]	1.51835			
		[4, 4]	1.52432			
1	0	[2, 2]	2.43084	2.43084	2.4552	–
		[3, 3]	2.42256			
		[4, 4]	2.48248			
	1	[2, 2]	2.06689	2.06689	2.0825	–
		[3, 3]	1.98361			
		[4, 4]	2.0067			
	2	[2, 2]	1.65093	1.62377	1.6345	–
		[3, 3]	1.50109			
		[4, 4]	1.62377			

5. The temperature and mass diffusion profiles increase with increasing Prandtl number and with increasing mass transfer parameters.

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