Chapter 31
Cyclic Contractions and Common Fixed Point Results of Integral Type Contractions in Multiplicative Metric Spaces

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Abstract The existence of common fixed points of cyclic contractive mappings satisfying generalized integral contractive conditions in multiplicative metric spaces is studied. The well-posedness of common fixed point results and periodic point results of cyclic contractions are also established. These results establish some of the general common fixed point theorems for self-mappings.

Keywords Common fixed point · Periodic point · Well-posedness · Contraction mappings · Multiplicative metric space

MSC 2010 Classification 47H09 · 47H10 · 54C60 · 54H25

31.1 Introduction. Cyclic Contraction Mappings with Restrictions of Integral Type

There are various extensions and generalization of Banach contraction principle [11] are established in [1–10, 21, 22, 24, 25, 27, 29–33, 37–42]. In 2003, Kirk et al. [28] studied the cyclical contractive condition for self-mappings and proved some fixed point results as an extension of Banach contraction principle [11]. It is important worth of cyclic contraction mappings that although a map that satisfy the Banach contraction is always continuous, but a map that satisfy cyclic contractive condition need not to be continuous. Păcurar and Rus [35] obtained some fixed point results for cyclic weak contraction operators. Piatek [36] studied various theorems for cyclic

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S. Silvestrov et al. (eds.), *Algebraic Structures and Applications*, Springer Proceedings in Mathematics & Statistics 317,
https://doi.org/10.1007/978-3-030-41850-2_31

Meir–Keeler type contractive maps. Karapinar [26] proved fixed point theorems of cyclic ϕ-weak contraction maps. Derafshpour and Rezapour [17] obtained best proximity points results for cyclic contractive nonself maps. Ozavsar and Cevikel [34] established a famous Banach contraction-principle in another setup known as the multiplicative metric space. They obtained the structure of topology and properties of multiplicative metric space. The multiplicative calculus concepts along with some fundamental results dealing with multiplicative calculus were created by Bashirov et al. [13]. The applicability of multiplicative calculus based problems in the field of biomedical-image analysis were shown by Florack and Assen [20]. Also, the common fixed point based theorems of maps under weak commutative conditions were proved by He et al. [23]. Recently, Yamaod and Sintunavarat [43] provided fixed point theorems of cyclic contractive based maps under (α, β)-admissible restriction in a multiplicative metric space structure.

In this work, we establish common fixed point results of cyclic contraction mappings with restrictions of integral type in multiplicative metric space. Well-posedness based results and periodic point results are also proved.

For a non-empty set X and selfmapping $f : X \to X$, a point u in X satisfy $f(u) = u$, then we call it a fixed point of f. We denote the collection of all fixed points of self-map $f : X \to X$ by $F(f)$. Also if a point u in X satisfy $f(u) = g(u) = u$, then we call it a common fixed point of f and g. We denote the collection of all common fixed points of self-map $f, g : X \to X$ by $F(f, g)$.

Definition 31.1 Let A and B be non-empty subsets in X and (X, d) be a metric space. A mapping $T : A \cup B \to A \cup B$ is called a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$.

Kirk et al. [28] showed the following interesting result for fixed point of map T.

Theorem 31.1 *Let A and B be nonempty closed subsets of X and (X, d) be a complete metric space. If the mapping $T : A \cup B \to A \cup B$ is cyclic map satisfying*

$$d(Tx, Ty) \leq \kappa d(x, y) \quad for\ \ all\ \ x \in A\ \ and\ \ y \in B \tag{31.1}$$

for $\kappa \in [0, 1)$, then T has a unique fixed point in $A \cap B$.

Definition 31.2 Let A and B be non-empty subsets in X with (X, d) a metric space and $f, g : A \cup B \to A \cup B$ be two maps. Then $A \cup B$ has a cyclic representation w.r.t. the pair (f, g) if $f(A) \subseteq B$, $g(B) \subseteq A$ and $X = A \cup B$.

31.2 Common Fixed Points of Cyclic Contraction Mappings

In this section, common fixed point results for cyclic self-mappings on $A \cup B$ are proved with restriction of Integral type conditions. We start with the following result.

Theorem 31.2 *Let (X, d) be complete multiplicative metric space and A and B be non-empty closed sets in X. Suppose that mappings $f, g : A \cup B \to A \cup B$ satisfy that*

(a) *$A \cup B$ has a cyclic representation w.r.t. the pair (f, g), that is, $f(A) \subseteq B$, $g(B) \subseteq A$ and $X = A \cup B$.*
(b) *for any real number $c \in [0, 1)$,*

$$\int_1^{d(f(\hat{x}),g(\hat{y}))} \psi(\lambda)\, d\lambda \leq \int_1^{M(\hat{x},\hat{y})} \psi(\lambda)\, d\lambda \quad \text{for all } \hat{x} \in A \text{ and } \hat{y} \in B,$$

where $M(\hat{x}, \hat{y}) = \max\{d^c(\hat{x}, \hat{y}), d^c(\hat{x}, f(\hat{x})), d^c(\hat{y}, g(\hat{y}))\}$ and $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.
Then there exists at most one $\hat{u} \in X$ satisfies $\hat{u} = f\hat{u} = g\hat{u}$. Moreover, $\hat{u} \in A \cap B$ and any fixed point of f is the fixed point of g and conversely.

Proof First, to prove that any fixed point of f is the fixed point of g, we suppose $\hat{u} \in A \cap B$ with $\hat{u} = f\hat{u}$. Then for $\hat{u} \in A$ as well as $\hat{u} \in B$,

$$\int_1^{d(\hat{u},g(\hat{u}))} \psi(\lambda)\, d\lambda = \int_1^{d(f(\hat{u}),g(\hat{u}))} \psi(\lambda)\, d\lambda \leq \int_1^{M(\hat{u},\hat{u})} \psi(\lambda)\, d\lambda,$$

$$M(\hat{u}, \hat{u}) = \max\{d^c(\hat{u}, \hat{u}), d^c(\hat{u}, f(\hat{u})), d^c(\hat{u}, g(\hat{u}))\} = d^c(\hat{u}, g(\hat{u})),$$

that is,

$$\int_1^{d(\hat{u},g(\hat{u}))} \psi(\lambda)\, d\lambda \leq \int_1^{d^c(\hat{u},g(\hat{u}))} \psi(\lambda)\, d\lambda.$$

As $c < 1$, so it follows that $d(\hat{u}, g(\hat{u})) = 1$ and hence $\hat{u} = g(\hat{u})$. Thus $\hat{u} = f(\hat{u}) = g(\hat{u})$. Similarly, for $\hat{u} = g(\hat{u})$ provides $\hat{u} = f(\hat{u}) = g(\hat{u})$.

Suppose x_0 is the arbitrary point in A. Let $x_1 = fx_0$ and as $f(A) \subseteq B$, so that we obtain $x_1 \in B$. Again let $x_2 = gx_1$ and since $g(B) \subseteq A$, so that $x_2 \in A$. Continuing this way, define the sequence $\{x_n\}$ in X with $fx_{2n} = x_{2n+1}$ and $gx_{2n+1} = x_{2n+2}$ for $n = 0, 1, 2, \ldots$.

For $x_{2n} \in A$ and $x_{2n+1} \in B$, we obtain

$$\int_1^{d(x_{2n+1},x_{2n+2})} \psi(\lambda)\, d\lambda = \int_1^{d(f(x_{2n}),g(x_{2n+1}))} \psi(\lambda)\, d\lambda \leq \int_1^{M(x_{2n},x_{2n+1})} \psi(\lambda)\, d\lambda,$$

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\{d^c(x_{2n}, x_{2n+1}), d^c(x_{2n}, f(x_{2n})), d^c(x_{2n+1}, g(x_{2n+1}))\} \\ &= \max\{d^c(x_{2n}, x_{2n+1}), d^c(x_{2n}, x_{2n+1}), d^c(x_{2n+1}, x_{2n+2})\} \\ &= \max\{d^c(x_{2n}, x_{2n+1}), d^c(x_{2n+1}, x_{2n+2})\}, \end{aligned}$$

that is,

$$\int_1^{d(x_{2n+1},x_{2n+2})} \psi(\lambda)\,d\lambda \le \int_1^{\max\{d^c(x_{2n},x_{2n+1}),d^c(x_{2n+1},x_{2n+2})\}} \psi(\lambda)\,d\lambda. \tag{31.2}$$

Now, if $\max\{d^c(x_{2n}, x_{2n+1}), d^c(x_{2n+1}, x_{2n+2})\} = d^c(x_{2n+1}, x_{2n+2})$, then

$$\int_1^{d(x_{2n+1},x_{2n+2})} \psi(\lambda)\,d\lambda \le \int_1^{d^c(x_{2n+1},x_{2n+2})} \psi(\lambda)\,d\lambda,$$

and so $d(x_{2n+1}, x_{2n+2}) \le d^c(x_{2n+1}, x_{2n+2})$. Since $c < 1$, so $d(x_{2n+1}, x_{2n+2}) = 1$ and hence $x_{2n+1} = x_{2n+2}$. As $gx_{2n+1} = x_{2n+2}$, so $gx_{2n+1} = x_{2n+1}$, that is, x_{2n+1} is the fixed point of g. So by above conclusion, x_{2n+1} is a common fixed point of f and g.

If $\max\{d^c(x_{2n}, x_{2n+1}), d^c(x_{2n+1}, x_{2n+2})\} = d^c(x_{2n}, x_{2n+1})$, then

$$\int_1^{d(x_{2n+1},x_{2n+2})} \psi(\lambda)\,d\lambda \le \int_1^{d^c(x_{2n},x_{2n+1})} \psi(\lambda)\,d\lambda.$$

By similar steps, we can show that

$$\int_1^{d(x_{2n+2},x_{2n+3})} \psi(\lambda)\,d\lambda \le \int_1^{d^c(x_{2n+1},x_{2n+2})} \psi(\lambda)\,d\lambda.$$

Thus for all $n \ge 0$,

$$\int_1^{d(x_n,x_{n+1})} \psi(\lambda)\,d\lambda \le \int_1^{d^c(x_{n-1},x_n)} \psi(\lambda)\,d\lambda.$$

Continuing this way, we obtain

$$\begin{aligned}\int_1^{d(x_n,x_{n+1})} \psi(\lambda)\,d\lambda &\le \int_1^{d^c(x_{n-1},x_n)} \psi(\lambda)\,d\lambda \\ &\le \int_1^{d^{c^2}(x_{n-2},x_{n-1})} \psi(\lambda)\,d\lambda \\ &\le \dots \le \int_1^{d^{c^n}(x_0,x_1)} \psi(\lambda)\,d\lambda.\end{aligned}$$

As $0 \le c < 1$, we further have

$$\int_1^{d(x_n,x_{n+1})} \psi(\lambda)\,d\lambda \to 0 \;\; as \;\; n \to \infty$$

and thus $d(x_n, x_{n+1}) \to 1$ as $n \to \infty$.

For $\grave{m}, n \in \mathbb{N}$ with $\grave{m} \ge n$,

$$\begin{aligned} d(x_n, x_{\grave{m}}) &\leq d(x_n, x_{n+1}) \cdot d(x_{n+1}, x_{n+2}) \cdots d(x_{\grave{m}-1}, x_{\grave{m}}) \\ &\leq d^{c^n}(x_0, x_1) \cdot d^{c^{n+1}}(x_0, x_1) \cdots d^{c^{\grave{m}-1}}(x_0, x_1) \\ &= (d(x_0, x_1))^{c^n + c^{n+1} + \cdots + c^{\grave{m}-1}} \\ &\leq (d(x_0, x_1))^{c^n(1+c+c^2+\cdots)} \\ &\leq (d(x_0, x_1))^{\frac{c^n}{1-c}} \to 1 \;\; as \;\; n, \grave{m} \to \infty. \end{aligned}$$

Thus

$$\int_1^{d(x_n, x_{\grave{m}})} \psi(\lambda)\, d\lambda \to 0 \;\; as \;\; \grave{m}, n \to \infty$$

and $\{x_n\}$ is Cauchy sequence in complete (X, d). We obtain element $z \in X$ for which $x_n \to z$ as $n \to \infty$, or $\lim_{n\to\infty} d(x_n, z) = 1$.

As $\{x_{2n}\}$ is a sequence in closed set A with $x_{2n} \to z$ as $n \to \infty$, so $z \in A$. Also, since the sequence $\{x_{2n+1}\}$ is contained in closed set B with $x_{2n+1} \to z$ as $n \to \infty$, so $z \in B$.

Now we are to show that $fz = z$. For $z \in A$ and $x_{2n+1} \in B$,

$$\int_1^{d(fz, x_{2n+2})} \psi(\lambda)\, d\lambda = \int_1^{d(fz, gx_{2n+1})} \psi(\lambda)\, d\lambda \leq \int_1^{M(z, x_{2n+1})} \psi(\lambda)\, d\lambda, \tag{31.3}$$

$$\begin{aligned} M(z, x_{2n+1}) &= \max\{d^c(z, x_{2n+1}), d^c(z, f(z)), d^c(x_{2n+1}, g(x_{2n+1}))\} \\ &= \max\{d^c(z, x_{2n+1}), d^c(z, f(z)), d^c(x_{2n+1}, x_{2n+2})\} \end{aligned}$$

which gives $\lim_{n\to\infty} M(z, x_{2n+1}) = \max\{1, d^c(z, f(z)), 1\} = d^c(z, f(z))$. Thus by passing to the limit as $n \to \infty$ in (31.3) implies that

$$\int_1^{d(fz, z)} \psi(\lambda)\, d\lambda \leq \int_1^{d^c(z, fz)} \psi(\lambda)\, d\lambda,$$

that is, $d(z, fz) = 1$, that is, $z = fz$. Hence by above conclusion, $z = fz = gz$.

Now we are two show the uniqueness. Suppose that, their exist $w_1, w_2 \in A \cap B$, where $w_1 = fw_1 = gw_1$ and $w_2 = fw_2 = gw_2$.

So that for $w_1 \in A$ and $w_2 \in B$, we have

$$\int_1^{d(w_1, w_2)} \psi(\lambda)\, d\lambda = \int_1^{d(f(w_1), g(w_2))} \psi(\lambda)\, d\lambda \leq \int_1^{M(w_1, w_2)} \psi(\lambda)\, d\lambda,$$

$$\begin{aligned} M(w_1, w_2) &= \max\{d^c(w_1, w_2), d^c(w_1, f(w_1)), d^c(w_2, g(w_2))\} \\ &= \max\{d^c(w_1, w_2), 1, 1\} = d^c(w_1, w_2), \end{aligned}$$

that is, $\int_1^{d(w_1,w_2)} \psi(\lambda)\, d\lambda \leq \int_1^{d^c(w_1,w_2)} \psi(\lambda)\, d\lambda$, which gives $d(w_1, w_2) = 1$ and hence $w_1 = w_2$. Thus common fixed point of f and g is unique in $A \cap B$. □

Corollary 31.1 *Let (X, d) be complete multiplicative metric space and A and B be non empty closed sets in X. If mappings $f, g : A \cup B \to A \cup B$ satisfy that*

(a) *$A \cup B$ has a cyclic representation w.r.t. the pair (f, g), that is, $f(A) \subseteq B$, $g(B) \subseteq A$ with $X = A \cup B$.*
(b) *for any real number $c \in [0, 1)$,*

$$\int_1^{d(f(\hat{x}),g(\hat{y}))} \psi(\lambda)\, d\lambda \leq \int_1^{d^c(\hat{x},\hat{y})} \psi(\lambda)\, d\lambda \quad \textit{for all } \hat{x} \in A \textit{ and } \hat{y} \in B,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.

Then there exists at most one $\hat{u} \in X$ satisfies $\hat{u} = f\hat{u} = g\hat{u}$. Moreover, $\hat{u} \in A \cap B$ and any fixed point of f is the fixed point of g and conversely.

In case $f = g$ in Theorem 31.2, we obtain following fixed point result of cyclic contraction under restriction of integral type contractive mapping.

Corollary 31.2 *Let (X, d) be complete multiplicative metric space and A and B be non-empty closed sets in X. Suppose that mapping $f : A \cup B \to A \cup B$ satisfies*

(a) *$f(A) \subseteq B$ and $g(B) \subseteq A$ with $X = A \cup B$; and*
(b) *for any real constant $c \in [0, 1)$,*

$$\int_1^{d(f(\hat{x}),g(\hat{y}))} \psi(\lambda)\, d\lambda \leq \int_1^{N(\hat{x},\hat{y})} \psi(\lambda)\, d\lambda \quad \textit{for all } \hat{x} \in A \textit{ and } \hat{y} \in B,$$

where $N(\hat{x}, \hat{y}) = \max\{d^c(\hat{x}, \hat{y}), d^c(\hat{x}, f(\hat{x})), d^c(\hat{y}, f(\hat{y}))\}$, and $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.

Then there exists at most one $\hat{u} \in X$ satisfies $\hat{u} = f\hat{u}$. Moreover, $\hat{u} \in A \cap B$.

Theorem 31.3 *Let (X, d) be complete multiplicative metric space and A and B be non-empty closed sets in X. If mappings $f, g : A \cup B \to A \cup B$ are satisfying that*

(a) *$A \cup B$ has a cyclic representation w.r.t. the pair (f, g), i.e., $f(A) \subseteq B$, $g(B) \subseteq A$ and $X = A \cup B$.*
(b) *for any real constants c_1, c_2, c_3 with $c_1 + c_2 + c_3 \in [0, 1)$,*

$$\int_0^{d(f(\hat{x}),g(\hat{y}))} \psi(\lambda)\, d\lambda \leq \int_0^{M^*(\hat{x},\hat{y})} \psi(\lambda)\, d\lambda \quad \textit{for all } \hat{x} \in A \textit{ and } \hat{y} \in B,$$

where $M^*(\hat{x},\hat{y}) = d^{c_1}(\hat{x},\hat{y}) \cdot d^{c_2}(\hat{x}, f(\hat{x})) \cdot d^{c_3}(\hat{y}, g(\hat{y}))$ *and* $\psi : [0,\infty) \to [0,\infty)$ *is a Lebesgue integrable function with finite integral on each compact set in* $[0,\infty)$*, and for each* $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.

Then there exists at most one $\hat{u} \in X$ *satisfies* $\hat{u} = f\hat{u} = g\hat{u}$. *Moreover,* $\hat{u} \in A \cap B$ *and any fixed point of* f *is the fixed point of* g *and conversely.*

Proof First we prove that any fixed point of f is the fixed point of g. Suppose that $\hat{u} \in A \cap B$ with $\hat{u} = f\hat{u}$. Then we have $\hat{u} \in A$ as well as $\hat{u} \in B$ and we have

$$\int_1^{d(\hat{u},g(\hat{u}))} \psi(\lambda)\, d\lambda = \int_1^{d(f(\hat{u}),g(\hat{u}))} \psi(\lambda)\, d\lambda \le \int_1^{M^*(\hat{u},\hat{u})} \psi(\lambda)\, d\lambda,$$

$$M^*(\hat{u},\hat{u}) = d^{c_1}(\hat{u},\hat{u}) \cdot d^{c_2}(\hat{u}, f(\hat{u})) \cdot d^{c_3}(\hat{u}, g(\hat{u})) = 1 \cdot 1 \cdot d^{c_3}(\hat{u}, g(\hat{u})),$$

that is,

$$\int_1^{d(\hat{u},g(\hat{u}))} \psi(\lambda)\, d\lambda \le \int_1^{d^{c_3}(\hat{u},g(\hat{u}))} \psi(\lambda)\, d\lambda.$$

As $c_3 < 1$, so it follows that $d(\hat{u}, g(\hat{u})) = 1$ and hence $\hat{u} = g(\hat{u})$. Thus $\hat{u} = f(\hat{u}) = g(\hat{u})$. Similarly, we can show that if $\hat{u} = g(\hat{u})$, then we have $\hat{u} = f(\hat{u}) = g(\hat{u})$.

Suppose x_0 is the arbitrary point in A. Let $x_1 = fx_0$ and as $f(A) \subseteq B$, so that $x_1 \in B$. Again let $x_2 = gx_1$ and as $g(B) \subseteq A$, so that $x_2 \in A$. Continuing this way, define a sequence $\{x_n\}$ in X with $fx_{2n} = x_{2n+1}$ and $gx_{2n+1} = x_{2n+2}$ for all $n = 0, 1, 2, \ldots$.

For $x_{2n} \in A$ and $x_{2n+1} \in B$, we obtain

$$\int_1^{d(x_{2n+1},x_{2n+2})} \psi(\lambda)\, d\lambda = \int_1^{d(f(x_{2n}),g(x_{2n+1}))} \psi(\lambda)\, d\lambda \le \int_1^{M^*(x_{2n},x_{2n+1})} \psi(\lambda)\, d\lambda,$$

$$\begin{aligned} M^*(x_{2n}, x_{2n+1}) &= d^{c_1}(x_{2n}, x_{2n+1}) \cdot d^{c_2}(x_{2n}, f(x_{2n})) \cdot d^{c_3}(x_{2n+1}, g(x_{2n+1})) \\ &= d^{c_1}(x_{2n}, x_{2n+1}) \cdot d^{c_2}(x_{2n}, x_{2n+1}) \cdot d^{c_3}(x_{2n+1}, x_{2n+2}) \\ &= (d(x_{2n}, x_{2n+1}))^{c_1+c_2} \cdot d^{c_3}(x_{2n+1}, x_{2n+2}), \end{aligned}$$

that is,

$$\int_1^{d(x_{2n+1},x_{2n+2})} \psi(\lambda)\, d\lambda \le \int_1^{(d(x_{2n},x_{2n+1}))^{c_1+c_2} \cdot d^{c_3}(x_{2n+1},x_{2n+2})} \psi(\lambda)\, d\lambda. \tag{31.4}$$

Now, if $(d(x_{2n}, x_{2n+1}))^{c_1+c_2} \le d^{c_3}(x_{2n+1}, x_{2n+2})$, then we obtain

$$\int_1^{d(x_{2n+1},x_{2n+2})} \psi(\lambda)\, d\lambda \leq \int_1^{(d(x_{2n+1},x_{2n+2}))^{c_1+c_2+c_3}} \psi(\lambda)\, d\lambda,$$

which implies that $d(x_{2n+1}, x_{2n+2}) \leq (d(x_{2n+1}, x_{2n+2}))^{c_1+c_2+c_3}$. Since $c_1 + c_2 + c_3 < 1$, so $d(x_{2n+1}, x_{2n+2}) = 1$ and hence $x_{2n+1} = x_{2n+2}$. As we have $gx_{2n+1} = x_{2n+2}$, so $gx_{2n+1} = x_{2n+1}$, that is, x_{2n+1} is the fixed point of g. Thus by above conclusion, $x_{2n+1} \in F(f, g)$.

Also, in case $(d(x_{2n}, x_{2n+1}))^{c_1+c_2} \geq d^{c_3}(x_{2n+1}, x_{2n+2})$, then

$$\int_1^{d(x_{2n+1},x_{2n+2})} \psi(\lambda)\, d\lambda \leq \int_1^{(d(x_{2n},x_{2n+1}))^{c_1+c_2+c_3}} \psi(\lambda)\, d\lambda,$$

that is,

$$\int_1^{d(x_{2n+1},x_{2n+2})} \psi(\lambda)\, d\lambda \leq \int_1^{(d(x_{2n},x_{2n+1}))^{\eta}} \psi(\lambda)\, d\lambda,$$

where $\eta = c_1 + c_2 + c_3 < 1$. Similarly, it can be showed that

$$\int_1^{d(x_{2n+2},x_{2n+3})} \psi(\lambda)\, d\lambda \leq \int_1^{(d(x_{2n+1},x_{2n+2}))^{\eta}} \psi(\lambda)\, d\lambda.$$

Thus for all $n \geq 0$,

$$\int_1^{d(x_n,x_{n+1})} \psi(\lambda)\, d\lambda \leq \int_1^{(d(x_{n-1},x_n))^{\eta}} \psi(\lambda)\, d\lambda. \tag{31.5}$$

Continuing this way, we obtain

$$\begin{aligned}\int_1^{d(x_n,x_{n+1})} \psi(\lambda)\, d\lambda &\leq \int_1^{(d(x_{n-1},x_n))^{\eta}} \psi(\lambda)\, d\lambda \\ &\leq \int_1^{(d(x_{n-2},x_{n-1}))^{\eta^2}} \psi(\lambda)\, d\lambda \\ &\leq \dots \leq \int_1^{(d(x_0,x_1))^{\eta^n}} \psi(\lambda)\, d\lambda.\end{aligned}$$

As $0 \leq \eta < 1$, it implies

$$\int_1^{d(x_n,x_{n+1})} \psi(\lambda)\, d\lambda \to 0 \ \ as \ \ n \to \infty$$

and we have $d(x_n, x_{n+1}) \to 1$ as $n \to \infty$.

For $\grave{m}, n \in \mathbb{N}$ with $\grave{m} \geq n$,

$$
\begin{aligned}
d(x_n, x_{\grave{m}}) &\le d(x_n, x_{n+1}) \cdot d(x_{n+1}, x_{n+2}) \cdots d(x_{\grave{m}-1}, x_{\grave{m}}) \\
&\le (d(x_0, x_1))^{\eta^n} \cdot (d(x_0, x_1))^{\eta^{n+1}} \cdots (d(x_0, x_1))^{\eta^{\grave{m}-1}} \\
&= (d(x_0, x_1))^{\eta^n + \eta^{n+1} + \ldots \eta^{\grave{m}-1}} \\
&\le (d(x_0, x_1))^{\eta^n (1+\eta+\eta^2+\ldots)} \\
&\le (d(x_0, x_1))^{\frac{\eta^n}{1-\eta}} \to 1 \ \ as \ \ n, \grave{m} \to \infty.
\end{aligned}
$$

Thus

$$
\int_1^{d(x_n, x_{\grave{m}})} \psi(\lambda)\, d\lambda \to 0 \ \ as \ \ \grave{m}, n \to \infty
$$

and $\{x_n\}$ is a Cauchy sequence in complete multiplicative space (X, d). And we obtain $z \in X$ for which $x_n \to z$ as $n \to \infty$, or $\lim_{n\to\infty} d(x_n, z) = 1$.

As $\{x_{2n}\}$ is a sequence in closed set A with $x_{2n} \to z$ as $n \to \infty$, so $z \in A$. Also, since the sequence $\{x_{2n+1}\}$ is contained in closed set B with $x_{2n+1} \to z$ as $n \to \infty$, so $z \in B$.

Now we are to show that $fz = z$. For $z \in A$ and $x_{2n+1} \in B$,

$$
\int_1^{d(fz, x_{2n+2})} \psi(\lambda)\, d\lambda = \int_1^{d(fz, gx_{2n+1})} \psi(\lambda)\, d\lambda \le \int_1^{M^*(z, x_{2n+1})} \psi(\lambda)\, d\lambda, \quad (31.6)
$$

$$
\begin{aligned}
M^*(z, x_{2n+1}) &= d^{c_1}(z, x_{2n+1}) \cdot d^{c_2}(z, f(z)) \cdot d^{c_3}(x_{2n+1}, g(x_{2n+1})) \\
&= d^{c_1}(z, x_{2n+1}) \cdot d^{c_2}(z, f(z)) \cdot d^{c_3}(x_{2n+1}, x_{2n+2})
\end{aligned}
$$

which gives $\lim_{n\to\infty} M^*(z, x_{2n+1}) = 1 \cdot d^{c_2}(z, f(z)) \cdot 1 = d^{c_3}(z, f(z))$. Thus by passing to the limit as $n \to \infty$ in (31.6) implies that

$$
\int_1^{d(fz, z)} \psi(\lambda)\, d\lambda \le \int_1^{d^{c_3}(z, fz)} \psi(\lambda)\, d\lambda,
$$

and so $d(z, fz) = 1$, that is, $z = fz$. By above conclusion, we obtain $z = fz = gz$.

Now we are two show that $F(f, g)$ is singleton. Suppose that, their exist $w_1, w_2 \in A \cap B$, where $w_1 = fw_1 = gw_1$ and $w_2 = fw_2 = gw_2$.

So that for $w_1 \in A$ and $w_2 \in B$, we have

$$
\int_1^{d(w_1, w_2)} \psi(\lambda)\, d\lambda = \int_1^{d(f(w_1), g(w_2))} \psi(\lambda)\, d\lambda \le \int_1^{M^*(w_1, w_2)} \psi(\lambda)\, d\lambda,
$$

$$
\begin{aligned}
M^*(w_1, w_2) &= d^{c_1}(w_1, w_2) \cdot d^{c_2}(w_1, f(w_1)) \cdot d^{c_3}(w_2, g(w_2)) \\
&= d^{c_1}(w_1, w_2) \cdot 1 \cdot 1 = d^{c_1}(w_1, w_2)
\end{aligned}
$$

that is,

$$\int_1^{d(w_1,w_2)} \psi(\lambda)\, d\lambda \le \int_1^{d^{c_1}(w_1,w_2)} \psi(\lambda)\, d\lambda, \tag{31.7}$$

giving $d(w_1, w_2) = 1$ and hence $w_1 = w_2$. Thus $F(f, g)$ is singleton in $A \cap B$. □

In case $f = g$ in Theorem 31.3, the following fixed point result follows under restriction of integral type cyclic contractive mapping.

Corollary 31.3 *Let (X, d) be complete multiplicative metric space and A and B be non-empty closed sets in X. Suppose a mapping $f : A \cup B \to A \cup B$ satisfies that*

(a) *$f(A) \subseteq B$ and $g(B) \subseteq A$ with $X = A \cup B$.*
(b) *for any real constants c_1, c_2, c_3 with $c_1 + c_2 + c_3 \in [0, 1)$,*

$$\int_0^{d(f(\hat{x}),f(\hat{y}))} \psi(\lambda)\, d\lambda \le \int_0^{M^*(\hat{x},\hat{y})} \psi(\lambda)\, d\lambda \quad \text{for all } \hat{x} \in A \text{ and } \hat{y} \in B,$$

where $M^*(\hat{x}, \hat{y}) = d^{c_1}(\hat{x}, \hat{y}) \cdot d^{c_2}(\hat{x}, f(\hat{x})) \cdot d^{c_3}(\hat{y}, f(\hat{y}))$ and $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.

Then there exists at most one $\hat{u} \in X$ satisfies $\hat{u} = f\hat{u}$. Moreover, $\hat{u} \in A \cap B$.

31.3 Well-Posedness Results for Cyclic Contraction Mappings

In this section, well-posedness of common fixed point problems for cyclic contraction maps are obtained under the restriction of integral type contraction in multiplicative metric spaces.

Theorem 31.4 *Let (X, d) be complete multiplicative metric space and A and B be non-empty closed sets in X. Suppose that mappings $f, g : A \cup B \to A \cup B$ satisfy that*

(a) *$A \cup B$ has a cyclic representation w.r.t. the pair (f, g), i.e., $f(A) \subseteq B$, $g(B) \subseteq A$ and $X = A \cup B$.*
(b) *for any real number $c \in [0, 1)$,*

$$\int_1^{d(f(\hat{x}),g(\hat{y}))} \psi(\lambda)\, d\lambda \le \int_1^{M(\hat{x},\hat{y})} \psi(\lambda)\, d\lambda \quad \text{for all } \hat{x} \in A \text{ and } \hat{y} \in B,$$

where $M(\hat{x}, \hat{y}) = \max\{d^c(\hat{x}, \hat{y}), d^c(\hat{x}, f(\hat{x})), d^c(\hat{y}, g(\hat{y}))\}$ and $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.

Then the common fixed point problem of f and g is well-posed.

Proof From Theorem 31.2, it follows that f and g have unique common fixed point, say $\hat{u} \in A \cap B$. Let $\{x_n\}$ be sequence in $A \cap B$ having $\lim_{n\to\infty} d(fx_n, x_n) = 1$ and $\lim_{n\to\infty} (gx_n, x_n) = 1$. We assume that $\hat{u} \neq x_n$ for each non-negative number n. By taking $\hat{u} \in A$ and $x_n \in B$, we obtain

$$\begin{aligned}\int_1^{d(\hat{u},x_n)} \psi(\lambda)\,d\lambda &\leq \int_1^{d(f(\hat{u}),g(x_n))\cdot d(g(x_n),x_n))} \psi(\lambda)\,d\lambda \\ &\leq \int_1^{M(\hat{u},x_n)\cdot d(g(x_n),x_n))} \psi(\lambda)\,d\lambda,\end{aligned}$$

$$\begin{aligned}M(\hat{u}, x_n) &= \max\{d^c(\hat{u}, x_n), d^c(\hat{u}, f(\hat{u})), d^c(x_n, g(x_n))\} \\ &= \max\{d^c(\hat{u}, x_n), 1, d^c(x_n, g(x_n))\} \\ &= \max\{d^c(\hat{u}, x_n), d^c(x_n, g(x_n))\},\end{aligned}$$

that is,

$$\int_1^{d(\hat{u},x_n)} \psi(\lambda)\,d\lambda \leq \int_1^{\max\{d^c(\hat{u},x_n),d^c(x_n,g(x_n))\}\cdot d(g(x_n),x_n))} \psi(\lambda)\,d\lambda. \tag{31.8}$$

On limiting as $n \to \infty$ implies that

$$\begin{aligned}\int_1^{\lim_{n\to\infty} d(\hat{u},x_n)} \psi(\lambda)\,d\lambda &\leq \int_1^{\lim_{n\to\infty}[\max\{d^c(\hat{u},x_n),d^c(x_n,g(x_n))\}\cdot d(g(x_n),x_n))]} \psi(\lambda)\,d\lambda \\ &= \int_1^{\lim_{n\to\infty} d^c(\hat{u},x_n)} \psi(\lambda)\,d\lambda,\end{aligned}$$

where $c < 1$, implies $d(x_n, \hat{u}) \to 1$, that is $\lim_{n\to\infty} x_n = \hat{u}$. This completes the proof. □

Corollary 31.4 *Let (X, d) be complete multiplicative metric space and A and B be non-empty closed sets in X. Suppose that mappings $f, g : A \cup B \to A \cup B$ satisfy that*

(a) *$A \cup B$ has a cyclic representation w.r.t. the pair (f, g), i.e., $f(A) \subseteq B$, $g(B) \subseteq A$ and $X = A \cup B$.*
(b) *for any real number $c \in [0, 1)$,*

$$\int_1^{d(f(\hat{x}),g(\hat{y}))} \psi(\lambda)\,d\lambda \leq \int_1^{d^c(\hat{x},\hat{y})} \psi(\lambda)\,d\lambda \quad \textit{for all } \hat{x} \in A \textit{ and } \hat{y} \in B, \tag{31.9}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.

Then the common fixed point problem of f and g is well-posed.

Theorem 31.5 *Let (X, d) be complete multiplicative metric space and A and B be non-empty closed sets in X. Suppose that mappings $f, g : A \cup B \to A \cup B$ satisfy that*

(a) *$A \cup B$ has a cyclic representation w.r.t. the pair (f, g), i.e., $f(A) \subseteq B$, $g(B) \subseteq A$ and $X = A \cup B$.*
(b) *for any real constants c_1, c_2, c_3 with $c_1 + c_2 + c_3 \in [0, 1)$,*

$$\int_0^{d(f(\hat{x}),g(\hat{y}))} \psi(\lambda)\, d\lambda \le \int_0^{M^*(\hat{x},\hat{y})} \psi(\lambda)\, d\lambda \quad \text{for all } \hat{x} \in A \text{ and } \hat{y} \in B,$$

where $M^*(\hat{x}, \hat{y}) = d^{c_1}(\hat{x}, \hat{y}) \cdot d^{c_2}(\hat{x}, f(\hat{x})) \cdot d^{c_3}(\hat{y}, g(\hat{y}))\}$ and $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.

Then the common fixed point problem of f and g is well-posed.

Proof From Theorem 31.3, it follows that f and g have the unique common fixed point, say $\hat{u} \in A \cap B$ such that $\hat{u} = f\hat{u} = g\hat{u}$. Let $\{x_n\}$ be a sequence in $A \cap B$ having $\lim_{n\to\infty} d(fx_n, x_n) = 1$ and $\lim_{n\to\infty} (gx_n, x_n) = 1$. We may assume that $\hat{u} \neq x_n$ for each non-negative number n. By taking $\hat{u} \in A$ and $x_n \in B$,

$$\begin{aligned}\int_1^{d(\hat{u},x_n)} \psi(\lambda)\, d\lambda &\le \int_1^{d(f(\hat{u}),g(x_n))\cdot d(g(x_n),x_n))} \psi(\lambda)\, d\lambda \\ &\le \int_1^{M^*(\hat{u},x_n)\cdot d(g(x_n),x_n))} \psi(\lambda)\, d\lambda,\end{aligned}$$

$$\begin{aligned}M^*(\hat{u}, x_n) &= d^{c_1}(\hat{u}, x_n) \cdot d^{c_2}(\hat{u}, f(\hat{u})) \cdot d^{c_3}(x_n, g(x_n)) \\ &= d^{c_1}(\hat{u}, x_n) \cdot 1 \cdot d^{c_3}(x_n, g(x_n)) = d^{c_1}(\hat{u}, x_n) \cdot d^{c_3}(x_n, g(x_n)),\end{aligned}$$

that is,

$$\int_1^{d(\hat{u},x_n)} \psi(\lambda)\, d\lambda \le \int_1^{d^{c_1}(\hat{u},x_n)\cdot d^{c_3}(x_n,g(x_n))\cdot d(g(x_n),x_n))} \psi(\lambda)\, d\lambda. \tag{31.10}$$

On limiting as $n \to \infty$ gives

$$\int_1^{\lim_{n\to\infty} d(\hat{u},x_n)} \psi(\lambda)\, d\lambda \le \int_1^{\lim_{n\to\infty} [d^{c_1}(\hat{u},x_n)\cdot d^{c_3}(x_n,g(x_n))\cdot d(g(x_n),x_n))]} \psi(\lambda)\, d\lambda$$
$$= \int_1^{\lim_{n\to\infty} d^{c_1}(\hat{u},x_n)} \psi(\lambda)\, d\lambda,$$

where $c_1 < 1$, implies $d(x_n, \hat{u}) \to 1$ as $n \to \infty$, that is $\lim_{n\to\infty} x_n = \hat{u}$. This completes the proof. □

31.4 Periodic Points of Cyclic Contractions

In, this section, periodic points results are establish for cyclic mappings under contraction with restrictions of integral type.

Theorem 31.6 *Let (X, d) be complete multiplicative metric space and A and B be non-empty closed sets in X. If mappings $f, g : A \cup B \to A \cup B$ satisfy that*

(a) *$A \cup B$ has a cyclic representation w.r.t. the pair (f, g), i.e., $f(A) \subseteq B$, $g(B) \subseteq A$ and $X = A \cup B$.*
(b) *for any real constant $c \in [0, 1)$,*

$$\int_1^{d(f(\hat{x}),g(\hat{y}))} \psi(\lambda)\, d\lambda \le \int_1^{M(\hat{x},\hat{y})} \psi(\lambda)\, d\lambda \quad \text{for all } \hat{x} \in A \text{ and } \hat{y} \in B,$$

where $M(\hat{x}, \hat{y}) = \max\{d^c(\hat{x}, \hat{y}), d^c(\hat{x}, f(\hat{x})), d^c(\hat{y}, g(\hat{y}))\}$ and $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.

Then f and g has property Q.

Proof From Theorem 31.2, it follows that $F(f, g)$ is singleton in $A \cap B$. Also as trivially $F(f, g) \subseteq F(f^n, g^n)$ for all $n \in \mathbb{N}$. Now we consider $\hat{u} \in F(f^n, g^n)$. By taking $f^{n-1}\hat{u} \in A$ and $\hat{u} \in B$, we have

$$\int_1^{d(\hat{u},g\hat{u})} \psi(\lambda)\, d\lambda \le \int_1^{d(f(f^{n-1}\hat{u}),g(\hat{u}))} \psi(\lambda)\, d\lambda \le \int_1^{M(f^{n-1}\hat{u},\hat{u})} \psi(\lambda)\, d\lambda,$$

$$\begin{aligned} M(f^{n-1}\hat{u}, \hat{u}) &= \max\{d^c(f^{n-1}\hat{u}, \hat{u}), d^c(f^{n-1}\hat{u}, f(f^{jn-1}\hat{u})), d^c(\hat{u}, g(\hat{u}))\} \\ &= \max\{d^c(f^{n-1}\hat{u}, \hat{u}), d^c(f^{n-1}\hat{u}, \hat{u}), d^c(\hat{u}, g(\hat{u}))\} \\ &= \max\{d^c(f^{n-1}\hat{u}, \hat{u}), d^c(\hat{u}, g(\hat{u}))\}, \end{aligned}$$

that is,

$$\begin{aligned}\int_1^{d(\hat{u},g\hat{u})}\psi(\lambda)\,d\lambda &\le \int_1^{\max\{d^c(f^{n-1}\hat{u},\hat{u}),d^c(\hat{u},g(\hat{u}))\}}\psi(\lambda)\,d\lambda\\ &\le \int_1^{d^c(f^{n-1}\hat{u},\hat{u})}\psi(\lambda)\,d\lambda\end{aligned}$$

and hence

$$\begin{aligned}\int_1^{d(\hat{u},g\hat{u})}\psi(\lambda)\,d\lambda &\le \int_1^{d^c(f^{n-1}\hat{u},\hat{u})}\psi(\lambda)\,d\lambda\\ &\le \int_1^{d^{c^2}(f^{n-2}\hat{u},\hat{u})}\psi(\lambda)\,d\lambda \le \cdots \le \int_1^{d^{c^n}(f\hat{u},\hat{u})}\psi(\lambda)\,d\lambda.\end{aligned}$$

In a similar way, by taking $\hat{u} \in A$ and $g^{n-1}\hat{u} \in B$, we have

$$\begin{aligned}\int_1^{d^{c^n}(f\hat{u},\hat{u})}\psi(\lambda)\,d\lambda &\le \int_1^{d^{c^n+1}(\hat{u},g^{n-1}\hat{u})}\psi(\lambda)\,d\lambda\\ &\le \int_1^{d^{c^n+2}(\hat{u},g^{n-2}\hat{u})}\psi(\lambda)\,d\lambda \le \cdots \le \int_1^{d^{c^{2n}}(\hat{u},g\hat{u})}\psi(\lambda)\,d\lambda.\end{aligned}$$

From (31.3) and (31.6), it follows that

$$\int_1^{d(\hat{u},g\hat{u})}\psi(\lambda)\,d\lambda \le \int_1^{d^{c^{2n}}(\hat{u},g\hat{u})}\psi(\lambda)\,d\lambda.$$

As $c^{2n} < 1$, so $d(\hat{u}, g\hat{u}) = 1$ that is, $\hat{u} = g\hat{u}$. Thus, $\hat{u} = f\hat{u} = g\hat{u}$ and hence $\hat{u} \in F(f, g)$, that is, $F(f^n, g^n) \subseteq F(f, g)$ and we conclude that $F(f^n, g^n) = F(f, g)$. □

Corollary 31.5 *Let (X, d) be complete multiplicative metric space and A and B be non-empty closed sets in X. If mappings $f, g : A \cup B \to A \cup B$ satisfy that*

(a) *$A \cup B$ has a cyclic representation w.r.t. the pair (f, g), i.e., $f(A) \subseteq B$, $g(B) \subseteq A$ and $X = A \cup B$.*
(b) *for any real number $c \in [0, 1)$,*

$$\int_1^{d(f(\hat{x}),g(\hat{y}))}\psi(\lambda)\,d\lambda \le \int_1^{d^c(\hat{x},\hat{y})}\psi(\lambda)\,d\lambda \quad \textit{for all } \hat{x} \in A \textit{ and } \hat{y} \in B,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon}\psi(\lambda)\,d(\lambda) > 0$.

Then f and g has property Q.

If we take $f = g$ in the Theorem 31.6, the conclude the following Corollary.

Corollary 31.6 *Let (X, d) be complete multiplicative metric space and A and B be non-empty closed sets in X. Suppose that mapping $f : A \cup B \to A \cup B$ satisfies that*

(a) $f(A) \subseteq B$, $g(B) \subseteq A$ *and* $X = A \cup B$.
(b) *for any real number* $c \in [0, 1)$,

$$\int_1^{d(f(\hat{x}), f(\hat{y}))} \psi(\lambda)\, d\lambda \leq \int_1^{M'(\hat{x}, \hat{y})} \psi(\lambda)\, d\lambda \quad \text{for all } \hat{x} \in A \text{ and } \hat{y} \in B,$$

where $M'(\hat{x}, \hat{y}) = \max\{d^c(\hat{x}, \hat{y}), d^c(\hat{x}, f(\hat{x})), d^c(\hat{y}, f(\hat{y}))\}$ *and* $\psi : [0, \infty) \to [0, \infty)$ *is a Lebesgue integrable function with finite integral on each compact set in* $[0, \infty)$, *and for each* $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.

Then f has property P.

Conclusion. We obtained the results for existence of common fixed points of cyclic contraction mappings that are satisfying the generalized integral contractive conditions in the structured of multiplicative metric spaces. We also presented the well-posedness of fixed and common fixed point results. The results related to the periodic point property of generalized integral contractive maps are also obtained in the setup of multiplicative metric spaces.

Acknowledgements Talat Nazir is grateful for support of EU Erasmus Mundus project FUSION and the research environment Mathematics and Applied Mathematics MAM, Division of Applied Mathematics, Mälardalen University for providing an excellent and inspiring environment during his visits.

References

1. Abbas, M., Ali, B., Suleiman, Y.I.: Common fixed points of locally contractive mappings in multiplicative metric spaces with application. Inter. J. Math. Math. Sci., Article ID 218683, 7 pp. (2015)
2. Abbas, M., Nazir, T., Romaguera, S.: Fixed point results for generalized cyclic contraction mappings in partial metric spaces. Rev. R. Acad. Sci. Exactas Fis. Nat. Ser. A Math. **106**, 287–297 (2012)
3. Abbas, M., Fisher, B., T, Nazir: Well-posedness and periodic point property of mappings satisfying a rational inequality in an ordered complex valued metric space. Sci. Stud. Res. Ser. Math. Inf. **22**(1), 5–24 (2012)
4. Abbas, M., Rhoades, B.E.: Fixed and periodic point results in cone metric spaces. Appl. Math. Lett. **22**, 511–515 (2009)
5. Agarwal, R.P., Alghamdi, M.A., Shahzad, N.: Fixed point theory for cyclic generalized contractions in partial metric spaces. Fixed Point Theory Appl. **2012**, 40, 11 pp. (2012)
6. Altun, I., Abbas, M., Simsek, H.: A fixed point theorem on cone metric spaces with new type contractivity. Banach J. Math. Anal. **5**(2), 15–24 (2011)
7. Altun, I., Türkoglu, D., Rhoades, B.E.: Fixed points of weakly compatible maps satisfying a general contractive of integral type. Fixed Point Theory Appl. Article ID **17301** (2007)
8. Alsulami, H.H., Karapınar, E., O'Regan, D., Shahi, P.: Fixed points of generalized contractive mappings of integral type. Fixed Point Theory Appl. **2014**, Article ID 213 (2014)

9. Akkouchi, M., Popa, V.: Well-posedness of fixed point problem for three mappings under strict contractive conditions. Bull. Math. Inf. Phys. Petroleum-Gash Univ. Ploiesti **61**(2), 1–10 (2009)
10. Aydi, H.,Vetro, C., Sintunavarat, W., Kumam, P.: Coincidence and fixed points for contractions and cyclical contractions in partial metric spaces. Fixed Point Theory Appl. **2012**, 124, 18 pp. (2012)
11. Banach, S.: Sur les operations dans les ensembles abstraits et leur application aux equations integrales. Fund. Math. **3**, 133–181 (1922)
12. Branciari, A.: A Fixed point theorem for mappings satisfying a general contractive condition of integral type. Int. J. Math. Sci. **29**(9), 531–536 (2002)
13. Bashirov, A.E., Kurpnar, E.M., Ozyapc, A.: Multiplicative calculus and its applications. J. Math. Anal. Appl. **337**, 36–48 (2008)
14. Bashirov, A.E., Misirli, E., Tandogdu, Y., Ozyapici, A.: On modeling with multiplicative differential equations. Appl. Math. J. Chin. Univ. **26**(4), 425–438 (2011)
15. Chaipunya, P., Cho, Y.J., Sintunavarat, W., Kumam, P.: Fixed point and common fixed point Theorems for cyclic quasi-contractions in metric and ultrametric spaces. Adv. Pure Math. **2**, 401–407 (2012)
16. Chaipunya, P., Cho, Y.J., Kumam, P.: A remark on the property ρ and periodic points of order ∞. Math. Vesnik **66**(4), 357–363 (2014)
17. Derafshpour, M., Rezapour, S.: On the existence of best proximity points of cyclic contractions. Adv. Dyn. Sys. Appl. **6**(1), 33–40 (2011)
18. Djoudi, A., Aliouche, A.: Common fixed point theorems of Greguš type for weakly compatible mappings satisfying contractive conditions of integral type. J. Math. Anal. Appl. **329**, 31–45 (2007)
19. Djoudi, A., Merghadi, F.: Common fixed point theorems for maps under a contractive condition of integral type. J. Math. Anal. Appl. **341**, 953–960 (2008)
20. Florack, L., Assen, H.V.: Multiplicative calculus in biomedical image analysis. J. Math. Imag. Vis. **42**(1), 64–75 (2012)
21. Feckan, M.: Nonnegative solutions of nonlinear integral equations. Comment. Math. Univ. Carol. **36**, 615–627 (1995)
22. Gornicki, J., Rhoades, B.E.: A general fixed point theorem for involutions. Indian J. Pure Appl. Math. **27**, 13–23 (1996)
23. He, X., Song, M., Chen, D.: Common fixed points for weak commutative mappings on a multiplicative metric space. Fixed Point Theory Appl. **2014**, 48, 9 pp. (2014)
24. Jachymski, J.: Remarks on contractive conditions of integral type. Nonlinear Anal. **71**, 1073–1081 (2009)
25. Jeong, G.S., Rhoades, B.E.: Maps for which $F(T) = F(T^n)$. Fixed Point Theory **6**, 87–131 (2005)
26. Karapinar, E.: Fixed point theory for cyclic weak ϕ-contraction. Appl. Math. Lett. **24**, 822–825 (2011)
27. Kumam, P., Rahimi, H., Rad, G.S.: The existence of fixed and periodic point Theorems in cone metric type spaces. J. Nonlinear Sci. Appl. **7**, 255–263 (2014)
28. Kirk, W.A., Srinivasan, P.S., Veeramini, P.: Fixed points for mappings satisfying cyclical contractive conditions. Fixed Point Theory **4**(1), 79–89 (2003)
29. Liu, Z., Li, X., Kang, S.M., Cho, S.Y.: Fixed point theorems for mappings satisfying contractive conditions of integral type and applications. Fixed Point Theory Appl. **2001**, 64 (2011)
30. Liu, Z., Zou, X., Kang, S.M., Ume, J.S.: Common fixed points for a pair of mappings satisfying contractive conditions of integral type. J. Inequal. Appl. **2014**, 394 (2014)
31. Murthy, P.P., Kumar, S., Tas, K.: Common fixed points of self maps satisfying an integral type contractive condition in fuzzy metric spaces. Math. Commun. **15**, 521–537 (2010)
32. Nashine, H.K., Sintunavarat, W., Kumam, P.: Cyclic generalized contractions and fixed point results with applications to an integral equation. Fixed Point Theory Appl. **2012**, 217, 13 pp. (2012)
33. Nashine, H.K., Samet, B., Vetro, C.: Fixed point theorems in partially ordered metric spaces and existence results for integral equations. Numer. Funct. Anal. Optim. **33**(11), 1304–1320 (2012)

34. Ozavsar, M., Cevikel, A.C.: Fixed point of multiplicative contraction mappings on multiplicative metric space (2012). arXiv:1205.5131v1 [matn.GN]
35. Păcurar, M., Rus, I.A.: Fixed point theory for cyclic ϕ-contractions. Nonlinear Anal. **72**(3–4), 1181–1187 (2010)
36. Piatek, B.: On cyclic Meir-Keeler contractions in metric spaces. Nonlinear Anal. **74**, 35–40 (2011)
37. Rahimia, H., Rhoades, B.E., Radenović, S., Rad, G.S.: Fixed and periodic point Theorems for T-contractions on cone metric spaces. Filomat **27**(5), 881–888 (2013)
38. Reich, S. Zaslawski, A.J.: Well posedness of fixed point problems. Far East J. Math. Sci. Special Volume Part 3, 393–401 (2001)
39. Rhoades, B.E., Abbas, M.: Maps satisfying a contractive condition of integral type for which fixed point and periodic point coincidence. Int. J. Pure Appl. Math. **45**(2), 225–231 (2008)
40. Rus, I.A.: Cyclic Representation and fixed points. In: Annals of the Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity, vol. 3, 171–178 (2005)
41. Sintunavarat, W., Kumam, P.: Gregus-type common fixed point theorems for tangential multivalued mappings of integral type in metric spaces. Int. J. Math. Math. Sci. **2011**, Article ID 923458 (2011)
42. Singh, K.L.: Sequences of iterates of generalized contractions. Fund. Math. **105**, 115–126 (1980)
43. Yamaod, O., Sintunavarat, W.: Some fixed point results for generalized contraction mappings with cyclic (α, β)-admissible mapping in multiplicative metric spaces. J. Inequal. Appl. **2014**, 488, 15 pp. (2014)