Chapter 30
Common Fixed Point for Integral Type Contractive Mappings in Multiplicative Metric Spaces

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Abstract The fixed and common fixed point problems of selfmappings that are satisfying certain type generalized integral contractions in the setup of multiplicative metric spaces are investigated. Well-posedness results for fixed point problem of maps under restrictions of integral type contractions are obtained. Moreover, the periodic points results of generalized integral type contraction mappings are also obtained.

Keywords Common fixed point · Periodic point · Well-posedness · Contraction mappings · Multiplicative metric space

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30.1 Introduction. Fixed and Common Fixed Points of Mappings

The study of common fixed point of maps with certain type of contractive restrictions is a powerful approach towards solving variety of scientific problems in various areas of Mathematics as well as important methodology for computational algorithms in Physics and other Natural sciences and Engineering subjects.

Banach contraction principle [8] is the simple and powerful theorem having variety of applicability and usage for the solutions of integral, differential, linear, non-linear, difference and homogenous equations. It was initially studied in the fixed point theory dealing with certain types of contractive mappings. In the current

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literature, various extensions and generalization of Banach contraction principle are established in [1, 4, 7, 11–13, 18, 22, 28, 29, 31, 33, 35]. Banach contraction principle was stated in [8] as follows.

Theorem 30.1 ([8]) *Let (X, d) be a complete metric space. If their is the constant $\alpha \in [0, 1)$ such that mapping $f : X \to X$ is satisfying*

$$d(fx_1, fx_2) \le \alpha\, d(x_1, x_2) \quad for \ \ all \ \ x_1, x_2 \in X. \tag{30.1}$$

Then there exists a unique $\hat{u} \in X$ satisfies $\hat{u} = f\hat{u}$. Moreover, for any initial value $x_0 \in X$, the iterative sequence $x_{n+1} = f(x_n)$ converges to $\hat{u}$.

Branciari [9] extended the Banach contraction principle and proved results of fixed points of maps in the complete metric space (X, d) satisfying general contraction based inequality of integral type. Branciari contraction principle with restrictions of integral type contractions states the following.

Theorem 30.2 ([9]) *Let (X, d) be a complete metric space. If for a $\alpha \in [0, 1)$, the mapping $f : X \to X$ satisfies*

$$\int_0^{d(fx_1, fx_2)} \psi(\lambda)\, d\lambda \le \alpha \int_0^{d(x_1,x_2)} \psi(\lambda)\, d\lambda, \quad for \ \ all \ \ x_1, x_2 \in X, \tag{30.2}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, nonnegative and for each $\varepsilon > 0$, $\int_0^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$; then a unique $\hat{u} \in X$ exists for $\hat{u} = f\hat{u}$. Moreover, for any initial value $x_0 \in X$, the iterative sequence $x_{n+1} = f(x_n)$ converges to $\hat{u}$.

Various other useful results related to the fixed point of mappings with restriction of integral type were proved in [5, 6, 17, 20, 23, 26]. In 2014, Liu et al. [24] provided the common fixed point results of certain maps that having contractive condition with restrictions of integral type. Common fixed point result for two maps having restriction of integral type defined as follows.

Theorem 30.3 ([8]) *Let (X, d) be a complete metric space. If for a constant $\alpha \in [0, 1)$, the mappings $f, g : X \to X$ satisfies*

$$\int_0^{d(fx_1, gx_2)} \psi(\lambda)\, d\lambda \le \alpha \int_0^{d(x_1,x_2)} \psi(\lambda)\, d\lambda, \quad for \ \ all \ \ x_1, x_2 \in X, \tag{30.3}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, nonnegative and each for $\varepsilon > 0$, $\int_0^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$; then a unique $\hat{u} \in X$ exists for $\hat{u} = f\hat{u} = g\hat{u}$. Moreover, for any initial value $x_0 \in X$, the iterative sequence defined as $x_{2n+1} = f(x_{2n})$ and $x_{2n+2} = g(x_{2n+1})$ converges to $\hat{u}$.

Other useful results regrading the common fixed point theorems satisfying contractive condition restrictions of integral type were established by [14, 15, 25, 34].

The present work is devoted to common fixed point problems for contractive mapping with restrictions of integral type in multiplicative metric spaces. In Sect. 30.2, some important basic notions and notations concerned with sequences and functions on multiplicative metric spaces are presented. In Sect. 30.3, common fixed points of mappings in multiplicative metric spaces are considered. In particular, the common fixed point result for mappings with restrictions of integral type in the setup of multiplicative metric space is described. In Sect. 30.4, several common fixed point results for self-mappings with restriction of Integral type conditions are obtained. Section 30.5 is devoted to well-posedness results for common fixed points. Well-posedness of common fixed point problems for multiplicative metric spaces is defined. Well-posedness results for fixed point problem of maps under restrictions of integral type contractions are obtained. In Sect. 30.6, periodic point results related to the mappings that are satisfying generalized integral type contraction conditions are established.

30.2 Multiplicative Metric Space

Ozavsar and Cevikel [27] established a famous Banach contraction-principle in another setup known as the multiplicative metric space. They obtained the structure of topology and properties of multiplicative metric space. The multiplicative calculus concepts along with some fundamental results dealing with multiplicative calculus were created by Bashirov et al. [10]. Various useful applications of multiplicative calculus were also established. They also established the value of multiplicative differential equations over the ordinary differential equations for solving variety of problems related to optimizations, economics as well as finance. Moreover, they defined the multiplicative absolute value function and used it for the multiplicative distance for non-negative real numbers as well as for the positive square matrices. The applicability of multiplicative calculus based problems in the field of biomedical-image analysis were shown by Florack and Assen [16]. Also, the common fixed point based theorems of maps under weak commutative conditions were proved by He et al. [19]. Recently, Yamaod and Sintunavarat [36] provided fixed point theorems of cyclic contractive based maps under (α, β)-admissible restriction in a multiplicative metric space structure.

We start with the definition of multiplicative metric space [27].

Definition 30.1 Let X be a non empty set. A multiplicative metric on X is a mapping $d : X \times X \to \mathbb{R}^+$ that satisfy that for any $x_1, x_2, x_3 \in X$,

(a) $d(x_1, x_2) \geq 1$ and $d(x_1, x_2) = 1$ if and only if $x = y$;
(b) $d(x_1, x_2) = d(x_2, x_1)$;
(c) $d(x_1, x_3) \leq d(x_1, x_2) \cdot d(x_2, x_3)$.

The pair (X, d) is called a multiplicative metric space.

The absolute valued function is defined for real numbers as follows.

Definition 30.2 The multiplicative absolute-value function $|\cdot| : \mathbb{R} \to \mathbb{R}^+$ is given by

$$|\alpha| = \begin{cases} \alpha & if\ \alpha \geq 1, \\ \frac{1}{\alpha} & if\ \alpha \in (0,1), \\ 1 & if\ \alpha = 0, \\ -\frac{1}{\alpha} & if\ \alpha \in (-1,0), \\ -\alpha & if\ \alpha \leq -1. \end{cases}$$

For the multiplicative absolute-value function, the following Proposition holds.

Proposition 30.1 *Multiplicative absolute value function, for all $u, v \in \mathbb{R}^+$, satisfies*

1. $|u| \geq 1$;
2. $u \leq |u|$;
3. $u \leq \frac{1}{|u|}$ *if* $u \leq 0$ *and* $\frac{1}{|u|} \leq u$ *if* $x > 0$;
4. $|u \cdot v| \leq |u|\,|v|$.

Example 30.1 ([27]) Let $X = \mathbb{R}^n_+$. Then

$$d_1(\mu, \upsilon) = \left|\frac{\mu_1}{\upsilon_1}\right| \cdot \left|\frac{\mu_2}{\upsilon_2}\right| \cdot \ldots \cdot \left|\frac{\mu_n}{\upsilon_n}\right| \text{ and } d_2(\mu, \upsilon) = \max\left\{\left|\frac{\mu_1}{\upsilon_1}\right|, \left|\frac{\mu_2}{\upsilon_2}\right|, \ldots, \left|\frac{\mu_n}{\upsilon_n}\right|\right\},$$
$$\mu = (\mu_1, \mu_2, \ldots, \mu_n), \upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n) \in \mathbb{R}^n_+,$$

define multiplicative metrics on X.

Example 30.2 For a fixed real number $\alpha > 1$, $d_\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d_\alpha(\mu, \upsilon) := \alpha^{\sum\limits_{i=1}^{n} |\mu_i - \upsilon_i|}, \text{ for } \mu = (\mu_1, \mu_2, \ldots, \mu_n), \upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n) \in \mathbb{R}^n, \quad (30.4)$$

multiplicative metric conditions hold.

Definition 30.3 ([27]) Let (X, d) be multiplicative metric space. For any $z_0 \in X$ with $\varepsilon > 1$, we define multiplicative open ball $B(z_0, \varepsilon)$ by $\{z \in X : d(z, z_0) < \varepsilon\}$. Also one can describe multiplicative closed ball by $\{z \in X : d(z, z_0) \leq \varepsilon\}$.

Definition 30.4 A sequence $\{x_n\}$ in a multiplicative metric space (X, d) is said to be multiplicative convergent to some point $x^* \in X$ if for any given $\varepsilon > 1$, an element k in $\mathbb{N}$ exists such that $x_n \in B(x^*, \varepsilon)$ for all $n \geq k$. If $\{x_n\}$ converges to x, we write $x_n \to x$ as $n \to \infty$.

Definition 30.5 ([27]) A sequence $\{x_n\}$ in a multiplicative metric space (X, d) is multiplicative convergent to x in X if and only if $d(x_n, x) \to 1$ as $n \to \infty$.

Definition 30.6 Let (X, d_X) and (Y, d_Y) be two multiplicative metric spaces, and x_0 an arbitrary but fixed element of X. A mapping $f : X \to Y$ is said to be multiplicative continuous at x_0 if and only if $x_n \to x_0$ in (X, d_X) implies that $f(x_n) \to f(x_0)$ in (Y, d_Y). That is, given arbitrary $\varepsilon > 1$, there exists $\delta > 1$ which depend on x_0 and ε such that $d_Y(fx_*, fx_0) < \varepsilon$ for all x_* in X having $d_X(x_*, x_0) < \delta$.

Definition 30.7 ([27]) A sequence $\{x_n\}$ in multiplicative metric space (X, d) is known as the multiplicative Cauchy sequence whenever for any $\varepsilon > 1$, there exists $n_0 \in \mathbb{N}$ such that $d(x_{\grave{m}}, x_n) < \varepsilon$ for all $n, \grave{m} \geq n_0$. Also (X, d) is known as a complete Multiplicative metric space whenever every multiplicative Cauchy sequence $\{x_n\}$ contained in X is the multiplicative convergent in X.

Theorem 30.4 ([27]) *A sequence $\{x_n\}$ in multiplicative metric space (X, d) is multiplicative Cauchy sequence whenever $d(x_k, x_n) \to 1$ as $k, n \to \infty$.*

Example 30.3 Let $C^*[\alpha, \beta]$ be a set of multiplicative continuous functions on $[\alpha, \beta]$ with values in $\mathbb{R}^+$, and the multiplicative metric d be defined as

$$d(f_1, f_2) = \sup_{c \in [\alpha, \beta]} \left| \frac{f_1(c)}{f_2(c)} \right| \text{ for } f_1, f_2 \in C^*[\alpha, \beta],$$

where $|\cdot|$ is the multiplicative absolute value function on $\mathbb{R}^+$. Then $(C^*[\alpha, \beta], d)$ is complete.

30.3 Common Fixed Points of Mappings in Multiplicative Metric Space

Let X be a non-empty set and $f : X \to X$ be any map. If a point p in X satisfies $f(p) = p$, then we call it a fixed point of f. We denote the collection of all fixed points of self-map $f : X \to X$ by $F(f)$. Also if a point p in X satisfies $f(p) = g(p) = p$, then we call it a common fixed point of f and g. We denote the collection of all common fixed points of self-map $f, g : X \to X$ by $F(f, g)$.

Bashirov et al. [10] defined multiplicative Banach contraction mappings and established the fixed point result in the multiplicative metric space setup.

Definition 30.8 Let (X, d) be multiplicative metric space. We say that $f : X \to X$ is multiplicative Banach contraction if for any real number $c^* \in [0, 1)$,

$$d(f(\hat{x}), f(\hat{y})) \leq d(\hat{x}, \hat{y})^{c^*}$$

is satisfied for all $\hat{x}, \hat{y} \in X$.

Theorem 30.5 ([10]) *Let (X, d) be multiplicative metric space and $f : X \to X$. Then $F(f) \neq \emptyset$ and singleton, provided that f is multiplicative Banach contraction on X.*

We will consider next the common fixed point result of mappings with restrictions of integral type in the setup of multiplicative metric space.

Definition 30.9 Let (X, d) be multiplicative metric space. We say that two mappings $f, g : X \to X$ are multiplicative Banach contraction with restrictions of integral type if for any real constant $c \in [0, 1)$,

$$\int_1^{d(f(\hat{x}),g(\hat{y}))} \psi(\lambda)\, d\lambda \leq \int_1^{d(\hat{x},\hat{y})^c} \psi(\lambda)\, d\lambda \tag{30.5}$$

is satisfied for all $\hat{x}, \hat{y} \in X$, where $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.

Definition 30.10 Let (X, d) be multiplicative metric space. We say that the mappings $f, g : X \to X$ are multiplicative generalized Banach contraction with restrictions of integral type if for any real constant $c \in [0, 1)$,

$$\int_1^{d(f(\hat{x}),g(\hat{y}))} \psi(\lambda)\, d\lambda \leq \int_1^{M(\hat{x},\hat{y})} \psi(\lambda)\, d\lambda \tag{30.6}$$

is satisfied for all $\hat{x}, \hat{y} \in X$, where $M(\hat{x}, \hat{y}) = \max\{d^c(\hat{x}, \hat{y}), d^c(\hat{x}, f(\hat{x})), d^c(\hat{y}, g(\hat{y}))\}$, and $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.

Now, we present a common fixed point result for mappings with restrictions of integral type in the setup of multiplicative metric space.

Theorem 30.6 *Let (X, d) be multiplicative metric space and $f, g : X \to X$. If f and g are multiplicative (generalized) Banach contraction with restrictions of integral type, then f and g have a unique common fixed point, i.e. $F(f, g) \neq \emptyset$ and singleton, provided that f is multiplicative Banach contraction on X.*

30.4 Common Fixed Point Results for Integral Type Contractions

In this section, several common fixed point results for self-mappings with restriction of Integral type conditions are obtained. Our first result is as follows.

Theorem 30.7 *Let (X, d) be complete multiplicative metric space. If the mappings $f, g : X \to X$ are multiplicative generalized Banach contraction with restrictions of integral type, that is, for any real constant $c \in [0, 1)$,*

$$\int_1^{d(f(\hat{x}),g(\hat{y}))} \psi(\lambda)\, d\lambda \leq \int_1^{M(\hat{x},\hat{y})} \psi(\lambda)\, d\lambda$$

is satisfied for all $\hat{x}, \hat{y} \in X$, where $M(\hat{x}, \hat{y}) = \max\{d^c(\hat{x}, \hat{y}), d^c(\hat{x}, f(\hat{x})), d^c(\hat{y}, g(\hat{y}))\}$ and $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on *each compact set in* $[0, \infty)$*, and for each* $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$*, then there exists a unique* $\hat{u} \in X$ *satisfying* $\hat{u} = f\hat{u} = g\hat{u}$*. Moreover, any fixed point of* f *is the fixed point of* g *and conversely.*

Proof First we are to show that any fixed point of f is the fixed point of g. Suppose that $\hat{u} \in X$ satisfies $\hat{u} = f\hat{u}$. Since f and g are multiplicative generalized Banach contraction pair, so that we have

$$\int_1^{d(\hat{u}, g(\hat{u}))} \psi(\lambda)\, d\lambda = \int_1^{d(f(\hat{u}), g(\hat{u}))} \psi(\lambda)\, d\lambda \leq \int_1^{M(\hat{u}, \hat{u})} \psi(\lambda)\, d\lambda,$$

where $M(\hat{u}, \hat{u}) = \max\{d^c(\hat{u}, \hat{u}), d^c(\hat{u}, f\left(\hat{u}\right)), d^c(\hat{u}, g\left(\hat{u}\right))\} = d^c\left(\hat{u}, g\left(\hat{u}\right)\right)$, that is,

$$\int_1^{d(\hat{u}, g(\hat{u}))} \psi(\lambda)\, d\lambda \leq \int_1^{d^c(\hat{u}, g(\hat{u}))} \psi(\lambda)\, d\lambda. \tag{30.7}$$

As $c < 1$, it follows that $d(\hat{u}, g(\hat{u})) = 1$, and hence $\hat{u} = g\left(\hat{u}\right)$. Thus $\hat{u} = f\left(\hat{u}\right) = g\left(\hat{u}\right)$. Similarly, $\hat{u} = g\left(\hat{u}\right)$ implies that $\hat{u} = f\left(\hat{u}\right) = g\left(\hat{u}\right)$.

Suppose x_0 is the arbitrary point of X. Define a sequence $\{x_n\}$ in X with $fx_{2n} = x_{2n+1}$ and $gx_{2n+1} = x_{2n+2}$ for all $n = 0, 1, 2, \ldots$. As the mappings f and g are multiplicative generalized Banach contraction pair, so that

$$\int_1^{d(x_{2n+1}, x_{2n+2})} \psi(\lambda)\, d\lambda = \int_1^{d(f(x_{2n}), g(x_{2n+1}))} \psi(\lambda)\, d\lambda \leq \int_1^{M(x_{2n}, x_{2n+1})} \psi(\lambda)\, d\lambda,$$

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\{d^c(x_{2n}, x_{2n+1}), d^c(x_{2n}, f(x_{2n})), d^c(x_{2n+1}, g(x_{2n+1}))\} \\ &= \max\{d^c(x_{2n}, x_{2n+1}), d^c(x_{2n}, x_{2n+1}), d^c(x_{2n+1}, x_{2n+2})\} \\ &= \max\{d^c(x_{2n}, x_{2n+1}), d^c(x_{2n+1}, x_{2n+2})\}, \end{aligned}$$

that is,

$$\int_1^{d(x_{2n+1}, x_{2n+2})} \psi(\lambda)\, d\lambda \leq \int_1^{\max\{d^c(x_{2n}, x_{2n+1}), d^c(x_{2n+1}, x_{2n+2})\}} \psi(\lambda)\, d\lambda. \tag{30.8}$$

Now, we have two possibilities. If

$$\max\{d^c(x_{2n}, x_{2n+1}), d^c(x_{2n+1}, x_{2n+2})\} = d^c(x_{2n+1}, x_{2n+2}),$$

then $\int_1^{d(x_{2n+1}, x_{2n+2})} \psi(\lambda)\, d\lambda \leq \int_1^{d^c(x_{2n+1}, x_{2n+2})} \psi(\lambda)\, d\lambda$, and we have

$$d(x_{2n+1}, x_{2n+2}) \leq d^c(x_{2n+1}, x_{2n+2}).$$

Since $c < 1$, so $d(x_{2n+1}, x_{2n+2}) = 1$ and hence $x_{2n+1} = x_{2n+2}$. As $gx_{2n+1} = x_{2n+2}$, so $gx_{2n+1} = x_{2n+1}$, that is, x_{2n+1} is the fixed point of g. Thus by above conclusion, x_{2n+1} is the common fixed point of f and g.

If $\max\{d^c(x_{2n}, x_{2n+1}), d^c(x_{2n+1}, x_{2n+2})\} = d^c(x_{2n}, x_{2n+1})$, then

$$\int_1^{d(x_{2n+1},x_{2n+2})} \psi(\lambda)\, d\lambda \leq \int_1^{d^c(x_{2n},x_{2n+1})} \psi(\lambda)\, d\lambda.$$

Similarly, $\int_1^{d(x_{2n+2},x_{2n+3})} \psi(\lambda)\, d\lambda \leq \int_1^{d^c(x_{2n+1},x_{2n+2})} \psi(\lambda)\, d\lambda$. Thus for all $n \geq 0$,

$$\int_1^{d(x_n,x_{n+1})} \psi(\lambda)\, d\lambda \leq \int_1^{d^c(x_{n-1},x_n)} \psi(\lambda)\, d\lambda. \tag{30.9}$$

Continuing this way, we obtain

$$\begin{aligned}\int_1^{d(x_n,x_{n+1})} \psi(\lambda)\, d\lambda &\leq \int_1^{d^c(x_{n-1},x_n)} \psi(\lambda)\, d\lambda \leq \int_1^{d^{c^2}(x_{n-2},x_{n-1})} \psi(\lambda)\, d\lambda \\ &\leq \cdots \leq \int_1^{d^{c^n}(x_0,x_1)} \psi(\lambda)\, d\lambda.\end{aligned}$$

As $0 \leq c < 1$, we further have

$$\int_1^{d(x_n,x_{n+1})} \psi(\lambda)\, d\lambda \to 0 \ \ as \ \ n \to \infty \tag{30.10}$$

and so $d(x_n, x_{n+1}) \to 1$ as $n \to \infty$.

For $n, \grave{m} \in \mathbb{N}$ with $m \geq n$,

$$\begin{aligned} d(x_n, x_{\grave{m}}) &\leq d(x_n, x_{n+1}) \cdot d(x_{n+1}, x_{n+2}) \cdots d(x_{\grave{m}-1}, x_{\grave{m}}) \\ &\leq d^{c^n}(x_0, x_1) \cdot d^{c^{n+1}}(x_0, x_1) \cdots d^{c^{\grave{m}-1}}(x_0, x_1) \\ &= (d(x_0, x_1))^{c^n + c^{n+1} + \cdots + c^{\grave{m}-1}} \\ &\leq (d(x_0, x_1))^{c^n(1+c+c^2+\dots)} \\ &\leq (d(x_0, x_1))^{\frac{c^n}{1-c}} \to 1 \ \ as \ \ n, \grave{m} \to \infty. \end{aligned}$$

Thus

$$\int_1^{d(x_n,x_{\grave{m}})} \psi(\lambda)\, d\lambda \to 0 \ \ as \ \ \grave{m}, n \to \infty \tag{30.11}$$

and $\{x_n\}$ becomes a Cauchy sequence in complete multiplicative space (X, d). So we obtain $z \in X$ for which $x_n \to z$ as $n \to \infty$, or $\lim_{n\to\infty} d(x_n, z) = 1$.

Now let us show that $fz = z$, that is, $d(fz, z) = 1$. For this, it follows that

$$\int_1^{d(fz, x_{2n+2})} \psi(\lambda)\, d\lambda = \int_1^{d(fz, gx_{2n+1})} \psi(\lambda)\, d\lambda \leq \int_1^{M(z, x_{2n+1})} \psi(\lambda)\, d\lambda, \quad (30.12)$$

$$\begin{aligned} M(z, x_{2n+1}) &= \max\{d^c(z, x_{2n+1}), d^c(z, f(z)), d^c(x_{2n+1}, g(x_{2n+1}))\} \\ &= \max\{d^c(z, x_{2n+1}), d^c(z, f(z)), d^c(x_{2n+1}, x_{2n+2})\}, \end{aligned}$$

which gives $\lim_{n\to\infty} M(z, x_{2n+1}) = \max\{1, d^c(z, f(z)), 1\} = d^c(z, f(z))$. Thus passing to the limit as $n \to \infty$ in (30.12) yields $\int_1^{d(fz,z)} \psi(\lambda)\, d\lambda \leq \int_1^{d^c(z,fz)} \psi(\lambda)\, d\lambda$, and so $d(z, fz) = 1$, that is, $z = fz$. Thus by above conclusion, $z = fz = gz$.

Now we are showing that f and g have a unique common fixed point. Suppose $w_1, w_2 \in X$ exist, such that $w_1 = fw_1 = gw_1$ and $w_2 = fw_2 = gw_2$.

As f and g are a multiplicative generalized Banach contraction pair, so that

$$\int_1^{d(w_1,w_2)} \psi(\lambda)\, d\lambda = \int_1^{d(f(w_1),g(w_2))} \psi(\lambda)\, d\lambda \leq \int_1^{M(w_1,w_2)} \psi(\lambda)\, d\lambda,$$

$$\begin{aligned} M(w_1, w_2) &= \max\{d^c(w_1, w_2), d^c(w_1, f(w_1)), d^c(w_2, g(w_2))\} \\ &= \max\{d^c(w_1, w_2), 1, 1\} = d^c(w_1, w_2), \end{aligned}$$

that is, $\int_1^{d(w_1,w_2)} \psi(\lambda)\, d\lambda \leq \int_1^{d^c(w_1,w_2)} \psi(\lambda)\, d\lambda$, which gives $d(w_1, w_2) = 1$ and hence $w_1 = w_2$. Thus the common fixed point of f and g is unique. □

Corollary 30.1 *Let (X, d) be complete multiplicative metric space. If the mappings $f, g : X \to X$ are multiplicative Banach contraction with restrictions of integral type, that is, for any real constant $c \in [0, 1)$,*

$$\int_1^{d(f(\hat{x}),g(\hat{y}))} \psi(\lambda)\, d\lambda \leq \int_1^{d^c(\hat{x},\hat{y})} \psi(\lambda)\, d\lambda \quad (30.13)$$

is satisfied for all $\hat{x}, \hat{y} \in X$, where $\psi : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0,\infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$, then there exists a unique $\hat{u} \in X$ satisfies $\hat{u} = f\hat{u} = g\hat{u}$. Moreover, any fixed point of f is the fixed point of g and conversely.

By taking $f = g$ in Theorem 30.7, we obtain the following fixed point result under restriction of integral type contractive mapping.

Corollary 30.2 *Let (X, d) be complete multiplicative metric space. Suppose that the mapping $f : X \to X$ satisfies the Banach contraction with restriction of integral type, that is, for any real constant $c \in [0, 1)$,*

$$\int_{1}^{d(f(\hat{x}),f(\hat{y}))} \psi(\lambda)\, d\lambda \leq \int_{1}^{M_1(\hat{x},\hat{y})} \psi(\lambda)\, d\lambda, \quad \textit{for all } \hat{x}, \hat{y} \in X, \tag{30.14}$$

where $M_1(\hat{x}, \hat{y}) = \max\{d^c(\hat{x}, \hat{y}), d^c(\hat{x}, f(\hat{x})), d^c(\hat{y}, f(\hat{y}))\}$, *and* $\psi : [0, \infty) \to [0, \infty)$ *is a Lebesgue integrable function with finite integral on each compact set in* $[0, \infty)$*, and for each* $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.

Then there exists a unique $\hat{u} \in X$ *satisfying* $\hat{u} = f\hat{u}$.

Theorem 30.8 *Let* (X, d) *be complete multiplicative metric space, and let mappings* $f, g : X \to X$ *satisfy for any real constants* c_1, c_2, c_3 *with* $c_1 + c_2 + c_3 \in [0, 1)$,

$$\int_{0}^{d(f(\hat{x}),g(\hat{y}))} \psi(\lambda)\, d\lambda \leq \int_{0}^{M^*(\hat{x},\hat{y})} \psi(\lambda)\, d\lambda, \quad \textit{for all } \hat{x}, \hat{y} \in X, \tag{30.15}$$

where $M^*(\hat{x}, \hat{y}) = d^{c_1}(\hat{x}, \hat{y}) \cdot d^{c_2}(\hat{x}, f(\hat{x})) \cdot d^{c_3}(\hat{y}, g(\hat{y}))$, *and* $\psi : [0, \infty) \to [0, \infty)$ *is a Lebesgue integrable function with finite integral on each compact set in* $[0, \infty)$*, and for each* $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.

Then there exists a unique $\hat{u} \in X$ *satisfying* $\hat{u} = f\hat{u} = g\hat{u}$. *Moreover, any fixed point of* f *is the fixed point of* g *and conversely.*

Proof First we are to show that any fixed point of f is the fixed point of g. Suppose that $\hat{u} \in X$ satisfies $\hat{u} = f\hat{u}$. Since f and g are satisfying the integral type restriction, so that we have

$$\int_{1}^{d(\hat{u},g(\hat{u}))} \psi(\lambda)\, d\lambda = \int_{1}^{d(f(\hat{u}),g(\hat{u}))} \psi(\lambda)\, d\lambda \leq \int_{1}^{M^*(\hat{u},\hat{u})} \psi(\lambda)\, d\lambda,$$

where $M^*(\hat{u}, \hat{u}) = d^{c_1}(\hat{u}, \hat{u}) \cdot d^{c_2}(\hat{u}, f(\hat{u})) \cdot d^{c_3}(\hat{u}, g(\hat{u})) = 1 \cdot 1 \cdot d^{c_3}(\hat{u}, g(\hat{u}))$, that is,

$$\int_{1}^{d(\hat{u},g(\hat{u}))} \psi(\lambda)\, d\lambda \leq \int_{1}^{d^{c_3}(\hat{u},g(\hat{u}))} \psi(\lambda)\, d\lambda. \tag{30.16}$$

As $c_3 < 1$, so it follows that $d(\hat{u}, g(\hat{u})) = 1$ and hence $\hat{u} = g(\hat{u})$. Thus $\hat{u} = f(\hat{u}) = g(\hat{u})$. Similarly, if $\hat{u} = g(\hat{u})$, then we have $\hat{u} = f(\hat{u}) = g(\hat{u})$.

Suppose x_0 is the arbitrary element in X. Define the sequence $\{x_n\}$ in X with $fx_{2n} = x_{2n+1}$ and $gx_{2n+1} = x_{2n+2}$ for all $n = 0, 1, 2, \ldots$.

As the mappings f and g are satisfying integral type restriction, we have

$$\int_{1}^{d(x_{2n+1},x_{2n+2})} \psi(\lambda)\, d\lambda = \int_{1}^{d(f(x_{2n}),g(x_{2n+1}))} \psi(\lambda)\, d\lambda \leq \int_{1}^{M^*(x_{2n},x_{2n+1})} \psi(\lambda)\, d\lambda,$$

$$\begin{aligned} M^*(x_{2n}, x_{2n+1}) &= d^{c_1}(x_{2n}, x_{2n+1}) \cdot d^{c_2}(x_{2n}, f(x_{2n})) \cdot d^{c_3}(x_{2n+1}, g(x_{2n+1})) \\ &= d^{c_1}(x_{2n}, x_{2n+1}) \cdot d^{c_2}(x_{2n}, x_{2n+1}) \cdot d^{c_3}(x_{2n+1}, x_{2n+2}) \\ &= (d(x_{2n}, x_{2n+1}))^{c_1+c_2} \cdot d^{c_3}(x_{2n+1}, x_{2n+2}), \end{aligned}$$

that is,

$$\int_1^{d(x_{2n+1},x_{2n+2})} \psi(\lambda)\,d\lambda \le \int_1^{(d(x_{2n},x_{2n+1}))^{c_1+c_2}\cdot d^{c_3}(x_{2n+1},x_{2n+2})} \psi(\lambda)\,d\lambda. \tag{30.17}$$

Now, we have two possibilities.

In case of $(d(x_{2n}, x_{2n+1}))^{c_1+c_2} \le d^{c_3}(x_{2n+1}, x_{2n+2})$,

$$\int_1^{d(x_{2n+1},x_{2n+2})} \psi(\lambda)\,d\lambda \le \int_1^{(d(x_{2n+1},x_{2n+2}))^{c_1+c_2+c_3}} \psi(\lambda)\,d\lambda,$$

implying that $d(x_{2n+1}, x_{2n+2}) \le (d(x_{2n+1}, x_{2n+2}))^{c_1+c_2+c_3}$. Since $c_1 + c_2 + c_3 < 1$, so $d(x_{2n+1}, x_{2n+2}) = 1$, and hence $x_{2n+1} = x_{2n+2}$. As we have $gx_{2n+1} = x_{2n+2}$, so $gx_{2n+1} = x_{2n+1}$, that is, x_{2n+1} is the fixed point of g. Thus by above conclusion, x_{2n+1} is a common fixed point of f and g.

In case of $(d(x_{2n}, x_{2n+1}))^{c_1+c_2} \ge d^{c_3}(x_{2n+1}, x_{2n+2})$,

$$\int_1^{d(x_{2n+1},x_{2n+2})} \psi(\lambda)\,d\lambda \le \int_1^{(d(x_{2n},x_{2n+1}))^{c_1+c_2+c_3}} \psi(\lambda)\,d\lambda,$$

that is,

$$\int_1^{d(x_{2n+1},x_{2n+2})} \psi(\lambda)\,d\lambda \le \int_1^{(d(x_{2n},x_{2n+1}))^{\eta}} \psi(\lambda)\,d\lambda, \tag{30.18}$$

where $\eta = c_1 + c_2 + c_3 < 1$. Similarly, we can show that

$$\int_1^{d(x_{2n+2},x_{2n+3})} \psi(\lambda)\,d\lambda \le \int_1^{(d(x_{2n+1},x_{2n+2}))^{\eta}} \psi(\lambda)\,d\lambda. \tag{30.19}$$

Thus for all $n \ge 0$,

$$\int_1^{d(x_n,x_{n+1})} \psi(\lambda)\,d\lambda \le \int_1^{(d(x_{n-1},x_n))^{\eta}} \psi(\lambda)\,d\lambda. \tag{30.20}$$

Continuing this way, we obtain

$$\begin{aligned}\int_1^{d(x_n,x_{n+1})} \psi(\lambda)\,d\lambda \le \int_1^{(d(x_{n-1},x_n))^{\eta}} \psi(\lambda)\,d\lambda \le \int_1^{(d(x_{n-2},x_{n-1}))^{\eta^2}} \psi(\lambda)\,d\lambda \\ \le \cdots \le \int_1^{(d(x_0,x_1))^{\eta^n}} \psi(\lambda)\,d\lambda.\end{aligned}$$

As $0 \le \eta < 1$, so that

$$\int_{1}^{d(x_n,x_{n+1})} \psi(\lambda)\,d\lambda \to 0 \;\; as \;\; n \to \infty \tag{30.21}$$

and we have $d(x_n, x_{n+1}) \to 1$ as $n \to \infty$.

For $\grave{m}, n \in \mathbb{N}$ with $\grave{m} \geq n$,

$$\begin{aligned} d(x_n, x_{\grave{m}}) &\leq d(x_n, x_{n+1}) \cdot d(x_{n+1}, x_{n+2}) \cdots d(x_{\grave{m}-1}, x_{\grave{m}}) \\ &\leq (d(x_0, x_1))^{\eta^n} \cdot (d(x_0, x_1))^{\eta^{n+1}} \cdots (d(x_0, x_1))^{\eta^{\grave{m}-1}} \\ &= (d(x_0, x_1))^{\eta^n+\eta^{n+1}+\cdots+\eta^{\grave{m}-1}} \\ &\leq (d(x_0, x_1))^{\eta^n(1+\eta+\eta^2+\dots)} \\ &\leq (d(x_0, x_1))^{\frac{\eta^n}{1-\eta}} \to 1 \;\; as \;\; n, \grave{m} \to \infty. \end{aligned}$$

Thus $\int_{1}^{d(x_n,x_{\grave{m}})} \psi(\lambda)\,d\lambda \to 0 \;\; as \;\; \grave{m}, n \to \infty$, and $\{x_n\}$ is the Cauchy sequence in complete multiplicative space (X, d). We obtain $z \in X$ for which $x_n \to z$ as $n \to \infty$, or $\lim_{n\to\infty} d(x_n, z) = 1$.

Now we are to prove $fz = z$, that is, $d(fz, z) = 1$. As

$$\begin{aligned} \int_{1}^{d(fz,x_{2n+2})} \psi(\lambda)\,d\lambda = \int_{1}^{d(fz,gx_{2n+1})} \psi(\lambda)\,d\lambda &\leq \int_{1}^{M^*(z,x_{2n+1})} \psi(\lambda)\,d\lambda, \quad (30.22) \\ M^*(z, x_{2n+1}) = d^{c_1}(z, x_{2n+1}) \cdot d^{c_2}(z, f(z)) &\cdot d^{c_3}(x_{2n+1}, g(x_{2n+1})) \\ = d^{c_1}(z, x_{2n+1}) \cdot d^{c_2}(z, f(z)) &\cdot d^{c_3}(x_{2n+1}, x_{2n+2}), \end{aligned}$$

it gives $\lim_{n\to\infty} M^*(z, x_{2n+1}) = 1 \cdot d^{c_2}(z, f(z)) \cdot 1 = d^{c_3}(z, f(z))$. Thus by passing to the limit as $n \to \infty$ in (30.22), holds $\int_{1}^{d(fz,z)} \psi(\lambda)\,d\lambda \leq \int_{1}^{d^{c_3}(z,fz)} \psi(\lambda)\,d\lambda$, and hence $d(z, fz) = 1$, that is, $z = fz$. Also by above conclusion, $z = fz = gz$.

Now we are two show that f and g have a unique common fixed point. If their exist $w_1, w_2 \in X$, such that $w_1 = fw_1 = gw_1$ and $w_2 = fw_2 = gw_2$, then by given restriction, we have

$$\begin{aligned} \int_{1}^{d(w_1,w_2)} \psi(\lambda)\,d\lambda = \int_{1}^{d(f(w_1),g(w_2))} \psi(\lambda)\,d\lambda &\leq \int_{1}^{M^*(w_1,w_2)} \psi(\lambda)\,d\lambda, \\ M^*(w_1, w_2) = d^{c_1}(w_1, w_2) \cdot d^{c_2}(w_1, f(w_1)) &\cdot d^{c_3}(w_2, g(w_2)) \\ = d^{c_1}(w_1, w_2) \cdot 1 \cdot 1 &= d^{c_1}(w_1, w_2) \end{aligned}$$

that is, $\int_{1}^{d(w_1,w_2)} \psi(\lambda)\,d\lambda \leq \int_{1}^{d^{c_1}(w_1,w_2)} \psi(\lambda)\,d\lambda$, which gives $d(w_1, w_2) = 1$ and hence $w_1 = w_2$. Thus the common fixed point of f and g is unique. □

In case of $f = g$ in Theorem 30.8, we get the following fixed point result under restriction of integral type contractive mapping.

Corollary 30.3 *Let (X, d) be complete multiplicative metric space. If the mapping $f : X \to X$ satisfying that for any real constants c_1, c_2, c_3 with $c_1+c_2+c_3 \in [0, 1)$,*

$$\int_0^{d(f(\hat{x}), f(\hat{y}))} \psi(\lambda)\, d\lambda \le \int_0^{M_2(\hat{x}, \hat{y})} \psi(\lambda)\, d\lambda, \quad \textit{for all } \hat{x}, \hat{y} \in X, \tag{30.23}$$

where $M_2(\hat{x}, \hat{y}) = d^{c_1}(\hat{x}, \hat{y}) \cdot d^{c_2}(\hat{x}, f(\hat{x})) \cdot d^{c_3}(\hat{y}, f(\hat{y}))\}$ and $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$, then there exists a unique $\hat{u} \in X$ satisfying $\hat{u} = f\hat{u}$.

30.5 Well-Posedness Results for Common Fixed Points

The notion of well-posedness for fixed point problems gets interest by various researchers. In this section, well-posedness of fixed point problem of maps under restrictions of integral type contractions are obtained. First, we define well-posedness of common fixed point problems for multiplicative metric space structure.

Definition 30.11 Let (X, d) be multiplicative metric space. Common fixed point problem of self-maps $f, g : X \to X$ is known as well-posed if set $F(f, g)$ is singleton with $x^* \in F(f, g)$ and for sequence $\{x_n\}$ in X with $\lim_{n\to\infty} d(fx_n, x_n) = 1$ and $\lim_{n\to\infty} d(gx_n, x_n) = 1$ holds $\lim_{n\to\infty} d(x_n, x^*) = 1$.

Theorem 30.9 *Let (X, d) be complete multiplicative metric space. Assume that the mappings $f, g : X \to X$ are multiplicative generalized Banach contraction with restrictions of integral type, that is, for any $c \in [0, 1)$,*

$$\int_1^{d(f(\hat{x}), g(\hat{y}))} \psi(\lambda)\, d\lambda \le \int_1^{M(\hat{x}, \hat{y})} \psi(\lambda)\, d\lambda, \quad \textit{for all } \hat{x}, \hat{y} \in X,$$

where $M(\hat{x}, \hat{y}) = \max\{d^c(\hat{x}, \hat{y}), d^c(\hat{x}, f(\hat{x})), d^c(\hat{y}, g(\hat{y}))\}$, and $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.

Then the common fixed point problem for f and g is well-posed.

Proof We obtain from Theorem 30.7 that f and g have the unique common fixed point, say $\hat{u} \in X$. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} d(fx_n, x_n) = 1$ and $\lim_{n\to\infty} (gx_n, x_n) = 1$. Without loss of generality, assume that $\hat{u} \neq x_n$ for every n. From triangle inequality

$$\begin{aligned}\int_1^{d(\hat{u},x_n)} \psi(\lambda)\,d\lambda &\le \int_1^{d(f(\hat{u}),g(x_n))\cdot d(g(x_n),x_n))} \psi(\lambda)\,d\lambda \\ &\le \int_1^{M(\hat{u},x_n)\cdot d(g(x_n),x_n))} \psi(\lambda)\,d\lambda,\end{aligned}$$

$$\begin{aligned}M(\hat{u},x_n) &= \max\{d^c(\hat{u},x_n), d^c(\hat{u}, f(\hat{u})), d^c(x_n, g(x_n))\} \\ &= \max\{d^c(\hat{u},x_n), 1, d^c(x_n, g(x_n))\} = \max\{d^c(\hat{u},x_n), d^c(x_n, g(x_n))\},\end{aligned}$$

that is,

$$\int_1^{d(\hat{u},x_n)} \psi(\lambda)\,d\lambda \le \int_1^{\max\{d^c(\hat{u},x_n),d^c(x_n,g(x_n))\}\cdot d(g(x_n),x_n))} \psi(\lambda)\,d\lambda. \tag{30.24}$$

Passing to the limit as $n \to \infty$, yields

$$\begin{aligned}\int_1^{\lim\limits_{n\to\infty} d(\hat{u},x_n)} \psi(\lambda)\,d\lambda &\le \int_1^{\lim\limits_{n\to\infty}[\max\{d^c(\hat{u},x_n),d^c(x_n,g(x_n))\}\cdot d(g(x_n),x_n))]} \psi(\lambda)\,d\lambda \\ &= \int_1^{\lim\limits_{n\to\infty} d^c(\hat{u},x_n)} \psi(\lambda)\,d\lambda, \quad \text{where } c < 1,\end{aligned}$$

implying $d(x_n, \hat{u}) \to 1$, that is $\lim\limits_{n\to\infty} x_n = \hat{u}$. This completes the proof. □

Corollary 30.4 *Let (X, d) be complete multiplicative metric space. Suppose that the mappings $f, g : X \to X$ are multiplicative Banach contraction with restrictions of integral type, that is, for any real number $c \in [0, 1)$,*

$$\int_1^{d(f(\hat{x}),g(\hat{y}))} \psi(\lambda)\,d\lambda \le \int_1^{d^c(\hat{x},\hat{y})} \psi(\lambda)\,d\lambda, \quad \textit{for all } \hat{x}, \hat{y} \in X,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\,d(\lambda) > 0$.

Then the common fixed point problem for f and g is well-posed.

Theorem 30.10 *Let (X, d) be complete multiplicative metric space. If the mappings $f, g : X \to X$ satisfy that for any real constants c_1, c_2, c_3 with $c_1 + c_2 + c_3 \in [0, 1)$,*

$$\int_0^{d(f(\hat{x}),g(\hat{y}))} \psi(\lambda)\,d\lambda \le \int_0^{M^*(\hat{x},\hat{y})} \psi(\lambda)\,d\lambda, \quad \textit{for all } \hat{x}, \hat{y} \in X,$$

where $M^*(\hat{x}, \hat{y}) = d^{c_1}(\hat{x}, \hat{y})\cdot d^{c_2}(\hat{x}, f(\hat{x}))\cdot d^{c_3}(\hat{y}, g(\hat{y}))\}$ and $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\,d(\lambda) > 0$.

Then the common fixed point problem for f and g is well-posed.

Proof We obtain from Theorem 30.8 that, f and g have the unique common fixed point, say $\hat{u} \in X$. Let $\{x_n\}$ be a sequence in X having $\lim\limits_{n\to\infty} d(fx_n, x_n) = 1$ and $\lim\limits_{n\to\infty} (gx_n, x_n) = 1$. We assume that $\hat{u} \neq x_n$ for every non-negative number n. By using triangle inequality,

$$\begin{aligned}
\int_1^{d(\hat{u},x_n)} \psi(\lambda)\, d\lambda &\leq \int_1^{d(f(\hat{u}),g(x_n))\cdot d(g(x_n),x_n))} \psi(\lambda)\, d\lambda \\
&\leq \int_1^{M^*(\hat{u},x_n)\cdot d(g(x_n),x_n))} \psi(\lambda)\, d\lambda, \\
M^*(\hat{u}, x_n) &= d^{c_1}(\hat{u}, x_n) \cdot d^{c_2}(\hat{u}, f(\hat{u})) \cdot d^{c_3}(x_n, g(x_n)) \\
&= d^{c_1}(\hat{u}, x_n) \cdot 1 \cdot d^{c_3}(x_n, g(x_n)) \\
&= d^{c_1}(\hat{u}, x_n) \cdot d^{c_3}(x_n, g(x_n)),
\end{aligned}$$

that is,

$$\int_1^{d(\hat{u},x_n)} \psi(\lambda)\, d\lambda \leq \int_1^{d^{c_1}(\hat{u},x_n)\cdot d^{c_3}(x_n,g(x_n))\cdot d(g(x_n),x_n))} \psi(\lambda)\, d\lambda. \tag{30.25}$$

Passing to the limit as $n \to \infty$ yields

$$\begin{aligned}
\int_1^{\lim\limits_{n\to\infty} d(\hat{u},x_n)} \psi(\lambda)\, d\lambda &\leq \int_1^{\lim\limits_{n\to\infty} [d^{c_1}(\hat{u},x_n)\cdot d^{c_3}(x_n,g(x_n))\cdot d(g(x_n),x_n))]} \psi(\lambda)\, d\lambda \\
&= \int_1^{\lim\limits_{n\to\infty} d^{c_1}(\hat{u},x_n)} \psi(\lambda)\, d\lambda,
\end{aligned}$$

where $c_1 < 1$, implies $d(x_n, \hat{u}) \to 1$, that is $\lim\limits_{n\to\infty} x_n = \hat{u}$. This completes the proof. □

30.6 Periodic Points of Contractive Mappings

It is noted that, if a point $\hat{u}$ is a fixed point of self-map f, then $\hat{u}$ is also fixed point of f^n for every $n \in \mathbb{N}$. But generally, the converse does not hold. For instance, take $X = [0, 1]$ with $f : X \to X$ defined as $f(\hat{u}) = 1 - \hat{u}$. Then $f\left(\frac{1}{2}\right) = \frac{1}{2}$, but every iterate for $n = 2, 3, 4, \ldots$, that is, $f^2, f^3, f^4, \ldots$ all are identity maps and so each point in $[0, 1]$ is the fixed point of $f^2, f^3, f^4, \ldots$. But in case of $X = [0, \pi]$, $f(t) = \cos t$, all iterates of f has same fixed point as f. Various useful results regarding periodic points of mappings were established in [2, 3, 21, 30, 32].

Definition 30.12 For a non-empty set X and for self-map $f : X \to X$, if $F(f) = F(f^n)$ hold for each $n \in \mathbb{N}$, then we say that f has property P.

Definition 30.13 For a non-empty set X and for self-map $f : X \to X$, if $F(f, g) = F(f^n, g^n)$ hold for each $n \in \mathbb{N}$, then we say that f and g has property Q.

Theorem 30.11 *Let (X, d) be complete multiplicative metric space. Suppose that the mappings $f, g : X \to X$ are a multiplicative generalized Banach contraction with restrictions of integral type, that is, for any real number $c \in [0, 1)$,*

$$\int_1^{d(f(\hat{x}),g(\hat{y}))} \psi(\lambda)\, d\lambda \le \int_1^{M(\hat{x},\hat{y})} \psi(\lambda)\, d\lambda, \quad \text{for all } \hat{x}, \hat{y} \in X,$$

where $M(\hat{x}, \hat{y}) = \max\{d^c(\hat{x}, \hat{y}), d^c(\hat{x}, f(\hat{x})), d^c(\hat{y}, g(\hat{y}))\}$, and $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.

Then f and g has property Q.

Proof We obtain from Theorem 30.7 that, the f and g have at most one common fixed point. Also trivially $F(f, g) \subseteq F(f^n, g^n)$ holds for every $n \in \mathbb{N}$. Let us consider $\hat{u} \in F(f^n, g^n)$. Then as $f, g : X \to X$ are a multiplicative generalized Banach contraction with restrictions of integral type, we have

$$\int_1^{d(\hat{u},g\hat{u})} \psi(\lambda)\, d\lambda \le \int_1^{d(f(f^{n-1}\hat{u}),g(\hat{u}))} \psi(\lambda)\, d\lambda \le \int_1^{M(f^{n-1}\hat{u},\hat{u})} \psi(\lambda)\, d\lambda,$$

$$\begin{aligned} M(f^{n-1}\hat{u}, \hat{u}) &= \max\{d^c(f^{n-1}\hat{u}, \hat{u}), d^c(f^{n-1}\hat{u}, f(f^{jn-1}\hat{u})), d^c(\hat{u}, g(\hat{u}))\} \\ &= \max\{d^c(f^{n-1}\hat{u}, \hat{u}), d^c(f^{n-1}\hat{u}, \hat{u}), d^c(\hat{u}, g(\hat{u}))\} \\ &= \max\{d^c(f^{n-1}\hat{u}, \hat{u}), d^c(\hat{u}, g(\hat{u}))\}, \end{aligned}$$

that is,

$$\int_1^{d(\hat{u},g\hat{u})} \psi(\lambda)\, d\lambda \le \int_1^{\max\{d^c(f^{n-1}\hat{u},\hat{u}),d^c(\hat{u},g(\hat{u}))\}} \psi(\lambda)\, d\lambda \le \int_1^{d^c(f^{n-1}\hat{u},\hat{u})} \psi(\lambda)\, d\lambda$$

and hence

$$\begin{aligned} \int_1^{d(\hat{u},g\hat{u})} \psi(\lambda)\, d\lambda &\le \int_1^{d^c(f^{n-1}\hat{u},\hat{u})} \psi(\lambda)\, d\lambda \le \int_1^{d^{c^2}(f^{n-2}\hat{u},\hat{u})} \psi(\lambda)\, d\lambda \\ &\le \cdots \le \int_1^{d^{c^n}(f\hat{u},\hat{u})} \psi(\lambda)\, d\lambda. \end{aligned} \tag{30.26}$$

In a similar way, we have

$$\begin{aligned} \int_1^{d^{c^n}(f\hat{u},\hat{u})} \psi(\lambda)\, d\lambda &\le \int_1^{d^{c^n+1}(\hat{u},g^{n-1}\hat{u})} \psi(\lambda)\, d\lambda \\ &\le \int_1^{d^{c^n+2}(\hat{u},g^{n-2}\hat{u})} \psi(\lambda)\, d\lambda \le \cdots \le \int_1^{d^{c^{2n}}(\hat{u},g\hat{u})} \psi(\lambda)\, d\lambda. \end{aligned} \tag{30.27}$$

From (30.26) and (30.27), it follows that

$$\int_1^{d(\hat{u},g\hat{u})} \psi(\lambda)\, d\lambda \le \int_1^{d^{c^{2n}}(\hat{u},g\hat{u})} \psi(\lambda)\, d\lambda.$$

As $c^{2n} < 1$, we obtain $d(\hat{u}, g\hat{u}) = 1$, that is, $\hat{u} = g\hat{u}$. Also by the concision of Theorem 30.7, we obtain $\hat{u} = f\hat{u} = g\hat{u}$ and so $\hat{u} \in F(f, g)$. Thus $F(f^n, g^n) \subseteq F(f, g)$ and the conclusion is $F(f^n, g^n) = F(f, g)$. □

Corollary 30.5 *Let (X, d) be complete multiplicative metric space. Suppose that the mappings $f, g : X \to X$ are multiplicative generalized Banach contraction with restrictions of integral type, that is, for any real constant $c \in [0, 1)$,*

$$\int_1^{d(f(\hat{x}),g(\hat{y}))} \psi(\lambda)\, d\lambda \le \int_1^{d^c(\hat{x},\hat{y})} \psi(\lambda)\, d\lambda$$

is satisfied for all $\hat{x}, \hat{y} \in X$, where $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.

Then f and g has property Q.

In case of $f = g$ in Theorem 30.11, we get the following fixed point result under restriction of integral type contractive mapping.

Corollary 30.6 *Let (X, d) be complete multiplicative metric space. Suppose that the mappings $f : X \to X$ is satisfying that for any real number $c \in [0, 1)$,*

$$\int_1^{d(f(\hat{x}),f(\hat{y}))} \psi(\lambda)\, d\lambda \le \int_1^{M(\hat{x},\hat{y})} \psi(\lambda)\, d\lambda, \quad \textit{for all } \hat{x}, \hat{y} \in X,$$

where $M(\hat{x}, \hat{y}) = \max\{d^c(\hat{x}, \hat{y}), d^c(\hat{x}, f(\hat{x})), d^c(\hat{y}, f(\hat{y}))\}$ and $\psi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function with finite integral on each compact set in $[0, \infty)$, and for each $\varepsilon > 1$, $\int_1^{\varepsilon} \psi(\lambda)\, d(\lambda) > 0$.

Then f has property P.

30.7 Conclusion

Recently many results appeared in the literature pertaining to the problems related to the fixed point and common fixed point problems and it applications. In the present work, we obtained several fixed point and common fixed point results that are satisfying generalized integral type contraction conditions in the setup of multiplicative metric spaces. We also obtained the well-posedness of these common fixed point problems. Furthermore, periodic point results related to the mappings that are satisfying generalized integral type contraction conditions are also established.

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