Chapter 3
Ternary Lie Superalgebras and Nambu-Hamilton Equation in Superspace

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Abstract In the present paper we give a survey of methods for constructing ternary Lie algebras and ternary Lie superalgebras. We also propose a generalization of Nambu-Hamilton equation to a superspace and show that this generalization induces a family of ternary Nambu-Poisson brackets of even degree functions on a superspace. Then we show that the construction of ternary quantum Nambu-Poisson bracket, based on the trace of a matrix, can be extended to matrix Lie superalgebra $\mathfrak{gl}(m, n)$ by means of the supertrace of a matrix. We propose a generalization of Nambu-Hamilton equation in superspace. We show that this generalization induces a family of ternary Nambu-Poisson brackets, which is defined with the help of Berezinian.

Keywords 3-Lie algebra · 3-Lie superalgebra · Superspace · Berezinian · Nambu-Poisson bracket · Nambu-Hamilton equation

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3.1 Introduction

In this paper, we consider n-Lie algebras, n-Lie superalgebras and their applications in Hamiltonian mechanics. The concept of n-Lie algebra was introduced by Filippov in [15] and then studied in a number of papers. The simplest example of an n-Lie algebra is the vector product of n vectors in $(n + 1)$-dimensional vector space [19]. The basic component of the structure of a Lie algebra is the Jacobi identity. This identity can be written either in the form of equality to zero of the sum of the double brackets of cyclic permutations of three elements, or in the form of a derivation of a Lie bracket. The latter can be extended to a bracket with an arbitrary number of

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arguments, which leads to a basic component of a concept of n-Lie algebra, which is now called the Filippov-Jacobi identity or the Fundamental Identity.

Independently from Filippov, Nambu proposed a generalization of Hamiltonian mechanics to phase spaces of odd dimensions [18], and in this generalization the Nambu-Hamilton equation induces a ternary operation on functions, which can be considered as a ternary (more generally, as n-ary) analog of the Poisson bracket. Later it turned out that the ternary bracket, which appeared in a generalization of Hamiltonian mechanics proposed by Nambu, satisfies the Filippov-Jacobi identity. It is interesting to note the fact that the structure of a Lie algebra plays an important role in Hamiltonian mechanics, where it enters through the Poisson bracket. Similarly, a generalization of the concept of Lie algebra to n-ary operations, proposed by Filippov, plays an important role in the generalized Nambu-Hamilton mechanics, where it enters by means of the n-ary Nambu-Poisson bracket.

Another area of theoretical physics, where the n-Lie algebras are applied, is a field theory. Particularly the authors of the paper [13] proposed a generalization of the Nahm's equation by means of quantum Nambu 4-bracket and showed that their generalization of the Nahm's equation describes M2 branes ending on M5 branes.

A quantization of Nambu-Hamiltonian mechanics is a problem that already Nambu mentioned and he also outlined possible solutions for this problem in his pioneering work. However, this problem is still unresolved. In [9] the authors set the task to find a quantum version of the ternary Nambu-Poisson bracket using a matrix algebra, where the matrices can be both ordinary (i.e. plane) square matrices and cubic (spatial) matrices. The motivation for this was the possible use of a Nambu-Poisson quantum bracket in M-theory. The authors of [9] proposed several realizations of a quantum Nambu-Poisson bracket using matrices and these realizations are based on a combination of the trace of a matrix and the commutator of two matrices. This construction was generalized and its various aspects were investigated in a number of papers [6–8, 16].

In the present paper we give a survey of methods for constructing ternary Lie algebras and ternary Lie superalgebras. We also propose a generalization of Nambu-Hamilton equation to a superspace and show that this generalization induces a family of ternary Nambu-Poisson brackets of even degree functions on a superspace. The present paper is organized as follows: First of all, in Sect. 3.2 we give the definition of n-Lie algebra, its particular case of ternary Lie algebra and few important well known examples of n-Lie algebras such as vector product and Jacobian n-Lie algebras. Then we describe the construction of ternary quantum Nambu-Poisson bracket (only in the case of (plane) square matrices), proposed in [9], and show that this construction can be generalized by means of 1-cochain of Lie algebra, which satisfies the condition written with the help of differential and wedge product of cochains. Then we describe the method of constructing ternary Lie algebras by means of involution and a derivation of a commutative, associative algebra, proposed in [12]. It should be mentioned that we slightly generalize the approach of [12] by using our approach based on a cochains, their wedge products and differential. Section 3.3 is devoted to 3-Lie superalgebras. At the beginning of this section we give the definition of n-Lie superalgebra and its particular case of ternary Lie superalgebra. We show that the system of two identities, which are equivalent to ternary Filippov-Jacobi identity [10], can be extended to ternary Lie superalgebras by means of a graded version of these identities. By other words, we prove that the graded version of the two identities, proposed in [10], is equivalent to graded Filippov-Jacobi identity, which is a basic component of the concept of ternary Lie superalgebra. Then we show that the construction of ternary quantum Nambu-Poisson bracket [9], based on the trace of a matrix, can be extended to matrix Lie superalgebra $\mathfrak{gl}(m, n)$ by means of the supertrace of a matrix. By other words, we show that if we have a matrix Lie superalgebra $\mathfrak{gl}(m, n)$, then we can construct a ternary graded bracket by means of the graded commutator of two supermatrices and the supertrace of a matrix and prove that this ternary graded bracket is graded skew-symmetric and satisfies the graded Filippov-Jacobi identity. In the last subsection of Sect. 3.3 we extend the method of constructing ternary Lie algebras with the help of a derivation and an involution of commutative, associative algebra to commutative superalgebra with superinvolution and even degree derivation. As an example of applications of this method, we construct the ternary Lie superalgebra of functions on a superspace, where the ternary graded Lie bracket is defined by means of superinvolution and even degree vector field. Finally, in Sect. 3.4 we propose a generalization of Nambu-Hamilton equation in superspace. We show that this generalization induces a family of ternary Nambu-Poisson brackets, which is defined with the help of Berezinian.

3.2 n-Lie Algebras

The notion of n-Lie algebra is an extension of the notion of Lie algebra to algebraic structures based on an n-ary law of multiplication. It is well known that Lie algebra is a vector space endowed with a binary multiplication law, which is bilinear, skew-symmetric and satisfies the Jacobi identity. If we wish to extend this structure onto n-ary law of multiplication, where $n \geq 2$, then we must formulate the above conditions in terms of an n-ary operation. It is clear that bilinearity and skew-symmetry are easily transferred to the case of an n-ary multiplication law by replacing them with multilinearity and skew-symmetry of a mapping with n arguments respectively. It is well known that the Jacobi identity can be written in the form of derivation of a Lie bracket. This makes it possible to extend the Jacobi identity to the case of an n-ary multiplication law. This approach was proposed by Filippov in [15] and the extension of the Jacobi identity to an n-ary multiplication law in the form of derivation, as it is mentioned above, is now called the Filippov-Jacobi identity. In this section we will give the definition of n-Lie algebra, and several examples of n-Lie algebras. We will also show how to construct 3-Lie algebra using the Lie bracket of a binary Lie algebra and an analog of the trace of a matrix.

Definition 3.1 Vector space $\mathfrak{g}$ endowed with a mapping $[\cdot, \ldots, \cdot] : \mathfrak{g}^n \to \mathfrak{g}$ is said to be a *n-Lie algebra*, if $[\cdot, \ldots, \cdot]$ is n-linear, skew-symmetric and satisfies the identity

$$[x_1, \ldots, x_{n-1}, [y_1, \ldots, y_n]] = \sum_{i=1}^{n} [y_1, \ldots, [x_1, \ldots, x_{n-1}, y_i], \ldots, y_n], \qquad (3.1)$$

where $x_1, \ldots, x_{n-1}, y_1, \ldots, y_n \in \mathfrak{g}$.

In the definition of n-Lie algebra the identity (3.1) is called the Filippov-Jacobi identity. It should be noted that in the papers, where n-Lie algebras are used for applications in field theories and Nambu's generalization of Hamiltonian mechanics, the identity (3.1) is also called Fundamental Identity. Particularly, if we take $n = 2$ in the above definition, then we have a vector space $\mathfrak{g}$ with a binary bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which is skew-symmetric and satisfies the Jacobi identity

$$[x, [y_1, y_2]] = [[x, y_1], y_2] + [y_1, [x, y_2]],$$

which can be also written in a more recognizable form as follows

$$[x, [y_1, y_2]] + [y_1, [y_2, x]] + [y_2, [x, y_1]] = 0.$$

Hence, in this case Definition 3.1 gives the definition of Lie algebra.

If we now take the first integer, which follows 2, i.e. if in Definition 3.1 we take $n = 3$, then we get the generalization of Lie algebra, which is called 3-Lie algebra or ternary Lie algebra. In this paper, we will pay special attention to these algebras and therefore, it is worthwhile to consider the general definition of n-Lie algebra in this particular case. A 3-Lie algebra is a vector space $\mathfrak{g}$ equipped with a ternary bracket $[\cdot, \cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which is skew-symmetric, i.e. it does not change under a cyclic permutation and change a sign under non-cyclic permutation of its arguments, and satisfies the ternary Filippov-Jacobi identity

$$\begin{aligned} &[x_1, x_2, [y_1, y_2, y_3]] = \\ &\qquad [[x_1, x_2, y_1], y_2, y_3]] + [y_1, [x_1, x_2, y_2], y_3]] + [y_1, y_2, [x_1, x_2, y_3]]. \end{aligned} \tag{3.2}$$

The following proposition gives the system of two identities, which is equivalent to the ternary Filippov-Jacobi identity.

Proposition 3.1 *Let $\mathfrak{g}$ be a vector space equipped with a skew-symmetric trilinear mapping $[\cdot, \cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Then $\mathfrak{g}$ is a 3-Lie algebra if and only if a trilinear mapping satisfies the following identities:*

$$[[x_1, x_2, x_3], u, v] = [[x_1, x_2, u], x_3, v] + [[x_1, u, x_3], x_2, v] + [[u, x_2, x_3], x_1, v],$$

and

$$\begin{aligned} [[x_1, x_2, u], y_1, y_2] + [[y_1, y_2, u], x_1, x_2] = {} & [[y_1, x_2, u], x_1, y_2] + [[x_1, y_1, u], x_2, y_2] \\ & + [[y_2, x_2, u], y_1, x_1] + [[x_1, y_2, u], y_1, x_2]. \end{aligned}$$

The proof can be found in [10].

Definition 3.2 Let $\mathfrak{g}$ be an n-Lie algebra and I be a subspace of vector space $\mathfrak{g}$. Then I is said to be an *ideal of n-Lie algebra* $\mathfrak{g}$ if for any $y \in I$ and $x_1, x_2, \ldots, x_{n-1} \in \mathfrak{g}$ it holds $[x_1, x_2, \ldots, x_{n-1}, y] \in I$. An n-Lie algebra $\mathfrak{g}$ is called *simple* if $[\mathfrak{g}, \mathfrak{g}, \ldots, \mathfrak{g}] \neq \{0\}$ and $\mathfrak{g}$ has no non-trivial ideals, i.e. n-Lie algebra $\mathfrak{g}$ has no ideals different from $\{0\}$ and $\mathfrak{g}$.

3.2.1 Vector Product and Jacobian n-Lie Algebras

As it is mentioned in Introduction, the concept of n-Lie algebra was introduced by Filippov in [15]. An example of such a structure is given by the vector product of vectors. Let $n \geq 3$ and consider an n-dimensional Euclidean vector space E^n with a metric g. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be a basis for E^n and $g_{\mu\nu} = g(\mathbf{e}_\mu, \mathbf{e}_\nu)$ be the metric tensor in this basis. One can define the vector product of $n-1$ vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$ as follows

$$[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}] = g^{\mu\tau} \epsilon_{\mu_1,\mu_2,\ldots,\mu_{n-1},\tau}\; v_1^{\mu_1}\, v_2^{\mu_2} \ldots v_{n-1}^{\mu_{n-1}}\, \mathbf{e}_\mu, \tag{3.3}$$

where $\mathbf{v}_i = v^\mu\, \mathbf{e}_\mu$, $(g^{\mu\tau})$ is the reciprocal matrix of $(g_{\mu\tau})$ and $\epsilon_{\mu_1,\mu_2,\ldots,\mu_{n-1},\tau}$ is a totally antisymmetric tensor in n-dimensional Euclidean space [19]. Euclidean space E^n endowed with the vector product (3.3) is an $(n-1)$-Lie algebra, i.e. the vector product (3.3) satisfies the Filippov-Jacobi identity. This algebra is called the vector product $(n-1)$-Lie algebra. It can be proved that the vector product $(n-1)$-Lie algebra is simple. Moreover, this $(n-1)$-Lie algebra is the only one simple finite-dimensional $(n-1)$-Lie algebra for $n > 3$ [17].

Another example of n-Lie algebra is related to the generalization of Hamiltonian mechanics proposed by Nambu in [18]. As previously, E^n is an n-dimensional Euclidean space, whose Cartesian coordinates will be denoted x^μ, $\mu = 1, 2, \ldots, n$. Let $F_1, F_2, \ldots, F_n$ be smooth functions on Euclidean space E^n. Define the n-ary bracket as follows

$$\{F_1, F_2, \ldots, F_n\} = \frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(x^1, x^2, \ldots, x^n)} = \mathrm{Det} \begin{pmatrix} \partial_{x^1} F_1 & \partial_{x^2} F_1 & \ldots & \partial_{x^n} F_1 \\ \partial_{x^1} F_2 & \partial_{x^2} F_2 & \ldots & \partial_{x^n} F_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial_{x^1} F_n & \partial_{x^2} F_n & \ldots & \partial_{x^n} F_n \end{pmatrix}. \tag{3.4}$$

This n-ary bracket of smooth functions is called the n-ary Nambu-Poisson bracket. Evidently, the n-ary Nambu-Poisson bracket is skew-symmetric and it can be proved that it satisfies the Filippov-Jacobi identity and, thus, determines the n-Lie algebra structure on the infinite dimensional space of smooth functions of Euclidean space E^n. Moreover, it is easily verified that the n-ary Nambu-Poisson bracket has the derivation property with respect to product of functions. If the algebra of smooth functions on a finite-dimensional smooth manifold is endowed with an n-ary bracket, which is skew-symmetric, has the derivation property with respect to product of functions and it satisfies the Filippov-Jacobi identity, then this manifold is called the Nambu-Poisson manifold [20].

It should be noted that the n-ary Nambu-Poisson bracket retains all its algebraic properties (skew-symmetry, derivation property, Filippov-Jacobi identity), if we replace in (3.4) the partial derivatives with commuting vector fields i.e. $\partial_{x^\mu} \to X_\mu$, where $X_\mu = X_\mu^\nu\,\partial_{x^\nu}$. This suggests an even more general structure. Assume $\mathcal{A}$ is a unital commutative associative algebra and $\delta_1, \delta_2, \ldots, \delta_n$ are derivations of this algebra, which commute $\delta_i\,\delta_j = \delta_j\,\delta_i$. Then the n-ary bracket

$$\{u_1, u_2, \ldots, u_n\} = \text{Det}\begin{pmatrix} \delta_1(u_1) & \delta_2(u_1) & \ldots & \delta_n(u_1) \\ \delta_1(u_2) & \delta_2(u_2) & \ldots & \delta_n(u_2) \\ \ldots & \ldots & \ldots & \ldots \\ \delta_1(u_n) & \delta_2(u_n) & \ldots & \delta_n(u_n) \end{pmatrix}, \tag{3.5}$$

is an n-ary Nambu-Poisson bracket. Hence, a unital commutative associative algebra $\mathcal{A}$ endowed with the n-ary Nambu-Poisson bracket (3.5) is an n-Lie algebra, which is called a *Jacobian algebra defined by* $\delta_1, \delta_2, \ldots, \delta_n$ [12].

3.2.2 Construction of 3-Lie Algebras Based on Trace

As it was mentioned before, Nambu proposed a generalization of Poisson bracket and Hamiltonian mechanics to spaces of odd dimensions and after that he tried to construct a quantum approach to this generalization of Poisson bracket. However, a problem of quantization of Nambu's generalization of Hamiltonian mechanics has proved difficult and is still considered to be unresolved. The problem can be formulated so that one has to construct an explicit quantum Nambu-Poisson bracket on an algebra of square or cubic matrices. In [9] the authors proposed several non-trivial examples of the quantum version of Nambu-Poisson bracket, constructed on an algebra of square and cubic matrices.

Let $\mathfrak{gl}(N)$ be the Lie algebra of Nth order square complex matrices. The commutator of two matrices will be denoted by $[A, B] = A\,B - B\,A$. The ternary bracket, proposed in [9], has the form

$$[A, B, C] = (\text{Tr}\,A)\,[B, C] + (\text{Tr}\,B)\,[C, A] + (\text{Tr}\,C)\,[A, B], \tag{3.6}$$

where $A, B, C \in \mathfrak{gl}(N)$. It is easy to verify that the ternary bracket (3.6) is totally skew-symmetric. Then, in [9] the authors prove that this ternary bracket satisfies the ternary Filippov-Jacobi identity. Hence, (3.6) is a ternary Lie bracket and the vector space $\mathfrak{gl}(N)$, endowed with this ternary Lie bracket, is a 3-Lie algebra. The matrix 3-Lie algebra constructed by means of the ternary Lie bracket (3.6) will be referred to as a *ternary matrix 3-Lie algebra induced by the matrix Lie algebra* $\mathfrak{gl}(N)$ *with the help of the trace*.

One can extend the above construction of the ternary Lie bracket from matrix Lie algebras to a more general class of n-Lie algebras by means of a linear function, which has a property of generalized trace [7, 11]. Indeed, assume that we have a

n-Lie algebra $\mathfrak{g}$ with an n-ary Lie bracket

$$(x_1, x_2, \ldots, x_n) \in \mathfrak{g} \times \ldots \mathfrak{g} \rightarrow [x_1, x_2, \ldots, x_n] \in \mathfrak{g}, \tag{3.7}$$

$\mathfrak{g}^*$ is the dual space of the vector space of $\mathfrak{g}$ and $\omega \in \mathfrak{g}^*$ is an element of the dual space, which, in analogy with the trace of a matrix, for any $x, y \in \mathfrak{g}$ satisfies $\omega([x, y]) = 0$. Then, it can be proved [7] that the $(n+1)$-ary bracket

$$[x_1, x_2, \ldots, x_{n+1}] = \sum_{k=1}^{n+1} (-1)^{k-1} \omega(x_k)\; [x_1, x_2, \ldots, \hat{x}_k, \ldots, x_{n+1}], \tag{3.8}$$

where hat over an element x_k means that this element is omitted, is an $(n+1)$-Lie bracket, i.e. totally skew-symmetric and satisfies the Filippov-Jacobi identity, and the vector space of $\mathfrak{g}$, endowed with (3.8), is an $(n+1)$-Lie algebra. This $(n+1)$-Lie algebra is referred to as an $(n+1)$-Lie algebra induced by an n-Lie algebra $\mathfrak{g}$ with the help of a generalized trace ω. Particularly, a Lie algebra $\mathfrak{g}$ induces the ternary Lie algebra if we endow the vector space of a Lie algebra $\mathfrak{g}$ with the following ternary Lie bracket

$$[x, y, z] = \omega(x)\; [y, z] + \omega(y)\; [z, x] + \omega(z)\; [x, y], \tag{3.9}$$

where $\omega \in \mathfrak{g}^*$ has the property of a generalized trace, i.e. $\omega([x, y]) = 0$ for any $x, y \in \mathfrak{g}$.

Thus, to construct an $(n+1)$-Lie algebra, we can use an n-Lie algebra and an element of its dual space, which has the property of a generalized trace. However, a more thorough analysis of the Filippov-Jacobi identity in the case of the ternary Lie bracket (3.9) shows that the totally skew-symmetric ternary bracket (3.9) satisfies the ternary Filippov-Jacobi identity if ω satisfies a more general equation than the property of generalized trace. In order to find this equation, we expand double ternary Lie brackets at the left and right hand side of the ternary Filippov-Jacobi identity

$$[x, y, [u, v, t]] = [[x, y, u], v, t] + [u, [x, y, v], t] + [u, v, [x, y, t]].$$

Part of the terms in the resulting expression vanish due to the skew symmetry and the Jacobi identity for the Lie bracket of a Lie algebra $\mathfrak{g}$. The remaining terms can be collected into expression by means of binary Lie brackets. For instance, all terms containing the binary Lie bracket $[x, y]$ can be collected into expression

$$(\omega(u)\, \omega([v, t]) + \omega(v)\, \omega([t, u]) + \omega(t)\, \omega([u, v]))\, [x, y]. \tag{3.10}$$

We get three more expressions of the same type if we collect all the terms containing one of the following Lie brackets: $[u, v]$, $[u, t]$, $[v, t]$. Hence the ternary bracket (3.9) will satisfy the ternary Filippov-Jacobi identity if for any $u, v, t \in \mathfrak{g}$ an element ω of the dual space $\mathfrak{g}^*$ satisfies the equation

$$\omega(u)\,\omega([v,t]) + \omega(v)\,\omega([t,u]) + \omega(t)\,\omega([u,v]) = 0. \tag{3.11}$$

Now, we can consider ω as a $\mathbb{C}$-valued 1-cochain of cochain complex of a Lie algebra $\mathfrak{g}$. Making use of the antiderivation $d : \wedge^k\mathfrak{g}^* \to \wedge^{k+1}\mathfrak{g}^*$, which corresponds to the Lie bracket of a Lie algebra $\mathfrak{g}$, we obtain the 2-cochain $d\omega$, where

$$d\omega(x, y) = -\omega([x, y]).$$

Finally, the wedge product $\omega \wedge d\omega$ is the 3-cochain, where

$$\begin{aligned}\omega \wedge d\omega(u, v, t) &= \omega(u)\,d\omega(v,t) + \omega(v)\,d\omega(t,u) + \omega(t)\,d\omega(u,v)\\ &= -\Big(\omega(u)\,\omega([v,t]) + \omega(v)\,\omega([t,u]) + \omega(t)\,\omega([u,v])\Big).\end{aligned}$$

Now, the Eq. (3.11) can be written in the form

$$\omega \wedge d\omega = 0. \tag{3.12}$$

Theorem 3.1 *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, $\mathfrak{g}^*$ be its dual space and $\omega \in \mathfrak{g}^*$. Define the ternary bracket by*

$$[x, y, z] = \omega(x)\;[y, z] + \omega(y)\;[z, x] + \omega(z)\;[x, y], \quad x, y, z \in \mathfrak{g}. \tag{3.13}$$

If ω satisfies the equation $\omega \wedge d\omega = 0$, then the ternary bracket (3.13) is a ternary Lie bracket, i.e. it satisfies the ternary Filippov-Jacobi identity, and the vector space of a Lie algebra $\mathfrak{g}$ equipped with the ternary Lie bracket (3.13) is the ternary Lie algebra.

Particularly, if ω is a 1-cocycle of Chevalley-Eilenberg complex, i.e. $d\omega = 0$, then (3.13) is a ternary Lie bracket. This is the case, when we construct ternary bracket for Nth order matrices by means of the trace of a matrix (3.6). Indeed, in this particular case ω is the trace of a matrix, i.e. we can consider the trace of a matric as an element of the dual space $\mathrm{Tr} : \mathfrak{gl}(N) \to \mathbb{C}$, and $d\,\mathrm{Tr}(x, y) = \mathrm{Tr}([x, y]) = 0$, i.e. Tr is a 1-cocycle.

3.2.3 Construction of 3-Lie Algebras Based on Derivation and Involution

In this section, we continue the description of methods for constructing 3-Lie algebras by means of the (binary) Lie bracket of a Lie algebra and additional structures defined on this algebra. The structure of the ternary Lie brackets, described in this section, is similar to the structure of the ternary Lie brackets constructed in the previous section with the help of the trace of a matrix and its analogues. As an initial Lie algebra for the construction of 3-Lie algebra, we take various Lie algebras constructed with the help of derivations and involutions of a commutative associative algebra. This section is based on the constructions of 3-Lie algebras proposed in [12], but in the present paper we propose a more general condition for the elements of the dual space of Lie algebra formulated ny means of the cochain complex of the Lie algebra and the antiderivation operator of this complex.

Let $\mathcal{A}$ be a commutative associative algebra over the field of complex numbers. A derivation δ of $\mathcal{A}$ is a linear mapping $\delta : \mathcal{A} \to \mathcal{A}$, which satisfies the Leibniz rule $\delta(xy) = \delta(x)\, y + x\, \delta(y)$, where $x, y \in \mathcal{A}$. An involution of an algebra $\mathcal{A}$ is a a mapping $x \in \mathcal{A} \mapsto x^* \in \mathcal{A}$, which satisfies the following conditions:

1. $(a\,x + y)^* = \bar{a}\,x^* + y^*$, where $a, b \in \mathbb{C}$, $x, y \in \mathcal{A}$ (antilinearity),
2. $(x^*)^* = x$, where $x \in \mathcal{A}$,
3. $(xy)^* = x^*\,y^*$.

Making use of a derivation δ and an involution $x \mapsto x^*$, one can construct the following (binary) Lie brackets on an algebra $\mathcal{A}$:

(a) $[x, y]_\delta = x\,\delta(y) - y\,\delta(x)$,
(b) $[x, y]_* = x^*\,y - y^*\,x$,
(c) $[x, y]_{*,\delta} = (x - x^*)\,\delta(y) - (y - y^*)\,\delta(x)$, where an involution and derivation satisfy the condition $(\delta(x))^* = -\delta(x^*)$ for any $x \in \mathcal{A}$.

An algebra $\mathcal{A}$ equipped with one of the Lie brackets (a), (b), (c) becomes the Lie algebra, which will be denoted by $\mathcal{A}_\delta, \mathcal{A}_*, \mathcal{A}_{*,\delta}$ respectively. Every Lie algebra $\mathcal{A}_\delta, \mathcal{A}_*, \mathcal{A}_{*,\delta}$ has its own cochain complex with operator of antiderivation, which will be denoted by $d_\delta, d_*, d_{*,\delta}$ respectively. In analogy with Theorem (3.1), we can prove the following theorem:

Theorem 3.2 *Let $\xi, \eta, \zeta \in \mathcal{A}^*$. If elements ξ, η of the dual space satisfy the equation*

$$\xi \wedge d_\delta(\xi) = 0,\ \ \eta \wedge d_*(\eta) = 0,$$

then ternary brackets

$$[x, y, z]_\xi = \xi(x)[y, z]_\delta + \xi(y)[z, x]_\delta + \xi(z)[x, y]_\delta,$$
$$[x, y, z]_\eta = \eta(x)[y, z]_* + \eta(y)[z, x]_* + \eta(z)[x, y]_*,$$

are ternary Lie brackets, i.e. they satisfy the Filippov-Jacobi identity, and an algebra $\mathcal{A}$ endowed with one of these ternary Lie brackets becomes 3-Lie algebra. If an element $\zeta \in \mathfrak{g}^*$ satisfies the equation

$$\zeta \wedge d_{*,\delta}(\zeta) = 0,$$

a derivation and an involution satisfy the relation $(\delta(x))^* = -\delta(x^*)$, then the ternary bracket

$$[x, y, z]_\zeta = \zeta(x)[y, z]_{*,\delta} + \zeta(y)[z, x]_{*,\delta} + \zeta(z)[x, y]_{*,\delta},$$

is a ternary Lie bracket and an algebra $\mathcal{A}$ equipped with this ternary Lie bracket is a 3-Lie algebra.

Another 3-Lie algebra that can be constructed with the help of the involution and the Lie bracket $[\ ,\ ]_\delta$, is defined by the following ternary bracket

$$[x, y, z]_{*,\delta} = x^*\ [y, z]_\delta + y^*\ [z, x]_\delta + z^*\ [x, y]_\delta. \tag{3.14}$$

Theorem 3.3 *If derivation δ and an involution $x \mapsto x^*$ of a commutative associative algebra $\mathcal{A}$ satisfy the relation*

$$(\delta(x))^* = -\delta(x^*),$$

then the ternary bracket (3.14) is ternary Lie bracket and an algebra $\mathcal{A}$ endowed with this ternary Lie bracket is a 3-Lie algebra.

3.3 *n*-Lie Superalgebras

As was shown in the previous section, the concept of a Lie algebra can be extended to n-ary laws of multiplication by means of a generalization of the Jacobi identity if it is written in the form of a derivation of a Lie bracket. A similar approach can be used to extend the notion of a Lie superalgebra to structures with n-ary laws of multiplication. Indeed, a definition of a Lie superalgebra is based on the graded Jacobi identity and, analogously to the ordinary Jacobi identity, the graded Jacobi identity can be written as a derivation of the graded Lie bracket. In this section, we consider the n-Lie superalgebras and various ways of constructing them. First of all, we propose two identities and prove that the graded ternary Filippov-Jacobi identity is equivalent to the system of these identities. Then, we propose a method for constructing ternary Lie superalgebras, which is based on the use of the supertrace and its analogues [1–3]. This method is similar to the method of constructing ternary Lie algebras using the trace and its analogs, described in the previous section. Next, we extend the method of constructing ternary Lie algebras with the help of involution and derivation of a commutative associative algebra to ternary Lie superalgebras and construct several ternary Lie superalgebras with the help of involution and even degree derivation of a commutative superalgebra. We apply this construction to the commutative superalgebra of functions on a superspace and construct by means of an involution and an even vector field the ternary Lie superalgebra of functions on a superspace.

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a super vector space. As usual, the elements of the subspace $\mathfrak{g}_0$ will be called even elements of $\mathfrak{g}$ and the elements of the subspace $\mathfrak{g}_1$ will be called odd elements of $\mathfrak{g}$. The degree of a homogeneous element $x \in \mathfrak{g}$ will be denoted by $|x|$. Thus, the value of a degree is a residue class modulo 2, i.e. $|x| \in \mathbb{Z}_2$.

Definition 3.3 An *n-Lie superalgebra* $\mathfrak{g}$ is a super vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ endowed with a n-ary bracket $(x_1, x_2, \ldots, x_n) \in \mathfrak{g} \times \mathfrak{g} \times \ldots \times \mathfrak{g} \mapsto [x_1, x_2, \ldots, x_n]_{\mathrm{gr}} \in \mathfrak{g}$, which satisfies the following conditions:

1. An n-ary bracket is linear in every argument of this bracket.
2. The degree of a bracket of n elements is equal to the sum of degrees of factors,

$$|[x_1, x_2, \ldots, x_n]_{\mathrm{gr}}| = \sum_{i=1}^{n} |x_i|$$

3. $[x_{i_1}, x_{i_2}, \ldots, x_{i_n}]_{\mathrm{gr}} = -(-1)^{\sigma}\ [x_1, x_2, \ldots, x_n]_{\mathrm{gr}}$, where $i_1, i_2, \ldots, i_n$ is a permutation of integers $1, 2, \ldots, n$ and σ is its parity. This property is called a graded skew-symmetry of n-ary bracket.
4.
$$[x_1, \ldots, x_{n-1}, [y_1, \ldots, y_n]_{\mathrm{gr}}]_{\mathrm{gr}} = \sum_{i=1}^{n} (-1)^{\mu_i} [y_1, \ldots, [x_1, \ldots, x_{n-1}, y_i]_{\mathrm{gr}}, \ldots, y_n]_{\mathrm{gr}}, \tag{3.15}$$

   where $\mu_1 = 0$ and $\mu_i = \sum_{k=1}^{n-1} |x_k| \sum_{l=1}^{i-1} |y_l|$ for $i > 1$.

An n-ary bracket, which satisfies the conditions 1 - 4, will be referred to as an *n-ary graded Lie bracket* and the identity (3.15) will be referred to as the *n-ary graded Filippov-Jacobi identity*.

This definition gives the definition of a Lie superalgebra, if we take $n = 2$. Indeed, according to Definition 3.3, in this case we will have binary bracket

$$(x, y) \in \mathfrak{g} \times \mathfrak{g} \mapsto [x, y]_{\mathrm{gr}} \in \mathfrak{g},$$

which is graded skew-symmetric, i.e. $[x, y]_{\mathrm{gr}} = -(-1)^{|x|\,|y|}[y, x]_{\mathrm{gr}}$, and satisfies the graded Jacobi identity

$$[x, [y, z]_{\mathrm{gr}}]_{\mathrm{gr}} = [[x, y]_{\mathrm{gr}}, z]_{\mathrm{gr}} + (-1)^{|x|\,|y|}[y, [x, z]_{\mathrm{gr}}]_{\mathrm{gr}}.$$

The next value of integer n is 3. In this case an n-Lie superalgebra $\mathfrak{g}$ will be called a *ternary Lie superalgebra* or *3-Lie superalgebra*. Since ternary Lie superalgebras play an important role in the present paper, we give a complete definition of such an algebra, which, of course, is a special case of Definition 3.3.

Definition 3.4 A *ternary Lie superalgebra* or *3-Lie superalgebra* is a super vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ endowed with a ternary brackets,

$$(x, y, z) \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \mapsto [x, y, z]_{\mathrm{gr}} \in \mathfrak{g},$$

which is trilinear, graded skew-symmetric, i.e.

$$[x,y,z]_{\rm gr} = -(-1)^{|x|\,|y|}[y,x,z]_{\rm gr}, \quad [x,y,z]_{\rm gr} = -(-1)^{|y|\,|z|}[x,z,y]_{\rm gr},$$

satisfies the condition $|[x, y, z]_{\rm gr}| = |x| + |y| + |z|$ and the ternary graded Filippov-Jacobi identity

$$[x,y,[u,v,w]_{\rm gr}]_{\rm gr} = [[x,y,u]_{\rm gr},v,w]_{\rm gr} + (-1)^{(|x|+|y|)\,|u|}\,[u,[x,y,v]_{\rm gr},w]_{\rm gr} + (-1)^{(|x|+|y|)(|u|+|v|)}\,[u,v,[x,y,w]_{\rm gr}]_{\rm gr}. \quad (3.16)$$

Proposition 3.2 *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{q}_1$ be a super vector space endowed with a ternary bracket*

$$(x,y,z) \in \mathfrak{g}\times\mathfrak{g}\times\mathfrak{g} \mapsto [x,y,z]_{gr} \in \mathfrak{g}, \quad (3.17)$$

which is a trilinear, graded skew-symmetric and satisfies $|[x, y, z]_{gr}| = |x| + |y| + |z|$. Then the ternary bracket (3.17) is a ternary graded Lie bracket, and a super vector space $\mathfrak{g}$ endowed with this ternary bracket is a ternary Lie superalgebra if and only if the ternary bracket (3.17) satisfies two identities

$$[[x,y,z]_{gr},u,v]_{gr} = (-1)^{\alpha}[[u,y,z]_{gr},x,v]_{gr} + (-1)^{\beta}[[x,u,z]_{gr},y,v]_{gr} + (-1)^{\gamma}[[x,y,u]_{gr},z,v]_{gr}, \quad (3.18)$$

where $\alpha = |u|(|x| + |y| + |z|) + |x|(|y| + |z|)$, $\beta = |u|(|y| + |z|) + |y||z|$, $\gamma = |u||z|$, and

$$[[x,y,z],u,v] + (-1)^{\mu}[[u,v,z],x,y] - (-1)^{\nu}[[x,u,z],y,v] - (-1)^{\lambda}[[v,y,z],u,x] - (-1)^{\rho}[[x,v,z],u,y] - (-1)^{\kappa}[[u,y,z],x,v] = 0, \quad (3.19)$$

where

$$\begin{aligned}
\mu &= (|x| + |y|)(|u| + |v|),\\
\nu &= |y||u| + |y||z| + |z||u|,\\
\lambda &= |v|(|y| + |z| + |u|) + |x|(|y| + |z| + |u|) + |x||v|,\\
\rho &= (|v| + |y|)(|z| + |u|) + |v||y|,\\
\kappa &= (|x| + |u|)(|y| + |z|) + |x||u|.
\end{aligned}$$

Proof Let us assume that (3.17) is a graded ternary Lie bracket, i.e. it satisfies the graded Filippov-Jacobi identity. Thus, we have

$$[x,y,[z,u,v]] = [[x,y,z],u,v] + (-1)^{(|x|+|y|)\,|z|}[z,[x,y,u],v] + (-1)^{(|x|+|y|)(|z|+|u|)}[z,u,[x,y,v]]. \quad (3.20)$$

Now, applying the graded Filippov-Jacobi identity to the last term at the right hand side of the above relation, we can write it as follows

$$\begin{aligned}\cancel{[x,y,[z,u,v]]} &= [[x,y,z],u,v] + (-1)^{(|x|+|y|)\,|z|}[z,[x,y,u],v] \\ &\quad +(-1)^{(|x|+|y|)(|z|+|u|)}\Big([[z,u,x],y,v] + (-1)^{|x|(|z|+|u|)}[x,[z,u,y],v] \\ &\qquad +(-1)^{(|z|+|u|)(|x|+|y|)}[x,y,[z,u,v]]\Big) \\ &= [[x,y,z],u,v] + (-1)^{(|x|+|y|)\,|z|}[z,[x,y,u],v] \\ &\quad +(-1)^{(|x|+|y|)(|z|+|u|)}[[z,u,x],y,v] + (-1)^{|y|(|z|+|u|)}[x,[z,u,y],v] \\ &\quad +\cancel{[x,y,[z,u,v]]}. \end{aligned} \tag{3.21}$$

Now, we interchange z and $[x, y, v]$ in the second term at the right hand side of the above expression. When performing this operation, we must multiply this term by $(-1)^{|z|(|x|+|y|+|u|)}$. As a result, this term will have a factor (-1) to power

$$|z|(|x|+|y|+|u|) + |z|(|x|+|y|) = |z||u|.$$

Thus

$$(-1)^{(|x|+|y|)\,|z|}[z,[x,y,u],v] = -(-1)^{|z||u|}[[x,y,u],z,v]. \tag{3.22}$$

Rearranging similarly the factors inside the brackets of the third and fourth terms at the right hand side of (3.21), we obtain

$$\begin{aligned}(-1)^{(|x|+|y|)(|z|+|u|)}[[z,u,x],y,v] &= -(-1)^{|y||u|+|z||u|+|y||z|}[[x,u,z],y,v], \\ (-1)^{|y|(|z|+|u|)}[x,[z,u,y],v] &= -(-1)^{|u|(|x|+|y|+|z|)+|x|(|y|+|z|)}[[u,y,z],x,v].\end{aligned}$$

Now, the relation (3.21) can be written in the form

$$\begin{aligned}[[x,y,z],u,v] &= (-1)^{|u|(|x|+|y|+|z|)+|x|(|y|+|z|)}[[u,y,z],x,v] \\ &\quad +(-1)^{|y||u|+|z||u|+|y||z|}[[x,u,z],y,v] + (-1)^{|z||u|}[[x,y,u],z,v],\end{aligned}$$

and this is the first identity (3.18). Analogously, the second identity (3.19) can be proved if we apply the graded Filippov-Jacobi identity (3.20) to the first and the second terms at the right hand side of (3.20).

3.3.1 Matrix 3-Lie Superalgebras Constructed by Means of Supertrace

In this section we show that the method of constructing 3-Lie algebras by means of the trace proposed in [9] and developed in the series of papers can be extended to 3-Lie superalgebras if we use the supertrace of a matrix [1–3]. It will be shown that given a matrix Lie superalgebra $\mathfrak{gl}(m,n)$ we can construct a ternary graded skew-symmetric bracket with the help of graded commutator of two matrices $X, Y \in \mathfrak{gl}(m,n)$ and the supertrace and prove that this ternary graded skew-symmetric bracket satisfies the graded ternary Filippov-Jacobi identity (3.16) and, hence, this ternary graded skew-symmetric bracket is a ternary graded Lie bracket.

An element of a matrix Lie superalgebra $\mathfrak{gl}(m,n)$ is a $(m+n)$ order square block matrix

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \tag{3.23}$$

where X_{11} is a mth order square matrix, X_{12} is an $m \times n$-dimensional matrix, X_{21} is a $n \times m$-dimensional matrix and X_{22} is a nth order square matrix. The vector space of these block matrices becomes a super vector space if the matrices with $X_{12} = X_{21} = 0$ will be given the degree zero and called even degree matrices and the matrices with $X_{11} = X_{22} = 0$ will be given the degree one and called odd degree matrices. If a matrix has certain degree, then, as usual in the supermathematics, it will be referred to as a homogeneous matrix. The degree of a matrix X will be denoted by $|X|$. In order to simplify notations, in this subsection we will also denote the degree of a matrix by the same, but small letter. For instance, the degree of a matrix X will be denoted by x. We will also denote the sum of two degrees x, y by $\overline{xy}$, i.e. $\overline{xy} = x + y$. The Lie superalgebra structure of $\mathfrak{gl}(m,n)$ is determined by the graded commutator of two matrices

$$[X, Y]_{\text{gr}} = X \cdot Y - (-1)^{xy}\, Y \cdot X, \tag{3.24}$$

which satisfies the graded Jacobi identity

$$[X, [Y, Z]_{\text{gr}}]_{\text{gr}} = [[X, Y]_{\text{gr}}, Z]_{\text{gr}} + (-1)^{xy}[Y, [X, Z]_{\text{gr}}]_{\text{gr}}.$$

If one of matrices (or both) is of even degree matrix, then the graded commutator is the usual commutator of two matrices, which will be denoted by $[X, Y] = X \cdot Y - Y \cdot X$. If X, Y are odd degree matrices, then the graded commutator becomes the anti-commutator which will be denoted by $\{X, Y\} = X \cdot Y + Y \cdot X$. We will denote the subspace of even degree matrices by $\mathfrak{gl}_0(m,n)$, and the subspace of odd degree matrices by $\mathfrak{gl}_1(m,n)$. Then $\mathfrak{g} = \mathfrak{gl}_0(m,n) \oplus \mathfrak{gl}_1(m,n)$. The supertrace of a matrix (3.23) is given by the formula

$$\text{Str}\, X = \text{Tr}\, X_{11} - \text{Tr}\, X_{22}. \tag{3.25}$$

The supertrace vanishes in the case of the graded commutator of two matrices

$$\text{Str}\,([X, Y]_{\text{gr}}) = 0. \tag{3.26}$$

From the formula for the supertrace it follows immediately that the supertrace of an odd degree matrix is zero, i.e. $\mathrm{Str}\, X = 0$ for any $X \in \mathfrak{gl}_1(m, n)$.

We define the ternary bracket of three matrices $X, Y, Z \in \mathfrak{g}l(m, n)$ by the formula

$$[X, Y, Z]_{\mathrm{gr}} = (\mathrm{Str}\, X)\ [Y, Z]_{\mathrm{gr}} + (-1)^{\overline{yz}\,x}(\mathrm{Str}\, Y)\ [Z, X]_{\mathrm{gr}} + (-1)^{z\,\overline{xy}}(\mathrm{Str}\, Z)\ [X, Y]_{\mathrm{gr}}. \tag{3.27}$$

First, we will show that in this way defined ternary bracket is a graded ternary bracket, i.e. for homogeneous matrices (of certain degree) it satisfies

$$\Big|[X, Y, Z]_{\mathrm{gr}}\Big| = x + y + z, \tag{3.28}$$

where x, y, z are degrees of matrices X, Y, Z respectively. If all three matrices X, Y, Z are even degree matrices, then

$$[X, Y, Z]_{\mathrm{gr}} = (\mathrm{Str}\, X)\ [Y, Z] + (\mathrm{Str}\, Y)\ [Z, X] + (\mathrm{Str}\, Z)\ [X, Y], \tag{3.29}$$

and, in this case, as it is easy to see, $[X, Y, Z]_{\mathrm{gr}}$ is the even degree matrix and the formula (3.28) is true. It is worth to note that in the case of even degree matrices the ternary bracket (3.29) is similar to the ternary Lie bracket (3.6) proposed in the paper [9], but it is not exactly the same if the block X_{22} in a matrix (3.23) is non trivial.

For other possible combinations of parities of matrices X, Y, Z, the formula (3.6) gives the following expressions for ternary bracket:

1. X is an odd degree and Y, Z are even degree matrices

$$[X, Y, Z]_{\mathrm{gr}} = (\mathrm{Str}\, Y)\ [Z, X] - (\mathrm{Str}\, Z)\ [Y, X]. \tag{3.30}$$

From the above formula, it follows that $[X, Y, Z]_{\mathrm{gr}}$ is the odd degree matrix, because the right hand side is the linear combination of two odd degree matrices $[Z, X]$ and $[Y, X]$. This is consistent with formula (3.28), which also gives the odd degree for $[X, Y, Z]_{\mathrm{gr}}$ ($x + y + z = 1 + 0 + 0 = 1$). It is interesting to note that the even degree term $(\mathrm{Str}\, X)\ [Y, Z]$, which would violate the consistency of the degrees of left and right hand sides in (3.6), vanishes due to the fact that the supertrace of an odd degree matrix X is 0.

2. X, Y are odd and Z is even degree matrix

$$[X, Y, Z]_{\mathrm{gr}} = \mathrm{Str}\, Z\ \{X, Y\}. \tag{3.31}$$

The degree of the matrix at the right hand side is even (anti-commutator of two odd degree matrices has even degree) and this is consistent with the formula (3.28), which gives for the left hand side of (3.31) $1 + 1 + 0 = 0$.

3. X, Y, Z are odd degree matrices

$$[X, Y, Z]_{\mathrm{gr}} = 0.$$

In this case the formula (3.28) is also true, because zero matrix can be given either even degree or odd degree.

Hence, (3.27) is a graded ternary bracket.

Next, we will show that (3.27) is a ternary graded skew-symmetric bracket. Indeed, interchanging matrices X, Y in the ternary graded bracket (3.27), we get

$$[Y, X, Z]_{gr} = (\mathrm{Str}\, Y)\ [X, Z]_{gr} + (-1)^{\overline{xz}\, y}(\mathrm{Str}\, X)\ [Z, Y]_{gr} + (-1)^{z\, \overline{xy}}(\mathrm{Str}\, Z)\ [Y, X]_{gr}. \tag{3.32}$$

Making use of the graded skew-symmetry of graded (binary) commutator, every term at the right-hand side of (3.32) can be written as follows:

$$\begin{aligned}
(-1)^{\overline{xz}\, y}(\mathrm{Str}\, X)\ [Z, Y]_{gr} &= -(-1)^{xy}\ (\mathrm{Str}\, X)\ [Y, Z]_{gr},\\
(\mathrm{Str}\, Y)\ [X, Z]_{gr} &= -(-1)^{xy}\big((-1)^{x\, \overline{yz}}(\mathrm{Str}\, Y)\ [Z, X]_{gr}\big),\\
(-1)^{z\, \overline{xy}}(\mathrm{Str}\, Z)\ [Y, X]_{gr} &= -(-1)^{xy}\big((-1)^{z\, \overline{xy}}(\mathrm{Str}\, Z)\ [X, Y]_{gr}\big).
\end{aligned}$$

Substituting the right-hand sides of these relations into (3.32) we obtain

$$[X, Y, Z]_{gr} = -(-1)^{xy}[Y, X, Z]_{gr},$$

and this proves the graded skew-symmetry of (3.27) for X, Y. Analogously we can verify the graded skew-symmetry for Y, Z.

Theorem 3.4 *A matrix Lie superalgebra $\mathfrak{gl}(m, n)$ endowed with the ternary graded bracket (3.27) is a 3-Lie superalgebra, i.e. the ternary graded bracket (3.27) satisfies the ternary graded Filippov-Jacobi identity*

$$\begin{aligned}
[X, Y, [Z, V, W]_{gr}]_{gr} = [[X, Y, Z]_{gr}, V, W]_{gr} + (-1)^{\overline{xy}\, z}[Z, [X, Y, V]_{gr}, W]_{gr}\\
+(-1)^{\overline{xy}\ \overline{zv}}[Z, V, [X, Y, W]_{gr}]_{gr}.
\end{aligned} \tag{3.33}$$

Thus, (3.27) is a ternary graded Lie bracket.

Proof In order to prove this theorem let us denote

$$A = [Z, V, W]_{gr}, \quad B = [X, Y, Z]_{gr}, C = [X, Y, V]_{gr}, \quad D = [X, Y, W]_{gr}.$$

Then, making use of these notations, we can write the ternary graded Filippov-Jacobi identity (3.33) in the form

$$[X, Y, A]_{gr} = [B, V, W]_{gr} + (-1)^{\overline{xy}\, z}[Z, C, W]_{gr} + (-1)^{\overline{xy}\ \overline{zv}}[Z, V, D]_{gr}. \tag{3.34}$$

We begin with the left hand side of ternary graded Filippov-Jacobi identity. The ternary graded bracket A is the linear combination of graded binary commutators

$$A = (\mathrm{Str}\, Z)\, [V, W]_{\mathrm{gr}} + (-1)^{z\, \overline{vw}} (\mathrm{Str}\, V)\, [W, Z]_{\mathrm{gr}} + (-1)^{w\, \overline{zv}} (\mathrm{Str}\, W)\, [Z, V]_{\mathrm{gr}}.$$

Because the supertrace vanishes on graded commutators, we have $\mathrm{Str}\, A = 0$. Thus the left hand side of the ternary graded Filippov-Jacobi identity (3.34) takes the form

$$\begin{aligned}
[X, Y, A]_{\mathrm{gr}} = (\mathrm{Str}\, X)\, [Y, A]_{\mathrm{gr}} + (-1)^{x\, \overline{ya}} (\mathrm{Str}\, Y)\, [A, X]_{\mathrm{gr}} = \\
= (\mathrm{Str}\, X)\, (\mathrm{Str}\, Z)\, [Y, [Z, W]_{\mathrm{gr}}]_{\mathrm{gr}} + \text{g.c.p.}(Z, V, W) \\
+ (-1)^{x\, \overline{ya}} \big((\mathrm{Str}\, Y)\, (\mathrm{Str}\, Z)\, [[Z, W]_{\mathrm{gr}}, X]_{\mathrm{gr}} \\
+ \text{g.c.p.}(Z, V, W) \big),
\end{aligned} \tag{3.35}$$

where a is the degree of the matrix A, and "g.c.p.(Z, V, W)" (graded cyclic permutations) means that the term $(\mathrm{Str}\, X)\, (\mathrm{Str}\, Z)\, [Y, [Z, W]_{\mathrm{gr}}]_{\mathrm{gr}}$ is followed by two more terms of the same kind, in which the arguments Z, V, W are cyclically permuted, i.e. $(Z, V, W) \to (V, W, Z)$ and $(Z, V, W) \to (W, Z, V)$ and multiplied by $(-1)^{z\, \overline{vw}}$ and $(-1)^{w\, \overline{zv}}$ respectively. Thus there are six terms at the left-hand side of the graded Filippov-Jacobi identity (3.34).

Analogously $\mathrm{Str}\, B = \mathrm{Str}\, C = \mathrm{Str}\, D = 0$, and the terms at the right hand side of the ternary graded Filippov-Jacobi identity (3.34) can be written as

$$\begin{aligned}
[B, V, W]_{\mathrm{gr}} &= (-1)^{b\, \overline{vw}} \big((\mathrm{Str}\, V)\, (\mathrm{Str}\, X)\, [W, [Y, Z]_{\mathrm{gr}}]_{\mathrm{gr}} + \text{g.c.p.}(X, Y, Z) \big) \\
&\quad + (-1)^{w\, \overline{bv}} \big((\mathrm{Str}\, W)\, (\mathrm{Str}\, X)\, [[Y, Z]_{\mathrm{gr}}, V]_{\mathrm{gr}} + \text{g.c.p.}(X, Y, Z) \big), \\
[Z, C, W]_{\mathrm{gr}} &= (\mathrm{Str}\, Z)\, (\mathrm{Str}\, X)\, [[Y, V]_{\mathrm{gr}}, W]_{\mathrm{gr}} + \text{g.c.p.}(X, Y, V) \\
&\quad + (-1)^{w\, \overline{cz}} \big((\mathrm{Str}\, W)\, (\mathrm{Str}\, X)\, [Z, [Y, V]_{\mathrm{gr}}]_{\mathrm{gr}} + \text{g.c.p.}(X, Y, V) \big), \\
[Z, C, W]_{\mathrm{gr}} &= (\mathrm{Str}\, Z)\, (\mathrm{Str}\, X)\, [V, [Y, W]_{\mathrm{gr}}]_{\mathrm{gr}} + \text{g.c.p.}(X, Y, W) \\
&\quad + (-1)^{z\, \overline{vd}} \big((\mathrm{Str}\, V)\, (\mathrm{Str}\, X)\, [[Y, W]_{\mathrm{gr}}, Z]_{\mathrm{gr}} + \text{g.c.p.}(X, Y, W) \big).
\end{aligned}$$

Thus there are totally 18 terms at the right-hand side of the ternary graded Filippov-Jacobi identity.

Now, let us consider the first term in (3.35)

$$\mathrm{Str}\, X\, \mathrm{Str}\, Z\, [Y, [Z, W]], \tag{3.36}$$

which is the first term of the left hand side of the ternary graded Filippov-Jacobi identity. There are two similar terms at the right hand side of the ternary graded Filippov-Jacobi identity, and they are the first terms in the expressions for $[B, V, W]$ and $[Z, C, W]$ multiplied by corresponding coefficients shown in (3.34))

$$(\mathrm{Str}\, Z)\, (\mathrm{Str}\, X)\, \big((-1)^{z\, \overline{xy}} [[Y, V]_{\mathrm{gr}}, W]_{\mathrm{gr}} + (-1)^{\overline{zv}\, \overline{xy}} [V, [Y, W]_{\mathrm{gr}}]_{\mathrm{gr}} \big). \tag{3.37}$$

The terms (3.36), (3.37) vanish if at least one of the matrices X, Z is odd degree matrix, because the supertrace of a matrix of odd degree is zero. Hence, what is remained is the case, when both matrices X, Z are matrices of even degree, that is $x = z = 0$. But in this case the expression (3.37) takes the form

$$(\mathrm{Str}\,Z)\,(\mathrm{Str}\,X)\left([[Y,V]_{\mathrm{gr}},W]_{\mathrm{gr}}+(-1)^{vy}[V,[Y,W]_{\mathrm{gr}}]_{\mathrm{gr}}\right). \tag{3.38}$$

We see that the term (3.36) at the left hand side of the ternary Filippov=Jacobi identity is canceled by two terms (3.38) of the right hand side of the identity, because all together they form the usual graded Jacobi identity. Analogously, it can be shown that all the rest five terms at the left hand side of the identity (3.35) are canceled by the corresponding terms of the right hand side of the identity.

Now, after these cancellations there are no more non-trivial terms at the left-hand side of the identity, and at the right-hand side there are $18-12=6$ terms. These terms can be split into pairs such that they cancel each other. For instance, in the first half of the expression $[B, V, W]$ there is the term

$$(-1)^{b\,\overline{vw}+z\,\overline{xy}}\big((\mathrm{Str}\,V)\,(\mathrm{Str}\,Z)\,[W,[X,Y]_{\mathrm{gr}}]_{\mathrm{gr}}, \tag{3.39}$$

where $b=x+y+z$, determined by the permutation $(X,Y,Z)\to(Z,X,Y)$. This term is non-trivial only when $v=z=0$, i. e.

$$(-1)^{w\,\overline{xy}}\big((\mathrm{Str}\,V)\,(\mathrm{Str}\,Z)\,[W,[X,Y]_{\mathrm{gr}}]_{\mathrm{gr}}. \tag{3.40}$$

The expression $[Z, C, W]_{\mathrm{gr}}$, accordingly to (3.34) multiplied by $(-1)^{z\,\overline{xy}}$, has the similar term

$$(-1)^{\overline{vz}\,\overline{xy}}(\mathrm{Str}\,V)\,(\mathrm{Str}\,Z)\,[[X,Y]_{\mathrm{gr}},W]_{\mathrm{gr}},$$

which is determined by the cyclic permutation $(X,Y,V)\to(V,X,Y)$ in the first half of the expression $[Z,C,W]_{\mathrm{gr}}$. Because this term is non-trivial only in the case $v=z=0$, we can write it as follows

$$(\mathrm{Str}\,V)\,(\mathrm{Str}\,Z)\,[[X,Y]_{\mathrm{gr}},W]_{\mathrm{gr}}=-(-1)^{w\,\overline{xy}}(\mathrm{Str}\,V)\,(\mathrm{Str}\,Z)\,[W,[X,Y]_{\mathrm{gr}}]_{\mathrm{gr}}. \tag{3.41}$$

Now, it is evident that the sum of two terms (3.40), (3.41) is zero, and this ends the proof.

This theorem can be formulated in a more general form if we use a notion of a generalized supertrace. Let $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ be a Lie superalgebra with graded Lie bracket denoted by $(x,y)\in\mathfrak{g}\times\mathfrak{g}\mapsto[x,y]_{\mathrm{gr}}\in\mathfrak{g}$. Let $\omega\in\mathfrak{g}^*$ is an element of the dual space. Define the trilinear function S_ω by

$$S_\omega(x,y,z)=\omega(x)\omega([y,z]_{\mathrm{gr}})+(-1)^{|x|(|y|+|z|)}\omega(y)\omega([z,x]_{\mathrm{gr}})$$
$$+(-1)^{|z|(|x|+|y|)}\omega(z)\omega([x,y]_{\mathrm{gr}}). \tag{3.42}$$

An element $\omega\in\mathfrak{g}^*$ is said to be a *generalized supertrace* of a Lie superalgebra $\mathfrak{g}$ if it satisfies the conditions:

1. $S_\omega(x,y,z)=0$, for any $x,y,z\in\mathfrak{g}$,
2. $\omega(x)=0$, for any odd degree element x, i.e. $x\in\mathfrak{g}_1$.

Evidently, the supertrace of a matrix satisfies the conditions of a generalized supertrace.

Theorem 3.5 *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and ω be its generalized supertrace. Then the ternary graded bracket*

$$[x, y, z]_\omega = \omega(x)\,[y, z]_{gr} + (-1)^{|x|(|y|+|z|)}\omega(y)\,[z, x]_{gr} + (-1)^{|z|(|x|+|y|)}\omega(z)\,[x, y]_{gr}, \tag{3.43}$$

is a ternary graded Lie bracket, and a Lie superalgebra $\mathfrak{g}$ endowed with the ternary graded Lie bracket (3.43) is a ternary Lie superalgebra.

3.3.2 Construction of 3-Lie Superalgebras Based on Derivation and Involution

Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a superalgebra. As before, the degree of a homogeneous element $u \in \mathcal{A}$ will be denoted by $|u|$. A superalgebra $\mathcal{A}$ is said to be commutative superalgebra if for any two homogeneous elements $u, v \in \mathcal{A}$ it holds $uv = (-1)^{|u||v|}vu$. A degree m derivation of a superalgebra $\mathcal{A}$, where m is either 0 (even degree derivation) or 1 (odd degree derivation), is a linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ such that $|\delta(u)| = |u| + m$ and it satisfies the graded Leibniz rule

$$\delta(uv) = \delta(u)\,v + (-1)^{m|u|}u\,\delta(v). \tag{3.44}$$

The degree of a derivation δ will be denoted by $|\delta|$. Hence if δ is an even degree derivation of a superalgebra $\mathcal{A}$, then $|\delta(u)|=|u|$, i.e. δ does not change the degree of a homogeneous element u, and for any two elements $u, v \in \mathcal{A}$ it satisfies the Leibniz rule

$$\delta(uv) = \delta(u)\,v + u\,\delta(v). \tag{3.45}$$

A mapping $* : u \in \mathcal{A} \mapsto u^* \in \mathcal{A}$ is said to be a superinvolution of a superalgebra $\mathcal{A}$ if it satisfies the following conditions:

1. a mapping $* : u \in \mathcal{A} \mapsto u^* \in \mathcal{A}$ is even degree mapping of a superalgebra $\mathcal{A}$, i.e. $* : u \in \mathcal{A}_0 \mapsto u^* \in \mathcal{A}_0$, $* : u \in \mathcal{A}_1 \mapsto u^* \in \mathcal{A}_1$ or $|u^*| = |u|$,
2. $(\lambda\, u + v)^* = \bar{\lambda}\, u^* + v^*$, $\lambda \in \mathbb{C}$, $u, v \in \mathcal{A}$, superinvolution is anti-linear,
3. $(u^*)^* = u$,
4. $(uv)^* = (-1)^{|u||v|}v^*u^*$.

In the case of a commutative superalgebra with superinvolution $*$ the fourth condition takes the form $(uv)^* = u^*v^*$. Any element x of superalgebra $\mathcal{A}$ with superinvolution $\mathcal{A}$ can be written in the form $x = x_1 + x_{-1}$, where $x_1^* = x_1,\ x_{-1}^* = -x_{-1}$ and

$$x_1 = \frac{1}{2}(x + x^*),\ x_{-1} = \frac{1}{2}(x - x^*). \tag{3.46}$$

It is worth to mention that the components x_1, x_{-1} have the same degree as x, i.e. $|x_1| = |x_{-1}| = |x|$.

A superinvolution and odd degree derivation can be used to construct binary graded Lie brackets on a superalgebra $\mathcal{A}$. Let us define

$$[u, v]_* = u^* v - (-1)^{|u||v|} v^* u, \tag{3.47}$$

$$[u, v]_\delta = u\,\delta(v) - (-1)^{|u||v|} v\,\delta(u), \tag{3.48}$$

$$[u, v]_{*,\delta} = (u - u^*)\,\delta(v) - (-1)^{|u||v|} (v - v^*)\delta(u). \tag{3.49}$$

We remind that a binary graded bracket is called a graded Lie bracket if it satisfies the graded Jacobi identity.

Lemma 3.1 *The binary graded brackets* (3.47)*,* (3.48) *are graded Lie brackets. The third graded bracket* (3.49) *is a graded Lie bracket if a superinvolution and an even degree derivation satisfy the condition* $(\delta(u))^* = -\delta(u^*)$.

Hence, every binary graded Lie bracket (3.47), (3.48), (3.49) determines the Lie superalgebra structure on a superalgebra $\mathcal{A}$. These Lie superalgebras will be denoted $\mathcal{A}_*, \mathcal{A}_\delta, \mathcal{A}_{*,\delta}$ respectively. It is should be mentioned that in the case of the latter Lie superalgebra we assume the condition $(\delta(u))^* = -\delta(u^*)$ for a superinvolution and even degree derivation to be satisfied.

Now, making use of the binary graded Lie brackets (3.47), (3.48), (3.49) and of a generalized supertrace for corresponding graded Lie brackets, we can construct ternary graded Lie brackets.

Theorem 3.6 *Let* $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ *be a commutative superalgebra,* $*$ *be its superinvolution and* δ *be its even degree derivation. If* ξ, η, χ *are generalized supertraces for Lie superalgebras* $\mathcal{A}_*$*,* $\mathcal{A}_\delta$*,* $\mathcal{A}_{*,\delta}$ *respectively, then the following ternary graded brackets*

$$[x, y, z]_* = \xi(x)\,[y, z]_* + (-1)^{|x|(|y|+|z|)}\xi(y)\,[z, x]_* + (-1)^{|z|(|x|+|y|)}\xi(z)\,[x, y]_*, \tag{3.50}$$

$$[x, y, z]_\delta = \eta(x)\,[y, z]_\delta + (-1)^{|x|(|y|+|z|)}\eta(y)\,[z, x]_\delta + (-1)^{|z|(|x|+|y|)}\eta(z)\,[x, y]_\delta, \tag{3.51}$$

$$[x, y, z]_{*,\delta} = \chi(x)\,[y, z]_{*,\delta} + (-1)^{|x|(|y|+|z|)}\chi(y)\,[z, x]_{*,\delta} + (-1)^{|z|(|x|+|y|)}\chi(z)\,[x, y]_{*,\delta}. \tag{3.52}$$

are ternary graded Lie brackets, i.e. they satisfy the ternary graded Filippov-Jacobi identity. Hence, the Lie superalgebras $\mathcal{A}_*, \mathcal{A}_\delta, \mathcal{A}_{*,\delta}$ *equipped with the ternary graded Lie brackets* (3.50)*,* (3.51)*,* (3.52) *are ternary Lie superalgebras.*

It is easy to see that all three ternary graded brackets (3.50), (3.51), (3.52) have the same structure as the ternary graded Lie bracket described in Theorem 3.5. Therefore, this theorem easily follows from Theorem 3.5.

Theorem 3.7 *Let* $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ *be a commutative superalgebra with superinvolution* $* : \mathcal{A} \to \mathcal{A}$ *and* δ *be an even degree derivation of* $\mathcal{A}$*. If for any* $u \in \mathcal{A}$ *a* *superinvolution and even degree derivation satisfy the condition* $(\delta(u))^* = -\delta(u^*)$*, then the ternary graded bracket*

$$[u, v, w]_{gr} = u^* [v, w]_\delta + (-1)^{|u|(|v|+|w|)} v^* [w, u]_\delta + (-1)^{|w|(|u|+|v|)} w^* [u, v]_\delta, \tag{3.53}$$

is a ternary graded Lie bracket and a superalgebra $\mathcal{A}$ *endowed with this ternary graded Lie bracket is a ternary Lie superalgebra.*

First of all, it should be noted that the proof of this theorem cannot be based on the statement of Theorem 3.5, because there is a significant difference in the structure of ternary brackets used in these theorems. Indeed, the ternary graded Lie bracket in Theorem 3.5 is constructed by means of a generalized supertrace ω, while the ternary graded bracket (3.53) is constructed by means of superinvolution. The proof of this theorem in the case of commutative algebra, proposed in [12], cannot also be automatically transferred to the case of a superalgebra, since in this case the order of the factors in a product plays a significant role due to the appearance of the factor -1, depending on the degrees of the elements. We checked the ternary graded Filippov-Jacobi identity for the ternary graded bracket (3.53) with the help of a computer program using noncommutative symbolic calculus. For this computer program we derived the formulae, which do not contain the factor -1, depending on the degrees of elements, but in this case the ordering of elements is essential. For instance, the ternary graded bracket (3.53) can be written in the form

$$[u, v, w]_{\mathrm{gr}} = u^* v\, \delta(w) - u^* \delta(v)\, w + \delta(u)\, v^* w - u\, v^* \delta(w) + u\, \delta(v)\, w^* - \delta(u)\, v\, w^*,$$

and our computer program expanded ternary graded brackets by means of this formula. One more useful formula, which we used in computer program, is

$$([u, v]_\delta)^* = [v^*, w^*]_\delta.$$

We can apply Theorem 3.7 to construct a ternary infinite dimensional Lie superalgebra of functions on a superspace. Let $\mathbb{R}^{n,2}$ be a superspace with n even degree coordinates $x^\mu, \mu = 1, 2, \ldots, n$ and two odd degree coordinates $\theta, \bar{\theta}$. We denote by $C^\infty(\mathbb{R}^{n,2})$ the superalgebra of smooth complex-valued functions on a superspace $\mathbb{R}^{n,2}$. This superalgebra is commutative superalgebra. A function $F(x, \theta, \bar{\theta})$ can be expanded in odd degree coordinates

$$F(x, \theta, \bar{\theta}) = F_0(x) + F_{10}(x)\, \theta + F_{01}(x)\, \bar{\theta} + F_{11}(x)\, \theta\bar{\theta}.$$

The degree of a homogeneous function F will be denoted by $|F|$. We endow this commutative superalgebra with superinvolution $F \mapsto F^*$, which is defined as follows:

$$F^*(x, \theta, \bar{\theta}) = \bar{F}_0(x) + \bar{F}_{10}(x)\, \bar{\theta} + \bar{F}_{01}(x)\, \theta + \bar{F}_{11}(x)\, \bar{\theta}\theta,$$

where bar over F_0, F_{10}, F_{01}, F_{11} stands for complex conjugation. Let X be an even degree vector field

$$X = X^{\mu}\ \frac{\partial}{\partial x^{\mu}} + \phi \frac{\partial}{\partial \theta} + \psi \frac{\partial}{\partial \bar{\theta}},$$

where every X^{μ} is an even degree function and ϕ, ψ are odd degree functions. Define the ternary graded bracket of three functions by

$$[F, G, H]_{\text{gr}} = F^{*}\ [G, H]_X + (-1)G^{*}\ [H, F]_X + (-1)H^{*}\ [F, G]_X, \tag{3.54}$$

where $[F, G]_X = F\ X(G) - (-1)^{|F||G|} G\ X(F)$.

Proposition 3.3 *The ternary graded bracket* (3.54) *for functions on a superspace* $\mathbb{R}^{n,2}$ *is a ternary graded Lie bracket if a vector field* X *has the form*

$$X = X^{\mu}\ \frac{\partial}{\partial x^{\mu}} + \phi \frac{\partial}{\partial \theta} - \phi^{*} \frac{\partial}{\partial \bar{\theta}},$$

where every function X^{μ} *satisfies the condition* $X_1^{\mu} = 0$ *(see the formula* (3.46)*) or, equivalently,* $(X^{\mu})^{*} = -X^{\mu}$.

Hence, the superalgebra of functions $C^{\infty}(\mathbb{R}^{n,2})$, endowed with the ternary graded Lie bracket (3.54), where an even degree vector field X satisfies the condition of Proposition 3.3, is an infinite dimensional ternary Lie superalgebra.

3.3.3 Classification of Low Dimensional 3-Lie Superalgebras

In this section we discuss a method that can be used to classify 3-Lie superalgebras, and afterwards the very same method is applied to give a classification of 3-Lie superalgebras of dimension $m|n$, where $m + n < 5$.

Definition 3.5 Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a n-Lie superalgebra, such that $\{e_1, e_2, \ldots, e_m\}$ and $\{f_1, f_2, \ldots, f_n\}$ span $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ respectively. Denote

$$\mathcal{B} = \{e_1, e_2, \ldots, e_m, f_1, f_2, \ldots, f_n\}.$$

Elements $K^{B}_{i_1 \ldots i_n}$ defined by

$$\left[b_{i_1}, \ldots, b_{i_n}\right] = K^{j}_{i_1 \ldots i_n} b_j,$$

where $b_{i_1}, \ldots, b_{i_n}, b_j \in \mathcal{B}$, are called *structure constants* of $\mathfrak{g}$ with respect to $\mathcal{B}$, and we say that the *super vector space dimension* of $\mathfrak{g}$ is $m|n$.

Assume $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ is a 3-Lie superalgebra and dimension of $\mathfrak{g}$ is $m|n$, that is, the dimensions of $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ are respectively m an d n. Denote

$$\mathcal{B} = \{e_1, e_2, \dots, e_m, f_1, f_2, \dots, f_n\}$$

and assume that e_α spans the even part of $\mathfrak{g}$ and f_i spans the odd part of g, where $1 \leq \alpha \leq m$ and $1 \leq i \leq m$. Moreover, let $b_i = e_i$, when $1 \leq i \leq m$, and let $b_i = f_{i-m}$ if $m < i \leq m+n$. As the bracket preserves gradings, then naturally

$$|[b_1, b_2, b_3]| = |b_1| + |b_2| + |b_3|,$$

which yields that we could use the structure constants to write down the values of brackets applied to generators of the algebra as follows

$$\begin{aligned}
\left[e_\alpha, e_\beta, e_\gamma\right] &= K^\lambda_{\alpha\beta\gamma} e_\lambda, \\
\left[e_\alpha, e_\beta, f_i\right] &= K^j_{\alpha\beta i} f_j, \\
\left[e_\alpha, f_i, f_j\right] &= K^\beta_{\alpha ij} e_\beta, \\
\left[f_i, f_j, f_k\right] &= K^l_{ijk} f_l,
\end{aligned} \tag{3.55}$$

where $\alpha \leq \beta \leq \gamma$ and $i \leq j \leq k$. As we can transform all other combinations of generators to one of these four variations listed above by applying the skew symmetric property of the bracket, then there is no need to take those other combinations into account.

By observing the left hand sides of relations (3.55) we can identify those combinations on which the bracket results as 0, or in other words, the combinations that are trivial in terms of the given 3-Lie superalgebra. To find those combinations of basis elements that give such result we need to check whether any permutation of the initial ordering yields the same bracket without preserving the sign. If this is the case, then naturally the bracket has to be equal to zero. Evidently

$$\begin{aligned}
[e_\alpha, e_\alpha, e_\alpha] &= -(-1)^{|e_\alpha||e_\alpha|}[e_\alpha, e_\alpha, e_\alpha] = -[e_\alpha, e_\alpha, e_\alpha], \\
\left[e_\alpha, e_\alpha, e_\beta\right] &= -(-1)^{|e_\alpha||e_\alpha|}\left[e_\alpha, e_\alpha, e_\beta\right] = -\left[e_\alpha, e_\alpha, e_\beta\right], \\
[e_\alpha, e_\alpha, f_i] &= -(-1)^{|e_\alpha||e_\alpha|}[e_\alpha, e_\alpha, f_i] = -[e_\alpha, e_\alpha, f_i],
\end{aligned}$$

meaning that triplets, regardless of the ordering, which always result in trivial bracket are

$$\{e_\alpha, e_\alpha, e_\alpha\}, \left\{e_\alpha, e_\alpha, e_\beta\right\}, \{e_\alpha, e_\alpha, f_i\},$$

where $\alpha \neq \beta, i \neq j, \alpha, \beta \in \{1, 2, \dots m\}$ and $i, j \in \{m+1, m+2, \dots, m+n\}$.

Our aim now is to put the graded Filippov-Jacobi identify into use. For that fix $1 \leq i \leq j \leq k \leq m+n, r, s \in \{1, 2, \dots, m+n\}$ and observe $\left[b_i, b_j, b_k\right] = K^l_{ijk} b_l$. Then nested bracket

$$\left[b_r, b_s, \left[b_i, b_j, b_k\right]\right] \tag{3.56}$$

can be calculated in two different ways. One way to do it is by direct computation using the information we already know, that is by plugging in the structure constants using the linearity of the bracket. This yields

$$\left[b_r, b_s, \left[b_i, b_j, b_k\right]\right] = K^l_{ijk}\left[b_r, b_s, b_l\right].$$

Now the remaining brackets on right hand side are to be reordered so that the indices of the triplets inside the brackets are nondecreasing. That is, transform $[b_r, b_s, b_l]$ to $(-1)^{\circlearrowleft_{rsl}}\left[b_{\hat{r}}, b_{\hat{s}}, b_{\hat{l}}\right]$, where $\{r, s, l\} = \left\{\hat{r}, \hat{s}, \hat{l}\right\}$, $\hat{r} \le \hat{s} \le \hat{l}$, and $(-1)^{\circlearrowleft_{rsl}}$ denotes the sign that comes from reordering elements in the bracket due to skew-symmetric properties. At this point we can express $\left[b_{\hat{r}}, b_{\hat{s}}, b_{\hat{l}}\right]$ yet again using structure constants as in $\left[b_{\hat{r}}, b_{\hat{s}}, b_{\hat{l}}\right] = K^t_{\hat{r}\hat{s}\hat{l}} b_t$.

Combining everything together we can see that

$$\left[b_r, b_s, \left[b_i, b_j, b_k\right]\right] = (-1)^{\circlearrowleft_{rsl}} K^l_{ijk} K^t_{\hat{r}\hat{s}\hat{l}} b_t.$$

Of course we can also apply Filippov-Jacobi identify to Eq. (3.56) to calculate the value. This gives us on the other hand

$$\begin{aligned}\left[b_r, b_s, \left[b_i, b_j, b_k\right]\right] = &\left[\left[b_r, b_s, b_i\right], b_j, b_k\right] + \\ &(-1)^{|b_i||b_r|+|b_i||b_s|}\left[b_i, \left[b_r, b_s, b_j\right], b_k\right] + \\ &(-1)^{|b_i||b_r|+|b_i||b_s|+|b_j||b_r|+|b_j||b_s|}\left[b_i, b_j, \left[b_r, b_s, b_k\right]\right].\end{aligned} \tag{3.57}$$

In Eq. (3.57) we can once again apply the algorithm described above to replace brackets in each summand with structure constants.

$$\left[b_r, b_s, \left[b_i, b_j, b_k\right]\right] =$$
$$K^t_{rsi} K^u_{jkt} b_u + (-1)^{|b_i b_r|+|b_i b_s|} K^t_{rsj} K^u_{ikt} b_u + (-1)^{|b_i b_r|+|b_i b_s|+|b_j b_r|+|b_j b_s|} K^t_{rsk} K^u_{ijt} b_u$$

Now let us use only such structure constants whose lower indices are in increasing order for consistency:

$$\begin{aligned}(-1)^{\circlearrowleft_{lrs}} K^l_{ijk} K_{\hat{l}\hat{r}\hat{s}^u} b_u = &(-1)^{\circlearrowleft_{rsi}+\circlearrowleft_{jkt}} K^t_{\hat{r}\hat{s}\hat{i}} K^u_{\hat{j}\hat{k}\hat{t}} b_u + \\ &(-1)^{\circlearrowleft_{rsj}+\circlearrowleft_{ikt}+|b_i b_r|+|b_i b_s|} K^t_{\hat{r}\hat{s}\hat{j}} K^u_{\hat{i}\hat{k}\hat{t}} b_u + \\ &(-1)^{\circlearrowleft_{rsk}+\circlearrowleft_{ijt}+|b_i b_r|+|b_i b_s|+|b_j b_r|+|b_j b_s|} K^t_{\hat{r}\hat{s}\hat{k}} K^u_{\hat{i}\hat{j}\hat{t}} b_u\end{aligned}$$

Since b_u are known structure constants, we are left with a set of quadratic equations for each $u \in \{1, 2, \ldots, m+n\}$

$$
\begin{aligned}
(-1)^{\circlearrowright_{lrs}} K^l_{ijk} K_{\hat{l}\hat{r}\hat{s}^u} = {} & (-1)^{\circlearrowright_{rsi}+\circlearrowright_{jkt}} K^t_{\hat{r}\hat{s}\hat{i}} K^u_{\hat{j}\hat{k}\hat{t}} + \\
& (-1)^{\circlearrowright_{rsj}+\circlearrowright_{ikt}+|b_i b_r|+|b_i b_s|} K^t_{\hat{r}\hat{s}\hat{j}} K^u_{\hat{i}\hat{k}\hat{t}} + \\
& (-1)^{\circlearrowright_{rsk}+\circlearrowright_{ijt}+|b_i b_r|+|b_i b_s|+|b_j b_r|+|b_j b_s|} K^t_{\hat{r}\hat{s}\hat{k}} K^u_{\hat{i}\hat{j}\hat{t}},
\end{aligned}
$$

where K^l_{ijk} are unknowns.

Altogether we are left with a system of quadratic equations whose solutions are structure constants of a $m|n$-dimensional 3-Lie superalgebra. However, as the structure constants are dependent of the choice of the basis of the underlying super vector space, these solutions are not unique up to isomorphism and duplicates need to be removed case-by-case.

In what follows, assume that super vector space $\mathfrak{g}$ is taken over filed $\mathbb{C}$. Now, if we put the above described algorithm in practice, following results emerge.

Theorem 3.8 *Let* $\mathfrak{g}$ *be a super vector space of dimension* $0|1$ *or* $1|1$. *Then 3-Lie superalgebra* $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ *is Abelian.*

Theorem 3.9 *Let* $\mathfrak{g}$ *be a super vector space of dimension* $0|2$ *or* $1|2$. *Then 3-Lie superalgebra* $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ *is either Abelian or it is isomorphic to 3-Lie superalgebra* $(\mathfrak{h}, [\cdot,\cdot,\cdot]_{\maltese})$ *whose non trivial brackets are either*

$$
\begin{cases}
[f_1, f_1, f_1] = -f_1 + f_2, \\
[f_1, f_1, f_2] = -f_1 + f_2, \\
[f_1, f_2, f_2] = -f_1 + f_2, \\
[f_2, f_2, f_2] = -f_1 + f_2,
\end{cases}
\quad \textit{or} \quad [f_1, f_1, f_1] = f_2,
$$

where f_1, f_2 *are odd generators of* $\mathfrak{h}$.

Theorem 3.10 *Let* $\mathfrak{g}$ *be a super vector space of dimension* $2|1$. *Then 3-Lie superalgebra* $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ *is either Abelian or it is isomorphic to 3-Lie superalgebra* $(\mathfrak{h}, [\cdot,\cdot,\cdot]_{\maltese})$ *whose non trivial brackets are either*

$$
\begin{cases}
[e_1, f_1, f_1] = -e_1 + e_2, \\
[e_2, f_1, f_1] = -e_1 - e_2
\end{cases}
\quad \textit{or} \quad [e_1, e_1, f_1] = f_1 \quad \textit{or} \quad [f_1, f_1, f_1] = f_1,
$$

where elements e_1, e_2 *and* f_1 *are respectively even and odd generators or* $\mathfrak{h}$.

3.4 Extension of Nambu Approach to Superspace

In this section, we show how to extend the Nambu approach to superspace. As is known, the generalization of Hamiltonian mechanics proposed by Nambu leads to the ternary bracket for functions on the phase space of odd dimension. This ternary bracket satisfies the Filippov – Jacobi identity, i.e. it defines the structure of a ternary

Lie algebra on a space of functions. Our extension of the Nambu approach to superspace leads to a family of ternary brackets on the algebra of even degree functions, parametrized by two odd degree functions.

We start with the superspace $\mathbb{R}^{3|2}$ with three real coordinates x, y, z and two Grassmann coordinates $\theta, \bar{\theta}$. In order to have compact notation for coordinates, we denote by r the collection of real coordinates x, y, z and by ξ the collection of Grassmann coordinates $\theta, \bar{\theta}$. The algebra of smooth functions on three dimensional space $\mathbb{R}^3$ will be denoted by $\mathfrak{C}$.

A smooth curve in the superspace $\mathbb{R}^{3|2}$ is given by $\alpha(t) = (r(t), \xi(t))$, where $\xi(t) = (\theta(t), \bar{\theta}(t))$, and

$$\begin{pmatrix} \theta(t) \\ \bar{\theta}(t) \end{pmatrix} = g(t) \begin{pmatrix} \theta \\ \bar{\theta} \end{pmatrix}, \tag{3.58}$$

where

$$g(t) = \begin{pmatrix} g_{11}(t) & g_{12}(t) \\ g_{21}(t) & g_{22}(t) \end{pmatrix}. \tag{3.59}$$

Then $\theta(t)\bar{\theta}(t) = \text{Det}(g(t))\, \theta\bar{\theta}$. Odd degree functions functions ϕ, ψ can be expand in Grassmann coordinates of superspace as follows

$$\phi(r, \xi) = \phi_1(r)\, \theta + \phi_2(r)\, \bar{\theta},$$
$$\psi(r, \xi) = \psi_1(r)\, \theta + \psi_2(r)\, \bar{\theta}.$$

The determinant of the second order matrix

$$\Psi(r) = \frac{\partial(\phi, \psi)}{\partial(\theta, \bar{\theta})} = \begin{pmatrix} \phi'_\theta & \phi'_{\bar{\theta}} \\ \psi'_\theta & \psi'_{\bar{\theta}} \end{pmatrix} = \begin{pmatrix} \phi_1(r) & \phi_2(r) \\ \psi_1(r) & \psi_2(r) \end{pmatrix}, \tag{3.60}$$

will be denoted by $\varDelta$. We will use $\frac{\partial(\phi,\psi)}{\partial(\theta,\bar{\theta})}$ to denote the matrix of partial derivatives of corresponding functions. The determinant of this matrix will be denoted by vertical lines. Hence

$$\varDelta = \left| \frac{\partial(\phi, \psi)}{\partial(\theta, \bar{\theta})} \right| = \begin{vmatrix} \phi'_\theta & \phi'_{\bar{\theta}} \\ \psi'_\theta & \psi'_{\bar{\theta}} \end{vmatrix} = \phi'_\theta\, \psi'_{\bar{\theta}} - \phi'_{\bar{\theta}}\, \psi'_\theta.$$

and we will assume that the functional matrix (3.60) is regular at any point r of $\mathbb{R}^3$, i.e. $\varDelta \neq 0$. It is useful to denote the algebra of second order matrices, whose entries are smooth functions on the three dimensional space $\mathbb{R}^3$, by $\text{Mat}_2(\mathfrak{C})$. Then the infinite dimensional matrices will be denoted by $\mathfrak{G}_2(\mathfrak{C})$, i.e.

$$\mathfrak{G}_2(\mathfrak{C}) = \{\Psi(r) \in \text{Mat}_2(\mathfrak{C}) : |\Psi(r)| \neq 0 \text{ at any point } r \in \mathbb{R}^3\}. \tag{3.61}$$

In [18] Nambu proposed a generalization of Hamilton mechanics for odd dimensional spaces. Particularly, he proposed the following system of equations in three dimensional space $\mathbb{R}^3$ (“Hamilton equations”)

$$\begin{aligned}\frac{dx}{dt} &= \left|\frac{\partial(H,G)}{\partial(x,y)}\right|,\\ \frac{dy}{dt} &= \left|\frac{\partial(H,G)}{\partial(y,z)}\right|,\\ \frac{dz}{dt} &= \left|\frac{\partial(H,G)}{\partial(z,x)}\right|,\end{aligned} \tag{3.62}$$

where H, G are two functions, which can be considered as Hamiltonians of a dynamical system. Then for any function F we have

$$\frac{dF}{dt} = \left|\frac{\partial(F,H,G)}{\partial(x,y,z)}\right|, \tag{3.63}$$

and the right hand side of this formula determines an analog of Poisson bracket in three dimensional space, which is now called Nambu-Poisson bracket.

We propose to extend this approach to the superspace $\mathbb{R}^{3|2}$ as follows: Consider a parametrized curve $\alpha(t) = (r(t), \xi(t))$ in the superspace $\mathbb{R}^{3|2}$, whose even degree part $r(t)$ is a solution of the system of equations

$$\begin{aligned}\frac{dx}{dt} &= \mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(y,z,\theta,\bar\theta)},\\ \frac{dy}{dt} &= \mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(z,x,\theta,\bar\theta)},\\ \frac{dz}{dt} &= \mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(x,y,\theta,\bar\theta)},\end{aligned} \tag{3.64}$$

where H, G are even degree functions, ϕ, ψ are odd degree functions and the right hand sides of the above equations are Berezinians. Comparing this extension of Nambu-Hamilton equations with the Nambu-Hamilton equations (3.62), we see that the structures of the right hand sides of the both systems of equations are very similar and the first difference is the Berezinian at the right hand side of (3.64), which replaces the determinant at the right hand sides of Nambu-Hamilton equations, i.e. when passing from space to superspace, we replace the determinant with its super analog, that is, the superdeterminant. Another difference is the appearance of odd degree functions ϕ, ψ. This is a peculiarity of the structure of superspace and we interpret these functions as additional parameters of the analogs of the Nambu-Hamilton equations that appear in superspace.

For instance

$$\mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(y,z,\theta,\bar\theta)} = \mathrm{Sdet}\frac{\partial(H,G,\phi,\psi)}{\partial(y,z,\theta,\bar\theta)} = \mathrm{Sdet}\begin{pmatrix} H'_y & H'_z & | & H'_\theta & H'_{\bar\theta}\\ G'_y & G'_z & | & G'_\theta & G'_{\bar\theta}\\ - & - & - & - & -\\ \phi'_y & \phi'_z & | & \phi'_\theta & \phi'_{\bar\theta}\\ \psi'_y & \psi'_z & | & \psi'_\theta & \psi'_{\bar\theta}\end{pmatrix}, \tag{3.65}$$

where Sdet stands for the superdeterminant of supermatrix and dotted lines show the structure of the supermatrix, i.e. they split the matrix into even degree and odd degree blocks. The elements of the upper-right block of this supermatrix are the right derivatives of functions H, G with respect to Grassmann variables $\theta, \bar{\theta}$, i.e.

$$H'_\theta = H\,\frac{\overleftarrow{\partial}}{\partial\theta},\quad H'_{\bar\theta} = H\,\frac{\overleftarrow{\partial}}{\partial\bar\theta},\quad G'_\theta = G\,\frac{\overleftarrow{\partial}}{\partial\theta},\quad G'_{\bar\theta} = G\,\frac{\overleftarrow{\partial}}{\partial\bar\theta}.$$

Thus according to the definition of superdeterminant [14] we have

$$\mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(y,z,\theta,\bar\theta)} = \Delta^{-1}\Big|\frac{\partial(H,G)}{\partial(y,z)} - \frac{\partial(H,G)}{\partial(\theta,\bar\theta)}\Big(\frac{\partial(\phi,\psi)}{\partial(\theta,\bar\theta)}\Big)^{-1}\frac{\partial(\phi,\psi)}{\partial(y,z)}\Big| \tag{3.66}$$

The odd degree part $\xi(t)$ of a curve α is a solution of the system of equations

$$\frac{d\theta}{dt} = \Delta^{-1}\Big(\Big|\frac{\partial(H,G)}{\partial(x,y)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(z,\bar\theta)}\Big| + \Big|\frac{\partial(H,G)}{\partial(y,z)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(x,\bar\theta)}\Big| + \Big|\frac{\partial(H,G)}{\partial(z,x)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(y,\bar\theta)}\Big|\Big), \tag{3.67}$$

$$\frac{d\bar\theta}{dt} = \Delta^{-1}\Big(\Big|\frac{\partial(H,G)}{\partial(x,y)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(z,\theta)}\Big| + \Big|\frac{\partial(H,G)}{\partial(y,z)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(x,\theta)}\Big| + \Big|\frac{\partial(H,G)}{\partial(z,x)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(y,\theta)}\Big|\Big), \tag{3.68}$$

The expression at the right-hand side of the Eq. (3.67) contains the determinants of matrices

$$\frac{\partial(\phi,\psi)}{\partial(z,\bar\theta)} = \begin{pmatrix}\phi'_z & \phi'_{\bar\theta}\\ \psi'_z & \psi'_{\bar\theta}\end{pmatrix},\quad \frac{\partial(\phi,\psi)}{\partial(y,\bar\theta)} = \begin{pmatrix}\phi'_y & \phi'_{\bar\theta}\\ \psi'_y & \psi'_{\bar\theta}\end{pmatrix},\quad \frac{\partial(\phi,\psi)}{\partial(x,\bar\theta)} = \begin{pmatrix}\phi'_x & \phi'_{\bar\theta}\\ \psi'_x & \psi'_{\bar\theta}\end{pmatrix}. \tag{3.69}$$

These matrices have no structure of supermatrices because their first columns consist of the odd degree elements while the second columns consist of the even degree elements. But determinants of these matrices are correctly defined because the elements of the main diagonal, as well as the elements of the secondary diagonal, commute. It is worth to mention that the values of these determinants are odd degree functions and this is consistent with the left hand side of (3.67), which is also the odd degree function. This also holds for the right hand side of the Eq. (3.68).

As it was mentioned before, we interpret odd degree functions ϕ, ψ in the Eqs. (3.64), (3.67), (3.68) as parameters of the system. It would be natural to expect that under certain conditions imposed on these parameters, we could obtain, as a special case of a system of Eqs. (3.64), (3.67), (3.68), the Nambu-Hamilton equations (3.62). It turns out that this is the case. In order to see this, we have to completely separate the system of functions H, G, ϕ, ψ according to the coordinates of superspace. To this end, we assume that two even degree functions (Hamiltonians)

H, G do not depend on Grassmann coordinates $\theta, \bar{\theta}$ and two odd degree functions $\phi(r,\xi), \psi(r,\xi)$ do not depend on real coordinates x, y, z of the superspace $\mathbb{R}^{3|2}$, i.e. we have

$$\phi(r,\xi) = \lambda_{11}\theta + \lambda_{12}\bar{\theta},$$
$$\psi(r,\xi) = \lambda_{21}\theta + \lambda_{22}\bar{\theta},$$

where λ_{ij} are real numbers. Then the matrix (3.60) takes the form

$$\Psi = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix},$$

i.e. it does not depend on a point $r \in \mathbb{R}^3$ and its determinant Δ is the non-zero real number. In this case the matrices (3.69) have zero column and their determinants vanish. Hence the right hand sides of Eqs. (3.67), (3.68) turn into zeros and we get $\theta'_t = \bar{\theta}'_t = 0$. Hence if $\alpha(t)$ is a solution of the system of Eqs. (3.64)–(3.68) in the case when odd degree functions $\phi(r,\xi), \psi(r,\xi)$ are constant functions in coordinates x, y, z, then Grassmann coordinates of solution $\alpha(t)$ do not depend on t and this solution can be considered as a parametrized curve $r(t)$ in the three dimensional space $\mathbb{R}^3$. Moreover, in this case the upper-right block of the supermatrix (3.65) is zero matrix and it follows immediately from the definition of superdeterminant (3.66) that the right-hand side of the first Eq. in (3.64) turns into ordinary determinant of the matrix $\frac{\partial(H,G)}{\partial(y,z)}$ with irrelevant numerical factor Δ^{-1}. The similar results hold in the case of the right hand sides of the second and third equations in (3.64). Thus the Eqs. (3.64) take on the form

$$\frac{dx}{dt} = \Delta^{-1}\left|\frac{\partial(H,G)}{\partial(y,z)}\right|, \quad \frac{dy}{dt} = \Delta^{-1}\left|\frac{\partial(H,G)}{\partial(z,x)}\right|, \quad \frac{dz}{dt} = \Delta^{-1}\left|\frac{\partial(H,G)}{\partial(x,y)}\right|, \tag{3.70}$$

and we see that in this particular case the system of Eqs. (3.64), (3.67), (3.68) reduces to the Nambu-Hamilton equations in three dimensional space (3.62). This gives us grounds to call the system of Eqs. (3.64), (3.67), (3.68) the generalization of Nambu-Hamilton equation in the superspace $\mathbb{R}^{3|2}$.

In order to write the generalization of Nambu-Hamilton equation in a more compact form we introduce the functions $\mathfrak{K}, \mathfrak{L}, \mathfrak{M}, \mathfrak{N}, \mathfrak{S}$, where $\mathfrak{K}, \mathfrak{L}, \mathfrak{M}$ are the functions at the right hand sides of the equations in (3.64) (from left to the right respectively), and $\mathfrak{N}, \mathfrak{S}$ are the right-hand sides of the Eqs. (3.67), (3.68) respectively. Thus

$$\mathfrak{K} = \text{Ber}\,\frac{(H,G,\phi,\psi)}{(y,z,\theta,\bar\theta)}, \qquad \mathfrak{L} = \text{Ber}\,\frac{(H,G,\phi,\psi)}{(z,x,\theta,\bar\theta)}, \qquad \mathfrak{M} = \text{Ber}\,\frac{(H,G,\phi,\psi)}{(x,y,\theta,\bar\theta)},$$
$$\mathfrak{R} = \frac{1}{\Delta^2}\Big(\Big|\frac{\partial(H,G)}{\partial(x,y)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(z,\bar\theta)}\Big| + \Big|\frac{\partial(H,G)}{\partial(y,z)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(x,\bar\theta)}\Big| + \Big|\frac{\partial(H,G)}{\partial(z,x)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(y,\bar\theta)}\Big|$$
$$\mathfrak{S} = \frac{1}{\Delta^2}\Big(\Big|\frac{\partial(H,G)}{\partial(x,y)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(z,\theta)}\Big| + \Big|\frac{\partial(H,G)}{\partial(y,z)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(x,\theta)}\Big| + \Big|\frac{\partial(H,G)}{\partial(z,x)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(y,\theta)}\Big|.$$

The right hand sides of the generalization of Nambu-Hamilton equation induce the even degree vector field on the superspace $\mathbb{R}^{3|2}$

$$\mathfrak{X} = \mathfrak{K}\,\frac{\partial}{\partial x} + \mathfrak{L}\,\frac{\partial}{\partial y} + \mathfrak{M}\,\frac{\partial}{\partial z} + \frac{\overleftarrow{\partial}}{\partial\theta}\,\mathfrak{R} + \frac{\overleftarrow{\partial}}{\partial\bar\theta}\,\mathfrak{S}. \tag{3.71}$$

It is worth to remind that the vector field induced by the right hand sides of Nambu-Hamilton equation is divergenceless [18] and this motivated Nambu to develop his approach, because the divergenceless of corresponding vector field is sufficient and necessary condition for Liouville theorem, which states that the volume of the flow generated by Hamiltonian vector field is constant in time. In analogy with Nambu-Hamilton equation (3.62) it can be shown by straightforward computations that the vector field of the generalization of Nambu-Hamilton equation (3.71) is also divergenceless in the superspace $\mathbb{R}^{3|2}$. Thus we have

$$\frac{\partial\mathfrak{K}}{\partial x} + \frac{\partial\mathfrak{L}}{\partial y} + \frac{\partial\mathfrak{M}}{\partial z} + \mathfrak{R}\,\frac{\overleftarrow{\partial}}{\partial\theta} + \mathfrak{S}\,\frac{\overleftarrow{\partial}}{\partial\bar\theta} = 0.$$

3.4.1 Extension of Nambu-Poisson Ternary Bracket to Superspace

The Nambu-Hamilton equations in three dimensional space $\mathbb{R}^3$ induce the ternary Nambu-Poisson bracket of smooth functions. This bracket is defined by means of the determinant of the matrix of partial derivatives of functions with respect to coordinates of $\mathbb{R}^3$. The Nambu-Poisson bracket is totally skew-symmetric, satisfies the Leibniz rule and the Filippov-Jacobi identity (Fundamental Identity) [20]. The aim of this section is to show that the generalization of Nambu-Hamilton equations (3.64), (3.67), (3.68) introduced in the previous section leads to ternary bracket of even degree functions, this ternary bracket depends on a pair of odd degree functions and can be defined by means of superdeterminant.

Let $F(r,\xi)$ be an even degree function, i.e. $F(r,\xi) = F_0(r) + F_1(r)\,\theta\bar\theta$. This function restricted to a curve $\alpha(t) = (x(t), y(t), z(t), \theta(t), \bar\theta(t))$, where

$$\theta(t) = f_{11}(t)\,\theta + f_{12}(t)\,\bar\theta, \;\; \bar\theta(t) = f_{21}(t)\,\theta + f_{22}(t)\,\bar\theta,$$

can be written in the form $F(t) = F_0(r(t)) + F_1(r(t))\,|f(t)|\,\theta\bar{\theta}$, where $|f(t)|$ is the determinant of the matrix (3.59). The derivative of this function can be written in the form

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} + F\,\frac{\overleftarrow{\partial}}{\partial\theta}\frac{d\theta}{dt} + F\,\frac{\overleftarrow{\partial}}{\partial\bar{\theta}}\frac{d\bar{\theta}}{dt}.$$

Indeed we have

$$\begin{aligned}\frac{dF}{dt} &= \frac{dF_0}{dt} + \frac{dF_1}{dt}\,|f(t)|\,\theta\bar{\theta} + F_1\,\frac{d}{dt}\Big(|f(t)|\Big)\,\theta\bar{\theta} \\ &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} - (F_1\,\bar{\theta}(t))\,\frac{d\theta}{dt} + (F_1\,\theta(t))\,\frac{d\bar{\theta}}{dt} \\ &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} + F\,\frac{\overleftarrow{\partial}}{\partial\theta}\frac{d\theta}{dt} + F\,\frac{\overleftarrow{\partial}}{\partial\bar{\theta}}\frac{d\bar{\theta}}{dt}. \end{aligned} \tag{3.72}$$

Next we assert that if $\alpha(t)$ is a solution of generalization of Nambu-Hamilton equation (3.64), (3.67), (3.68) in superspace then the derivative of any even degree function F can be expressed by means of Berezinian as follows

$$\frac{dF}{dt} = \mathrm{Ber}\,\frac{(F,H,G,\phi,\psi)}{(x,y,z,\theta,\bar{\theta})} = \mathrm{Sdet}\begin{pmatrix} F'_x & F'_y & F'_z & | & F'_\theta & F'_{\bar{\theta}} \\ H'_x & H'_y & H'_z & | & H'_\theta & H'_{\bar{\theta}} \\ G'_x & G'_y & G'_z & | & G'_\theta & G'_{\bar{\theta}} \\ - & - & - & - & - & - \\ \phi'_x & \phi'_y & \phi'_z & | & \phi'_\theta & \phi'_{\bar{\theta}} \\ \psi'_x & \psi'_y & \psi'_z & | & \psi'_\theta & \psi'_{\bar{\theta}} \end{pmatrix}. \tag{3.73}$$

This formula suggests that it is natural to introduce a new bracket, which can be considered as an analogue of the Nambu-Poisson ternary bracket [18, 20] in the superspace $\mathbb{R}^{3|2}$. We consider even degree functions F, H, G in (3.73) as arguments and two odd degree functions ϕ, ψ as parameters of this new ternary bracket. Evidently these two functions can be identified with the matrix $\Psi(r)$ (3.60). We denote this new ternary bracket by bold curly brackets and define it by

$$\{F,H,G\}_\Psi = \mathrm{Ber}\,\frac{(F,H,G,\phi,\psi)}{(x,y,z,\theta,\bar{\theta})}, \tag{3.74}$$

where F, H, G are even degree functions on the superspace $\mathbb{R}^{3|2}$ and Ψ shows dependence of ternary bracket on matrix $\Psi \in \mathfrak{G}_2(\mathfrak{C})$ associated to odd degree functions ϕ, ψ. Thus we have associated to each element Ψ of the infinite dimensional group of invertible matrices $\mathfrak{G}_2(\mathfrak{C})$ the ternary bracket (3.74) of even degree functions on the superspace $\mathbb{R}^{3|2}$, that is

$$\Psi \in \mathfrak{G}_2(\mathfrak{C}) \;\mapsto\; \{\ ,\ ,\ \}_\Psi.$$

Now our aim is to prove the formula (3.73). In order to simplify the form of formulae we introduce the following notations

$$\epsilon^{H,G}_{x,y}=\left|\frac{\partial(H,G)}{\partial(x,y)}\right|,\quad \epsilon^{H,G}_{y,z}=\left|\frac{\partial(H,G)}{\partial(y,z)}\right|,\quad \epsilon^{H,G}_{z,x}=\left|\frac{\partial(H,G)}{\partial(z,x)}\right|$$
$$\delta_{x,\theta}=\left|\frac{\partial(\phi,\psi)}{\partial(x,\theta)}\right|,\quad \delta_{y,\theta}=\left|\frac{\partial(\phi,\psi)}{\partial(y,\theta)}\right|,\quad \delta_{z,\theta}=\left|\frac{\partial(\phi,\psi)}{\partial(z,\theta)}\right|,$$
$$\delta_{x,\bar\theta}=\left|\frac{\partial(\phi,\psi)}{\partial(x,\bar\theta)}\right|,\quad \delta_{y,\bar\theta}=\left|\frac{\partial(\phi,\psi)}{\partial(y,\bar\theta)}\right|,\quad \delta_{z,\bar\theta}=\left|\frac{\partial(\phi,\psi)}{\partial(z,\bar\theta)}\right|.$$

Then the Berezinian of the supermatrix at the left-hand side of (3.73) can be written in the form of ordinary determinant

$$\Delta^{-1}\begin{vmatrix} F'_x-F'_\theta\frac{\delta_{x,\bar\theta}}{\Delta}+F'_{\bar\theta}\frac{\delta_{x,\theta}}{\Delta} & F'_y-F'_\theta\frac{\delta_{y,\bar\theta}}{\Delta}+F'_{\bar\theta}\frac{\delta_{y,\theta}}{\Delta} & F'_z-F'_\theta\frac{\delta_{z,\bar\theta}}{\Delta}+F'_{\bar\theta}\frac{\delta_{z,\theta}}{\Delta} \\ H'_x-H'_\theta\frac{\delta_{x,\bar\theta}}{\Delta}+H'_{\bar\theta}\frac{\delta_{x,\theta}}{\Delta} & H'_y-H'_\theta\frac{\delta_{y,\bar\theta}}{\Delta}+H'_{\bar\theta}\frac{\delta_{y,\theta}}{\Delta} & H'_z-H'_\theta\frac{\delta_{z,\bar\theta}}{\Delta}+H'_{\bar\theta}\frac{\delta_{z,\theta}}{\Delta} \\ G'_x-G'_\theta\frac{\delta_{x,\bar\theta}}{\Delta}+G'_{\bar\theta}\frac{\delta_{x,\theta}}{\Delta} & G'_y-G'_\theta\frac{\delta_{y,\bar\theta}}{\Delta}+G'_{\bar\theta}\frac{\delta_{y,\theta}}{\Delta} & G'_z-G'_\theta\frac{\delta_{z,\bar\theta}}{\Delta}+G'_{\bar\theta}\frac{\delta_{z,\theta}}{\Delta}\end{vmatrix}, \tag{3.75}$$

where $H'_\theta, H'_{\bar\theta}, G'_\theta, G'_{\bar\theta}$ are right derivatives. If we expand this determinant along the first row we get

$$\begin{aligned} F'_x\,\mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(y,z,\theta,\bar\theta)}&+F'_y\,\mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(z,x,\theta,\bar\theta)}+F'_z\,\mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(x,y,\theta,\bar\theta)}\\ &+F'_\theta\,\frac{1}{\Delta^2}(\epsilon^{H,G}_{y,z}\,\delta_{x,\bar\theta}+\epsilon^{H,G}_{z,x}\,\delta_{y,\bar\theta}+\epsilon^{H,G}_{z,x}\,\delta_{y,\bar\theta})+F'_{\bar\theta}\,\frac{1}{\Delta^2}(\epsilon^{H,G}_{y,z}\,\delta_{x,\theta}\\ &+\epsilon^{H,G}_{z,x}\,\delta_{y,\theta}+\epsilon^{H,G}_{z,x}\,\delta_{y,\theta}). \end{aligned} \tag{3.76}$$

Now making use of the system of Eqs. (3.64), (3.67), (3.68) and the Eq. (3.72), we get the Eq. (3.73).

Every column of the determinant (3.75) is the linear combination of five columns

$$\mathfrak{R}_x=\begin{pmatrix}F'_x\\H'_x\\G'_x\end{pmatrix},\ \mathfrak{R}_y=\begin{pmatrix}F'_y\\H'_y\\G'_y\end{pmatrix},\ \mathfrak{R}_z=\begin{pmatrix}F'_z\\H'_z\\G'_z\end{pmatrix},$$
$$\mathfrak{R}_\theta=\begin{pmatrix}F'_\theta\\H'_\theta\\G'_\theta\end{pmatrix},\ \ \mathfrak{R}_{\bar\theta}=\begin{pmatrix}F'_{\bar\theta}\\H'_{\bar\theta}\\G'_{\bar\theta}\end{pmatrix}, \tag{3.77}$$

with corresponding coefficients. Hence we can write the determinant (3.75) in the form

$$\Delta^{-1}\left|\mathfrak{R}_x-\mathfrak{R}_\theta\frac{\delta_{x,\bar\theta}}{\Delta}+\mathfrak{R}_{\bar\theta}\frac{\delta_{x,\theta}}{\Delta}\quad \mathfrak{R}_y-\mathfrak{R}_\theta\frac{\delta_{y,\bar\theta}}{\Delta}+\mathfrak{R}_{\bar\theta}\frac{\delta_{y,\theta}}{\Delta}\quad \mathfrak{R}_z-\mathfrak{R}_\theta\frac{\delta_{z,\bar\theta}}{\Delta}+\mathfrak{R}_{\bar\theta}\frac{\delta_{z,\theta}}{\Delta}\right|. \tag{3.78}$$

Now using the properties of ordinary determinant and taking all possible combinations of columns, we can write the determinant (3.78) as the sum of determinants, where every determinant is determined by a corresponding combination of columns (3.77). It follows from the property $\theta^2 = \bar{\theta}^2 = 0$ of Grassmann coordinates that determinant of a combination of columns, which includes at least two columns $\mathfrak{R}_\theta, \mathfrak{R}_{\bar{\theta}}$, vanishes. Altogether we have seven non-trivial combinations of columns (i.e. the determinant of this combination of columns does not vanish), which give the following expression for the ternary bracket (3.74)

$$\begin{aligned}\{\!\!\{F, H, G,\}\!\!\}_\Psi = \frac{1}{\Delta}\,|\mathfrak{R}_x\ \ \mathfrak{R}_y\ \ \mathfrak{R}_z| \\ -\frac{1}{\Delta^2}\Big(|\mathfrak{R}_x\ \ \mathfrak{R}_y\ \ \mathfrak{R}_\theta|\,\delta_{z,\bar{\theta}} + |\mathfrak{R}_x\ \ \mathfrak{R}_\theta\ \ \mathfrak{R}_z|\,\delta_{y,\bar{\theta}} + |\mathfrak{R}_\theta\ \ \mathfrak{R}_y\ \ \mathfrak{R}_z|\,\delta_{x,\bar{\theta}} \\ -|\mathfrak{R}_x\ \ \mathfrak{R}_y\ \ \mathfrak{R}_{\bar{\theta}}|\,\delta_{z,\theta} - |\mathfrak{R}_x\ \ \mathfrak{R}_{\bar{\theta}}\ \ \mathfrak{R}_z|\,\delta_{y,\theta} - |\mathfrak{R}_{\bar{\theta}}\ \ \mathfrak{R}_y\ \ \mathfrak{R}_z|\,\delta_{x,\theta}\Big).\end{aligned} \tag{3.79}$$

The first term at the right-hand side of the above relation is the usual Nambu-Poisson ternary bracket of even degree functions F, H, G

$$\{F, H, G\} = |\mathfrak{R}_x\ \ \mathfrak{R}_y\ \ \mathfrak{R}_z| = \begin{vmatrix} F'_x & F'_y & F'_z \\ H'_x & H'_y & H'_z \\ G'_x & G'_y & G'_z \end{vmatrix}. \tag{3.80}$$

In addition to the usual Nambu-Poisson ternary bracket, the expression at the right-hand side of relation (3.79) also includes terms enclosed in parentheses. These terms depend on the derivatives of odd degree functions ϕ, ψ. This suggests us to introduce one more ternary bracket as follows

$$\begin{aligned}\{F, H, G\}_\Psi = \left|\frac{\partial(F,H,G)}{\partial(x,y,\theta)}\right|\left|\frac{\partial(\phi,\psi)}{\partial(z,\bar{\theta})}\right| + \left|\frac{\partial(F,H,G)}{\partial(x,\theta,z)}\right|\left|\frac{\partial(\phi,\psi)}{\partial(y,\bar{\theta})}\right| \\ +\left|\frac{\partial(F,H,G)}{\partial(\theta,y,z)}\right|\left|\frac{\partial(\phi,\psi)}{\partial(x,\bar{\theta})}\right| - \left|\frac{\partial(F,H,G)}{\partial(x,y,\bar{\theta})}\right|\left|\frac{\partial(\phi,\psi)}{\partial(z,\theta)}\right| \\ -\left|\frac{\partial(F,H,G)}{\partial(x,\bar{\theta},z)}\right|\left|\frac{\partial(\phi,\psi)}{\partial(y,\theta)}\right| - \left|\frac{\partial(F,H,G)}{\partial(\bar{\theta},y,z)}\right|\left|\frac{\partial(\phi,\psi)}{\partial(x,\theta)}\right|.\end{aligned} \tag{3.81}$$

It should be noted that the order of cofactors in every product at the right-hand side of (3.81) is important, because cofactors are odd degree functions.

Now we can express the ternary bracket (3.74) as the sum of the usual Nambu-Poisson bracket (3.80) and the ternary bracket (3.81). Hence

$$\{\!\!\{F, H, G\}\!\!\}_\Psi = \frac{1}{\Delta}\,\{F, H, G\} - \frac{1}{\Delta^2}\,\{F, H, G\}_\Psi. \tag{3.82}$$

The formula (3.82) gives grounds to consider the ternary bracket (3.74) introduced by means of superdeterminant as an extension of usual Nambu-Poisson bracket to the superspace $\mathbb{R}^{3|2}$. It can be proved that this extension preserves all the algebraic properties of the Nambu-Poisson bracket such as skew-symmetry, the Leibniz rule and the Filippov-Jacobi identity (Fundamental Identity) [4, 5].

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